Chapter 2

The Space of Hausdorff Continuous Interval Valued Functions

2.1 Introduction

Historically, the interval analysis, or the analysis of interval valued functions, is associated with the so called validated computing where algorithms generating validated bounds for the exact solutions of mathematical problems are designed and investigated, see [2] and [81]. The Hausdorff continuous functions, being a particular class of interval valued functions, belong to interval analysis, see Moore [107]. However, interest in the interval valued functions comes also from other branches of mathematics such as nonlinear partial differential equations, see [13] which strengthens the results in [109], and approximation theory. In fact, Hausdorff continuous functions of one variable were introduced first by Sendov in connection with Hausdorff approximations of real functions of real argument, see [120]. The concept was further developed in [10] as part of the analysis of interval valued functions and extended to interval valued functions defined on a topological space [4]. The name Hausdorff continuous is due to a characterization of these functions in terms of the Hausdorff distance between the graphs of real functions as defined in [119]. Recently, it is shown by Anguelov and Rosinger [12] that very large classes of nonlinear partial differential equations have solutions which can be assimilated with Hausdorff continuous functions. Further applications of the concept of Hausdorff continuity are presented in [10]. For recent advances in theory of Hausdorff approximations, see [3] and the references therein.

This chapter will serve as an introduction to the Hausdorff continuous functions and the stated results will be used in the sequel. The Baire operators and the graph completion operator which are instrumental for the definition and the properties of Hausdorff continuous functions are discussed in Section 2.2. Section 2.3 deals with Hausdorff continuous functions. Two very important theorems are presented. The first theorem shows how to construct Hausdorff continuous functions, and the second gives useful necessary and sufficient conditions for an interval valued function to be Hausdorff continuous function. In Section 2.4, Theorem 2.4.1 shows that the set of Hausdorff continuous functions is Dedekind order complete while Theorem 2.4.2 gives a useful representation of supremum of a subset of a set of all Hausdorff continuous functions. The generalized lower, upper Baire and graph completion operators are given in Section 2.5.

2.2 Baire Operators and Graph Completion Operator

In this section, we recall the upper Baire operator S, lower Baire operator I and the graph completion operator F for interval valued function. Some properties of I, S, and F are given in this section.

Let X be an arbitrary topological space.

Identifying $a \in \mathbb{R}$ with the point interval $[a,\ a] \in \mathbb{IR}$, we consider $\mathbb{R}$ as a subset of $\mathbb{IR}$. From this, it follows that the set $\mathbb{A}(X)$ contains the set of functions with real $\mathcal{A}(X)$. Hence,

$$C(X) \subseteq \mathcal{A}(X) \subseteq \mathbb{A}(X).$$

A partial order which extends the total order on $\mathbb{R}$ can be defined on $\mathbb{IR}$ in more than one way. However, it will prove useful to consider on $\mathbb{IR}$ the partial order $\leq$ defined by

$$[\underline{a},\ \overline{a}] \leq [\underline{b},\ \overline{b}] \Longleftrightarrow \underline{a} \leq \underline{b},\ \overline{a} \leq \overline{b}. \tag{2.2.1}$$

The partial order, given in (2.2.1), is introduced and studied by Markov, see [99, 100]. The inclusion in $\mathbb{A}(X)$ is defined by

$$[\underline{a}, \overline{a}] \subseteq [\underline{b}, \overline{b}] \Longleftrightarrow \underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}. \tag{2.2.2}$$

The partial order induced in $\mathbb{A}(X)$ by (2.2.1) in a point-wise way, i.e., for $u, v \in \mathbb{A}(X)$,

$$u \leq v \Longleftrightarrow u(x) \leq v(x), \ x \in X, \tag{2.2.3}$$

is an extension of the usual point-wise order in the set of real valued functions $\mathcal{A}(X)$. Let $u \in \mathbb{A}(X)$. For every $x \in X$, the value of u is interval $[\underline{u}(x), \overline{u}(x)] \in \mathbb{IR}$. Hence, the function u can be written in the form $u = [\underline{u}, \overline{u}]$, where $\underline{u}$, $\overline{u} \in \mathcal{A}(X)$ and $\underline{u}(x) \leq \overline{u}(x)$.

The inclusion induced in $\mathbb{A}(X)$ by (2.2.2) in a point-wise way is given by

$$u \subseteq v \Leftrightarrow \underline{v}(x) \leq \underline{u}(x) \leq \overline{u}(x) \leq \overline{v}(x), \ x \in X, \tag{2.2.4}$$

where $u = [\underline{u}, \overline{u}] \in \mathbb{A}(X)$ and $v = [\underline{v}, \overline{v}] \in \mathbb{A}(X)$.

The definition of upper semicontinuous envelope, the lower semicontinuous envelope given in (1.3.1) and (1.3.2), respectively, for $u \in \mathcal{A}(X)$, can be extended to functions $u = [\underline{u}, \overline{u}] \in \mathbb{A}(X)$ as follows [4]:

Definition 2.2.1 *The mappings $S, I : \mathbb{A}(X) \to \mathcal{A}(X)$, defined by*

$$S(u)(x) = \inf_{\delta>0} \sup\{z \in u(y) : y \in B_\delta(x) \cap X\}, \ x \in X, \tag{2.2.5}$$
$$I(u)(x) = \sup_{\delta>0} \inf\{z \in u(y) : y \in B_\delta(x) \cap X\}, \ x \in X, \tag{2.2.6}$$

are called upper Baire and lower Baire operators, respectively.

In [15], the mappings (2.2.5) and (2.2.6) were defined and studied in the particular case of functions $u \in \mathcal{A}(X)$ when X is a subset of $\mathbb{R}$.

The lower and upper Baire operators of $u = [\underline{u}, \overline{u}] \in \mathbb{A}(X)$ can be conveniently represented in terms of the functions $\underline{u}$ and $\overline{u}$. Indeed, from (2.2.5) and (2.2.6), it is easy to see that

$$I(u) = I(\underline{u}), \ S(u) = S(\overline{u}). \tag{2.2.7}$$

Clearly, for every $u \in \mathbb{A}(X)$, we have

$$I(u)(x) \leq u(x) \leq S(u)(x), \ x \in X. \tag{2.2.8}$$

Hence, the graph completion operator is defined on $\mathbb{A}(X)$ as follows [4]:

Definition 2.2.2 *The mapping*

$$F : \mathbb{A}(X) \to \mathbb{A}(X),$$

defined by

$$F(u)(x) := [I(u)(x),\ S(u)(x)],\ u \in \mathbb{A}(X),\ x \in X,$$

is called a graph completion operator.

The name of this operator, given by Sendov [119], is derived from the fact that, considering the graphs of u and $F(u)$ as subsets of the topological space $X \times \mathbb{R}$, the graph of $F(u)$ is the minimal closed set, which is a graph of interval valued function on X and contains the graph of u.

For every $u = [\underline{u}, \overline{u}] \in \mathbb{A}(X)$, the property (2.2.7) implies that $F(u)$ can be written in the form

$$F(u) = [I(\underline{u}), S(\overline{u})], \tag{2.2.9}$$

thus

$$F(u) = u \Leftrightarrow \underline{u} = I(\underline{u}),\ \overline{u} = S(\overline{u})].$$

In other words, $u = [\underline{u}, \overline{u}]$ is a fixed point of the operator F if and only if $\underline{u}$ is a fixed point of the operator I, while $\overline{u}$ is a fixed point of the operator S.

By (2.2.9) and (2.2.8), we have

$$u(x) \subseteq F(u)(x),\ x \in X.$$

There is an alternative characterization of lower and upper semicontinuity that involves the fixed points of the lower and upper Baire operators for $u \in \mathcal{A}(X)$. Indeed, we have

$$u \in LSC(X) \Leftrightarrow I(u) = u, \tag{2.2.10}$$

$$u \in USC(X) \Leftrightarrow S(u) = u. \tag{2.2.11}$$

In the sequel, we will use the following properties of operators I, S, and F, which were proved in [4].

Theorem 2.2.1 *(a) The operators I, S, and F are all monotone increasing with respect to the partial order* (2.2.3)*, that is, for any two functions u, $v \in \mathbb{A}(X)$,*

$$u \leq v \Longrightarrow I(u) \leq I(v),\ S(u) \leq S(v),\ F(u) \leq F(v). \tag{2.2.12}$$

(b) The operator F is monotone with respect to the relation inclusion (2.2.4), that is, for any two functions u, $v \in \mathbb{A}(X)$,

$$u(x) \subseteq v(x),\ x \in X \Rightarrow F(u)(x) \subseteq F(v)(x),\ x \in X.$$

(c) The operators I, S and F are all idempotent, that is, for any function $u \in \mathbb{A}(X)$, we have

$$I(I(u)) = I(u),\ S(S(u)) = S(u),\ F(F(u)) = F(u). \tag{2.2.13}$$

The property (c) of Theorem 2.2.1 and the properties (2.2.10) and (2.2.11) imply that for $u \in \mathbb{A}(X)$, we have

$$I(u) \in LSC(X), \tag{2.2.14}$$

$$S(u) \in USC(X). \tag{2.2.15}$$

Upon an obvious extension of the respective result in [15] we have the following lemma about semicontinuous functions.

Lemma 2.2.1 *We have the following:*
(i) Let $\mathcal{L} \subset LSC(X)$, $\mathcal{L} \neq \emptyset$ be bounded from above at each $x \in \Omega$. Then the function l defined by

$$l(x) = \sup\{v(x) : v \in \mathcal{L}\}$$

is lower semicontinuous.
(ii) Let $\mathcal{U} \subset USC(X)$, $\mathcal{U} \neq \emptyset$ be bounded from below at each $x \in \Omega$. Then the function u defined by

$$u(x) = \inf\{v(x) : v \in \mathcal{U}\}$$

is upper semicontinuous.

2.3 Hausdorff Continuous Functions

This section introduces the concept of Hausdorff continuous interval valued functions and discusses their properties.

Definition 2.3.1 *A function $u \in \mathbb{A}(X)$ is called Hausdorff continuous, or for short, H-continuous, if and only if for every function $v \in \mathbb{A}(X)$, we have satisfied the following minimality condition on u*

$$v(x) \subseteq u(x),\ x \in X \Rightarrow F(v)(x) = u(x),\ x \in X. \tag{2.3.1}$$

The minimality condition in (2.3.1) with respect to their graph completion operator F plays a fundamental role. At first it may appear that it applies to each individual point x of X, not involving neighborhoods. However, the graph completion operator F does appear in this condition and this operator according to definition of F and therefore to definition of I and S does certainly refer to neighborhoods of points in X, a situation typical, among others, for the concept of continuity.

We recall here the concept of segment continuity associated with the graph completion operator, [120].

Definition 2.3.2 *A function $u \in \mathbb{A}(X)$ is called segment continuous, or S-continuous, if $F(u) = u$.*

It follows from the idempotence of a function F, see (2.2.13), that for any function $u \in \mathbb{A}(X)$

$$F(u) \text{ is an S-continuous function.} \tag{2.3.2}$$

Furthermore,

$$u \text{ is S-continuous } \Leftrightarrow F(u) = u. \tag{2.3.3}$$

It is easy to see that on the set $\mathbb{A}(X)$

$$\text{H-continuity } \Rightarrow \text{ S-continuity.} \tag{2.3.4}$$

Indeed, if u is H-continuous from the inclusion $u(x) \subseteq u(x)$, $x \in X$, and Definition 2.3.1, it follows that $F(u)(x) = u(x)$, $x \in X$. Thus u is S-continuous on X.

The inverse implication in (2.3.4) is not true. Indeed, consider the example when $X = [0, 1]$. Then, the function u, defined by

$$u(x) = \begin{cases} [-1, 1], \ x = 0 \\ 1 \qquad\quad , \ 0 < x \leq 1, \end{cases}$$

is S-continuous on $[0, 1]$, but it is not H-continuous on $[0, 1]$. Indeed, it is clear that $F(u)(x) = u(x), x \in [0, 1]$. Now consider interval valued function v defined on $[0, 1]$ by

$$v(x) = \begin{cases} [0, 1], \ x = 0 \\ 1 \qquad , \ 0 < x \leq 1. \end{cases}$$

Clearly, the inclusion $v(x) \subseteq u(x)$ holds for every $x \in [0, 1]$ and v is S-continuous on $[0, 1]$ because $F(v)(x) = v(x), x \in [0, 1]$. But $F(v)(0) \neq u(0)$ so u cannot be H-continuous function on $[0, 1]$.

However, through the concept of S-continuity, H-continuous functions can be characterized in the following way, [120].

Theorem 2.3.1 *A function $u \in \mathbb{A}(X)$ is H-continuous if and only if u satisfies the following two conditions:*
(i) u is S-continuous on X,
(ii) for every S-continuous function v, the inclusion $v(x) \subseteq u(x)$, $x \in X$ implies $v(x) = u(x)$, $x \in X$.

In general, H-continuity is not preserved on subsets of the domain X. Precisely, if $K \subset X$ and $u \in \mathbb{H}(X)$, then we cannot conclude that $u|_K \in \mathbb{H}(K)$, where $u|_K$ is the restriction of u to K. For example, consider $X = \mathbb{R}, K = [0, +\infty)$, and u defined by

$$u(x) = \begin{cases} -1 \qquad , x < 0 \\ [-1, 1], \ x = 0 \\ 1 \qquad\quad , x > 0. \end{cases} \tag{2.3.5}$$

Obviously, $u \in \mathbb{H}(\mathbb{R})$ and $u|_K$ is defined on K by

$$u|_K(x) = \begin{cases} [-1, 1], \ x = 0 \\ 1 \qquad\quad , \ x > 0. \end{cases}$$

To prove that $u|_K \notin \mathbb{H}(K)$, we consider the interval valued function w defined on K by

$$w(x) = \begin{cases} [0, 1], \ x = 0 \\ 1 \qquad , x > 0. \end{cases}$$

Clearly, the inclusion $w(x) \subseteq u|_K(x)$ holds for every $x \in K$ and w is S-continuous on K. Therefore $F(w)(x) = w(x), x \in K$. But $w(0) \neq u|_K(0)$, so $u|_K$ cannot be H-continuous function in K although $K \subset X$.

We know that every S-continuous function is defined in terms of a lower semicontinuous function and an upper semicontinuous function, but there is more.

The following theorem shows how to construct an H-continuous function.

Theorem 2.3.2 [4] *Every pair of a lower semicontinuous function $\underline{u}$ and an upper semicontinuous function $\overline{u}$ such that $\underline{u} \leq \overline{u}$ on X defines an S-continuous function $u(x) = [\underline{u}(x), \overline{u}(x)]$, $x \in X$. Furthermore, if the set*

$$\{\varphi \in \mathbb{A}(X) : \underline{u} \leq \varphi \leq \overline{u}\}$$

does not contain any lower or upper semicontinuous functions, except for $\underline{u}$, respectively $\overline{u}$, then the function u is H-continuous.

The minimality condition given in the above theorem associated with the Hausdorff continuous functions can also be formulated in terms of semicontinuous functions, namely, if $u = [\underline{u}, \overline{u}]$ is S-continuous, then u is H-continuous if and only if

$$\{\varphi \in \mathcal{A}(\Omega) : \varphi \text{ is semicontinuous}, \underline{u} \leq \varphi \leq \overline{u}\} = \{\underline{u}, \overline{u}\}.$$

Theorem 2.3.2 and property (2.3.4) imply that

$$C(X) = USC(X) \cap LSC(X) \subset \mathbb{H}(X) \subset \mathbb{F}(X).$$

Let us note that the set $\mathbb{H}(X)$ is certainly wider than $C(X)$. An example of H-continuous function which is not continuous is given in (2.3.5).

The concept of H-continuity can be considered as a generalization of the concept of continuity of real functions in the sense that the only real (point valued) functions contained in $\mathbb{H}(X)$ are the continuous functions, that is,

$$u \in \mathcal{A}(X) \cap \mathbb{H}(X) \Rightarrow u \in C(X).$$

The following theorem, proved in [4], generalizes some results discussed in [120] to the case when $X \subseteq \mathbb{R}$. It gives useful necessary and sufficient conditions for an interval valued function to be H-continuous.

Theorem 2.3.3 *Let* $u = [\underline{u}, \overline{u}] \in \mathbb{A}(X)$*. The following conditions are equivalent :*

(a) $u \in \mathbb{H}(X)$*,*

(b) $F(\overline{u}) = F(\underline{u}) = u$*,*

(c) $S(\underline{u}) = \overline{u}$*,* $I(\overline{u}) = \underline{u}$*, and* $u \in \mathbb{F}(X)$*.*

With every interval function u one can associate H-continuous functions as stated in the next theorem which is proved in [4].

Theorem 2.3.4 *Let* $u \in \mathbb{A}(X)$*. Both functions* $F(S(I(u)))$ *and* $F(I(S(u)))$ *are Hausdorff continuous and we have*

$$F(S(I(u))) \leq F(I(S(u))).$$

This theorem is illustrated by the following example.

Example 2.3.1 *Consider the function* $u \in \mathbb{A}(\mathbb{R})$ *given by*

$$u(x) = \begin{cases} [-1, 1] & , \quad x \in \mathbb{Z} \\ 0 & , \quad x \in (-\infty, 0) \setminus \mathbb{Z} \\ [0, 1] & , \quad x \in (0, +\infty) \setminus \mathbb{Z}, \end{cases}$$

where $\mathbb{Z}$ *denotes the set of integers. We have* $F(u) = u$ *meaning that* u *is S-continuous. We have the H-continuous functions*

$$F(S(I(u)))(x) = 0, x \in \mathbb{R}$$

and

$$F(I(S(u)))(x) = \begin{cases} 0 & , x \in (-\infty, 0) \\ [0, 1], & x = 0 \\ 1 & , x \in (0, +\infty). \end{cases}$$

It is quite interesting that pairing a lower semicontinuous function $\underline{u}$ with an upper semi-continuous function $\overline{u}$ such that $\underline{u} \leq \overline{u}$ produces a completely new concept from both algebraic and topological points of view, namely the concept of S-continuous interval functions. The concept of Hausdorff continuity is closely connected with the Hausdorff distance between functions as introduced by Sendov [120]. The Hausdorff distance $\rho(u, v)$ between two functions $u,\ v \in \mathbb{A}(X)$ is defined as the Hausdorff distance between the graphs of the functions $F(u)$ and $F(v)$ considered as subsets of $\mathbb{R}^{n+1}$. More precisely, we have

$$\begin{aligned}\rho(u, v) &= \max\{ \sup_{x_1 \in X} \sup_{y_1 \in F(u)(x_1)} \inf_{x_2 \in X} \inf_{y_2 \in F(v)(x_2)} ||(x_1 - x_2, y_1 - y_2)||, \\ &\sup_{x_2 \in X} \sup_{y_2 \in F(v)(x_2)} \inf_{x_1 \in X} \inf_{y_1 \in F(u)(x_1)} ||(x_1 - x_2, y_1 - y_2)||\}, \end{aligned} \tag{2.3.6}$$

where $||.||$ is a given norm in $\mathbb{R}^{n+1}$.

Condition (b) in Theorem 2.3.3 implies that for any H-continuous function $u = [\underline{u}, \overline{u}]$, the Hausdorff distance between the functions $\underline{u}$ and $\overline{u}$ is zero. More precisely, we have

$$u = [\underline{u}, \overline{u}] \in \mathbb{H}(X) \Leftrightarrow \begin{cases} u \in \mathbb{F}(X) \\ \rho(\underline{u}, \overline{u}) = 0. \end{cases}$$

We should note that any Hausdorff continuous function is "essentially" point valued in the sense that it assumes point values everywhere except on a small set. The next theorem shows that this set is a set of first Baire category, that is, a countable union of closed and nowhere dense sets.

Theorem 2.3.5 [4] *Let $u = [\underline{u}, \overline{u}]$ be an H-continuous function on X. The set*

$$W_u = \{x \in X : \overline{u}(x) - \underline{u}(x) > 0\} \tag{2.3.7}$$

is of first Baire category.

Through an application of the Baire category theorem [132], the above theorem implies that the complement of W_u in X is a set of second category. Hence,

$$D_u = X \setminus W_u = \{x \in X : \overline{u}(x) = \underline{u}(x)\} \text{ is dense in } X.$$

Since a finite or countable union of sets of first Baire category is also a set of first Baire category [130], we have that for every finite or countable set $\mathcal{F}$ of Hausdorff continuous functions, the set

$$\begin{aligned} D_{\mathcal{F}} &= \{x \in X : \underline{u}(x) = \overline{u}(x),\ u = [\underline{u}, \overline{u}] \in \mathcal{F}\} \\ &= X \setminus \left(\bigcup_{u \in \mathcal{F}} W_u \right) \end{aligned}$$

is dense in X.

In the following theorem, it is shown that for H-continuous functions interval values are used in an 'economical' way, namely only at points of discontinuity.

Theorem 2.3.6 [5] *Let* $u = [\underline{u}, \overline{u}]$ *be an H-continuous function on* X.
(a) If $\underline{u}$ *or* $\overline{u}$ *is continuous at a point* $a \in X$, *then* $\underline{u}(a) = \overline{u}(a)$.
(b) If $\underline{u}(a) = \overline{u}(a)$ *for some* $a \in X$, *then both* $\underline{u}$ *and* $\overline{u}$ *are continuous at* a.

The above theorem implies that for every $u \in \mathbb{A}(X)$ the set W_u defined by (2.3.7) has the following representations:

$$\begin{aligned} W_u &= \{x \in X : I(u) \text{ is discontinuous at } x\} \\ &= \{x \in X : S(u) \text{ is discontinuous at } x\}. \end{aligned}$$

In important similarity between continuous, and on other hand, H-continuous functions is that both of them are determined uniquely if they are known a dense subset of their domains. This property comes in spite of the fact that, as seen above, H-continuous functions can have discontinuities on sets of first Baire category, and such sets can have arbitrary large positive Lebesgue measure, see [112]. Indeed we have the following Theorem [5].

Theorem 2.3.7 *Let* $u, v \in \mathbb{H}(X)$ *and let* D *be a dense subset of* X. *Then*

$$(a)\ u(x) \leq v(x),\ x \in D \Rightarrow u(x) \leq v(x),\ x \in X, \tag{2.3.8}$$

and

$$(b)\ u(x) = v(x),\ x \in D \Rightarrow u(x) = v(x),\ x \in X.$$

2.4 The Set $\mathbb{H}(X)$ is Dedekind Order Complete

One of the most surprising and useful properties of the set $\mathbb{H}(X)$ of all H-continuous functions is its Dedekind order completeness with respect to the partial order given in (2.2.3). What makes this property so significant is the fact that, with very few exceptions, the usual spaces in Real Analysis or Functional Analysis, e.g., space of continuous functions, is not Dedekind order complete. In this way, the class of Hausdorff continuous functions can provide solutions to open problems or improve earlier results related to order. This section discusses Dedekind order completeness of $\mathbb{H}(X)$. The representation of supremum (resp. infimum) in $\mathbb{H}(X)$ trough the point-wise supremum (resp. infimum) is given.

Let us recall the concept of Dedekind order completeness [109].

Definition 2.4.1 *A partial ordered set $(X, \leq)$ is called Dedekind order complete, if and only if every nonempty subset A of X which is bounded from above has a supremum in X and every nonempty subset B of X which is bounded from below has an infimum in X.*

A general result on Dedekind order completion of partially ordered sets was established by MacNeille in 1937, see [98]. The problem of order completion of $C(X)$ is particulary addressed in [97].

The following theorem states one of the most amazing properties of the set $\mathbb{H}(X)$, namely its Dedekind order completeness.

Theorem 2.4.1 [5] *(a) For every nonempty subset $\mathcal{F}$ of $\mathbb{H}(X)$ which is bounded from above there exist $u \in \mathbb{H}(X)$ such that*

$$u = \sup \mathcal{F}. \tag{2.4.1}$$

(b) For every nonempty subset $\mathcal{Z}$ of $\mathbb{H}(X)$ which is bounded from below there exists $v \in \mathbb{H}(X)$ such that

$$v = \inf \mathcal{Z}. \tag{2.4.2}$$

The supremum in (2.4.1) and infimum in (2.4.2) are not taken point-wise but according the partial order (2.2.3). This result indicates that indeed the partial order (2.2.3) induced point-wise by (2.2.1) is an appropriate partial order to be associated with the Hausdorff continuous interval valued function.

Remark 2.4.1 *The concept of viscosity solution is defined through order. Hence the order properties of $\mathbb{H}(X)$, in particular its Dedekind completeness, play an important role. Topological structures have been defined on $\mathbb{H}(X)$ in different ways. For example, in [6] there is a convergence structure defined through the partial order (2.2.3) on the set $\mathbb{H}(X)$ which is generally not a topology. The supremum norm was considered in [11] on the set of bounded H-continuous functions.*

The following theorem gives useful representation of supremum (resp. infimum) of a subset of $\mathbb{H}(X)$ in terms of the point-wise supremum (resp. infimum).

Theorem 2.4.2 [5] *(a) Let the set $\mathcal{F} \subseteq \mathbb{H}(X)$ be bounded from above and let the function $\psi \in \mathcal{A}(X)$ be defined by*

$$\psi(x) := \sup\{\overline{f}(x) : f = [\underline{f}, \overline{f}] \in \mathcal{F}\},\ x \in X.$$

Then

$$\sup \mathcal{F} = F(S(\psi)) = [I(S(\psi)), S(\psi)].$$

(b) Let the set $\mathcal{Z} \subseteq \mathbb{H}(X)$ be bounded from below and let the function $\varphi \in \mathcal{A}(X)$ be defined by

$$\varphi(x) := \inf\{\underline{f}(x) : f = [\underline{f}, \overline{f}] \in \mathcal{Z}\}, x \in X.$$

Then

$$\inf \mathcal{Z} = F(I(\varphi)) = [I(\varphi), S(I(\varphi))].$$

Theorem 2.4.1 establishes a close connection between the supremum (resp. infimum) in $\mathbb{H}(X)$ about the partial order (2.2.3) and the point-wise supremum (resp. infimum). These two functions are not the same, that is, for a set $\mathcal{F} \subseteq \mathbb{H}(X)$, in general, there exists $x \in X$ such that

$$\begin{aligned} (\inf \mathcal{F})(x) &= (\inf_{f \in \mathcal{F}} f)(x) \neq \inf_{f \in \mathcal{F}}(f(x)), \\ (\sup \mathcal{F})(x) &= (\sup_{f \in \mathcal{F}} f)(x) \neq \sup_{f \in \mathcal{F}}(f(x)). \end{aligned}$$

Example 2.4.1 *Let $X = \mathbb{R}^n$. Consider the set $\mathcal{F} = \{f_\delta : \delta > 0\}$, where*

$$f_\delta(x) = \begin{cases} 1 - \frac{|x|}{\delta}, & |x| \leq \delta \\ 0 &, \textit{otherwise}. \end{cases}$$

The point-wise infimum of $\mathcal{F}$ is

$$\varphi(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0. \end{cases}$$

The function φ is not an H-continuous function on $\mathbb{R}^n$. Indeed, consider the continuous function $v(x) = 0, x \in \mathbb{R}^n$. Obviously, we have $\varphi(x) \supseteq v(x), x \in \mathbb{R}^n$ and $v(0) \neq \varphi(0)$ so φ cannot be H-continuous function in $\mathbb{R}^n$. The infimum of $\mathcal{F}$ in $\mathbb{H}(\mathbb{R}^n)$ is $u(x) = 0$, $x \in \mathbb{R}^n$.

Example 2.4.2 *Let $X = \mathbb{R}$. Consider the set $\mathcal{F} = \{g_n : n \in \mathbb{N}\}$, where*

$$g_n(x) = \begin{cases} x^{-2n-1} & , \quad x \neq 0 \\ [-\infty, +\infty] & , \quad x = 0. \end{cases}$$

The point-wise supremum of $\mathcal{F}$ is

$$\psi(x) = \begin{cases} 0 & , \quad x < -1 \\ x^{-1} & , \quad -1 \leq x < 0 \\ \infty & , \quad 0 \leq x < 1 \\ x^{-1} & , \quad x \geq 0. \end{cases}$$

Hence

$$(\sup \mathcal{F})(x) = \begin{cases} 0 & , \quad x < -1 \\ [-1, 0] & , \quad x = -1 \\ x^{-1} & , \quad -1 < x < 0 \\ [-\infty, +\infty] & , \quad x = 0 \\ \infty & , \quad 0 < x < 1 \\ [1, \infty] & , \quad x = 1 \\ x^{-1} & , \quad x > 1. \end{cases}$$

We will use the following theorem given in [4].

Theorem 2.4.3 *Let D be dense subset of X, we have the following:*

$$u \in \mathbb{H}(D) \Rightarrow F(u) \in \mathbb{H}(X).$$

Since X is always dense in $\overline{X}$, we have the following corollary.

Corollary 2.4.1 *Let $u \in \mathbb{A}(X)$. Then*

$$u \in \mathbb{H}(X) \Rightarrow F(u) \in \mathbb{H}(\overline{X}).$$

Further properties of the H-continuous functions are discussed in [120], [10], [4], where it is shown, among others, that they retain some of the essential characteristics of the usual continuous functions.

2.5 Generalized Baire Operators and Graph Completion Operator

Let D be a dense subset of X. For any $u \in \mathbb{A}(D)$ we can consider the following generalization of the upper and lower Baire operators as well as the graph completion operator, given in Section 2.2.

Definition 2.5.1 *The mapping $S(D, X, .), I(D, X, .) : \mathbb{A}(D) \to \mathcal{A}(X)$ defined for $\mathbb{A}(D)$ by*

$$S(D, X, u)(x) = \inf_{\delta>0} \sup\{z \in u(y) : y \in B_\delta(x) \cap D\},\ x \in X,$$
$$I(D, X, u)(x) = \sup_{\delta>0} \inf\{z \in u(y) : y \in B_\delta(x) \cap D\},\ x \in X,$$

are called generalized upper Baire and generalized lower Baire operators, respectively.

Clearly, for every $u \in \mathbb{A}(D)$, we have

$$I(D, X, u)(x) \leq S(D, X, u)(x),\ x \in X \tag{2.5.1}$$

and

$$I(D, X, u)(x) \leq u(x) \leq S(D, X, u)(x),\ x \in D.$$

Definition 2.5.2 *The mapping $F(D, X, .) : \mathbb{A}(D) \to \mathbb{A}(X)$, defined for $u \in \mathbb{A}(D)$ by*

$$F(D, X, u)(x) = [I(D, X, u)(x), S(D, X, u)(x)],\ x \in D,$$

is called generalized graph completion operator.

Note that the usual Baire operators and graph completion operator obtained from the above definitions using $D = X$, i.e., if $u \in \mathbb{A}(X)$ then

$$I(u) = I(X, X, u),\ S(u) = S(X, X, u),\ F(u) = F(X, X, u).$$

The generalized lower and upper Baire operators as well as the graph completion operator of an interval valued function $u = [\underline{u}, \overline{u}] \in \mathbb{A}(D)$ can be conveniently represented in terms of the functions $\underline{u}$ and $\overline{u}$:

$$I(D, X, u) = I(D, X, \underline{u}),\ S(D, X, u) = S(D, X, \overline{u}),\ F(D, X, u) = [I(D, X, \underline{u}), S(D, X, \overline{u})].$$

In the sequel, we will use the following properties of operators $I(D, X, .)$, S(D,X,.), and F(D,X,.), which were proved in [4].

Theorem 2.5.1 *(i) Let D be a dense subset of X. Then*

$$I(D, X, u) \in LSC(X) \text{ and } S(D, X, u) \in USC(X). \tag{2.5.2}$$

(ii) If $u, v \in \mathbb{A}(D)$, where D is dense in X, then

$$u(x) \leq v(x), x \in D \Rightarrow I(D, X, u)(x) \leq I(D, X, u)(x), x \in X; \tag{2.5.3}$$

$$u(x) \leq v(x), x \in D \Rightarrow S(D, X, u)(x) \leq S(D, X, u)(x), x \in X; \tag{2.5.4}$$

$$u(x) \leq v(x), x \in D \Rightarrow F(D, X, u)(x) \leq F(D, X, u)(x), x \in X. \tag{2.5.5}$$

(iii) The generalized graph completion operator is monotone about inclusion with respect to the functional argument, that is, if $u, v \in \mathbb{A}(D)$, where D is dense in X then

$$u(x) \subseteq v(x), x \in X \Rightarrow F(D, X, u)(x) \subseteq F(D, X, v)(x), x \in X.$$

(iv) If D_1 and D_2 are dense subsets of X and $u \in \mathbb{A}(D_1 \cup D_2)$, then

$$D_1 \subseteq D_2 \Rightarrow I(D_2, X, u)(x) \leq I(D_1, X, u)(x), x \in X, \tag{2.5.6}$$

$$D_1 \subseteq D_2 \Rightarrow S(D_1, X, u)(x) \leq S(D_2, X, u)(x), x \in X. \tag{2.5.7}$$

The generalized lower and upper Baire operators have the following optimality property.

Theorem 2.5.2 [4] *Let D be a dense subset of X. For any function $u \in \mathbb{A}(D)$ we have*

$$(i) \quad v \in LSC(X), v(x) \leq u(x), x \in D \Rightarrow v(x) \leq I(D, X, u)(x), x \in X, \tag{2.5.8}$$

$$(ii) \quad v \in USC(X), u(x) \leq v(x), x \in D \Rightarrow S(D, X, u)(x) \leq v(x), x \in X. \tag{2.5.9}$$

The above theorem explains why $I(D, X, .), S(D, X, .)$ are called, respectively, lower and upper semicontinuous envelope of the function u.

We will use the following Theorem [4]:

Theorem 2.5.4 *Let D be a dense subset of X. If $u \in C(D)$, then*

$$F(D, X, u) \in \mathbb{H}(X)$$

and

$$F(D, X, u)(x) = u(x), x \in D.$$