Chapter 1

Introduction

1.1 The Hamilton-Jacobi Equations

The theory of viscosity solutions was developed for certain types of first and second order partial differential equations. It has been particularly useful in describing the solutions of partial differential equations associated with deterministic and stochastic optimal control problems [16], [53]. In this thesis, we are interested in the theory of viscosity solutions of first-order Hamilton-Jacobi equation in its general form associated with boundary condition, namely

$$H(x, u(x), Du(x)) = 0, x \in \Omega, \tag{1.1.1}$$

$$u(x) = g(x), x \in \partial\Omega. \tag{1.1.2}$$

Here,

Ω is an arbitrary nonempty open subset of $\mathbb{R}^n$
$H : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ (Hamiltonian) is jointly continuous function in all its arguments,
$g : \partial\Omega \to \mathbb{R}$ is a given (not necessarily continuous) function,
$u : \overline{\Omega} \to \mathbb{R}$ is the unknown function, and
$Du(x)$ is the gradient of the function u at a point $x = (x_1, ..., x_n) \in \Omega$.

The evolutionary Hamilton-Jacobi equations of the form

$$u_t(y,t) + H(y, t, u(y,t), D_y u(y,t)) = 0, \quad (y,t) \in D \times (0,T), \tag{1.1.3}$$

where $D \subset \mathbb{R}^n$, the t subscript denotes a temporal partial derivatives in the time variable t, and $D_y u$ is the gradient of u with respect to y, are reduced to the form (1.1.1) by the substitutions

$$x = (y, t) \in \Omega = D \times (0, T) \subseteq \mathbb{R}^{n+1}, \;\; \widetilde{H}(x, r, q) = H(x, r, q_1, ..., q_n) + q_{n+1}$$

with

$$q = (q_1, ..., q_n, q_{n+1}) \in \mathbb{R}^{n+1}.$$

The Hamilton-Jacobi equations play an important role in many fields of mathematics and physics, as for instance, calculus of variations [29], combustion [18], computer graphics [117], optimal control theory [93], differential games [55], image processing [122], quantum mechanics [114], and geometric optics [26]. For this reason, many theoretical and numerical studies have been devoted to solving the Hamilton-Jacobi equations.

Let us remark that the equation (1.1.1) is global nonlinear equation. Classical approach to the study of the problem (1.1.1)-(1.1.2) is the method of characteristics. This technique gives an elementary way a local existence result for smooth solutions, and at the same time shows that no global smooth solution exists in general.

By a classical solution of the problem (1.1.1)-(1.1.2), we mean a function $u \in C^1(\Omega) \cap C(\overline{\Omega})$ satisfying (1.1.1) and (1.1.2). Nonlinear partial differential equations of the form (1.1.1) do not, in general, possess classical solutions as can be seen in the following example.

Example 1.1.1 *Consider the following Dirichlet problem*

$$(u_x(x))^2 = 1, x \in (0, 1), \tag{1.1.4}$$

$$u(0) = u(1) = 0. \tag{1.1.5}$$

There cannot exist a classical solution of (1.1.4) which satisfies (1.1.5). Indeed, assume there is $u \in C^1((0, 1)) \cap C([0, 1])$ *satisfying (1.1.5). Then there exists* $x_0 \in (0, 1)$ *such that* $u_x(x_0) = 0$. *This is a contradiction to (1.1.4).*

For more general results about the method of characteristics applied to first-order Hamilton-Jacobi equations, we refer to R. Courant and D. Hilbert [35], L. C. Evans [46], F. John [75, 76], P. D. Lax [88] and H. Rund [118].

The equation (1.1.1) has been approached by looking for generalized solutions.

Definition 1.1.1 *A function $u : \overline{\Omega} \to \mathbb{R}$ is said to be a generalized solution of equation (1.1.1) if u is locally Lipschitz in Ω, continuous on $\overline{\Omega}$, and satisfies (1.1.1) almost everywhere on Ω.*

The generalized solution is almost everywhere differentiable by the well-known classical Rademacher's Theorem, see [47], on the almost everywhere differentiability of Lipschitz continuous functions, and (1.1.1)-(1.1.2) are to be understood as to hold almost everywhere. The existence results have been obtained by several authors, see for examples A. Douglis [43, 44], S. N. Kruzkov [82, 83, 84], W. H. Fleming [49, 50, 51], Hopf [68], E. D. Conway and Hopf [34], and A. Friedman [56]. For more complete references, we refer to some recent monographs by Benton [27] and P. -L. Lions [93].

The reader can also find an extensive presentation of the results on the solvability of the problem (1.1.1)-(1.1.2) in the books by Gilbarg and Trudinger [58, 59] and Ladyzhenskaya and Uraltseva [86].

The difficulty with the above concept of generalized solutions is that the equation (1.1.1), together with the boundary condition (1.1.2), typically has many generalized solutions. This is shown by Example 1.1.1. Obviously, the function

$$u_1(x) = \begin{cases} x & , \quad 0 \le x \le \frac{1}{2} \\ 1 - x & , \quad \frac{1}{2} \le x \le 1 \end{cases}$$

satisfies (1.1.5) and solves the equation (1.1.4) almost everywhere, in fact everywhere except on the point $x = \frac{1}{2}$. But u_1 is not unique and there exist infinitely many generalized solutions. In particular, we may build a sequence of generalized solutions as follows : for $m \ge 2$

$$u_m(x) = \begin{cases} x - \frac{2j}{2^m} & , \quad \frac{2j}{2^m} \le x \le \frac{2j+1}{2^m} \\ \\ \frac{2j+2}{2^m} - x & , \quad \frac{2j+1}{2^m} \le x \le \frac{2j+2}{2^m}, \end{cases} \tag{1.1.6}$$

for $j = 0, 1, ..., 2^{m-1} - 1$ and $x \in (0, 1)$.

It is evident that $u_m(0) = u_m(1) = 0$ and $((u_m)_x(x))^2 = 1$ everywhere on $(0, 1)$ except at the corners of its graph. The function u_m is bounded, Lipschitz continuous and piecewise analytic. Thus u_m is a generalized solution of (1.1.4) and satisfies (1.1.5).

One sees immediately that one has

$$0 \leq u_m(x) \leq \frac{1}{2^m},\ x \in [0,1],\ \forall m \geq 1. \tag{1.1.7}$$

Thus, u_m converges to 0 uniformly as $m \to +\infty$, but $u \equiv 0$ does not satisfy (1.1.4) anywhere on $(0,1)$. Therefore, a stability property is false for generalized solutions.

Similarly, consider the equation $u_t + (u_y)^2 = 0$ for $y \in \mathbb{R}, t > 0$, coupled with the initial condition $u(y,0) = 0$. The function

$$v(y,t) = \begin{cases} 0 & , \quad |y| \geq t > 0 \\ -t + |y| & , \quad t \geq |y| \end{cases}$$

satisfies the initial condition, is continuous and has all the regularity one desires off the lines $y = 0, t = |y|$, and satisfies the equation off these lines. Thus $u \equiv 0$ and v are distinct generalized solutions of the above Cauchy problem and satisfy the initial condition.

The above two examples show that the notion of generalized solution, in terms of Definition 1.1.1, is too weak and in order to obtain an uniqueness result one needs to restrict the class of solutions by adding some suitable admissibility condition.

1.2 The Classical Theory of Viscosity Solutions

To solve uniqueness (and stability) question given in Section 1.1, in the early 1980s, M. G. Crandall and P.-L. Lions [38, 39] introduced a class of continuous generalized solutions of (1.1.1), called *viscosity solutions* (for reasons detailed below) which need not be differentiable anywhere, as the only regularity required in the definition is continuity.

To motivate the definition of continuous viscosity solution of equation (1.1.1), let us consider the approximate equation for (1.1.1), namely,

$$H(x, u_\varepsilon(x), Du_\varepsilon(x)) - \varepsilon\nabla^2 u_\varepsilon(x) = 0,\ x \in \Omega, \tag{1.2.1}$$

were $\varepsilon > 0$ is a small parameter. The equations (1.2.1) are quasilinear elliptic and have been studied for a long time (see in particular O. Ladyzenskaya and N. N. Uraltseva [87], D. Gilbarg and N. S. Trudinger [58], J. Serrin [121]). It is shown in [59, 85] that the equation (1.2.1) together with boundary condition has a unique classical solution. We hope that as $\varepsilon \to 0$ the solution $u_\varepsilon \in C^2(\Omega)$ of (1.2.1) will converge to some sort of weak solution of (1.1.1). This technique is the method of *vanishing viscosity.* It comes from a well known method in fluid dynamics where the coefficient ε represents physically the *viscosity* of the fluid and explains the name of solutions.

The vanishing viscosity method works as follows. Suppose that the family of solutions of (1.2.1), namely, $\{u_\varepsilon\}_{\varepsilon>0}$, is uniformly bounded and equicontinuous on compact set $\overline{\Omega}$. Consequently, the Arzela-Ascoli's [79] compactness criterion, ensures that there exists a function
$u \in C(\overline{\Omega})$ and a subsequence $\{u_{\varepsilon_j}\}_{j=1}^{\infty}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ such that

$$u_{\varepsilon_j} \to u \text{ uniformly on } \overline{\Omega},\ \text{ as } \varepsilon_j \to 0. \tag{1.2.2}$$

Fix any $\varphi \in C^1(\Omega)$ and suppose

$$u - \varphi \text{ has a local maximum at some point } x_0 \in \Omega. \tag{1.2.3}$$

For $\delta > 0$, consider the function $\psi \in C(\Omega)$ defined by $\psi(x) := \varphi(x) + \delta(|x - x_0|^2)$. Then we have $\psi \in C^1(\Omega)$ and $D\psi(x_0) = D\varphi(x_0)$.

Obviously, the function

$$u - \psi \text{ has a strict local maximum at } x_0. \tag{1.2.4}$$

By (1.2.2) and (1.2.4), for each sufficiently small $\varepsilon_j > 0$, there exists a point $x_{\varepsilon_j} \in \Omega$ such that $x_{\varepsilon_j} \to x_0$ as $\varepsilon_j \to 0$ and

$$u_{\varepsilon_j} - \psi \text{ has a local maximum at } x_{\varepsilon_j}. \tag{1.2.5}$$

Now owing to (1.2.5), by elementary calculus,

$$Du_{\varepsilon_j}(x_{\varepsilon_j}) = D\psi(x_{\varepsilon_j}) \tag{1.2.6}$$

and

$$\nabla^2(u_{\varepsilon_j} - \psi)(x_{\varepsilon_j}) \leq 0 \tag{1.2.7}$$

hold. We consequently can calculate

$$\begin{aligned} H(x_{\varepsilon_j}, u(x_{\varepsilon_j}), D\psi(x_{\varepsilon_j})) &= H(x_{\varepsilon_j}, u(x_{\varepsilon_j}), Du_{\varepsilon_j}(x_{\varepsilon_j})) && \text{by (1.2.6)} \\ &= \varepsilon_j \nabla^2 u_{\varepsilon_j}(x_{\varepsilon_j}) && \text{by (1.2.1)} \\ &\leq \varepsilon_j \nabla^2 \psi(x_{\varepsilon_j}) && \text{by (1.2.7).} \end{aligned} \tag{1.2.8}$$

Since $x_{\varepsilon_j} \to x_0$ as $\varepsilon_j \to 0$, we can pass to the limit in (1.2.8) using that $u_{\varepsilon_j}(x_{\varepsilon_j}) \to u(x_0)$, $D\psi(x_{\varepsilon_j}) \to D\psi(x_0) = D\varphi(x_0)$, $\varepsilon_j \nabla^2 \varphi(x_{\varepsilon_j}) \to 0$, and H is continuous to have

$$H(x_0, u(x_0), D\varphi(x_0)) \leq 0. \tag{1.2.9}$$

Consequently, condition (1.2.3) implies inequality (1.2.9). Similarly, we deduce the reverse inequality

$$H(x_0, u(x_0), D\varphi(x_0)) \geq 0 \tag{1.2.10}$$

provided

$$u - \varphi \text{ has a local minimum at } x_0. \tag{1.2.11}$$

The proof is exactly similar to that mentioned before, except that the inequalities (1.2.7) and thus in (1.2.8), are reversed. In summary, for any $x_0 \in \Omega$ and $\varphi \in C^1(\Omega)$ such that inequality (1.2.9) follows from (1.2.3), and (1.2.10) from (1.2.11). We have in effect put the derivatives onto φ, at the expense of certain inequalities holding.

The properties (1.2.9) and (1.2.10) of the limit u of the subsequence u_{ε_j} motivate the following concept of weak solution [36].

Definition 1.2.1 *A function $u \in C(\Omega)$ is a viscosity subsolution of (1.1.1) in Ω if, for any $\varphi \in C^1(\Omega)$, we have*

$$H(x_0, u(x_0), D\varphi(x_0)) \leq 0 \tag{1.2.12}$$

at any local maximum point $x_0 \in \Omega$ of $u - \varphi$.
Similarly, $u \in C(\Omega)$ is a viscosity supersolution of (1.1.1) in Ω if, for any $\varphi \in C^1(\Omega)$, we have

$$H(x_0, u(x_0), D\varphi(x_0)) \geq 0 \tag{1.2.13}$$

at any local minimum point $x_0 \in \Omega$ of $u - \varphi$.
Finally, $u \in C(\Omega)$ is a viscosity solution of (1.1.1) if it is both a viscosity subsolution and a viscosity supersolution of (1.1.1) in Ω.

The notion of viscosity solution is a notion of "weak " solution of Hamilton-Jacobi equation, since u is assumed to be only continuous and the existence of Du is not necessary. But, in some sense, at a point of maximum of $u - \varphi$, where $\varphi \in C^1(\Omega)$, a good candidate to replace Du is $D\varphi$.

Remark 1.2.1 *In the definition of viscosity subsolution one can always assume that $u-\varphi$ has a local strict maximum at x_0 (otherwise, replace $\varphi(x)$ by $\varphi(x)+|x-x_0|^2$). Moreover, since (1.2.12) depends only on the value of $D\varphi$ at x_0, it is not restrictive to assume that $u(x_0) = \varphi(x_0)$. Similar remarks apply of course to the definition of supersolution. Geometrically, this means that the validity of the subsolution condition (1.2.12) for u is tested on smooth functions "touching from above " the graph of u at x_0 and the validity of the supersolution condition (1.2.13) for u is tested on smooth functions "touching from below" the graph of u at x_0.*

There is an alternative way of defining viscosity solutions of (1.1.1) which is equivalent of Definition 1.2.1. Let us associate with a function $u \in C(\Omega)$ and $x \in \Omega$ the sets

$$D^+u(x) := \{p \in \mathbb{R}^n : \limsup_{y \to x,\ y \in \Omega} \frac{u(y) - u(x) - p.(y-x)}{|x-y|} \leq 0\},$$
$$D^-u(x) := \{p \in \mathbb{R}^n : \liminf_{y \to x,\ y \in \Omega} \frac{u(y) - u(x) - p.(y-x)}{|x-y|} \geq 0\}.$$

These sets are called, respectively, the superdifferential and the subdifferential (or semidifferentials) of a function u at a point x.

From the definition of subdifferentials and superdifferentials it follows that, for any $x \in \Omega$, $D^-(-u)(x) = -D^+u(x)$.

Some properties of semidifferentials are collected in the following lemma.

Lemma 1.2.1 [16] *Let $u \in C(\Omega)$ and $x \in \Omega$. Then,*
(a) $D^+u(x)$ and $D^-u(x)$ are closed convex (possibly empty) subsets of $\mathbb{R}^n$;
(b) if u is differentiable at x, then $D^+u(x) = D^-u(x) = \{Du(x)\}$;
(c) if for x both $D^+u(x), D^-u(x)$ are nonempty, then $D^+u(x) = D^-u(x) = \{Du(x)\}$.

The following new definition of continuous viscosity solution of (1.1.1) is equivalent of Definition 1.2.1, see [16].

Definition 1.2.2 *A function $u \in C(\Omega)$ is a viscosity subsolution of (1.1.1) in Ω if, for any $x \in \Omega$, it satisfies*

$$H(x, u(x), p) \leq 0, \ \forall p \in D^+u(x). \tag{1.2.14}$$

A function $u \in C(\Omega)$ is a viscosity supersolution of (1.1.1) in Ω if, for any $x \in \Omega$, we have

$$H(x, u(x), p) \geq 0, \forall p \in D^-u(x). \tag{1.2.15}$$

A function $u \in C(\Omega)$ is a viscosity solution of (1.1.1) if it is a viscosity subsolution and supersolution.

Example 1.2.1 *Consider the following equation*

$$(u_x(x))^2 - 1 = 0, \ x \in (-1, 1). \tag{1.2.16}$$

Then, the function $u(x) = 1 - |x|$ is a continuous viscosity solution of (1.2.16) but the function $v(x) = -u(x) = |x| - 1$ is a viscosity subsolution but not a viscosity supersolution of (1.2.16). To check this, notice first that if $x \neq 0$, u and v are classical solutions of (1.2.16). Therefore, at those points both the supersolution and the subsolution conditions (1.2.15) and (1.2.14), respectively, are trivially satisfied.

However, $D^+u(0) = [-1,1]$ *and* $D^-u(0) = \emptyset$; *thus, the requirement in (1.2.15) is empty, while (1.2.14) holds since* $p^2 - 1 \leq 0$ *for all* $p \in D^+u(0)$. *So* u *is a continuous viscosity solution of (1.2.16). On the other hand, since* $p = 0$ *belongs to* $D^-v(0) = [-1,1]$, *then the condition of supersolution (1.2.15) is not satisfied at* $x = 0$ *and* v *is not a continuous viscosity solution of (1.2.16).*

Note that viscosity solutions are not preserved by changing the sign in the equation. Indeed, $v(x) = |x| - 1$ is a viscosity solution of $-(u_x(x))^2 + 1 = 0$ in $(-1,1)$ but in Example 1.2.1, v is not a continuous viscosity solution of $((u_x(x))^2 - 1 = 0$ in $(-1,1)$.

In general, if $H(x,r,p)$ is nondecreasing in r, a function u is a viscosity subsolution of $H(x,u,Du) = 0$ if and only if $v = -u$ is a viscosity supersolution of $-H(x,-v,-Dv) = 0$ in Ω; similarly, u is a viscosity supersolution of $H(x,u,Du) = 0$ if and only if $v = -u$ is a viscosity subsolution of $-H(x,-v,-Dv) = 0$ in Ω.

The following theorem, given in [16], establishes the consistency of the notion of continuous viscosity solutions and classical solutions.

Theorem 1.2.1 *(i) If* $u \in C^1(\Omega)$ *is a classical solution of (1.1.1), then* u *is a viscosity solution of (1.1.1).*
(ii) If $u \in C(\Omega)$ *is a viscosity solution of (1.1.1), then* $H(x_0, u(x_0), Du(x_0)) = 0$ *at any point* $x_0 \in \Omega$ *where* u *is differentiable.*
(iii) If $u \in C(\Omega)$ *is locally Lipschitz continuous and it is a viscosity solution of (1.1.1), then* $H(x, u(x), Du(x)) = 0$ *almost everywhere in* Ω, *and thus* u *is a generalized solution of (1.1.1) in terms of Definition 1.1.1.*

The converse of property (*iii*) in Theorem 1.2.1 is false in general. There are many generalized solutions which are not viscosity solutions. As an example, observe that $v(x) = |x| - 1$ is a generalized solution of equation $(v_x(x))^2 = 1$ in $(-1,1)$, because it satisfies the equation in $(-1,1)$ except at $x = 0$, but it is shown in Example 1.2.1 that v is not a viscosity supersolution of the same equation in $(-1,1)$.

The next theorem is a stability result for viscosity solutions.

Theorem 1.2.2 [16] *Let for every $m \in \mathbb{N}$ the function $u_m \in C(\Omega)$ be a viscosity solution of an equation of the form*

$$H_m(x, u(x), Du(x)) = 0, \ x \in \Omega,$$

where $\{H_m\}_{m\geq 1}$ is the sequence of Hamiltonians. If

$$\begin{aligned} u_m &\to u \text{ locally uniformly in } \Omega, \\ H_m(x,r,p) &\to H(x,r,p) \text{ locally uniformly in } \Omega \times \mathbb{R} \times \mathbb{R}^n, \end{aligned}$$

then u is a continuous viscosity solution of (1.1.1).

Theorem 1.2.2 does not hold for generalized solutions in general. As an example, the property (1.1.7) implies that the uniform limit of the sequence $\{u_m\}$, given in (1.1.6) for $m \geq 2$, is identically zero and does not satisfy the equation (1.1.4) at any point of $(0, 1)$. Therefore, the functions u_m, $m \geq 2$, are not viscosity solutions of (1.1.4).

The following theorem, given in [40], gives very general existence, uniqueness and continuous dependence results for viscosity solutions of the problem of the form (1.1.3)-(1.1.2).

Theorem 1.2.3 *Consider the equation*

$$u_t + H(Du) = 0, (x,t) \in \mathbb{R}^n \times (0, \infty) \tag{1.2.17}$$

with initial conditions

$$u(x, 0) = u_0(x), x \in \mathbb{R}^n. \tag{1.2.18}$$

Let $H \in C(\mathbb{R}^n)$ and u_0 be uniformly continuous in $\mathbb{R}^n$. Then there is a unique continuous function $u : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ with the following properties: u is uniformly continuous in x uniformly in t, u is a continuous viscosity solution of (1.2.17) and u satisfies (1.2.18).

The following theorem, given in [36], shows that continuous viscosity solution of the problem (1.2.17)-(1.2.18) depends monotonically on the initial value.

Theorem 1.2.4 *Let $0 < T < \infty$ and let u, v be bounded and uniformly continuous real functions in $(\mathbb{R}^n \times [0, T])$. If u and v are continuous viscosity solutions of the equation (1.2.17), with initial conditions u_0 and v_0, respectively, then*

$$\sup_{\mathbb{R}^n \times [0,T]} (\max(u - v, 0)) \leq \sup_{\mathbb{R}^n} (\max(u_0 - v_0, 0)).$$

In general, for any given Hamilton-Jacobi equation of the form (1.1.3), where the Hamiltonian $H(y, t, u, Du)$ is continuous, nondecreasing in u, there exists a unique uniformly continuous viscosity solution if the initial data is bounded and uniformly continuous [128].

The theory of continuous viscosity solutions has been extensively studied in many relevant articles such as Crandall-Lions [39], Crandall-Evans-Lions [36]. For more complete references, we refer to some monographs by Lions [93], Bardi-Capuzzo-Dolcetta [16], Fleming-Soner [53], and Barles [19]. In [93], existence and uniqueness results for many classes of Hamilton-Jacobi equations are given . The existence and uniqueness of continuous viscosity solution of Hamilton-Jacobi equation in infinite dimensions are given in [41] and [42].

1.3 Discontinuous Viscosity Solutions

According to definition of viscosity solution, introduced by Crandall and Lions, when the value function of optimal control problem is uniformly continuous, it is then a viscosity solution of the associated partial differential equation [93].

However, many optimal control problems, such as the exit time problems, have discontinuous value functions [16]. For this reason, the concept of classical viscosity solutions was further generalized to include solutions that are not necessarily continuous.

The notion of viscosity solutions in the context of semicontinuous solutions has been introduced first by Ishii [71].

We recall that a function $u : \Omega \to \mathbb{R}$ is upper (respectively, lower) semicontinuous if for any $x \in \Omega$ and $\varepsilon > 0$ there is $\delta > 0$ such that $u(y) < u(x) + \varepsilon$ (respectively, $u(y) > u(x) - \varepsilon$) for all $y \in \Omega \cap B_\delta(x)$; Weierstrass' Theorem on the existence of maxima (respectively, minima) on compact sets holds for upper (respectively, lower) semicontinuous functions.

The definition of continuous viscosity subsolution and supersolution of (1.1.1) extends naturally to semicontinuous functions as follows [16].

Definition 1.3.1 *A function $u \in USC(\Omega)$ is called a viscosity subsolution of the Hamilton-Jacobi equation (1.1.1) if for any $\varphi \in C^1(\Omega)$ we have*

$$H(x, u(x), D\varphi(x)) \leq 0$$

at any local maximum point x of $u - \varphi$.
A function $u \in LSC(\Omega)$ is called a viscosity supersolution of the Hamilton-Jacobi equation (1.1.1) if for any $\varphi \in C^1(\Omega)$ we have

$$H(x, u(x), D\varphi(x)) \geq 0$$

at any local minimum point x of $u - \varphi$.

Notice that for any $\varphi \in C^1(\overline{\Omega})$ and $u \in USC(\overline{\Omega})$ the difference $u - \varphi \in USC(\overline{\Omega})$. Hence, $u - \varphi$ attains its maximum on $\overline{\Omega}$. This is an indication that the upper semicontinuous function is used in the definition of a viscosity subsolution. Similarly, if $u \in LSC(\overline{\Omega})$ then $u - \varphi \in LSC(\overline{\Omega})$ attains its minimum on $\overline{\Omega}$, and thus this explains the use of lower semicontinuous function in the definition of a viscosity supersolution.

Since $D^+u(x)$ and $D^-u(x)$ make sense for any real valued function u, see [14], we have the following result.

Lemma 1.3.1 [14] *(i) Let $u \in USC(\Omega)$. $p \in D^+u(x) \Leftrightarrow$ there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $u - \varphi$ has a local maximum at a point $x \in \Omega$.*
(ii) Let $v \in LSC(\Omega)$. $p \in D^-v(x) \Leftrightarrow$ there exists $\varphi \in C^1(\Omega)$ such that $D\varphi(x) = p$ and $v - \varphi$ has a local minimum at a point $x \in \Omega$.

As for continuous viscosity subsolutions and supersolutions, there is an equivalent definition by means of semidifferentials of u instead of test functions.

Definition 1.3.2 [16] *A function $u \in USC(\Omega)$ is a viscosity subsolution of (1.1.1) in Ω if, for any $x \in \Omega$, it satisfies*

$$H(x, u(x), p) \leq 0, \ \forall p \in D^+u(x).$$

A function $u \in LSC(\Omega)$ is a viscosity supersolution of (1.1.1) in Ω if, for any $x \in \Omega$, we have

$$H(x, u(x), p) \geq 0, \forall p \in D^-u(x).$$

Naturally, a solution should be required somehow to incorporate the properties of both a subsolution and a supersolution. In the classical viscosity solutions theory, a viscosity solution is a function u which is both a subsolution and a supersolution.
Since $USC(\Omega) \cap LSC(\Omega) = C(\Omega)$ this clearly implies that the viscosity solutions defined in this way are all continuous functions.

The concept of viscosity solution for functions which are not necessarily continuous is introduced by using the upper and lower semicontinuous envelopes, see [71]. Let us recall that the upper semicontinuous envelope of a function u which we denote by $S(u)$ is the least upper semicontinuous function which is greater than or equal to u. In a similar way, the lower semicontinuous envelope of a function u, denoted by $I(u)$, is the largest lower semicontinuous function not greater than u.

For a locally bounded function $u : \Omega \to \mathbb{R}$, we have the following representations of $S(u)$ and $I(u)$:

$$S(u)(x) = \min\{f(x) : f \in USC(\Omega), u \leq f\} = \inf_{\delta>0} \sup\{u(y) : y \in B_\delta(x)\}, \quad (1.3.1)$$

$$I(u)(x) = \max\{f(x) : f \in LSC(\Omega), u \geq f\} = \sup_{\delta>0} \inf\{u(y) : y \in B_\delta(x)\}. \quad (1.3.2)$$

It is clear that $I(u) \leq u \leq S(u)$ in Ω. Note that

$$u \in USC(\Omega) \text{ iff } u = S(u) \text{ and } u \in LSC(\Omega) \text{ iff } u = I(u) \text{ in } \Omega.$$

Using the fact that for any locally bounded function $u : \Omega \to \mathbb{R}$, the functions $S(u)$ and $I(u)$ are always, respectively, upper semicontinuous and lower semicontinuous, see (2.2.15) and (2.2.14), a viscosity solution for u can be defined as follows, see [16].

Definition 1.3.3 *A locally bounded function $u : \Omega \to \mathbb{R}$ is a (discontinuous) viscosity solution of the Hamilton-Jacobi equation (1.1.1) if $S(u)$ is a viscosity subsolution of (1.1.1) and $I(u)$ is a viscosity supersolution of (1.1.1).*

Note that the definition of viscosity subsolution and viscosity supersolution for semicontinuous functions is consistent with the concept of continuous viscosity solutions, because a function that is simultaneously a viscosity subsolution and viscosity supersolution is automatically continuous. Thus, if $u \in C(\Omega)$, then $I(u) = S(u) = u$ and Definition 1.2.1 coincides with Definition 1.3.3. From now on the expressions viscosity subsolution and supersolution are used in the sense of Definition 1.3.1 and viscosity solution is used in the sense of Definition 1.3.3.

The direct method of proving the existence of viscosity solutions of the Hamilton-Jacobi equation (1.1.1) is the *Perron's method* [71]. The idea is to build a viscosity solution as the supremum of viscosity subsolutions. This is an analogue for Hamilton-Jacobi equations to the well-known method of finding solutions of the Laplace equation due to O. Perron [113]. The Perron method is the following:
Let $\mathcal{U}$ be a nonempty set of viscosity subsolutions of (1.1.1) having the following two properties:
(i) if $v \in \mathcal{U}$ is not a viscosity solution of (1.1.1), then there is a function $w \in \mathcal{U}$ such that $w(y) > v(y)$ for some $y \in \Omega$,
(ii) if $u(x) = \sup\{v(x) : v \in \mathcal{U}\}$ for $x \in \Omega$, then $u \in \mathcal{U}$.
Set $u(x) = \sup\{v(x) : v \in \mathcal{U}\}$ for $x \in \Omega$. Then u is a viscosity solution of (1.1.1). Moreover, if $S(u) \leq I(u)$ on Ω, then $u \in C(\Omega)$.

Thus the existence problem of viscosity solutions is reduced to that of finding an appropriate set $\mathcal{U}$ of viscosity subsolutions of (1.1.1) which satisfies (i) and (ii), depending on boundary conditions, assumptions on Hamiltonians H and so on. Also, the continuity of viscosity solutions may follow from the special structure of (1.1.1) or from the comparison principle between viscosity subsolutions and supersolutions.

We will use the following result which was provided in [16].

Theorem 1.3.1 *(a) Let $\mathcal{U}$ be a set of functions such that $S(w)$ is a viscosity subsolution of (1.1.1) for all $w \in \mathcal{U}$, and define*

$$u(x) := \sup_{w \in \mathcal{U}} w(x), \; x \in \Omega.$$

If u is locally bounded, then $S(u)$ is a viscosity subsolution of (1.1.1).
(b) Let $\mathcal{Z}$ be a set of functions such that $I(w)$ is a viscosity supersolution of (1.1.1) for all $w \in \mathcal{Z}$, and define

$$v(x) := \inf_{w \in \mathcal{Z}} w(x), \; x \in \Omega.$$

If v is locally bounded, then $I(v)$ is a viscosity supersolution of (1.1.1).

The following general existence theorem for equation (1.1.1) using Perron's method is given in [16].

Theorem 1.3.2 *Assume there exists a viscosity subsolution u_1 and a viscosity supersolution u_2 of the Hamilton-Jacobi equation (1.1.1) such that $u_1 \leq u_2$. Then the functions*

$$(a) \; u(x) := \sup\{w(x) : u_1 \leq w \leq u_2, \; S(w) \textit{ viscosity subsolution of (1.1.1)}\},$$

$$(b) \; v(x) := \inf\{w(x) : u_1 \leq w \leq u_2, \; I(w) \textit{ viscosity supersolution of (1.1.1)}\},$$

are (discontinuous) viscosity solutions of (1.1.1).

Since we deal, in some sense, with two functions $I(u)$ and $S(u)$, it is not immediately clear how to interpret uniqueness with discontinuous viscosity solutions. For example, if u is equal to 1 in Ω except at x_0 where is equal to 0, then $S(u) \equiv 1$ and the subsolution condition does not differentiate between the function u and a function constantly equal to $f \equiv 1$. The usual way to get an uniqueness result is to obtain a comparison principle between viscosity supersolutions and viscosity subsolutions [21].

For uniqueness of viscosity solutions, let us specify what is meant by comparison principle for semicontinuous functions.

Definition 1.3.4 *We say that the Dirichlet problem for equation (1.1.1) satisfies the comparison principle or the comparison principle holds for Hamiltonians H if for any function $u \in BUSC(\overline{\Omega})$ and any function $v \in BLSC(\overline{\Omega})$ which are, respectively, a viscosity subsolution and a viscosity supersolution of (1.1.1) such that $u \leq v$ on $\partial\Omega$, we have $u \leq v$ in Ω.*

The next theorem shows that the comparison principle between viscosity subsolutions and viscosity supersolutions, given in Definition 1.3.4, implies the continuity of the unique viscosity solution.

Theorem 1.3.3 *[16] Assume that the Dirichlet problem for (1.1.1) satisfies a comparison principle. If $u : \overline{\Omega} \to \mathbb{R}$ is a viscosity solution of (1.1.1) which is bounded on Ω and continuous at all $x \in \partial\Omega$, then $u \in C(\overline{\Omega})$. In particular, for any $g \in C(\partial\Omega)$ in* (1.1.2), *there is at most one such viscosity solution of (1.1.1) satisfying $u = g$ on $\partial\Omega$.*

Let u be a viscosity solution of (1.1.1) which is bounded on Ω and $u \in C(\partial\Omega)$. Then $S(u) \in BUSC(\overline{\Omega})$ is a viscosity subsolution and $I(u) \in BLSC(\overline{\Omega})$ is a viscosity supersolution of (1.1.1). Since u is continuous at $\partial\Omega$, we have $S(u)(x) = I(u)(x) = u(x), x \in \partial\Omega$. Therefore, by the comparison principle, we have $S(u) \leq I(u)$ in $\overline{\Omega}$. Since by construction $I(u) \leq S(u)$, we have $S(u) = I(u) = u$ is a continuous viscosity solution, but it is a too restrictive conclusion since a lot of applications has naturally discontinuous solutions. Therefore, the above comparison principle adds nothing to the problem of uniqueness of discontinuous viscosity solution.

Next we give some simple examples showing that the notion of viscosity solution, given in Definition 1.3.3, is rather weak. The first one shows that nowhere continuous functions can be solutions, in terms of Definition 1.3.3, of very simple equations such as

$$u_x = 0 \qquad x \in \mathbb{R}. \tag{1.3.3}$$

Example 1.3.1 *Let u be function defined by*

$$u(x) = \begin{cases} 1 & , \quad x \in \mathbb{Q} \\ 0 & , \quad x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $S(u) \equiv 1$ and $I(u) \equiv 0$, so u is a viscosity solution of (1.3.3).

The second example explains the nonuniqueness of Ishii's results, namely a boundary value problem satisfying comparison principle, given in Definition 1.3.4, may have infinitely many discontinuous viscosity solutions satisfying boundary conditions.

Example 1.3.2 *The Dirichlet problem*

$$u_x(x) = 1,\ x \in (0,1) \tag{1.3.4}$$

$$u(0) = 1,\ u(1) = 2 \tag{1.3.5}$$

has the continuous viscosity solution $u(x) = x+1$ *by Theorem 1.3.3 and* u *satisfies (1.3.5). In addition, for any dense subset* X *of* $[0,1]$ *such that* 0 *and* 1 *belong to* X*, the function*

$$v(x) = \begin{cases} x \quad\quad , x \in [0,1]\backslash X \\ x+1, x \in [0,1] \cap X \end{cases}$$

is also a viscosity solution because $S(v)(x) = x+1$ *and* $I(v)(x) = x$ *are both classical solutions of the equation (1.3.4). Moreover,* v *satisfies the boundary conditions (1.3.5).*

Refer to Theorem 1.3.3, if we require continuity at all boundary points, then any function which is viscosity solution in the sense of Definition 1.3.3 is automatically continuous in $\overline{\Omega}$ and this rules out many interesting problems.

Then for Hamiltonians satisfying the structural assumptions of any comparison theorems, a viscosity solution of (1.1.1) may be a discontinuous function only if it is discontinuous at some boundary point. This fact is a general property of Hamilton-Jacobi equations.

Due to the nonuniqueness in Ishii's result, see Example 1.3.2, other notions of discontinuous solutions were proposed by various authors. Barles and Perthame [20], Barron and Jensen [23], Subbotin [131] made efforts in studying the discontinuous solutions. Their notions are in the context of semicontinuous solutions. But the definition of Ishii, given in [71], played a pivotal role. Other notions of discontinuous solutions are also introduced by Giga Sato [57], C. Guiqiang and S. Bo [62], Siconolfi [127], etc.

For presentation of discontinuous viscosity solution, the reader can consult Crandall-Ishii-Lions [37], Fleming-Soner [53], Bardi-Capuzzo-Dolcetta [16], Bardi et al. [17], and Barles [19] and the references there in.

The application of Perron's method for establishing an existence result for L^p-viscosity solutions of fully nonlinear second-order elliptic equations is given in [78].

1.4 Objectives of this Thesis

The first main goal of this thesis is to propose a new approach to the treatment of discontinuous solutions for first-order Hamilton-Jacobi equations by involving Hausdorff continuous interval valued functions. The only assumption is a continuity of Hamiltonian.

Applications of Hausdorff continuous functions are relevant because it has been shown recently by R. Anguelov and E. E. Rosinger [12] that the solutions of large classes of non-linear partial differential equations can be assimilated with Hausdorff continuous functions on the open domains.

To reach this goal, the following objectives are identified:

• to show that interval valued functions can be considered as discontinuous viscosity solutions;

• to prove that the main ideas within the classical theory of continuous viscosity solutions can be extended almost unchanged to the wider space of Hausdorff continuous functions;

• to prove that the Hausdorff continuous viscosity solutions have a more clear interpretation than the existing concepts of discontinuous solutions, e.g., envelope solutions;

• to show that the value function of optimal control problem as solution of associated Hamilton-Jacobi-Bellman equation typically belongs to the wider space of Hausdorff continuous functions.

The second main goal is the design of numerical schemes for Hamilton-Jacobi equations and for conservation laws which preserve essential properties of the exact solutions. This is typically expressed through the concept of qualitative stability. To achieve this we use the nonstandard finite difference method to design

• a scheme for Hamilton-Jacobi equation which is qualitatively stable with respect to monotonicity property;

• schemes for conservation laws which are qualitatively stable to respect to total variation diminishing property.

1.5 Outline of this Thesis

From Definition 1.3.3, interval valued functions appear naturally in the context of discontinuous viscosity solutions. Indeed, Definition 1.3.3 places requirements not on the function u itself but on its lower and upper semicontinuous envelopes or, in other words, on the interval valued function

$$F(u)(x) = [I(u)(x), S(u)(x)],\ x \in \Omega,$$

which is called the *graph completion* of u, see [15]. Clearly, Definition 1.3.3 treats functions which have the same upper and lower semicontinuous envelopes, that is, have the same graph completion, as identical functions. On the other hand, since different functions can have the same graph completion, a function can not in general be identified from its graph completion, that is, functions with the same graph completion are indistinguishable. Therefore, no generality will be lost if only interval valued functions representing graph completions are considered.

The second chapter introduces the concept of Hausdorff continuous (H-continuous) interval valued functions and discusses their properties. The applications of H-continuous functions to problems in Analysis [5] and to nonlinear partial differential equation [13] are based on the quite extraordinary fact that the set of all Hausdorff continuous functions on open domains is Dedekind order complete. This property is given in this chapter.

In Chapter 3, which contains the first main result of this thesis, we discuss Hausdorff continuous viscosity solutions of Hamilton-Jacobi equations and we prove that the notion of Hausdorff continuous viscosity solution is stronger than the notion of (discontinuous) viscosity solution, see Section 3.2. In the second place, we show that when the H-continuous viscosity solution is a supremum of viscosity subsolutions or infimum of viscosity supersolutions, it can be linked to an envelope viscosity solution. This is given in Section 3.3. In Section 3.4, we begin with some properties of Hausdorff continuous viscosity subsolutions and supersolutions. Moreover, we formulate and prove an existence theorem for Hamilton-Jacobi equations using Perron's method for H-continuous viscosity solutions. Section 3.5 deals with uniqueness of H-continuous viscosity solution. We formulate the comparison principle for Hausdorff continuous functions and give sufficient conditions implying it in a weaker form. The uniqueness result, see Theorem 3.5.1, is given under the assumption that comparison principle, given in Definition 3.5.1, is satisfied.

Finally we express the H-continuous viscosity solutions of Hamilton-Jacobi equation as solutions to an operator equation involving the extended Hamiltonian operator in the same way as the classical solutions of Hamilton-Jacobi equation are solutions of operator equation associated of this Hamilton-Jacobi equation.

In Chapter 4, the theory of Hausdorff continuous viscosity solutions is applied to optimal control problem in particular discounted minimum time problem. We show that the value function of discounted minimum time problem is an envelope viscosity solution of an associated Hamilton-Jacobi-Bellman equation. This is illustrated by a Zermelo navigation problem given in Section 4.3.

The second main result is given in Chapter 5. We consider two approaches to numerical solutions for Hamilton-Jacobi equations. The first one, given in Section 5.2, is a monotone scheme for Hamilton-Jacobi equation. The study of numerical approximation to multi-dimensional Hamilton-Jacobi equations was started by Crandall and Lions [40]. They presented monotone finite difference schemes on rectangular domains. The numerical solutions in [40, Theorem 1] indicate convergence to the viscosity solutions of Hamilton-Jacobi equations. Finite difference methods were developed for solving Hamilton-Jacobi equations [128, 92, 111]. However, the schemes applying standard finite difference techniques are typically monotone under some restriction on the time step sizes. We motivate by the paper [91] where a severe restriction on the time step size is imposed for the numerical scheme for Hamilton-Jacobi equations obtained through the coupling of the finite difference method (in space) and the finite element method (in time) to be monotone. In Section 5.2, we relax this restriction by using Micken's nonstandard finite difference method [103]. More precisely, Micken's rule of nonlocal approximation is exploited and this leads to a nonstandard scheme that replicates the monotonicity property of the Hamilton-Jacobi equations for all positive time step sizes. Furthermore, the superiority of the nonstandard method to the standard one is confirmed by numerical results at the end of this section.

The second approach is based on preserving total variation diminishing property of conservation laws. It has been shown that schemes with such qualitative stability resolve discontinuities in the solution without spurious oscillations which are often displayed by numerical solutions [124], [125]. These schemes are called total variation diminishing (TVD).

One problem associated with the explicit total variation diminishing methods is a restriction on the time step-size which in some cases could be rather severe. This is particularly pronounced in high-order methods, e.g., methods of Runge-Kutta type [60, 61]. On the other hand, the computational complexity of total variation diminishing implicit methods is significantly higher particularly when nonlinear functions are involved.

In Section 5.3, we construct (i) an implicit nonstandard finite difference scheme using nonlocal approximation of nonlinear terms and (ii) explicit nonstandard finite difference schemes where renormalization of the denominator is used. Numerical results demonstrating the properties of the methods are presented.

It is well known that the Hamilton-Jacobi equations are closely related to conservation laws [77], hence successful numerical methods for conservation laws are adapted for solving the Hamilton-Jacobi equations. A long this line, we mention the early work of Osher and Sethian [110] and Osher and Shu [111] in constructing high-order essentially non-oscillatory (ENO) schemes for solving the Hamilton-Jacobi equations. These ENO schemes for solving the Hamilton-Jacobi equations were based on ENO schemes for solving hyperbolic conservation laws in [65, 125, 126]. We mention also the weighted (ENO) (WENO) schemes for solving the Hamilton-Jacobi equations by Zhang and Shu [137] and by Jiang and Peng [72], based on the WENO schemes for solving conservation laws [95, 74]. Adapted from the discontinuous Galerkin methods for solving hyperbolic conservation laws [33], a discontinuous Galerkin method for solving Hamilton-Jacobi equations was developed by Hu and Shu in [70].

At the end of Section 5.3, we use a discontinuous Galerkin finite element method of Hu and Shu [70] to solve the one dimensional Hamilton-Jacobi equation which applies the discontinuous Galerkin framework on the conservations laws. Namely, since the derivative of the solution u of Hamilton-Jacobi equation satisfies a conservation law, we apply the usual discontinuous Galerkin method on this conservation law to advance the derivative of u. The solution u itself is then recovered from this derivative computed using nonstandard total variation diminishing method.

Finally, Chapter 6 summarizes the results that have been done in the thesis, highlights the most outstanding results and gives some directions for future research.

1.6 Summary of Contributions

The main contributions of this thesis are:

• The concept of Hausdorff continuous viscosity solution is given in Section 3.2, see Definition 3.2.1.

• Existence theorem for H-continuous viscosity solutions using Perron's method. The solution is constructed as a supremum of a set of viscosity subsolutions in the set of Hausdorff continuous functions, Theorem 3.4.2.

• The relation between the H-continuous viscosity solution and the existing theory of discontinuous viscosity solutions. Namely, any H-continuous viscosity solution is discontinuous viscosity solution as defined by Ishii, and it is typically also an envelope viscosity solution. The H-continuous viscosity solution is a stronger concept than the concept of discontinuous viscosity solution given by Ishii and as well as the concept of envelope viscosity solution. Yet, the existence is proved under the same assumptions, see Sections 3.2 and 3.3.

• In Section 3.5, the Hausdorff continuous viscosity solutions of Hamilton-Jacobi equations are expressed as the solutions to an operator equation involving the extended operator in the same way as the classical solutions of Hamilton-Jacobi equations are solutions of this operator equation.

• The value function of the optimal control problem can be considered as H-continuous viscosity solution of associated Hamilton-Jacobi-Bellman equation, see Chapter 4.

• The design of nonstandard finite difference scheme for Hamilton-Jacobi equation preserving the monotonicity property for any time step size is given in Section 5.2.

• The design of nonstandard finite difference schemes for conservation laws preserving the total variation diminishing property for all positive time step sizes is given in Section 5.3.