CHAPTER 7: CORRELATING THE ELEMENTS

This chapter presents the correlation of the elements from the first and the second runs of the questionnaire with 37 students and 122 students respectively, as well as from the examination analysis with 151 students that were analysed qualitatively in Chapter 6. Kendall tau and Pearson correlation coefficients are used to determine the level of significance of the correlations. The null hypothesis H_o in this study is that: there are no associations in performance between the different elements. Depending on the p-value, relating to the significance level, the null hypothesis is either rejected or not rejected. The ranked marks for each respondent (0 to 4) obtained for the 11 elements, from the 37 responses from April 2007 results; and four elements from the 122 responses from the October 2007 results and five elements from the 151 responses from August 2007 examination analysis are correlated using a non-parametric test, Kendall tau correlation coefficient. The marks obtained by the 151 students from Question 5 out of 40 are correlated with the overall examination marks out of 100 using scatter plots and the Pearson correlation coefficient as a parametric test. A histogram is used to display the distribution of the students' marks and the scatter plot is used to identify students' performance in terms of four quadrants before the correlations are done.

7.1 NON-PARAMETRIC TESTS: KENDALL TAU (τ)

When using non-parametric tests, the rank scores obtained by the students are correlated. The correlations are done based on the average ranks that each student got for all questions under each of the 11 elements. For example, the average rank of the two questions under graphing skills is correlated to the average rank of the two questions under consolidation and general level of cognitive development. In the Questionnaire 1st run, all the students were given all 23 questions to respond to. For the Questionnaire 2nd run, two groups of students were given different questions. It was only for one group where all questions under four elements were given. In other elements, students were given only one question to respond to. For the examination analysis, the questions that the students responded to could only be classified under five elements. The students' performance based on the elements from the Questionnaire 1st run with 37 responses, the Questionnaire 2nd run with 122 responses and the examination analysis with 151 responses are correlated separately using Kendall tau (τ) correlation coefficients. Kendall tau is used since most of the results contained tied ranks between 0 and 4, with more than one student having the same score.

The data are presented in tables where the pairs of correlations are indicated. The level of significance shown by no asterisk is not significant (p-value is > 0.05); one asterisk (*) indicates significance where $0.01 < p < 0.05$ and two asterisks (**) indicates where the p-value is < 0.01 as highly significant. The diagonal row gives a correlation of 1, since the averages from the same elements are correlated, e.g. graphing skills is correlated with graphing skills. The correlation discussed here is below the diagonal. In correlating the elements the direction of the association of the elements is not known. My research hypothesis (H_o) is: *There are no associations in performance between the different elements*. It does not state the direction of the difference or association among the elements. Therefore I used a two-tailed test of significance. The null hypothesis is either rejected or not rejected, depending on the p value, relating to the significance level at 0.01 (1%) or at 0.05 (5%) levels.

7.1.1 Correlations for the Questionnaire 1st run

Table 7.1: Kendall tau for the Questionnaire 1st run

		GR	AV2D	VA2D	AV3D	VA3D	2D-3D	3D-2D	CD(V)	DC-CD(A)	GMNP	CGLCD
GR	Correlation Coefficient	1.000										
	Sig. (2-tailed)	.										
AV2D	Correlation Coefficient	.112	1.000									
	Sig. (2-tailed)	.407	.									
VA2D	Correlation Coefficient	.028	-.071	1.000								
	Sig. (2-tailed)	.834	.607	.								
AV3D	Correlation Coefficient	.168	**-.029**	.324*	1.000							
	Sig. (2-tailed)	.201	.833	.016	.							
VA3D	Correlation Coefficient	**.416****	.174	.084	.294*	1.000						
	Sig. (2-tailed)	.002	.202	.536	.026	.						
2D-3D	Correlation Coefficient	.291*	.227	.275*	.144	.271*	1.000					
	Sig. (2-tailed)	.025	.091	.039	.268	.039	.					
3D-2D	Correlation Coefficient	**.384****	.207	.206	.198	.248	**.463****	1.000				
	Sig. (2-tailed)	.003	.127	.123	.131	.060	.000	.				
CD(V)	Correlation Coefficient	.314*	.048	-.040	.178	.319*	**.437****	**.407****	1.000			
	Sig. (2-tailed)	.018	.724	.769	.180	.017	.001	.002	.			
DC-CD(A)	Correlation Coefficient	.180	.161	.209	**.368****	.261	.314*	.328*	**.450****	1.000		
	Sig. (2-tailed)	.174	.237	.122	.005	.051	.017	.013	.001	.		
GMNP	Correlation Coefficient	.145	.081	-.174	.224	**.252**	-.135	-.183	.071	-.079	1.000	
	Sig. (2-tailed)	.269	.547	.192	.087	.055	.295	.161	.592	.547	.	
CGLCD	Correlation Coefficient	**.374****	.306*	.069	.319*	**.436****	**.455****	**.444****	**.593****	**.502****	.138	1.000
	Sig. (2-tailed)	.006	.027	.615	.018	.001	.001	.001	.000	.000	.303	.

** Correlation is significant at the 0.01 level (2-tailed).

* Correlation is significant at the 0.05 level (2-tailed)

From Table 7.1, the correlations for the 11 elements are displayed and interpreted from Kendall's tau correlation coefficient. All correlations in Table 7.1, which are significant at 0.01 and 0.05 levels, are positive, while none of the negative correlations are significant.

7.1.1.1 Correlating the skill factor consolidation and general level of cognitive development and the other elements

The two highly significant correlations $\tau = 0.593$, $p < 0.001$, and $\tau = 0.502$, $p < 0.001$ are correlating the consolidation and general level of cognitive development to two different elements, translation from continuous to discrete (visually) and translation from discrete to continuous and from continuous to discrete (algebraically). Such an association between the consolidation and general level of cognitive development and elements in the continuous to discrete representations, points out to how important the selection of representative strip is to the ability to perform better in a question that requires consolidation and general level of cognitive development.

The correlation of the consolidation and general level of cognitive development is also highly significant to four other elements: graphing skills, $\tau = 0.374$, $p = 0.006$; translation from visual to algebraic in 3D, $\tau = 0.436$, $p = 0.001$; translation from 2D to 3D, $\tau = 0.455$, $p = 0.001$; and the translation from 3D to 2D, $\tau = 0.444$, $p = 0.001$. Such an association between the consolidation and general level of cognitive development and graphing skills; translation from visual to algebraic in 3D; translation from 2D to 3D and translation from 3D to 2D also points out the strong correspondence between consolidation and general level of cognitive development and performance in these elements. The performance in these elements is related to how graphs are drawn and interpreted. It relates to how the region bounded by the drawn graphs or a given diagram is rotated about any axis, especially in three-dimensions. It also relates to having to draw a 2D diagram from the given 3D solid.

The correlation of the consolidation and general level of cognitive development is significant to the two elements: translation from algebraic to visual in 2D and translation from algebraic to visual in 3D, and not significant to the two elements: translation from visual to algebraic in 2D and general manipulation skills. Overall, consolidation and general level of cognitive development is strongly associated with these six elements, graphing skills, translation from visual to algebraic in 3D, translation from 2D to 3D, translation from 3D to 2D, translation from continuous to discrete (visually) and translation from discrete to continuous and from continuous to discrete (algebraically).

7.1.1.2 Correlating general manipulation skills to other elements

The element, general manipulation skills is the only element which does not show any significant correlations to the other elements. It also has a high number (4) of negative correlations in relation to other elements. General manipulation skills correlates negatively to the elements: translation from visual to algebraic in 2D; translation from 2D to 3D; translation from 3D to 2D and translation from discrete to continuous and from continuous to discrete algebraically. The non-significant correlations of manipulation skills to the other elements reveals that lack of manipulation skills does not impact on the skills in the other elements.

7.1.1.3 Correlating translation from discrete to continuous and from continuous to discrete algebraically to other elements

In addition to the consolidation and general level of cognitive development, the element translation from discrete to continuous and from continuous to discrete algebraically is highly correlated to translation from continuous to discrete (visually) with $\tau = 0.450$, $p = 0.001$ and translation from algebraic to visual in 3D with $\tau = 0.368$, $p = 0.005$, respectively. Both elements and use the strip as the main focus, visually and algebraically. This shows that algebraic thinking is related to visual thinking when translating from continuous to discrete and vice versa.

The element translation from discrete to continuous and from continuous to discrete algebraically has significant correlations between the elements translation from 2D to 3D and translation from 3D to 2D.

7.1.1.4 Correlating translation from continuous to discrete (visually) to other elements

Apart from the element consolidation and general level of cognitive development and the element translation from discrete to continuous and from continuous to discrete algebraically, the element translation from continuous to discrete (visually) is highly correlated to the two elements, translation from 3D to 2D with $\tau = 0.407$, $p = 0.002$ and translation from 2D to 3D $\tau = 0.437$, $p = 0.001$. This reveals that the selection of the strip and approximating the bounded region impacts on how one translates between 2D and 3D, be it from area to volume or from volume to area. The element translation from continuous to discrete (visually) shows significant correlations with the two elements, graphing skills and translation from visual to algebraic in 3D with non-significant correlations to the other remaining elements.

7.1.1.5 Correlating translation from 3D to 2D to other elements

In addition to the elements consolidation and general level of cognitive development and translation from continuous to discrete (visually), the element translation from 3D to 2D is highly correlated to the two elements, translation from 2D to 3D with $\tau = 0.463$, $p < 0.001$ and graphing skills with $\tau = 0.384$, $p = 0.003$. This reveals that the ability to translate from 3D to 2D is associated with the skills that one has in drawing graphs and how the drawn 2D diagrams are rotated to 3D. Earlier on we observed that this element was significantly correlated to the element translation from discrete to continuous and from continuous to discrete algebraically. It also shows no significant correlations with the other elements.

7.1.1.6 Correlating translation from 2D to 3D to other elements

As previously discussed, the element, translation from 2D to 3D, was highly correlated to the three elements, translation from 3D to 2D, translation from continuous to discrete (visually) and consolidation and general level of cognitive development. This means that translation from 2D to 3D is associated with translation from 3D to 2D, to the way in which the strip is being selected and requires a level of cognitive development.

In addition to the element translation from discrete to continuous and from continuous to discrete (algebraically), this element is significantly correlated to the three elements graphing skills; translation from visual to algebraic in 2D and translation from visual to algebraic in 3D There is no significant correlation of this element to the elements, translation from algebraic to visual in 2D translation from algebraic to visual in 3D and general manipulation skills.

7.1.1.7 Correlating translation from visual to algebraic in 3D to other elements

In addition to being highly correlated with the element consolidation and general level of cognitive development, this element is also highly correlated to graphing skills with $\tau = 0.416$, $p = 0.002$. This reveals the relationship between the skills that one has in drawing graphs with the skill of interpreting the drawn graphs, resulting in the formula for volume. Apart from the translation from 2D to 3D and translation from continuous to discrete visually translation from visual to algebraic in 3D has a significant correlation to translation from algebraic to visual in 3D. The rest of the correlations (translation from algebraic to visual in 2D, translation from algebraic to visual in 3D, translation from 2D to 3D, translation from discrete to continuous and from continuous to discrete algebraically and general manipulation skills are non-significant.

7.1.1.8 Correlating translation from algebraic to visual in 3D to other elements

Looking at the remaining correlations for this element, it is highly correlated with translation from discrete to continuous and from continuous to discrete algebraically. These two elements are highly associated because the equations given are to be translated and represented visually in both elements.

This element is significantly correlated to the translation from visual to algebraic in 3D, the translation from visual to algebraic in 2D and the consolidation and general level of cognitive development.

7.1.1.9 Correlating translation from visual to algebraic in 2D to other elements

This element is not highly associated with any elements. It does not have any significant correlation with 8 elements. It only has significant correlations with the elements translation from algebraic to visual in 3D and translation from 2D to 3D. Of the eight non-significant correlations that this element has with the other elements, three are negative. These correlations reveal that the ability to solve problems involving translation from visual to algebraic in 2D is not associated with the other elements.

7.1.1.10 Correlating translation from algebraic to visual in 2D to other elements

Similar to the element translation from visual to algebraic in 2D, this element does not have any significant correlations with many elements. It also has a high number of non-significant correlations (9) of which two are negative. This element is only significantly correlated to consolidation and general level of cognitive development. The correlation here reveals that drawing diagrams in 2D is associated with the cognitive demands of the task.

7.1.1.11 Summary for the Questionnaire 1st run

The conclusions that can be drawn from Kendall's tau correlation coefficient suggest that the consolidation and general level of cognitive development as well as the translation from continuous to discrete visually has the highest significant correlations with most of the elements. The element involving general manipulation skills shows all correlations that are not significant in relation to all 9 elements. These non-significant correlations mean that performance in general manipulation skills does not have any impact on how one performs in VSOR. Similarly all the correlations for the translation from algebraic to visual in 2D, except consolidation and general level of cognitive development has non-significant correlations to the other elements.

The conclusions above imply that in order to do well in VSOR, the students must be competent in the skill that involves the consolidation and general level of cognitive development as it is strongly associated with other elements as well as translation between discrete and continuous (the proper identification of the correct strip). The results also reveal that the skills that involve general manipulation do not have any impact on other elements, while the translation from algebraic to visual in 2D have animpact only on the element consolidation and general level of cognitive development.

In the section that follows, the results involving the Questionnaire 2nd run (122 respondents) are presented and analysed, again using Kendall tau to establish similar or different trends from the Questionnaire 1st run (37 respondents) discussed above.

7.1.2 Correlations for the Questionnaire 2nd run

Table 7.2: Kendall tau for overall 122 responses

		GR	AV3D	3D-2D	GMNP
GR	Correlation Coefficient	1.000			
	Sig. (2-tailed)	.			
AV3D	Correlation Coefficient	.228**	1.000		
	Sig. (2-tailed)	.002	.		
3D-2D	Correlation Coefficient	.106	.129	1.000	
	Sig. (2-tailed)	.139	.074	.	
GMNP	Correlation Coefficient	.140	.284**	.099	1.000
	Sig. (2-tailed)	.050	.000	.158	.

** Correlation is significant at the 0.01 level (2-tailed).

In Table 7.2 all correlations are seen to be positive. An element that has highly significant correlations to other elements is the element, translation from algebraic to visual in 3D, with the highest correlation of $\tau = 0.284$, $p < 0.001$ with general manipulation skills and the correlation of $\tau = 0.228$, $p = 0.002$ with graphing skills. The correlations obtained imply that translation from algebraic to visual in 3D is strongly associated with general manipulation skills and with graphing skills. In both cases the association between translation from algebraic to visual in 3D to these other elements is statistically significant at the 1% level ($p < 0.01$). The correlations between graphing skills and the other two elements translation from 3D to 2D and general manipulation skills are not significant. This means that a skill in drawing graphs appears not to be associated with how one performs calculations or how one translates from 3D to 2D.

7.1.2.1 Summary for the Questionnaire 2nd run

The conclusions can be drawn that overall the translation from algebraic to visual in 3D has a high correlation with general manipulation skills and graphing skills in relation to other elements that are correlated. The conclusions above imply that in order to do well in VSOR, the students must be competent in the skill that involves the translation from algebraic to visual in 3D , general manipulation skills and graphing skills.

7.1.3 Conclusion for the correlations from the questionnaires

In the Questionnaire 2nd run the element translation from algebraic to visual in 3D was highly correlated to the elements graphing skills and general manipulation skills, but in the Questionnaire 1st run the correlations were not significant. In the Questionnaire 1st run, the elements graphing skills and the elements translation from 3D to 2D were highly correlated, whereas in Questionnaire 2nd run, their correlations were not significant. Similar results were found between the following correlations:

- Translation from algebraic to visual in 3D and translation from 3D to 2D.
- General manipulation skills and graphing skills.
- General manipulation skills and translation from 3D to 2D.

The general conclusion that could be made is that general manipulation skills do not have any impact on most of the elements, whereas graphing skill does.

It must however be noted that in the Questionnaire 1st all 11 elements were correlated and that in the Questionnaire 2nd run only four elements graphing skills, translation from algebraic to visual in 3D, translation from 3D to 2D and general manipulation skills were correlated. In both runs of the questionnaires the students were different, the lecturers who taught them where different and probably their level of preparedness were different. However, one can make some inferences about the recurring trends in the correlations despite their different circumstances, as it is evident from similar results that were found above from the correlations of the elements.

7.1.4 Correlations for the examinations analysis

In Table 7.3 the five elements from the 11 elements used in the main instrument are correlated.

Table 7.3: Correlations from Kendall's tau

		GMNPav	GRav	CDav	VA2Dav	VA3Dav
GMNPav	Correlation Coefficient	1.000				
	Sig. (2-tailed)	.				
	N	151				
GRav	Correlation Coefficient	.444**	1.000			
	Sig. (2-tailed)	.000	.			
	N	151	151			
CDav	Correlation Coefficient	.366**	**.561****	1.000		
	Sig. (2-tailed)	.000	.000	.		
	N	151	151	151		
VA2Dav	Correlation Coefficient	.342**	**.575****	.415**	1.000	
	Sig. (2-tailed)	.000	.000	.000	.	
	N	151	151	151	151	
VA3Dav	Correlation Coefficient	.412**	.480**	.428**	.370**	1.000
	Sig. (2-tailed)	.000	.000	.000	.000	.
	N	151	151	151	151	151

**. Correlation is significant at the 0.01 level (2-tailed).

In Table 7.3 the correlations of the five elements, general manipulation skills, graphing skills, translation from continuous to discrete, translation from visual to algebraic in 2D and translation from visual to algebraic in 3D are given and based on Kendall's tau coefficient of correlation (τ). Kendall's coefficient of correlation takes into account the ranks that have ties (more than one student having the same score). The correlations found in Table 7.3 show a highly significant association between the five elements from Question 5 at 1% level. The highest correlations are for the element graphing skills to the elements translation from visual to algebraic in 2D with $\tau = 0.575$, $p < 0.001$ and translating from continuous to discrete visually with $\tau = 0.561$, $p < 0.001$.

The correlations in Table 7.3 mean that for all the elements, the higher a student scores in one element, the higher a student will score in the other element since all correlations are positive. The lower a student scores on graphing skills, the lower a student will score on problems requiring the translation from visual to algebraic in 2D and vice versa. This reveals that the examination paper is assessed in such a way that the responses to the five different elements are strongly associated with one another.

7.1.5 Summary for the examination correlations

Interpretations of the Kendall's tau correlation coefficient for the 151 students for the 5 elements reveals that, overall, the graphing skills and translation from visual to algebraic in 2D are highly correlated as well as the association of the graphing skills and translating from continuous to discrete. The other correlations are highly significant as well. One can therefore conclude that in learning VSOR, the consolidation and general level of cognitive development appears to be significant as it depends on all five elements correlated. The five elements being correlated (graphing skills, translation from visual to algebraic in 2D, translation from visual to algebraic in 3D, translating from continuous to discrete visually and general manipulation skills) are therefore fundamental in learning VSOR in relation to the way in which the final N6 mathematics examination paper is prepared.

7.2 PARAMETRIC TESTS: PEARSON (*r*)

In the section that follows, the marks obtained by 151 students in Question 5 (out of 40 with a passing mark of 16) are correlated to the marks that they obtained for the whole paper (out of 100 with a passing mark of 40). The parametric tests are used, using graphs and tables. Under the parametric tests, histograms; a scatter plot; the Pearson correlation coefficient (r) and the level of significance are discussed based on the overall numerical value that each student obtained for Question 5 and comparing it to the mark for the whole paper. The five elements are: general manipulation skills; graphing skills; translation from continuous to discrete; translation from visual to algebraic in 2D and translation from visual to algebraic in 3D.

My research hypothesis does not state the direction of the difference or association among variables. Therefore I used a two-tailed test of significance. The null hypothesis H_o in relation to the examination analysis and the elements is that: *There are no associations in performance between Question 5 and the whole paper*. The null hypothesis is either rejected or not rejected depending on the p-value, relating to the significance level.

Before correlating the scores obtained by students in Question 5 to those in the whole paper and determining the level of significance using Pearson's correlation coefficient, a histogram is used to display the distribution of the students' marks. Thereafter a scatter plot is used to identify where each student lies in terms of the marks obtained for Question 5 and the whole paper. The distribution of the results is shown for Question 5 in Figure 7.1 and for the whole paper in Figure 7.2 for the 151 students. The data are almost symmetrically distributed.

7.2.1. The histogram for students' performance

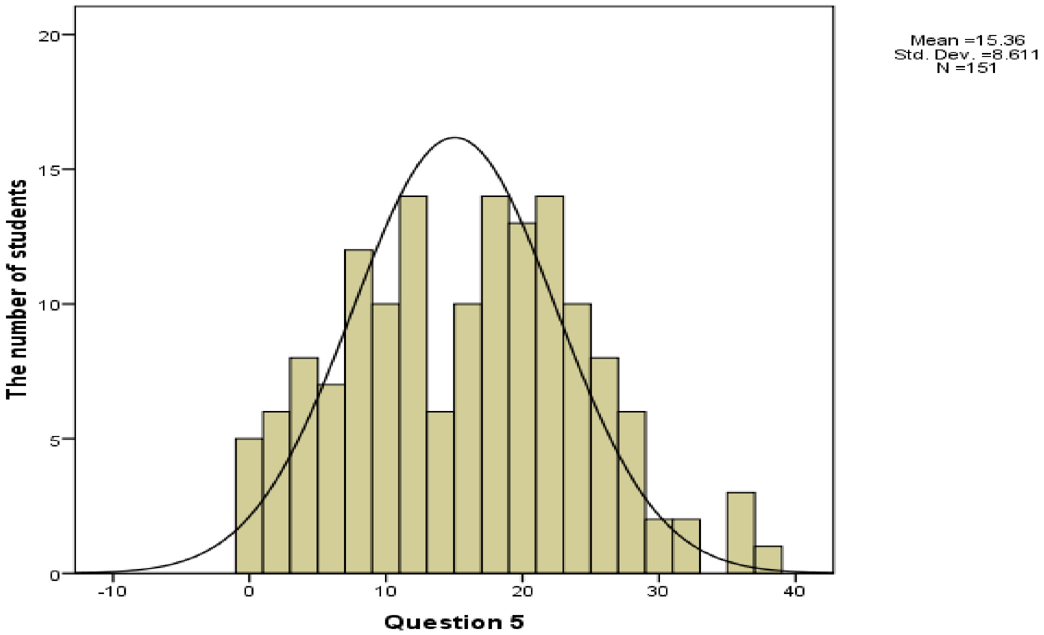

Figure 7.1: Performance in Question 5

Figure 7.1 shows how many students obtained a particular score, ranging from zero to 40. It is evident from the histogram that 5 students scored zero for Question 5 whereas none of the students scored the total mark of 40. The highest number of students (from the bars) scored 12; 18 and 24 respectively with no students scoring 34. Ten students got 16 marks, the passing mark for Question 5 and 78 (51.6%) students passed the Question 5. The mean for Question 5 is 15.4 (below 16), resulting a mean percentage of 38.5%, with the standard deviation of 8.611.

The coefficient of variation for this data is $\frac{8.611}{15.36} \times 100 =$ 56.1%.

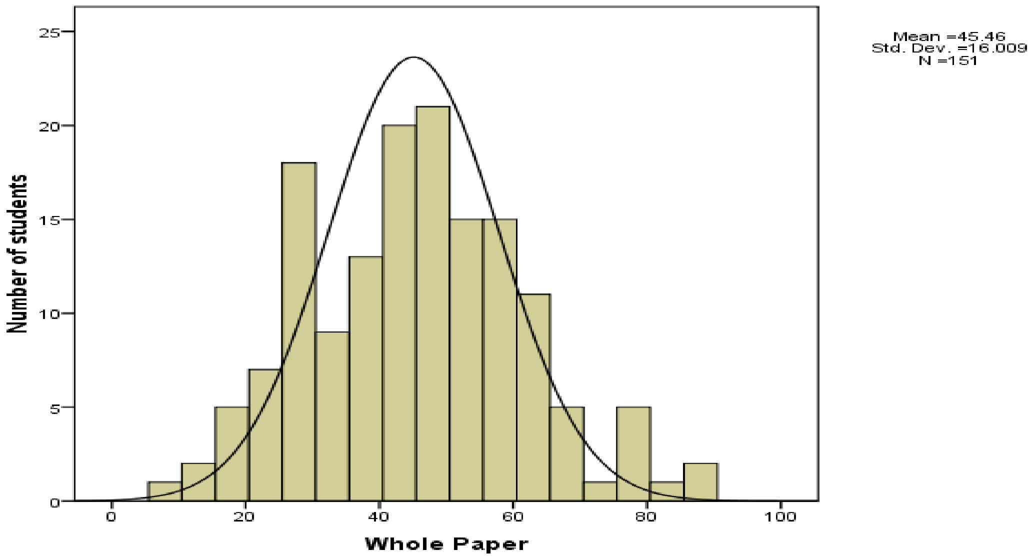

Figure 7.2: Performance in the whole paper

Figure 7.2 shows how many students obtained a particular score for the whole paper, ranging from zero to 100. It is evident from the histogram that no students scored zero for the whole paper whereas none of the students scored the total mark of 100. The lowest mark is below ten (8) obtained by one student and the highest mark is 90, also obtained by one student. The highest number of students (21) scored between 45 and 50 with no students scoring below 5 and above 90. The majority of the students, 99 (65.6%) passed the whole paper. Students performed better in the whole examination (65.6%) than in Question 5 (51.6%). The low percentage obtained in Question 5 in relation to the whole examination implies that Question 5 was difficult.

The mean for the whole paper is 45.5, resulting in a mean percentage of 45.5% (higher than that of question 5) with a standard deviation of 16.009. The coefficient of variation for this is $\frac{16.009}{45.46}\times 100 = 35.2\%$. Compared to the coefficient of variation obtained in Question 5, it shows that there is less variability (data less dispersed) 35.2% in the marks obtained in the whole paper compared to those obtained in Question 5 with the coefficient of variation being 56.1%.

7.2.2 The scatter plot for students' performance

In Figure 7.3 a scatterplot is used to show the association between the marks obtained in Question 5 (out of 40 with a passing mark of 16) and the marks obtained in the whole papers (out of 100 with a passing mark of 40) which are presented in Appendix 6B. The passing mark is used to separate the students according to specified quadrants A, B, C and D.

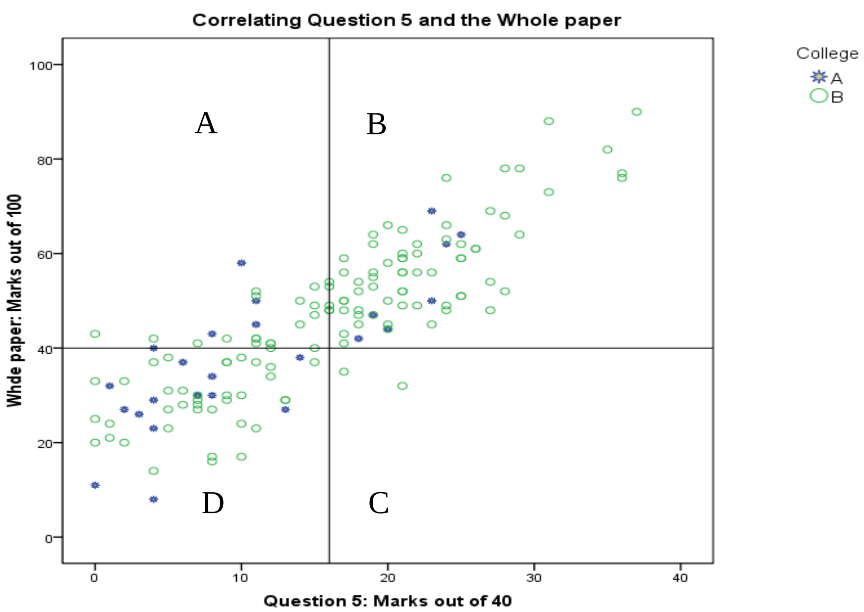

Figure 7.3: Scatterplot on Question 5 and the whole paper

Comments on the performance in terms of the quadrants A, B,C and D are made in Table 7.4 (* representing students from College A and o representing students from College B).

Table 7.4: Displaying performance in the four quadrants

Quadrants	Students performance	Ratio and % of students
A	Those who failed Question 5 and passed the whole paper.	23 = $\frac{23}{151}\times 100$ = 15.2%
B	Those who passed Question 5 and passed the whole paper.	76 = $\frac{76}{151}\times 100$ = 50.3%
C	Those who passed Question 5 and failed the whole paper.	2 = $\frac{2}{151}\times 100$ = 1.3%
D	Those who failed Question 5 and failed the whole paper	50 = $\frac{50}{151}\times 100$ = 33.1%
		100%

The majority of the students (76), that is 50.3%, fall in quadrant B, which means that there is a tendency that those who pass Question 5 tend to pass the whole paper and vice versa. The second group, comprising 50 students, that is 33.1% fall in quadrant D, which means that there is again a tendency that those who fail Question 5 tend to fail the whole paper and vice versa. The third group is in quadrant A (23) that is 15.2% representing those students who failed Question 5, yet passed the whole paper. In quadrant C there are two outliers, about 1.3% (a very small percentage), who passed Question 5 but failed the whole paper, scoring less than 40 per cent in the whole examination.

On the scatter plot there are scores that lie on top of one another and cannot be identified individually on the scatter plot as some students obtained the same marks. For example two students obtained 25 out of 40 in Question 5 and 51 out of 100 in the whole paper from quadrant B and two students obtained 12 out of Question 5 and 41 out of the whole paper from quadrant A (refer to Appendix 6B). There are also those students that lie on the border line. There are four students that are on the border line of quadrant A and B, representing about 3%. These students are included in quadrant B since they obtained 16 marks for Question 5 and passed the whole paper. The two students that lie on the border of quadrant A and D representing about 1% are included in quadrant A because they failed Question 5, yet scored 40% for the whole paper, which is regarded as a passing mark.

The majority of the students in quadrants B and D (83.4%) justifies that performance in Question 5 and performance in the whole paper are related. That means those who perform well in in this question are likely to perform well in the whole paper and vice versa. The opposite is also true in that those who perform poorly in this question are likely to perform poorly in the whole paper and vice versa. Only in exceptional cases, like the two outliers, was it found that such students passed Question 5 but failed the whole paper. In addition to that, those outliers scored very low marks, 17 and 21 respectively out of 40 for Question 5 (refer to Appendix 6B). The marks obtained by the best performing student, 37 for Question 5 and 90 for the whole paper and the worst performing student, 0 for Question 5 and 11 for the whole paper, also justifies the association between Question 5 and the whole paper. Up to this point the direction of this association of Question 5 and the whole paper is not known, until tests for causality are done.

In the section that follows, the marks obtained by the individual students in Question 5 are correlated with the whole paper and the level of significance of that correlation is discussed.

7.3 The Pearson's correlation and the level of significance for the 151 students

In correlating the marks obtained by a student out of 40 in Question 5 and the marks out of 100 obtained in the whole paper, it was found that there was a very strong positive correlation, $r = 0.852$, $p < 0.001$ between the marks obtained in Question 5 and the whole paper, which is highly significant. What this correlation means is that high marks obtained in Question 5 are associated with high marks obtained in the whole paper. It also means that low marks obtained in Question 5 are associated with low marks in the whole examination paper.

This correlation means that performance in Question 5 is strongly positively correlated to the performance in the whole paper, hence their association is statistically significant at the 1% level. There is therefore convincing evidence against the null hypothesis that the correlation coefficient is zero. We have confidence that such a big correlation coefficient for the 151 students did not occur by chance, hence such an association is genuine. An assumption that could be made based on the above correlation and its significance is that the majority of the students who pass Question 5 tend to pass the whole paper and that the majority of the students who fail Question 5 tend to fail the whole paper, rejecting the null hypothesis that: *There are no associations in performance between Question 5 and the whole paper*. We can conclude that there is a strong association between Question 5 and the whole paper. There is overwhelming evidence against the null hypothesis.

In order to measure how much variability in the performance in the whole paper may be explained by the performance in Question 5, the coefficient of determination is used. The coefficient of determination (R^2) is a measure of the amount of variability in one variable that is explained by the other variable (Field, 2005: 128), which is always positive. In this case it is $(0.852)^2$, which is equal to 0.73. This value (converted to percentage) means that performance in Question 5 accounts for 73% of the variability in the performance for the whole paper. The other 27% may be accounted for by other variables that do not relate to performance. For an example, the socio-economic factors that affect the student might affect how one performed during the examination.

7.3.1 Conclusion from the parametric tests

Conclusions can be made for the analysis of the 151 responses from the above sections. It can be concluded from the scatter plots, from the correlations of Question 5 and the whole paper, where $r = 0.852$, $p < 0.001$ and the correlation of the five elements as well from their significance level, that there is convincing evidence that (a) There are associations in performance between Question 5 and the whole paper and that (b) There are associations in performance between the different elements, from Question 5. The findings are that the performance in Question 5 accounts for 73% of how one performs in the whole paper. Half of the students, 50.3% who passed Question 5 passed the whole paper and 33.1% of the students who failed Question 5 failed the paper as evident from the four quadrants. This justifies without any doubt that 83.4% of the associations results in a strong positive linear relationship.

This justification is further emphasised through the correlation of Question 5 to the whole paper using Pearson's correlation coefficient and the average ranks of the five elements from Question 5 using Kendall's tau correlation coefficients, which also showed statistically significant correlations at 1% level. These highly significant correlations are as result of the subquestions in question 5 where solution for the first subquestion is used in finding the solution for the subsequent questions. As a result, failure in the first question, which in most cases involves drawing graphs, leads to failure in the subsequent questions, relating to interpretations of the drawn graphs.

CHAPTER 8: OBSERVATIONS AND AN INTERVIEW

This chapter presents the description and analysis of the data collected from the classroom observations focusing mainly on the lecturer and a group of students selected, including an interview with a former N6 student. The results of the observations and the interview confirm or contrast the results obtained in Chapter 6. The data collected is described qualitatively in terms of what was said or done (narratives, verbal or written) by the lecturer and the selected group of students as well as the single interviewed student in relation to the five skill factors discussed in Chapter 3. The results are presented with selected extracts from the classroom observations and excerpts from the interview. For the classroom observations, the way in which the content learnt was introduced, the use of procedural knowledge and conceptual knowledge, the level of difficulty of the content and the assessment strategies are discussed. The interpretations of these results are presented in Chapter 9.

8.1 CLASSROOM OBSERVATIONS

One N6 classroom with about 40 students from College A was observed for 5 days from Monday 15th October to Tuesday 23 October 2007, focusing on what was said and done. In the presentation of the results, extracts from the classroom observations are used and analysed in terms of the five skill factors. In-depth discussions are presented for lessons one and five only, since they are regarded as the main lessons. In the first lesson the foundation for VSOR was laid and in the fifth lesson students were working in groups throughout the lesson, thus indicating what was learnt from the previous lessons. Lessons 2, 3 and 4 are discussed with few extracts where necessary to corroborate, justify, contradict or augment what happened in the first lesson. The lecturer is referred to as L, students as STs and single students as S_1, S_2, … and the researcher as R. In the extracts that follow, for example, 01:44 indicates that an event occurred 1minute 44 seconds after the lesson was introduced. The words used by the lecturer and the researcher appear in normal font. The students' responses are in italic format, while the reactions and comments for all participants are in square brackets.

8.1.1 The first lesson

8.1.1.1 Observing the lecturer and the students in lesson 1

- **Introducing the content to be learnt**

The lesson was introduced with the topic 'Application of integration' written on the board.
Below an extract of the introduction of the lesson is presented.

> 00:00 to 01:44 (74 seconds)
> L: …you did application of integration. You apply integration to calculate the 'area under a curve' [repeats] a curve… you did that at the previous level [referring to N5 level] … at N1 at N2 level, you did the chapter called mensuration ... you were measuring, calculating, perimeter, area, volume, and distance around the figure … volume = Area x height. That is the sequence, you do perimeter, area, and you do volume … but please allow me to start with volume, we will do area after volumes, neh!

The lecturer referred the students to the graphical and visual representation when saying that "you apply integration to calculate the area under a curve". The curve refers to what has being drawn, and can be visualised. The lecturer referred the students to a section on mensuration from the previous year, elaborating on the formula for volume, as volume = Area x height, hence emphasising the procedural skills.

The lecturer introduced a section on volume by introducing students to shapes.

> 01:44 to 02:40 (56 seconds)
> L: Now, for us to can do this section on volumes … we need to understand just few things, 'the shapes' [repeats three times]… we will be doing volumes, since we apply this on graphs, it will be a volume of a solid which is rotating about the *x*-axis or the *y*-axis. But as this area is rotating it formulates a particular shape, hence I want us to look at the shapes first, because those shapes inform us of the formula to use when we calculate the volume.

The lecturer emphasised that the students must understand shapes, pointing to the fact that shapes are important in learning volumes and that the formulae that the students will use in calculating the volumes will be derived from those shapes. The shapes in this case relate to the visual representation and they can be presented as diagrams. The lecturer mentioned that, rotating the area about the *x*-axis or the *y*-axis gives rise to a particular shape. During this rotation the translation is from 2D to 3D. From his statement: "shapes inform us of the formula to use when we calculate the volume", the lecturer was relating to the translation from the visual graphs to the algebraic equations for volume.

The lecturer also made use of the terminology and concepts required in this section, such as 'application of graphs', 'volume of a solid' and 'rotation' about the *x*-axis or the *y*-axis. The lecturer did not ask students how volumes are formulated, but explained to them that the shapes that one gets after rotation are important. The word, 'volume of a solid' was used but was not explained to the students, nor demonstrated. Application of graphs may be related to the way in which students interpret graphs (translating them from visual to algebraic or translating them from continuous to discrete). The volume of a solid is related to a 3D diagram that could be formulated by translating from 2D to 3D, while rotation refers to a skill that the students can use when they translate a 2D diagram to formulate a 3D diagram visually or by imagining it.

The lesson continued with the lecturer writing 'Disc' on the board. An example used involved a graph of a parabola and a straight line graph on the same Cartesian plane drawn on the board with a Δy representative strip as shown in Figure 8.1. During this lesson the lecturer was asking questions and the students were responding. The students' responses as a chorus are given in the normal brackets in italic format followed by the lecturer's responses in regular font if he repeats what the students said, while the reactions and comments for either the students or the lecturer are given in the square brackets.

02:40 to 04:36

L: [Write on the board 'Disc' and draw the two graphs with a horizontal strip, Δy]. If we have this two functions, $f(x)$ and $g(x)$ [facing the class] …what type of graph is the … $g(x)$? (*parabola*) parabola…and $f(x)$? (*straight line*) straight line graph… what are the co-ordinates of the turning point of the parabola (*4 and zero*, [*4;0*]), 4 and zero [4;0] neh!? [lecturer not puzzeled] … hhh!, (*zero and zero*, [*0;0*]) zero and zero [0;0]. Is this turning point having a minimum or maximum value? (*minimum*), [pointing on the graph's turning point] is the a value positive or negative? (*positive*) positive.

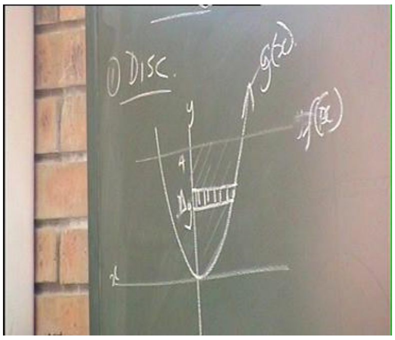

Figure 8.1: Example 1 graphs

The drawn graphs represent the graphing skills even though the equations of the graphs were not given; hence there was no translation from algebraic to visual in 2D. The Δy strip (drawn to represent the shaded area) indicates the translation from continuous to discrete representations. The reason why a Δy strip was drawn was also not discussed with the students. It was impossible in this case to detect whether the students were able to draw those graphs or either identify the correct strip. The lecturer did not start by giving the equations of the graphs and translating from algebraic to visual skills in 2D or using the general manipulation skills to find the important points on the graphs. It is not explicit whether the students still remembered what the word 'disc' meant as the lecturer did not explain what it meant or revising it since it is a concept that was learnt from the previous level. The general manipulation skills were applied when students gave the coordinates of the turning point of the parabola as 4 and 0 before a correct answer of 0 and 0 was given. A question relating to visual skills in 2D was also asked when the students were asked to determine whether the turning point of the given parabola was a minimum or maximum value.

In relation to the representative strip, representing the shaded area, the lecturer asked the students whether the representative strip drawn, is parallel or perpendicular to the *y*-axis. The lecturer referred to it as a Δy strip and related it to transformation as it rotates. The use of transformation was to enhance the visual skills and translation from 2D to 3D for the strip only and encouraging a form of imagination.

04:36 to 05:36
L: …if we have the area bounded by the graphs $f(x)$ and $g(x)$ and the *y*-axis, this is the ... in the 1st quadrant …and we have a representative strip [show them on the board]. This representative strip, is it parallel or perpendicular to the *y*-axis? (*perpendicular to the y-axis*); we call it a Δy strip. If it is perpendicular to the *x*-axis we call it? … (Δx) Δx, ok. Now if this representative strip, which is representing this area is rotating [demonstrate with a finger pointing up, and rotating anticlockwise] about the *y*-axis we talk about transformation, do you know transformation. [No response from students].

The lesson continued with the lecturer demonstrating transformation to the students as shown in Figure 8.2.

05:36 to 06:40
L: If for an example, this hyperbola, [shown alongside].A hyperbola is made up of two curves [pointing to the top graph and the bottom graph] this curve [pointing to the bottom one] is the mirror image of this one [pointing to the top graph]. Meaning that this reference line [referring to a dotted line passing through the origin] is the mirror image line, meaning that the distance from this point [showing points on the top graph and the bottom graph] up to this point will be the same to the distance from this point to that point … we have equal distance… now, that is transformation.

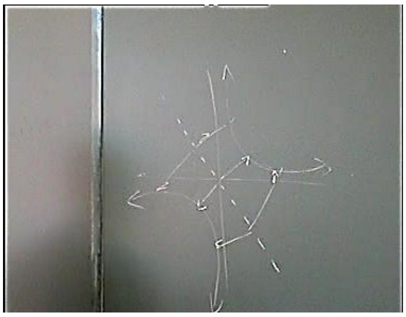

Figure 8.2: Transformation

From the above extract the lecturer was trying to emphasise the visual representation using transformation of the hyperbola showing the line $y = x$ as an axis of rotation as well as the graphing skills, without starting with the equation for the hyperbola.

In Figure 8.3, the lecturer reverted to the first example.

06:38 to 07:30
L: Now when we rotate for example a representative strip [back to the drawn graphs], if you rotate it about the *y*-axis, it means that the length of this representative strip from the ... this point [finger pointing at the origin as in the diagram], will be the same. Now if it is rotating, it is rotating like this, [with a finger moving anticlockwise and pointing up]. What are we going to have (*a circle*), [many students whispering] circle …but this axis [pointing on the *y*-axis] as our reference line, we will be having sort of a hole like this [pointing at the strip on the y-axis] … it will form sort of a ... [no response, lecturer finishes up] … sort of a disc [lecturer draws a disc alongside]. It rotates about our reference line.

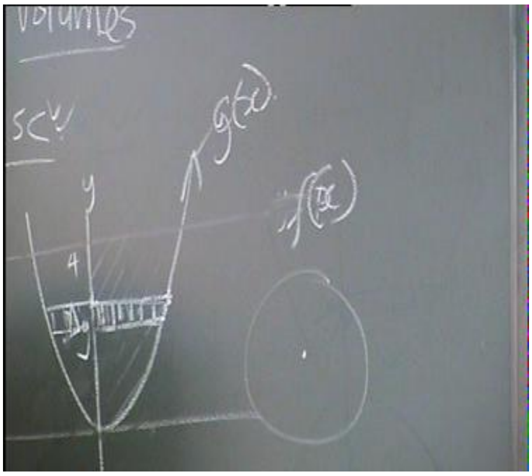

Figure 8.3: The disc method for example 1

The formation of the disc resulting from rotating (demonstrating with the finger) the given Δy strip about the *y*-axis relates to the translation from 2D to 3D, where visual skills are used to aid imagination in order to demonstrate a ‘circle’ referring to a disc in relation to the length of the representative strip.

In the section that follows the lecturer continued with the same graphs and the same Δ*y* strip but then rotated about the *x*-axis, as the second example as in Figure 8.4.

> 07:30 to 10:50
> L: … if we can take this parabola and this other function [draw the original graphs with a Δ*y* strip] this representative strip, we rotated it about the *y*-axis [referring to the first example] and a disc, was formed. What then happens if the same representative strip is rotated now about the *x*-axis? …if it is rotating about the *x*-axis [lecturer demonstrates a cylinder shape with fingers being horizontal and rotating, both left and right fingers pointing towards one another and drew a Δ*y* strip below the *x*-axis]. Do you see what will happen? The distance from ... [showing distance from the strip to the bottom of the parabola at turning point (0;0)] … we are having our mirror image line, our reference line is our *x*-axis, meaning that this graph [draw a parabola facing downwards] … the distance from this point to that point [pointing on the top graph and then on the bottom graph, are equal distances from the graph to the *y*-axis], … the distance from the strip to the *x*-axis must be equal distance [shows below the *x*-axis] and the length of the representative strip must also be the same. Now it must rotate. What is going to be formed here? (*a cylinder*) a cylinder [also demonstrating with a glass and a diagram].

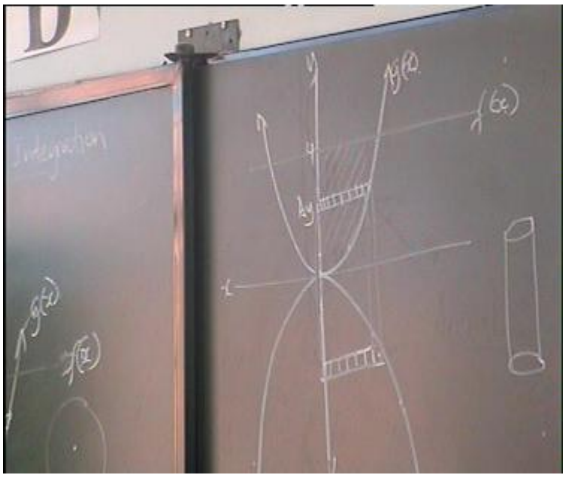

Figure 8.4: The shell method for example 2

The lecturer related the first example, about rotation of the Δ*y* strip about the *y*-axis to the second example, about rotation of the same strip about the *x*-axis. The lecturer demonstrated the rotation by hands and through transformations and mirror images. When asking students what shape would be formed (emphasising the visual skills), they shouted out ‘cylinder’, while translating from 2D to 3D. An error was picked up as the lecturer drew the rotated cylinder on the board. Rightfully, the lecturer was supposed to have drawn a cylinder lying horizontally, not vertically, as if a Δ*x* strip was rotated about the *y*-axis. The students where then referred to the question papers that were given to them in the previous lesson to refer to the formulae from the formula sheet that would be used in this section. The fact that the students were given question papers before the lesson justifies why some students were able to shout out the correct answers. Probably, the students prepared before the lesson, hence knew the answers. The lecturer emphasised that the shape that is generated after the rotation of the strip, informs the students on which formula to choose from the formula sheet.

> 10:50 to 12:59
> If you went through the question papers you were always finding the question: Use the so called shell method to calculate the volume generated when the area bounded by ... so so so … and the graph ... so … is rotating about the *x*-axis or the *y*-axis, what then will be that shell method, it will be the method that we use to calculate the volume generated when rotating [refers to strip in the second example].This means that same graph, you can calculate the area, you can calculate the volume generated … if it is rotating about the *y*-axis, the disc is formed [relating to the first example]. From page no. 3 of your formula sheet (students are paging) there are lists of formulae, for volume… so that we must know, where to get what? Now if the question says, calculate the volume generated … you must pick up the formula, what informs you what formula you must choose, must be the shape after drawing the graph.

The lecturer referred the students to the way in which the questions are asked and in conjunction with the students chose the formula from the formula sheet to be used to calculate the volume. From the second example he referred to rotating a Δy strip about the x-axis resulting in a shell and in the first example related to rotating a Δy strip about the y-axis resulting in a disc method from the shape formed, hence emphasising the visual skills. The lecturer advised the students to draw the graph(s), relating to the graphing skills, and then rotated the area bounded by the drawn graphs after having drawn the representative strip, relating to translating from visual 2D to 3D, before they could choose the formula from the shape formulated after rotation, relating to the translation of the shape (visual) to the algebraic equation (for volume) in 3D. In this case, the lecturer was leading the students to get engaged in the graphical representation and the algebraic representations. Below an example of how the equation was selected from the formula sheet when translating from the visual graphs (not drawn in 3D) to the algebraic equation for volume is given.

> 13:23 to 14:13
> L: … if the question says, calculate the volume generated when the representative strip is rotated about, one the y-axis [referring back to the first example] … $V_y = \pi\int_a^b x^2 dy$ [from the students]. Why Δy? Because your representative strip is perpendicular to the y-axis, neh!. Are you fine with that? (*yes*)

The lecturer used the first and the second examples to demonstrate rotation from a Δy strip about the y-axis and about the x-axis respectively, relating to translation from 2D to 3D only visually from the rotated strip without drawing the solid of revolution generated. The formula given, $V_y = \pi\int_a^b x^2 dy$ illustrating the translation from visual to algebraic in 3D, was demonstrated by the lecturer as a circle drawn next to the graph as in Figure 8.3. When identifying a representative strip, it is not explicit whether the students knew when to use a Δy strip or a Δx strip, or both or when a certain strip cannot be used.

These differences were not clarified to the students during the lesson that was done on the previous level. Perhaps the lecturer could have used the same graphs (as in the first and the second examples), to demonstrate what happens if a Δx strip was rotated about the y-axis and about the x-axis, stemming from the Riemann sums, hence enhancing the translation from continuous to discrete representations visually in approximating the shaded area. If a Δx strip was rotated, from the graphs based on the first example, students could have probably been introduced nicely to the washer method upon rotation about the x-axis, since the strip does not touch any axis. They could also relate to the cylinder upon rotation about the y-axis. This type of in-depth way of teaching by using different strips on the same graphs and using different rotations might help students in knowing why a certain strip might fail in certain graphs and that the way you choose and rotate a strip determines which method you will use to calculate the volume generated, be it disc, washer or shell.

When using a Δx strip, the lecturer used a different example, an exponential function given below in Figure 8.5 as the third example. The third example was used to aid students in selecting a formula for volume using a Δx strip upon rotation about the x-axis, resulting in a disc method.

14:14 to 16:32

L: Now … when the representative strip is parallel to the *y*-axis or is perpendicular to the *x*-axis, … see what our formula will be like. If it were an exponential function [draws an exponential function]. Now, look at this figure. If this representative strip [referring to the drawn Δx strip] … are we going to write Δy or Δx? (Δx), why Δx?… because the representative strip is perpendicular to the ….[student finish up] (*x-axis*), then we have Δx. If representative strip is rotated about y-axis, what shape will be formed? (*cylinder*) … and which method will you use to calculate volume (*shell method*). If the representative strip is rotated about the *x*-axis, which shape will be formed? (*disc*). It will be a disc. Then what will be the formula? You are calculating the volume which will be generated when the representative strip is rotated about the *x*-axis. The formula to calculate volume, it will be V_x because the representative strip will be rotating about the *x*-axis it will be $V_x = \pi\int_a^b y^2 dx$ … Δx lower limit (*a*) upper limit (*b*). Why Δx, ... ok ... now why y here? ... the length of the representative strip, why *x* here, the length of the representative strip (referring to the first example).

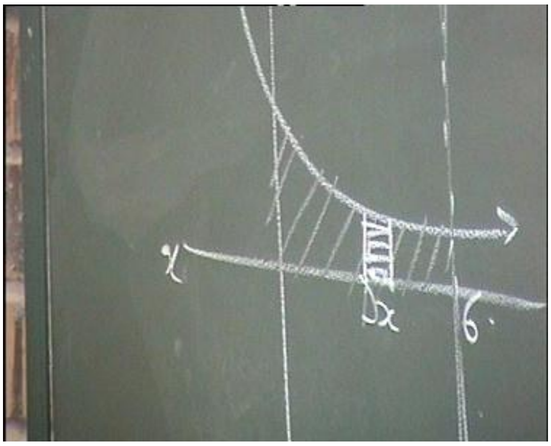

Figure 8.5: The disc method for example 3

From the above extract, again graphical skills were displayed by the lecturer for drawing the two graphs (the exponential function and the line $x = 6$) and again without the translation from algebraic to visual in 2D since no equations were given. The Δx representative strip drawn indicates the translation from continuous to discrete representations. The lecturer emphasised the location of this particular strip in relation to the *x*-axis or the *y*-axis. That is, if the strip is parallel to the *y*-axis, it is at the same time perpendicular to the *x*-axis, hence enhancing the visual skills. Further emphasis on what shape was to be formed as the representative strip is rotated about the *y*-axis and about the *x*-axis, relates to translation from 2D to 3D also enhancing the visual skills. It was never mentioned why a Δy strip was not used and what it means in terms of approximating the bounded region. The formula $V_x = \pi\int_a^b y^2 dx$ for a disc was used to represent rotation about the *x*-axis, hence translating from the visual graph to an algebraic equation in 3D.

The lecturer reverted to the second example (Figure 8.4) to select a formula for volume using a Δy strip rotated about the *x*-axis, resulting in a shell method after rotation, hence translating from visual to algebraic in 3D.

> 16:40 to 18:48
> L: To calculate volume here [referring to the second example] you will use shell method, which formula are you going to use? … If it is rotating about the (*x-axis*) the volume will be $V_x = 2\pi \int_a^b xydy$ why Δy…because the representative strip is parallel to the *y*-axis [an error from the lecturer]. If this strip [referring to the third example] is rotated about the *y*-axis, to calculate the volume, what will be the formula $V_y = 2\pi \int_a^b xydx$ ...

In introducing the washer method, the lecturer referred the students to the distance between the strip and the other axis and not touching a particular axis.

> L: For all … graphs … the representative strip is on the axis [showing students examples 1, 2 and 3 used that the representative strip is, either on the *x*-axis or on the *y*-axis].

The fourth example involving the 1st quadrant region bounded by a hyperbola of the form xy = k and a straight line y = mx + c was used, where the Δx strip drawn does not touch the *x*-axis or the *y*-axis, resulting in a washer method after rotation about the *x* - axis, as shown in Figure 8.6.

> 18:49 to 19:46
> L: Sometimes, the representative strip can be lifted, it can be on the Cartesian plane, not on the axis, neh!… [Draw two graphs]. We have our area there [pointing on the shaded area]. Then we can draw our representative strip either perpendicular to the *x*-axis or parallel to the *x*-axis. Let's do it this way…we are going to have here [drew a Δx strip on the graph perpendicular to the *x*-axis] Δx neh! [pauses].

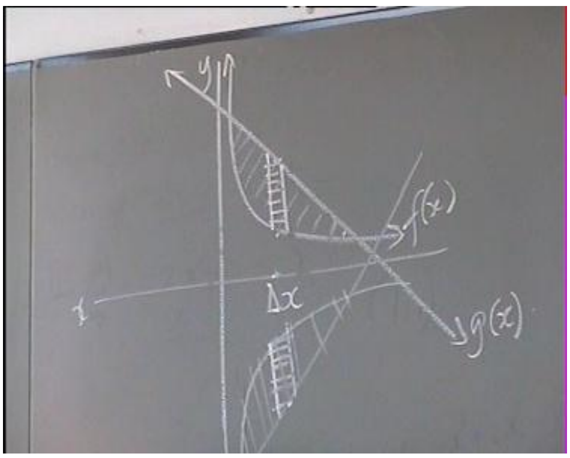

Figure 8.6: The washer method for example 4

The lecturer articulated that the drawn representative strip can be rotated about any axis, also emphasising the new shape to be formed.

> 19:50 to 21:25
> L: … this representative strip can be rotated either about *x*-axis or *y*-axis. What will be the shape formed when … rotated about the *y*-axis? (*cylinder*), a cylinder … meaning that it will be the shell method, neh!. Now, if it rotates about the *x*-axis? [as shown in the figure, demonstrating with a finger], which shape will be formed? (*washer*) washer. Meaning that the distance from here up to here [*x*-axis to the bottom of the strip] will be the same as the distance from the … [pointing below the *x*-axis, and draws the rotated graph]. This is the mirror image as it is rotating.

The lecturer demonstrated how an annulus (a disc with a hole), shown in Figure 8.7 is formed, when a strip is lifted, enhancing the visual skills and translation from 2D to 3D.

> 21:28 to 22:25
> Somewhere … there is this very small hole [referring to the first example for the disc] …but what then happens… because it rotates this way [demonstrating rotation about the *x*-axis with hands], that means that there will be a bigger hole [repeats] … is the length of the representative strip. Meaning that [shade] this is our area [pointing at the shaded area], what do we call this shape [no response] this shape is called an 'Annulus' ok.

Figure 8.7: The annulus

The drawn annulus was further used to come up with the formula for a washer, discussed below, hence translating from the visual graph to the algebraic equation in 3D.

> 22:42 to 24:08
> … how do we get the magnitude of this area [referring to the shaded area] we get the of the area of the bigger circle, minus the area of the smaller circle. The area of the annulus is … area is pie r squared [πr^2] = area of bigger circle – minus area of smaller circle … [$\pi R^2 - \pi r^2$] you take out pie as a common factor … and we say $V = \pi(R^2 - r^2)$, that volume is area multiplied by the perpendicular height, where in the perpendicular height in this case will be the length of the representative strip … that will be Δy or Δx. That is where you do the volume. But this is a constant [referring π] we take it out of the integral sign, when we do the volume. If it is rotating about the *x* axis, it will be upper limit, lower limit, $V = \pi\int_a^b (y_1^2 - y_2^2)\,dx$ … the length of the representative strip [referring to $R^2 - r^2$] … can be, change in the *y* value or change in the *x* value.

The lecturer worked towards the formula for volume of a washer in conjunction with the students and explained what the different variables in the formula meant, by interpreting the drawn graph. In interpreting the graph, they translated from visual graph to an algebraic equation. The lecturer was referring the students to the different types of graphs (the top as y_1 and the bottom as y_2) relating to the representative strip and the names of the graphs, hence translating from the visual graph to the algebraic equation resulting in the formation of the annulus as a result of rotating the representative strip from 2D to 3D. The annulus was later referred to as a washer. As it was the case in the previous exercises, the lecturer emphasised the axis of rotation as the reference line. Students were referred to the formula sheet on a regular basis to choose the formula to use relating to the washer. The lecturer explained the washer method by demonstrating the location of the strip in relation to the top graph and the bottom graph, hence translating from the visual graph to the algebraic equation in 3D as discussed in the extract below.

> 24:10 to 27:32
>
> L: So it can be $V_x = \pi\int_a^b (y_1^2 - y_2^2)\,dx$ … which one is y_1… which one is y_2, you are going to say the top minus bottom …you subtract the smaller one from the bigger one ... if it is rotating about the *x*-axis, it means that the *x*-axis must be your reference line … you are going to move from the *x*-axis you going up … the 1st graph you are going to meet will be the smaller one , so it will be the top one minus the bottom one [referring to example 4, top being straight line and bottom being hyperbola] … so it will be top minus bottom. That formula is it on page 3? or 4?.(*page 5*) page 5 [moves toward the students].. [the lecturer looks at the formula sheet and pauses].If… if [moves back to the board] this representative strip is perpendicular to the y-axis [drawing a Δy strip on the diagram used for example 4]. If it rotates about the *x*-axis, which shape is going to be formed? …. (*cylinder*) cylinder. Which method do we use? (*shell method*) shell method. But, if it is rotating about the y-axis are we going to have a shhhh, a disc or cylinder? [no response, lecturer repeats] (*none of the above*) none of the above, neh! What are we going to have? (*annulus*) We are going to have an annulus, which is also called (*washer*) [few students responded] a washer , are you fine with that. What will be the formula, if it is rotating about the y-axis.
>
> $V_y = \pi\int_a^b (x_1^2 - x_2^2)\,dy$ [students telling the lecturer what to write].You have this formula on page no5 [moving towards the students]. I think this are the basics that you have to know to answer the questions, on this section, the volumes [pause, and moves towards the board].

An example was given (from the question paper), where the graph had to be drawn first before calculating the volume (graphing skills). The representative strip had to be selected first before rotation (translation from continuous to discrete). The lecturer worked with the students to find the intercepts and the turning points of the graph (general manipulation skills).

> 27:53 to 30:50
>
> L: If we have the question which goes [writes the question on the board]. Determine the volume of a solid generated when the curve $y = 2x - x^2$ and the *x*-axis rotates … when area between the curve , this one and this one [pointing at the given equations, on the board] rotates about one(a) the *x*-axis and two (b) the *y*-axis ... where do you start? (*draw the graph*) you draw the graph [emphasising]. You all know how to draw a parabola, neh!….is that a parabola [students hesitant, not clear what they say]. Is it a parabola? (*yes*) …are you sure (yes) yes it is a parabola. Now, what do we then do, to determine your intercepts, *x*-intercepts and *y*-intercepts? *y* intercept:0, *x*-intercept: you let $y = 0$ therefore you take it to the right and take out the common factor, meaning that your intercepts will be (*x*-intercept: 0 and 2) [students and lecturer responded simultaneously]. Coordinates of the turning points? There are about 2 methods which you can use; one, you find the derivative of a function, you let $\frac{dy}{dx} = 0$ and then you find the value of *x* and you substitute in the original equation. Or you can use the formula $-\frac{b}{2a}$ and $f\left(-\frac{b}{2a}\right)$. What you get …*x*-axis ($\frac{1}{2}$) *x* is 1?, ($\frac{1}{2}$ *and 2*) [another student] 2 …1; and what? ($\frac{1}{2}$) [One student says it loudly, female], *x* is $\frac{1}{2}$? (*1 and 1*) [the argument with turning point carried on for about 37 seconds, then it was finally agreed that $x = 1$ and $y = 1$] turning point at (1;1). So basically, we try and draw the graph.

The lecturer drew the graph shown in Figure 8.8 on the board without involving the students. The whole class contributed towards the calculation of volume, before the students could work individually or in groups. That was done from 31:30 to 35:42 (4 minutes 12 seconds)

> L: So this area between this curve and the *x*-axis … we can have our representative strip Δx neh? [no explanation why Δx is used and not Δy]. Now the first question is (a) if it rotates about the *x*-axis … what will be the volume it will be $V_x =$ [the students and the lecture together] it will be … what will be the shape formed … (*a disc*) a disc. Then you choose the formula. What will be the formula? What are our limits? [0, 2]

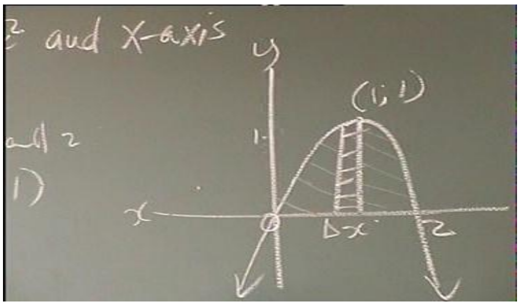

Figure 8.8: The parabola using a Δx strip

The lecturer explained to the students how the *y* value from the formula is replaced through substitution (translating from visual to algebraic). The lecturer in conjunction with the students used general manipulation skills while squaring the *y* in order to evaluate the whole integral, replacing *y* by *x*.

> L: The fact of the matter is that we cannot integrate this *y* with respect to the variable *x*, … $y\Delta x$ but only if we have *x* here [meaning substitute *y* for *x*]. Now it will be pie into *f* (upper limit) – *f* (lower limit). F of (lower limit will be zero because of this 0 [referring to the *x*-axis]. Then … we have (a) $V_x = \pi\int_0^2 y^2 dx$ can you square that *y*?, $2x - x^2$ all squared [lecturer writes, while it is not clear what the students says], $\pi\int_0^2 (4x^2 - 4x^3 + x^4)\, dx$, the fact of the matter is that we cannot integrate this *y*, with the variable *x*, meaning that we have to express this y in terms of *x*.

The lecturer in conjunction with the students continued to solve the problem (using general manipulation skills), applying the rules of integration. The application was done successfully until the volume was obtained using the disc method for problem (a), as in Figure 8.8.

> L: Then lets integrate [students shout] $= \pi\left[\frac{4}{3}x^3 - x^4 + \frac{x^5}{5}\right]_0^2 =$ now … upper limit minus lower limit … Can someone do it for us? … you should get the answer as 3.351, did you get it? 3.351 neh! after multiplying by pie (*yes*) … it is volume the unit must be cubed, if it is area it was going to be squared.

The lecturer started the second question (b) where the shell was formed after rotation (translating from 2D to 3D), but did not complete it. The students had to substitute for *y* (translating from visual to algebraic) as they were shown in Question (a) with the disc method. One is not sure whether they succeeded or not since the solution was not done in class.

> L:The next question [b] says calculate volume when this representative strip is rotating about the *y*-axis. What is the shape that is going to be formed? (*Cylinder*), cylinder neh! the formula? $V_y = 2\pi\int_0^2 xy\,dx$, but he points at *y*, must be expressed in terms of *x*, so there is where you are going to put in this $2x - x^2$ and multiply then integrate, you use upper limit minus lower limit, you find the volume, and it must be in cubic units, neh!

In the section that follows, other aspects that were evident during the lesson and had an impact on the 11 elements are discussed as follows: procedural knowledge; conceptual knowledge and the level of difficulty of the content and assessment strategies.

- **The use of procedural and conceptual knowledge, level of difficulty of the content and assessment strategies**

All problems relating to procedural knowledge involved general manipulation skills and graphing skills. The lecturer in conjunction with the students had to find the *x* and the *y* intercepts as well as the coordinates of the turning points in cases where a parabola was used. After selecting the formula from the formula sheet, the substitution method was used when replacing *x* by *y* or replacing *y* by *x*, before evaluating the integral. The last part of the lesson was based on evaluating the integral, with the lecturer taking a leading role. For graphing skills, everything was procedural in nature since the graphs used during the lesson were ready made. The graphs used were familiar graphs like parabolas, straight lines and an exponential graph which are easy to draw without complex calculations. However, one cannot comment whether students would have drawn them correctly or not, since the lecturer drew them on the board.

Problems relating to conceptual knowledge involved graphing skills, translating from continuous to discrete, translating from 2D to 3D and translating from visual to algebraic in 3D. It was not possible to comment about the graphing skills and the translation from continuous to discrete (visually) since the lecturer drew the graphs and identified the strip without involving the students. There were many instances where the lecturer emphasised visual skills when relating to the position of the strip, whether the strip was parallel or perpendicular the a particular axis, and how it rotates about a certain axis, hence enhancing the conceptual understanding in relation to the representative strip by translating from 2D to 3D.

In more than two thirds of the lesson, the lecturer demonstrated the conceptual ideas through rotation from 2D to 3D by showing them a disc (circle), a shell (cylinder) and a washer (annulus); even though the students were not given an opportunity/time to think critically about the different shapes after rotation and to draw a solid of revolution generated. The lecturer emphasised how the formula was derived, resulting from the diagram (translating from visual to algebraic), only for the washer. In other cases, the lecturer explained to the students without deriving the formulae visually that after rotating a representative strip, a formula can be selected from the formula sheet based on the shape formulated, for a disc and for a shell.

From the way in which the students responded to questions posed by the lecturer, an assumption was made that the students understood the lesson. From the way in which the lesson was presented, it seemed as if the lesson was not cognitively demanding, even though some gaps were evident. The gaps that I picked up during the lesson were that the lecturer did not involve students in drawing the first four graphs as well as explaining why a Δx strip or a Δy strip was used. While students were working in groups and individually, it was evident that some students had problems with the selection of the strip. Drawing of graphs and identifying the representative strip might have impacted on the level of cognitive difficulty in learning VSOR. One student asked the lecturer a question, in Setswana, translated in brackets in italic, relating to how a strip is selected, revealing the difficulties encountered. Comments are given in square brackets.

> 36:54 to 40:21
> S: (*how do I know that*) this [pointing at the strip on his diagram] … (*what do you call it … that the strip that I use, is like this or like this?*) [demonstrating with hands, vertical or horizontal]. What is it that is going to tell me (*that my representative strip is*) perpendicular to the *x*-axis or to the *y*-axis?
> L: I gave you a bundle of question papers. I was telling you … can you take out one.
> S: No I do not have one [this student did not look at, or use question papers before the lesson like other students did].

The lecturer moved away [apparently looking for question papers, and came back later]. I approached the student, in order to find out what exactly transpired.

> R: You are worried about this question? Both of them?
> S: (*My stress is the strip*), how do I know (*that the strip that I use, will be positioned like this or like this?*) [show with lands again, vertical and horizontal]. [Lecturer is back with the question paper and read the question to the student]
> L: Make a neat sketch of the graph … and show the representative strip that you will use to calculate the area. Now after drawing this two graphs, then your strip will either be perpendicular to the *y*-axis or to the *x*-axis: If you draw … and rotates about *y*-axis and then about *x*-axis you will find two different values, but if you draw it perpendicularly, if you rotate about the *y*-axis, when you rotate about the *x*-axis, you will find two values, these two values, rotate by *x* [referring to rotations with different strips, Δx or Δy] and these two values, rotate by *y* [referring to rotations with different strips, Δx or Δy] will be the same.
> S: (*it is up to me how I position the strip*) unless stated
> L: Yes, unless otherwise stated but usually they don't state.

The lecturer made it explicit to the student what the question entails, but still did not relate to the shaded region on how a strip is selected in order to approximate it. The lecturer continued to explain the difference between the strip being parallel or perpendicular to a certain axis (visual skills) as he did during the lesson and how it is rotated (translating from 2D to 3D) and the different values that one will obtain when calculating volume as a result of different rotations. The advice given to this student was that one can decide which strip one wants to use (meaning that any representative can be used, be it vertical or horizontal), hence ignoring the translation from continuous to discrete (visually) based on the approximation of the shaded region stemming from the Riemann sums. The lecturer did not relate the strip to be the one that would best approximate the area (the region bounded by the graphs) that would result in volume after rotation about a particular axis based on how the graphs are drawn or on the given equations, but rather emphasised the order in which the questions are asked. The lecturer also emphasised that when calculating area, using Δy or Δx strip, area remains the same, but if you rotate (translating from 2D to 3D), there will be different answers for volume. In this case the lecturer was coaching the students on how to attend to the examination questions. Below an extract is given.

> L: What the question will say, it will ask you to indicate the representative strip, after drawing the graph. …you find there are the 3 questions: One [1], the first question, will say calculate the point of intersection. Two [2], draw the graph, draw the sketch indicating the two graphs, clearly … indicate the representative strip that you will use to calculate the volume generated as it rotates about the y-axis or the *x*-axis. Let's go through this one, sketch the graph, 'show the representative strip or element that you will use' [emphasised and repeated]…to calculate the volume generated when the area bounded by the graph or *x*-axis …rotates about the *y*-axis: In other words you draw your own representative strip, so they do not indicate whether it is perpendicular to the *x*-axis or to the *y*-axis, hence they say: you will use. Area stays the same. But with volume is different, if it rotates this way [referring to rotation about the *x*-axis] is different shape, then it will have its own volume.

Despite the fact that the students were given the question papers before the lesson and were shouting out the answers, some students were not participating and seemed puzzled during the lesson. That could have emanated from the fact that some of the students did not know why a strip was drawn as Δy or as Δx. This was not discussed and clarified during the lesson. The emphasis was that the representative strip should be perpendicular or parallel to a certain axis. The emphasis was on the shape that the chosen strip generates after rotation about a particular axis, representing a disc, shell or washer, rather than how a particular strip approximates the region bounded by the graphs. The solid of revolution generated after rotating the bounded region was not drawn. Even if the lecturer showed in-depth knowledge of the content and explained this section properly, it was evident after the lesson that not all students had a proper basic knowledge from the previous levels.

The lesson ended while the students were working independently on four questions, while the lecturer was moving around to monitor their work. I observed that many students used a table method to plot graphs, which demonstrates the use of procedural skills. The lecturer gave students activities to work on in groups, but did not wrap up the lesson to summarise what was done. As a result there was no evidence of feedback and reflection.

In lesson one the students were taught for 36 minutes and 30 seconds in English only before they could work independently. The lecturer took an authoritative role and the students were very attentive in class. That was the same throughout the other lessons. The students were only able to speak in other languages, mainly Setswana, when they worked in groups.

8.1.1.2 The five skill factors for the first lesson

The first skill factor involving graphing skills and translation between visual graphs and algebraic equations was addressed only when the lecturer drew graphs on the board (graphing skills) without involving the students hence there was no opportunity for graphing skills to develop and translation from algebraic equations to visual graphs in 2D. The translation from visual graphs to algebraic equations in 3D was evident when the lecturer worked with the students to select the equations to use when calculating volume based on the different shapes, disc, washer or shell. The second skill factor, three-dimensional thinking involving the translation from 2D to 3D was addressed from the strip that was rotated to form different shapes, disc, washer or shell, which were drawn on the graphs without drawing a solid of revolution formed. The third skill factor, moving between discrete and continuous, was not well addressed since the lecturer drew the representative strip on the board without relating to the Riemann sums. The reason why a Δy or Δx strip was used was not explained or reinforced. The fourth skill factor involving general manipulation skills was used minimally. It was only evident when the students calculated the turning points of the parabolas. The fifth skill factor, the consolidation and general level of cognitive development, which involves all the four skill factors was not high enough, since for example most of the students failed to draw strips correctly.

8.1.2 Observing the second lesson

At the beginning of this lesson students were working in groups for about 14 minutes. The lecturer then taught for about 48 minutes. A problem was encountered during this lesson, where the lecturer could not solve a question that required integration from the substitution method and ended up asking the researcher to intervene.

8.1.2.1 Observing the students in lesson 2

The students were evaluating volume based on the graphs $y = e^{2x}$ and $x = 1$. They were seen to draw the graph of $y = e^{2x}$ using a table method. When I asked, they said that they find the table method easier and that they prefer it. From the use of the table method, it seems as if these students did not know the general characteristics of such an exponential graph, hence resorted to the use of procedural knowledge, by using the table method. Even though the students drew the two graphs, the graph of an exponential function $y = e^{2x}$ was not correct. As you can see in Figure 8.9, the graph drawn does not show exponential growth. It starts at the origin instead of intercepting the *y*-axis at 1. The graph drawn resembles half of a parabola, having a maximum turning point and facing downwards. However, it can still be used for evaluation of area or volume correctly.

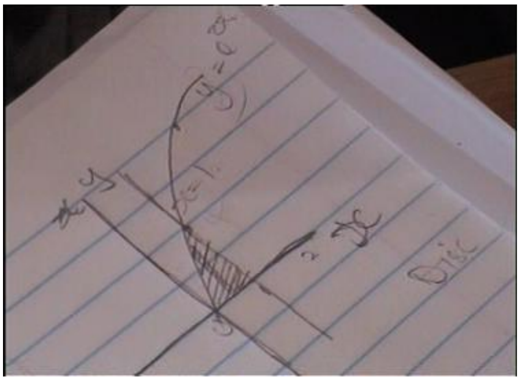

Figure 8.9: The exponential graph

After drawing the two graphs, relating to graphing skills and translation from algebraic to visual, the area was shaded (visual skills) and a Δx strip was selected (translation from continuous to discrete (visually)) as shown in Figure 8.9. The strip was then rotated about the *x*-axis (translating from 2D to 3D). It seemed as if the students understood what the strip would form after rotation as well as the formula to be used, since they wrote a disc next to the graph. The formula was selected from the worksheet that their lecturer gave them as a summary for the section on rotation of strips and the formulae to be used.

In translating from the visual graph to the algebraic equation for volume after rotation of the drawn strip, a disc method was used. The students were able to substitute correctly from the equations of the two graphs (translation from visual to algebraic in 3D and using general manipulation skills) and to evaluate the volume. The students evaluated the integral correctly and demonstrated good manipulation skills. The volume was evaluated correctly as 13.399.

When the students were asked why they did not use the cylinder method, one student said:

> 10:32 to 10:35
> S1: I think the disc is simpler than the cylinder

Even though some students were able to use the correct methods upon rotation of the selected strip, some students were still asking for clarity as to how one knows which method to use, pointing out to the problems they had in translating from 2D to 3D.

> 11:14 to 13:29
> S4: How do you know that this is a disc and this a cylinder and this is a washer?
> S1: This thing, when you look at it like this… this is your strip … If you rotate this thing you'll rotate this way (rotating anti-clockwise … the strip will rotate forming a circle, and in case of the cylinder you'll see the y-axis … Since this part (the strip) it rotates that means it's no longer a straight line it's a cylinder.

The dominant student (S1) explained to the other students using a demonstration, of a disc method (refer to Figure 8.10) and a shell method by rotating a pen and using the diagram as shown in Figure 8.11. A circle was drawn and a cross section of a shell was drawn upon rotation of a Δx strip about the y–axis (calling it a cylinder) as follows:

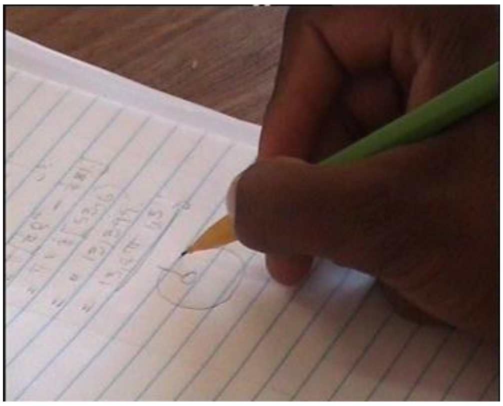

Figure 8.10: Rotating anti-clockwise

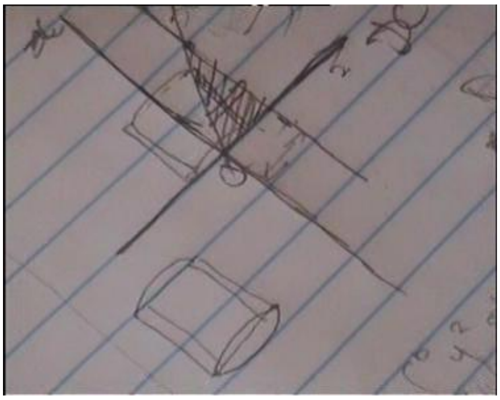

Figure 8.11: Cross section of a shell

S1emphasised the importance of the sketch (graphing skills) as a starting point of knowing what needs to be rotated and the role of visualisation of the different methods from the graphs. However, some students mentioned that they have problems when it comes to drawing graphs. This is what S1 said:

> 13.31 to 13:42
> S1: Obviously if you want to solve it you have to have a sketch.
> R: If you don't have the sketch?
> S4:You can't see if it's a cylinder or a shell, you can't see.
> S1: If you can work with things on your mind then it's okay but it will be easier if you draw it or work with something that you see.

8.1.2.2 Observing the lecturer in Lesson 2

- **Introducing the content to be learnt**

After observing the students working in groups, the lecturer introduced the lesson by emphasising how different hyperbolas are drawn. A hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ was drawn while discussing it with the students. The students were shown how to find the intercepts and the asymptotes (using general manipulation skills) and to represent them graphically as shown in Figure 8.12, hence emphasising the graphing skills. Later a line $x=7$ was drawn (graphing skills) on the same set of axes as shown in Figure 8.13 with a Δx strip (translating from continuous to discrete) located on the shaded region. In drawing both graphs, the lecturer involved students critically based on the shape of the graphs and their intercepts. The question required that a shell method be used to find the volume generated if the area bounded by the graphs in the 1st quadrant is rotated about the *y*-axis.

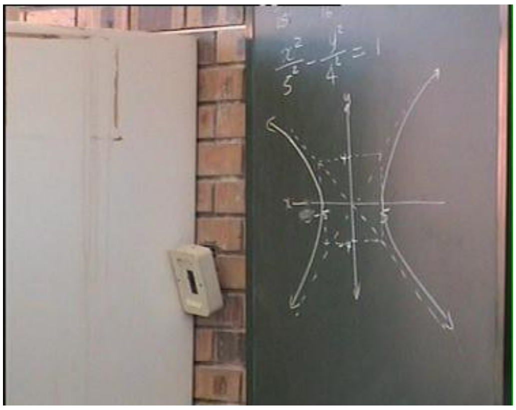

Figure 8.12: The rectangular hyperbola

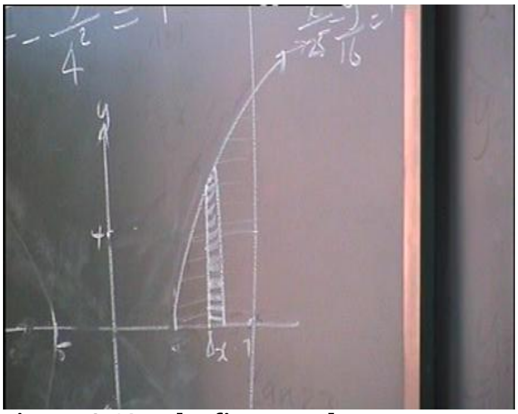

Figure 8.13: The first quadrant

The lecturer explained to the students how the graph will be drawn, and how a strip will be selected.

> 25:39 to 26:34
> L: It means that you're going to ignore the curves in the 4th quadrant. So we're going to concentrate in the 1st quadrant. Now the question is saying use shell method. How are we going to draw a representative strip? Is it going to be perpendicular to the *y*-axis or to the *x*-axis?
> Sts: ... to the *y*-axis
> L: If your representative strip is perpendicular to the *y*-axis. If it rotates which shape will that going to be? … it will not be perpendicular to the *y*-axis but perpendicular to the *x*-axis. So that when it rotates it will form a shell.

The problem that they had to solve stipulated that a shell method was to be used. The students were saying that the representative strip must be drawn perpendicular to the *y*-axis, referring to a Δy strip. The lecturer discussed with the students why a Δy strip could not be used since upon rotation about the *y*-axis a shell will not be formed but that a Δx strip would be used since upon rotation about the *y*-axis, it gives rise to a shell. Since a shell was not drawn, there was no translation from 2D to 3D. In translating from visual to algebraic in 3D, a formula for the shell method $2\pi\int_5^7 xy\,dx$ was selected from the formula sheet. When evaluating this formula in order to calculate the volume upon rotation about the *y*-axis (V_y), the lecturer got stuck at this step $V_y = 2\pi\int_5^7 x\sqrt{\frac{16x^2}{25}-16}\,dx$ after substituting for the *y* value, using general manipulation skills. This is as a result of the complexity of some problems involving integration techniques in VSOR. Suggestions were given from the students as to how to simplify before integrating. One student said that the inside of the square root becomes, $\frac{4x}{5}-4$ (taking the square roots of each term). The lecturer explained to the students that if they take square roots like that, it means that $\left(\frac{4x}{5}-4\right)^2$ must give back what is inside the square root sign, which is not possible. There was a pause for some time while the lecturer was asking for more suggestions. I probed the chosen group as a way of scaffolding in order to provoke the students' thinking processes in relation to the given suggestion of $\sqrt{\frac{16x^2}{25}-16}$ simplifying to become $\frac{4x}{5}-4$.

33:24 to 33:49

R: What is it that makes it not to be possible to say whatever he was saying? (referring to $\frac{4x}{5}-4$)

What is the main thing? … What makes his answer incorrect?

S1: … there is a negative sign

R: and what if it wasn't a negative, what was supposed to be there?

S1: multiplication or division

In that way I was trying to draw attention to the very important basic rules of mathematics (the mathematics register), that one can take the square root only if what is inside the square root is to be multiplied or divided, and that one cannot take the square root if it involves the sum or the difference. Another suggestion from the students was that they have to multiply the inside of the square root with *x* from the step $x\sqrt{\frac{16x^2}{25}-16}$. The lecturer explained to them that it would not be possible since there is an exponent $\frac{1}{2}$. The errors that were made here relate to the students' failure to use general manipulation skills.

As the lesson continued, there was a pause for a long time and the lecturer asked students for further suggestions. There were no suggestions and the lecturer suggested that they try the product rule (asking students to do it on the board but nobody volunteered). It seemed as if no one knew what needs to be done next as they struggled for about 4 minutes. The lecturer ended up asking the researcher to assist in solving the problem and I solved the whole problem on the board in collaboration with the whole class. In this case, I temporarily relinquished my role after being asked to assist in solving the problem, from that of the researcher to being a lecturer.

36:30 to 38:01
L: How do we integrate that? Mam how can we integrate that one? [referring to the researcher]
R: You know I am hearing interesting things here. They wanted to multiply *x* and he was explaining it that we cannot …
L: There is an exponent
R: … those are the common errors that you always do … you cannot multiply *x* if there's a power $\frac{1}{2}$

I explained to the students that if it was $2\pi\int\sqrt{\frac{16}{25}x^2-16}\ dx$, they would use the formula $\sqrt{x^2-b^2}$ (identified by the students) from the formula sheet, since there is no *x* outside the square root sign. I further explained that, because there is an *x* outside the square root sign, then the substitution method would be used in order to eliminate that *x* after substitution, before evaluating the integral. I solved the problem on the board in collaboration with the whole class showing them how the *x* will be eliminated as $\frac{16}{25}x^2-16$ is equated to u and then differentiated to be $\frac{32}{25}x=\frac{du}{dx}$, so as to replace $\frac{16}{25}x^2-16$ in the formula by u and to integrate further with respect to u. From the way in which the students were responding, it seemed as if the students followed what I was doing. They finally calculated the volume as 196.996 cubic units. The skills involved here were the general manipulation skills, where integration rules were used and calculations were done.

The lecturer then continued with the second problem where it was expected that the point of intersection be calculated (general manipulation skills) before the graphs of $y=x+4$ and $xy=-3$ were drawn (graphing skills) in collaboration with the students. The students suggested that they wanted to use a Δy strip (translating from continuous to discrete) as in the Figure 8.14 that would be to rotated about the *y*-axis. The use of a Δy strip, upon rotation about the *y*-axis, resulted in a washer being formed. The students avoided using the shell method and opted for the washer method as a result of rotating a Δy strip about the *y*-axis. When substituting into the formula for volume, students had problems identifying the limits of the Δy strip as the *y* values from the points of intersection as 3 being the upper limit and 1 being the lower limit, hence failed to translate from visual to algebraic. Through the lecturer questioning and directing, the students finally managed to use the correct limits for integration. The correct formula was used and the volume evaluated correctly to be 8.388 cubic units. The washer used was not drawn and there was no evidence of translation from 2D to 3D. However, the translation from visual to algebraic in 3D was done successfully from the representative strip and the correct formula for the washer was used. The students demonstrated the ability to use general manipulation skills throughout as they solved the problem.

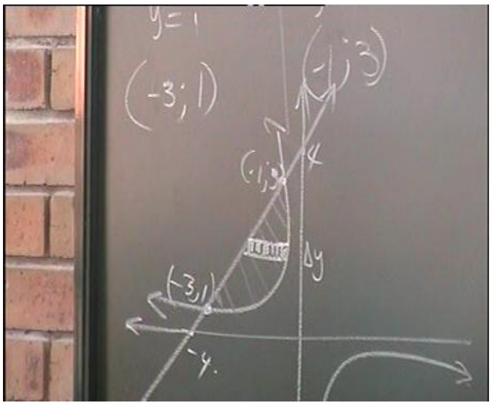

Figure 8.14: The second quadrant

- **The use of procedural and conceptual knowledge, level of difficulty of the content and assessment strategies**

The lecturer emphasised the drawing of graphs and on calculations as well as on how to use the formulae when solving problems. The conceptual knowledge was not displayed. The lecturer taught the students how to calculate and how to draw graphs also selecting a strip, without thorough explanation. The level of difficulty was evidently too high as it was evident when the lecturer asked the researcher to intervene (relinquishing roles and assisting for a short period of time). It was clear that the lecturer was unable to solve the problem involving a certain integration technique, called the substitution method. The lecturer gave students questions and they worked in groups to solve problems given from the previous lesson before the lesson began.

8.1.2.3 The five skill factors for the second lesson

The first skill factor involving graphing skills and translation between visual graphs and algebraic equations was addressed when the students drew the graphs of $y = e^{2x}$ and $x = 1$ (graphing skills) even if the exponential function was not fully correct; drew the graphs as a result of the translation from the algebraic equations to visual graphs and selected the correct formula for disc from the formula sheet, addressing the translation from visual graphs to algebraic equations in 3D. The third skill factor, moving between discrete and continuous, was evident when the students selected a Δx strip, which was problematic to most of the students. The Δx strip selected was rotated about the x-axis to form a disc (translation from 2D to 3D), addressing the second skill factor, three-dimensional thinking. The fourth skill factor, general manipulation skill, was evident when students used a table method in plotting graphs as they calculated the coordinates. It was also evident when the students calculated the volume correctly after substituting the equations of the two graphs correctly in the formula they selected from the formula sheet. The performance in the fifth skill factor, the consolidation and general level of cognitive development, which involves all the four skill factors, was fair since the problem was solved fully correct but some of the students still did not know which strip to use. Only as a result of working as a group, the students were competent in the skill, level of cognitive development.

In relation to the lecturer, the first skill factor involving graphing skills and translation between visual graphs and algebraic equations was addressed when the lecturer drew a hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ on the board (graphing skills) involving the students, mainly on how the intercepts (x and y) are plotted. In that way there was an opportunity for graphing skills to develop critically and translation from algebraic to visual in 2D. In the second problem, the lecturer involved the students in drawing the graphs. The translation from visual graphs to algebraic equations in 3D was evident when the lecturer worked with the students to select the equation for volume to substitute based on the shell method as requested from the question, and for the washer method as requested for the second question, but students struggled to use the y values as the limits of integration for the second question, hence failed partially to translate from visual to algebraic. The second skill factor, three-dimensional thinking (translation from 2D to 3D), was not addressed since the cylinder that could be formed through the rotation of the selected Δx strip and the washer for the second question were not drawn. The solids of revolution formed for both graphs were also not drawn. The third skill factor, moving between discrete and continuous, was addressed when the students suggested that a Δy strip be used and rotated about the y-axis to form a cylinder, which was incorrect, revealing that the students had problems in translating from continuous to discrete. For the second question, the selected Δy strip was correct.

Addressing the fourth skill factor, general manipulation skills was evident when the lecturer in conjunction with the students calculated the intercepts of the hyperbola and the asymptotes.

General manipulation skills were also not adequate, as became evident when students made mathematical errors in solving problems involving a square root, where $\sqrt{\frac{16x^2}{25}-16}$ was simplified as $\frac{4x}{5}-4$. The whole class also failed in general manipulation skills when a problem involving integration through the substitution method was difficult to solve. General manipulation skills were only achieved when the students gave correct answers to the solution to evaluate the volume after the substitution method was used properly. In the second problem the students were able to manipulate correctly. Due to the problems encountered in solving the first problem based on the hyperbola in relation to failure to the selection the correct strip, absence of the cylinder formulated after rotation, failure to use general manipulation skills in finding the square root and failure to use the substitution method for integration, one can argue that there was no competency with consolidation and general level of cognitive development since the tasks were too difficult for the students. However, in the second problem, the students achieved the desired level of cognitive development except that they did not show the solids formed as they translated from 2D to 3D.

8.1.3 Observing the third lesson

8.1.3.1 Observing the students and the lecturer in Lesson 3

The students were solving problems before their lecturer came in. The graphs of $y=3x$ and, $y=7x-x^3$, were drawn, successfully displaying graphing skills by students. They started by finding the intercepts of the graph of $y=3x$ (general manipulation skills). In drawing the graph of $y=7x-x^3$, they used the table method for points on the graph and used the first derivative to find the coordinates of the turning points (general manipulation skills). In substituting the *x* values from the turning points in order to find the coordinates of the turning points some students made errors as the answers they gave from their calculators were different. Students made errors in calculating the coordinates of the turning points, before they could get the correct answer. Initially they used the *y* values 7.128 and -7.126 they found after substituting the *x* values of the turning points $\pm\sqrt{\frac{7}{3}}$ in the original equation incorrectly as the coordinates (7.128; -7.126) of the turning points (general manipulation skills). The students made an error when trying to draw the graphs before they could calculate the *x* and the *y* intercepts (graphing skills). I intervened and they ended up calculating the intercepts. Finally the students used the correct points (intercepts and turning points) and the graphs were drawn correctly as shown in Figure 8.15.

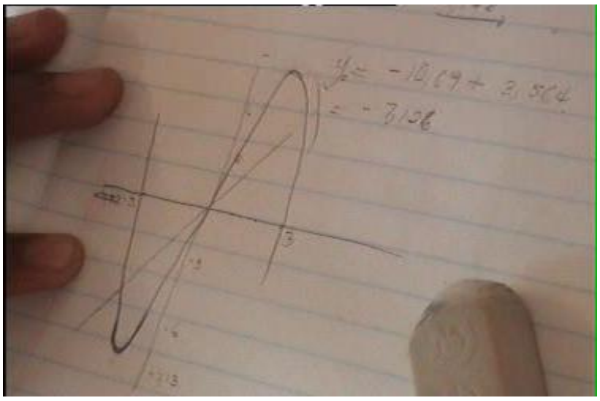

Figure 8.15: The cubic and straight line graphs

The next graphs to be drawn were the graphs of $y = x^2$ and the graph of $y^2 = 8x$.

The students did not know how to show the points of intersection graphically as the points where the two graphs meet, but rather used them as turning points on the graphs, hence drawing something like a cubic function graph as shown in Figure 8.16. The misconception of drawing a cubic graph might have emanated from a step $x^3 = 8$ (after substitution), where they were calculating the point of intersection. When realising that they are failing to draw the graphs properly, some students suggested that they use the table (refer to Figure 8.17) to draw the graph of $y^2 = 8x$, which displays the use of procedural knowledge and general manipulation skills. Finally, the graphs of $y = x^2$ and the graph of $y^2 = 8x$ were drawn as in Figure 8.17. One of the students called the graph of $y^2 = 8x$ exponential graph and $y = x^2$ parabola. The other students did not respond. It seems as if the students were not aware that that the graph of $y^2 = 8x$ is also a parabola.

07:40 to 08:42
S1: … the graph will be looking like this [turning point shown on the graph, as in Figure 8.16, the graph looks like a cubic function]. [One student disagreed and erased the part that turns after the point [2; 4], … it has a minimum turning point [referring to the graph of $y = x^2$], for the second graph, it says $y^2 = 8x$, it will be simple if we use the table, we will have to make y the subject of the formula … so let us do the table. By the way there is no square root of a negative number so we start at 0. [a table drawn].

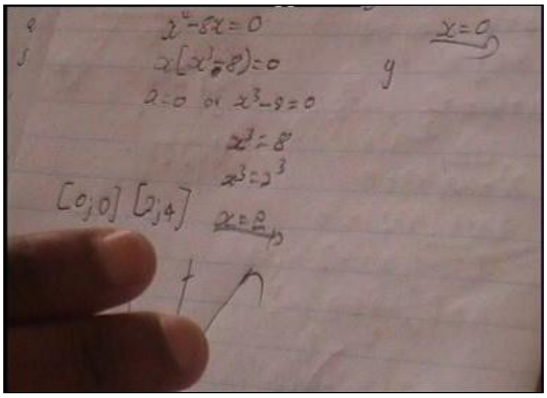

Figure 8. 16: The intersection points

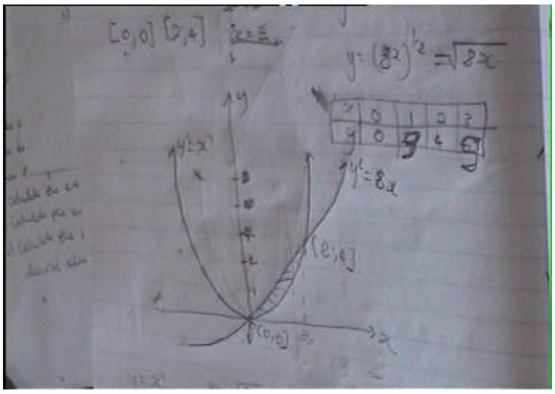

Figure 8.17: The two parabolas

In the section that follows, the lecturer carried on the third lesson after the group work.

- **Introducing the content to be learnt**

The lecturer solved the same problem on the board in collaboration with the students. The lecturer was asking questions about the point of intersection and the students responded.

11.46 to 14:48
Sts: … coordinates of the points of intersections are (0; 0) and (2;4) [lecturer writes on the board]
L: What type of graph is this [referring to y = x^2]
Sts: Parabola
S: What type of graph is it? [referring to $y^2 = 8x$]
Sts: Exponential graph
L: exponential graph? Exponential graph? [puzzled] Exponential goes this way, goes this way demonstrating using hands]
S: I don't know its name but it's not hyperbola. I don't know its name.
L: Usually you may find this function $y^2 = 8x$, written as $y = \sqrt{8x}$. … this …it's like a parabola neh? This 'side way parabola', it's either it opens to the right or to the left neh!.

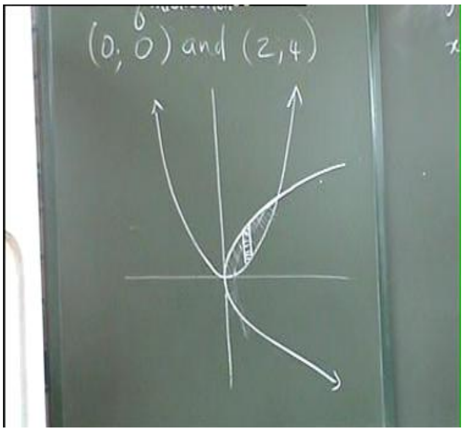

Figure 8.18: The parabolas drawn

The lecturer drew the graphs on the board also showing the points of intersections as shown in Figure 8.18, hence enhancing graphical skills and general manipulation skills. The lecturer showed the students a different equation of the graph of $y^2 = 8x$ as $y = \sqrt{8x}$, also referred to as a horizontal parabola. However, in the selection of the strip, the lecturer did not collaborate with the students. As a result, the translation from continuous to discrete (visually) was compromised. He simply drew a Δx strip on the shaded area.

The lecturer continued working with the students for about 25 minutes teaching them what the coordinates of the centroid are. This was done visually on the graph, translating from the visual graph to the algebraic equation in 2D. Both coordinates were calculated collaboratively, enhancing general manipulation skills, with full participation from the students and guidance from the lecturer. The equation to determine the x co-ordinate $\bar{x}$ was given, relating to the distance from the y-axis and was calculated to be $\frac{2.4u^3}{0.667u^2} = 0.899u$. Even if some errors were made before the correct answer was obtained. The students had to finish up the last steps in order to find $\bar{y}$, the distance from the x-axis. Below, the lecturer coached the students on how to approach these problems.

L: … so you get 9 marks for that. It is important that even if you cannot calculate the coordinates of that centroid, you must get marks for points of intersection. You must get points for drawing the graphs neh!

From the above statement, it is clear that there was an emphasis on marks, rather than on learning. After the lesson presentation, the students were able to demonstrate the coordinates of the centroid visually and algebraically on the other graphs they have as shown in Figure 8.19.

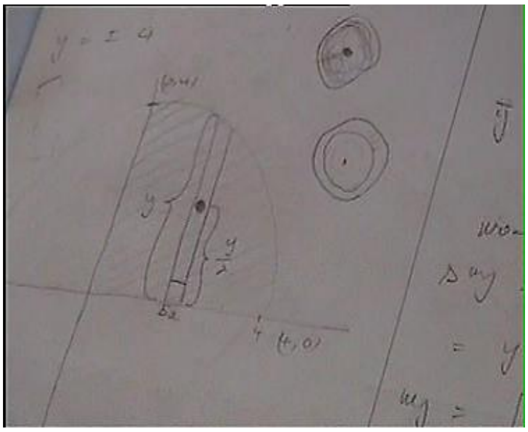

Figure 8.19: Locating the centroid

The students were also able to calculate the *y* or the *x* ordinates of the centroid, displaying general manipulation skills from the given graphs, hence also translating from visual to algebraic in 2D, resulting in the coordinates of the centroid as $(\bar{x};\bar{y})$. The substitution method was used correctly in the problem that they solved, since it involved the equation of the circle, where *y* was made the subject of the formula. In this case students demonstrated the ability to use general manipulation skills.

- **The use of procedural and conceptual knowledge, level of difficulty of the content and assessment strategies**

The lesson is based on calculations, and the use of table method in plotting graphs. The derivation of formulae was emphasised. The lesson involved conceptual knowledge to a lesser extent, only with the selection of the representative strip, where some visual skills were used as well as demonstration of the coordinates of the centroid on the selected strip. The problem involved integration of roots and was a bit long. The lecturer as well made some errors at the end of the second co-ordinate of the centriod, leaving out some multiplications. The lecturer ended asking the students to complete the problem on their own. These kinds of errors are also common with students. The lecturer gave students feedback from the previous activities, but he did not complete the second part of the problem. In most cases, students were solving problems as a group.

8.1.3.2 The five skill factors for the third lesson

The first skill factor involving graphing skills and translation between and algebraic equation and visual graphs in 2D and 3D, was addressed when the students drew graphs (graphing skills) and translating from algebraic to visual in 2D, even though errors were made before the students could draw the correct graphs. The translation from visual graphs to algebraic equations in 3D was evident when the lecturer worked with the students to select the equations of the centroid based on the location of the centroid of the strip. The second skill factor, three-dimensional thinking (translation from 2D to 3D), was not addressed since the centroid does not involve three-dimensional thinking. The third skill factor, moving between discrete and continuous, was not well addressed since the lecturer drew the representing strip on the board without relating to the Riemann sums, even explaining his choice to the students. The reason why a Δx strip was used was not explained or reinforced. The fourth skill factor general manipulation skill was evident when students were calculating the intercepts of the graphs, the points of intersection and the coordinates of the centroid in collaboration with the lecturer. The fifth skill factor, the consolidation and general level of cognitive development, which involves five elements from the four skill factors was not as yet attained as the tasks require a higher level of cognitive development. Most of the students struggled to draw graphs correctly, the choice of the strip was not well clarified and errors were made in calculating the coordinates of the centroid.

8.1.4 Observing the fourth lesson

8.1.4.1 Observing the lecturer in Lesson 4

- **Introducing the content to be learnt**

The lecturer began the lesson by writing the topic ‘centre of gravity’ on the board and an activity as follows:

> 00:00 to 00:40
> L: Calculate the distance of the centre of gravity from the x-axis of a solid generated when the area bounded by $y^2 = 4ax$, $x = 0$ and $y = b$ is rotated about the y-axis.

The lecturer continued to explain the difference between the centre of gravity and the centroid. The lecturer did not demonstrate the difference visually from graphs but referred the students to the formula sheet. The emphasis was on using the formula sheet to select the formula and then calculating the centre of gravity required, hence promoting procedural knowledge.

> 01.05 to 01:49
> L: Eh centre of gravity. The only difference between centre of gravity and the centroid is that the centroid is … the area about the x-axis … over the area. So with centre of gravity it will be the volume about eh

> volume multiplied by moment about eh … a particular axis above the volume. In the formula sheet the page that I've just gave out.

The lecturer continued to draw the graphs and selecting the strip without justifying why such a strip was used. That might have impacted on in-depth knowledge, especially based on the Riemann sum relating to how a chosen strip approximates the given area.

- **The use of procedural and conceptual knowledge, level of difficulty of the content and assessment strategies**

The lecturer emphasised the derivation of formulae for the centre of gravity procedurally. The lesson did not cater for conceptual knowledge. The lesson was more on calculations, with less focus on the development of visual skills. The content introduced was difficult. The centre of gravity was not well demonstrated. Students were given problems that they solved as a group.

8.1.4.2 Observing the students in Lesson 4

There was an interesting situation when the students struggled to draw the graphs of $\frac{x^2}{9}-\frac{y^2}{4}=1$ without showing the asymptotes until they finally succeeded as shown from Figure 8.20 to Figure 8.22 in the next page when a Δy representative strip was chosen. The question required that they shade the region bounded by the graphs of $\frac{x^2}{9}-\frac{y^2}{4}=1$, the x-axis and y = 3 as well as showing the strip that would be used when rotating about the y-axis. In selecting the strip, a Δy strip was drawn. When I asked why a Δy strip was used, one of the students argued that a Δy strip is used because rotation is about the y-axis and that if rotation was about the x-axis, a Δx strip would be used. This misconception points to the fact that the students do not have in-depth knowledge on how the strip is selected, it is shallow.

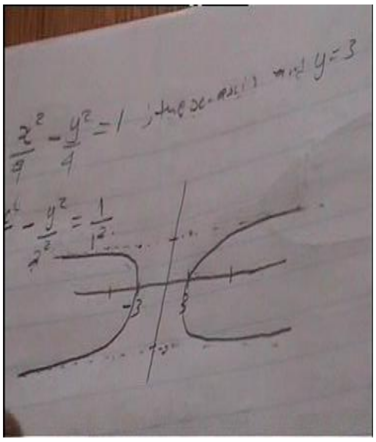

Figure 8.20: The 1st attempt

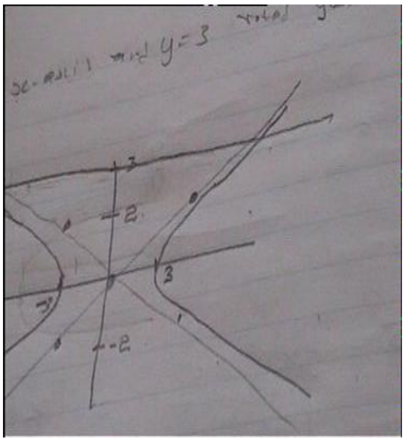

Figure 8.21: The 2nd attempt

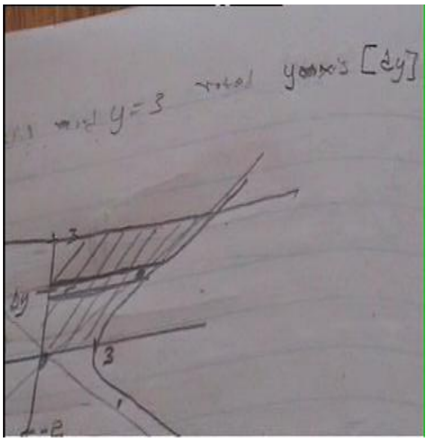

Figure 8.22: The last attempt

The extract below justifies what transpired during the selection of the representative strip.

> 35:30 to 35:51
> R: why are you choosing that one? [referring to a Δy strip]?
> S1 and S2: … it rotates about the *y*-axis
> R: So when they say rotated about the *y*-axis. You're going to choose Δy?
> S1: yah, the strip must be …
> S3: with respect to *y*
> R: then what if we change the question and say rotates at the *x*-axis
> S1: we must change the strip
> R: and put it like?
> S1: like this (referring to Δx strip)
> R: Okay

During the group work, when finding the intercepts of the hyperbola, the students had problems in interpreting $\sqrt{-4}$ as being undefined, to justify that the hyperbola did not have the *y* intercepts. They were also not sure on how the representative strip should be selected. The other problem encountered was that the students were struggling to solve this activity when expected to evaluate the integral even if it was the same as the one that I did in class involving the use of the substitution method. The errors that these students made until they obtained the correct answer revealed that the student had made mathematical errors (due to lack in manipulation skills). They were a bit confused. Their work was not logical and in an orderly manner. They later obtained it right with my assistance through scaffolding. Generally the students lacked the skills of drawing graphs as well as interpreting those graphs. During the group discussions, it was evident that the students still struggled to draw the correct graphs and made errors in calculations. The students lacked the graphical skills and the general manipulation skills.

8.1.4.3 The five skill factors for the fourth lesson

The first skill factor involving graphing skills and translation between visual graphs and algebraic equations in 2D and 3D, was addressed when the students drew graphs (graphing skills) and translating from algebraic equations to visual graphs in 2D. Students' performance in drawing graphs improved, even though they were in some instances still making errors before they could draw the correct graphs. The translation from visual graphs to algebraic equations in 3D was evident when the lecturer worked with the students to select the equations of the centre of gravity even though it was done procedurally. The lecturer explained the formula to calculate the distance of the centre of gravity from a particular axis from the formula sheet, explaining the relationship between moments of volume and volume. The second skill factor, three-dimensional thinking (translation from 2D to 3D), was not addressed since the 3D diagrams were not drawn. The third skill factor, moving between discrete and continuous, was not well addressed since the lecturer drew the representing strip on the board without relating to the Riemann sums. The reason why a Δx or a Δy strip was used was not explained or reinforced.

The fourth skill factor general manipulation skill was evident when students were evaluating moments of volume and volumes from the integrals in collaboration with the lecturer. The fifth skill factor, the consolidation and general level of cognitive development, which involves five elements from the four skill factors, was not attained since the tasks were cognitively demanding. There was an improvement in drawing graphs and translation from visual graphs to algebraic equations from 2D to 3D was not achieved. Most the students struggled to do general manipulation skills properly, since they struggled in most cases with problems that involved the substitution method even if it was explained by the researcher in one lesson. This indicates that the students find VSOR challenging and need more time to grasp the concepts.

8.1.5 Observing the fifth lesson

8.1.5.1 Lesson 5: Group work

In this lesson, the students were working in different groups. The discussion that follows is of a group of eight students who were working on four activities from one of the previous question paper during the fifth lesson, with the researcher scaffolding. During this lesson, students were working as a group to consolidate what was done throughout the past four lessons. During their discussion, I asked students questions to justify what they were doing as well as probing their responses. S1 was dominant and the one who was writing the solutions down during this discussion. The students managed to solve the first three problems out of four before the end of the lesson. A detailed discussion of the first two questions is given with a summary for the third question.

- **The first question**

The first question that the students answered was as follows:

5.1.1 Make a neat sketch of $\frac{x^2}{25}-\frac{y^2}{16}=1$ and show the area bounded by the graph and the line $x=7$. Show the representative strip/element that you will use to calculate the volume (by using the SHELL-METHOD only) of the solid generated when the area in the first quadrant is rotated about the *y*-axis.

5.1.2 Use the SHELL-METHOD to calculate the volume described in QUESTION 5.1.1 above.

The students started by drawing the graphs correctly in the first quadrant showing all the important points as shown in Figure 8.23, as well as identifying a Δx strip. The students were competent in graphing skills. The discussion below follows from the drawn graph, which was correct. It involves both students and the researcher in with the conservation translated from Setswana.

S's: From, $\frac{x^2}{25} - \frac{y^2}{16} = 1$ we have $\frac{y^2}{16} = \frac{x^2}{25} - 1$,

then $y = \sqrt{\frac{16x^2}{25} - 16}$

$$V_y = 2\pi\int_5^7 x\,y\,dx = V_y = 2\pi\int_5^7 x\sqrt{\frac{16x^2}{25} - 16}\,dx$$

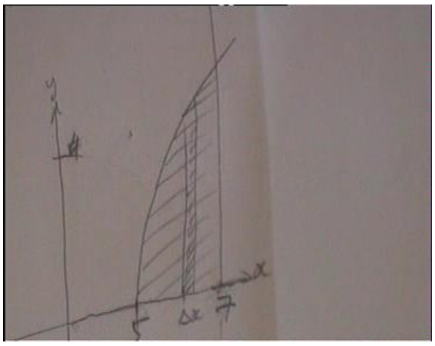

Figure 8.23: The graphing skills

S3: *Let* $u = \frac{16x^2}{25} - 16$, *then* $\frac{du}{dx} = 32x$, *then* Δx
R: 32*x*? How did you get 32*x*?
S1: We differentiated $16x^2$ and we get 32*x*…and 16 is 0.
R: Then 25. Is it lost…where is it?
S1: No, it is not lost. I think we differentiated in terms of *x*
S2: …where is 25?
R: By the way what is the answer when you differentiate $3x^2$
S1: $3x^2$ becomes 6*x*…ok $\frac{16x^2}{25}$ becomes $2 \times \frac{16x}{25}$, then $\frac{32x}{25}$
S1: … we want Δx so $\frac{du}{dx} = \frac{32x}{25}$ and $dx = \frac{25}{32}du$
R: mmh! where is *x* ?
S2 & S3: where is *x* ?
S1: Oh, $dx = \frac{25x}{32}\,du$, I forgot
S2: This *x* is for 32; it must be written below so as to see that it is $\frac{1}{x}$ not *x*.
S1; So what next?
S3: Now we use the formula $2\pi\int_5^7 x\sqrt{u}\ \ dx$ *(students stuck)*
R: Say u to the power half [scaffolding].
S5: We get $2\pi\int_5^7 x \cdot u^{\frac{1}{2}}\ \frac{25}{32x}du$

S3: $\frac{50\pi}{32}\left[\frac{(u)^{\frac{3}{2}}}{\frac{3}{2}}\right]_5^7$

S2: $\frac{50\pi}{32}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_5^7$ [some students were saying multiply by $\frac{2}{3}$ instead of division]

S1: $\frac{100\pi}{96}\left[\left(\frac{16x^2}{96} - 16\right)^{\frac{3}{2}}\right]_5^7 =$

S1: $\frac{100\pi}{96}[60,199 - 0] = 198u^3$

In the above extract students were seen helping one another, while I was helping out only when they got stuck, in a way of cuing or leading step-by-step. Some students were seen in some instances helping step-by-step when others got stuck like S1 and S3. These students were able to recognise when they went astray, and had good manipulation skills. The students were able to solve the problem through scaffolding, until they reached the final solution. They were able to draw the graphs properly and by helping one another they were able to evaluate the integral correctly. When using the definite integrals, the students substituted *x* by *u* with boundaries of 8 and 5. However, this did not affect the answer as they substituted back the *u* value with the *x* value.

- **The second question**

5.2.1 Calculate the point of intersection of the graphs of $y = 2x + 2$ and $y = x^2 + 2$
Sketch the graphs and show the representative strip/element that you will use to calculate the area bounded by the graphs.

5.2.2 Calculate the area described in QUESTION 5.2.1.

5.2.3 Calculate the area moment of the bounded area about the *x*-axis as well as the distance of the centroid from the x-axis described in QUESTION 5.2.1.

In the second question that they solved, they were asked to first calculate the point of intersection of the graphs of $y = 2x + 2$ and $y = x^2 + 2$ before they could draw the graphs, which were calculated to be (0,2) and (2,6). After drawing the graphs, they realised that the drawn graphs do not intersect at the points they found as intersection points. They finally realised that for one of the graphs $y = 2x + 2$, the *x* and the *y* intercepts were swapped, as indicated in Figure 8.24. These students lacked graphing skills.

S1: If intersection is at (0,2) and (2,6) so why the graph is like that (referring to the first picture below), it means that this graph is on the incorrect position.
S5: *y* is 2, not *x* = 2
S3: You mean that *y* = 2 and *x* = -1, so graph is …
S: So graph must be like this (drawing the correct graph figure 8.26)
S3: ..any way this graph(referring to the incorrect graph) is the symmetrical graph of the other one (meaning the correct one as in the middle picture below).

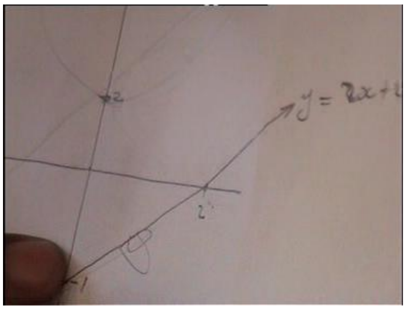

Figure 8.24: The straight line

The other graph of $y = x^2 + 2$ also gave them problems as they could not show or understand that it does not have the *x*-intercepts, they failed to understand what $\sqrt{-2}$ becomes as they solved $x^2 = -2$. They argued that it gives an error as indicated from their calculator, without interpreting what that really meant. Only after scaffolding through probing from the researcher, they realised that the roots were non-real, and as a result there would not be any *x*-intercepts. The discussion that follows justifies what happened.

14: 48 to 16:52
S3: … we are dealing with the shell method, then what is rotating?
S1: Whoo!! whoo! Where are the *x*-intercepts? [the student asks the researcher] … Mam, do we have *x* intercepts?
S3: It is 0 and 2
R: *x* intercepts? For what?
S1: For the graph $y = x^2 + 2$. For *x*-intercept this is zero (referring to 0) then we will have x^2 equals to (pause)…2 will be this side, … becomes negative, there way that we can get the square root, unless!
S3: Let's factorise $y = x^2 + 2$.
S2: How are we going to factorise that; it can't be factorised
S1: Maybe if it was negative maybe [referring that if it was $y = x^2 - 2$]
S1: Then we calculate area with the formula [they want to avoid] the problem $y = x^2 + 2$
R: Please continue, you said if you take 2 to the other side?
S1: Whoo! If there is $x^2 = -2$, then root -2 is error.
R: Then what does that mean?
S1: It means that it won't touch the *x*-axis, so we leave it like that
R: [smiling] so you were opposing that?
S1: No, I want to know whether it will touch the *x*-axis. So if it is like that it won't touch.
R: So what does it mean?
S1: It means that there is error …
S2: So let it not touch
S: So it faces up without touching the *x*-axis. [referring to the graph $y = x^2 + 2$.]

Students continued to solve the question but got confused in the process of drawing the second graph, $y = x^2 + 2$, on the same set of axis. They knew that the parabola faces upwards, but struggled to draw it. The students lacked graphing skills. Initially students started by drawing an incorrect graph. Refer to Figure 8.25 and the correct graph Figure 8.26.

16:55 to 18:58
S5: So why does it pass there?
S1: No, this is not the turning point, it turns at (0, 2) [Graph erased].
S2: You may draw an absolute value.
S1: No
S2: Our points of intersection are (2,6)
[finally the correct graph was drawn]

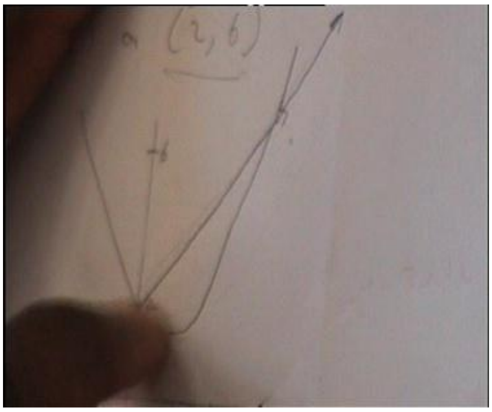

Figure 8.25: The incorrect graphs

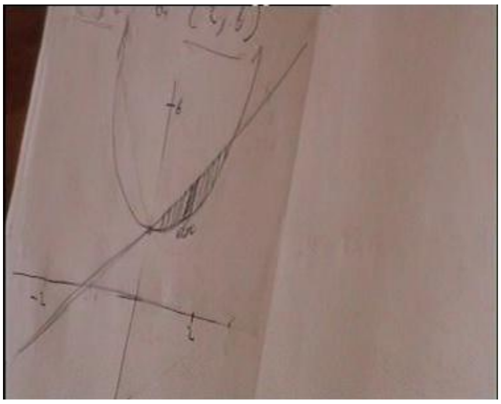

Figure 8.26: The correct graphs

The problem then arose when students had to select the strip to use. The students continued to read the question and progressed as follows

19: 05 to 20:42
S2: Question says calculate the area
S3: The next question says calculate the moment of the area …
S1: So this one will be rotating about which axis?
S2: You just choose
R: Mmh! [surprised]
S1: It won't be the same if we chose the *y* axis they won't be the same.
S2: Then lets choose …
R: What is your question?
S2: Calculate the area bounded
R: Ya! And you [referring to S1]? what were you saying
S1: I was saying they will be equal [referring to answers] if we rotate about the *y*-axis and about the *x*-axis?
R: Read the question … is there anywhere where they talk about rotating?
S1: [read the Question 5.2.3], they say calculate the area moment of the bounded area about the *x*-axis, as well as the distance of the centriod from the … axis, so before you start you must read the whole question?
R: I think it is important.
S1: Then, we will rotate it about the *x*-axis [student confused again].
R: Do they say rotate?
S2: We will rotate it, about the *x* – axis [another student also confused].
S1: It means that we do not have to rotate
R: They say area moment
S1: There is no way that it won't rotate
R: But the question says area moment, do you rotate in area moment? If they say area, do you rotate?
Sts: [Argue about which strip to use] Lets find the formula
S3: Lets chose the *x* one (as shown in Figure 8.39)
S1: Let's use Δx strip

The discussion above highlights that the students do not know how to select the strip and neither when to rotate. They do not know that with area one does not rotate. They seem to prefer the Δx strip as shown in Figure 8.26. They fail to explain the translation from continuous to discrete (visually) based on the strips used. The terminology use in the question does not make sense to them.

After selection of the strip, the students selected the formula to use and continued with the calculation. Through continuous scaffolding, step-by-step and through hints, they managed to get the solution correct. When calculating area, the students wanted to use the centroid formula, I advised them to use the area formula not that of the centroid. When I asked them what the formula for area is this is what one student said.

22:04 to 26:55
S1: The formula for area is length times breadth that will be a change the *x* multiply by a change in *y*
R: Why do you go back there, what did you do in the first lesson?
S3: We will use $A = \int_c^d (x_2 - x_1)\, dy$
S2: It is the formula for the washer
S3: The washer – the washer

S1: The question says area of this strip ok? ... this is our strip, which formula do we use? We use $A = \int_a^b (y_2 - y_1)\, dx$

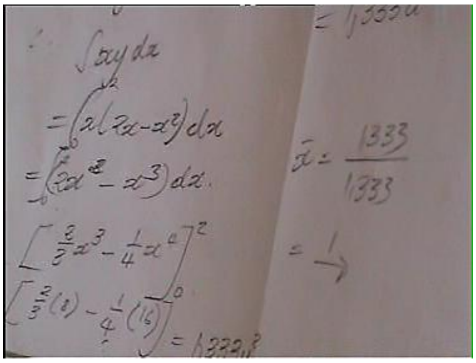

Figure 8.27: The centroid

The students substituted correctly, integrated correctly and obtained the answer for area as 1.333, demonstrating appropriate general manipulation skills as shown in Figure 8.27. They continued with the second part of the question and calculated the centroid correctly as $\left(1; \frac{y}{2}\right)$ as shown in the calculation.

- **The third question**

5.3.1 Make a neat sketch of the graph of $y = 3e^{2x}$ and show the area bounded by $y = 3e^{2x}$ and the lines x = 0, y = 0 and x = 2. Show the representative strip/element that you will use to calculate this area.

5.3.2 Calculate the area described in QUESTION 5.3.1.

5.3.3 Calculate the second moment of area about the y-axis of the area described in QUESTION 5.3.1

The third question was solved in a similar way through scaffolding. The students drew graphs as shown in Figure 8.28 but the exponential graph was drawn as a decreasing function. Throughout the problem-solving situation, if students experience problems, I probed until they reflected on their work and exchanged ideas to reach the correct solution. The discussions helped the students to reflect critically on their work in order to make informative decisions. Some of the problems encountered were that students drew incorrect graphs and mainly used a Δx strip. However, in some instances the incorrect graph did not make the solution incorrect. In terms of integration techniques the students were doing well, with minor errors especially in calculations after substitutions from the limits as well as using integration by parts. That was perhaps due to the fact that some students were not able to use the calculators properly. Throughout the recording process, student one was dominating the discussions and able to pick up many errors, with scaffolding from the researcher and assistance from other students.

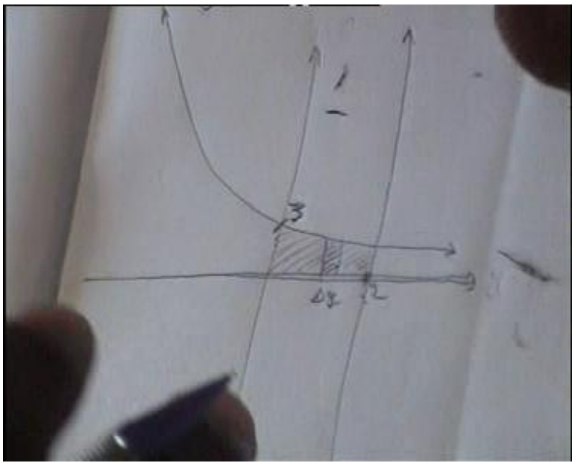

Figure 8.28: A decreasing exponential graph

From the above discussion one can conclude that the students were not fully competent in drawing graphs (failing to translate from algebraic to visual). They were also not competent in translating from continuous to discrete since they did not know why a particular strip was used and failed to translate from 2D to 3D as they did not draw such diagrams after rotation. They, however, demonstrated some capabilities in translating from visual to algebraic both in 2D and 3D, as they were always able to select the correct formula (disc, washer or shell) and substituted correctly from the graphs. The general manipulation skills in most cases were affected by the errors that they made, but besides those errors, one could say that they were partially competent.

8.1.5.2 The five skill factors for the fifth lesson

The fifth lesson can be summarised in terms of the five skill factors. The first skill factor involving graphing skills and translation between and algebraic in 2D and 3D, was not adequately developed as the students struggled to draw most of the graphs, unless I assisted. The only graph that the students were able to draw without problems was $\frac{x^2}{25}-\frac{y^2}{16}=1$, probably because they did it in class during the second lesson, even if they could not remember completely. The students were seen to rely on the formula sheet to get their formula and not from their drawn graphs, hence that did not improve on their ability to translate visual to algebraic in 2D and 3D. The 3D solids generated when translating from 2D to 3D were not drawn. The students were not competent in the second skill factor, three-dimensional thinking (translation from 2D to 3D). The different shapes, disc, washer or shell, were not drawn. The third skill factor, moving between discrete and continuous, was not well addressed since the strips drawn were drawn without relating to the Riemann sums. The reason why a Δy or Δx

strip was used was not clear to most students. The fourth skill factor, general manipulation skill was developed even though some errors were made when calculating points of intersections and other important points as well as and when evaluating the integrals. The fifth skill factor, the consolidation and general level of cognitive development, was not well developed, since for example most of the students failed to draw strips correctly, they could not interpret those graphs. They could not identify the correct representative strip and they did not draw the 3D solids.

8.1.6 Summary of the classroom observations

For the lessons observed, the lecturer did not teach the students how to draw graphs and how to select the rectangular strip, probably because they had been dealt with at previous levels. The lecturer focused mainly on finding the important points of the graphs like the turning points and the intercepts, the rotation of the strip and using the strip to select the formula from the formula sheet and to do calculations for area, volume, centroid and the centre of gravity. The students were somewhat competent in calculating the necessary points and drawing graphs, but had problems in selecting the correct representative strip and rotating it. In cases where the strip was rotated correctly, a correct formula was selected and substituted correctly. However, when having to evaluate the integral, students' performance was merely satisfactory. They struggled at times to use integration techniques (especially the substitution method) and showed good performance only when using simple rules for integration. What was also observed was that students were not using textbooks in class. They used notes and questions compiled by the lecturer as hand-outs.

8.2 INTERVIEW WITH ONE STUDENT

8.2.1 Presentation of the interview results

The student was interviewed based on her general perceptions about the mathematics at N6 level. The interview was done in Setswana and translated into English. Excerpts from the interview are used in the discussion. In the following excerpts R refers to the researcher and S refers to the student being interviewed.

The discussion reveals that the student interviewed is very much aware that in order to do well in VSOR, one should be able to draw the graphs and that in order to calculate the volume, one needs to interpret the drawn graph. What the student emphasised is that without the correct graph, one will not be able to calculate the volume. According to this student, drawing graphs for her was never a problem.

> R: …My interview is on the section of areas and volumes. …just tell me your experience from N5, maybe starting from N4 as well.
> Int: Oh…basically…what can I say….I got lecturers who taught me …I was satisfied. And I did not have problems with areas and volumes. I knew them and I liked them...because as you draw number one, you will get number 2.
> R: What do you mean, if you draw number one, you will get number 2
> S: Is like if they give you something like, they ask you about volume. You have to draw 1-sketch ok! … then from that sketch is where you will be able to see that you will have to calculate something like what…, so that is why I like it. It is different from if you are given a question and asked to solve it. That means that this one (referring to the question on areas and volumes) can be solved by looking at the sketch … that is why I like them. You can answer the question based on the sketch. So if you fail to draw the sketch, you won't get it. That is why I like them, because they give you an idea, themselves, by just drawing you can see how to get this length
> R: So let's say you fail to draw the sketch.
> S: I fail to draw the sketch?
> R: Mm
> S: Eish! ... that never occurred to me, where I failed to draw the sketch…in most cases I can draw.

The student also believes that most of the students think that this section is difficult and that prevents them from doing well in this section, and believes that you can still get the answer (by just writing) even if the graph is not drawn.

> R: … and then… what is your experience from other students about that section?
> S: They say that it is difficult… and my belief is that if you say that something is difficult … and tell your mind that, you will believe that ... that thing is difficult. I believed that they will be simple despite what other student told us, making us to believe that they are difficult. But they are interesting ... and what thing else … if you can fail to draw the graph you can still get the answer.
> R: How?
> S: If you fail to draw the sketch ... what can you do… you can just write... but is not always that you fail to draw the sketch.

The student stated that they did not have a lecturer at N5 level and that they studied on their own. The difficulty that the students had with VSOR might be because the students from her class studied on their own at N5 level and that they carried on to the N6 level without the proper foundation from N5, since this section on VSOR starts at N5 level where they use the disc method and the washer method. However, the student believed that there was an improvement at N6 level as they had a good lecturer.

> R: …and what is the relationship, between N5 and N6, based on that section?
> S: At N5, I struggled a bit, because we did not have a class lecturer. Basically we worked in groups to assist one another.
> R: Is it possible?
> S: ….there were other students who were bright, then they would come with information from other places, maybe from brothers… and we worked in the afternoon, or during maths period. With N5, I struggled a bit.
> R: …and N6?
> S: At N6…that is where I started to know them.
> R: ..and, were they interesting?
> S: Yes:

This student pointed out a form of advice to other students, that in order to do well in this question, their lecturer emphasised that the students work from the graphs that they have drawn. No questions are given with graphs. They have to draw before they calculate the area or the volume, they must have a sketch.

> R: Then what is your advice to other students, for that section, you know how much it weighs? That is why it is my concern as well.
> S: Yes, … you know eish! That one! I advise them to draw a sketch, even if they do not have a clue. But at least somebody might see what you want to say, that is where the idea come from.
> R: So, what if you cannot draw a sketch.
> S: But there is no way that you can avoid to draw.
> R: So how did other students managed… to draw sketches in general.
> S: They taught us from the beginning of the block... telling us that you have to draw a sketch, if you cannot (shake head disagreement), you must know that you are lost, and we put that in our heads, that you have to draw a sketch, so as to find the answer. Even if it is incorrect, but that could be better because maybe graph was supposed to be here, or here, that is where we see that the sketch helps. That is sketch, sketch, sketch.

The student believes that the question is graded according to the N6 level.

> R: So, the way the question is asked, I mean Question 5, what can you say about the way in which they ask? Are you happy about the way this question is being asked, especially at N6 level?
> S: Yes, it goes according to the standard.
> R: Which standard?
> S: The N6 standard, … it suits the N6 level.

The interviewed student had a strong belief that most students cannot draw graphs, they can select the strip correctly if the graph is given and are proficient when doing calculations. The student believes that if the graph is given, most students will be able to select the strip correctly and do the calculations correctly. This student also pointed to the fact that some students do not draw graphs because they are lazy and believe that the graphs waste their time. She also pointed out that that she believes in sketching graphs and that failing to draw a graph will lead to failure in answering the question correctly. The message is that students must sketch.

> R: Let's talk about the graph, you have been emphasising that if you do not have a sketch, you cannot do anything. Now my question is what if the sketch is given?
> S: You will be able to get it right, and then, you also know your area.
> R: Then, what if the question comes with a sketch.
> S: (looks excited) If they give me the, sketch, … in that case I will finish quickly… that means if they can improve and give us sketches, it will be much better, because other students are lazy to draw sketches, they say that the sketch waste their time.
> R: Lets us say that they are not lazy, what if they do not have a background on graphs. By the way at which level did you do the graphs?
> S: From N1
> R: What if others are not good in drawing sketches?
> S: Eish! There forget … I don't know, because I believe in sketches, you have to draw a sketch.

After probing from the researcher to find out how the student feels if a graph is given and students are asked to calculate the area and volume, the student convincingly highlighted that that would be a bonus, and that every student would pass N6 mathematics. That is what the student believed in, which is not the real situation, since the students struggle even when graphs are drawn. In addition to that it becomes easy since the formulae are given and they can use the knowledge they have from other subjects. In general, she highlights that selection of the strip might sometimes be a problem to other students, if they cannot put it in the right place, or they draw it on the graph that is drawn incorrectly. The student also believes that the paper is too long.

R: So what I am saying is that, what if the question is changed a little bit. By the way that question has about five subquestions. If part of the questions comes with a sketch?
S: It will be much better, we will succeed. All the people will manage to pass the N6 maths, because the 3 hours allocated is not enough.
R: So, if the sketch is given, what becomes a problem now?
S: ..if you do not know your formula, you do not how to calculate area and volume… normally the challenge is there but normally, they give formulae on the formula sheet, but sometimes they expect you to have done it in mechanical or electrotechnics. You have to know it, you have to know about area and volume... and about the strip. So you see if you did not draw a sketch. So it's hard to say you can't.
R: So, how would that improve, you have said it in a way, so according to you, what is the problem with this section, is it the graph or the strip or the calculation?
S: It is the strip. If you can't put it in the right place, you won't get the right, answer.
R: Then what if you have drawn the incorrect graph? Can you put it in the right place?
S: No.

The student believes that in order to do well in this section, one must practise enough. It was also pointed out that the error that you make at the beginning, like drawing the incorrect graph and selecting the strip incorrectly, affects everything that you do thereafter, resulting in incorrect responses. According to the student, the selection of the strip and the calculations based on formulae from the formula sheet will be correct based on the incorrect graph drawn.

R: What is the main problem here?
S: The main problem is with the question itself. ... the way in which they ask, but if you practise. …
R: Let us say that you make an error with the graph?
S: What? Everything will be incorrect, let us say 4.1 ;4.2 and 4.3
R: And what is that? What does it tell you if it is like that?
S: That tells me that I have failed.
R: And, in reality, did you fail or not?
S: I won't fail because this question has 40 marks, and I still have 60 on the other side.
R: Now the question where you say 4.1; 4.2; 4.3
S: Those ones I won't get correct, so you see if I fail to draw the graph, I won't get it.
R: So what will your calculation be based on?
S: On the graph …the one drawn incorrectly.
R: Is it incorrect?
S: Mm
R: So, you cannot calculate.
S: No, in my mind I will be telling myself that this graph, is right.
R: … let us say that I check your incorrect graph, and your calculation based on your incorrect graph, what do I get?
S: Incorrect answer.
R: I am talking about the answer; I am talking about your calculation based on the incorrect drawn graph. …are you going to be able to select the strip correctly on you graph, even if that graph is incorrect?
S: Yes, I am going to put the strip correctly and calculate based on that strip, whether it will be in the *y*-axis or in the *x*-axis and calculate according to my mind. I will take the formula sheet, and chose the formula.
R: So what is the main problem?
S: Is the graph, calculation is not a problem, the graph is a problem.

When asked about the different formulae for washer, cylindrical shell or disc, the interviewed student believed that the students were competent in that, since they would have practised, and that the style in which the questions are asked must be changed, so that the students could do well.

> R: Will the students be able to see the washer….the cylinder?
> S: Yes, plus because one would have practised, basically the problem is the graph.
> R: What can be done to improve this section; is the problem with the students, or the section or the question paper:
> S: Question paper … the way in which the examiner asks questions, like give us more information like at matric level, they give you a lot of information that helps you, like if the graph is given, even if the strip is not given. If you are asked to calculate this and this and put the strip on the given graph, we will be more encouraged ... I wish that the person whose doing the question paper can realise that and improve it.

8.2.2 Analysis of the interview results

In terms of the five skill factors, the interviewed student emphasised the importance of graphing skills, where the students must be competent in drawing graphs. She pointed out that some students do not like drawing graphs and that graphs are crucial since the drawn graphs are starting points to calculating areas and volumes. That was captured from her statement as she said: “You can answer the question based on the sketch. So if you fail to draw the sketch, you won’t get it”. She also emphasised that the students must select the correct strip (translation between continuous and discrete) and be able to rotate the selected strip correctly (pointing at the translation from 2D to 3D. She argued that what one calculates will be incorrect if the graphs were drawn incorrectly and the strip selected and rotated incorrectly relating to the general manipulation skills and the consolidation and general level of cognitive development. From what is gathered from the interview, graphing skills, selection of the strip and rotation thereof are prerequisites in calculating areas and volumes as one has to comply with them first before choosing the formula sheet and do the manipulations.

8.3 CONCLUSION

The conclusions that can be made from the classroom observations and the interview are that students in general lack graphing skills. The students make many errors in calculating the intercepts and turning points of the graphs, pointing also to a lack of general manipulation skills. In other instances even if the intercepts are correct, they fail to use them to draw graphs. Another problem that arises after having drawn the graphs (that they also struggle with) relates to selection of the representative strip. The students tend to draw the strip based on what they prefer, not based on what the question requires, hence failing to translate from continuous to discrete. Some students relate the position of the strip to how it rotates. For example, if they are asked to rotate about the *x*-axis they draw a Δx strip. Some students also talk about the disc method and rotation even when they are asked to find the centroid which depends on area only, without rotation.

Even if the graph was drawn, the students in most cases failed to show the diagrams when translating the drawn strip from 2D to 3D after rotation. The students were successful in translating from visual graphs to algebraic equations as they were able to select the correct equations from the formula sheet and substituted correctly, for disc washer, shell centroid and centre of gravity. The difficulties that these students have, might emanate from the way in which they were taught. The lecturer drew graphs without involving the students and without translating from algebraic equations to visual representation and identified the strips on those graphs without making any link to the approximation of the bounded region, hence failing to translate from continuous to discrete. The students in this case did not know why a particular strip was used. The lecturer only emphasised that the strip had to be either parallel or perpendicular to a certain axis and how it rotated as well as which formula was used upon rotation. Even though the lecturer was adequately qualified and knew most of the VSOR content well, procedural skills were more emphasised during the lesson, rather than the conceptual skills. It was also evident that most students, including the lecturer had problems in solving problems that involved the use of the substitution method, hence pointing to the complexity added to VSOR through integrating techniques. The fact the students observed did not have a lecturer at N5 level and that they had to do the VSOR content on their own, adds to the complexity since they lack the adequate background knowledge. It is therefore necessary that the preknowledge required for the VSOR content be revised at N6 level, emphasising the development of conceptual skills. That should be done despite the fact that the concepts to be learnt were done at previous levels. In that case the content may be accessible to the students.