Hausdorff continuous viscosity solutions of Hamilton-Jacobi equations and their numerical analysis

by

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Submitted in partial fulfillment
of the requirements for the degree

Philosophiae Doctor

in the Faculty of Natural and Agricultural Sciences

University of Pretoria

Pretoria

September 2007

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Acknowledgements

I extend my sincerest gratitude towards to following individuals and organizations, all of whom have play some part in the successful completion of this work.

• To my supervisor Prof. R. Anguelov my deepest and heartfelt gratitude. He is the person who made it possible for me to undertake and complete these doctoral studies.

• I would like to thank Prof. J. M.-S. Lubuma and Prof. E. E. Rosinger for their enthusiasm about and their interest in my research.

• Thank you Prof. J. P. Luhandjula from University of South Africa and all friends for their encouragements.

• I thank my family, especially my mother Melanie Nyiranvano, my wife Devote Muyisenge and my children for their kindness and tolerance, and for never losing faith in me trough very difficulty times.

• The Ministry of Education of Rwanda under the World Bank Human Resource Development Project provided financial support for which I am grateful.

Dedicated to the memory of my father and brothers

TO THE GLORY OF GOD

Hausdorff continuous viscosity solutions of Hamilton-Jacobi equations and their numerical analysis

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Degree: PhD

Abstract

The theory of viscosity solutions was developed for certain types of nonlinear first-order and second-order partial differential equations. It has been particularly useful in describing the solutions of partial differential equations associated with deterministic and stochastic optimal control problems [16], [53]. In its classical formulation, see [16], the theory deals with solutions which are continuous functions. The concept of continuous viscosity solutions was further generalized in various ways to include discontinuous solutions with the definition of Ishii given in [71] playing a pivotal role. In this thesis we propose a new approach for the treatment of discontinuous solutions of first-order Hamilton-Jacobi equations, namely, by involving Hausdorff continuous interval valued functions.

The advantages of the proposed approach are justified by demonstrating that the main ideas within the classical theory of continuous viscosity solutions can be extended almost unchanged to the wider space of Hausdorff continuous functions and the existing theory of discontinuous viscosity solutions is a particular case of that developed in this thesis in terms of Hausdorff continuous interval valued functions.

Two approaches to numerical solutions for Hamilton-Jacobi equations are presented. The first one is a monotone scheme for Hamilton-Jacobi equations while the second is based on preserving total variation diminishing property for conservation laws.

In the first approach, we couple the finite element method with the nonstandard finite difference method which is based on the Mickens' rule of nonlocal approximation [9]. The scheme obtained in this way is unconditionally monotone.

In the second approach, computationally simple implicit schemes are derived by using nonlocal approximation of nonlinear terms. Renormalization of the denominator of the discrete derivative is used for deriving explicit schemes of first or higher order. Unlike the standard explicit methods, the solutions of these schemes have diminishing total variation for any time step size.

DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

Signature:

Name: Froduald Minani **Date: 2007/December/06**

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$\mathbb{R}^n$	$n-$dimensional Euclidean space
$\mathbb{R} = \mathbb{R}^1$	set of all real numbers
$\mathbb{Z}$	set of all integers
$\mathbb{N}$	set of all positive integers
$\mathbb{Q}$	set of rational numbers
Ω	subset of $\mathbb{R}^n$ (open, in most cases)
$\partial\Omega$	boundary of the set Ω
$\overline{\Omega} = \Omega \cup \partial\Omega$	closure of the set Ω
$x = (x_1, ..., x_n)$	a typical point in $\mathbb{R}^n$
$x.y = \sum\limits_{i=1}^{n} x_i\, y_i$	scalar product of elements x and y of $\mathbb{R}^n$
$\lvert x\rvert = \lvert x\rvert_2 = (\sum\limits_{i=1}^{n} x_i^2)^{1/2}$	Euclidean norm of x in $\mathbb{R}^n$
$[a, b]$	closed real interval
(a, b)	open real interval
$B_r(a)$	open ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$
$\overline{B_r}(a)$	closed ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$
$\alpha = (\alpha_1, ..., \alpha_n)$	multi-index, $\alpha_i \in \mathbb{N}, i = 1, ..., n$
$\lvert\alpha\rvert = \sum\limits_{j=1}^{n} \alpha_j$	length of multi-index α
$C^m(\Omega)$	space of functions having m continuous derivatives on $\Omega, m \in \mathbb{N}$
$C(\Omega) = C^0(\Omega)$	space of continuous functions on Ω
	and uniformly continuous on bounded subset of Ω
$D^\alpha u(x)$	partial derivative :$D^\alpha u(x) = \frac{\partial^{\lvert\alpha\rvert} u(x)}{\partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}}$
$D^m u(x)$	set of all partial derivatives of order m: $Du(x) = \{D^\alpha u(x) : \lvert\alpha\rvert = m\}$
$Du(x) = \nabla u(x)$	gradient of u at a point x in $\mathbb{R}^n$: $Du(x) = (u_{x_1}, ..., u_{x_n})$
$\nabla^2 u(x) = \sum\limits_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$	Laplacian of u at x
$C^\infty(\Omega) = \bigcap\limits_{k\geq 0} C^k(\Omega)$	space of infinitely differentiable functions on Ω

supp(u)	support of the function u
$C_0(\Omega)$	space of continuous functions having compact support in Ω
$C_0^m(\Omega)$	space $C_0(\Omega) \cap \mathcal{C}^m(\Omega)$
$C_0^\infty(\Omega)$	space $C_0(\Omega) \cap C^\infty(\Omega)$
$L^p(\Omega)$	usual space of measurable functions
	whose pth power is Lebesgue integrable on Ω
$L^\infty(\Omega)$	space of measurable functions
	which are bounded almost everywhere on Ω
$\|\|u\|\|_\infty = \{\sup_{x\in\Omega} \|u(x)\| < \infty\}$	norm of the function u in $L^\infty(\Omega)$
$\|u\|_{m,\infty} = \|\|D^m u\|\|_\infty$	norm of the function $D^m u$ in $L^\infty(\mathbb{R}^2)$
$f\|_A$	restriction of the function f to the set A
$C^{0,1}(\Omega)$	space of Lipshitz continuous functions on Ω
$USC(\Omega)$	set of upper semicontinuous functions on Ω
$LSC(\Omega)$	set of lower semicontinuous functions on Ω
$BUSC(\Omega)$	set of bounded upper semicontinuous functions on Ω
$BLSC(\Omega)$	set of bounded lower semicontinuous functions on Ω
$BUC(\Omega)$	set of bounded and uniformly continuous functions on Ω
$\mathbb{IR}$	set of finite closed real intervals: $\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \leq \overline{a}\}$
$\mathbb{A}(\Omega)$	set of locally bounded functions on Ω with values
	which are finite closed real intervals
$\mathcal{A}(\Omega)$	set of locally bounded functions with real values
$u = [\underline{u}, \overline{u}]$	function in $\mathbb{A}(\Omega)$, where $\underline{u}, \overline{u} \in \mathcal{A}(\Omega)$
$\mathbb{F}(\Omega)$	set of segment continuous functions on Ω
$\mathbb{H}(\Omega)$	set of Hausdorff continuous functions on Ω