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APPENDIX A

Derivations for chapter 1

A.1 Background (dynamic stiffness of the relaxation model)

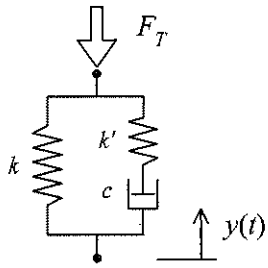

Figure A.1: Mechanical model of a relaxation isolator

Since the spring k' is in series with the dashpot, the force generated by each must be equal:

$$\begin{aligned} c(\dot{y}-\dot{u}) &= k'u \\ i\omega c(Y-U) &= k'U \\ i\omega cY &= (k'+i\omega c)U \\ U &= \frac{i\omega c}{k'+i\omega c}Y \end{aligned} \tag{A.1}$$

The total force of the two elements in parallel is:

$$\begin{aligned} F_T &= kY + k'U \\ &= kY + k'\frac{i\omega c}{k'+i\omega c}Y \end{aligned} \tag{A.2}$$

The above equation can be written in non-dimensional form by dividing with k:

$$\begin{aligned} \frac{F_T}{kY} &= 1+\frac{k'}{k}\frac{i\omega\frac{c}{k'}}{1+i\omega\frac{c}{k'}} \\ &= 1+\frac{k'}{k}\frac{i\omega\frac{c}{k}}{\frac{k'}{k}+i\omega\frac{c}{k}} \end{aligned} \tag{A.3}$$

A.2 Isolators

A.2.1 Passive isolator (intermediate mass isolator)

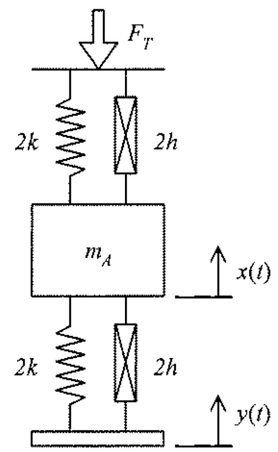

Figure A.2: Mechanical model of an intermediate mass isolator

The equation of motion is:

$$m_A\ddot{x}+2k(1+i\eta)x=-2k(1+i\eta)(x-y)$$
$$m_A\ddot{x}+4k(1+i\eta)x=2k(1+i\eta)y \qquad (A.4)$$

The equation of motion can be rewritten in the frequency domain:

$$\left[4k(1+i\eta)-\omega^2 m_A\right]X=2k(1+i\eta)Y \qquad (A.5)$$

The transmissibility is now:

$$\frac{X}{Y}=\frac{2k(1+i\eta)}{4k(1+i\eta)-\omega^2 m_A} \qquad (A.6)$$

The dynamic stiffness is a function of the displacement of mass m_A:

$$F_T=2k(1+i\eta)X \qquad (A.7)$$

Using the transmissibility the equation can be written in terms of the input displacement amplitude Y:

$$F_T=2k(1+i\eta)\left[\frac{2k(1+i\eta)}{4k(1+i\eta)-\omega^2 m}\right]Y \qquad (A.8)$$

When normalised, the above equation becomes:

$$\frac{F_T}{kY} = \frac{(1+i\eta)^2}{1+i\eta-\left(\frac{\omega}{\omega_n}\right)^2} \tag{A.9}$$

$$\text{where:} \quad \omega_n = \sqrt{\frac{4k}{m_A}}$$

A.2.2 Active isolator (absolute velocity feedback isolator)

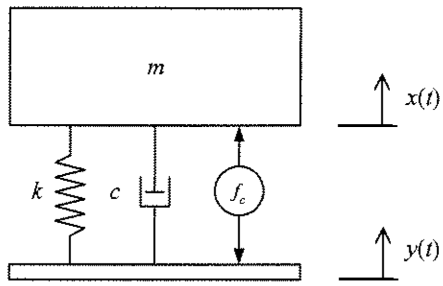

Figure A.3: Mechanical model with skyhook damping

The control force is proportional to the velocity, but in the opposite direction:

$$f_c(t) = -\beta\dot{x} \tag{A.10}$$

The equation of motion including the control force is:

$$\begin{aligned} m\ddot{x}+c\dot{x}+kx &= c\dot{y}+ky-\beta\dot{x} \\ m\ddot{x}+(c-\beta)\dot{x}+kx &= c\dot{y}+ky \end{aligned} \tag{A.11}$$

Transforming the equation to the frequency domain:

$$\left[-\omega^2 m+i\omega(c+\beta)+k\right]X = (i\omega c+k)Y \tag{A.12}$$

The transmissibility is:

$$\frac{X}{Y} = \frac{k+i\omega c}{k+i\omega(c+\beta)-\omega^2 m} \tag{A.13}$$

When normalised, the above equation becomes:

$$\frac{X}{Y} = \frac{1+i2\frac{\omega}{\omega_n}\zeta}{1+i2\frac{\omega}{\omega_n}(\zeta+\zeta_\beta)-\left(\frac{\omega}{\omega_n}\right)^2} \tag{A.14}$$

$$\text{where:} \quad \omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2m\omega_n}, \quad \zeta_\beta = \frac{\beta}{2m\omega_n}$$

A.2.3 Active isolator (general feedforward active isolator)

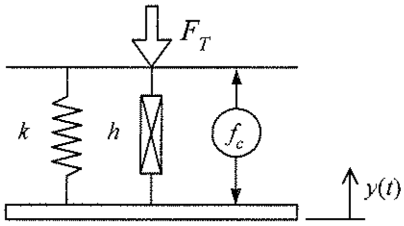

Figure A.4: Mechanical model of an active isolator

The control force is:

$$f_c(t) = \alpha\ddot{y} + (\gamma + i\beta)y \tag{A.15}$$

The total force transmitted is the sum of the spring, damper and control force:

$$\begin{aligned} f_t &= k(1+i\eta)y + \alpha\ddot{y} + (\gamma + i\beta)y \\ &= (k+\gamma)y + i(k\eta+\beta)y + \alpha\ddot{y} \end{aligned} \tag{A.16}$$

When transformed to the frequency domain the above equation becomes:

$$F_T = (k+\gamma)Y + i(k\eta+\beta)Y - \alpha\omega^2 Y \tag{A.17}$$

When normalised, the above equation becomes:

$$\begin{aligned} \frac{F_T}{(k+\gamma)Y} &= 1 + i\frac{k\eta+\beta}{k+\gamma} - \frac{\alpha}{k+\gamma}\omega^2 \\ &= 1 + i\frac{k\eta+\beta}{k+\gamma} - \left(\frac{\omega}{\omega_i'}\right)^2 \\ \text{where: } \omega_i' &= \sqrt{\frac{k+\gamma}{\alpha}} \end{aligned} \tag{A.18}$$

A.3 Vibration-absorbing isolators

A.3.1 Passive vibration-absorbing isolator

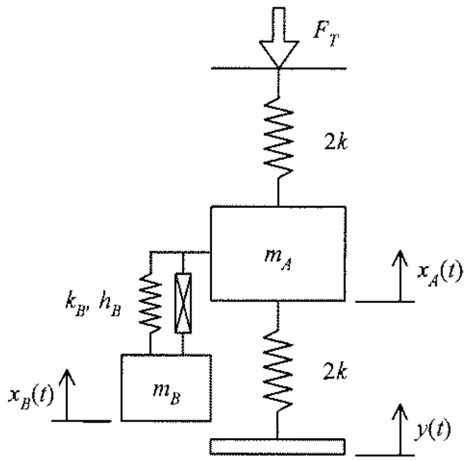

Figure A.5: Mechanical model of a passive vibration-absorbing isolator

The kinetic energy of the system is:

$$T = \frac{1}{2}\left(m_A \dot{x}_A^2 + m_B \dot{x}_B^2\right) \tag{A.19}$$

From the above equation the derivatives can be found:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_A}\right) &= m_A \ddot{x}_A \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_B}\right) &= m_B \ddot{x}_B \end{aligned} \tag{A.20}$$

The potential energy is:

$$\begin{aligned} V &= \frac{1}{2}\left[2k\left(x_A - y\right)^2 + 2kx_A^2 + k_B\left(x_A - x_B\right)^2\right] \\ &= \frac{1}{2}\left[2k\left(x_A^2 - x_A y + y^2\right) + 2kx_A^2 + k_B\left(x_A^2 - x_A x_B + x_B^2\right)\right] \end{aligned} \tag{A.21}$$

From the above equation the derivatives can be found:

$$\begin{aligned} \frac{\partial V}{\partial x_A} &= 2kx_A - 2ky + 2kx_A - k_B x_B - k_B x_A = \left(4k - k_B\right)x_A - k_B x_B - 2ky \\ \frac{\partial V}{\partial x_B} &= -k_B x_A + k_B x_B \end{aligned} \tag{A.22}$$

The equation of motion can be found by substituting the derivates in Lagrange's equations and incorporating a hysteretic damping model:

$$\begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix}\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix} + \begin{bmatrix} 4k(1+i\eta)+k_B(1+i\eta_B) & -k_B(1+i\eta_B) \\ -k_B(1+i\eta_B) & k_B(1+i\eta_B) \end{bmatrix}\begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 2k(1+i\eta)y \\ 0 \end{bmatrix} \quad \text{(A.23)}$$

The second equation can be rewritten in the frequency domain as:

$$\begin{gathered} -k_B(1+i\eta_B)X_A + \left[k_B(1+i\eta_B)-\omega^2 m_B\right]X_B = 0 \\ X_B = \frac{k_B(1+i\eta_B)}{k_B(1+i\eta_B)-\omega^2 m_B}X_A \end{gathered} \quad \text{(A.24)}$$

By substituting for X_B in the first equation the response of X_A to the input Y can be found:

$$\begin{gathered} \left[4k(1+i\eta)+k_B(1+i\eta_B)-\omega^2 m_A\right]X_A - k_B(1+i\eta_B)\frac{k_B(1+i\eta_B)}{k_B(1+i\eta_B)-\omega^2 m_B}X_A = 2k(1+i\eta)Y \\ \left\{\left[4k(1+i\eta)+k_B(1+i\eta_B)-\omega^2 m_A\right]\left[k_B(1+i\eta_B)-\omega^2 m_B\right]-k_B^2(1+i\eta_B)^2\right\}X_A \\ = 2k(1+i\eta)\left[k_B(1+i\eta_B)-\omega^2 m_B\right]Y \\ \frac{X_A}{Y} = \frac{2k(1+i\eta)\left[k_B(1+i\eta_B)-\omega^2 m_B\right]}{\left[4k(1+i\eta)+k_B(1+i\eta_B)-\omega^2 m_A\right]\left[k_B(1+i\eta_B)-\omega^2 m_B\right]-k_B^2(1+i\eta_B)^2} \end{gathered} \quad \text{(A.25)}$$

The transmitted force can now be found in terms of the input by using the above transmissibility:

$$\begin{aligned} F_T &= 2k(1+i\eta)X_A \\ &= \frac{4k^2(1+i\eta)^2\left[k_B(1+i\eta_B)-\omega^2 m_B\right]}{\left[4k(1+i\eta)+k_B(1+i\eta_B)-\omega^2 m_A\right]\left[k_B(1+i\eta_B)-\omega^2 m_B\right]-k_B^2(1+i\eta_B)^2}Y \end{aligned} \quad \text{(A.26)}$$

Rearranging the above equation gives the normalised dynamic stiffness:

$$\frac{F_T}{kY} = \frac{4k(1+i\eta)k_B(1+i\eta_B)\left(1+i\eta_B-\omega^2\frac{m_B}{k_B}\right)}{4k(1+i\eta)k_B(1+i\eta_B)\left\{\left[1+i\eta+\frac{1}{4}\frac{k_B}{k}(1+i\eta_B)-\omega^2\frac{m_A}{4k}\right]\left(1+i\eta_B-\omega^2\frac{m_B}{k_B}\right)-\frac{1}{4}\frac{k_B}{k}(1+i\eta_B)\right\}} \quad \text{(A.27)}$$

This equation can further be manipulated to be a function of non-dimensional terms only by introducing the natural frequencies:

$$\frac{F_T}{kY} = \frac{1+i\eta_B-\left(\frac{\omega}{\omega_B}\right)^2}{\left[1+i\eta+\frac{1}{4}\frac{k_B}{k}(1+i\eta_B)-\left(\frac{\omega}{\omega_A}\right)^2\right]\left[1+i\eta_B-\left(\frac{\omega}{\omega_B}\right)^2\right]-\frac{1}{4}\frac{k_B}{k}(1+i\eta_B)} \quad \text{(A.28)}$$

where: $\omega_A = \sqrt{\frac{4k_A}{m_A}} \quad \omega_B = \sqrt{\frac{k_B}{m_B}}$

The frequencies at which the dynamic stiffness is equal to 1 can be calculated to define the bandwidth of the device. For an undamped system the condition is:

$$\left|\frac{F_T}{kY}\right| = 1$$
$$\therefore \frac{F_T}{kY} = \pm 1 \tag{A.29}$$

The stiffness ratio is given by $\mu_k = k_B/k_A$. For $F_T/kY = 1$:

$$\frac{F_T}{kY} = \frac{1-\left(\frac{\omega}{\omega_B}\right)^2}{\left[1+\frac{1}{4}\mu_k-\left(\frac{\omega}{\omega_A}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_B}\right)^2\right]-\frac{1}{4}\mu_k} = 1$$
$$1-\left(\frac{\omega}{\omega_B}\right)^2 = 1-\left(\frac{\omega}{\omega_B}\right)^2+\frac{1}{4}\mu_k-\frac{1}{4}\mu_k\left(\frac{\omega}{\omega_B}\right)^2-\left(\frac{\omega}{\omega_A}\right)^2+\left(\frac{\omega}{\omega_A}\right)^2\left(\frac{\omega}{\omega_B}\right)^2-\frac{1}{4}\mu_k$$
$$\frac{1}{4}\mu_k\left(\frac{1}{\omega_B}\right)^2+\left(\frac{1}{\omega_A}\right)^2=\left(\frac{1}{\omega_A}\right)^2\left(\frac{1}{\omega_B}\right)^2\omega^2 \tag{A.30}$$
$$\left[\frac{1}{4}\mu_k\frac{1}{\omega_B^2}+\frac{1}{\omega_A^2}\right]\omega_B^2\omega_A^2=\omega^2$$
$$\omega^2=\left[\frac{1}{4}\mu_k\omega_A^2+\omega_B^2\right]$$
$$\omega_1=0,\quad \frac{\omega_2}{\omega_A}=\sqrt{\frac{1}{4}\mu_k+\left(\frac{\omega_B}{\omega_A}\right)^2}$$

For $F_T/kY = -1$:

$$\frac{F_T}{kY}\frac{1-\left(\frac{\omega}{\omega_B}\right)^2}{\left[1+\frac{1}{4}\mu_k-\left(\frac{\omega}{\omega_A}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_B}\right)^2\right]-\frac{1}{4}\mu_k} = -1$$
$$1-\left(\frac{\omega}{\omega_B}\right)^2 = -1+\left(\frac{\omega}{\omega_B}\right)^2-\frac{1}{4}\mu_k+\frac{1}{4}\mu_k\left(\frac{\omega}{\omega_B}\right)^2+\left(\frac{\omega}{\omega_A}\right)^2-\left(\frac{\omega}{\omega_A}\right)^2\left(\frac{\omega}{\omega_B}\right)^2+\frac{1}{4}\mu_k \tag{A.31}$$
$$0=\omega^4-\left(\frac{1}{4}\mu_k\omega_A^2+2\omega_A^2+\omega_B^2\right)\omega^2+2\omega_A^2\omega_B^2$$

The solution for the above equation is:

$$\omega^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{\frac{1}{4}\mu_k\omega_A^2 + 2\omega_A^2 + \omega_B^2 \pm \sqrt{\left(\frac{1}{4}\mu_k\omega_A^2 + 2\omega_A^2 + \omega_B^2\right)^2 - 8\omega_A^2\omega_B^2}}{2}$$
$$\frac{\omega_3}{\omega_A} = \sqrt{\frac{\frac{1}{4}\mu_k + 2 + \left(\frac{\omega_B}{\omega_A}\right)^2 - \sqrt{\left(\frac{1}{4}\mu_k + 2 + \left(\frac{\omega_B}{\omega_A}\right)^2\right)^2 - 8\left(\frac{\omega_B}{\omega_A}\right)^2}}{2}} \tag{A.32}$$
$$\frac{\omega_4}{\omega_A} = \sqrt{\frac{\frac{1}{4}\mu_k + 2 + \left(\frac{\omega_B}{\omega_A}\right)^2 + \sqrt{\left(\frac{1}{4}\mu_k + 2 + \left(\frac{\omega_B}{\omega_A}\right)^2\right)^2 - 8\left(\frac{\omega_B}{\omega_A}\right)^2}}{2}}$$

A.3.2 Passive vibration-absorbing isolator (multiple-absorber VAI)

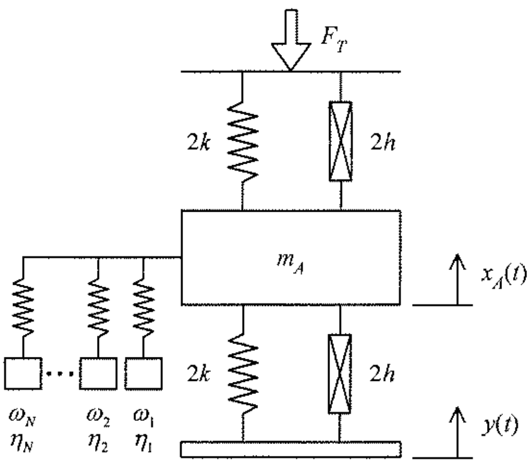

Figure A.6: Mechanical model of a multiple-absorber VAI

The derivation follows from the previous case where one absorber was considered. The kinetic energy is:

$$T = \frac{1}{2}\left(m_A\dot{x}_A^2 + \sum_{q=1}^{N} m_i\dot{x}_i^2\right) \tag{A.33}$$

From the above equation the derivatives can be found:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_A}\right) = m_A\ddot{x}_A$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_q}\right) = m_q\ddot{x}_q \tag{A.34}$$

The potential energy is:

$$\begin{aligned} V &= \frac{1}{2}\left[2k\left(x_A - y\right)^2 + 2kx_A^2 + \sum_{q=1}^{N} k_q\left(x_A - x_q\right)^2\right] \\ &= \frac{1}{2}\left[4kx_A^2 - 4kx_A y + 2ky^2 + \sum_{q=1}^{N} k_q x_A^2 - 2k_q x_q x_A + k_q x_q^2\right] \end{aligned} \tag{A.35}$$

From the above equation the derivatives can be found:

$$\begin{aligned} \frac{\partial V}{\partial x_A} &= \left(4k + \sum_{q=1}^{N} k_q\right)x_A - \sum_{q=1}^{N} k_q x_q - 2ky \\ \frac{\partial V}{\partial x_q} &= -k_q x_A + k_q x_q \end{aligned} \tag{A.36}$$

The equation of motion can be found by substituting the derivates in Lagrange's equations and incorporating a hysteretic damping model:

$$\begin{bmatrix} m_A & 0 & 0 & \cdots & 0 \\ 0 & m_1 & 0 & \cdots & 0 \\ 0 & 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & m_N \end{bmatrix} \begin{bmatrix} \ddot{x}_A \\ \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_N \end{bmatrix} +$$

$$\begin{bmatrix} 4k(1+i\eta) + \sum_{q=1}^{N} k_q(1+i\eta_q) & -k_1(1+i\eta_1) & -k_2(1+i\eta_2) & \cdots & -k_N(1+i\eta_N) \\ -k_1(1+i\eta_1) & k_1(1+i\eta_1) & 0 & \cdots & 0 \\ -k_2(1+i\eta_2) & 0 & k_2(1+i\eta_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -k_N(1+i\eta_N) & 0 & 0 & 0 & k_N(1+i\eta_N) \end{bmatrix} \begin{bmatrix} x_A \\ x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} 2k(1+i\eta) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} y \tag{A.37}$$

The first equation can be transformed to the frequency domain:

$$\left[1 + i\eta + \frac{1}{4}\sum_{q=1}^{N} \frac{k_q}{k}(1+i\eta_q) - \left(\frac{\omega}{\omega_A}\right)^2\right] X_A - \frac{1}{4}\sum_{q=1}^{N} \frac{k_q}{k}(1+i\eta_q) X_q = \frac{1}{2}(1+i\eta) Y \tag{A.38}$$

The q^{th} equation in the frequency domain is:

$$-k_q\left(1+i\eta_q\right)X_A + \left[k_q\left(1+i\eta_q\right) - \omega^2 m_q\right]X_q = 0 \tag{A.39}$$

The displacement X_q can now be found:

$$X_q = \frac{k_q\left(1+i\eta_q\right)}{k_q\left(1+i\eta_q\right) - \omega^2 m_q} X_A \tag{A.40}$$

This equation can be used to eliminate the X_q degree of freedom from Equation (A.38):

$$\left[1 + i\eta + \frac{1}{4}\sum_{q=1}^{N} \frac{k_q}{k}(1+i\eta_q) - \left(\frac{\omega}{\omega_A}\right)^2 - \frac{1}{4}\sum_{q=1}^{N} \frac{k_q}{k} \frac{\left(1+i\eta_q\right)^2}{1 + i\eta_q - \left(\frac{\omega}{\omega_q}\right)^2}\right] X_A = \frac{1}{2}(1+i\eta) Y \tag{A.41}$$

The transmissibility between the intermediate mass and the excitation point is therefore:

$$\frac{X_A}{Y}=\frac{\frac{1}{2}(1+i\eta)}{1+i\eta+\frac{1}{4}\sum_{q=1}^{N}\frac{k_q}{k}(1+i\eta_q)-\left(\frac{\omega}{\omega_A}\right)^2-\frac{1}{4}\sum_{q=1}^{N}\frac{k_q}{k}\frac{(1+i\eta_q)^2}{1+i\eta_q-\left(\frac{\omega}{\omega_q}\right)^2}} \tag{A.42}$$

The force transmitted is:

$$F_T=2k(1+i\eta)X_A \tag{A.43}$$

From which the normalised dynamic stiffness can be found:

$$\frac{F_T}{kY}=\frac{(1+i\eta)^2}{\left[1+i\eta+\frac{1}{4}\sum_{q=1}^{N}\frac{k_q}{k}(1+i\eta_q)-\left(\frac{\omega}{\omega_A}\right)^2\right]-\frac{1}{4}\sum_{q=1}^{N}\frac{k_q}{k}\frac{(1+i\eta_q)^2}{1+i\eta_q-\left(\frac{\omega}{\omega_q}\right)^2}} \tag{A.44}$$

A.3.3 Passive vibration-absorbing isolator (non-linear VAI)

For this case the auxiliary spring in Figure A.5 is assumed to exhibit Duffing non-linearity and the auxiliary system is viscously damped while the primary system is undamped. The equation of motion for the intermediate mass is:

$$\begin{aligned}
m_A\ddot{x}_A+c_B(\dot{x}_A-\dot{x}_B)+4kx_A+k_B(x_A-x_B)+\alpha k_B(x_A-x_B)^3&=2ky\\
\frac{m_A}{4k}\ddot{x}_A+\frac{c_B}{4k}(\dot{x}_A-\dot{x}_B)+x_A+\frac{k_B}{4k}(x_A-x_B)+\alpha\frac{k_B}{4k}(x_A-x_B)^3&=\frac{1}{2}y\\
\frac{1}{\omega_A^2}\ddot{x}_A+\frac{1}{4}\frac{k_B}{k}\frac{2\zeta_B}{\omega_B}(\dot{x}_A-\dot{x}_B)+x_A+\frac{1}{4}\frac{k_B}{k}(x_A-x_B)+\alpha\frac{1}{4}\frac{k_B}{k}(x_A-x_B)^3&=\frac{1}{2}y\\
\ddot{x}_A+\frac{1}{4}\frac{k_B}{k}\frac{2\zeta_B}{\omega_B}\omega_A^2(\dot{x}_A-\dot{x}_B)+\omega_A^2x_A+\frac{1}{4}\frac{k_B}{k}\omega_A^2(x_A-x_B)+\alpha\frac{1}{4}\frac{k_B}{k}\omega_A^2(x_A-x_B)^3&=\frac{1}{2}\omega_A^2y
\end{aligned} \tag{A.45}$$

The auxiliary system equation of motion is:

$$\begin{aligned}
m_B\ddot{x}_B-c_B(\dot{x}_A-\dot{x}_B)-k_B(x_A-x_B)-\alpha k_B(x_A-x_B)^3&=0\\
\ddot{x}_B-2\zeta_B\omega_B(\dot{x}_A-\dot{x}_B)-\omega_B^2(x_A-x_B)-\alpha\omega_B^2(x_A-x_B)^3&=0\\
\ddot{x}_B-2\zeta_B\omega_B(\dot{x}_A-\dot{x}_B)-\omega_B^2\left[(x_A-x_B)+\alpha(x_A-x_B)^3\right]&=0
\end{aligned} \tag{A.46}$$

The force transmitted can be calculated by solving for x_A in the above equations and then applying:

$$\frac{f_T}{k}=2x_A \tag{A.47}$$

A.3.4 Adaptive vibration-absorbing isolator

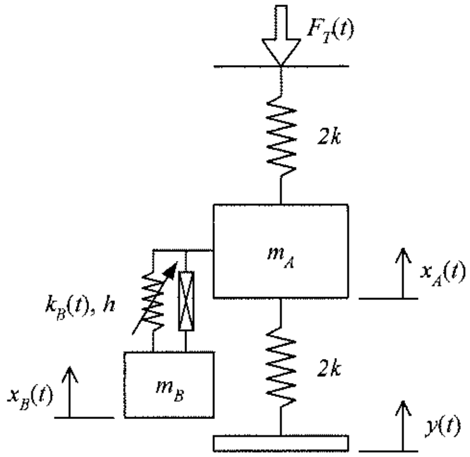

Figure A.7: Mechanical model of a tuneable VAI

The equation of motion for mass m_A can be written using Equation (A.45):

$$m_A\ddot{x}_A + c_B\left(\dot{x}_A - \dot{x}_B\right) + 4kx_A + k_B\left(x_A - x_B\right) = 2ky$$
$$\frac{m_A}{4k}\ddot{x}_A + \frac{c_B}{4k}\left(\dot{x}_A - \dot{x}_B\right) + x_A + \frac{k_B}{4k}\left(x_A - x_B\right) = \frac{1}{2}y \tag{A.48}$$

The damping terms can be written in terms of the primary system stiffness so that it is independent of the stiffness ratio:

$$\frac{c_B}{4k} = \frac{1}{4}\frac{2\bar{\zeta}_B m_B \bar{\omega}_B}{k} = \frac{1}{2}\frac{\bar{\zeta}_B}{\bar{\omega}_B}$$
$$\text{where: } \bar{\zeta}_B = \frac{c_B}{2m_B\bar{\omega}_B}, \quad \bar{\omega}_B = \sqrt{\frac{k}{m_B}} \tag{A.49}$$

Introducing this relationship and the primary system natural frequency into the equation of motion produces:

$$\frac{1}{\omega_A^2}\ddot{x}_A + \frac{1}{2}\frac{\bar{\zeta}_B}{\bar{\omega}_B}\left(\dot{x}_A - \dot{x}_B\right) + x_A + \frac{1}{4}\mu_k\left(x_A - x_B\right) = \frac{1}{2}y$$
$$\ddot{x}_A + \frac{1}{2}\frac{\bar{\zeta}_B}{\bar{\omega}_B}\omega_A^2\left(\dot{x}_A - \dot{x}_B\right) + \omega_A^2 x_A + \frac{1}{4}\mu_k\omega_A^2\left(x_A - x_B\right) = \frac{1}{2}\omega_A^2 y \tag{A.50}$$
$$\text{where: } \omega_A = \sqrt{\frac{4k}{m_A}}$$

The damping term can further be simplified in terms of the mass ratio:

$$\frac{1}{2}\bar{\zeta}_B\frac{\omega_A}{\bar{\omega}_B}\omega_A=\frac{1}{2}\bar{\zeta}_B 2\sqrt{\mu_m}\omega_A=\bar{\zeta}_B\sqrt{\mu_m}\omega_A$$
$$\text{where: }\frac{\omega_A}{\bar{\omega}_B}=\frac{\sqrt{\frac{4k}{m_A}}}{\sqrt{\frac{k}{m_B}}}=2\sqrt{\frac{k}{m_A}\frac{m_B}{k}}=2\sqrt{\mu_m} \tag{A.51}$$

The equation of motion is now:

$$\ddot{x}_A+\sqrt{\mu_m}\bar{\zeta}_B\omega_A\left(\dot{x}_A-\dot{x}_B\right)+\omega_A^2x_A+\frac{1}{4}\mu_k\omega_A^2\left(x_A-x_B\right)=\frac{1}{2}\omega_A^2y \tag{A.52}$$

where the stiffness ratio is the only parameter that is a function of time.

The equation of motion for the auxiliary mass can be found from Equation (A.46):

$$m_B\ddot{x}_B-c_B\left(\dot{x}_A-\dot{x}_B\right)-k_B\left(x_A-x_B\right)=0$$
$$\frac{m_B}{k_B}\ddot{x}_B-\frac{c_B}{k_B}\left(\dot{x}_A-\dot{x}_B\right)-\left(x_A-x_B\right)=0 \tag{A.53}$$

The damping term can be simplified as follows:

$$\frac{c_B}{k_B}=4\frac{k}{k_B}\frac{c_B}{4k}=\frac{4}{\mu_k}\frac{1}{2}\frac{\bar{\zeta}_B}{\bar{\omega}_B}=\frac{2}{\mu_k}\frac{\bar{\zeta}_B}{\bar{\omega}_B} \tag{A.54}$$

Substitution in the equation of motion yields:

$$\frac{1}{\omega_B^2}\ddot{x}_B-\frac{2}{\mu_k}\frac{\bar{\zeta}_B}{\bar{\omega}_B}\left(\dot{x}_A-\dot{x}_B\right)-\left(x_A-x_B\right)=0$$
$$\ddot{x}_B-\frac{2}{\mu_k}\frac{\bar{\zeta}_B}{\bar{\omega}_B}\omega_B^2\left(\dot{x}_A-\dot{x}_B\right)-\omega_B^2\left(x_A-x_B\right)=0 \tag{A.55}$$

The following two natural frequency ratios can be written in terms of the mass and stiffness ratios:

$$\frac{\omega_B^2}{\bar{\omega}_B^2}=\frac{\frac{k_B}{m_B}}{\frac{k}{m_B}}=\frac{k_B}{k}=\mu_k$$
$$\frac{\omega_A^2}{\omega_B^2}=4\frac{\frac{k}{m_A}}{\frac{k_B}{m_B}}=4\frac{m_B}{m_A}\frac{k}{k_B}=4\frac{\mu_m}{\mu_k} \tag{A.56}$$

Substitution in the equation of motion yields:

$$\ddot{x}_B - \frac{2}{\mu_k}\bar{\zeta}_B \frac{\omega_B}{\bar{\omega}_B}\omega_B\left(\dot{x}_A - \dot{x}_B\right) - \frac{1}{4}\frac{\mu_k}{\mu_m}\omega_A^2\left(x_A - x_B\right) = 0$$

$$\ddot{x}_B - \frac{2}{\mu_k}\bar{\zeta}_B\sqrt{\mu_k}\,\frac{1}{2}\frac{\sqrt{\mu_k}}{\sqrt{\mu_m}}\omega_A\left(\dot{x}_A - \dot{x}_B\right) - \frac{1}{4}\frac{\mu_k}{\mu_m}\omega_A^2\left(x_A - x_B\right) = 0 \qquad \text{(A.57)}$$

$$\ddot{x}_B - \frac{\bar{\zeta}_B}{\sqrt{\mu_m}}\omega_A\left(\dot{x}_A - \dot{x}_B\right) - \frac{1}{4}\frac{\mu_k}{\mu_m}\omega_A^2\left(x_A - x_B\right) = 0$$

In the frequency domain, when using the hysteretic damping, the normalised dynamic stiffness can be written in terms of the mass and stiffness ratios. The equation for a passive VAI (Equation (A.28)) can be rewritten using the relationship in Equation (A.56), yielding:

$$\frac{F_T}{kY} = \frac{1 + i\eta_B - \frac{\mu_m}{\mu_k}\left(\frac{\omega}{\omega_A}\right)^2}{\left[1 + \frac{1}{4}\mu_k\left(1 + i\eta_B\right) - \left(\frac{\omega}{\omega_A}\right)^2\right]\left[1 + i\eta_B - \frac{\mu_m}{\mu_k}\left(\frac{\omega}{\omega_A}\right)^2\right] - \frac{1}{4}\mu_k\left(1 + i\eta_B\right)^2} \qquad \text{(A.58)}$$

where: $\mu_m = \frac{m_B}{m_A}, \quad \mu_k = \frac{k_B}{k}$

The undamped frequencies of maximum dynamic stiffness can be found by equating the denominator of Equation (A.58) to zero:

$$\left[1 + \frac{1}{4}\mu_k - \left(\frac{\omega}{\omega_A}\right)^2\right]\left[1 - 4\frac{\mu_m}{\mu_k}\left(\frac{\omega}{\omega_A}\right)^2\right] - \frac{1}{4}\mu_k = 0$$

$$1 + \frac{1}{4}\mu_k - \left(\frac{\omega}{\omega_A}\right)^2 - 4\frac{\mu_m}{\mu_k}\left(\frac{\omega}{\omega_A}\right)^2 - 4\frac{\mu_m}{\mu_k}\left(\frac{\omega}{\omega_A}\right)^2\frac{1}{4}\mu_k + 4\frac{\mu_m}{\mu_k}\left(\frac{\omega}{\omega_A}\right)^2\left(\frac{\omega}{\omega_A}\right)^2 - \frac{1}{4}\mu_k = 0$$

$$1 - \left(\frac{\omega}{\omega_A}\right)^2 - 4\frac{\mu_m}{\mu_k}\left(\frac{\omega}{\omega_A}\right)^2 - \mu_m\left(\frac{\omega}{\omega_A}\right)^2 + 4\frac{\mu_m}{\mu_k}\left(\frac{\omega}{\omega_A}\right)^2\left(\frac{\omega}{\omega_A}\right)^2 = 0 \qquad \text{(A.59)}$$

$$\omega_A^4 - \omega^2\omega_A^2 - 4\frac{\mu_m}{\mu_k}\omega^2\omega_A^2 - \mu_m\omega^2\omega_A^2 + 4\frac{\mu_m}{\mu_k}\omega^4 = 0$$

$$4\frac{\mu_m}{\mu_k}\omega^4 - \left(\omega_A^2 + 4\frac{\mu_m}{\mu_k}\omega_A^2 + \mu_m\omega_A^2\right)\omega^2 + \omega_A^4 = 0$$

$$\omega^4 - \left(\frac{1}{4}\frac{\mu_k}{\mu_m}\omega_A^2 + \omega_A^2 + \frac{1}{4}\mu_k\omega_A^2\right)\omega^2 + \frac{1}{4}\frac{\mu_k}{\mu_m}\omega_A^4 = 0$$

The positive roots of the above polynomial are:

$$\Omega_1^2, \Omega_2^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Omega_1^2, \Omega_2^2 = \frac{\frac{1}{4}\frac{\mu_k}{\mu_m}\omega_A^2 + \omega_A^2 + \frac{1}{4}\mu_k\omega_A^2 \pm \sqrt{\left(\frac{1}{4}\frac{\mu_k}{\mu_m}\omega_A^2 + \omega_A^2 + \frac{1}{4}\mu_k\omega_A^2\right)^2 - \frac{\mu_k}{\mu_m}\omega_A^4}}{2} \qquad \text{(A.60)}$$

$$\Omega_1^2, \Omega_2^2 = \omega_A^2\frac{\frac{1}{4}\frac{\mu_k}{\mu_m} + 1 + \frac{1}{4}\mu_k \pm \sqrt{\left(\frac{1}{4}\frac{\mu_k}{\mu_m} + 1 + \frac{1}{4}\mu_k\right)^2 - \frac{\mu_k}{\mu_m}}}{2}$$

The above equation can be normalised as follows:

$$\frac{\Omega_1}{\omega_A},\frac{\Omega_2}{\omega_A}=\sqrt{\frac{\frac{1}{4}\frac{\mu_k}{\mu_m}+1+\frac{1}{4}\mu_k\pm\sqrt{\left(\frac{1}{4}\frac{\mu_k}{\mu_m}+1+\frac{1}{4}\mu_k\right)^2-\frac{\mu_k}{\mu_m}}}{2}} \quad \text{(A.61)}$$

A.3.5 Active vibration-absorbing isolator (acceleration and displacement feedback)

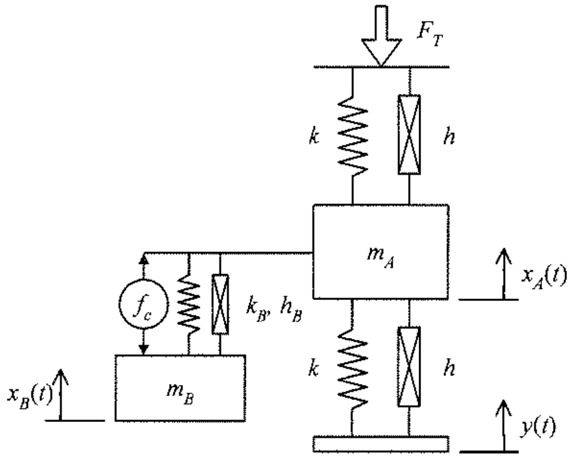

Figure A.8: Mechanical model of an active vibration-absorbing isolator

The control force is using relative acceleration and displacement feedback with gains α and γ:

$$f_c(t)=\alpha(\ddot{x}_B-\ddot{x}_A)+\gamma(x_B-x_A) \quad \text{(A.62)}$$

The force can be introduced on the right hand side of Equation (A.23). This leads to:

$$\begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix}\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix}+\begin{bmatrix} 4k(1+i\eta)+k_B(1+i\eta_B) & -k_B(1+i\eta_B) \\ -k_B(1+i\eta_B) & k_B(1+i\eta_B) \end{bmatrix}\begin{bmatrix} x_A \\ x_B \end{bmatrix}=\begin{bmatrix} 2k(1+i\eta)y+f_c \\ -f_c \end{bmatrix} \quad \text{(A.63)}$$

When simplified, the above equation becomes:

$$\begin{bmatrix} m_A+\alpha & -\alpha \\ -\alpha & m_B+\alpha \end{bmatrix}\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix}+\begin{bmatrix} 4k(1+i\eta)+k_B(1+i\eta_B)+\gamma & -k_B(1+i\eta_B)-\gamma \\ -k_B(1+i\eta_B)-\gamma & k_B(1+i\eta_B)+\gamma \end{bmatrix}\begin{bmatrix} x_A \\ x_B \end{bmatrix}=\begin{bmatrix} 2k(1+i\eta)y \\ 0 \end{bmatrix} \quad \text{(A.64)}$$

In the frequency domain the above equation yields:

$$\begin{aligned} &\left[4k(1+i\eta)+k_B(1+i\eta_B)+\gamma-\omega^2(m_A+\alpha)\right]X_A-\left[k_B(1+i\eta_B)+\gamma\right]X_B=2k(1+i\eta)Y \\ &-\left[k_B(1+i\eta_B)+\gamma\right]X_A+\left[k_B(1+i\eta_B)+\gamma-\omega^2(m_B+\alpha)\right]X_B=0 \end{aligned} \quad \text{(A.65)}$$

The second equation can be simplified to give an expression for X_B:

$$X_B=\frac{k_B(1+i\eta_B)+\gamma}{k_B(1+i\eta_B)+\gamma-\omega^2(m_B+\alpha)}X_A \quad \text{(A.66)}$$

Introducing the above equation into the first equation yields the transmissibility:

$$\left[4k(1+i\eta)+k_B(1+i\eta_B)+\gamma-\omega^2(m_A+\alpha)\right]X_A-\left[k_B(1+i\eta_B)+\gamma\right]X_B=2k(1+i\eta)Y$$

$$\left[4k(1+i\eta)+k_B(1+i\eta_B)+\gamma-\omega^2(m_A+\alpha)\right]X_A-\left[k_B(1+i\eta_B)+\gamma\right]\frac{k_B(1+i\eta_B)+\gamma}{k_B(1+i\eta_B)+\gamma-\omega^2(m_B+\alpha)}X_A=2k(1+i\eta)Y$$

$$\frac{X_A}{Y}=\frac{2k(1+i\eta)\left[k_B(1+i\eta_B)+\gamma-\omega^2(m_B+\alpha)\right]}{\left[4k(1+i\eta)+k_B(1+i\eta_B)+\gamma-\omega^2(m_A+\alpha)\right]\left[k_B(1+i\eta_B)+\gamma-\omega^2(m_B+\alpha)\right]-\left[k_B(1+i\eta_B)+\gamma\right]^2} \tag{A.67}$$

For the undamped case the force transmitted is:

$$F_T=2kX_A \tag{A.68}$$

Substituting for the displacement of the intermediate mass in terms of the excitation:

$$\frac{F_T}{Y}=\frac{4k^2\left[k_B+\gamma-\omega^2(m_B+\alpha)\right]}{\left[4k+k_B+\gamma-\omega^2(m_A+\alpha)\right]\left[k_B+\gamma-\omega^2(m_B+\alpha)\right]-\left[k_B+\gamma\right]^2} \tag{A.69}$$

The above equation can be non-dimensionalised:

$$\frac{F_T}{kY}=\frac{1-\omega^2\frac{m_B+\alpha}{k_B+\gamma}}{\left(1+\frac{k_B+\gamma}{4k}-\omega^2\frac{m_A+\alpha}{4k}\right)\left(1-\omega^2\frac{m_B+\alpha}{k_B+\gamma}\right)-\frac{k_B+\gamma}{4k}}$$

$$=\frac{1-\left(\frac{\omega}{\omega_B'}\right)^2}{\left[1+\frac{1}{4}\frac{k_B}{k}\left(1+\frac{\gamma}{k_B}\right)-\left(\frac{\omega}{\omega_A}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_B'}\right)^2\right]-\frac{1}{4}\frac{k_B}{k}\left(1+\frac{\gamma}{k_B}\right)} \tag{A.70}$$

$$\text{where: }\quad \omega_A=\sqrt{\frac{4k}{m_A+\alpha}},\quad \omega_B'=\sqrt{\frac{k_B\left(1+\frac{\gamma}{k_B}\right)}{m_B\left(1+\frac{\alpha}{m_B}\right)}}=\omega_B\sqrt{\frac{1+\frac{\gamma}{k_B}}{1+\frac{\alpha}{m_B}}}$$

The Routh-Hurwitz stability criterion is evaluated using the characteristic equation and the sub-determinants defined by the coefficients of the characteristic equation. The characteristic equation can be found from the determinant of the equation of motion, which for the undamped case is:

$$D(s)=\begin{vmatrix}4k+k_B+\gamma+s^2(m_A+\alpha) & -k_B-\gamma-s^2\alpha\\ -k_B-\gamma-s^2\alpha & k_B+\gamma+s^2(m_B+\alpha)\end{vmatrix}$$

$$=\left[4k+k_B+\gamma+s^2(m_A+\alpha)\right]\left[k_B+\gamma+s^2(m_B+\alpha)\right]-\left[-k_B-\gamma-s^2\alpha\right]^2 \tag{A.71}$$

$$=\left[m_Bm_A+\alpha(m_A+m_B)\right]s^4+\left[4m_Bk+4k\alpha+(m_B+m_A)(k_B+\gamma)\right]s^2+4kk_B+4k\gamma$$

The Routh-Hurwitz criterion requires that the coefficients of the characteristic equation be larger than zero (Rao, 1990):

$$\begin{aligned}
&a_0 = m_B m_A + \alpha(m_A + m_B) > 0\\
&\quad \therefore \alpha > -\frac{m_B m_A}{m_A + m_B}\\
&a_2 = 4m_B k + 4k\alpha + k_B m_B + \gamma m_B + k m_B + m_A \gamma > 0\\
&\quad \therefore 4km_B^2 > 0\\
&a_4 = 4kk_B + 4k\gamma > 0\\
&\quad \therefore \frac{\gamma}{k_B} > -1
\end{aligned} \tag{A.72}$$

Additionally for a two degree of freedom system the following inequality must be satisfied:

$$a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2 \tag{A.73}$$

The above condition is not satisfied and it is concluded that the system is marginally stable.

A.3.6 Active vibration-absorbing isolator (relative velocity feedback)

For a control force defined by the relative velocity feedback is:

$$f_c(t) = \beta(\dot{x}_B - \dot{x}_A) \tag{A.74}$$

The equation of motion must necessarily include a viscous damping model:

$$\begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix}\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix} + \begin{bmatrix} 4c + c_B & -c_B \\ -c_B & c_B \end{bmatrix}\begin{bmatrix} \dot{x}_A \\ \dot{x}_B \end{bmatrix} + \begin{bmatrix} 4k + k_B & -k_B \\ -k_B & k_B \end{bmatrix}\begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 2ky + 2c\dot{y} + \beta(\dot{x}_B - \dot{x}_A) \\ -\beta(\dot{x}_B - \dot{x}_A) \end{bmatrix} \tag{A.75}$$

When transformed to the frequency domain the above equation becomes:

$$\begin{bmatrix} 4k + k_B + i\omega(4c + c_B + \beta) - \omega^2 m_A & -k_B - i\omega(c_B + \beta) \\ -k_B - i\omega(c_B + \beta) & k_B + i\omega(c_B + \beta) - \omega^2 m_B \end{bmatrix}\begin{bmatrix} X_A \\ X_B \end{bmatrix} = \begin{bmatrix} 2(k + i\omega c) \\ 0 \end{bmatrix} Y \tag{A.76}$$

The second equation can be used to find an expression for the absorber displacement:

$$\begin{aligned}
&-\left[k_B + i\omega(c_B + \beta)\right]X_A + \left[k_B + i\omega(c_B + \beta) - \omega^2 m_B\right]X_B = 0\\
&X_B = \frac{k_B + i\omega(c_B + \beta)}{k_B + i\omega(c_B + \beta) - \omega^2 m_B} X_A
\end{aligned} \tag{A.77}$$

When the absorber displacement is eliminated from the first equation of Equation (A.76):

$$\left[4k+k_B+i\omega\left(4c+c_B+\beta\right)-\omega^2 m_A\right]X_A-\left[k_B+i\omega\left(c_B+\beta\right)\right]X_B=2\left(k+i\omega c\right)Y$$

$$\left[4k+k_B+i\omega\left(4c+c_B+\beta\right)-\omega^2 m_A\right]X_A-\left[k_B+i\omega\left(c_B+\beta\right)\right]\frac{k_B+i\omega\left(c_B+\beta\right)}{k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B}X_A=2\left(k+i\omega c\right)Y$$

$$\left\{\left[4k+k_B+i\omega\left(4c+c_B+\beta\right)-\omega^2 m_A\right]\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]-\left[k_B+i\omega\left(c_B+\beta\right)\right]^2\right\}X_A= \qquad \text{(A.78)}$$

$$2\left(k+i\omega c\right)\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]Y$$

$$X_A=\frac{2\left(k+i\omega c\right)\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]}{\left[4k+k_B+i\omega\left(4c+c_B+\beta\right)-\omega^2 m_A\right]\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]-\left[k_B+i\omega\left(c_B+\beta\right)\right]^2}Y$$

The force transmitted is:

$$F_T=2\left(k+i\omega c\right)X_A \qquad \text{(A.79)}$$

When substituting the intermediate mass displacement, this equation becomes:

$$\frac{F_T}{Y}=2\left(k+i\omega c\right)\frac{2\left(k+i\omega c\right)\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]}{\left[4k+k_B+i\omega\left(4c+c_B+\beta\right)-\omega^2 m_A\right]\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]-\left[k_B+i\omega\left(c_B+\beta\right)\right]^2} \qquad \text{(A.80)}$$

The equation can be normalised as follows:

$$\frac{F_T}{Y}=\frac{4k^2k_B\left(1+i2\frac{\omega}{\omega_A}\zeta\right)^2\left[1+i2\frac{\omega}{\omega_B}\zeta_B-\left(\frac{\omega}{\omega_B}\right)^2\right]}{4k\left[1+\frac{1}{4}\frac{k_B}{k}+i2\left(\frac{\omega}{\omega_A}\zeta+\frac{\omega}{\omega_B}\frac{k_B}{k}\zeta_B\right)-\left(\frac{\omega}{\omega_A}\right)^2\right]k_B\left[1+i2\frac{\omega}{\omega_B}\zeta_B-\left(\frac{\omega}{\omega_B}\right)^2\right]-k_B^2\left(1+i2\frac{\omega}{\omega_B}\zeta_B\right)^2}$$

$$\frac{F_T}{kY}=\frac{\left(1+i2\frac{\omega}{\omega_A}\zeta\right)^2\left[1+i2\frac{\omega}{\omega_B}\zeta_B-\left(\frac{\omega}{\omega_B}\right)^2\right]}{\left[1+\frac{1}{4}\frac{k_B}{k}+i2\left(\frac{\omega}{\omega_A}\zeta+\frac{\omega}{\omega_B}\frac{k_B}{k}\zeta_B\right)-\left(\frac{\omega}{\omega_A}\right)^2\right]\left[1+i2\frac{\omega}{\omega_B}\zeta_B-\left(\frac{\omega}{\omega_B}\right)^2\right]-\frac{1}{4}\frac{k_B}{k}\left(1+i2\frac{\omega}{\omega_B}\zeta_B\right)^2} \qquad \text{(A.81)}$$

where: $\zeta=\frac{c}{2m_A\omega_A}$ $\quad \zeta_B=\frac{c_B+\beta}{2m_B\omega_B}$

The characteristic equation is:

$$D(s)=\begin{vmatrix}4k+k_B+s\left(4c+c_B+\beta\right)+s^2m_A & -k_B-s\left(c_B+\beta\right)\\ -k_B-s\left(c_B+\beta\right) & k_B+s\left(c_B+\beta\right)+s^2m_B\end{vmatrix}$$

$$=\left[4k+k_B+s\left(4c+c_B+\beta\right)+s^2m_A\right]\left[k_B+s\left(c_B+\beta\right)+s^2m_B\right]-\left[k_B+s\left(c_B+\beta\right)\right]^2 \qquad \text{(A.82)}$$

$$=m_Am_Bs^4+\left(4cm_B+c_Bm_B+m_Ac_B+m_A\beta+\beta m_B\right)s^3+\left(4km_B+m_Ak_B+4cc_B+4c\beta+k_Bm_B\right)s^2$$

$$+\left(4ck_B+4kc_B+4k\beta\right)s+4kk_B$$

The Routh-Hurwitz criterion requires that the coefficients of the characteristic equation be larger than zero:

$$\begin{aligned}
&a_0 = m_A m_B > 0 \\
&a_1 = 4cm_B + c_B m_B + m_A c_B + m_A \beta + \beta m_B > 0 \\
&\quad \therefore \beta > -c_B - 4c\frac{m_B}{m_A + m_B} \\
&a_2 = 4km_B + m_A k_B + 4cc_B + 4c\beta + k_B m_B > 0 \\
&\quad \therefore \beta > -c_B - \frac{k_B m_B}{4c} - \frac{km_B}{c} - \frac{m_A k_B}{4c} \\
&a_3 = 4ck_B + 4kc_B + 4k\beta > 0 \\
&\quad \therefore \beta > -c_B - c\frac{k_B}{k} \\
&a_4 = 4kk_B > 0
\end{aligned} \tag{A.83}$$

Additionally for a two degree of freedom system the following inequality must be satisfied:

$$\begin{aligned}
&a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2 \\
&\left(4cm_B + c_B m_B + m_A c_B + m_A \beta + \beta m_B\right)\left(4km_B + m_A k_B + 4cc_B + 4c\beta + k_B m_B\right)\left(4ck_B + 4kc_B + 4k\beta\right) \\
&> m_A m_B \left(4ck_B + 4kc_B + 4k\beta\right)^2 + 4kk_B\left(4cm_B + c_B m_B + m_A c_B + m_A \beta + \beta m_B\right)^2
\end{aligned} \tag{A.84}$$

The worst case occurs when the intermediate mass damping is equal to zero and then:

$$\beta > -c_B \tag{A.85}$$

A.3.7 Active vibration-absorbing isolator (absolute velocity feedback)

The control force when using absolute absorber velocity feedback is:

$$f_c(t) = \beta \dot{x}_B \tag{A.86}$$

The equation of motion is:

$$\begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix}\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \end{bmatrix} + \begin{bmatrix} 4c + c_B & -c_B \\ -c_B & c_B \end{bmatrix}\begin{bmatrix} \dot{x}_A \\ \dot{x}_B \end{bmatrix} + \begin{bmatrix} 4k + k_B & -k_B \\ -k_B & k_B \end{bmatrix}\begin{bmatrix} x_A \\ x_B \end{bmatrix} = \begin{bmatrix} 2ky + 2c\dot{y} + \beta \dot{x}_B \\ -\beta \dot{x}_B \end{bmatrix} \tag{A.87}$$

When transformed to the frequency domain, the above equation becomes:

$$\begin{bmatrix} 4k + k_B + i\omega\left(4c + c_B\right) - \omega^2 m_A & -k_B - i\omega\left(c_B + \beta\right) \\ -k_B - i\omega c_B & k_B + i\omega\left(c_B + \beta\right) - \omega^2 m_B \end{bmatrix}\begin{bmatrix} X_A \\ X_B \end{bmatrix} = \begin{bmatrix} 2\left(k + i\omega c\right) \\ 0 \end{bmatrix} Y \tag{A.88}$$

The second equation yields the absorber displacement:

$$X_B = \frac{k_B + i\omega c_B}{k_B + i\omega\left(c_B + \beta\right) - \omega^2 m_B} X_A \tag{A.89}$$

Introducing the above equation into the first equation yields the transmissibility:

$$\left[4k+k_B+i\omega\left(4c+c_B\right)-\omega^2 m_A\right]X_A-\left[k_B+i\omega\left(c_B+\beta\right)\right]X_B=2\left(k+i\omega c\right)Y$$

$$\left[4k+k_B+i\omega\left(4c+c_B\right)-\omega^2 m_A\right]X_A-\left[k_B+i\omega\left(c_B+\beta\right)\right]\frac{k_B+i\omega c_B}{k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B}X_A=2\left(k+i\omega c\right)Y$$

$$\left\{\left[4k+k_B+i\omega\left(4c+c_B\right)-\omega^2 m_A\right]\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]-\left[k_B+i\omega\left(c_B+\beta\right)\right]\left(k_B+i\omega c_B\right)\right\}X_A \qquad \text{(A.90)}$$
$$=2\left(k+i\omega c\right)\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]Y$$

$$\frac{X_A}{Y}=\frac{2\left(k+i\omega c\right)\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]}{\left[4k+k_B+i\omega\left(4c+c_B\right)-\omega^2 m_A\right]\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]-\left[k_B+i\omega\left(c_B+\beta\right)\right]\left(k_B+i\omega c_B\right)}$$

The force transmitted is:

$$F_T=2\left(k+i\omega c\right)X_A \qquad \text{(A.91)}$$

Substituting for the displacement of the intermediate mass in terms of the excitation:

$$\frac{F_T}{Y}=\frac{4\left(k+i\omega c\right)^2\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]}{\left[4k+k_B+i\omega\left(4c+c_B\right)-\omega^2 m_A\right]\left[k_B+i\omega\left(c_B+\beta\right)-\omega^2 m_B\right]-\left[k_B+i\omega\left(c_B+\beta\right)\right]\left(k_B+i\omega c_B\right)}$$

$$\frac{F_T}{kY}=\frac{\left(1+i\omega\frac{c}{k}\right)^2\left[1+i\omega\frac{c_B+\beta}{k_B}-\omega^2\frac{m_B}{k_B}\right]}{\left[1+\frac{1}{4}\frac{k_B}{k}+i\omega\left(\frac{4c+c_B}{4k}\right)-\omega^2\frac{m_A}{4k}\right]\left[1+i\omega\frac{c_B+\beta}{k_B}-\omega^2\frac{m_B}{k_B}\right]-\frac{1}{4}\frac{k_B}{k}\left[1+i\omega\frac{c_B+\beta}{k_B}\right]\left(1+i\omega\frac{c_B}{k_B}\right)} \qquad \text{(A.92)}$$

The above equation can be non-dimensionalised as follows:

$$\frac{F_T}{kY}=\frac{\left(1+i2\zeta\frac{\omega}{\omega_A}\right)^2\left[1+i2\frac{\omega}{\omega_B}\zeta_B\left(1+\frac{\beta}{c_B}\right)-\left(\frac{\omega}{\omega_B}\right)^2\right]}{\left[1+\frac{1}{4}\frac{k_B}{k}+i2\left(\frac{\omega}{\omega_A}\zeta+\frac{1}{4}\frac{k_B}{k}\frac{\omega}{\omega_B}\zeta_B\right)-\left(\frac{\omega}{\omega_A}\right)^2\right]\left[1+i2\frac{\omega}{\omega_B}\zeta_B\left(1+\frac{\beta}{c_B}\right)-\left(\frac{\omega}{\omega_B}\right)^2\right]-\frac{1}{4}\frac{k_B}{k}\left[1+i2\frac{\omega}{\omega_B}\zeta_B\left(1+\frac{\beta}{c_B}\right)\right]\left(1+i2\frac{\omega}{\omega_B}\zeta_B\right)} \qquad \text{(A.93)}$$

where: $\zeta_B=\frac{c_B}{2m_B\omega_B}$, $\zeta=\frac{c}{2m_A\omega_A}$

The characteristic equation is:

$$D(s)=\begin{vmatrix}4k+k_B+s\left(4c+c_B\right)+s^2 m_A & -k_B-s\left(c_B+\beta\right)\\ -k_B-sc_B & k_B+i\omega\left(c_B+\beta\right)+s^2 m_B\end{vmatrix}$$
$$=\left[4k+k_B+s\left(4c+c_B\right)+s^2 m_A\right]\left[k_B+s\left(c_B+\beta\right)+s^2 m_B\right]-\left(k_B+sc_B\right)\left[k_B+s\left(c_B+\beta\right)\right] \qquad \text{(A.94)}$$
$$=s^4 m_A m_B+(4cm_B+c_B m_B+m_A c_B+m_A\beta)s^3+(4km_B+k_B m_B+4cc_B+4c\beta+m_A k_B)s^2$$
$$+(4kc_B+4k\beta+4ck_B)s+4kk_B$$

The Routh-Hurwitz criterion requires that the coefficients of the characteristic equation be larger than zero:

$$\begin{aligned}
&a_0 = m_A m_B > 0 \\
&a_1 = 4cm_B + c_B m_B + m_A c_B + m_A\beta > 0 \\
&\quad \therefore \beta > -c_B\left(1+\frac{m_B}{m_A}\right) - 4c\frac{m_B}{m_A} \\
&a_2 = 4km_B + k_B m_B + 4cc_B + 4c\beta + m_A k_B > 0 \\
&\quad \therefore \beta > -c_B - \frac{km_B}{c} - \frac{k_B m_B}{4c} - \frac{m_A k_B}{4c} \\
&a_3 = 4kc_B + 4k\beta + 4ck_B > 0 \\
&\quad \therefore \beta > -c_B - c\frac{k_B}{k} \\
&a_4 = 4kk_B > 0
\end{aligned} \tag{A.95}$$

Additionally for a two degree of freedom system the following inequality must be satisfied:

$$\begin{aligned}
&a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2 \\
&\left(4cm_B + c_B m_B + m_A c_B + m_A\beta\right)\left(4km_B + k_B m_B + 4cc_B + 4c\beta + m_A k_B\right)\left(4kc_B + 4k\beta + 4ck_B\right) \\
&> m_A m_B\left(4kc_B + 4k\beta + 4ck_B\right)^2 + 4kk_B\left(4cm_B + c_B m_B + m_A c_B + m_A\beta\right)^2
\end{aligned} \tag{A.96}$$

The worst case occurs when the intermediate mass damping is equal to zero and then:

$$\beta > -c_B \tag{A.97}$$

A.4 Amplified vibration-absorbing isolators

A.4.1 Passive amplified vibration-absorbing isolator

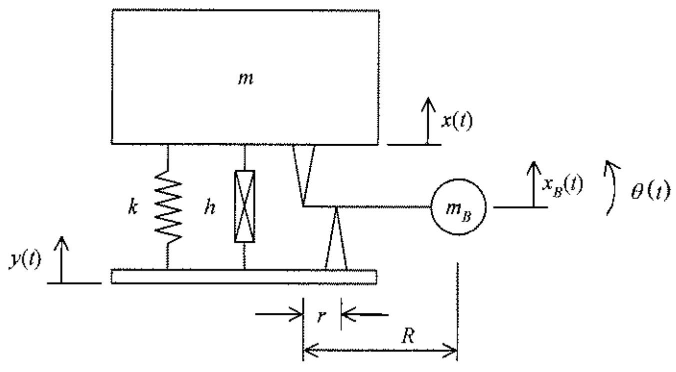

Figure A.9: Mechanical model of a pendulum AVAI

The continuity between the response excitation and the pendulum displacement can be written by considering Figure A.10.

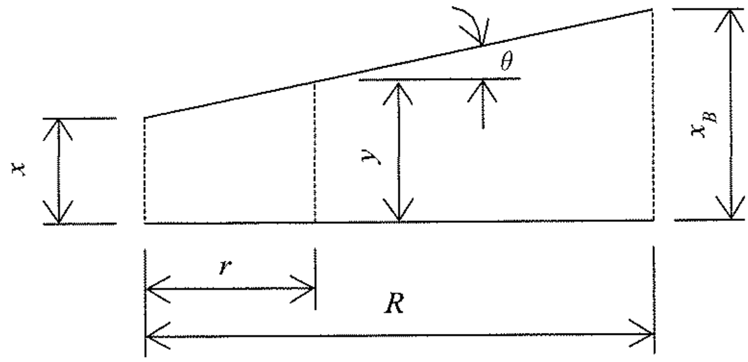

Figure A.10: Continuity description

The distance travelled by the absorber mass and its angle of rotation can be written as:

$$x_B = \left(1-\frac{R}{r}\right)x + \frac{R}{r}y \qquad \text{(A.98)}$$
$$\theta = \frac{y-x}{r}$$

The kinetic energy can be found in terms of the excitation and response degrees of freedom by substituting the absorber degrees of freedom using the continuity equations:

$$\begin{aligned} T &= \frac{1}{2}\left(m\dot{x}^2 + m_B\dot{x}_B^2 + I_G\dot{\theta}^2\right) \\ &= \frac{1}{2}\left[m + m_B\left(1-\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\dot{x}^2 + \left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r} - \frac{I_G}{r^2}\right]\dot{x}\dot{y} + \frac{1}{2}\left[m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\dot{y}^2 \end{aligned} \qquad \text{(A.99)}$$

From the above equation the derivatives can be found:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) &= \left[m + m_B\left(1-\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\ddot{x} + \left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r} - \frac{I_G}{r^2}\right]\ddot{y} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) &= \left[m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\ddot{y} + \left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r} - \frac{I_G}{r^2}\right]\ddot{x} \end{aligned} \qquad \text{(A.100)}$$

The potential energy is:

$$V = \frac{1}{2}k\left(x-y\right)^2 \qquad \text{(A.101)}$$

From the above equation the derivatives can be found:

$$\begin{aligned} \frac{\partial V}{\partial x} &= k\left(x-y\right) \\ \frac{\partial V}{\partial y} &= k\left(y-x\right) \end{aligned} \qquad \text{(A.102)}$$

The Rayleigh term is:

$$R=\frac{1}{2}c\left(\dot{x}-\dot{y}\right)^2 \tag{A.103}$$

From the above equation the derivatives can be found:

$$\begin{aligned}\frac{\partial R}{\partial \dot{x}}&=c\left(\dot{x}-\dot{y}\right)\\ \frac{\partial R}{\partial \dot{y}}&=c\left(\dot{y}-\dot{x}\right)\end{aligned} \tag{A.104}$$

The most general case including both viscous and hysteretic damping gives:

$$\begin{bmatrix} m+m_B\left(1-\frac{R}{r}\right)^2+\frac{I_G}{r^2} & m_B\left(1-\frac{R}{r}\right)\frac{R}{r}-\frac{I_G}{r^2} \\ m_B\left(1-\frac{R}{r}\right)\frac{R}{r}-\frac{I_G}{r^2} & m_B\left(\frac{R}{r}\right)^2+\frac{I_G}{r^2}\end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix}+\begin{bmatrix} c & -c\\ -c & c\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}+\begin{bmatrix} k(1+\eta) & -k(1+\eta)\\ -k(1+\eta) & k(1+\eta)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}f_x\\ f_y\end{bmatrix} \tag{A.105}$$

If the y degree of freedom is prescribed and no external forces are acting on the system the second equation must be neglected and the equation of motion becomes:

$$\left[m+m_B\left(1-\frac{R}{r}\right)^2+\frac{I_G}{r^2}\right]\ddot{x}+c\dot{x}+k\left(1+\eta\right)x=-\left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r}-\frac{I_G}{r^2}\right]\ddot{y}+c\dot{y}+k\left(1+\eta\right)y \tag{A.106}$$

In the frequency domain the above equation can be used to give the transmissibility:

$$\frac{X}{Y}=\frac{k\left(1+\eta\right)+i\omega c-\omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right]}{k\left(1+\eta\right)+i\omega c-\omega^2\left[m+m_B\left(1-\frac{R}{r}\right)^2+\frac{I_G}{r^2}\right]} \tag{A.107}$$

The above equation can be non-dimensionalised by introducing the isolation and natural frequencies:

$$\frac{X}{Y}=\frac{1+i\eta+i2\frac{\omega}{\omega_n}\zeta-\left(\frac{\omega}{\omega_i}\right)^2}{1+i\eta+i2\frac{\omega}{\omega_n}\zeta-\left(\frac{\omega}{\omega_n}\right)^2}$$

where: $\omega_n=\sqrt{\frac{k}{m+m_B\left(\frac{R}{r}-1\right)^2+\frac{I_G}{r^2}}}$, $\omega_i=\sqrt{\frac{k}{m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}}}$, $\zeta=\frac{c}{2\left[m+m_B\left(\frac{R}{r}-1\right)^2+\frac{I_G}{r^2}\right]\omega_n}$ (A.108)

The invariant frequency can be calculated by setting the absolute value of the transmissibility equal to 1. For transmissibility equal to 1 the trivial solution of zero results. When the transmissibility is equal to -1:

$$\frac{1-\left(\frac{\omega_A}{\omega_i}\right)^2}{1-\left(\frac{\omega_A}{\omega_n}\right)^2}=-1$$

$$\omega_n^2\omega_i^2-\omega_n^2\omega_A^2=-\omega_n^2\omega_i^2+\omega_i^2\omega_A^2$$

$$2\omega_n^2\omega_i^2=\left(\omega_n^2+\omega_i^2\right)\omega_A^2 \tag{A.109}$$

$$\frac{\omega_A^2}{\omega_i^2}=\frac{2}{1+\frac{\omega_i^2}{\omega_n^2}}$$

$$\frac{\omega_A}{\omega_i}=\sqrt{\frac{2}{1+\left(\frac{\omega_i}{\omega_n}\right)^2}}$$

The force transmitted can be calculated by using the first equation of Equation (A.105) and calculating the force f_x needed to restraint the displacement x to zero:

$$\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right]\ddot{y}-c\dot{y}-k\left(1+\eta\right)y=f_x \tag{A.110}$$

When transformed to the frequency domain, substituting the external force (F_x) with the transmitted force ($-F_T$) and neglecting the viscous damping, the above equation becomes:

$$\left\{\omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right]-k\left(1+i\eta\right)\right\}Y=F_x$$

$$\frac{F_T}{Y}=k\left(1+i\eta\right)-\omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right] \tag{A.111}$$

Which can then be non-dimensionalised as follows:

$$\frac{F_T}{kY}=1+i\eta-\left(\frac{\omega}{\omega_i}\right)^2 \tag{A.112}$$

The damped natural and isolation frequencies can be calculated setting the derivative of the transmissibility equal to zero:

$$\frac{\partial}{\partial\omega}\left|\frac{X}{Y}\right| = \frac{\partial}{\partial\omega}\left\{\frac{\left[1-\left(\frac{\omega}{\omega_i}\right)^2\right]^2+4\left(\frac{\omega}{\omega_n}\zeta\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\left(\frac{\omega}{\omega_n}\zeta\right)^2}\right\}^{\frac{1}{2}} = 0$$

$$\frac{1}{2}\left\{\frac{\left[1-\left(\frac{\omega}{\omega_i}\right)^2\right]^2+4\left(\frac{\omega}{\omega_n}\zeta\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\left(\frac{\omega}{\omega_n}\zeta\right)^2}\right\}^{-\frac{1}{2}}\left\{\frac{\frac{\partial}{\partial\omega}\left\{\left[1-\left(\frac{\omega}{\omega_i}\right)^2\right]^2+4\left(\frac{\omega}{\omega_n}\zeta\right)^2\right\}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\left(\frac{\omega}{\omega_n}\zeta\right)^2\right\}-\frac{\partial}{\partial\omega}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\left(\frac{\omega}{\omega_n}\zeta\right)^2\right\}\left\{\left[1-\left(\frac{\omega}{\omega_i}\right)^2\right]^2+4\left(\frac{\omega}{\omega_n}\zeta\right)^2\right\}}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\left(\frac{\omega}{\omega_n}\zeta\right)^2\right\}^2}\right\} = 0$$

$$\frac{\partial}{\partial\omega}\left\{1-2\left(\frac{\omega}{\omega_i}\right)^2+\left(\frac{\omega}{\omega_i}\right)^4+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}\left\{1-2\left(\frac{\omega}{\omega_n}\right)^2+\left(\frac{\omega}{\omega_n}\right)^4+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}-\frac{\partial}{\partial\omega}\left\{1-2\left(\frac{\omega}{\omega_n}\right)^2+\left(\frac{\omega}{\omega_n}\right)^4+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}\left\{1-2\left(\frac{\omega}{\omega_i}\right)^2+\left(\frac{\omega}{\omega_i}\right)^4+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\} = 0$$

$$\left\{\frac{8\zeta^2}{\omega_n^2}-\frac{4}{\omega_i^2}+\frac{4}{\omega_i^4}\omega^2\right\}\left\{1+\frac{4\zeta^2-2}{\omega_n^2}\omega^2+\frac{1}{\omega_n^4}\omega^4\right\}-\left\{\frac{8\zeta^2}{\omega_n^2}-\frac{4}{\omega_n^2}+\frac{4}{\omega_n^4}\omega^2\right\}\left\{1+\left(\frac{4\zeta^2}{\omega_n^2}-\frac{2}{\omega_i^2}\right)\omega^2+\frac{1}{\omega_i^4}\omega^4\right\} = 0$$

$$\frac{1}{\omega_n^2}-\frac{1}{\omega_i^2}+\left(\frac{1}{\omega_i^4}-\frac{1}{\omega_n^4}\right)\omega^2+\left(-\frac{2\zeta^2}{\omega_n^6}+\frac{1}{\omega_i^2\omega_n^4}-\frac{1}{\omega_n^2\omega_i^4}+\frac{2\zeta^2}{\omega_i^4\omega_n^2}\right)\omega^4 = 0$$

$$\frac{\Omega_n}{\omega_n},\frac{\Omega_i}{\omega_n} = \sqrt{\frac{-\left(\frac{\omega_n}{\omega_i}\right)^2-1\pm\sqrt{\left[\left(\frac{\omega_n}{\omega_i}\right)^2-1\right]^2+8\zeta^2\left[\left(\frac{\omega_n}{\omega_i}\right)^2+1\right]}}{4\zeta^2+4\zeta^2\left(\frac{\omega_n}{\omega_i}\right)^2-2\left(\frac{\omega_n}{\omega_i}\right)^2}}$$

(A.113)

A.4.2 Passive amplified vibration-absorbing isolator (multiple absorbers fitted)

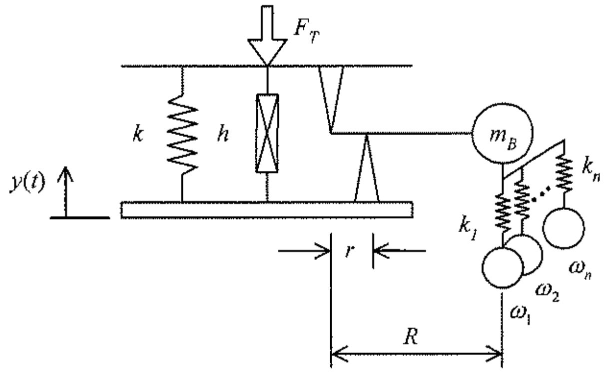

Figure A.11: Mechanical model of multiple absorbers fitted to a pendulum

The continuity is again given by Equation (A.98). The kinetic energy is:

$$T = \frac{1}{2}\left(m\dot{x}^2 + m_B\dot{x}_B^2 + I_G\dot{\theta}^2 + \sum_{i=1}^{n} m_q\dot{x}_q^2 \right)$$
$$\frac{1}{2}\left\{\left[m + m_B\left(1-\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\dot{x}^2 + 2\left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r} - \frac{I_G}{r^2}\right]\dot{x}\dot{y} + \left[m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\dot{y}^2 + \sum_{q=1}^{n} m_q\dot{x}_q^2\right\} \quad (A.114)$$

From the above equation the derivatives can be found:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \left[m + m_B\left(1-\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\ddot{x} + \left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r} - \frac{I_G}{r^2}\right]\ddot{y}$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = \left[m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\ddot{y} + \left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r} - \frac{I_G}{r^2}\right]\ddot{x} \quad (A.115)$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_q}\right) = m_q\ddot{x}_q$$

The potential energy is:

$$V = \frac{1}{2}\left[k(x-y)^2 + \sum_{q=1}^{n} k_q\left(x_B - x_q\right)^2\right]$$
$$= \frac{1}{2}\left\{k\left(x^2 - 2xy - y^2\right) + \sum_{q=1}^{n} k_q\left[\left(1-\frac{R}{r}\right)x + \frac{R}{r}y - x_q\right]^2\right\}$$
$$= \frac{1}{2}\left\{k\left(x^2 - 2xy - y^2\right) + \sum_{q=1}^{n} k_q\left[\left(1-\frac{R}{r}\right)^2 x^2 + 2\left(1-\frac{R}{r}\right)\frac{R}{r}xy + \left(\frac{R}{r}\right)^2 y^2 - 2\left(1-\frac{R}{r}\right)xx_q - \frac{R}{r}yx_q + x_q^2\right]\right\} \quad (A.116)$$
$$= \frac{1}{2}\left\{k\left(x^2 - 2xy - y^2\right) + \left(1-\frac{R}{r}\right)^2 x^2\sum_{q=1}^{n} k_q + 2\left(1-\frac{R}{r}\right)\frac{R}{r}xy\sum_{q=1}^{n} k_q + \left(\frac{R}{r}\right)^2 y^2\sum_{q=1}^{n} k_q - 2\left(1-\frac{R}{r}\right)x\sum_{q=1}^{n} k_q x_q \right.$$
$$\left. - \frac{R}{r}y\sum_{q=1}^{n} k_q x_q + \sum_{q=1}^{n} k_q x_q^2\right\}$$

From the above equation the derivatives can be found:

$$\begin{aligned}
\frac{\partial V}{\partial x} &= \left(k+\left(1-\frac{R}{r}\right)^2\sum_{q=1}^{n}k_q\right)x+\left(-k+\left(1-\frac{R}{r}\right)\frac{R}{r}\sum_{q=1}^{n}k_q\right)y-\left(1-\frac{R}{r}\right)\sum_{q=1}^{n}k_q x_q \\
\frac{\partial V}{\partial y} &= \left[-k+\left(1-\frac{R}{r}\right)\frac{R}{r}\sum_{q=1}^{n}k_q\right]x+\left[k+\left(\frac{R}{r}\right)^2\sum_{q=1}^{n}k_q\right]y-\frac{R}{r}\sum_{q=1}^{n}k_q x_q \\
\frac{\partial V}{\partial x_q} &= -k_q\left(1-\frac{R}{r}\right)x-k_q\frac{R}{r}y+k_q x_q
\end{aligned} \tag{A.117}$$

The complete undamped equation of motion is:

$$\begin{bmatrix} m + m_B\left(\frac{R}{r}-1\right)^2 + \frac{I_G}{r^2} & 0 & 0 & \cdots & 0 \\ 0 & m_1 & 0 & \cdots & 0 \\ 0 & 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & m_q \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_q \end{bmatrix} + \begin{bmatrix} k + i\eta + \left(\frac{R}{r}-1\right)^2 \sum_{q=1}^{n} k_q + i\eta_q & \left(\frac{R}{r}-1\right)k_1\left(1+i\eta_1\right) & \left(\frac{R}{r}-1\right)k_2\left(1+i\eta_2\right) & \cdots & \left(\frac{R}{r}-1\right)k_q\left(1+i\eta_q\right) \\ \left(\frac{R}{r}-1\right)k_1\left(1+i\eta_1\right) & k_1 & 0 & \cdots & 0 \\ \left(\frac{R}{r}-1\right)k_2\left(1+i\eta_2\right) & 0 & k_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \left(\frac{R}{r}-1\right)k_q\left(1+i\eta_q\right) & 0 & 0 & 0 & k_q \end{bmatrix} \begin{bmatrix} x \\ x_1 \\ x_2 \\ \vdots \\ x_q \end{bmatrix} = \begin{bmatrix} m_B\left(\frac{R}{r}-1\right)\frac{R}{r} + \frac{I_G}{r^2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ddot{y} + \begin{bmatrix} k + \left(\frac{R}{r}-1\right)\frac{R}{r}\sum_{q=1}^{n} k_q \\ k_1\frac{R}{r} \\ k_2\frac{R}{r} \\ \vdots \\ k_q\frac{R}{r} \end{bmatrix} y + \begin{bmatrix} f_x \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{A.118}$$

With the x degree of freedom forced to zero by the force f_x the first equation of the above set of equations can be transformed to the frequency domain:

$$F_x = \sum_{q=1}^{n}\left(\frac{R}{r}-1\right)k_q\left(1+i\eta_q\right)X_q - \left\{k + i\eta + \left(\frac{R}{r}-1\right)\frac{R}{r}\sum_{q=1}^{n}k_q\left(1+i\eta_q\right) - \omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right]\right\}Y \tag{A.119}$$

The 2nd to q^{th} equation is given by:

$$\begin{aligned} \left[k_q\left(1+i\eta_q\right) - \omega^2 m_q\right]X_q &= \frac{R}{r}k_q\left(1+i\eta_q\right)Y \\ X_q &= \frac{\frac{R}{r}\left(1+i\eta_q\right)}{1+i\eta_q - \left(\frac{\omega}{\omega_q}\right)^2}Y \end{aligned} \tag{A.120}$$

By substituting the above equation into Equation (A.119) and noting that $F_T = -F_x$ the normalised dynamic stiffness can be written as:

$$\frac{F_T}{kY} = 1 + i\eta + \left(\frac{R}{r}-1\right)\frac{R}{r}\sum_{q=1}^{n}\frac{k_q}{k}\left(1+i\eta_q\right) - \left(\frac{\omega}{\omega_i}\right)^2 - \frac{R}{r}\left(\frac{R}{r}-1\right)\sum_{q=1}^{n}\frac{k_q}{k}\frac{\left(1+i\eta_q\right)^2}{1+i\eta_q-\left(\frac{\omega}{\omega_q}\right)^2} \tag{A.121}$$

The undamped frequencies of zero dynamic stiffness can be found by setting the above equation to zero. For one absorber attached to the pendulum mass the two frequencies are:

$$1+\left(\frac{R}{r}-1\right)\frac{R}{r}\sum_{q=1}^{n}\frac{k_1}{k}-\left(\frac{\omega}{\omega_i}\right)^2-\frac{R}{r}\left(\frac{R}{r}-1\right)\frac{k_1}{k}\frac{1}{1-\left(\frac{\omega}{\omega_1}\right)^2}=0$$

$$\omega^4-\left[\omega_i^2+\left(\frac{R}{r}-1\right)\frac{R}{r}\frac{k_1}{k}\omega_i^2+\omega_1^2\right]\omega^2+\omega_i^2\omega_1^2=0 \tag{A.122}$$

$$(\omega_1')^2,(\omega_2')^2=\frac{\left(\frac{R}{r}-1\right)\frac{R}{r}\frac{k_1}{k}\omega_i^2+\omega_i^2+\omega_1^2\pm\sqrt{\left[\left(\frac{R}{r}-1\right)\frac{R}{r}\frac{k_1}{k}\omega_i^2+\omega_i^2+\omega_1^2\right]^2-4\omega_i^2\omega_1^2}}{2}$$

A.4.3 Passive amplified vibration-absorbing isolator (non-linear)

For a system with Duffing type non-linearity the equation of motion is:

$$\left[m+m_B\left(1-\frac{R}{r}\right)^2+\frac{I_G}{r^2}\right]\ddot{x}+\left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r}-\frac{I_G}{r^2}\right]\ddot{y}+c(\dot{x}-\dot{y})+k(x-y)+\alpha k_B(x-y)^3=0 \tag{A.123}$$

This equation can be non-dimensionalised as follows:

$$\frac{\left[m+m_B\left(1-\frac{R}{r}\right)^2+\frac{I_G}{r^2}\right]}{k}\ddot{x}+\frac{\left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r}-\frac{I_G}{r^2}\right]}{k}\ddot{y}+\frac{c}{k}(\dot{x}-\dot{y})+(x-y)+\alpha(x-y)^3=0$$

$$\frac{1}{\omega_n^2}\ddot{x}-\frac{1}{\omega_i^2}\ddot{y}+\frac{c}{k}(\dot{x}-\dot{y})+(x-y)+\alpha(x-y)^3=0 \tag{A.124}$$

$$\ddot{x}-\left(\frac{\omega_n}{\omega_i}\right)^2\ddot{y}+2\zeta\omega_n(\dot{x}-\dot{y})+\omega_n^2\left[(x-y)+\alpha(x-y)^3\right]=0$$

A.4.4 Passive amplified vibration-absorbing isolator (motion transformation system)

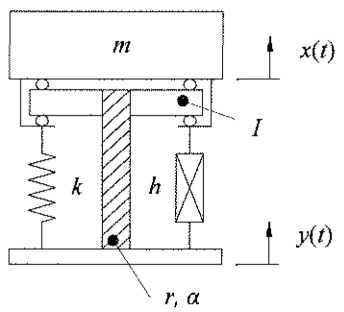

Figure A.12: Mechanical model of motion transformation system

Continuity between the input, output and the rotation angle of the mass can be written as:

$$\phi = \frac{x-y}{r\tan\alpha} \tag{A.125}$$

where r is the mean thread radius and α the helix angle of the thread.

The kinetic energy is:

$$\begin{aligned} T &= \frac{1}{2}\left[m\dot{x}^2 + I\dot{\phi}^2\right] \\ &= \frac{1}{2}\left[m\dot{x}^2 + \frac{I}{r^2\tan^2\alpha}\left(\dot{x}^2 - 2\dot{x}\dot{y} + \dot{y}^2\right)\right] \end{aligned} \tag{A.126}$$

From the above equation the derivative is:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = \left(m + \frac{I}{r^2\tan^2\alpha}\right)\ddot{x} - \frac{I}{r^2\tan^2\alpha}\ddot{y} \tag{A.127}$$

The potential energy is simply:

$$V = \frac{1}{2}k\left(x-y\right)^2 \tag{A.128}$$

From the above equation the derivative is:

$$\frac{\partial V}{\partial x} = kx - ky \tag{A.129}$$

The equation of motion can now be found:

$$\left(m + \frac{I}{r^2\tan^2\alpha}\right)\ddot{x} + k\left(1+i\eta\right)x = \frac{I}{r^2\tan^2\alpha}\ddot{y} + k\left(1+i\eta\right)y \tag{A.130}$$

The equation of motion can be transformed to the frequency domain:

$$\left[k\left(1+i\eta\right) - \omega^2\left(m + \frac{I}{r^2\tan^2\alpha}\right)\right]X = \left[k\left(1+i\eta\right) - \omega^2\frac{I}{r^2\tan^2\alpha}\right]Y \tag{A.131}$$

Form the above equation the transmissibility can be found:

$$\frac{X}{Y} = \frac{k\left(1+i\eta\right) - \omega^2\dfrac{I}{r^2\tan^2\alpha}}{k\left(1+i\eta\right) - \omega^2\left(m + \dfrac{I}{r^2\tan^2\alpha}\right)} \tag{A.132}$$

The undamped isolation frequency is found by equating the numerator to zero:

$$\omega_i = \sqrt{\frac{k}{\dfrac{I}{r^2\tan^2\alpha}}} \tag{A.133}$$

A.4.5 Adaptive amplified vibration-absorbing isolator

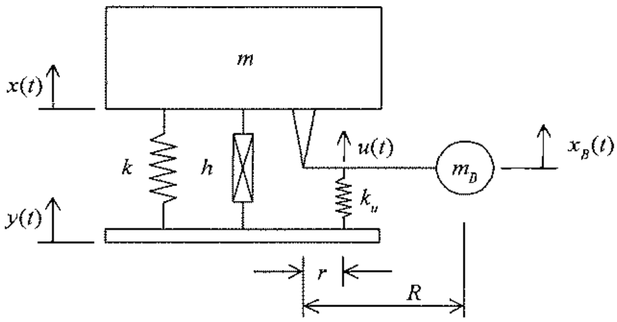

Figure A.13: Mechanical model of an AVAI with flexible fulcrum

The continuity between the x_B, x and u can be written as:

$$\begin{aligned} x_B &= \frac{R}{r}u + \left(1 - \frac{R}{r}\right)x \\ \theta &\approx \frac{u - x}{r} \end{aligned} \tag{A.134}$$

The kinetic energy is:

$$\begin{aligned} T &= \frac{1}{2}\left\{m\dot{x}^2 + m_B\dot{x}_B^2 + I_G\dot{\theta}^2\right\} \\ &= \frac{1}{2}\left\{m\dot{x}^2 + m_B\left[\left(1 - \frac{R}{r}\right)^2\dot{x}^2 + 2\left(1 - \frac{R}{r}\right)\frac{R}{r}\dot{x}\dot{u} + \left(\frac{R}{r}\right)^2\dot{u}^2\right] + \frac{I_G}{r^2}\left(\dot{u}^2 - 2\dot{u}\dot{x} + \dot{x}^2\right)\right\} \end{aligned} \tag{A.135}$$

From the above equation the derivates can be found:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) &= \left[m_B\left(1 - \frac{R}{r}\right)\frac{R}{r} - \frac{I_G}{r^2}\right]\ddot{x} + \left[m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\ddot{u} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) &= \left[m + m_B\left(1 - \frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\ddot{x} + \left[m_B\left(1 - \frac{R}{r}\right)\frac{R}{r} - \frac{I_G}{r^2}\right]\ddot{u} \end{aligned} \tag{A.136}$$

The potential energy is:

$$\begin{aligned} V &= \frac{1}{2}\left[k(x - y)^2 + k_u(u - y)^2\right] \\ &= \frac{1}{2}\left(kx^2 - 2kxy + (k + k_u)y^2 - 2k_u uy + k_u u^2\right) \end{aligned} \tag{A.137}$$

From the above equation the derivates can be found:

$$\begin{aligned}\frac{\partial V}{\partial y} &= (k+k_u)y - k_u u - kx \\ \frac{\partial V}{\partial u} &= k_u u - k_u y \\ \frac{\partial V}{\partial x} &= kx - ky\end{aligned} \tag{A.138}$$

The complete equation of motion (assuming excitation at the y-degree of freedom) is:

$$\begin{bmatrix} m + m_B\left(1-\frac{R}{r}\right)^2 + \frac{I_G}{r^2} & m_B\left(1-\frac{R}{r}\right)\frac{R}{r} - \frac{I_G}{r^2} \\ m_B\left(1-\frac{R}{r}\right)\frac{R}{r} - \frac{I_G}{r^2} & m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{u}\end{bmatrix} + \begin{bmatrix} k(1+\eta) & 0 \\ 0 & k_u(1+\eta_u)\end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix} = \begin{bmatrix} k(1+\eta)y + f_x \\ k_u(1+\eta_u)y\end{bmatrix} \tag{A.139}$$

In the frequency domain the U degree of freedom can be found in terms of X using the second equation in the set shown above:

$$\begin{aligned} -\omega^2\left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r} - \frac{I_G}{r^2}\right]X + \left\{-\omega^2\left[m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right] + k_u(1+i\eta_u)\right\}U &= k_u(1+i\eta_u)Y \\ U = \frac{k_u(1+i\eta_u)Y - \omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r} + \frac{I_G}{r^2}\right]X}{k_u(1+i\eta_u) - \omega^2\left[m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]} \end{aligned} \tag{A.140}$$

The U degree of freedom can now be eliminated from the first equation, yielding the transmissibility between the excitation and the response (assuming the force acting on m is zero):

$$\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r} + \frac{I_G}{r^2}\right]U + \left\{k(1+i\eta) - \omega^2\left[m + m_B\left(1-\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\right\}X = k(1+i\eta)Y$$

$$\begin{aligned} &\left\{k_u(1+i\eta_u) - \omega^2\left[m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\right\}\left\{k(1+i\eta) - \omega^2\left[m + m_B\left(1-\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\right\}X - \left\{\omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r} + \frac{I_G}{r^2}\right]\right\}^2 X \\ &= \left\{k_u(1+i\eta_u) - \omega^2\left[m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\right\}k(1+i\eta)Y - \omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r} + \frac{I_G}{r^2}\right]k_u(1+i\eta_u)Y \end{aligned}$$

$$\frac{X}{Y} = \frac{k(1+i\eta)\left\{k_u(1+i\eta_u) - \omega^2\left[m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\right\} - \omega^2 k_u(1+i\eta_u)\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r} + \frac{I_G}{r^2}\right]}{\left\{k_u(1+i\eta_u) - \omega^2\left[m_B\left(\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\right\}\left\{k(1+i\eta) - \omega^2\left[m + m_B\left(1-\frac{R}{r}\right)^2 + \frac{I_G}{r^2}\right]\right\} + \omega^4\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r} + \frac{I_G}{r^2}\right]^2} \tag{A.141}$$

The dynamic stiffness can be calculated by finding the force acting on the mass m that will force it to zero from the first equation defined by Equation (A.139):

$$\left\{k_u\left(1+i\eta_u\right)-\omega^2\left[m_B\left(\frac{R}{r}\right)^2+\frac{I_G}{r^2}\right]\right\}k\left(1+i\eta\right)Y-\omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right]k_u\left(1+i\eta_u\right)Y$$
$$+F_x\left\{k_u\left(1+i\eta_u\right)-\omega^2\left[m_B\left(\frac{R}{r}\right)^2+\frac{I_G}{r^2}\right]\right\}=0 \tag{A.142}$$
$$\frac{F_T}{Y}=k\left(1+i\eta\right)-\frac{\omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right]k_u\left(1+i\eta_u\right)}{k_u\left(1+i\eta_u\right)-\omega^2\left[m_B\left(\frac{R}{r}\right)^2+\frac{I_G}{r^2}\right]}$$

The above equation can be normalised as follows:

$$\frac{F_T}{kY}=1+i\eta-\frac{\left(\frac{\omega}{\omega_i}\right)^2\left(1+i\eta_u\right)}{1+i\eta_u-\left(\frac{\omega}{\omega_u}\right)^2}$$
$$\text{where:}\quad \omega_i=\sqrt{\frac{k}{m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}}}\qquad \omega_u=\sqrt{\frac{k_u}{m_B\left(\frac{R}{r}\right)^2+\frac{I_G}{r^2}}} \tag{A.143}$$

The frequency ω_u can be eliminated by introducing the stiffness ratio k_u/k:

$$\frac{F_T}{kY}=1+i\eta-\frac{\left(\frac{\omega}{\omega_i}\right)^2\left(1+i\eta_u\right)}{1+i\eta_u-\frac{k}{k_u}\left(\frac{\frac{R}{r}}{\frac{R}{r}-1}\right)\left(\frac{\omega}{\omega_i}\right)^2} \tag{A.144}$$

A.4.6 Active AVAI (acceleration and displacement feedback)

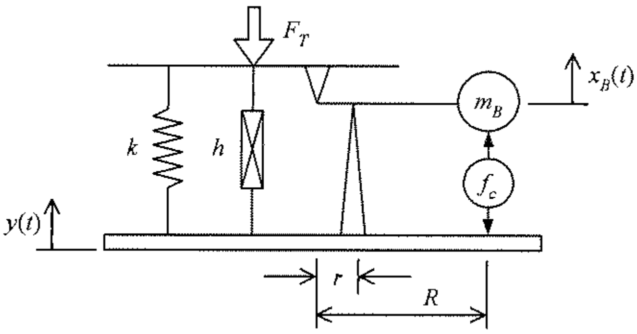

Figure A.14: Mechanical model of an active AVAI

For a control force given by the acceleration and displacement of the input y:

$$f_c = \alpha \ddot{y} + \gamma y \tag{A.145}$$

The dynamic stiffness can be found by using the first equation of the set defined by Equation (A.105) and adding the control force to the right hand side. The control force contribution to the transmitted force is calculated by taking the mechanical advantage of the pendulum into account. The equation is now:

$$\left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r}-\frac{I_G}{r^2}\right]\ddot{y}-k(1+\eta)y = f_x - f_c\left(\frac{R}{r}-1\right) \tag{A.146}$$

On substitution of the control force the above equation becomes:

$$\begin{aligned}&\left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r}-\frac{I_G}{r^2}\right]\ddot{y}-k(1+i\eta)y = f_x - (\alpha\ddot{y}+\gamma y)\left(\frac{R}{r}-1\right)\\ &\left[m_B\left(1-\frac{R}{r}\right)\frac{R}{r}-\frac{I_G}{r^2}+\alpha\left(\frac{R}{r}-1\right)\right]\ddot{y}-\left[k(1+i\eta)-\gamma\left(\frac{R}{r}-1\right)\right]y = f_x\end{aligned} \tag{A.147}$$

By substituting the transmitted force and transforming to the frequency domain the above equation becomes:

$$\frac{F_T}{Y} = k-\gamma\left(\frac{R}{r}-1\right)+ik\eta-\omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}-\alpha\left(\frac{R}{r}-1\right)\right] \tag{A.148}$$

The equation can be normalised if divided by the static stiffness:

$$\frac{F_T}{\left[k-\gamma\left(\frac{R}{r}-1\right)\right]Y}=1+i\frac{\eta}{1-\frac{\gamma}{k}\left(\frac{R}{r}-1\right)}-\left(\frac{\omega}{\omega_i'}\right)^2$$
$$\omega_i'=\sqrt{\frac{k-\gamma\left(\frac{R}{r}-1\right)}{m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}-\alpha\left(\frac{R}{r}-1\right)}} \qquad \text{(A.149)}$$

The stability of the system must be analysed by using the equation of motion when a mass is attached to the system. The equation of motion is (Equation (A.152)):

$$\left[m+m_B\left(\frac{R}{r}-1\right)^2+\frac{I_G}{r^2}\right]\ddot{x}+c\dot{x}+kx=\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right]\ddot{y}+c\dot{y}+ky-\left(\alpha\ddot{y}+\gamma y\right)\left(\frac{R}{r}-1\right) \qquad \text{(A.150)}$$

Since no gains appear on the left hand side of the above equation the system is unconditionally stable.

A.4.7 Active AVAI (relative velocity feedback)

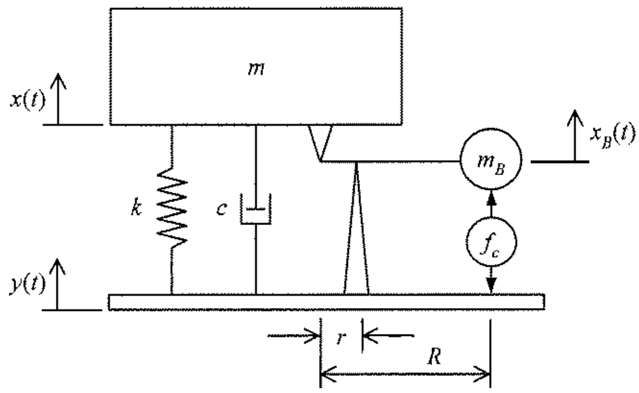

Figure A.15: Mechanical model of an active AVAI for transmissibility

The control force is:

$$f_c=\beta\left(\dot{x}-\dot{y}\right) \qquad \text{(1.151)}$$

Equation (A.105) can be rewritten to include the control force:

$$\left[m+m_B\left(\frac{R}{r}-1\right)^2+\frac{I_G}{r^2}\right]\ddot{x}+c\dot{x}+kx=\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right]\ddot{y}+c\dot{y}+ky-f_c\left(\frac{R}{r}-1\right) \qquad \text{(A.152)}$$

By substituting the control force it follows that:

$$\left[m+m_B\left(\frac{R}{r}-1\right)^2+\frac{I_G}{r^2}\right]\ddot{x}+c\dot{x}+kx=\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right]\ddot{y}+c\dot{y}+ky-\beta(\dot{x}-\dot{y})\left(\frac{R}{r}-1\right)$$
$$\left[m+m_B\left(\frac{R}{r}-1\right)^2+\frac{I_G}{r^2}\right]\ddot{x}+\left[c+\beta\left(\frac{R}{r}-1\right)\right]\dot{x}+kx=\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right]\ddot{y}+\left[c+\beta\left(\frac{R}{r}-1\right)\right]\dot{y}+ky \tag{A.153}$$

By transforming to the frequency domain:

$$\frac{X}{Y}=\frac{k+i\omega\left[c+\beta\left(\frac{R}{r}-1\right)\right]-\omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r}+\frac{I_G}{r^2}\right]}{k+i\omega\left[c+\beta\left(\frac{R}{r}-1\right)\right]-\omega^2\left[m+m_B\left(\frac{R}{r}-1\right)^2+\frac{I_G}{r^2}\right]} \tag{A.154}$$

By introducing some non-dimensional parameters:

$$\frac{X}{Y}=\frac{1+i2\frac{\omega}{\omega_n}\left[\zeta+\zeta_\beta\left(\frac{R}{r}-1\right)\right]-\left(\frac{\omega}{\omega_i}\right)^2}{1+i2\frac{\omega}{\omega_n}\left[\zeta+\zeta_\beta\left(\frac{R}{r}-1\right)\right]-\left(\frac{\omega}{\omega_n}\right)^2}$$
$$\text{where:}\quad \zeta_\beta=\frac{\beta}{2\left[m+m_B\left(\frac{R}{r}-1\right)^2+\frac{I_G}{r^2}\right]\omega_n} \tag{A.155}$$

The stability can be analysed by considering the characteristic equation:

$$\left[m+m_B\left(\frac{R}{r}-1\right)^2+\frac{I_G}{r^2}\right]s^2+\left[c+\beta\left(\frac{R}{r}-1\right)\right]s+k=0 \tag{A.156}$$

The roots are:

$$s_1,s_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{-\left[c+\beta\left(\frac{R}{r}-1\right)\right]\pm\sqrt{\left[c+\beta\left(\frac{R}{r}-1\right)\right]^2-4\left[m+m_B\left(\frac{R}{r}-1\right)^2+\frac{I_G}{r^2}\right]k}}{2\left[m+m_B\left(\frac{R}{r}-1\right)^2+\frac{I_G}{r^2}\right]} \tag{A.157}$$

The above equation will have at least one positive real part if $-b > 0$. To be stable the real part must be negative:

$$-\left[c+\beta\left(\frac{R}{r}-1\right)\right]<0$$
$$\beta>-\frac{c}{\left(\frac{R}{r}-1\right)} \tag{A.158}$$

A.4.8 Active AVAI (absolute velocity feedback)

The system is described by Figure A.15 in §A.4.7. The control force is:

$$f_c = \beta \dot{x} \tag{A.159}$$

Equation (A.105) can be rewritten to include the control force:

$$\left(m + m_B\left(\frac{R}{r}-1\right)^2 + \frac{I_G}{r^2}\right)\ddot{x} + c\dot{x} + kx = \left(m_B\left(\frac{R}{r}-1\right)\frac{R}{r} + \frac{I_G}{r^2}\right)\ddot{y} + c\dot{y} + ky - f_c\left(\frac{R}{r}-1\right) \tag{A.160}$$

By substituting the control force it follows that:

$$\begin{aligned}&\left[m + m_B\left(\frac{R}{r}-1\right)^2 + \frac{I_G}{r^2}\right]\ddot{x} + c\dot{x} + kx = \left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r} + \frac{I_G}{r^2}\right]\ddot{y} + c\dot{y} + ky - \beta\dot{x}\left(\frac{R}{r}-1\right)\\&\left[m + m_B\left(\frac{R}{r}-1\right)^2 + \frac{I_G}{r^2}\right]\ddot{x} + \left[c + \beta\left(\frac{R}{r}-1\right)\right]\dot{x} + kx = \left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r} + \frac{I_G}{r^2}\right]\ddot{y} + c\dot{y} + ky\end{aligned} \tag{A.161}$$

By transforming to the frequency domain:

$$\frac{X}{Y} = \frac{k + i\omega c - \omega^2\left[m_B\left(\frac{R}{r}-1\right)\frac{R}{r} + \frac{I_G}{r^2}\right]}{k + i\omega\left[c + \beta\left(\frac{R}{r}-1\right)\right] - \omega^2\left[m + m_B\left(\frac{R}{r}-1\right)^2 + \frac{I_G}{r^2}\right]} \tag{A.162}$$

By introducing some non-dimensional parameters:

$$\begin{aligned}&\frac{X}{Y} = \frac{1 + i2\frac{\omega}{\omega_n}\zeta - \left(\frac{\omega}{\omega_i}\right)^2}{1 + i2\frac{\omega}{\omega_n}\left[\zeta + \zeta_\beta\left(\frac{R}{r}-1\right)\right] - \left(\frac{\omega}{\omega_n}\right)^2}\\&\text{where: }\ \zeta_\beta = \frac{\beta}{2\left[m + m_B\left(\frac{R}{r}-1\right)^2 + \frac{I_G}{r^2}\right]\omega_n}\end{aligned} \tag{A.163}$$

The stability criterion is the same as for the relative feedback case, but is irrelevant in any case since positive feedback will be used and stability will therefore not be a concern.

APPENDIX B

Derivations for chapter 2

B.1 Adaptive AVAI with variable reservoir wall flexibility (Type I)

B.1.1 Reservoir flexibility covering full wall

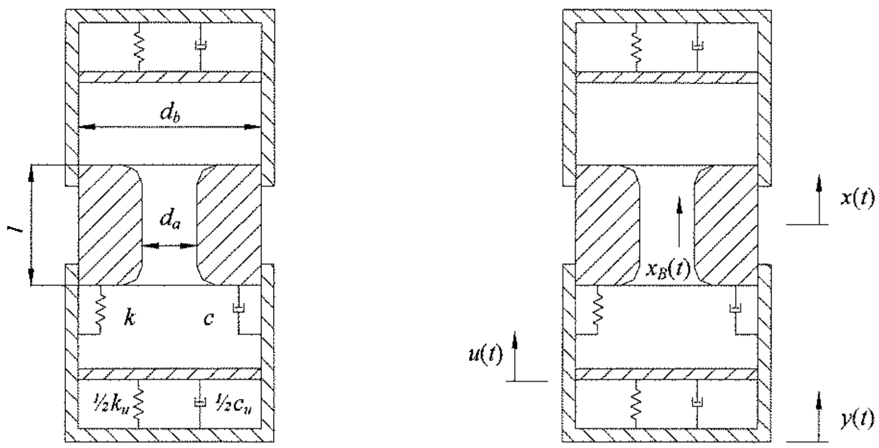

Figure B.1 Mechanical model of an adaptive AVAI with flexibility covering the complete reservoir wall

In Figure B.1 the displacement of the top reservoir wall is equal to the bottom reservoir wall (u), because the fluid is incompressible. Continuity gives the displacement of the fluid in the port:

$$
\begin{aligned}
uA_b &= \left(A_b - A_a\right)x + A_a x_B \\
x_B &= \left(1 - \frac{A_b}{A_a}\right)x + \frac{A_b}{A_a}u \\
\text{where:}\quad & A_a = \pi\frac{d_a^2}{4},\quad A_b = \pi\frac{d_b^2}{4},
\end{aligned}
\tag{B.1}
$$

The total kinetic energy is:

$$
\begin{aligned}
T &= \frac{1}{2}\left(m_x\dot{x}^2 + m_y\dot{y}^2 + m_B\dot{x}_B^2\right) \\
&= \frac{1}{2}\left\{m_x\dot{x}^2 + m_y\dot{y}^2 + m_B\left[\left(1-\frac{A_b}{A_a}\right)^2\dot{x}^2 + 2\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}\dot{x}\dot{u} + \left(\frac{A_b}{A_a}\right)^2\dot{u}^2\right]\right\}
\end{aligned}
\tag{B.2}
$$

From the above equation the derivatives can be calculated:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) &= m_y\ddot{y} \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) &= m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}\ddot{x} + m_B\left(\frac{A_b}{A_a}\right)^2\ddot{u} \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) &= \left[m_x + m_B\left(1-\frac{A_b}{A_a}\right)^2\right]\ddot{x} + m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}\ddot{u}
\end{aligned}
\tag{B.3}
$$

The potential energy is:

$$\begin{aligned} V &= \frac{1}{2}\left[k\left(x-y\right)^2 + k_u\left(u-y\right)^2\right] \\ &= \frac{1}{2}\left(kx^2 - 2kxy + \left(k+k_u\right)y^2 - 2k_u uy + k_u u^2\right) \end{aligned} \tag{B.4}$$

From the above equation the derivatives can be calculated:

$$\begin{aligned} \frac{\partial V}{\partial y} &= \left(k+k_u\right)y - k_u u - kx \\ \frac{\partial V}{\partial u} &= k_u u - k_u y \\ \frac{\partial V}{\partial x} &= kx - ky \end{aligned} \tag{B.5}$$

The Rayleigh term is:

$$\begin{aligned} R &= \frac{1}{2}\left[c\left(\dot{x}-\dot{y}\right)^2 + c_u\left(\dot{u}-\dot{x}\right)^2\right] \\ &= \frac{1}{2}\left[c\dot{x} - 2c\dot{x}\dot{y} + c\dot{y}^2 + c_u\dot{u} - 2c_u\dot{u}\dot{y} + c_u\dot{y}^2\right] \end{aligned} \tag{B.6}$$

From the above equation the derivatives can be calculated:

$$\begin{aligned} \frac{\partial R}{\partial \dot{y}} &= \left(c+c_u\right)\dot{y} - c_u\dot{u} - c\dot{x} \\ \frac{\partial R}{\partial \dot{u}} &= c_u\dot{u} - c_u\dot{y} \\ \frac{\partial R}{\partial \dot{x}} &= c\dot{x} - c\dot{y} \end{aligned} \tag{B.7}$$

Lagrange's equations are defined as (Rao, 1990):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) - \frac{\partial T}{\partial x_j} + \frac{\partial V}{\partial x_j} = Q_j^{(n)} \quad j = 1,2,\ldots,n \tag{B.8}$$

The equation of motion can be derived by substituting the derivatives into Lagrange's equations:

$$\begin{bmatrix} m_y & 0 & 0 \\ 0 & m_x + m_B\left(1-\frac{A_b}{A_a}\right)^2 & m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a} \\ 0 & m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a} & m_B\left(\frac{A_b}{A_a}\right)^2 \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{x} \\ \ddot{u} \end{bmatrix} + \begin{bmatrix} c+c_u & -c & -c_u \\ -c & c & 0 \\ -c_u & 0 & c_u \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{x} \\ \dot{u} \end{bmatrix} + \begin{bmatrix} k+k_u & -k & -k_u \\ -k & k & 0 \\ -k_u & 0 & k_u \end{bmatrix} \begin{bmatrix} y \\ x \\ u \end{bmatrix} = \begin{bmatrix} f_y \\ f_x \\ 0 \end{bmatrix} \tag{B.9}$$

If the forces f_y and f_x are zero and the y-displacement is prescribed, the equation for the y degree of freedom can be eliminated from the equation of motion:

$$\begin{bmatrix} m_x + m_B\left(1-\frac{A_b}{A_a}\right)^2 & m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a} \\ m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a} & m_B\left(\frac{A_b}{A_a}\right)^2 \end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{u}\end{bmatrix}+\begin{bmatrix} c & 0\\ 0 & c_u\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{u}\end{bmatrix}+\begin{bmatrix} k & 0\\ 0 & k_u\end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix}=\begin{bmatrix} c\dot{y}+ky\\ c_u\dot{y}+k_u y\end{bmatrix} \tag{B.10}$$

The above equation can be transformed to the frequency domain by assuming harmonic excitation:

$$\begin{bmatrix} k+i\omega c-\omega^2\left[m_x+m_B\left(1-\frac{A_b}{A_a}\right)^2\right] & -\omega^2 m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a} \\ -\omega^2 m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a} & k_u+i\omega c_u-\omega^2 m_B\left(\frac{A_b}{A_a}\right)^2 \end{bmatrix}\begin{bmatrix}X\\ U\end{bmatrix}=\begin{bmatrix} i\omega c+k\\ i\omega c_u+k_u\end{bmatrix}Y \tag{B.11}$$

The second equation in the set defined above can be used to find an expression for the U degree of freedom:

$$U=\frac{\left(i\omega c_u+k_u\right)Y+\omega^2 m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}X}{k_u+i\omega c_u-\omega^2 m_B\left(\frac{A_b}{A_a}\right)^2} \tag{B.12}$$

The above equation can be used to eliminate the U degree of freedom from the first equation in the set defined by Equation (B.11), which leads to the transmissibility:

$$\frac{X}{Y}=\frac{\left(k+i\omega c\right)\left[k_u+i\omega c_u-\omega^2 m_B\left(\frac{A_b}{A_a}\right)^2\right]+\omega^2\left(k_u+i\omega c_u\right)m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}}{\left[k_u+i\omega c_u-\omega^2 m_B\left(\frac{A_b}{A_a}\right)^2\right]\left\{k+i\omega c-\omega^2\left[m_x+m_B\left(1-\frac{A_b}{A_a}\right)^2\right]\right\}-\left[\omega^2 m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}\right]^2} \tag{B.13}$$

The next section will assume that the excitation occurs at the x degree of freedom as this was the preferred orientation as explained in Chapter 2. If the forces f_y and f_x are zero and the x-displacement is prescribed, the equation for the x degree of freedom can be eliminated from Equation (B.9):

$$\begin{bmatrix} m_y & 0\\ 0 & m_B\left(\frac{A_b}{A_a}\right)^2\end{bmatrix}\begin{bmatrix}\ddot{y}\\ \ddot{u}\end{bmatrix}+\begin{bmatrix} c+c_u & -c_u\\ -c_u & c_u\end{bmatrix}\begin{bmatrix}\dot{y}\\ \dot{u}\end{bmatrix}+\begin{bmatrix} k+k_u & -k_u\\ -k_u & k_u\end{bmatrix}\begin{bmatrix}y\\ u\end{bmatrix}=\begin{bmatrix} c\dot{x}+kx\\ -m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}\ddot{x}\end{bmatrix} \tag{B.14}$$

The above equation can be transformed to the frequency domain by assuming harmonic excitation:

$$\begin{bmatrix} k+k_u+i\omega\left(c+c_u\right)-\omega^2 m_y & -\left(k_u+i\omega c_u\right)\\ -\left(k_u+i\omega c_u\right) & k_u+i\omega c_u-\omega^2 m_B\left(\frac{A_b}{A_a}\right)^2\end{bmatrix}\begin{bmatrix}Y\\ U\end{bmatrix}=\begin{bmatrix} k+i\omega c\\ \omega^2 m_B\left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}\end{bmatrix}X \tag{B.15}$$

The second equation in the set defined above can be used to find an expression for the U degree of freedom:

$$U = \frac{\omega^2 m_B \left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a} X + \left(k_u + i\omega c_u\right) Y}{k_u + i\omega c_u - \omega^2 m_B \left(\frac{A_b}{A_a}\right)^2} \tag{B.16}$$

The above equation can be used to eliminate the U degree of freedom from the first equation in the set defined by Equation (B.15), which leads to the transmissibility:

$$\frac{Y}{X} = \frac{\left(k + i\omega c\right)\left[k_u + i\omega c_u - \omega^2 m_B \left(\frac{A_b}{A_a}\right)^2\right] + \left(k_u + i\omega c_u\right)\omega^2 m_B \left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}}{\left[k_u + i\omega c_u - \omega^2 m_B \left(\frac{A_b}{A_a}\right)^2\right]\left[k + k_u + i\omega\left(c + c_u\right) - \omega^2 m_y\right] - \left(k_u + i\omega c_u\right)^2} \tag{B.17}$$

For the undamped case non-dimensionalisation of Equation (B.17) yields:

$$\frac{Y}{X} = \frac{k\left[k_u - \omega^2 m_B \left(\frac{A_b}{A_a}\right)^2\right] + k_u \omega^2 m_B \left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}}{\left[k_u - \omega^2 m_B \left(\frac{A_b}{A_a}\right)^2\right]\left[k + k_u - \omega^2 m_y\right] - k_u^2}$$

$$= \frac{k k_u \left\{1 - \omega^2 \frac{m_B \left(\frac{A_b}{A_a}\right)^2}{k_u} + \omega^2 \frac{m_B \left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}}{k}\right\}}{k k_u \left\{\left[1 - \omega^2 \frac{m_B \left(\frac{A_b}{A_a}\right)^2}{k_u}\right]\left(1 + \frac{k_u}{k} - \omega^2 \frac{m_y}{k}\right) - \frac{k_u}{k}\right\}}$$

$$= \frac{1 - \omega^2 \frac{m_B \left(\frac{A_b}{A_a}\right)^2}{k_u} + \omega^2 \frac{m_B \left(1-\frac{A_b}{A_a}\right)\frac{A_b}{A_a}}{k}}{\left[1 - \omega^2 \frac{m_B \left(\frac{A_b}{A_a}\right)^2}{k_u}\right]\left(1 + \frac{k_u}{k} - \omega^2 \frac{m_y}{k}\right) - \frac{k_u}{k}}$$

$$= \frac{1 - \left(\frac{\omega}{\omega_2}\right)^2 - \left(\frac{\omega}{\bar{\omega}_i}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right]\left[1 + \mu_k - \left(\frac{\omega}{\omega_1}\right)^2\right] - \mu_k}$$

$$\text{where:} \quad \omega_1^2 = \frac{k}{m_y}, \quad \omega_2^2 = \mu_k \frac{k}{m_B \left(\frac{A_b}{A_a}\right)^2}, \quad \bar{\omega}_i^2 = \frac{k}{m_B \left(\frac{A_b}{A_a} - 1\right)\frac{A_b}{A_a}} \tag{B.18}$$

If the area ratio is much larger than 1 then ω_2 and $\bar{\omega}_i$ are related:

$$\omega_2^2 = \mu_k \frac{k}{m_B\left(\frac{A_b}{A_a}\right)^2} = \mu_k \bar{\omega}_2^2 \approx \mu_k \bar{\omega}_i^2 \tag{B.19}$$

The isolation frequency can be calculated by setting the numerator in Equation (B.18) equal to zero:

$$\begin{aligned} 1-\left(\frac{\omega}{\omega_2}\right)^2-\left(\frac{\omega}{\bar{\omega}_i}\right)^2 &= 0 \\ \omega^2\left[\frac{\omega_1^2}{\bar{\omega}_i^2}+\frac{\omega_1^2}{\omega_2^2}\right] &= \omega_1^2 \\ \frac{\omega_i^2}{\omega_1^2} &= \frac{1}{\left(\frac{\omega_1}{\bar{\omega}_i}\right)^2+\frac{1}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2} \end{aligned} \tag{B.20}$$

The frequency ratio $\omega_1/\bar{\omega}_i$ can be written in terms of the mass (μ_m) and area (μ_A) ratios:

$$\left(\frac{\omega_1}{\bar{\omega}_i}\right)^2 = \frac{\frac{k}{m_y}}{\frac{k}{m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}} = \frac{k}{m_y}\frac{m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}{k} = \frac{m_B}{m_y}\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a} = \mu_m\left(\mu_A-1\right)\mu_A \tag{B.21}$$

The frequency ratio $\omega_1/\bar{\omega}_2$ can be written in terms of the mass and area ratios:

$$\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2 = \frac{\frac{k}{m_y}}{\frac{k}{m_B\left(\frac{A_b}{A_a}\right)^2}} = \frac{k}{m_y}\frac{m_B\left(\frac{A_b}{A_a}\right)^2}{k} = \frac{m_B}{m_y}\left(\frac{A_b}{A_a}\right)^2 = \mu_m\mu_A^2 \tag{B.22}$$

Using the above two relations the isolation frequency can be rewritten as:

$$\begin{aligned} \left(\frac{\omega_i}{\omega_1}\right)^2 &= \frac{1}{\frac{m_B}{m_y}\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}+\frac{1}{\mu_k}\frac{m_B}{m_y}\left(\frac{A_b}{A_a}\right)^2} \\ &= \frac{1}{\mu_m\left(\mu_A-1\right)\mu_A+\frac{\mu_m\mu_A^2}{\mu_k}} \end{aligned} \tag{B.23}$$

The undamped frequencies of maximum transmissibility can be found by equating the denominator of Equation (B.18) to zero:

$$\left[1-\left(\frac{\omega}{\omega_2}\right)^2\right]\left[1+\mu_k-\left(\frac{\omega}{\omega_1}\right)^2\right]-\mu_k=0$$

$$\frac{1}{\omega_2^2\omega_1^2}\omega^4-\left[\left(\frac{1}{\omega_2}\right)^2(1+\mu_k)+\left(\frac{1}{\omega_1}\right)^2\right]\omega^2+1=0$$

$$\left(\frac{\omega_1}{\omega_2}\right)^2\left(\frac{\omega}{\omega_1}\right)^4-\left[\left(\frac{\omega_1}{\omega_2}\right)^2(1+\mu_k)+1\right]\left(\frac{\omega}{\omega_1}\right)^2+1=0 \tag{B.24}$$

$$\left(\frac{\Omega_1}{\omega_1}\right)^2,\left(\frac{\Omega_2}{\omega_1}\right)^2=\frac{\left(\frac{\omega_1}{\omega_2}\right)^2(1+\mu_k)+1\mp\sqrt{\left[\left(\frac{\omega_1}{\omega_2}\right)^2(1+\mu_k)+1\right]^2-4\left(\frac{\omega_1}{\omega_2}\right)^2}}{2\left(\frac{\omega_1}{\omega_2}\right)^2}$$

The following two relations will assist in writing the equation in terms of the stiffness ratio ($\mu_k = k_u/k$) and constant frequency ratios:

$$\frac{\omega_1}{\omega_2}=\frac{1}{\sqrt{\mu_k}}\frac{\omega_1}{\bar{\omega}_2} \tag{B.25}$$

$$\zeta_2=\frac{1}{\sqrt{\mu_k}}\frac{c_u}{2m_B\left(\frac{A_b}{A_a}\right)^2\bar{\omega}_2}=\frac{1}{\sqrt{\mu_k}}\bar{\zeta}_2 \tag{B.26}$$

When introducing Equation (B.25) in Equation (B.24) the frequencies of maximum transmissibility can be written in terms of the stiffness ratio:

$$\begin{aligned}\left(\frac{\Omega_1}{\omega_1}\right)^2,\left(\frac{\Omega_2}{\omega_1}\right)^2&=\frac{\frac{1}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2(1+\mu_k)+1\mp\sqrt{\left[\frac{1}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2(1+\mu_k)+1\right]^2-4\frac{1}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2}}{2\frac{1}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2}\\&=\frac{\frac{1+\mu_k}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2+1\mp\sqrt{\left[\frac{1+\mu_k}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2+1\right]^2-\frac{4}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2}}{\frac{2}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2}\\&=\frac{(1+\mu_k)\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2+\mu_k\mp\sqrt{\left[(1+\mu_k)\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2+\mu_k\right]^2-4\mu_k\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2}}{2\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2}\end{aligned} \tag{B.27}$$

For the damped case Equation (B.17) can be non-dimensionalised as follows:

$$\frac{Y}{X}=\frac{\left(1+i\omega\frac{c}{k}\right)\left[1+i\omega\frac{c_u}{k_u}-\omega^2\frac{m_B\left(\frac{A_b}{A_a}\right)^2}{k_u}\right]-\left(1+i\omega\frac{c_u}{k_u}\right)\omega^2\frac{m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}{k}}{\left[1+i\omega\frac{c_u}{k_u}-\omega^2\frac{m_B\left(\frac{A_b}{A_a}\right)^2}{k_u}\right]\left[1+\frac{k_u}{k}+i\omega\frac{\left(c+c_u\right)}{k}-\omega^2\frac{m_y}{k}\right]-\frac{k_u}{k}\left(1+i\omega\frac{c_u}{k_u}\right)^2}$$

$$=\frac{\left(1+i2\frac{\omega}{\omega_1}\zeta_1\right)\left[1+i2\frac{\omega}{\omega_2}\zeta_2-\left(\frac{\omega}{\omega_2}\right)^2\right]-\left(1+i2\frac{\omega}{\omega_2}\zeta_2\right)\left(\frac{\omega}{\bar{\omega}_i}\right)^2}{\left[1+\frac{k_u}{k}+i2\left(\frac{\omega}{\omega_1}\zeta_1+\frac{k_u}{k}\frac{\omega}{\omega_2}\zeta_2\right)-\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1+i2\frac{\omega}{\omega_2}\zeta_2-\left(\frac{\omega}{\omega_2}\right)^2\right]-\frac{k_u}{k}\left(1+i2\frac{\omega}{\omega_2}\zeta_2\right)^2} \qquad \text{(B.28)}$$

where: $\zeta_1=\frac{c}{2m_y\omega_1}$, $\zeta_2=\frac{c_u}{2m_B\left(\frac{A_b}{A_a}\right)^2\omega_2}$

Equation (B.28) can be written in terms of frequency and damping ratios that are independent of the stiffness ratio:

$$\frac{Y}{X}=\frac{\left(1+i2\frac{\omega}{\omega_1}\zeta_1\right)\left[1+i\frac{2}{\mu_k}\frac{\omega_1}{\bar{\omega}_2}\frac{\omega}{\omega_1}\bar{\zeta}_2-\frac{1}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2\left(\frac{\omega}{\omega_1}\right)^2\right]-\left(1+i\frac{2}{\mu_k}\frac{\omega_1}{\bar{\omega}_2}\frac{\omega}{\omega_1}\bar{\zeta}_2\right)\left(\frac{\omega_1}{\bar{\omega}_i}\right)^2\left(\frac{\omega}{\omega_1}\right)^2}{\left[1+\mu_k+i2\frac{\omega}{\omega_1}\left(\zeta_1+\frac{\omega_1}{\bar{\omega}_2}\bar{\zeta}_2\right)-\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1+i\frac{2}{\mu_k}\frac{\omega_1}{\bar{\omega}_2}\frac{\omega}{\omega_1}\bar{\zeta}_2-\frac{1}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2\left(\frac{\omega}{\omega_1}\right)^2\right]-\mu_k\left(1+i\frac{2}{\mu_k}\frac{\omega_1}{\bar{\omega}_2}\frac{\omega}{\omega_1}\bar{\zeta}_2\right)^2} \qquad \text{(B.29)}$$

Equation (B.14) can be non-dimensionalised as follows for the first equation:

$$m_y\ddot{y}+\left(c+c_u\right)\dot{y}-c_u\dot{u}+\left(k+k_u\right)y-k_uu=c\dot{x}+kx$$

$$\frac{m_y}{k}\ddot{y}+\left(\frac{c}{k}+\mu_k\frac{c_u}{k_u}\right)\dot{y}-\mu_k\frac{c_u}{k_u}\dot{u}+\left(1+\mu_k\right)y-\mu_ku=\frac{c}{k}\dot{x}+x$$

$$\frac{1}{\omega_1^2}\ddot{y}+2\left(\frac{\zeta_1}{\omega_1}+\mu_k\frac{\zeta_2}{\omega_2}\right)\dot{y}-2\mu_k\frac{\zeta_2}{\omega_2}\dot{u}+\left(1+\mu_k\right)y-\mu_ku=2\frac{\zeta_1}{\omega_1}\dot{x}+x$$

$$\ddot{y}+2\left(\zeta_1+\mu_k\frac{\omega_1}{\omega_2}\zeta_2\right)\omega_1\dot{y}-2\mu_k\frac{\omega_1}{\omega_2}\zeta_2\omega_1\dot{u}+\omega_1^2\left(1+\mu_k\right)y-\omega_1^2\mu_ku-2\zeta_1\omega_1\dot{x}-\omega_1^2x=0 \qquad \text{(B.30)}$$

$$\ddot{y}+2\left(\zeta_1+\frac{\omega_1}{\bar{\omega}_2}\bar{\zeta}_2\right)\omega_1\dot{y}-2\frac{\omega_1}{\bar{\omega}_2}\bar{\zeta}_2\omega_1\dot{u}+\omega_1^2\left(1+\mu_k\right)y-\omega_1^2\mu_ku-2\zeta_1\omega_1\dot{x}-\omega_1^2x=0$$

where: $\frac{\zeta_2}{\omega_2}=\frac{\frac{1}{\sqrt{\mu_k}}\bar{\zeta}_2}{\sqrt{\mu_k}\bar{\omega}_2}=\frac{1}{\sqrt{\mu_k}}\bar{\zeta}_2\frac{1}{\sqrt{\mu_k}\bar{\omega}_2}=\frac{1}{\mu_k}\frac{\bar{\zeta}_2}{\bar{\omega}_2}$

And as follows for the second equation:

$$
\begin{aligned}
& m_B\left(\frac{A_b}{A_a}\right)^2 \ddot{u}+c_u(\dot{u}-\dot{y})+k_u(u-y)-m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}\ddot{x}=0 \\
& \frac{m_B\left(\frac{A_b}{A_a}\right)^2}{k_u}\ddot{u}+\frac{c_u}{k_u}(\dot{u}-\dot{y})+u-y-\frac{m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}{k_u}\ddot{x}=0 \\
& \frac{1}{\omega_2^2}\ddot{u}+2\frac{\zeta_2}{\omega_2}(\dot{u}-\dot{y})+u-y-\frac{1}{\mu_k}\frac{1}{\bar{\omega}_i^2}\ddot{x}=0 \\
& \ddot{u}+2\zeta_2\omega_2(\dot{u}-\dot{y})+\omega_2^2(u-y)-\frac{1}{\mu_k}\left(\frac{\omega_2}{\bar{\omega}_i}\right)^2\ddot{x}=0 \\
& \ddot{u}+2\bar{\zeta}_2\bar{\omega}_2(\dot{u}-\dot{y})+\mu_k\bar{\omega}_2^2(u-y)-\left(\frac{\bar{\omega}_2}{\bar{\omega}_i}\right)^2\ddot{x}=0 \\
& \text{where:}\quad \zeta_2\omega_2=\frac{1}{\sqrt{\mu_k}}\bar{\zeta}_2\sqrt{\mu_k}\,\bar{\omega}_2=\bar{\zeta}_2\bar{\omega}_2
\end{aligned}
\tag{B.31}
$$

B.1.2 Reduced-area reservoir wall stiffness

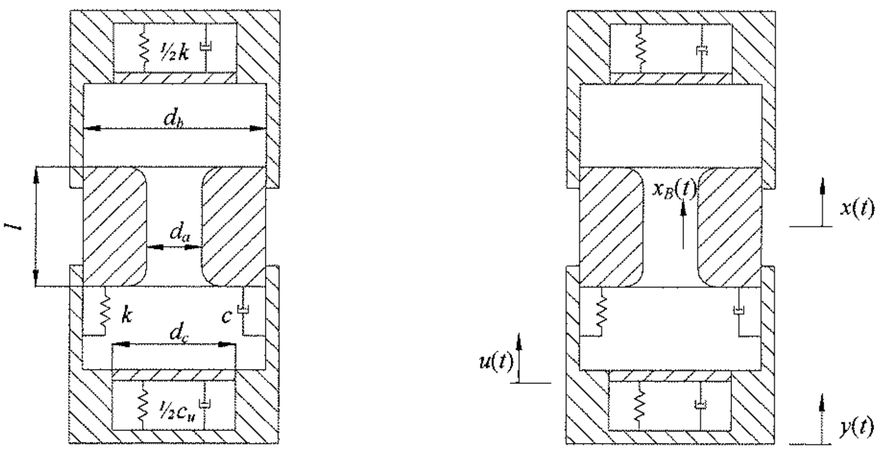

Figure B.2: Mechanical model of an AVAI with reduced-area reservoir wall stiffness

The fluid continuity through the port is changed with the reduced area give by A_c:

$$
\begin{aligned}
& y(A_b-A_c)+uA_c=(A_b-A_a)x+A_a x_B \\
& x_B=\left(1-\frac{A_b}{A_a}\right)x+\frac{A_b-A_c}{A_a}y+\frac{A_c}{A_a}u \\
& \text{where:}\quad A_a=\pi\frac{d_a^2}{4},\quad A_b=\pi\frac{d_b^2}{4},\quad A_c=\pi\frac{d_c^2}{4}
\end{aligned}
\tag{B.32}
$$

The above equation reduces to Equation (B.1) when the reduced area covers the full wall (i.e. $A_c = A_b$).

The continuity equation will be used to eliminate the fluid motion (x_B) from the equations of motion. The equation of motion is derived using Lagrange's equations. The kinetic energy is:

$$\begin{aligned} T &= \frac{1}{2}\left\{ m_y \dot{y}^2 + m_x \dot{x}^2 + m_B \dot{x}_B^2 \right\} \\ &= \frac{1}{2}\left\{ m_y \dot{y}^2 + m_x \dot{x}^2 + m_B \left[\left(1 - \frac{A_b}{A_a}\right)^2 \dot{x}^2 + 2\left(1 - \frac{A_b}{A_a}\right)\frac{A_b - A_c}{A_a}\dot{x}\dot{y} + \left(\frac{A_b - A_c}{A_a}\right)^2 \dot{y}^2 \right.\right. \\ &\qquad \left.\left. + 2\left(1 - \frac{A_b}{A_a}\right)\frac{A_c}{A_a}\dot{x}\dot{u} + 2\left(\frac{A_b - A_c}{A_a}\right)\frac{A_c}{A_a}\dot{u}\dot{y} + \left(\frac{A_c}{A_a}\right)^2 \dot{u}^2 \right] \right\} \end{aligned} \tag{B.33}$$

The derivates are:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) &= \left[m_y + m_B \left(\frac{A_b - A_c}{A_a}\right)^2 \right]\ddot{y} + m_B \left[\left(1 - \frac{A_b}{A_a}\right)\left(\frac{A_b - A_c}{A_a}\right)\ddot{x} + \left(\frac{A_b - A_c}{A_a}\right)\frac{A_c}{A_a}\ddot{u} \right] \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) &= \left[m + m_B \left(1 - \frac{A_b}{A_a}\right)^2 \right]\ddot{x} + m_B \left[\left(1 - \frac{A_b}{A_a}\right)\frac{A_b - A_c}{A_a}\ddot{y} + \left(1 - \frac{A_b}{A_a}\right)\frac{A_c}{A_a}\ddot{u} \right] \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) &= m_B \left[\frac{A_c}{A_a}\left(1 - \frac{A_b}{A_a}\right)\ddot{x} + \frac{A_c}{A_a}\left(\frac{A_b - A_c}{A_a}\right)\ddot{y} + \left(\frac{A_c}{A_a}\right)^2 \ddot{u} \right] \end{aligned} \tag{B.34}$$

The potential energy and Rayleigh terms do not depend on the continuity equation and are therefore exactly as before (refer §B.1.1). The equation of motion can be derived using Lagrange's equations and by assuming that the x degree of freedom is prescribed:

$$\begin{aligned} &\begin{bmatrix} m_y + m_B \left(\frac{A_b - A_c}{A_a}\right)^2 & m_B \left(\frac{A_b - A_c}{A_a}\right)\frac{A_c}{A_a} \\ m_B \left(\frac{A_b - A_c}{A_a}\right)\frac{A_c}{A_a} & m_B \left(\frac{A_c}{A_a}\right)^2 \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{u} \end{bmatrix} + \begin{bmatrix} c + c_u & -c_u \\ -c_u & c_u \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{u} \end{bmatrix} + \begin{bmatrix} k + k_u & -k_u \\ -k_u & k_u \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \\ &= \begin{bmatrix} m_B \left(\frac{A_b}{A_a} - 1\right)\left(\frac{A_b - A_c}{A_a}\right)\ddot{x} + c\dot{x} + kx \\ m_B \left(\frac{A_b}{A_a} - 1\right)\frac{A_c}{A_a}\ddot{x} \end{bmatrix} \end{aligned} \tag{B.35}$$

Transforming the above equation to the frequency domain yields:

$$\begin{aligned} &\begin{bmatrix} k + k_u + i\omega(c + c_u) - \omega^2 \left[m_y + m_B \left(\frac{A_b - A_c}{A_a}\right)^2 \right] & -k_u - i\omega c_u - \omega^2 m_B \left(\frac{A_b - A_c}{A_a}\right)\frac{A_c}{A_a} \\ -k_u - i\omega c_u - \omega^2 m_B \left(\frac{A_b - A_c}{A_a}\right)\frac{A_c}{A_a} & k_u + i\omega c_u - \omega^2 m_B \left(\frac{A_c}{A_a}\right)^2 \end{bmatrix} \begin{bmatrix} Y \\ U \end{bmatrix} \\ &= \begin{bmatrix} \omega^2 m_B \left(1 - \frac{A_b}{A_a}\right)\left(\frac{A_b - A_c}{A_a}\right) + i\omega c + k \\ \omega^2 m_B \left(1 - \frac{A_b}{A_a}\right)\frac{A_c}{A_a} \end{bmatrix} X \end{aligned} \tag{B.36}$$

The second equation in the set described above can be used to eliminate the U degree of freedom.

The non-dimensional transmissibility can be found as follows:

$$\frac{Y}{X}=\frac{\left[k+i\omega c-\omega^2 m_B\left(\frac{A_b}{A_a}-1\right)\left(\frac{A_b-A_c}{A_a}\right)\right]\left[k_u+i\omega c_u-\omega^2 m_B\left(\frac{A_c}{A_a}\right)^2\right]-\omega^2 m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_c}{A_a}\left[k_u+i\omega c_u+\omega^2 m_B\left(\frac{A_b-A_c}{A_a}\right)\frac{A_c}{A_a}\right]}{\left\{k+k_u+i\omega\left(c+c_u\right)-\omega^2\left[m_y+m_B\left(\frac{A_b-A_c}{A_a}\right)^2\right]\right\}\left[k_u+i\omega c_u-\omega^2 m_B\left(\frac{A_c}{A_a}\right)^2\right]-\left[k_u+i\omega c_u+\omega^2 m_B\left(\frac{A_b-A_c}{A_a}\right)\frac{A_c}{A_a}\right]^2}$$

$$=\frac{\left[1+i\omega\frac{c}{k}-\omega^2(1-\lambda)\frac{m_B\left(\frac{A_b}{A_a}-1\right)\left(\frac{A_b}{A_a}\right)}{k}\right]\left[1+i\omega\frac{c_u}{k_u}-\omega^2\lambda^2\frac{m_B\left(\frac{A_b}{A_a}\right)^2}{k_u}\right]-\omega^2\lambda\frac{m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}{k}\left[1+i\omega\frac{c_u}{k_u}+\omega^2(1-\lambda)\lambda\frac{m_B\left(\frac{A_b}{A_a}\right)^2}{k_u}\right]}{\left\{1+\frac{k_u}{k}+i\omega\frac{c+c_u}{k}-\omega^2\left[\frac{m_y}{k}+\frac{m_B(1-\lambda)^2\left(\frac{A_b}{A_a}\right)^2}{k}\right]\right\}\left[1+i\omega\frac{c_u}{k_u}-\omega^2\lambda^2\frac{m_B\left(\frac{A_b}{A_a}\right)^2}{k_u}\right]-\frac{k_u}{k}\left[1+i\omega\frac{c_u}{k_u}+\omega^2(1-\lambda)\lambda\frac{m_B\left(\frac{A_b}{A_a}\right)^2}{k_u}\right]^2}$$

$$=\frac{\left[1+i2\frac{\omega}{\omega_1}\zeta_1-(1-\lambda)\left(\frac{\omega}{\bar{\omega}_i}\right)^2\right]\left[1+i2\frac{\omega}{\omega_2}\zeta_2-\lambda^2\left(\frac{\omega}{\omega_2}\right)^2\right]-\lambda\left(\frac{\omega}{\bar{\omega}_i}\right)^2\left[1+i2\frac{\omega}{\omega_2}\zeta_2+(1-\lambda)\lambda\left(\frac{\omega}{\omega_2}\right)^2\right]}{\left\{1+\frac{k_u}{k}+i2\left(\frac{\omega}{\omega_1}\zeta_1+\frac{k_u}{k}\frac{\omega}{\omega_2}\zeta_2\right)-\left[\left(\frac{\omega}{\omega_1}\right)^2+(1-\lambda)^2\frac{k_u}{k}\left(\frac{\omega}{\omega_2}\right)^2\right]\right\}\left[1+i2\frac{\omega}{\omega_2}\zeta_2-\lambda^2\left(\frac{\omega}{\omega_2}\right)^2\right]-\frac{k_u}{k}\left[1+i2\frac{\omega}{\omega_2}\zeta_2+(1-\lambda)\lambda\left(\frac{\omega}{\omega_2}\right)^2\right]^2}$$

where: $\lambda=\frac{A_c}{A_b}$ (B.37)

The transmissibility can be written in terms of the stiffness ratio and constants defined by Equation (B.25) and (B.26):

$$\frac{Y}{X}=\frac{\left[1+i2\frac{\omega}{\omega_1}\zeta_1-(1-\lambda)\left(\frac{\omega_1}{\bar{\omega}_i}\right)^2\left(\frac{\omega}{\omega_1}\right)^2\right]\left[1+i\frac{2}{\mu_k}\frac{\omega_1}{\bar{\omega}_2}\frac{\omega}{\omega_1}\bar{\zeta}_2-\frac{\lambda^2}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2\left(\frac{\omega}{\omega_1}\right)^2\right]-\lambda\left(\frac{\omega_1}{\bar{\omega}_i}\right)^2\left(\frac{\omega}{\omega_1}\right)^2\left[1+i\frac{2}{\mu_k}\frac{\omega_1}{\bar{\omega}_2}\frac{\omega}{\omega_1}\bar{\zeta}_2+(1-\lambda)\frac{\lambda}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2\left(\frac{\omega}{\omega_1}\right)^2\right]}{\left\{1+\mu_k+i2\frac{\omega}{\omega_1}\left(\zeta_1+\frac{\omega_1}{\bar{\omega}_2}\bar{\zeta}_2\right)-\left[\left(\frac{\omega}{\omega_1}\right)^2+(1-\lambda)^2\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2\left(\frac{\omega}{\omega_1}\right)^2\right]\right\}\left[1+i\frac{2}{\mu_k}\frac{\omega_1}{\bar{\omega}_2}\frac{\omega}{\omega_1}\bar{\zeta}_2-\frac{\lambda^2}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2\left(\frac{\omega}{\omega_1}\right)^2\right]-\mu_k\left[1+i\frac{2}{\mu_k}\frac{\omega_1}{\bar{\omega}_2}\frac{\omega}{\omega_1}\bar{\zeta}_2+(1-\lambda)\frac{\lambda}{\mu_k}\left(\frac{\omega_1}{\bar{\omega}_2}\right)^2\left(\frac{\omega}{\omega_1}\right)^2\right]^2}\quad\text{(B.38)}$$

B.2 Adaptive AVAI with slug (Type II)

B.2.1 Slug springs

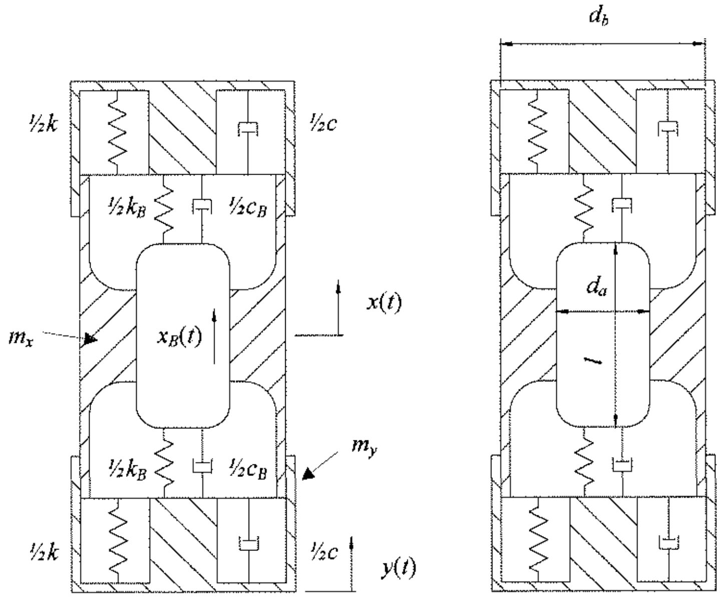

Figure B.3: Mechanical model of an adaptive AVAI with a slug

The equations of motion will be derived using Lagrange's equations. The equations of motion for this configuration can also be derived by considering force balance as shown by Halwes (1981a). The fluid continuity through the port is:

$$\begin{aligned} yA_b &= (A_b - A_a)x + A_a x_B \\ x_B &= \left(1 - \frac{A_b}{A_a}\right)x + \frac{A_b}{A_a}y \end{aligned} \tag{B.39}$$

The kinetic energy is:

$$\begin{aligned} T &= \frac{1}{2}\left(m_x\dot{x}^2 + m_B\dot{x}_B^2 + m_y\dot{y}^2\right) \\ &= \frac{1}{2}\left\{m_x\dot{x}^2 + m_B\left[\left(\frac{A_b}{A_a} - 1\right)^2\dot{x}^2 - 2\left(\frac{A_b}{A_a} - 1\right)\frac{A_b}{A_a}\dot{x}\dot{y} + \left(\frac{A_b}{A_a}\right)^2\dot{y}^2\right] + m_y\dot{y}^2\right\} \end{aligned} \tag{B.40}$$

where: $m_B = \rho A_a l$

From the previous equation the derivatives can be found:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) &= \left[m_x + m_B\left(\frac{A_b}{A_a}-1\right)^2\right]\ddot{x} + m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}\ddot{y} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) &= m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}\ddot{x} + \left[m_y + m_B\left(\frac{A_b}{A_a}\right)^2\right]\ddot{y}\end{aligned} \tag{B.41}$$

The potential energy is:

$$\begin{aligned}V &= \frac{1}{2}\left[k(x-y)^2 + k_B(x_B-y)^2\right] \\ &= \frac{1}{2}\left[k + k_B\left(1-\frac{A_b}{A_a}\right)^2\right]\left(x^2 - 2xy + y^2\right)\end{aligned} \tag{B.42}$$

From the previous equation the derivatives can be found:

$$\begin{aligned}\frac{\partial V}{\partial x} &= \left[k + k_B\left(1-\frac{A_b}{A_a}\right)^2\right]x - \left[k + k_B\left(1-\frac{A_b}{A_a}\right)^2\right]y \\ \frac{\partial V}{\partial y} &= -\left[k + k_B\left(1-\frac{A_b}{A_a}\right)^2\right]x + \left[k + k_B\left(1-\frac{A_b}{A_a}\right)^2\right]y\end{aligned} \tag{B.43}$$

The complete equation of motion can be found using Lagrange's equations:

$$\begin{aligned}&\begin{bmatrix} m_x + m_B\left(\frac{A_b}{A_a}-1\right)^2 & -m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a} \\ -m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a} & m_y + m_B\left(\frac{A_b}{A_a}\right)^2 \end{bmatrix}\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} + \begin{bmatrix} c + c_B\left(\frac{A_b}{A_a}-1\right)^2 & -\left[c + c_B\left(\frac{A_b}{A_a}-1\right)^2\right] \\ -\left[c + c_B\left(\frac{A_b}{A_a}-1\right)^2\right] & c + c_B\left(\frac{A_b}{A_a}-1\right)^2 \end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} \\ &+ \begin{bmatrix} k + k_B\left(\frac{A_b}{A_a}-1\right)^2 & -\left[k + k_B\left(\frac{A_b}{A_a}-1\right)^2\right] \\ -\left[k + k_B\left(\frac{A_b}{A_a}-1\right)^2\right] & k + k_B\left(\frac{A_b}{A_a}-1\right)^2 \end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned} \tag{B.44}$$

Assuming that excitation occurs at the y degree of freedom, the second equation is eliminated:

$$\begin{aligned}&\left[m_x + m_B\left(\frac{A_b}{A_a}-1\right)^2\right]\ddot{x} + \left[c + c_B\left(\frac{A_b}{A_a}-1\right)^2\right]\dot{x} + \left[k + k_B\left(\frac{A_b}{A_a}-1\right)^2\right]x \\ &= m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}\ddot{y} + \left[c + c_B\left(\frac{A_b}{A_a}-1\right)^2\right]\dot{y} + \left[k + k_B\left(\frac{A_b}{A_a}-1\right)^2\right]y\end{aligned} \tag{B.45}$$

The equation of motion can be non-dimensionalised by introducing the damping ratio and the isolation and natural frequencies:

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x=\left(\frac{\omega_n}{\omega_i}\right)^2\ddot{y}+2\zeta\omega_n\dot{y}+\omega_n^2 y$$

$$\text{where: }\quad \omega_i=\sqrt{\frac{k+k_B\left(\frac{A_b}{A_a}-1\right)^2}{m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}},\quad \omega_n=\sqrt{\frac{k+k_B\left(\frac{A_b}{A_a}-1\right)^2}{m_x+m_B\left(\frac{A_b}{A_a}-1\right)^2}},\quad \zeta=\frac{c+c_B\left(\frac{A_b}{A_a}-1\right)^2}{2\left[m_x+m_B\left(\frac{A_b}{A_a}-1\right)^2\right]\omega_n} \tag{B.46}$$

The transmissibility is:

$$\frac{X}{Y}=\frac{1+i2\frac{\omega}{\omega_n}\zeta-\left(\frac{\omega}{\omega_i}\right)^2}{1+i2\frac{\omega}{\omega_n}\zeta-\left(\frac{\omega}{\omega_n}\right)^2} \tag{B.47}$$

Assuming that excitation occurs at the x degree of freedom, the first equation is eliminated:

$$\begin{aligned}&\left[m_y+m_B\left(\frac{A_b}{A_a}\right)^2\right]\ddot{y}+\left[c+c_B\left(\frac{A_b}{A_a}-1\right)^2\right]\dot{y}+\left[k+k_B\left(\frac{A_b}{A_a}-1\right)^2\right]y\\&=m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}\ddot{x}+\left[c+c_B\left(\frac{A_b}{A_a}-1\right)^2\right]\dot{x}+\left[k+k_B\left(\frac{A_b}{A_a}-1\right)^2\right]x\end{aligned} \tag{B.48}$$

The equation of motion can be non-dimensionalised by introducing the damping ratio and the isolation and natural frequencies:

$$\ddot{y}+2\zeta\omega_n\dot{y}+\omega_n^2 y=\left(\frac{\omega_n}{\omega_i}\right)^2\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x$$

$$\text{where: }\quad \omega_i=\sqrt{\frac{k+k_B\left(\frac{A_b}{A_a}-1\right)^2}{m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}},\quad \omega_n=\sqrt{\frac{k+k_B\left(\frac{A_b}{A_a}-1\right)^2}{m_y+m_B\left(\frac{A_b}{A_a}\right)^2}},\quad \zeta=\frac{c+c_B\left(\frac{A_b}{A_a}-1\right)^2}{2\left[m_y+m_B\left(\frac{A_b}{A_a}\right)^2\right]\omega_n} \tag{B.49}$$

The equation is the same as when the x degree of freedom is the excitation degree of freedom, except for the definition of the natural frequency and damping ratio. In the analysis that follows the ratio X/Y can therefore be inverted as long as the corresponding definition of the natural frequency and damping ratio is used. The transmissibility can be rewritten in terms of the isolation frequency by introducing the frequency ratio:

$$\frac{X}{Y}=\frac{1+i2\frac{\omega_i}{\omega_n}\frac{\omega}{\omega_i}\zeta-\left(\frac{\omega}{\omega_i}\right)^2}{1+i2\frac{\omega_i}{\omega_n}\frac{\omega}{\omega_i}\zeta-\left(\frac{\omega_i}{\omega_n}\right)^2\left(\frac{\omega}{\omega_i}\right)^2} \tag{B.50}$$

The frequency ratio is a function of the mass and area ratio only. For excitation at y:

$$\left(\frac{\omega_n}{\omega_i}\right)^2=\frac{m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}{m_x+m_B\left(\frac{A_b}{A_a}-1\right)^2}=\frac{\frac{m_B}{m_x}\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}{1+\frac{m_B}{m_x}\left(\frac{A_b}{A_a}-1\right)^2}=\frac{\mu_m\left(\mu_A-1\right)\mu_A}{1+\mu_m\left(\mu_A-1\right)^2} \tag{B.51}$$

$$\text{where:}\quad \mu_m=\frac{m_B}{m_x},\quad \mu_A=\frac{A_b}{A_a}$$

and for excitation at x:

$$\left(\frac{\omega_n}{\omega_i}\right)^2=\frac{m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}{m_y+m_B\left(\frac{A_b}{A_a}\right)^2}=\frac{\frac{m_B}{m_y}\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}{1+\frac{m_B}{m_y}\left(\frac{A_b}{A_a}\right)^2}=\frac{\mu_m\left(\mu_A-1\right)\mu_A}{1+\mu_m\mu_A^2} \tag{B.52}$$

$$\text{where:}\quad \mu_m=\frac{m_B}{m_y},\quad \mu_A=\frac{A_b}{A_a}$$

The frequency ratio must be as small as possible, which can be achieved if the denominator is much larger than the numerator of the previous 2 equations. Considering the first equation:

$$\begin{gathered}\frac{\mu_m\left(\mu_A-1\right)\mu_A}{1+\mu_m\left(\mu_A-1\right)^2}\ll 1\\ \mu_m\left(\mu_A-1\right)\mu_A-\mu_m\left(\mu_A-1\right)^2\ll 1\\ \mu_m\left(\mu_A-1\right)\left[\mu_A-\left(\mu_A-1\right)\right]\ll 1\\ \mu_m\left(\mu_A-1\right)\ll 1\\ \text{condition met if } \mu_m\rightarrow 0 \quad \text{and/or } \mu_A\rightarrow 1\end{gathered} \tag{B.53}$$

The second equation follows similarly. The current natural and isolation frequencies can be written in terms of their initial values (before changes to the stiffness ratio) indicated by the prime:

$$\begin{gathered}\omega_n^2=\frac{k+k_B\left(\frac{A_b}{A_a}-1\right)^2}{k'+k_B'\left(\frac{A_b}{A_a}-1\right)^2}\frac{k'+k_B'\left(\frac{A_b}{A_a}-1\right)^2}{m+m_B\left(\frac{A_b}{A_a}-1\right)^2}=\mu_k\omega_n'^2\\ \omega_i^2=\frac{k+k_B\left(\frac{A_b}{A_a}-1\right)^2}{k'+k_B'\left(\frac{A_b}{A_a}-1\right)^2}\frac{k'+k_B'\left(\frac{A_b}{A_a}-1\right)^2}{m+m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}=\mu_k\omega_i'^2\\ \zeta=\frac{\omega_n'}{\omega_n}\frac{c}{2\left[m_x+m_B\left(\frac{A_b}{A_a}-1\right)^2\right]\omega_n'}=\frac{1}{\sqrt{\mu_k}}\zeta'\end{gathered} \tag{B.54}$$

The transmissibility can conveniently be written in terms of the stiffness ratio:

$$\frac{X}{Y}=\frac{1+i2\frac{\omega}{\omega_n'}\frac{\omega_n'}{\omega_n}\zeta-\left(\frac{\omega_n'}{\omega_i}\right)^2\left(\frac{\omega}{\omega_n'}\right)^2}{1+i2\frac{\omega}{\omega_n'}\frac{\omega_n'}{\omega_n}\zeta-\left(\frac{\omega_n'}{\omega_n}\right)^2\left(\frac{\omega}{\omega_n'}\right)^2}$$

$$=\frac{1+i2\frac{\omega}{\omega_n'}\left(\frac{\omega_n'}{\omega_n}\right)^2\zeta'-\frac{\omega_n'^2}{\omega_i^2}\left(\frac{\omega}{\omega_n'}\right)^2}{1+i2\frac{\omega}{\omega_n'}\left(\frac{\omega_n'}{\omega_n}\right)^2\zeta'-\frac{\omega_n'^2}{\omega_n^2}\left(\frac{\omega}{\omega_n'}\right)^2}$$

$$=\frac{1+i2\frac{\omega}{\omega_n'}\frac{1}{\mu_k}\zeta'-\frac{\omega_n'^2}{\mu_k\omega_i'^2}\left(\frac{\omega}{\omega_n'}\right)^2}{1+i2\frac{\omega}{\omega_n'}\frac{1}{\mu_k}\zeta'-\frac{\omega_n'^2}{\mu_k\omega_n'^2}\left(\frac{\omega}{\omega_n'}\right)^2}$$

$$=\frac{1+i2\frac{\omega}{\omega_n'}\frac{1}{\mu_k}\zeta'-\frac{1}{\mu_k}\left(\frac{\omega_n}{\omega_i}\right)^2\left(\frac{\omega}{\omega_n'}\right)^2}{1+i2\frac{\omega}{\omega_n'}\frac{1}{\mu_k}\zeta'-\frac{1}{\mu_k}\left(\frac{\omega}{\omega_n'}\right)^2}$$

$$\text{where: }\quad \mu_k=\frac{k+k_B\left(\frac{A_b}{A_a}-1\right)^2}{k'+k_B'\left(\frac{A_b}{A_a}-1\right)^2},\quad \zeta'=\frac{c}{2\left[m_x+m_B\left(\frac{A_b}{A_a}-1\right)^2\right]\omega_n'} \qquad \text{(B.55)}$$

The dimensions used in the design of the device are shown in Figure B.4.

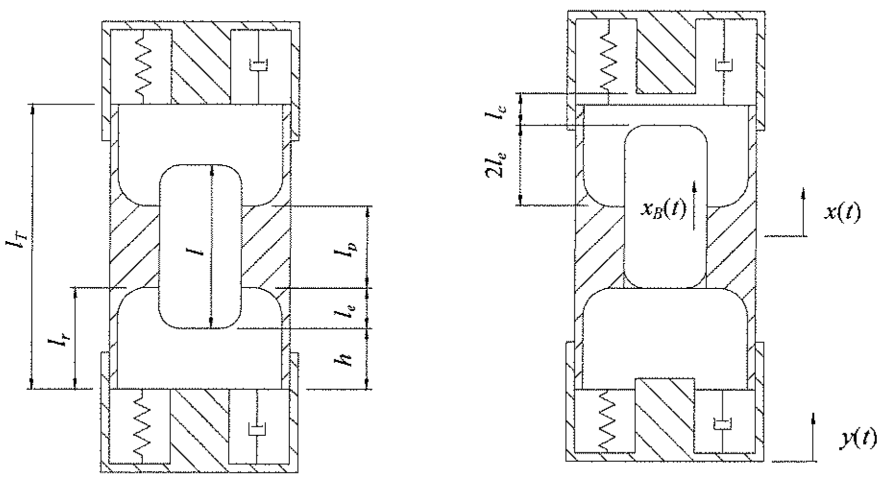

Figure B.4: Definition of dimensions

The isolation frequency can be rewritten to find the slug length:

$$l = \frac{k + k_B \left(\frac{A_b}{A_a} - 1 \right)^2}{\omega_i^2 \left(\frac{A_b}{A_a} - 1 \right) \frac{A_b}{A_a} \rho A_a} \tag{B.56}$$

The slug displacement for an undamped device can be found from the continuity equation:

$$X_B = \frac{A_b}{A_a} Y \tag{B.57}$$

The total length required is a function of the reservoir length (l_r) the protrusion length (l_e) and the slug length (l):

$$l_T = 2l_r - 2l_e + l \tag{B.58}$$

If the length of the protrusion (l_e) is assumed to be equal to the slug displacement:

$$l_T = 2l_r - 2\frac{A_b}{A_a} Y + l \tag{B.59}$$

The length of the reservoir is a function of the compressed length of the spring (l_c):

$$l_r = l_c + 2l_e - Y \tag{B.60}$$

The total length required can be rewritten using the expressions in Equation (B.59) and Equation (B.60):

$$l_T = 2l_c + 2\left(\frac{A_b}{A_a} - 1 \right) Y + l \tag{B.61}$$

The port area can now be found in terms of the outside dimensions of the device using the expressions for the slug length Equation (B.56) and the total length Equation (B.61):

$$\begin{aligned}
& l_T \omega_i^2 \left(\frac{A_b}{A_a} - 1 \right) \frac{A_b}{A_a} \rho A_a = 2l_c \omega_i^2 \left(\frac{A_b}{A_a} - 1 \right) \frac{A_b}{A_a} \rho A_a + 2\left(\frac{A_b}{A_a} - 1 \right) Y \omega_i^2 \left(\frac{A_b}{A_a} - 1 \right) \frac{A_b}{A_a} \rho A_a + k + k_B \left(\frac{A_b}{A_a} - 1 \right)^2 \\
& l_T \omega_i^2 \frac{A_b}{A_a} A_b \rho - l_T \omega_i^2 A_b \rho = 2l_c \omega_i^2 \frac{A_b}{A_a} A_b \rho - 2l_c \omega_i^2 A_b \rho + 2Y \omega_i^2 A_b \rho \left[\left(\frac{A_b}{A_a} \right)^2 - 2\frac{A_b}{A_a} + 1 \right] + k + k_B \left[\left(\frac{A_b}{A_a} \right)^2 - 2\frac{A_b}{A_a} + 1 \right] \\
& l_T \omega_i^2 \frac{A_b^2}{A_a} \rho - l_T \omega_i^2 A_b \rho = 2l_c \omega_i^2 \frac{A_b^2}{A_a} \rho - 2l_c \omega_i^2 A_b \rho + 2Y \omega_i^2 A_b \rho \frac{A_b^2}{A_a^2} - 4Y \omega_i^2 A_b \rho \frac{A_b}{A_a} + 2Y \omega_i^2 A_b \rho + k + k_B \frac{A_b^2}{A_a^2} - 2k_B \frac{A_b}{A_a} + k_B \\
& l_T \omega_i^2 \rho A_b^2 A_a - l_T \omega_i^2 A_b \rho A_a^2 - 2l_c \omega_i^2 \rho A_b^2 A_a + 2l_c \omega_i^2 A_b \rho A_a^2 - 2Y \omega_i^2 A_b \rho A_b^2 + 4Y \omega_i^2 A_b \rho A_b A_a \\
& \qquad - 2Y \omega_i^2 A_b \rho A_a^2 - kA_a^2 - k_B A_b^2 + 2k_B A_b A_a - k_B A_a^2 = 0 \\
& \left(2l_c \omega_i^2 A_b \rho - 2Y \omega_i^2 A_b \rho - l_T \omega_i^2 A_b \rho - k - k_B \right) A_a^2 + \left(l_T \omega_i^2 \rho A_b^2 - 2l_c \omega_i^2 \rho A_b^2 + 4Y \omega_i^2 A_b^2 \rho + 2k_B A_b \right) A_a - 2Y \omega_i^2 \rho A_b^3 - k_B A_b^2 = 0
\end{aligned} \tag{B.62}$$

The current isolation frequency can be written in terms of the initial isolation frequency by equating the numerator to zero:

$$\begin{aligned}\left(\frac{\omega_i}{\omega_n'}\right)^2 &= \mu_k\left(\frac{\omega_i'}{\omega_n'}\right)^2\\ &= \mu_k\left(\frac{\omega_i}{\omega_n}\right)^2\\ &= \mu_k\frac{1+\mu_m\left(\mu_A-1\right)^2}{\mu_m\left(\mu_A-1\right)\mu_A} \quad \text{(excitation at } x\text{)}\\ &\text{or}\\ &= \mu_k\frac{1+\mu_m\left(\mu_A-1\right)^2}{\mu_m\mu_A^2} \quad \text{(excitation at y)}\end{aligned} \tag{B.63}$$

The damped isolation frequency can be written in terms of the stiffness ratio:

$$\left(\frac{\Omega_i}{\omega_n}\right)^2 = \frac{-\left(\frac{\omega_n}{\omega_i}\right)^2-1-\sqrt{\left[\left(\frac{\omega_n}{\omega_i}\right)^2-1\right]^2+8\zeta^2\left[\left(\frac{\omega_n}{\omega_i}\right)^2+1\right]}}{4\zeta^2+4\zeta^2\left(\frac{\omega_n}{\omega_i}\right)^2-2\left(\frac{\omega_n}{\omega_i}\right)^2}$$

$$\left(\frac{\Omega_i}{\sqrt{\mu_k}\omega_n'}\right)^2 = \frac{-\left(\frac{\sqrt{\mu_k}\omega_n'}{\sqrt{\mu_k}\omega_i'}\right)^2-1-\sqrt{\left[\left(\frac{\sqrt{\mu_k}\omega_n'}{\sqrt{\mu_k}\omega_i'}\right)^2-1\right]^2+8\left(\frac{\zeta'}{\sqrt{\mu_k}}\right)^2\left[\left(\frac{\sqrt{\mu_k}\omega_n'}{\sqrt{\mu_k}\omega_i'}\right)^2+1\right]}}{4\left(\frac{\zeta'}{\sqrt{\mu_k}}\right)^2+4\left(\frac{\zeta'}{\sqrt{\mu_k}}\right)^2\left(\frac{\sqrt{\mu_k}\omega_n'}{\sqrt{\mu_k}\omega_i'}\right)^2-2\left(\frac{\sqrt{\mu_k}\omega_n'}{\sqrt{\mu_k}\omega_i'}\right)^2}$$

$$\frac{1}{\mu_k}\left(\frac{\Omega_i}{\omega_n'}\right)^2 = \frac{-\left(\frac{\omega_n'}{\omega_i'}\right)^2-1-\sqrt{\left[\left(\frac{\omega_n'}{\omega_i'}\right)^2-1\right]^2+8\frac{\zeta'^2}{\mu_k}\left[\left(\frac{\omega_n'}{\omega_i'}\right)^2+1\right]}}{4\frac{\zeta'^2}{\mu_k}+4\frac{\zeta'^2}{\mu_k}\left(\frac{\omega_n'}{\omega_i'}\right)^2-2\left(\frac{\omega_n'}{\omega_i'}\right)^2}$$

$$\begin{aligned}\left(\frac{\Omega_i}{\omega_n'}\right)^2 &= \mu_k\frac{-\left(\frac{\omega_n'}{\omega_i'}\right)^2-1-\sqrt{\left[\left(\frac{\omega_n'}{\omega_i'}\right)^2-1\right]^2+8\frac{\zeta'^2}{\mu_k}\left[\left(\frac{\omega_n'}{\omega_i'}\right)^2+1\right]}}{4\frac{\zeta'^2}{\mu_k}+4\frac{\zeta'^2}{\mu_k}\left(\frac{\omega_n'}{\omega_i'}\right)^2-2\left(\frac{\omega_n'}{\omega_i'}\right)^2}\\ &= \mu_k\frac{-\left(\frac{\omega_n}{\omega_i}\right)^2-1-\sqrt{\left[\left(\frac{\omega_n}{\omega_i}\right)^2-1\right]^2+8\frac{\zeta'^2}{\mu_k}\left[\left(\frac{\omega_n}{\omega_i}\right)^2+1\right]}}{4\frac{\zeta'^2}{\mu_k}+4\frac{\zeta'^2}{\mu_k}\left(\frac{\omega_n}{\omega_i}\right)^2-2\left(\frac{\omega_n}{\omega_i}\right)^2}\end{aligned} \tag{B.64}$$

B.2.2 Slug stops

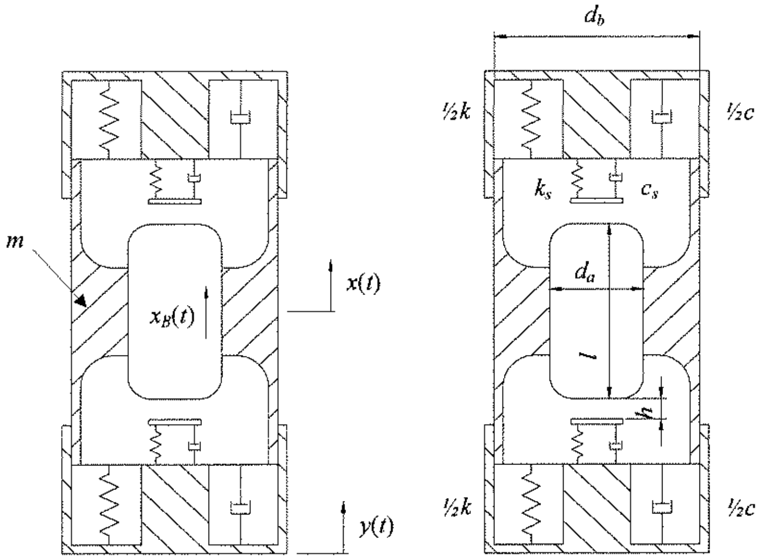

Figure B.5: Mechanical model of a type II AVAI with slug stops

Continuity is described by Equation (B.39). For this derivation it is assumed that the stop will stay in contact with the slug if the relative displacement between the slug and the reservoir is larger than the gap (h). This might not be the situation for high damping ratios, in which case the approach taken by Luo and Hanagud (1998) is more appropriate. For this derivation it will be more convenient to use force balance on the various components rather than Lagrange's equations as was done up till now.

Graphically the stiffness can be represented as shown in Figure B.6, where f_s is the force acting on the slug.

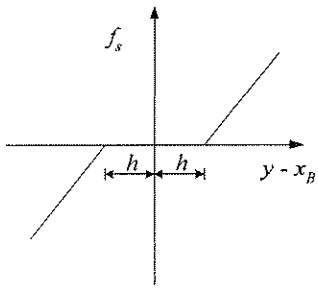

Figure B.6: Graphical representation of stop force

The stop force is:

$$\begin{aligned} f_s &= \beta k_s\left[y - x_B - \operatorname{sign}(y - x_B)h\right] \\ &= \beta k_s\left\{y - x + \frac{A_b}{A_a}(x-y) - \operatorname{sign}\left[y - x + \frac{A_b}{A_a}(x-y)\right]h\right\} \\ \text{where:}\quad \beta &= \begin{cases} 0 \text{ if } \left|y - x + \frac{A_b}{A_a}(x-y)\right| < h \\ 1 \text{ if } \left|y - x + \frac{A_b}{A_a}(x-y)\right| \geq h \end{cases} \end{aligned} \tag{B.65}$$

The stop damping force is:

$$\begin{aligned} f_d &= \beta c_s(\dot{y} - \dot{x}_B) \\ &= \beta c_s\left[\dot{y} - \dot{x} + \frac{A_b}{A_a}(\dot{x} - \dot{y})\right] \end{aligned} \tag{B.66}$$

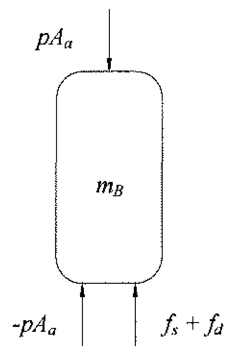

Figure B.7: Forces acting on the slug when in contact with the bottom stop

The forces acting on the slug are:

$$\begin{aligned} &-2pA_a + f_s + f_d = m_B\ddot{x}_B \\ &p = \frac{k_s}{2A_a}\Delta_d + \frac{c_s}{2A_a}\Delta_v - \frac{m_B}{2A_a}\left[\left(1 - \frac{A_b}{A_a}\right)\ddot{x} + \frac{A_b}{A_a}\ddot{y}\right] \\ &\text{where:} \\ &\Delta_d = \beta\left\{y - x + \frac{A_b}{A_a}(x-y) - \operatorname{sign}\left[y - x + \frac{A_b}{A_a}(x-y)\right]h\right\} \\ &\Delta_v = \beta\left[\dot{y} - \dot{x} + \frac{A_b}{A_a}(\dot{x} - \dot{y})\right] \end{aligned} \tag{B.67}$$

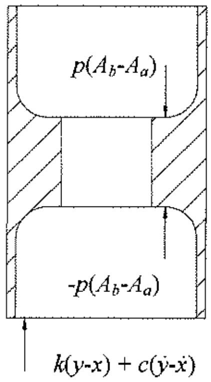

Figure B.8: Forces acting on the port

The forces acting on the port can be found by substituting the expression found for the pressure in Equation (B.67):

$$-2p\left(A_b - A_a\right) + k\left(y - x\right) + c\left(\dot{y} - \dot{x}\right) = m\ddot{x}$$

$$-\left\{\frac{k_s}{A_a}\Delta_d + \frac{c_s}{A_a}\Delta_v - \frac{m_B}{A_a}\left[\left(1 - \frac{A_b}{A_a}\right)\ddot{x} + \frac{A_b}{A_a}\ddot{y}\right]\right\}\left(A_b - A_a\right) + k\left(y - x\right) + c\left(\dot{y} - \dot{x}\right) = m\ddot{x}$$

$$k\left(y - x\right) + c\left(\dot{y} - \dot{x}\right) - k_s\Delta_d\left(\frac{A_b}{A_a} - 1\right) - c_s\Delta_v\left(\frac{A_b}{A_a} - 1\right) - m_B\left[\left(\frac{A_b}{A_a} - 1\right)^2\ddot{x} - \frac{A_b}{A_a}\left(\frac{A_b}{A_a} - 1\right)\ddot{y}\right] = m\ddot{x} \tag{B.68}$$

$$\left[m + m_B\left(\frac{A_b}{A_a} - 1\right)^2\right]\ddot{x} = m_B\left(\frac{A_b}{A_a} - 1\right)\frac{A_b}{A_a}\ddot{y} + k\left(y - x\right) + c\left(\dot{y} - \dot{x}\right) - k_s\Delta_d\left(\frac{A_b}{A_a} - 1\right) - c_s\Delta_v\left(\frac{A_b}{A_a} - 1\right)$$

The above equation can be written in terms of non-dimensional parameters:

$$\ddot{x} = \left(\frac{\omega_n}{\omega_i}\right)^2\ddot{y} - 2\zeta\omega_n\left(\dot{x} - \dot{y}\right) - \omega_n^2\left(x - y\right) - \omega_n^2\frac{k_s}{k}\Delta_d\left(\frac{A_b}{A_a} - 1\right) - 2\zeta_s\omega_n\Delta_v\left(\frac{A_b}{A_a} - 1\right)$$

$$\text{where:}\quad \omega_n = \sqrt{\frac{k}{m + m_B\left(\frac{A_b}{A_a} - 1\right)^2}},\quad \omega_i = \sqrt{\frac{k}{m_B\left(\frac{A_b}{A_a} - 1\right)\frac{A_b}{A_a}}},\quad \zeta_s = \frac{c_s}{2\left[m + m_B\left(\frac{A_b}{A_a} - 1\right)^2\right]\omega_n} \tag{B.69}$$

B.2.3 Leakage

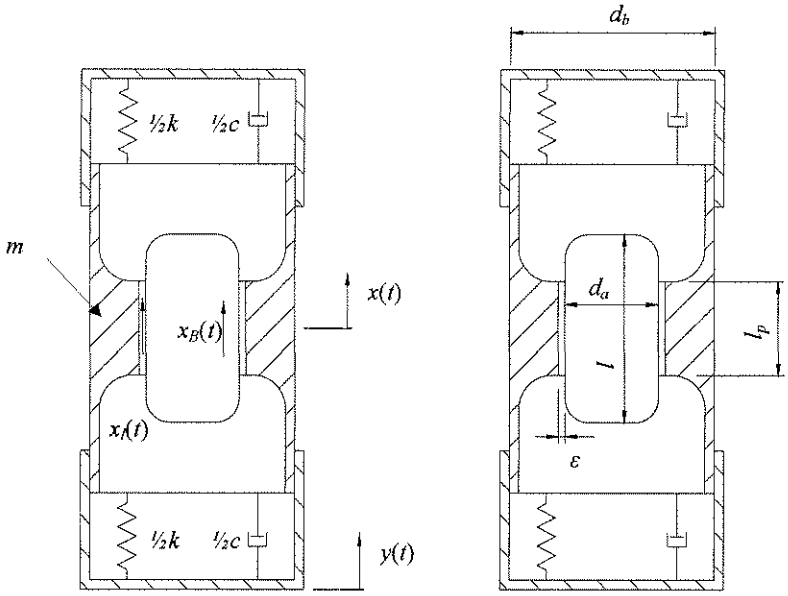

Figure B.9: Mechanical model of a type II AVAI with leakage

The effect of leakage will be studied with this model. It is assumed that the slug is connected to the port through discrete dashpots as shown in Figure B.9. The exact properties of these dashpots can be determined by finding the velocity-dependent shear force acting on the port and the slug. The shear force is a function of the viscosity and the slope of the radial velocity profile in the annulus. The dashpot properties will, however, not be calculated and this model will only be used to make a qualitative assessment of the effect of modelling leakage.

If it is assumed that the gap between the port and the slug is small the leakage area is:

$$A_\varepsilon = \varepsilon \pi d_a \tag{B.70}$$

The continuity equation is:

$$\begin{aligned} yA_b &= x_B A_a + \varepsilon \pi d_a x_l + \left(A_b - A_a - \varepsilon \pi d_a\right) x \\ x_B &= \frac{A_b}{A_a} y + \left(1 + \frac{\varepsilon \pi d_a - A_b}{A_a}\right) x - \frac{\varepsilon \pi d_a}{A_a} x_l \end{aligned} \tag{B.71}$$

If the gap is zero then the above equation reduces to that found in Equation (B.39).

The kinetic energy is:

$$
\begin{aligned}
T &= \frac{1}{2}\left(m\dot{x}^2 + m_B\dot{x}_B^2 + m_l\dot{x}_l^2\right) \\
&= \frac{1}{2}\left\{m\dot{x}^2 + m_B\left[\frac{A_b}{A_a}\dot{y} - \left(\frac{A_b - \varepsilon\pi d_a}{A_a} - 1\right)\dot{x} - \frac{\varepsilon\pi d_a}{A_a}\dot{x}_l\right]^2 + m_l\dot{x}_l^2\right\} \\
&= \frac{1}{2}\left\{m\dot{x}^2 + m_B\left[\left(\frac{A_b}{A_a}\right)^2\dot{y}^2 + \left(\frac{A_b - \varepsilon\pi d_a}{A_a} - 1\right)^2\dot{x}^2 + \left(\frac{\varepsilon\pi d_a}{A_a}\right)^2\dot{x}_l^2 - 2\frac{A_b}{A_a}\left(\frac{A_b - \varepsilon\pi d_a}{A_a} - 1\right)\dot{x}\dot{y}\right.\right. \\
&\qquad \left.\left. - 2\frac{A_b}{A_a}\frac{\varepsilon\pi d_a}{A_a}\dot{x}_l\dot{y} + 2\left(\frac{A_b - \varepsilon\pi d_a}{A_a} - 1\right)\frac{\varepsilon\pi d_a}{A_a}\dot{x}_l\dot{x}\right] + m_l\dot{x}_l^2\right\}
\end{aligned}
\tag{B.72}
$$

where: $m_B = \rho A_a l, \quad m_l = \rho_f \varepsilon\pi d_a l_p$

The derivatives are:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) &= \left[m + m_B\left(1 + \frac{\varepsilon\pi d_a - A_b}{A_a}\right)^2\right]\ddot{x} + m_B\left(\frac{A_b - \varepsilon\pi d_a}{A_a} - 1\right)\frac{\varepsilon\pi d_a}{A_a}\ddot{x}_l - m_B\frac{A_b}{A_a}\left(\frac{A_b - \varepsilon\pi d_a}{A_a} - 1\right)\ddot{y} \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_l}\right) &= m_B\left(\frac{A_b - \varepsilon\pi d_a}{A_a} - 1\right)\frac{\varepsilon\pi d_a}{A_a}\ddot{x} + \left[m_l + m_B\left(\frac{\varepsilon\pi d_a}{A_a}\right)^2\right]\ddot{x}_l - m_B\frac{A_b}{A_a}\frac{\varepsilon\pi d_a}{A_a}\ddot{y} \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) &= -m_B\frac{A_b}{A_a}\left(\frac{A_b - \varepsilon\pi d_a}{A_a} - 1\right)\ddot{x} - m_B\frac{A_b}{A_a}\frac{\varepsilon\pi d_a}{A_a}\ddot{x}_l + m_B\left(\frac{A_b}{A_a}\right)^2\ddot{y}
\end{aligned}
\tag{B.73}
$$

The potential energy is:

$$
\begin{aligned}
V &= \frac{1}{2}k(x - y)^2 \\
&= \frac{1}{2}k\left(x^2 - 2xy + y^2\right)
\end{aligned}
\tag{B.74}
$$

The derivatives are:

$$
\begin{aligned}
\frac{\partial V}{\partial x} &= kx - ky \\
\frac{\partial V}{\partial x_l} &= 0 \\
\frac{\partial V}{\partial y} &= -kx + ky
\end{aligned}
\tag{B.75}
$$

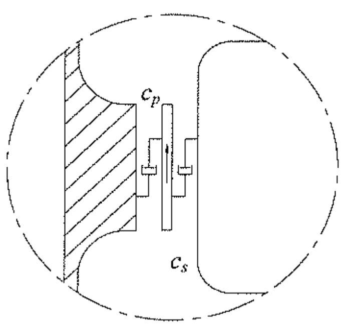

Figure B.10: Fluid damping

The Rayleigh term is:

$$
\begin{aligned}
R &= \frac{1}{2}\left[c\left(\dot{x}-\dot{y}\right)^2 + c_p\left(\dot{x}_l-\dot{x}\right)^2 + c_s\left(\dot{x}_s-\dot{x}_l\right)^2\right] \\
&= \frac{1}{2}\left\{c\left(\dot{x}-\dot{y}\right)^2 + c_p\left(\dot{x}_l-\dot{x}\right)^2 + c_s\left[\frac{A_b}{A_a}\dot{y} - \left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\dot{x} - \left(\frac{\varepsilon\pi d_a}{A_a}+1\right)\dot{x}_l\right]^2\right\} \\
&= \frac{1}{2}\left\{c\left(\dot{x}^2-\dot{x}\dot{y}+\dot{y}^2\right) + c_p\left(\dot{x}_l^2-\dot{x}_l\dot{x}+\dot{x}^2\right) + c_s\left[\left(\frac{A_b}{A_a}\right)^2\dot{y}^2 + \left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)^2\dot{x}^2 + \left(\frac{\varepsilon\pi d_a}{A_a}+1\right)^2\dot{x}_l^2 - 2\frac{A_b}{A_a}\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\dot{x}\dot{y} - 2\frac{A_b}{A_a}\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)\dot{x}_l\dot{y} + 2\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)\dot{x}_l\dot{x}\right]\right\}
\end{aligned}
\tag{B.76}
$$

From the above equation the derivatives can be found:

$$
\begin{aligned}
\frac{\partial R}{\partial \dot{x}} &= \left[c + c_p + c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)^2\right]\dot{x} + \left[c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\left(\frac{\varepsilon\pi d_a}{A_a}+1\right) - c_p\right]\dot{x}_l - \left[c + c_s\frac{A_b}{A_a}\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\right]\dot{y} \\
\frac{\partial R}{\partial \dot{x}_l} &= \left[c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\left(\frac{\varepsilon\pi d_a}{A_a}+1\right) - c_p\right]\dot{x} + \left[c_s\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)^2 + c_p\right]\dot{x}_l - c_s\frac{A_b}{A_a}\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)\dot{y} \\
\frac{\partial R}{\partial \dot{y}} &= \left[c + c_s\frac{A_b}{A_a}\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\right]\dot{x} - c_s\frac{A_b}{A_a}\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)\dot{x}_l + \left[c + c_s\left(\frac{A_b}{A_a}\right)^2\right]\dot{y}
\end{aligned}
\tag{B.77}
$$

The complete equation of motion can now be found by substituting the derivates found above in Lagrange's equations:

$$
\begin{aligned}
&\left[m + m_B\left(1+\frac{\varepsilon\pi d_a - A_b}{A_a}\right)^2\right]\ddot{x} + \left[c + c_p + c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)^2\right]\dot{x} + m_B\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\frac{\varepsilon\pi d_a}{A_a}\ddot{x}_l + \left[c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\left(\frac{\varepsilon\pi d_a}{A_a}+1\right) - c_p\right]\dot{x}_l + kx \\
&\qquad = m_B\frac{A_b}{A_a}\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\ddot{y} + \left[c + c_s\frac{A_b}{A_a}\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\right]\dot{y} + ky \\
&m_B\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\frac{\varepsilon\pi d_a}{A_a}\ddot{x} + \left[c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\left(\frac{\varepsilon\pi d_a}{A_a}+1\right) - c_p\right]\dot{x} + \left[m_l + m_B\left(\frac{\varepsilon\pi d_a}{A_a}\right)^2\right]\ddot{x}_l + \left[c_s\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)^2 + c_p\right]\dot{x}_l \\
&\qquad = m_B\frac{A_b}{A_a}\frac{\varepsilon\pi d_a}{A_a}\ddot{y} + c_s\frac{A_b}{A_a}\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)\dot{y}
\end{aligned}
\tag{B.78}
$$

By transforming to the frequency domain:

$$\begin{aligned}
&\left\{k+i\omega\left[c+c_p+c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)^2\right]-\omega^2\left[m+m_B\left(1+\frac{\varepsilon\pi d_a-A_b}{A_a}\right)^2\right]\right\}X+\left\{i\omega\left[c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)-c_p\right]-\omega^2 m_B\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\frac{\varepsilon\pi d_a}{A_a}\right\}X_l \\
&\qquad=\left\{k+i\omega\left[c+c_s\frac{A_b}{A_a}\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\right]-\omega^2 m_B\frac{A_b}{A_a}\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\right\}Y \\
&\left\{i\omega\left[c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)-c_p\right]-\omega^2 m_B\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\frac{\varepsilon\pi d_a}{A_a}\right\}X+\left\{i\omega\left[c_s\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)^2+c_p\right]-\omega^2\left[m_l+m_B\left(\frac{\varepsilon\pi d_a}{A_a}\right)^2\right]\right\}X_l \\
&\qquad=\left\{i\omega c_s\frac{A_b}{A_a}\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)-\omega^2 m_B\frac{A_b}{A_a}\frac{\varepsilon\pi d_a}{A_a}\right\}Y
\end{aligned} \tag{B.79}$$

The second of Equation (B.79) can be used to find X_l:

$$X_l=\frac{\left\{i\omega c_s\frac{A_b}{A_a}\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)-\omega^2 m_B\frac{A_b}{A_a}\frac{\varepsilon\pi d_a}{A_a}\right\}Y-\left\{i\omega\left[c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)-c_p\right]-\omega^2 m_B\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\frac{\varepsilon\pi d_a}{A_a}\right\}X}{i\omega\left[c_s\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)^2+c_p\right]-\omega^2\left[m_l+m_B\left(\frac{\varepsilon\pi d_a}{A_a}\right)^2\right]} \tag{B.80}$$

Back substitution into the first of Equation (B.79) yields:

$$\begin{aligned}
&\left\{k+i\omega\left[c+c_p+c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)^2\right]-\omega^2\left[m+m_B\left(1+\frac{\varepsilon\pi d_a-A_b}{A_a}\right)^2\right]\right\}\left\{i\omega\left[c_s\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)^2+c_p\right]-\omega^2\left[m_l+m_B\left(\frac{\varepsilon\pi d_a}{A_a}\right)^2\right]\right\}X \\
&-\left\{i\omega\left[c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)-c_p\right]-\omega^2 m_B\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\frac{\varepsilon\pi d_a}{A_a}\right\}\left\{i\omega\left[c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)-c_p\right]-\omega^2 m_B\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\frac{\varepsilon\pi d_a}{A_a}\right\}X \\
&=\left\{i\omega\left[c_s\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)^2+c_p\right]-\omega^2\left[m_l+m_B\left(\frac{\varepsilon\pi d_a}{A_a}\right)^2\right]\right\}\left\{k+i\omega\left[c+c_s\frac{A_b}{A_a}\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\right]-\omega^2 m_B\frac{A_b}{A_a}\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\right\}Y \\
&-\left\{i\omega\left[c_s\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)-c_p\right]-\omega^2 m_B\left(\frac{A_b-\varepsilon\pi d_a}{A_a}-1\right)\frac{\varepsilon\pi d_a}{A_a}\right\}\left\{i\omega c_s\frac{A_b}{A_a}\left(\frac{\varepsilon\pi d_a}{A_a}+1\right)-\omega^2 m_B\frac{A_b}{A_a}\frac{\varepsilon\pi d_a}{A_a}\right\}Y
\end{aligned} \tag{B.81}$$

The following natural frequencies and damping ratios are now defined:

$$\omega_l=\sqrt{\frac{k}{m_l+m_B\left(4\frac{\varepsilon}{d_a}\right)^2}},\quad \omega_i=\sqrt{\frac{k}{m_B\frac{A_b}{A_a}\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)}},\quad \omega_\varepsilon=\sqrt{\frac{k}{m_B\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)4\frac{\varepsilon}{d_a}}},$$
$$\omega_\varepsilon'=\sqrt{\frac{k}{m_B\frac{A_b}{A_a}4\frac{\varepsilon}{d_a}}},\quad \omega_n=\sqrt{\frac{k}{m+m_B\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)^2}},\quad \zeta_s=\frac{c_s}{2\left[m+m_B\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)^2\right]\omega_n},$$
$$\zeta_p=\frac{c_p}{2\left[m+m_B\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)^2\right]\omega_n} \tag{B.82}$$

The transmissibility can be found by using the above non-dimensional relationships:

$$\frac{X}{Y}=\frac{AB-CD}{EF-G^2}$$
$$\frac{A}{k}=i2\frac{\omega}{\omega_n}\left[\zeta_s\left(4\frac{\varepsilon}{d_a}+1\right)^2+\zeta_p\right]-\left(\frac{\omega_n}{\omega_l}\right)^2\left(\frac{\omega}{\omega_n}\right)^2$$
$$\frac{B}{k}=1+i2\frac{\omega}{\omega_n}\left[\zeta+\zeta_s\frac{A_b}{A_a}\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)\right]-\left(\frac{\omega_n}{\omega_i'}\right)^2\left(\frac{\omega}{\omega_n}\right)^2$$
$$\frac{C}{k}=i2\frac{\omega}{\omega_n}\left[\zeta_s\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)\left(4\frac{\varepsilon}{d_a}+1\right)-\zeta_p\right]-\left(\frac{\omega_n}{\omega_\varepsilon}\right)^2\left(\frac{\omega}{\omega_n}\right)^2$$
$$\frac{D}{k}=i2\frac{\omega}{\omega_n}\zeta_s\frac{A_b}{A_a}\left(4\frac{\varepsilon}{d_a}+1\right)-\left(\frac{\omega_n}{\omega_\varepsilon'}\right)^2\left(\frac{\omega}{\omega_n}\right)^2$$
$$\frac{E}{k}=1+i2\frac{\omega}{\omega_n}\left[\zeta+\zeta_p+\zeta_s\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)^2\right]-\left(\frac{\omega}{\omega_n}\right)^2$$
$$\frac{F}{k}=i2\frac{\omega}{\omega_n}\left[\zeta_s\left(4\frac{\varepsilon}{d_a}+1\right)^2+\zeta_p\right]-\left(\frac{\omega_n}{\omega_l}\right)^2\left(\frac{\omega}{\omega_n}\right)^2$$
$$\frac{G}{k}=i2\frac{\omega}{\omega_n}\left[\zeta_s\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)\left(4\frac{\varepsilon}{d_a}+1\right)-\zeta_p\right]-\left(\frac{\omega_n}{\omega_\varepsilon}\right)^2\left(\frac{\omega}{\omega_n}\right)^2$$
$$\frac{\omega_n}{\omega_l}=\frac{\frac{m_l}{m_B}+\left(4\frac{\varepsilon}{d_a}\right)^2}{\frac{m}{m_B}+\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)^2},\quad \frac{\omega_n}{\omega_i'}=\frac{\frac{A_b}{A_a}\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)}{\frac{m}{m_B}+\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)^2},$$
$$\frac{\omega_n}{\omega_\varepsilon}=\frac{4\frac{\varepsilon}{d_a}\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)}{\frac{m}{m_B}+\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)^2},\quad \frac{\omega_n}{\omega_\varepsilon'}=\frac{4\frac{\varepsilon}{d_a}\frac{A_b}{A_a}}{\frac{m}{m_B}+\left(\frac{A_b}{A_a}-4\frac{\varepsilon}{d_a}-1\right)^2} \tag{B.83}$$

B.2.4 Slug with diaphragm seal

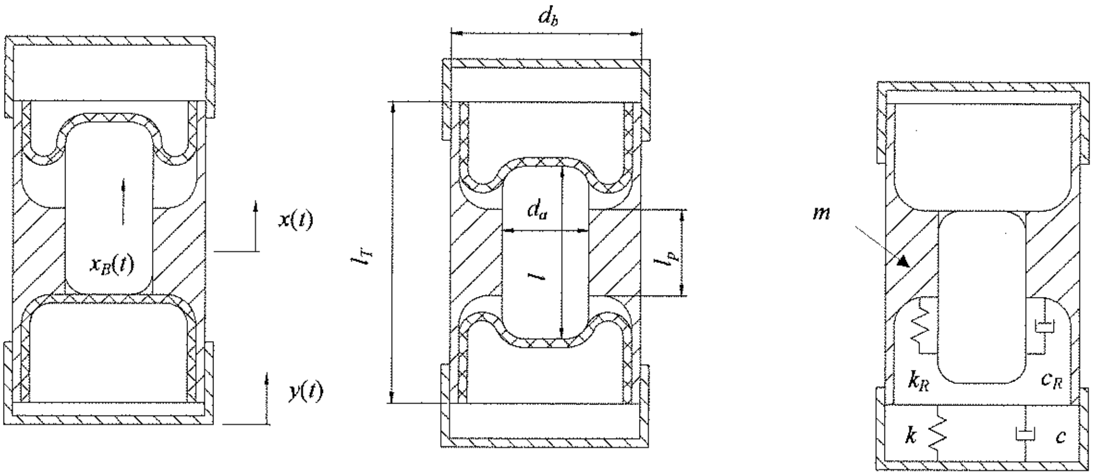

Figure B.11 Mechanical model of a type II AVAI with a rolling diaphragm seal

To derive the continuity equation it must be recognised that the diaphragm cannot stretch and that the relative motion between the slug and the port is:

$$x_R = \frac{1}{2}(x_B + x) \tag{B.84}$$

Considering continuity gives:

$$\begin{aligned} yA_b &= x_B A_a + \frac{1}{2}(x_B + x)(A_b - A_a) \\ x_B &= -\left(\frac{A_b - A_a}{A_b + A_a}\right)x + \frac{2A_b}{A_b + A_a}y \end{aligned} \tag{B.85}$$

The total kinetic energy is:

$$\begin{aligned} T &= \frac{1}{2}\left(m\dot{x}^2 + m_B\dot{x}_B^2\right) \\ T &= \frac{1}{2}\left\{\left[m + m_B\left(\frac{A_b - A_a}{A_b + A_a}\right)^2\right]\dot{x}^2 - m_B\frac{4A_b(A_b - A_a)}{(A_b + A_a)^2}\dot{x}\dot{y} + 4m_B\left(\frac{A_b}{A_b + A_a}\right)^2\dot{y}^2\right\} \end{aligned} \tag{B.86}$$

From the above equation the derivatives can be found:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) &= \left[m + m_B\left(\frac{A_b - A_a}{A_b + A_a}\right)^2\right]\ddot{x} + m_B\frac{4A_b(A_a - A_b)}{(A_b + A_a)^2}\ddot{y} \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) &= m_B\frac{4A_b(A_a - A_b)}{(A_b + A_a)^2}\ddot{x} + 4\left(\frac{A_b}{A_b + A_a}\right)^2\ddot{y} \end{aligned} \tag{B.87}$$

The potential energy is (k_R represents the total stiffness of both diaphragms):

$$\begin{aligned} V &= \frac{1}{2}\left[k(x-y)^2 + k_R(x_B - x)^2\right] \\ &= \frac{1}{2}\left[k(x-y)^2 + k_R\left(\frac{2A_b}{A_b+A_a}y - \left(\frac{A_b-A_a}{A_b+A_a}+1\right)x\right)^2\right] \end{aligned} \tag{B.88}$$

From the above equation the derivatives can be found:

$$\begin{aligned} \frac{\partial V}{\partial x} &= \left[k + k_R\left(\frac{A_b-A_a}{A_b+A_a}\right)^2 - 2k_R\frac{A_b-A_a}{A_b+A_a} + k_R\right]x - \left[k + k_R\frac{2A_b(A_b-A_a)}{(A_b+A_a)^2} + k_R\frac{2A_b}{A_b+A_a}\right]y \\ \frac{\partial V}{\partial y} &= -\left[k + k_R\frac{2A_b(A_b-A_a)}{(A_b+A_a)^2} + k_R\frac{2A_b}{A_b+A_a}\right]x + \left[k + k_R\left(\frac{2A_b}{A_b+A_a}\right)^2\right]y \end{aligned} \tag{B.89}$$

The Rayleigh term is:

$$\begin{aligned} R &= \frac{1}{2}\left[c(\dot{x}-\dot{y})^2 + c_R(\dot{x}_B - \dot{x})^2\right] \\ &= \frac{1}{2}\left[c(\dot{x}-\dot{y})^2 + c_R\left(\frac{2A_b}{A_b+A_a}\dot{y} - \left(\frac{A_b-A_a}{A_b+A_a}+1\right)\dot{x}\right)^2\right] \end{aligned} \tag{B.90}$$

From the above equation the derivatives can be found:

$$\begin{aligned} \frac{\partial R}{\partial \dot{x}} &= \left[c + c_R\left(\frac{A_b-A_a}{A_b+A_a}\right)^2 - 2c_R\frac{A_b-A_a}{A_b+A_a} + c_R\right]\dot{x} - \left[c + c_R\frac{2A_b(A_b-A_a)}{(A_b+A_a)^2} + c_R\frac{2A_b}{A_b+A_a}\right]\dot{y} \\ \frac{\partial R}{\partial \dot{y}} &= -\left[c + c_R\frac{2A_b(A_b-A_a)}{(A_b+A_a)^2} + c_R\frac{2A_b}{A_b+A_a}\right]\dot{x} + \left[c + c_R\left(\frac{2A_b}{A_b+A_a}\right)^2\right]\dot{y} \end{aligned} \tag{B.91}$$

The complete equation of motion can now be found by substituting the derivates found above in Lagrange's equations:

$$\begin{aligned} &\left[m + m_B\left(\frac{A_b-A_a}{A_b+A_a}\right)^2\right]\ddot{x} + \left[c + c_R\left(\frac{A_b-A_a}{A_b+A_a}\right)^2 - 2c_R\frac{A_b-A_a}{A_b+A_a} + c_R\right]\dot{x} + \left[k + k_R\left(\frac{A_b-A_a}{A_b+A_a}\right)^2 - 2k_R\frac{A_b-A_a}{A_b+A_a} + k_R\right]x \\ &= -m_B\frac{4A_b(A_a-A_b)}{(A_b+A_a)^2}\ddot{y} + \left[c + c_R\frac{2A_b(A_b-A_a)}{(A_b+A_a)^2} + c_R\frac{2A_b}{A_b+A_a}\right]\dot{y} + \left[k + k_R\frac{2A_b(A_b-A_a)}{(A_b+A_a)^2} + k_R\frac{2A_b}{A_b+A_a}\right]y \end{aligned} \tag{B.92}$$

The transmissibility can now be calculated by transforming the above equation to the frequency domain:

$$\frac{X}{Y} = \frac{k + k_R\frac{2A_b(A_b-A_a)}{(A_b+A_a)^2} + k_R\frac{2A_b}{A_b+A_a} + i\omega\left[c + c_R\frac{2A_b(A_b-A_a)}{(A_b+A_a)^2} + c_R\frac{2A_b}{A_b+A_a}\right] + \omega^2 m_B\frac{4A_b(A_a-A_b)}{(A_b+A_a)^2}}{k + k_R\left(\frac{A_b-A_a}{A_b+A_a}\right)^2 - 2k_R\frac{A_b-A_a}{A_b+A_a} + k_R + i\omega\left[c + c_R\left(\frac{A_b-A_a}{A_b+A_a}\right)^2 - 2c_R\frac{A_b-A_a}{A_b+A_a} + c_R\right] + \omega^2\left[m + m_B\left(\frac{A_b-A_a}{A_b+A_a}\right)^2\right]} \tag{B.93}$$

The isolation frequency is:

$$\omega_i = \sqrt{\frac{k + k_R \dfrac{2A_b\left(A_b - A_a\right)}{\left(A_b + A_a\right)^2} + k_R \dfrac{2A_b}{A_b + A_a}}{m_B \dfrac{4A_b\left(A_a - A_b\right)}{\left(A_b + A_a\right)^2}}} \tag{B.94}$$

The effective mass term can be rewritten as follows:

$$m_B \frac{4A_b\left(A_a - A_b\right)}{\left(A_b + A_a\right)^2}\ddot{x} = m_B \frac{4A_a^2}{\left(A_b + A_a\right)^2}\frac{A_b\left(A_b - A_a\right)}{A_a^2} = \frac{4}{\left(\dfrac{A_b}{A_a} + 1\right)^2} m_B \left(1 - \frac{A_b}{A_a}\right)\frac{A_b}{A_a} \tag{B.95}$$

By comparing the effective absorber mass term in Equation (B.94) with the isolation frequency of a system without a rolling diaphragm an effective mass ratio can be found:

$$M_{ratio} = \frac{4}{\left(\dfrac{A_b}{A_a} + 1\right)^2} \tag{B.96}$$

APPENDIX C

Derivations for chapter 3

C.1 Type I AVAI (equation of motion)

The equation of motion from Appendix B can be non-dimensionalised and written in terms of the stiffness ratio as follows for the first equation:

$$m_y\ddot{y}+(c+c_u)\dot{y}-c_u\dot{u}+(k+k_u)y-k_u u=c\dot{x}+kx$$

$$\frac{m_y}{k}\ddot{y}+\left(\frac{c}{k}+\mu_k\frac{c_u}{k_u}\right)\dot{y}-\mu_k\frac{c_u}{k_u}\dot{u}+(1+\mu_k)y-\mu_k u=\frac{c}{k}\dot{x}+x$$

$$\frac{1}{\omega_1^2}\ddot{y}+2\left(\frac{\zeta_1}{\omega_1}+\mu_k\frac{\zeta_2}{\omega_2}\right)\dot{y}-2\mu_k\frac{\zeta_2}{\omega_2}\dot{u}+(1+\mu_k)y-\mu_k u=2\frac{\zeta_1}{\omega_1}\dot{x}+x$$

$$\ddot{y}+2\left(\zeta_1+\mu_k\frac{\omega_1}{\omega_2}\zeta_2\right)\omega_1\dot{y}-2\mu_k\frac{\omega_1}{\omega_2}\zeta_2\omega_1\dot{u}+\omega_1^2(1+\mu_k)y-\omega_1^2\mu_k u-2\zeta_1\omega_1\dot{x}-\omega_1^2 x=0 \tag{C.1}$$

$$\ddot{y}+2\left(\zeta_1+\frac{\omega_1}{\bar{\omega}_2}\bar{\zeta}_2\right)\omega_1\dot{y}-2\frac{\omega_1}{\bar{\omega}_2}\bar{\zeta}_2\omega_1\dot{u}+\omega_1^2(1+\mu_k)y-\omega_1^2\mu_k u-2\zeta_1\omega_1\dot{x}-\omega_1^2 x=0$$

where: $$\frac{\zeta_2}{\omega_2}=\frac{\frac{1}{\sqrt{\mu_k}}\bar{\zeta}_2}{\sqrt{\mu_k}\bar{\omega}_2}=\frac{1}{\sqrt{\mu_k}}\bar{\zeta}_2\frac{1}{\sqrt{\mu_k}\bar{\omega}_2}=\frac{1}{\mu_k}\frac{\bar{\zeta}_2}{\bar{\omega}_2}$$

And as follows for the second equation:

$$m_B\left(\frac{A_b}{A_a}\right)^2\ddot{u}+c_u(\dot{u}-\dot{y})+k_u(u-y)-m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}\ddot{x}=0$$

$$\frac{m_B\left(\frac{A_b}{A_a}\right)^2}{k_u}\ddot{u}+\frac{c_u}{k_u}(\dot{u}-\dot{y})+u-y-\frac{m_B\left(\frac{A_b}{A_a}-1\right)\frac{A_b}{A_a}}{k_u}\ddot{x}=0$$

$$\frac{1}{\omega_2^2}\ddot{u}+2\frac{\zeta_2}{\omega_2}(\dot{u}-\dot{y})+u-y-\frac{1}{\mu_k}\frac{1}{\bar{\omega}_i^2}\ddot{x}=0 \tag{C.2}$$

$$\ddot{u}+2\zeta_2\omega_2(\dot{u}-\dot{y})+\omega_2^2(u-y)-\frac{1}{\mu_k}\left(\frac{\omega_2}{\bar{\omega}_i}\right)^2\ddot{x}=0$$

$$\ddot{u}+2\bar{\zeta}_2\bar{\omega}_2(\dot{u}-\dot{y})+\mu_k\bar{\omega}_2^2(u-y)-\left(\frac{\bar{\omega}_2}{\bar{\omega}_i}\right)^2\ddot{x}=0$$

where: $$\zeta_2\omega_2=\frac{1}{\sqrt{\mu_k}}\bar{\zeta}_2\sqrt{\mu_k}\bar{\omega}_2=\bar{\zeta}_2\bar{\omega}_2$$

C.2 Type I AVAI (quadrature objective function)

The quadrature objective function can be evaluated by finding the point product of the input and output signals. This must be done over a predetermined time period T. Assuming that the input and output are given by two harmonic functions:

$$\begin{aligned} x(t) &= X\cos(\omega t+\theta) \\ y(t) &= Y\cos(\omega t) \end{aligned} \tag{C.3}$$

where θ is the angle between the response of the system and the input.

The input can be rewritten as:

$$\begin{aligned} x(t) &= X\cos(\omega t+\theta) \\ &= X\cos(\omega t)\cos(\theta) - X\sin(\omega t)\sin(\theta) \end{aligned} \tag{C.4}$$

The point product is (Long *et al.* 1994):

$$\begin{aligned} f &= x(t)\bullet y(t) \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \left[X\cos(\omega t)\cos(\theta) - X\sin(\omega t)\sin(\theta)\right] Y\cos(\omega t+\phi)\, dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} X\cos(\omega t)\cos(\theta) Y\cos(\omega t+\phi) - X\sin(\omega t)\sin(\theta) Y\cos(\omega t+\phi)\, dt \\ &= \frac{XY\cos(\theta)}{2} \\ &= 0 \qquad \text{when} \quad \theta = (2n+1)\frac{\pi}{2} \quad n = 0,1,2... \end{aligned} \tag{C.5}$$

C.3 Type II AVAI (equation of motion)

The equation of motion from Appendix B can be written in terms of the stiffness ratio to make tuning possible. The equation of motion is:

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \left(\frac{\omega_n}{\omega_i}\right)^2 \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x \tag{C.6}$$

As before, the current natural frequency and damping ratio can be written in terms of the initial value as follows:

$$\begin{aligned} \zeta &= \frac{1}{\sqrt{\mu_k}}\zeta' \\ \omega_n &= \sqrt{\mu_k}\,\omega_n' \end{aligned} \tag{C.7}$$

Substituting the above relations in the equation of motion:

$$\ddot{y} + 2\zeta'\omega_n'\dot{y} + \mu_k\omega_n' y = \left(\frac{\omega_n'}{\omega_i'}\right)^2 \ddot{x} + 2\zeta'\omega_n'\dot{x} + \mu_k\omega_n' x \tag{C.8}$$

APPENDIX D

Derivations for chapter 4

D.1 Vibration measurement of a Boart Longyear S250 rock drill (calibration factors)

Table D.1: Calibration factors

	x-direction	*y*-direction	*z*-direction
Calibration factor [V/g]	0.029176	0.031604	0.031604

D.2 Type I AVAI design (air spring stiffness)

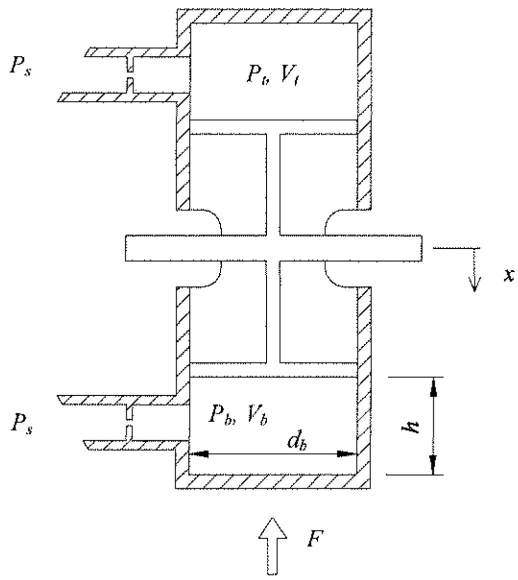

Figure D.1: Double sided air spring

The force balance acting on the piston is:

$$F = (P_b - P_t)A \tag{D.1}$$

Assuming adiabatic compression the pressure in the top and bottom chambers can be written in terms of the initial pressure (P_i) and volume (V_i):

$$P_b V_b^n = P_i V_i^n \tag{D.2}$$

where n is the ratio of specific heats (1.4 for air).

The force relationship of Equation (D.1) can now be written in terms of the initial pressure and volume values:

$$\begin{aligned} F &= P_i A\left[\left(\frac{V_i}{V_b}\right)^n - \left(\frac{V_i}{V_t}\right)^n\right] \\ &= P_i A h^n\left[\left(\frac{1}{h-x}\right)^n - \left(\frac{1}{h+x}\right)^n\right] \end{aligned} \qquad \text{(D.3)}$$

where h is the initial height.

The stiffness of the spring is the derivative of the force with respect to displacement:

$$k = \frac{dF}{dx} = nP_i A h^n\left[\left(\frac{1}{h-x}\right)^{n+1} + \left(\frac{1}{h+x}\right)^{n+1}\right] \qquad \text{(D.4)}$$

At small displacements this reduces to:

$$k = \frac{2nP_i A}{h} \qquad \text{(D.5)}$$

D.3 Type I AVAI design (heavy liquid properties)

Table 3.2 Summary of liquid properties at 25°C

Liquid		ρ [kg/m^3]	μ [N.s/m^2]	T_m [°C]	T_b [°C]	Hazard rating*
Water	H_2O	998	1.00×10^{-3}	0	100	0
Bromine	Br_2	3113	0.91×10^{-3}	-7	59	4
Bromoform	$CHBr_3$	2894		N/A	150	3
Carbon tetrachloride	CCl_4	1590	0.97×10^{-3}	-23	76	3
Lead tetrachloride	$PbCl_4$	3174		-15	105	
LST		2954				
Mercury	Hg	13550	1.56×10^{-3}	-38	356	4
Phosphorous tribromide	PBr_3	2846		-40	173	3
Selenium bromide	Se_2Br_2	3597			227	
Selenium monochloride	Se_2Cl_2	2764		-85	130	
Tetrabromoacetylene	$Br_2CHCHBr_2$	2954		0	135	2
Thionyl bromide	$SOBr_2$	2675		-52	138	
Thiophosphorylbromidechloride	$PSBr_2Cl$	2475		-60	95	
Tindibromidedichloride	$SnBr_2Cl_2$	2814		-20	65	

* Baker SAF-T-DATA™ health rating, 0 = no hazard, 4 = extreme hazard.

D.4 Type I AVAI design (forces acting on the drill)

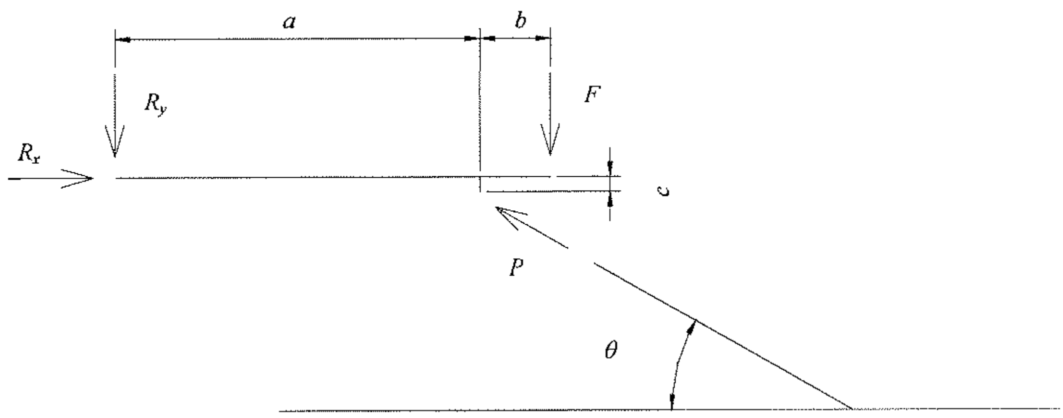

Figure D.2: Forces acting on the drill

The forces acting in the x-direction:

$$\sum_{\rightarrow} F_x = 0 \\ = R_x - P\cos(\theta) \tag{D.6}$$

The forces acting in the y-direction:

$$\sum_{\uparrow} F_y = 0 \\ = -R_y - F + P\sin(\theta) \tag{D.7}$$

The moments about R:

$$\sum M = 0 \\ = F(a+b) - P\sin(\theta)a + P\cos(\theta)c \\ F = \frac{P\sin(\theta)a - P\cos(\theta)c}{a+b} \tag{D.8}$$

Table D.2: Drill dimensions

Dimension	Value [mm]
a	1690
b	330
c	70
Piston diameter	63

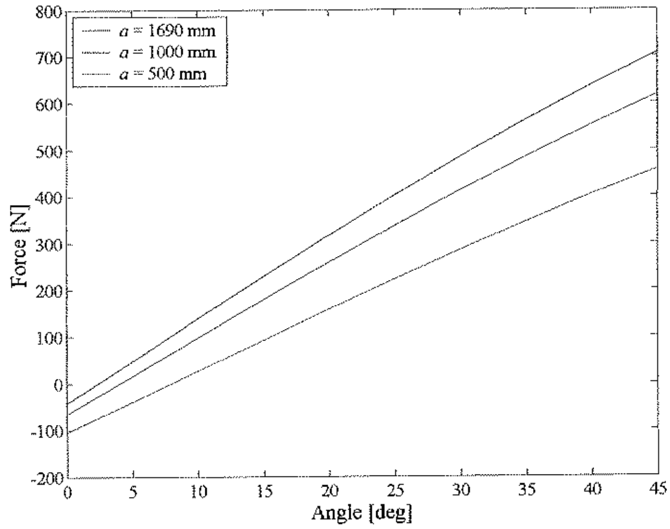

Figure D.3: Force *vs.* angle for a supply pressure of 400 kPa

D.5 Type I AVAI design (forces acting on the handle)

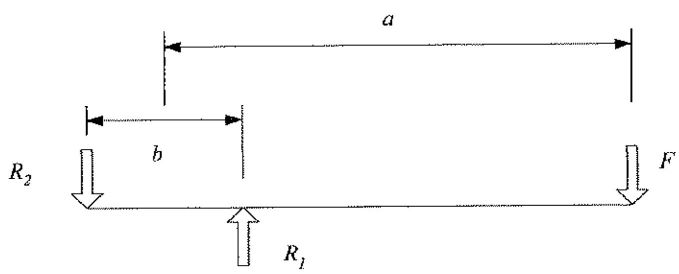

Figure D.4: Moment acting on the handle

The deflection at point R_1 can be calculated using the sum of the moments about R_2:

$$\begin{aligned} \sum M &= 0 \\ &= F\left(a+\frac{b}{2}\right)-R_1 b \\ R_1 &= \frac{F\left(a+\frac{b}{2}\right)}{b} \\ \delta_1 &= \frac{F}{k_1}\left(\frac{a}{b}+\frac{1}{2}\right) \end{aligned} \quad \text{(D.9)}$$

The deflection at point R_2 can be calculated using the sum of the moments and force balance in the y-direction:

$$\begin{aligned}\sum_{\uparrow} F_y &= 0 \\ &= -R_2 - R_1 - F \\ R_2 &= -\frac{F\left(a+\frac{b}{2}\right)}{b} - F \\ \delta_2 &= -\frac{F}{k_2}\left(\frac{a}{b}-\frac{1}{2}\right)\end{aligned} \tag{D.10}$$

The rotation angle can now be calculated by using Equation (D.9) and Equation (D.10):

$$\begin{aligned}\theta &= \sin^{-1}\left(\frac{\delta_2-\delta_1}{b}\right) \\ &= \sin^{-1}\left[\frac{-\frac{F}{k_2}\left(\frac{a}{b}-\frac{1}{2}\right)-\frac{F}{k_1}\left(\frac{a}{b}+\frac{1}{2}\right)}{b}\right]\end{aligned} \tag{D.11}$$

D.6 Type II AVAI design (effective area calculation)

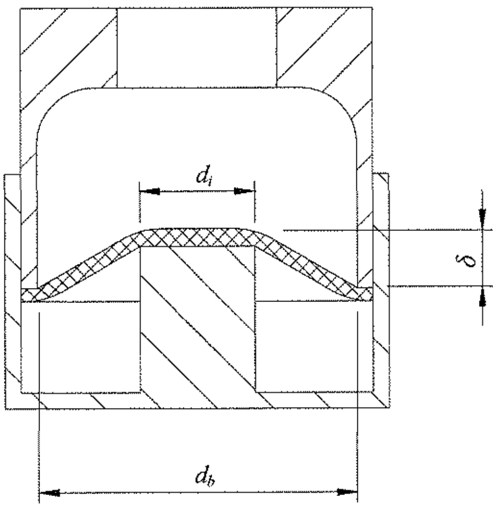

Figure D.5: Definition of dimensions

The volume change due to relative displacement δ is:

$$\Delta V = \frac{1}{3}A_b h - \frac{1}{3}A_i\left(h-\delta\right) \tag{D.12}$$

The relationship between h and δ is:

$$\frac{\frac{1}{2}d_b}{h} = \frac{\frac{1}{2}d_b - \frac{1}{2}d_i}{\delta}$$
$$h = \frac{d_b}{d_b - d_i}\delta \qquad \text{(D.13)}$$

The volume change can now be written as:

$$\begin{aligned} \Delta V &= \frac{1}{3}A_b h - \frac{1}{3}A_i(h-\delta) \\ &= \frac{1}{3}A_b \frac{d_b}{d_b - d_i}\delta - \frac{1}{3}A_i\left(\frac{d_b}{d_b - d_i}\delta - \delta\right) \\ &= \frac{\pi}{12}\left(\frac{d_b^3}{d_b - d_i} - \frac{d_b d_i^2}{d_b - d_i} + d_i^2\right)\delta \end{aligned} \qquad \text{(D.14)}$$

D.7 Type II AVAI design (damped design method)

The ratio of natural to isolation frequency is independent of the change in stiffness:

$$\frac{\omega_n}{\omega_i} = \sqrt{\frac{m_b \dfrac{A_b}{A_a}\left(\dfrac{A_b}{A_a} - 1\right)}{m + m_b\left(\dfrac{A_b}{A_a}\right)^2}} \qquad \text{(D.15)}$$

The damped isolation frequency is:

$$\left(\frac{\Omega_i}{\omega_n}\right)^2 = \frac{-\left(\dfrac{\omega_n}{\omega_i}\right)^2 - 1 - \sqrt{\left[\left(\dfrac{\omega_n}{\omega_i}\right)^2 - 1\right]^2 + 8\zeta^2\left[\left(\dfrac{\omega_n}{\omega_i}\right)^2 + 1\right]}}{4\zeta^2 + 4\zeta^2\left(\dfrac{\omega_n}{\omega_i}\right)^2 - 2\left(\dfrac{\omega_n}{\omega_i}\right)^2} \qquad \text{(D.16)}$$

The right hand side of the equation is constant:

$$C = \frac{-\left(\dfrac{\omega_n}{\omega_i}\right)^2 - 1 - \sqrt{\left[\left(\dfrac{\omega_n}{\omega_i}\right)^2 - 1\right]^2 + 8\zeta^2\left[\left(\dfrac{\omega_n}{\omega_i}\right)^2 + 1\right]}}{4\zeta^2 + 4\zeta^2\left(\dfrac{\omega_n}{\omega_i}\right)^2 - 2\left(\dfrac{\omega_n}{\omega_i}\right)^2} \qquad \text{(D.17)}$$

The isolation frequency can be written in terms of the constant C:

$$\Omega_i^2 = \omega_n^2 C$$
$$\Omega_i^2 = \frac{k}{m + m_b\left(\dfrac{A_b}{A_a}\right)^2}C \qquad \text{(D.18)}$$

The device has to be designed such that the excitation frequency coincides with the isolation frequency (i.e. $\Omega_i = \Omega_e$). If it is assumed that the stiffness consists of a spring in parallel with the air spring, the stiffness can be written in terms of the pressure:

$$k = k_p P_s + k_c \tag{D.19}$$

Since there are two unknowns, two sets of excitation and pressure values are needed to solve for k_p and k_s. In matrix format the set of equations are:

$$\begin{bmatrix} P_1 & 1 \\ P_2 & 1 \end{bmatrix} \begin{bmatrix} k_p \\ k_s \end{bmatrix} = \frac{\left[m + m_B \left(\frac{A_b}{A_a} \right)^2 \right]}{C} \begin{bmatrix} \Omega_{e1}^2 \\ \Omega_{e2}^2 \end{bmatrix} \tag{D.20}$$

To solve for the stiffness:

$$\begin{bmatrix} k_p \\ k_c \end{bmatrix} = \frac{\left[m + m_b \left(\frac{A_b}{A_a} \right)^2 \right]}{C} \begin{bmatrix} P_1 & 1 \\ P_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \Omega_{e1}^2 \\ \Omega_{e2}^2 \end{bmatrix} \tag{D.21}$$

APPENDIX E

Derivations for chapter 5

E.1 Refined model for a type I AVAI

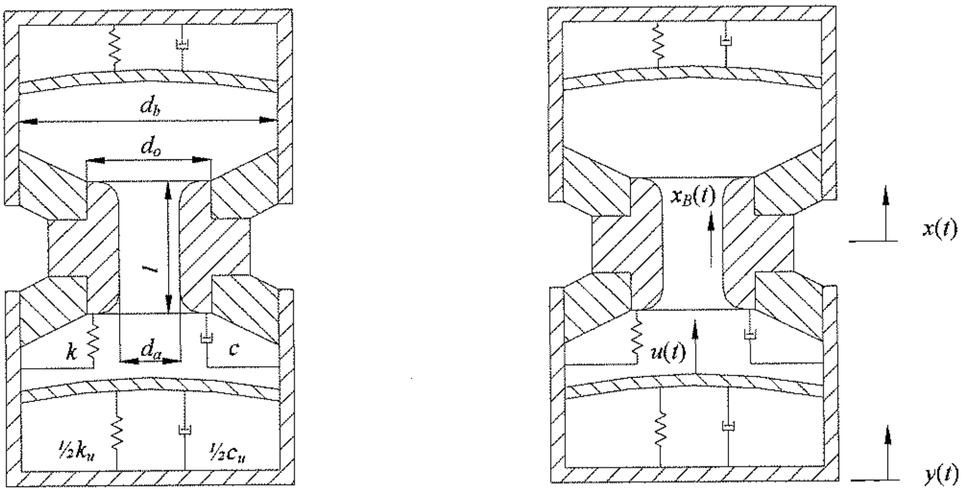

Figure E.1: A liquid vibration absorber system with base excitation

The displacement of the top and bottom elements must be equal under steady-state conditions because the fluid is incompressible. To find the continuity it is necessary to calculate the fluid displaced by the membrane. To do this it is necessary to know the shape of the membrane when deflected. If it is assumed that the displacement of the membrane assumes the shape of a paraboloid, the volume displaced can conveniently be written as:

$$V = \frac{1}{2} A_b h \tag{E.1}$$

with h the height of the paraboloid, which in this case is the relative displacement between u and y. Alternatively, if the membrane behaves as if fixed at the boundary, no rotation is possible and the deflected shape can be calculated (Young & Budynas, 2002).

The shape is a function of the radius:

$$\begin{aligned} y(r) &= y_c + \frac{M_c r^2}{2D(1+\nu)} + LT_y \\ \text{where:}\quad LT_y &= -\frac{qr^4}{64D} \\ y_c &= -\frac{qa^4}{64D} \\ M_c &= \frac{qa^2(1+\nu)}{16} \\ D &= \frac{Et^3}{12(1-\nu^2)} \\ \text{for } r_o &= 0 \end{aligned} \tag{E.2}$$

a is the disk radius, v Poison's ratio, q the distributed load, E is the Young's modulus, t the membrane thickness and y_c the displacement of the centre of the disk. From the above y can be rewritten:

$$\begin{aligned} y &= y_c + \frac{qa^2}{32D}r^2 - \frac{qr^4}{64D} \\ &= y_c\left(1 - \frac{2}{a^2}r^2 + \frac{r^4}{a^4}\right) \end{aligned} \tag{E.3}$$

The displaced volume can now be calculated:

$$\begin{aligned} V &= \int_0^a \int_0^{2\pi} y_c\left(1 - \frac{2}{a^2}r^2 + \frac{r^4}{a^4}\right) r\,d\theta\,dr \\ &= 2\pi y_c \int_0^a r - \frac{2}{a^2}r^3 + \frac{r^5}{a^4}dr \\ &= 2\pi y_c\left(\frac{1}{2}r^2 - \frac{1}{2}\frac{r^4}{a^2} + \frac{r^6}{6a^4}\right)\Bigg|_0^a \\ &= \frac{1}{3}\pi a^2 y_c \\ &= \frac{1}{3}A_b y_c \end{aligned} \tag{E.4}$$

To make provision to test assumption regarding membrane shape the derivation of the equation of motion will be done in terms of a shape factor S_f. The volume is therefore:

$$V = S_f A_b h \tag{E.5}$$

The change in volume can be calculated by subtracting the initial volume from the current volume. In Figure E.2 the initial values are denoted by the subscript i, while the current values are without subscript.

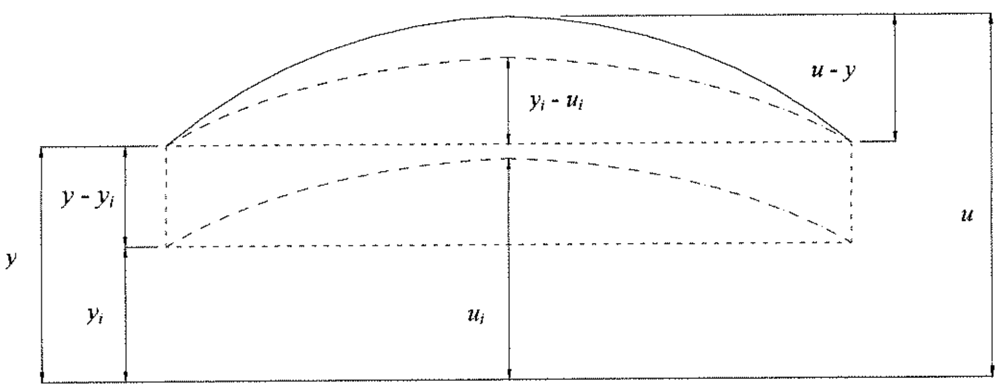

Figure E.2: Membrane displacement

The volume after relative displacement δ is:

$$\begin{aligned} V_s &= \frac{1}{3}A_b h_o - A_o\left(h-\delta\right) - \frac{1}{3}A_o\left(h_o - h + \delta\right) \\ h_o &= \frac{\delta - h}{\frac{d_o}{d_b} - 1} \\ &= \left(\frac{1}{3}A_b\frac{d_b}{d_b - d_o} - A_o - \frac{1}{3}A_o\frac{d_o}{d_b - d_o}\right)\left(h-\delta\right) \end{aligned} \tag{E.8}$$

The change in volume due to the relative displacement δ is:

$$\begin{aligned} \Delta V_\delta &= V_i - V_s \\ &= \left[\frac{A_b d_b - A_o d_o}{3\left(d_b - d_o\right)} - A_o\right]\delta \\ &= A_e\delta \end{aligned} \tag{E.9}$$

The total displaced volume is a function of the absolute and relative displacements of both x and y as explained in Figure E.4.

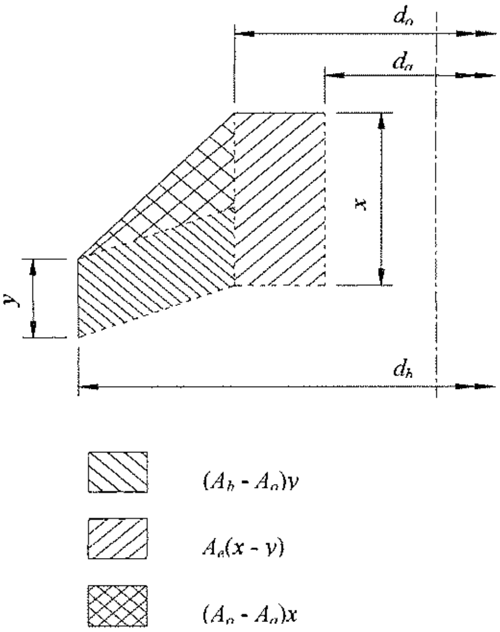

Figure E.4: Total displaced volume

The continuity is now given by:

$$\left(1-S_f\right)A_b y+S_f A_b u=\left[\frac{A_b d_b-A_o d_o}{3\left(d_b-d_o\right)}-A_a\right](x-y)+\left(A_b-A_a\right)y+A_a x_B$$

$$\begin{aligned}
x_B&=\left[1-\frac{A_b d_b-A_o d_o}{3A_a\left(d_b-d_o\right)}\right](x-y)+\left(1-\frac{A_b}{A_a}\right)y+\left(1-S_f\right)\frac{A_b}{A_a}y+S_f\frac{A_b}{A_a}u\\
&=\left[\frac{A_b d_b-A_o d_o}{3A_a\left(d_b-d_o\right)}-S_f\frac{A_b}{A_a}\right]y+\left[1-\frac{A_b d_b-A_o d_o}{3A_a\left(d_b-d_o\right)}\right]x+S_f\frac{A_b}{A_a}u\\
&=C_1 y-C_2 x+C_3 u
\end{aligned}\tag{E.10}$$

$$C_1=\frac{A_b d_b-A_o d_o}{3A_a\left(d_b-d_o\right)}-S_f\frac{A_b}{A_a}=\frac{1}{3}\frac{A_b}{A_a}\left[\left(\frac{d_o}{d_b}\right)^2+\frac{d_o}{d_b}+1-S_f\right]$$

$$C_2=1-\frac{A_b d_b-A_o d_o}{3A_a\left(d_b-d_o\right)}=\frac{1}{3}\frac{A_b}{A_a}\left[\left(\frac{d_o}{d_b}\right)^2+\frac{d_o}{d_b}+1\right]-1$$

$$C_3=S_f\frac{A_b}{A_a}$$

The kinetic energy is:

$$\begin{aligned}
T&=\frac{1}{2}m_x\dot{x}^2+\frac{1}{2}m_y\dot{y}^2+\frac{1}{2}m_B\dot{x}_B^2\\
&=\frac{1}{2}m_x\dot{x}^2+\frac{1}{2}m_y\dot{y}^2+\frac{1}{2}m_B\left(C_1^2y^2+C_2^2x^2+C_3^2u^2-2C_1C_2\dot{x}\dot{y}+2C_1C_3\dot{y}\dot{u}-2C_2C_3\dot{x}\dot{u}\right)
\end{aligned}\tag{E.11}$$

The derivatives are:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}}\right)&=m_x\ddot{x}+m_B\left(C_2^2\ddot{x}-C_1C_2\ddot{y}-C_2C_3\ddot{u}\right)\\
&=-m_BC_1C_2\ddot{y}+\left(m_x+m_BC_2^2\right)\ddot{x}-m_BC_2C_3\ddot{u}\\
\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{y}}\right)&=m_y\ddot{y}+m_B\left(C_1^2\ddot{y}-C_1C_2\ddot{x}+C_1C_3\ddot{u}\right)\\
&=\left(m_y+m_BC_1^2\right)\ddot{y}-m_BC_1C_2\ddot{x}+m_BC_1C_3\ddot{u}\\
\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{u}}\right)&=m_B\left(C_3^2\ddot{u}+C_1C_3\ddot{y}-C_2C_3\ddot{x}\right)\\
&=m_BC_1C_3\ddot{y}-m_BC_2C_3\ddot{x}+m_BC_3^2\ddot{u}
\end{aligned}\tag{E.12}$$

The mass matrix is:

$$\begin{bmatrix}M_1&M_2&M_3\\M_4&M_5&M_6\\M_7&M_8&M_9\end{bmatrix}=\begin{bmatrix}m_y+m_BC_1^2&-m_BC_1C_2&m_BC_1C_3\\-m_BC_1C_2&m_x+m_BC_2^2&-m_BC_2C_3\\m_BC_1C_3&-m_BC_2C_3&m_BC_3^2\end{bmatrix}\tag{E.13}$$

The potential energy and Raleigh terms are the same as before (refer to Appendix B). The equation of motion with y prescribed and no external forces acting on the system is:

$$\begin{bmatrix}M_5&M_6\\M_8&M_9\end{bmatrix}\begin{bmatrix}\ddot{x}\\\ddot{u}\end{bmatrix}+\begin{bmatrix}c&0\\0&c_u\end{bmatrix}\begin{bmatrix}\dot{x}\\\dot{u}\end{bmatrix}+\begin{bmatrix}k&0\\0&k_u\end{bmatrix}\begin{bmatrix}x\\u\end{bmatrix}=\begin{bmatrix}ky+c\dot{y}-M_4\ddot{y}\\k_uy+c_u\dot{y}-M_7\ddot{y}\end{bmatrix}\tag{E.14}$$

By transforming to the frequency domain:

$$\begin{bmatrix} k+i\omega c-\omega^2 M_5 & -\omega^2 M_6 \\ -\omega^2 M_8 & k_u+i\omega c_u-\omega^2 M_9 \end{bmatrix}\begin{bmatrix} X \\ U \end{bmatrix}=\begin{bmatrix} k+i\omega c+\omega^2 M_4 \\ k_u+i\omega c_u+\omega^2 M_7 \end{bmatrix}Y \tag{E.15}$$

The membrane displacement is:

$$U=\frac{\left(k+i\omega c+\omega^2 M_4\right)Y-\left(k+i\omega c-\omega^2 M_5\right)X}{-\omega^2 M_6} \tag{E.16}$$

The transmissibility is:

$$\frac{X}{Y}=\frac{\omega^2 M_6\left(k_u+i\omega c_u+\omega^2 M_3\right)+\left(k_u+i\omega c_u-\omega^2 M_9\right)\left(k+i\omega c+\omega^2 M_2\right)}{\left(k_u+i\omega c_u-\omega^2 M_9\right)\left(k+i\omega c-\omega^2 M_5\right)-\left(\omega^2 M_6\right)^2} \tag{E.17}$$

The following natural frequencies can be defined:

$$\omega_1^2=\frac{k}{M_1}=\frac{k}{m_y+m_B C_1^2} \tag{E.18}$$

The transmissibility can now be non-dimensionalised:

$$\frac{X}{Y}=\frac{\omega^2\frac{M_6}{k_u}\left(1+i\omega\frac{k_u}{k}\frac{c_u}{k_u}+\omega^2\frac{M_3}{k}\right)+\left(1+i\omega\frac{c_u}{k_u}-\omega^2\frac{M_9}{k_u}\right)\left(1+i\omega\frac{c}{k}+\omega^2\frac{M_2}{k}\right)}{\left(1+i\omega\frac{c_u}{k_u}-\omega^2\frac{M_9}{k_u}\right)\left[1+i\omega\frac{c}{k}-\omega^2\frac{M_5}{k}\right]-\frac{k_u}{k}\left(\omega^2\frac{M_6}{k_u}\right)^2}$$

$$=\frac{-\omega^2\frac{m_B C_2 C_3}{k_u}\left(1+i\omega\frac{k_u}{k}\frac{c_u}{k_u}+\omega^2\frac{m_B C_1 C_3}{k}\right)+\left(1+i\omega\frac{c_u}{k_u}-\omega^2\frac{m_B C_3^2}{k_u}\right)\left(1+i\omega\frac{c}{k}-\omega^2\frac{m_B C_1 C_2}{k}\right)}{\left(1+i\omega\frac{c_u}{k_u}-\omega^2\frac{m_B C_3^2}{k_u}\right)\left[1+i\omega\frac{c}{k}-\omega^2\frac{m_x+m_B C_2^2}{k}\right]-\frac{k_u}{k}\left(\omega^2\frac{m_B C_2 C_3}{k_u}\right)^2}$$

$$=\frac{\frac{C_2}{C_3}\left(\frac{\omega}{\omega_9'}\right)^2\left[1+i2\frac{\omega}{\omega_9'}\frac{k_u}{k}\zeta_u+\frac{k_u}{k}\frac{C_1}{C_3}\left(\frac{\omega}{\omega_9'}\right)^2\right]+\left[1+i2\frac{\omega}{\omega_9'}\zeta_u-\left(\frac{\omega}{\omega_9'}\right)^2\right]\left[1+i2\frac{\omega}{\omega_2}\zeta_2-\left(\frac{\omega}{\omega_2}\right)^2\right]}{\left[1+i2\frac{\omega}{\omega_9'}\zeta_u-\left(\frac{\omega}{\omega_9'}\right)^2\right]\left[1+i2\frac{\omega}{\omega_5}\zeta_5-\left(\frac{\omega}{\omega_5}\right)^2\right]-\frac{k_u}{k}\left[\frac{C_2}{C_3}\left(\frac{\omega}{\omega_9'}\right)^2\right]^2}$$

$$\omega_3^2=\frac{k}{M_3}=\frac{k}{m_B C_1 C_3},\quad \omega_2^2=\frac{k}{M_2}=\frac{k}{m_B C_1 C_2},\quad \omega_6'^2=\frac{k_u}{M_6}=\frac{k_u}{m_B C_2 C_3}$$

$$\omega_9'^2=\frac{k_u}{M_9}=\frac{k_u}{m_B C_3^2},\quad \omega_3^2=\frac{k}{k_u}\frac{C_3}{C_1}\omega_9'^2,\quad \omega_1^2=\frac{k}{M_5}=\frac{k}{m_x+m_B C_2^2},\quad \omega_6'^2=\frac{C_3}{C_2}\omega_9'^2$$

$$\zeta_u=\frac{c_u}{2m_B C_3^2\omega_9'},\quad \zeta_5=\frac{c}{2\left(m_x+m_B C_2^2\right)\omega_5},\quad \zeta_2=\frac{c}{2m_B C_1 C_2\omega_2} \tag{E.19}$$

The equation of motion with x prescribed and no external forces acting on the system is:

$$\begin{bmatrix} M_1 & M_3 \\ M_7 & M_9 \end{bmatrix}\begin{bmatrix} \ddot{y} \\ \ddot{u} \end{bmatrix}+\begin{bmatrix} c+c_u & -c_u \\ -c_u & c_u \end{bmatrix}\begin{bmatrix} \dot{y} \\ \dot{u} \end{bmatrix}+\begin{bmatrix} k+k_u & -k_u \\ -k_u & k_u \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix}=\begin{bmatrix} kx+c\dot{x}-M_2\ddot{x} \\ -M_8\ddot{x} \end{bmatrix} \tag{E.20}$$

By transforming to the frequency domain:

$$\begin{bmatrix} k + k_u + i\omega(c + c_u) - \omega^2 M_1 & -k_u - i\omega c_u - \omega^2 M_3 \\ -k_u - i\omega c_u - \omega^2 M_7 & k_u + i\omega c_u - \omega^2 M_9 \end{bmatrix} \begin{bmatrix} Y \\ U \end{bmatrix} = \begin{bmatrix} k + i\omega c + \omega^2 M_2 \\ \omega^2 M_8 \end{bmatrix} X \tag{E.21}$$

The membrane displacement can be found from the second equation in the set defined above:

$$U = \frac{\left(k_u + i\omega c_u + \omega^2 M_2\right) X - \left[k + k_u + i\omega\left(c + c_u\right) - \omega^2 M_1\right] Y}{-\left(k_u + i\omega c_u + \omega^2 M_3\right)} \tag{E.22}$$

The transmissibility is:

$$\frac{Y}{X} = \frac{\left(k_u + i\omega c_u + \omega^2 M_3\right)\omega^2 M_8 + \left(k_u + i\omega c_u - \omega^2 M_9\right)\left(k + i\omega c + \omega^2 M_2\right)}{\left(k_u + i\omega c_u - \omega^2 M_9\right)\left[k + k_u + i\omega\left(c + c_u\right) - \omega^2 M_1\right] - \left(k_u + i\omega c_u + \omega^2 M_3\right)^2} \tag{E.23}$$

The transmissibility can now be non-dimensionalised:

$$\frac{Y}{X} = \frac{\omega^2 \frac{M_6}{k_u}\left(1 + i\omega \frac{k_u}{k}\frac{c_u}{k_u} + \omega^2 \frac{M_3}{k}\right) + \left(1 + i\omega \frac{c_u}{k_u} - \omega^2 \frac{M_9}{k_u}\right)\left(1 + i\omega \frac{c}{k} + \omega^2 \frac{M_2}{k}\right)}{\left(1 + i\omega \frac{c_u}{k_u} - \omega^2 \frac{M_9}{k_u}\right)\left[1 + \frac{k_u}{k} + i\omega\left(\frac{c}{k} + \frac{k_u}{k}\frac{c_u}{k_u}\right) - \omega^2 \frac{M_1}{k}\right] - \frac{k_u}{k}\left(1 + i\omega \frac{c_u}{k_u} + \omega^2 \frac{M_3}{k_u}\right)^2}$$

$$= \frac{-\omega^2 \frac{m_B C_2 C_3}{k_u}\left(1 + i\omega \frac{k_u}{k}\frac{c_u}{k_u} + \omega^2 \frac{m_B C_1 C_3}{k}\right) + \left(1 + i\omega \frac{c_u}{k_u} - \omega^2 \frac{m_B C_3^2}{k_u}\right)\left(1 + i\omega \frac{c}{k} - \omega^2 \frac{m_B C_1 C_2}{k}\right)}{\left(1 + i\omega \frac{c_u}{k_u} - \omega^2 \frac{m_B C_3^2}{k_u}\right)\left[1 + \frac{k_u}{k} + i\omega\left(\frac{c}{k} + \frac{k_u}{k}\frac{c_u}{k_u}\right) - \omega^2 \frac{m_y + m_B C_1^2}{k}\right] - \frac{k_u}{k}\left(1 + i\omega \frac{c_u}{k_u} + \omega^2 \frac{m_B C_1 C_3}{k_u}\right)^2}$$

$$= \frac{-\frac{C_2}{C_3}\left(\frac{\omega}{\omega_9'}\right)^2\left[1 + i2\frac{\omega}{\omega_9'}\frac{k_u}{k}\zeta_u + \frac{k_u}{k}\frac{C_1}{C_3}\left(\frac{\omega}{\omega_9'}\right)^2\right] + \left[1 + i2\frac{\omega}{\omega_9'}\zeta_u - \left(\frac{\omega}{\omega_9'}\right)^2\right]\left[1 + i2\frac{\omega}{\omega_2}\zeta_2 - \left(\frac{\omega}{\omega_2}\right)^2\right]}{\left[1 + i2\frac{\omega}{\omega_9'}\zeta_u - \left(\frac{\omega}{\omega_9'}\right)^2\right]\left[1 + \frac{k_u}{k} + i2\left(\frac{\omega}{\omega_1}\zeta_1 + \frac{k_u}{k}\frac{\omega}{\omega_9'}\zeta_u\right) - \left(\frac{\omega}{\omega_1}\right)^2\right] - \frac{k_u}{k}\left[1 + i2\frac{\omega}{\omega_9'}\zeta_u + \frac{C_1}{C_3}\left(\frac{\omega}{\omega_9'}\right)^2\right]^2}$$

$$\omega_3^2 = \frac{k}{M_3} = \frac{k}{m_B C_1 C_3}, \quad \omega_2^2 = \frac{k}{M_2} = \frac{k}{m_B C_1 C_2}, \quad \omega_6'^2 = \frac{k_u}{M_6} = \frac{k_u}{m_B C_2 C_3}, \quad \omega_9'^2 = \frac{k_u}{M_9} = \frac{k_u}{m_B C_3^2}$$

$$\omega_3^2 = \frac{k}{k_u}\frac{C_3}{C_1}\omega_9'^2, \quad \omega_3'^2 = \frac{C_3}{C_1}\omega_9'^2, \quad \omega_1^2 = \frac{k}{M_1} = \frac{k}{m_y + m_B C_1^2}, \quad \omega_6'^2 = \frac{C_3}{C_2}\omega_9'^2$$

$$\zeta_u = \frac{c_u}{2 m_B C_3^2 \omega_9'}, \quad \zeta_1 = \frac{c}{2\left(m_y + m_B C_1^2\right)\omega_1}, \quad \zeta_2 = \frac{c}{2 m_B C_1 C_2 \omega_2} \tag{E.24}$$