CHAPTER 6: QUESTIONNAIRE AND EXAMINATIONS

This chapter presents Phase II, being the main study. Section 1 consists of the students' written responses that were gathered through two investigations from the developed 23-item instrument (the questionnaire), with 11 elements grouped under the five skill factors. The first investigation involves the first run of the questionnaire that was administered in April 2007 to 37 respondents, while the second investigation involves the second run of the questionnaire that was administered in October 2007 to 122 respondents and later again in April 2008 to 54 respondents. Section 2 consists of two investigations from the August 2007 mathematics N6 examinations. The first investigation involves the analysis of the examination results for 151 respondents while the second investigation involves detailed written examination responses with seven respondents from College A. The students' written responses were marked according to the rank scores and classified into 5 elements under Skill factor V. The data are presented in tables and multiple bar graphs and described quantitatively and qualitatively. The skill factors are further classified as requiring conceptual skills or procedural skills or both. The interpretation of the data presented and analysed in this chapter is done in Chapter 9. It is located within the conceptual framework of this study discussed in Chapter 3 and related to existing studies, discussed in Chapter 2.

6.1 PRESENTATION AND ANALYSIS OF THE RESULTS FROM THE 23-item INSTRUMENT (QUESTIONNAIRE)

The presentation and analysis of the responses for the 23 questions are classified and discussed under the 11 elements and the five skill factors and summarised. Similar elements, for example, Elements 6 and 7, both relating to translation between 2D and 3D, are discussed together. Tables and multiple bar graphs are used to display rank scores for the responses as *fully correct* (FC:4), *almost correct* (AC:3), *traces of understanding* (TU:2), *no understanding* (NU:1) and *not done* (ND:0), shown in Appendix 4A for the Questionnaire 1st run and Appendix 5A and 5C for the Questionnaire 2nd run. The response percentage per rank score with the raw score in the brackets for the Questionnaire 1st run in April 2007 with 23 questions and 37 respondents; the Questionnaire 2nd run in October 2007 with 16 questions and 122 respondents and again in April 2008 with 7 questions and 54 respondents is given. The description of performance is discussed in terms of the proportion of the acceptably correct responses (sum of fully correct and almost correct responses) at different performance levels. A summary of students' written responses are presented under the rankings, almost correct; traces of understanding and no understanding, with some examples of students' actual written responses showing traces of understanding and no understanding.

6.1.1 Skill factor I: Graphing skills and translating between visual graphs and algebraic equations/expressions in 2D and 3D

- **Element 1: Graphing skills**

Two questions were given:

1 A: Draw a line with a positive gradient passing through the origin for $x \in [0,3]$.

1 B: Sketch the graphs and shade the first quadrant area bounded by $x^2 - y^2 = 9$ and $x = 5$.

<u>Questionnaire 1st run</u>

Table 6.1: Element 1 for the Questionnaire 1st run as Question 1

RESPONSES	Q1A	%	Q1B	%
Fully correct	4	10.8	9	24.3
Almost correct	6	16.2	10	27.0
Traces of understanding	0	0	8	21.6
No understanding	17	46	10	27.0
Not done	10	27.0	0	0
% (FC + AC)		**27**		**51.3**

N = 37

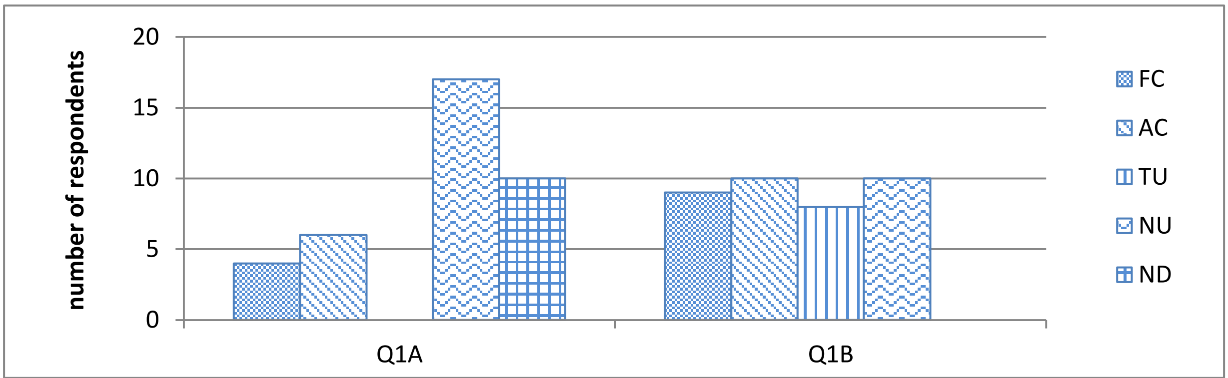


Figure 6.1: Questionnaire 1st run for Question 1

<u>Questionnaire 2nd run</u>

Table 6.2: Element 1 questions

RESPONSES	Q1A	%	Q1B	%
Fully correct	8	6.6	54	44.3
Almost correct	17	13.9	16	13.1
Traces of understanding	32	26.2	19	15.6
No understanding	58	47.5	31	25.4
Not done	7	5.7	2	1.6
% (FC + AC)		**20.5**		**57.4**

N = 122

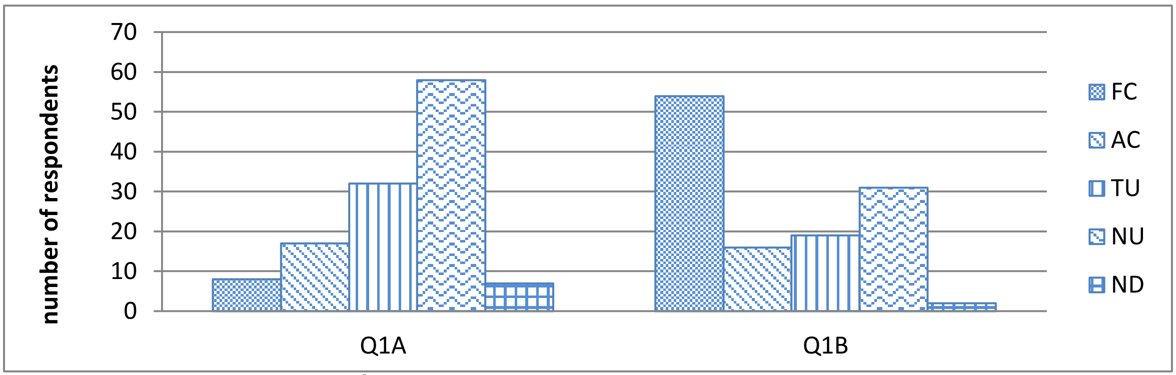


Figure 6.2: Questionnaire 2nd run for Question 1

From Tables 6.1 and 6.2 as well as Figures 6.1 and 6.2 for both questionnaire runs, it is evident that the performance was low for Question 1A (drawing a line with a positive gradient), with a large proportion of responses, 46 % (17) and 47.5% (58), for the Questionnaire 1st run and for the Questionnaire 2nd run respectively, showing no understanding. There was no student from the Questionnaire 1st run that showed traces of understanding in this question, while for the Questionnaire 2nd run, only 26.2% (32) of the responses revealed some traces of understanding. It is also evident from both bar graphs (Figures 6.1 and 6.2), which are negatively skewed that most responses are not acceptably correct, 27% (10) and 20.5% (25) for the Questionnaire 1st run and for the Questionnaire 2nd run respectively (given in Tables 6.1 and 6.2). This proportion of responses that are not acceptably correct indicates that the performance in drawing a line with a positive gradient passing through the origin for a certain interval in both runs of the questionnaire was not satisfactory.

In Question 1B for the Questionnaire 1st run, there was the same trend in the responses, 24.3%; 27%; 21;6% and 27% (9, 10, 8 and 10) respectively for all categories, fully correct; almost correct; traces of understanding and no understanding, with all students attempting this question. The students were required to draw a hyperbola and a straight line. The same trend in responses is represented from the nearly symmetric bar graph (Figure 6.1), where 51.3% (19) of the responses were acceptably correct. However, for the Questionnaire 2nd run, 44.3% (54) of the responses were fully correct while 25.4% (31) of the responses revealed that the students did not understand the question, with only 1.6% (2) of the responses not done. It is also indicated from the bar graph (Figure 6.2), which is positively skewed, that a large proportion of the responses 57.4% (70) were acceptably correct. The results reveal that the performance in drawing a rectangular hyperbola and a straight line graph, as well as shading the bounded area in both runs of the questionnaire was satisfactory.

Below some examples of what students actually did in Question 1A and 1B are given.

Question 1A: Typical students' errors when drawing a line with a positive gradient

Almost correct responses: Students drew

- A line $y = x$ and $y = 1$ with a Δy strip labelled Δx
- A line $y = x$ and $y = 3$

Traces of understanding: Students drew

- A line with a positive gradient passing through the negative *x*–axis and passing through the *y*- axis at 3

No understanding: Students drew

- A vertical line on the *y*-axis ending at point 3
- A vertical line on the *y*-axis with an arrow passing point 3 and going down below pointing down below zero
- A vertical line pointing up on the *y*-axis with an arrow at point 3
- A line $y = 3$ showing coordinates (0;3)
- Point 3 on the y – axis
- A line with a negative gradient intersecting the y – axis and the x – axis
- Point 3 on the x – axis
- A line $x = 3$ drawn going up starting from the x – axis
- A line $x = 3$
- A line with a negative gradient passing through the y – axis at 3
- A number line from 0 to 3
- A line $y = 3$

The responses given above reveal that the students did not know what a positive gradient means. The students drew different kinds of lines including those with a negative gradient as well as horizontal and vertical lines passing through 3. The students were also not able to use the given interval of $x \in [0, 3]$. The lines passing through 3 might be an indication that the students were not able to interpret and use the given interval of $x \in [0, 3]$, which was seemingly interpreted to mean the *x* and the *y* intercepts.

In Figures 6.3 and 6.4, examples of actual written responses are given for the responses showing *no understanding*. Figure 6.3 shows an example of a horizontal line passing through 3, while Figure 6.4 shows a line with a negative gradient.

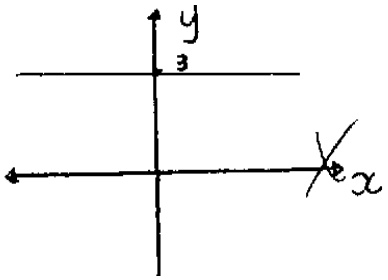


Figure 6.3: A line passing through *y* = 3

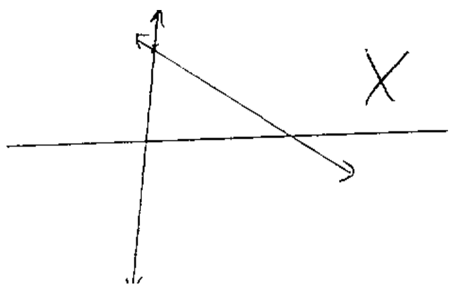


Figure 6.4: A line with a negative gradient

Question 1B: Typical students' errors when drawing a hyperbola $x^2 - y^2 = 9$ and $x = 5$

Almost correct responses: Students drew

- Both graphs correctly, but did not shade anything
- Both graphs correctly, but shaded quadrant 1 and 2
- Both graphs correctly with a Δy strip in the 1st quadrant
- Both graphs correctly with a Δy strip in the 1st and the 4th quadrants

Traces of understanding: Students drew

- $x = 5$ correctly, but had problems with the graph of $x^2 - y^2 = 9$, which was represented differently as half an ellipse with intercepts on the y – axis as ± 3
- A circle with intercepts on the y – axis and on the x – axis as ± 3, with a Δx strip in the circle's 1st quadrant
- The graphs $x = 5$ and $x^2 - y^2 = 9$ not intersecting since the hyperbola was not extended
- A circle with intercepts on the y – axis as ± 3
- Half a circle with intercepts on the y – axis as 3 and on the x-axis as ± 3
- A full rectangular hyperbola with both x - intercepts
- A semicircle or a quarter of a circle

No understanding: Students drew

- Half an ellipse intersecting the x– axis at ± 5 and the y-axis at – 3
- Different graphs like a parabola or an incorrect rectangular hyperbola
- An ellipse with x- intercepts as ± 3 and y – intercepts as ± 4
- An ellipse with x- intercepts as ± 5 and y – intercepts as ± 4
- Half a circle and the line $y = x$
- A line passing through $x = -3$ and $y = 3$

The responses given above reveal that most of the students were able to draw the line $x = 5$, but had problems drawing the graph of $x^2 - y^2 = 9$. Some students were seen to draw ellipses, circles, even semicircles when attempting to draw the hyperbola. The students also had problems in showing the correct intercepts of the hyperbola on the x-axis only. In many instances the x – intercepts were incorrect. The y – intercepts were also given even though they were supposed to be non-real roots. Figures 6.5 and 6.6 are examples of actual written responses showing *no understanding*, showing half of an ellipse with both intercepts on both axes.

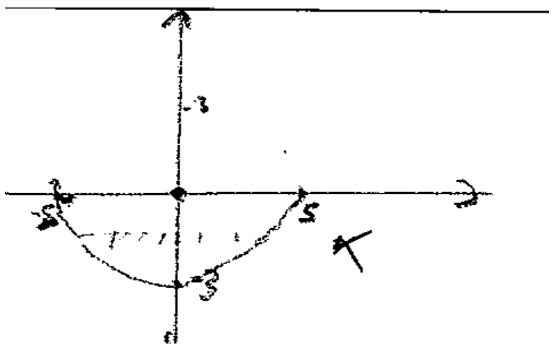


Figure 6.5: Half an ellipse below the x-axis

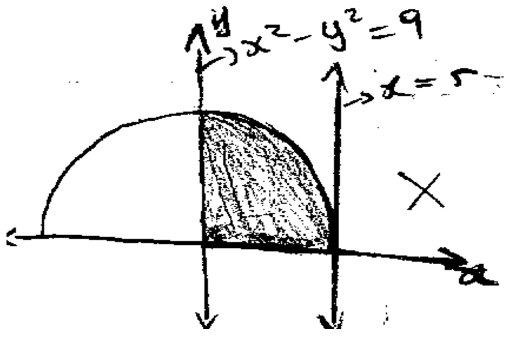


Figure 6.6: Half an ellipse above the x-axis

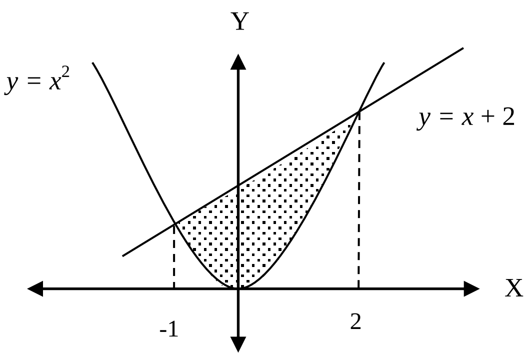
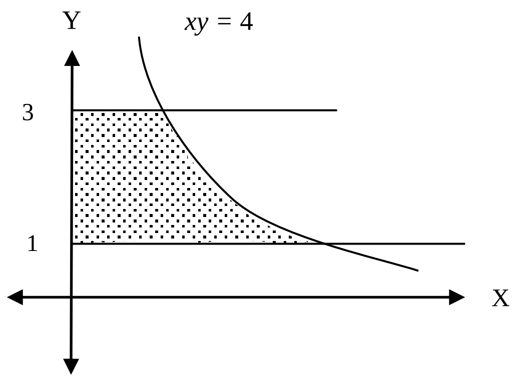
Discussion on Element 1

From Question 1A, drawing a straight line graph given as word problem, seemed simpler than Question 1B where a graph of a hyperbola $x^2 - y^2 = 9$ and a straight line $x = 5$ were required, yet a large proportion of the students, 51.3% (19) and 57.4% (70) respectively from Questionnaire 1st run and Questionnaire 2nd run were able to draw both graphs in Question 1B, regarded as satisfactory performance, compared with only 27% (10) from Questionnaire 1st run and 20.5% (25) from Questionnaire 2nd run who could not draw the straight line in Question 1A, resulting in performance that is not satisfactory for this question. This might be because Question 1B was familiar since it is similar to some questions in past examination papers, unlike Question 1A which was given as a word problem. In drawing a line with a positive gradient passing through the origin for $x \in [0,3]$ some students drew vertical lines ending at 3 or lines $y = 3$ or $x = 3$ and other different lines. Those who tried to draw lines with a positive gradient had their lines not through the origin, sometimes passing through 3 on the *x*-axis or on the *y*-axis. The students who drew a line *y* = 3, may have done that because they misinterpreted $x \in [0,3]$ to mean the coordinates (0;3) which means that the *y* value is 3 while the *x* value is 0. The students had difficulty in interpreting a verbal description such as "a line with a positive gradient" and did not know what $x \in [0,3]$ means.

- **Element 2: (algebraic to visual in 2D) and Element 3 (visual to algebraic in 2D)**

The questions for Elements 2 and 3 were as follows:

Table 6.3: Element 2 and 3 questions

<table>
<tr><td>2A: Represent $x^2 + y^2 \leq 9$ by a picture.</td><td>2B: Sketch the area represented by $\int_0^1 (x - x^2)\, dx$.</td></tr>
<tr><td>3A: Substitute the equations of the given graphs in a suitable formula to represent the area of the shaded region.

</td><td>3B: Substitute the equations of the given graphs in a suitable formula to represent the area of the shaded region.

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Questionnaire 1st run

Table 6.4: Element 2 and 3 for the Questionnaire 1st run as Question 2 and Question 3

RESPONSES	Q2A	%	Q2B	%	Q3A	%	Q3B	%
Fully correct	3	8.1	13	35.1	31	83.8	16	43.2
Almost correct	28	75.7	8	21.6	2	5.4	4	10.8
Traces of understanding	4	10.8	4	10.8	3	8.1	8	21.6
No understanding	1	2.7	10	27.0	1	2.7	**8**	21.6
Not done	1	2.7	2	5.4	0	0	1	2.7
%(FC + AC)		**83.8**		**56.7**		**89.2**		**54**

N = 37

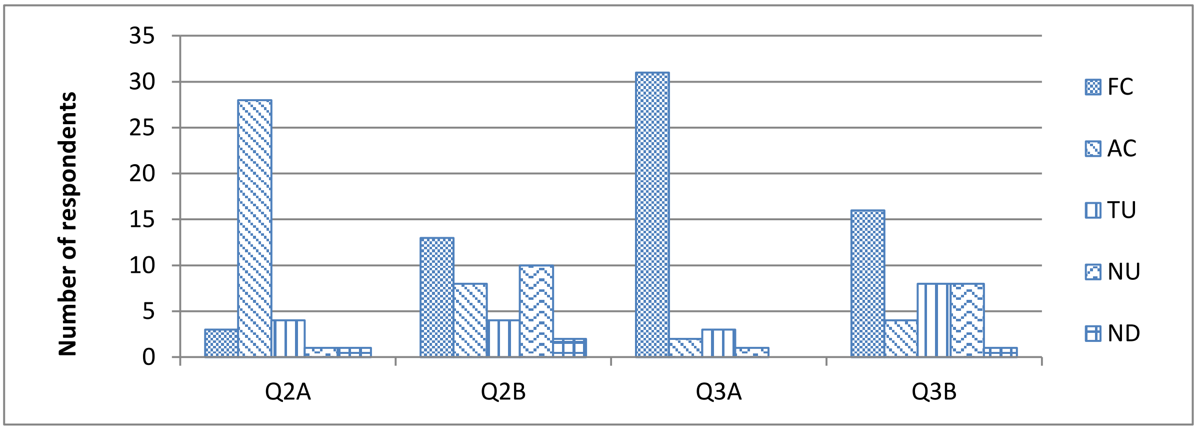


Figure 6.7: Questionnaire 1st run for Question 2 and Question 3

Questionnaire 2nd run

Table 6.5 Element 2 and 3 for the Questionnaire 2nd run as Question 2A and Question 3A

RESPONSES	Q2A	%	Q3A	%
Fully correct	0	0	30	55.6
Almost correct	29	53.7	17	31.5
Traces of understanding	18	33.3	0	0
No understanding	5	9.3	5	9.3
Not done	2	3.7	2	3.7
% (FC + AC)	**53.7**		**87.1**	

N = 54

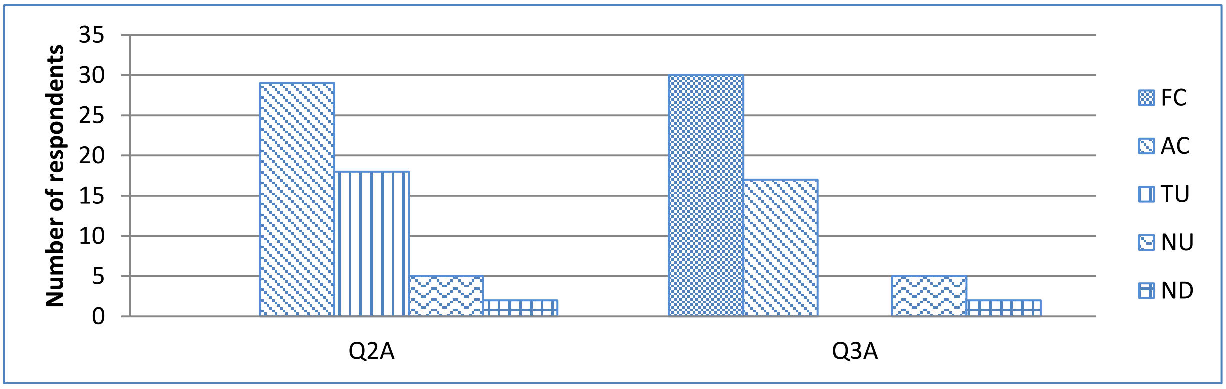


Figure 6.8: Questionnaire 2nd run for Question 2A and Question 3A

Table 6.6 Element 2 and 3 for the Questionnaire 2nd run as Question 2B and Question 3B

RESPONSES	Q2B	%	Q3B	%
Fully correct	9	7.4	43	35.3
Almost correct	28	23	11	9.0
Traces of understanding	30	24.6	44	36.1
No understanding	48	39.3	23	18.9
Not done	7	5.7	1	0.8
% (FC + AC)		**30.4**		**44.3**

N = 122

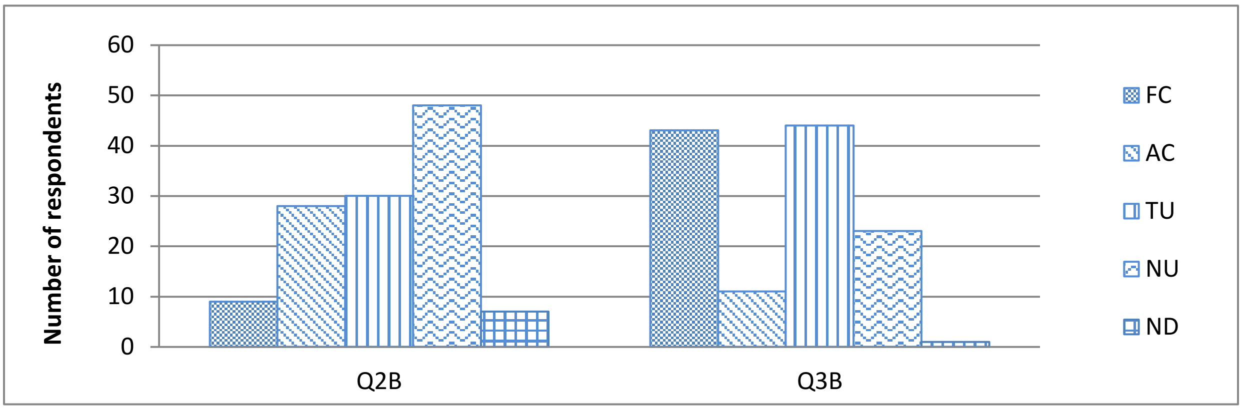


Figure 6.9: Questionnaire 2nd run for Question 2B and Question 3B

Tables 6.4, 6.5 and 6.6 and Figures 6.7, 6.8 and 6.9 display the pattern of responses for the four questions, Question 2A, 2B, 3A and 3B. In Question 2A from Questionnaire 1st run, a large proportion of almost correct responses, 75.7% (28) were the students who were able to in draw a circle but did not shade inside it to represent the inequality. For the Questionnaire 2nd run, 53.7% (29), was the highest proportion of almost correct responses, in relation to the other rank scores, being those students who did not shade inside the circle to represent the inequality. The results reveal that from the proportion of acceptably correct responses, the performance was excellent, 83.8% (31) for the Questionnaire 1st run, and satisfactory for the Questionnaire 2nd run 53.7% (29), evident from the positively skewed bar graph in Figures 6.7 and 6.9.

For Question 2B, the highest proportion of responses, 35.1% (13), from the Questionnaire 1st run, showed that the responses were fully correct, with only 7.4% (9) from the Questionnaire 2nd run. Despite the fact that most of the responses from the Questionnaire 1st run were fully correct, there was also a higher proportion of responses, 27% (10), indicating that the students did not understand the question. From Questionnaire 2nd run, most of the responses, 39.3% (48), reveal that the students struggled with this question, showing no understanding. The students were expected to sketch a graph represented by an integral formula for area representing a parabola $y = x - x^2$ or a straight line $y = x$ and a parabola $y = x^2$. The results reveal that the performance was satisfactory, with 56.7% (13) of the responses being acceptably correct in the Questionnaire 1st run, evident from the positively skewed bar graph in Figure 6.7. For the Questionnaire 2nd run, the performance was not satisfactory, with only 30.4% (37) of the responses being acceptably correct, evident from the negatively skewed bar graph in Figure 6.8. Overall, in answering Question 2, the students performed better in the Questionnaire 1st run than in the Questionnaire 2nd run, where none of the students were able to give fully correct responses for Question 2A, with only 7.4% (9) of the responses being fully correct in Question 2B.

For Question 3A a large proportion of the responses, 83.8% (31) from the Questionnaire 1st run were fully correct, where all the students responded to the question. The students were successful in substituting correctly from the drawn graphs when representing area as an integral where a Δx strip was appropriate. For the Questionnaire 2nd run, even if most of the responses, 55.6% (30) are fully correct, this percentage is very low when compared with the 83.8% from the Questionnaire 1st run. However, both graphs representing Question 3A (Figures 6.7 and 6.9) from the questionnaire runs are positively skewed. The performance in this question was excellent, with 89.2% (33) and 87.1% (47) of the acceptably correct responses in the Questionnaire 1st run and Questionnaire 2nd run respectively.

For Question 3B, which required the use of a Δy strip when representing the formula for area, the highest proportion of responses, 43.2% (16) for the Questionnaire 1st run represent fully correct responses, with equal proportion of responses, 21.6% of the responses showing traces of understanding and no understanding. However, as represented from the bar graph (see Figure 6.7), the data are positively skewed, where most of the responses, 54% (20) are acceptably correct, indicating satisfactory performance. For the Questionnaire 2nd run, almost the same proportion of responses, 35.3% (43) and 36.1% (44) respectively were fully correct and showing traces of understanding, evident from the bi-modal graph in Figure 6.8. The performance from the Questionnaire 2nd run was also satisfactory, with 44.3% (54) of acceptably correct responses.

All four positively skewed graphs in Figure 6.7 from the Questionnaire 1st run reveal that overall, most of the responses were acceptably correct, 83.8% (31), 56.7% (21), 89.2% (33) and 54% (20) in Questions 2A, 2B, 3A and 3B respectively. For the Questionnaire 2nd run, the graphs are also positively skewed as in Figure 6.9, indicating that there were most acceptably correct responses, 53.7% (29) and 87.1% (47) in Questions 2A and 3A respectively. However, from all bar graphs for Question 2 and 3 (Figures 6.7, 6.8 and 6.9), the lowest bar is for not done, representing that the majority of the students, more than 94% (Tables 6.4, 6.5 and 6.6) attempted Question 2 and 3.

Below some examples of what students actually did in Question 2A and 2B are given.

Question 2A: Typical students' errors in representing an inequality for a circle

Almost correct: Students drew

- A circle with intercepts ± 3 for x and y

Traces of understanding: Students drew

- A circle with intercepts ± 9 for x and y
- Semicircle with x intercepts ± 3 and y intercept of 3

No understanding: Students drew

- A quarter of a circle in quadrant 1

The responses reveal that the majority of students were able to draw the graph but did not use the inequality by shading inside the circle. The performance was relatively good.

Question 2B: Typical students' errors in drawing the graph represented by an integral

Almost correct: Students drew

- A correct graph but shaded even below the x –axis
- $y = 1 - x^2$ and shaded the first quadrant
- $y = 1 - x^2$ with a Δy strip and labelled it as $y = x - x^2$
- $y = x$ and the Δx strip
- A correct graph with one x – intercept incorrect
- $y = x$ and $y = x^2$ a Δx strip
- The parabola $y = x^2 - x$
- The opposite of the correct graph

Traces of understanding: Students drew the graph of

- $y = x$ and $y = -x^2$ and interpreted – sign incorrectly
- $y = x$ and part of a circle in the 1st quadrant only
- $y = x$ and $y = x^2 - x$

No understanding: Students drew

- A graph similar to $x = \sqrt{y}$ and shaded above the x axis for $y \in (0;2)$
- $y = -x^2$ with the shading below it in the 4th quadrant
- $y = -x^2$ in the 3rd and the 4th quadrant with a Δx strip selected on top of the graph between 0 and 1
- A line having the x intercept as 1 and the y intercept as 1
- $y = x + 1$ with a Δx strip
- A line having the x intercept as 1 and the y intercept as 1
- $y = -x + x^2$ with a Δx strip

Even if the performance for Question 2B was not satisfactory, most of the students had some idea about what the question entailed, that of drawing a parabola facing downwards and shading the first quadrant area between the x values of 0 and 1. However, most of the students who succeeded in this question did not identify the integral given to relate to the region bounded by the straight line $y = x$ and the parabola $y = x^2$. The integral was in most cases identified to represent the parabola $y = x - x^2$, the way in which it was asked. The students did not interpret the formula to identify the region bounded by the two graphs, being the straight line $y = x$ and the parabola $y = x^2$. They interpreted the question as if they were asked to draw the parabola, which means that the integral formula representing the region bounded by the two graphs was not identified. In Figures 6.10 and 6.11, examples of actual written responses are given showing *no understanding*, where in some cases straight lines and parabolas facing upwards were drawn.

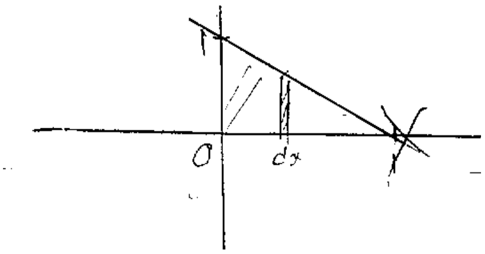


Figure 6.10: A line with a negative slope

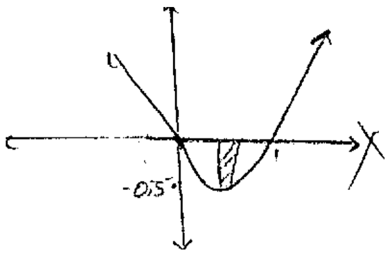


Figure 6.11: A parabola $y = x - x^2$

Discussion on Element 2

In Question 2A, most of the students with acceptably correct responses, 75.7% (28) from Questionnaire 1st run and 53.7% (29) from Questionnaire 2nd run, were able to draw the required circle but did not indicate the inequality involved by shading the inside of the drawn circle. However, the performance in the Questionnaire 1st run was excellent, 83.8% (31) while the performance in the Questionnaire 2nd run was satisfactory, 53.7% (29). The performance in both questionnaire runs was lower in Question 2B requiring that the students must sketch the area represented by an indefinite integral formula, representing a parabola, or a straight line and a parabola. This might be as a result of students being familiar to the way in which this question is assessed in past examination papers and in their textbooks, where they are used to writing down the formula that represents the area of the region bounded by the graphs and not the other way round. The performance in the Questionnaire 1st run in Question 2B was satisfactory, 56.7% (21), while the performance in the Questionnaire 2nd run was not satisfactory, 30.4% (37). The successes in Question 2, was mainly based on the responses, showing traces of understanding, and not on the fully correct responses.

Below some examples of what students actually did in Question 3A and 3B are given

Question 3A: Typical errors when performing the substitution requiring a Δx strip

Almost correct: Students
- Made mistakes with one graph from correct limits
- Substituted correctly but did not specify the limits used

Traces of understanding: Students
- Wrote the correct equation but did not substitute
- Integrated incorrectly

No understanding: Students
- Took moments about the x – axis as if they were calculating the centroids

The students performed extremely well in this section, no actual written examples are given.

Question 3B: Typical errors when performing the substitution requiring a Δy strip

Almost correct: Students
- Chose a correct formula and a Δy strip but did not substitute
- Chose a correct formula and substituted correctly, but drew a Δx strip which was labelled incorrectly as Δy and did not substitute the limits

Traces of understanding: Students
- Drew a Δy strip but used an incorrect formula as $\int (y_1 - y_2)\, dy$, then $\int_4^3 \left(3 - \frac{4}{x}\right) dy$
- Drew a Δy strip but used an incorrect formula $\int_1^3 3^2 - \left(\frac{4}{y}\right)^2 dy$
- Drew a Δx strip but used y values as boundaries
- Drew a Δy strip but used Δx incorrectly in the formula as $\int_1^3 \frac{4}{x} dx$
- Drew a Δy strip and the correct formula but did not substitute in the formula
- Used formula as $\int_1^3 \left(\frac{4}{x}\right) dx$ with limits for y
- Used formula as $\int_1^4 \left(\frac{4}{x}\right) dx$

No understanding: Students
- Did not substitute nor draw the strip
- Only wrote down equations as $y = \frac{4}{x}$; $y = 3$ and $y = 1$
- Wrote only $y = \frac{4}{x}$
- Wrote $y = 0$, $x = y$ and $y = e^{2x}$
- Used moment about the x – axis and formula $\int \left(\frac{4}{x}\right) - 3 - 1\, dx$

Despite the fact that on the graph, the y values that would serve as limits were given, some students abandoned them and did not use them or used them with Δx instead of Δy. Some students drew a Δy strip, but could not represent it correctly when translating from the graph (visual) to an algebraic equation for area, especially when having to make x the subject of the formula in order to substitute. However, in some cases, the students were successful in making x the subject of the formula, in expressing $x = \frac{4}{y}$ and substituted with it correctly, to give the expression for area, with incorrect limits, while in most cases the incorrect formula for substitution, was used as $y = \frac{4}{x}$ with a Δx strip, which reveals the preference of these students in using a Δx strip even if it is not possible.

In Figures 6.12, 6.13, 6.14 and 6.15, examples actual written responses are given for the responses showing *no understanding* and *traces of understanding*.

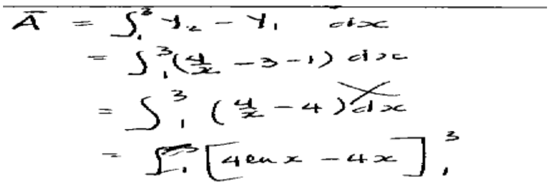


Figure 6.12: Δx with y limits

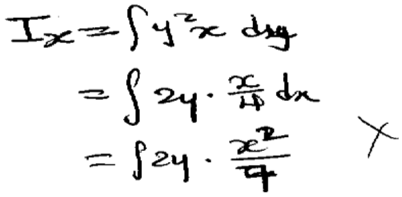


Figure 6.13: Formula for moment of inertia

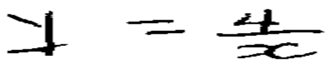


Figure 6.14: A hyperbolic equation

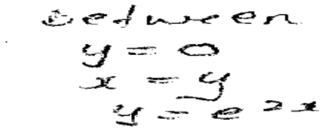


Figure 6.15: An exponential equation

Discussion on Element 3

In Element 3 the questions required that students must substitute the equations of given graphs into a suitable formula for area. Most of the students, 89.2% (33) from the Questionnaire 1st run and 87.1% (47) from the Questionnaire 2nd run were successful in Question 3A, which reveals that they could substitute correctly from the correct formula for area when a Δx strip was appropriate, regarded as excellent performance. The students successfully translated from the visual graph to the algebraic equation for area. However, the level of success was lower in Question 3B, in the Questionnaire 1st run with only with 54% (20) successes and 44.3% (54) successes in the Questionnaire 2nd run, regarded as satisfactory performance. The reason might perhaps be based on the fact that the students were now in Question 3B required to use a Δy strip, which they do not normally prefer to work with.

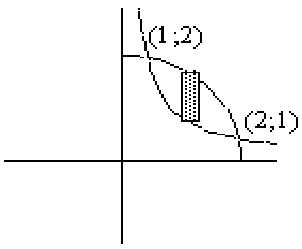
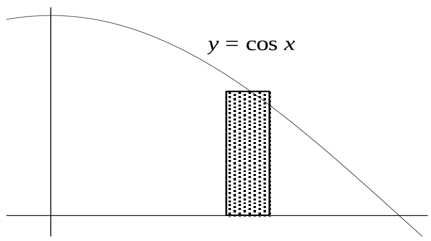
Summary for Element 2 and 3

The results for Element 2 and 3 reveal that in some instances the students were able to translate from algebraic equations/expressions to visual graphs (Element 2) to a lesser extent than when translation from visual graphs to algebraic equations (Element 3). It seems as if the students are not good at drawing graphs, that represent integral formula for area, but if a graph is drawn, they are able to translate appropriately to the formula for area, using integrals, preferably if the Δx strip is required. The reason for using a Δy strip for those who used it might be because the y values were given in the question, as it was for Question 3A where the x values were given.

- **Element 4 (algebraic to visual in 3D) and Element 5 (visual to algebraic in 3D)**

The questions for Element 4 and 5 were as follows:

Table 6.7: Element 4 and 5 questions

4A: Draw the 3-D solid of which the volume is given by $V = \pi\int_0^1 (1-x)^2\, dx$ and show the representative strip.	4B: Draw the 3-D solid of which the volume is given by $V = 2\pi\int_0^1 x(1-x^2)\, dx$ and show the representative strip.
5A: The figure below represents the first quadrant area bounded by the graphs of $x^2 + y^2 = 5$ and $xy = 2$. Using the selected strip, substitute the equations of the given graphs in a suitable formula to represent the volume generated if the selected area is rotated **about the *x*-axis.** Do not calculate the volume. 	5B: The figure below represents the area bounded by the graphs of $y = \cos x$, the *x*-axis and the *y*-axis. Using the selected strip, substitute the equations of the given graphs in a suitable formula to represent the volume generated when this area is rotated **about the *y*-axis.** Do not calculate the volume. 

<u>Questionnaire 1st run</u>

Table 6.8: Element 4 and 5 for the Questionnaire 1st run as Question 4 and Question 5

RESPONSES	Q4A	%	Q4B	%	Q5A	%	Q5B	%
Fully correct	4	10.8	4	10.8	23	62.2	4	10.8
Almost correct	8	21.6	18	48.7	6	16.2	6	16.2
Traces of understanding	3	8.1	10	27.0	5	13.5	0	0
No understanding	11	29.7	2	5.4	3	8.1	18	48.7
Not done	11	29.7	3	8.1	0	0	9	24.3
% (FC + AC)		**32.4**		**59.5**		**78.4**		**27**

N = 37

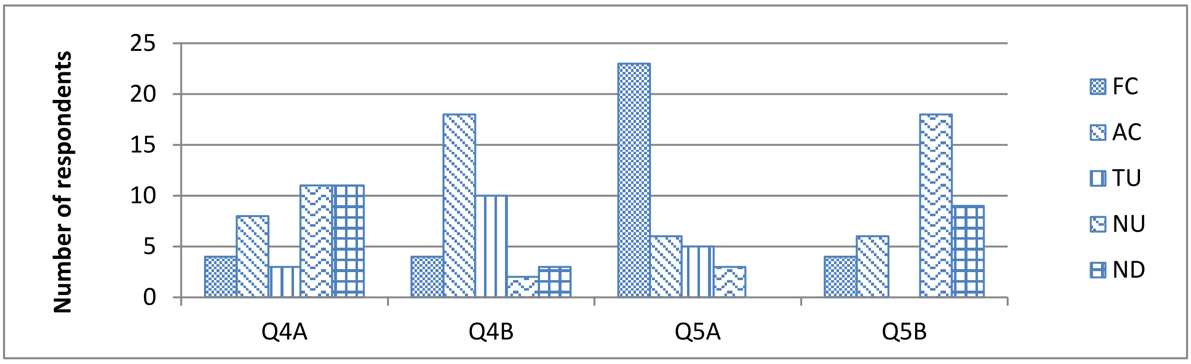


Figure 6.16: Questionnaire 1st run for Question 4 and Question 5

Questionnaire 2nd run

Table 6.9: Element 4 for the Questionnaire 2nd run as Question 4

RESPONSES	Q4A	%	Q4B	%
Fully correct	0	0	2	1.6
Almost correct	5	4.1	16	13.1
Traces of understanding	25	20.5	15	12.3
No understanding	61	50	75	61.5
Not done	31	25.4	14	11.5
% (FC + AC)		4.1		14.7

N = 122

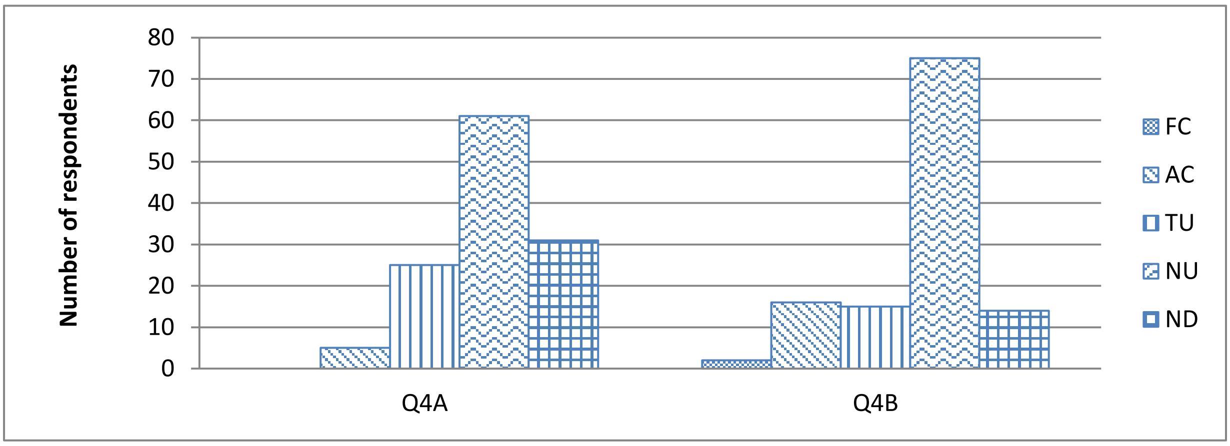


Figure 6.17: Questionnaire 2nd run for Question 4

Table 6.10: Element 5 for the Questionnaire 2nd run as Question 5A

Responses	Q5A	%
Fully correct	32	59.3
Almost correct	4	7.4
Traces of understanding	9	16.7
No understanding	9	16.7
Not done	0	0
% (FC + AC)		66.7

N = 54

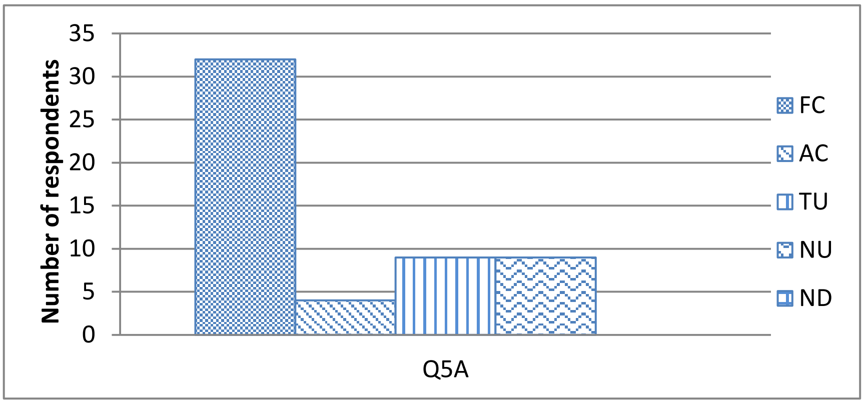


Figure 6.18: Questionnaire 2nd run for Question 5A

Table 6.11: Element 5 for the Questionnaire 2nd run as Question 5B

RESPONSES	Q5B	%
Fully correct	18	14.8
Almost correct	23	18.9
Traces of understanding	14	11.5
No understanding	57	46.7
Not done	10	8.2
% (FC + AC)		33.7

N = 122

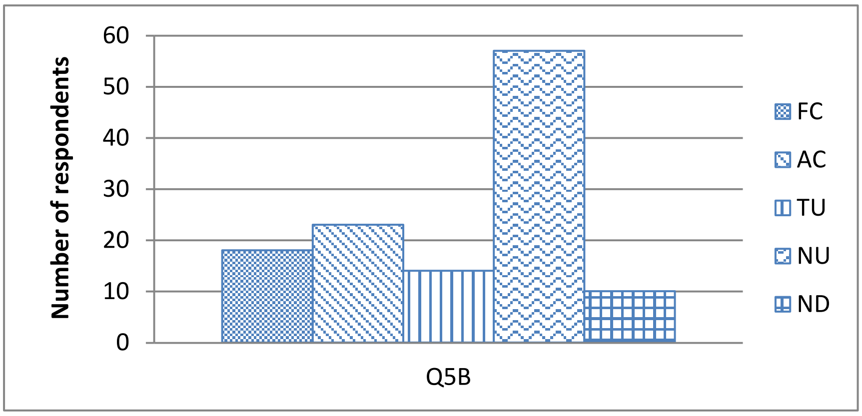


Figure 6.19: Questionnaire 2nd run for Question 5B

All the questions under Elements 4 and 5 involve 3D solids resulting from rotation of 2D graphs. The results for Question 4 (the translation from algebraic to visual) in both runs of the questionnaire are discussed from Tables 6.8 and 6.9 as well as Figures 6.16 and 6.17. In the Questionnaire 1st run, students performed better in Question 4B where they were expected to draw a 3D solid (representing a shell method) from the formula for volume given as an integral, compared with Question 4A where they were expected to draw a 3D solid (representing a disc method) from the formula for volume given as an integral. In Question 4A the highest proportion of the responses 29.7% (11) showed no understanding and not done, with only 10.8% (4) of the responses being fully correct and 21.6% (8) showing almost correct responses, while for Question 4B, 10.8% (4) of the responses were fully correct and 48.7% (18) showing almost correct responses. For the Questionnaire 2nd run, the performance for Questions 4A and 4B was poor. In Question 4A, only 4.1% (5) of the responses were almost correct, 50% (61) showing no understanding with no fully correct responses. In Question 4B only 1.6% (2) of the responses were fully correct, with most of the responses 61.5% (75) showing no understanding.

A higher proportion of acceptably correct responses (59.5%) to Question 4B from the Questionnaire 1st run reveals that the students were able to relate the graph from the 2π to the shell method, regarded as satisfactory performance, compared to Question 4A, where only 32.4% of the responses were acceptably correct, regarded as performance that was not satisfactory. The problem encountered by most of the students in Question 4A and in Question 4B, as it was evident from the Questionnaire 2nd run was that the students were not able to draw a 3D solid, evident from the low proportion of acceptably correct responses in these questions as 4.1% (5) and 14.7% (18) respectively, revealing that the performance was poor. Few acceptably correct responses in Question 4A (32.4%) in the Questionnaire 1st run and Question 4 (4A: 4.1% and 4B: 14.7%) in the Questionnaire 2nd run are displayed from all three negatively skewed graphs in Figures 6.16 and 6.17 respectively.

The results for Question 5 (the translation from visual to algebraic) in both runs of the questionnaire are discussed from Tables 6.8, 6.10 and 6.11 as well as Figures 6.16, 6.18 and 6.19. From both questionnaire runs, it is evident that the students did better in Question 5A, than in Question 5B. In Question 5A, most of the responses were fully correct, 62.2% (23) for Questionnaire 1st run and 59.3% (32) for Questionnaire 2nd run, with all students attempting this question. The students did not do well in Question 5B with the highest proportion of responses, 48.7% (18) showing no understanding and 24.3% (9) of the students not responding for the Questionnaire 1st run, while for the Questionnaire 2nd run, the highest proportion of responses, 46.7% (57) showed that the students did not understand this question. In Question 5A, a Δx strip was drawn on the bounded region by the graphs of a circle and a hyperbola and students were asked to come up with a formula for volume upon rotation of this region about the *x*-axis, resulting in a washer. In Question 5B, a Δx strip was drawn on the bounded region by the graph of $y = \cos x$ and students were asked to come up with a formula for volume, resulting in a shell upon rotation about the *y*-axis.

In Question 5A the performance was excellent in the Questionnaire 1st run, with 78.4% (29) of the acceptably correct responses and good in the Questionnaire 2nd run with 66.7% (36) of the acceptably correct responses, as shown in Figures 6.16 and 6.18 from the positively skewed graphs. For Question 5B the performance was not satisfactory with 27% (10) of the responses being correct from the Questionnaire 1st run and 33.7% (41) of the responses being correct for the Questionnaire 2nd run, as shown in Figures 6.16 and 6.19 from the negatively skewed graphs. The results in Question 5 reveal that most of the students were able to translate from visual graphs to algebraic equations in 3D when rotating a Δx strip (drawn on the diagram) about the x-axis (Question 5A), resulting in a washer, but encountered difficulties when rotation was about the y-axis (Question 5B), resulting in a shell. A large proportion of acceptably correct responses in both questionnaire runs reveal that the students were able to come up with the correct formula for the volume and they also substituted correctly.

Overall, from both runs of the questionnaires, the students had difficulty with Questions 4A and 5B. For the Questionnaire 2nd run the results reveal that the students had difficulty in translating from algebraic equations to visual graphs in 3D, where a solid of revolution was to be drawn as required in Question 4A and 4B, than having to translate from visual graphs to algebraic equations in 3D resulting in a washer method. For the Questionnaire 2nd run, all the graphs representing Questions 4A, 4B and 5B are negatively skewed (Figures 6.17 and 6.19).

Below some examples of what students actually did in Question 4A and 4B are given.

Question 4A: Typical students' errors in drawing a 3D-solid represented by a disc from an equation

Almost correct: Students drew the graph of

- $y = 1 - x$ with a Δx strip but not a 3D solid
- $y = 1 - x$ without a strip also not a 3D solid

No understanding: Students drew

- A line with a positive gradient passing through the origin
- A parabola $y = x - x^2$ with a Δx strip labelled Δy
- A parabola $y = 1 - x^2$ with a Δx strip
- A parabola $y = 1 - x^2$ with a Δy strip
- A parabola similar to $y = x^2 - 1$ and the other one as $y = 1 - x^2$
- $y = x$ and Δx strips below it
- $y = x$ and $y = 1$, then selected a Δx strip, similar to washer as $V = \pi \int_0^1 (1^2 - x^2)\ dx$
- A parabola $y = (x-1)^2$ with a Δx strip between 1 and 0
- A line with x- intercept of -1 and y - intercept of 2
- A line with x - intercept of -1 and y – intercept of 1

The responses in the above examples reveal that the students did not understand Question 4A. It seems as if many students did not recognise the disc method in the formula, and therefore did not recognise that the straight line $y = 1 - x$ in this case was rotated about the *x*-axis, resulting in a disc. These students were not relating the square to the disc formula but instead they used it to draw different types of parabolas. One student related the given formula to a washer by drawing the graphs $y = x$ and $y = 1$, that represented a washer upon rotation about the *x*-axis. A 3D solid was not drawn as required to represent a solid of revolution from the integral equation given. Instead graphs given in 2D were drawn. Ordinary graphs, passing through the *x*-axis, sometimes even graphs that did not resemble a parabola or a straight line were drawn.

In Figures 6.20 and 6.21, actual written responses showing *no understanding* are indicated.

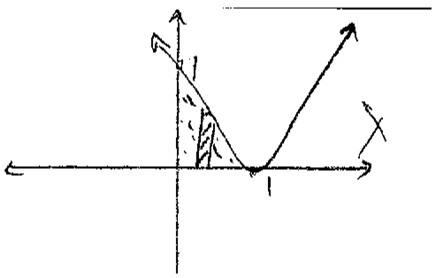


Figure 6.20: A positive parabola

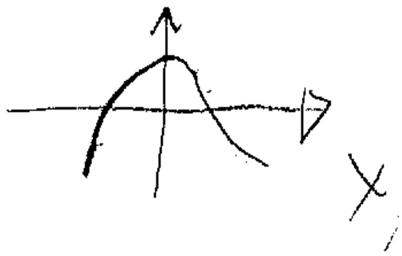


Figure 6.21: a negative parabola

Question 4B: Typical students' errors in drawing a 3D-solid represented by a shell from an equation

Almost correct: Students drew

- A correct graph, but passing down below the *y*-axis, showing a Δx strip and showing a disc
- A correct graph drawn, but not representing a 3D solid

Traces of understanding: Students drew

- Graphs of $y = 1 - x^2$ and $y = x$

No understanding: Students drew

- A graph similar to $y = -x^3$ with a Δx strip in the 4th quadrant
- A parabola with *y* intercepts only and a Δx strip below it between limits 1 and 0

The above actual responses give examples of the incorrect graphs showing how some of the students failed to draw a 3D solid from the integral equation (formula) for volume given. The students were not able to interpret the given equation appropriately. This was evident from the 2D graphs that were drawn. Some of the students drew two graphs separately as $y = x$ and $y = 1 - x^2$, since they saw them as separate graphs, while some drew the graph of $y = -x^3$, probably because they multiplied the expression $x(1 - x^2)$ from the given integral formula.

In Figures 6.22 and 6.23 examples of actual written responses are given showing *traces of understanding*.

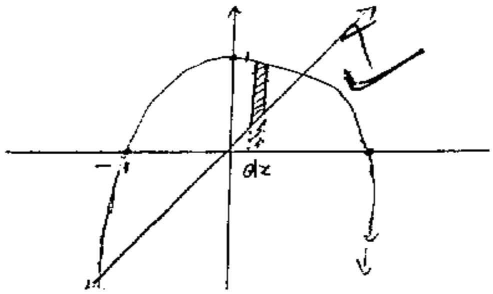


Figure 6.22: A complete parabola

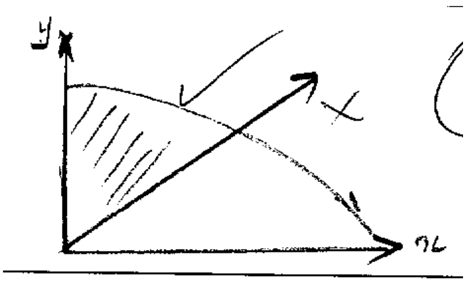


Figure 6.23: Half a parabola

Discussion on Element 4

In Element 4 the students were required to draw a 3D solid from the given formula for volume. Students in Questionnaire 1st run performed better than those in the Questionnaire 2nd run in Question 4B, with mainly almost correct responses. Question 4B employed a shell method. The success (mainly from the almost correct responses) might probably be because the 2π was related to the shell method. However the students were unable to draw a 3D solid, but were seen to draw ordinary parabolas passing through the *x*-axis. There were also some instances where some students misinterpreted the x in the formula, and in addition to the correct parabola, drew a graph of $y = x$ as well. In Question 4A most of the students did not associate the square on $1 - x$ with a disc. They interpreted the question incorrectly by drawing parabolas, instead of drawing the graph of a straight line $y = 1 - x$, which was rotated about the *x*-axis using a Δx strip on the interval [0,1]. Other students did not use this interval when drawing their graphs. Again, same as in Question 2A, where the formula for area was given as an integral, the students were given the formula for volume as an integral, and asked to represent it as a solid of revolution, which they were possibly not familiar with, since they are more familiar with finding the formula for volume after drawing the graphs and not the other way round.

Those students, who drew parabolas instead of straight lines, had difficulty in translating the given algebraic equation to the visual 3D graph, which represents a solid of revolution. The students failed to relate the given equations in 3D (one as a disc and the other one as a shell) to the graphs in 2D that they represent, perhaps because they are normally asked to come up with the equation for volume from the rotated region bounded by graphs, which is the opposite of the Questions 4A and 4B. The incorrect graphs drawn by students reveal that the students were not familiar with this type of questions.

Below some examples of what students actually did in Question 5A and 5B are given.

Question 5A: Typical students' errors in substituting, when rotation is about the *x*-axis

Almost correct: Students

- Substituted one equation correctly, with correct boundaries, at times π missing
- Substituted one equation correctly, but the integral sign and boundaries were missing

Traces of understanding: Students

- Wrote the correct formula for a washer with limits as 1 and 0, but did not substitute into it; while the other students used the limits as b and a, but did not substitute into it
- Solved the substituted equations incorrectly

No understanding: Students

- Redrew the graph and changed the strip into the Δy strip
- Used the formula $\pi \int_0^1 y^2 \, dx$
- Calculated area instead of volume

Those students, who failed in this question, were in most cases using an incorrect strip or the disc method instead of the washer method.

Question 5B: Typical students' errors in substituting, when rotation is about the *y*-axis

Almost correct: Students used

- Correct formula as $2\pi \int_0^{\frac{\pi}{2}} xy\,dx$ but did not finish
- Incorrect upper limits $\frac{\pi}{3}$ and $\frac{\pi}{4}$
- Wrote $2\pi \int_a^b x \cos x \, dx$

No understanding: Students

- Used formula for a disc using Δy in the formula even if a Δx strip was given in the diagram and continued to substitute, representing an inverse of a cosine $\pi \int_a^b x^2 \, dy = \pi \int_0^1 \cos^{-1} y \, dy$
- Used formula for a disc $\pi \int_a^b x^2 dy = \pi \int_0^{\frac{\pi}{2}} \cos^{-1} y \, dy$ even if a Δx strip was given
- Used $V = \pi \int \cos x \, dy$; $\pi \int_a^b \cos^2 x \, dx$; $\pi \int_{-1}^1 \cos^2 x \, dy$ or $\pi \int_0^{\frac{\pi}{2}} x \sin x + \cos x \, dx$;
- Drew the graph with a Δy strip
- Used different formulae were used for volume as $\int_a^b y^2 dx$; $\pi \int_0^1 \cos^2 x \, dx$; $\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx$;
- Used $V = \pi \int_a^b x^2 dy$ then $V = \pi \int \sec x \, dy$ while other used $2\pi \int_a^b y \cos^{-1} y \, dx$

In Figures 6.24 and 6.25, some examples of actual written responses are given where there was *no understanding*. The majority of the students were unable to evaluate the integral.

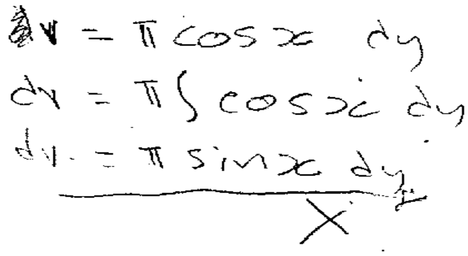


Figure 6.24: Cos*x* and a Δ*y* strip

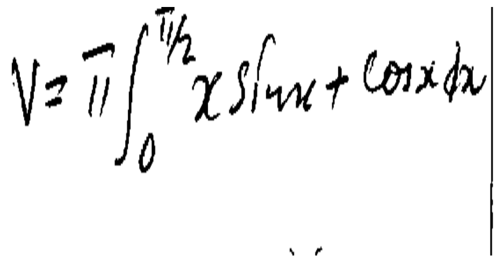


Figure 6.25: Integration by parts

Discussion on Element 5

From the Questionnaire 1st run, a large proportion of students (78.4%) were able to substitute correctly for Question 5A that did not require the use of a shell method, while a large proportion of students (73%) failed in Question 5B where a Δ*x* strip resulted in a shell upon rotation. With the Questionnaire 2nd run, similar results were found with 66.7% of the responses being acceptably correct for Question 5A and only 33.7% of the responses being acceptably correct for Question 5B. In responding to Question 5B, a large number of students used a Δ*y* in the formula along with a disc method (probably because they find it easy to work with), squaring the cos*x* and using other methods including the inverse of a cos*x* graph having a π (used for a disc method) outside the integral sign instead of a 2π (used for a shell method). The upper and lower limits were incorrect in most cases or not given.

Summary for Element 4 and 5

The results for these elements reveal that the students do not know what a 3D solid is and that they prefer to use a disc method or washer method despite the strip that is used. Students showed competency in questions that required the straight forward substitution, especially if the question required the use of a washer method. It is evident from the substitutions that a Δ*x* strip is preferred, and that students avoid using the shell method even if the given strip results in a shell upon rotation.

Discussion on Skill factor I (Elements 1-5)

The conclusion to be made on Skill factor I is that students struggle to draw graphs that they are not familiar with. If a graph that they are familiar with for an example, the graph of $y = x$ is asked differently, given as a word problem (using the mathematics register), most of them fail. In other instances it was also revealed that the students prefer using a Δ*x* strip as well as a disc or washer method, avoiding a shell method even if the region bounded by the drawn graphs results in a shell upon rotation. It was also evident that even if the graphs obviously required the use of Δy strip, some students were seen to use Δx in the formulae but using the y values as limits for integration. Most of the students were able to substitute the given equations into the formula for area or volume, but failed in most cases to interpret the visual graphs and to translate the information from the drawn graphs to the algebraic equations. There were instances where a Δx strip was drawn for the students and some students translated it correctly if it resulted in a disc or washer method upon rotation, but used a Δy in their formulae (with a disc) if upon rotation the drawn Δx strip resulted in a shell.

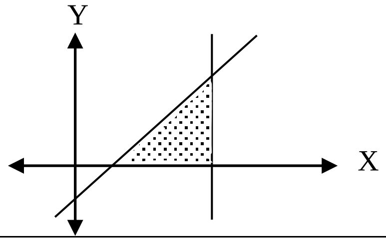
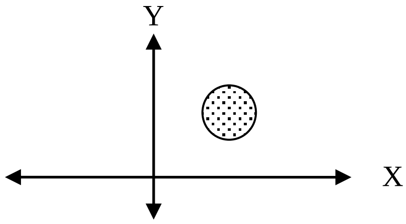
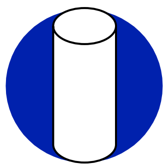
In discussions with students as they were solving problems, it appeared that most students had problems understanding what a 3D-diagram means. Many students also believe that when asked to rotate about the y-axis one must use a Δy strip and when rotating about the x-axis, one must use a Δx strip. This conception justifies why students use the disc method often. If one rotates a Δx strip about the x-axis one will always get a disc or a washer and the same applies if a Δy strip is rotated about the y-axis. Overall the results reveal that the students avoid using a shell. The results also reveal that the students cannot draw 3D solids.

6.1.2 Skill factor II: Three-dimensional thinking

- **Element 6 (2D to 3D) and Element 7 (3D to 2D)**

The questions for Element 6 and 7 were as follows:

Table 6.12: Element 6 and 7 questions

6A: Draw the 3-dimensional solid that is generated when the shaded area below is rotated about the ***x*-axis.** 	6 B: Draw a 3-dimensional solid that will be generated if you rotate the circle below about the ***y*-axis.** 
7 A: Sketch a graph that will generate half a sphere when rotated about the y- axis.	7 B: A hole is drilled through the centre of the sphere as in the picture. Sketch the graphs that were rotated to generate the solid as in the picture below. 

Questionnaire 1st run

Table 6.13: Element 6 and 7 for the Questionnaire 1st run as Question 6 and Question 7

RESPONSES	Q6A	%	Q6B	%	Q7A	%	Q7B	%
Fully correct	15	40.5	3	8.1	21	56.8	3	8.1
Almost correct	1	2.7	2	5.4	4	10.8	2	5.4
Traces of understanding	2	5.4	10	27	2	5.4	14	37.8
No understanding	8	21.6	19	51.4	7	18.9	4	10.8
Not done	11	29.7	3	8.1	3	8.1	14	37.8
% (FC + AC)		43.2		13.5		67.6		13.5

N = 37

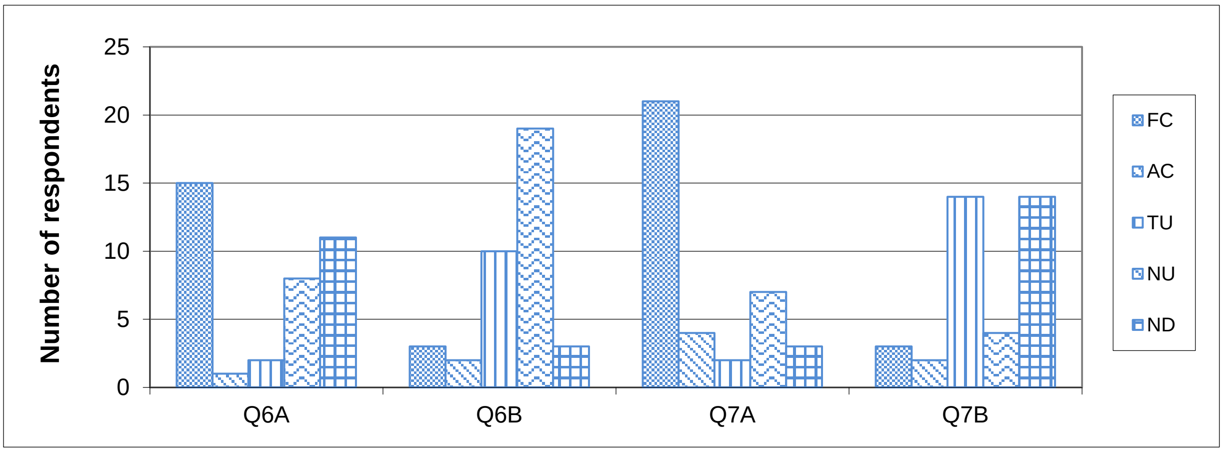


Figure 6.26: Questionnaire 1st run for Question 6 and Question 7

Questionnaire 2nd run

Table 6.14: Element 6 for the Questionnaire 2nd run as Question 6A

RESPONSES	Q6A	%
Fully correct	0	0
Almost correct	15	12.3
Traces of understanding	19	15.6
No understanding	66	54.1
Not done	22	18
% (FC + AC)		12.3

N = 122

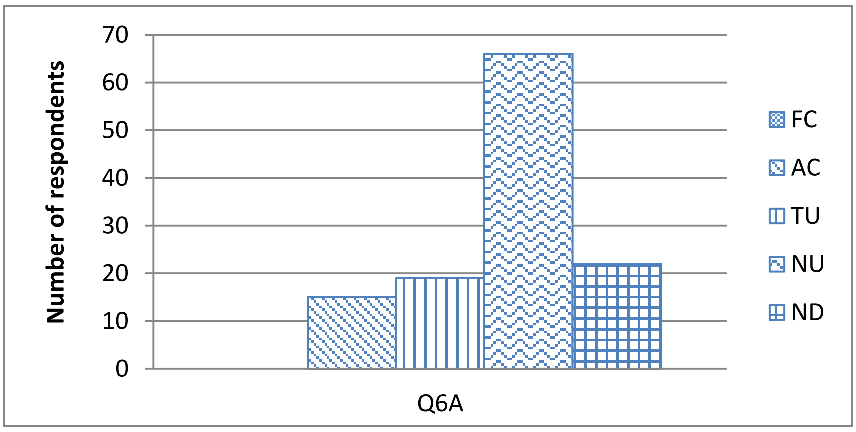


Figure 6.27: Questionnaire 2nd run for Question 6A

Table 6.15: Element 6 for the Questionnaire 2nd run as Question 6B

RESPONSES	Q6B	%
Fully correct	9	16.7
Almost correct	8	14.8
Traces of understanding	18	33.3
No understanding	18	33.3
Not done	1	1.9
% (FC + AC)		31.5

N = 54

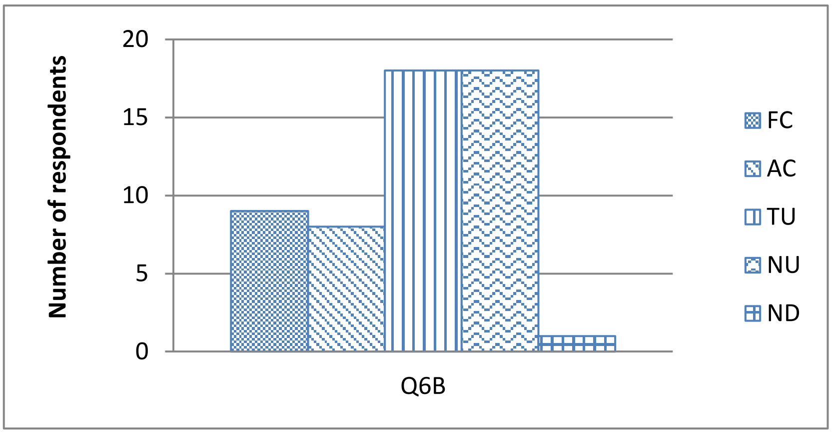


Figure 6.28: Questionnaire 2nd run for Question 6B

Table 6.16: Element 7 for the Questionnaire 2nd run as Question 7

RESPONSES	Q7A	%	Q7B	%
Fully correct	28	23.0	4	3.3
Almost correct	8	6.6	1	0.8
Traces of understanding	13	10.7	32	26.2
No understanding	57	46.7	46	37.7
Not done	16	13.1	39	32
% (FC + AC)		29.6		4.1

N = 122

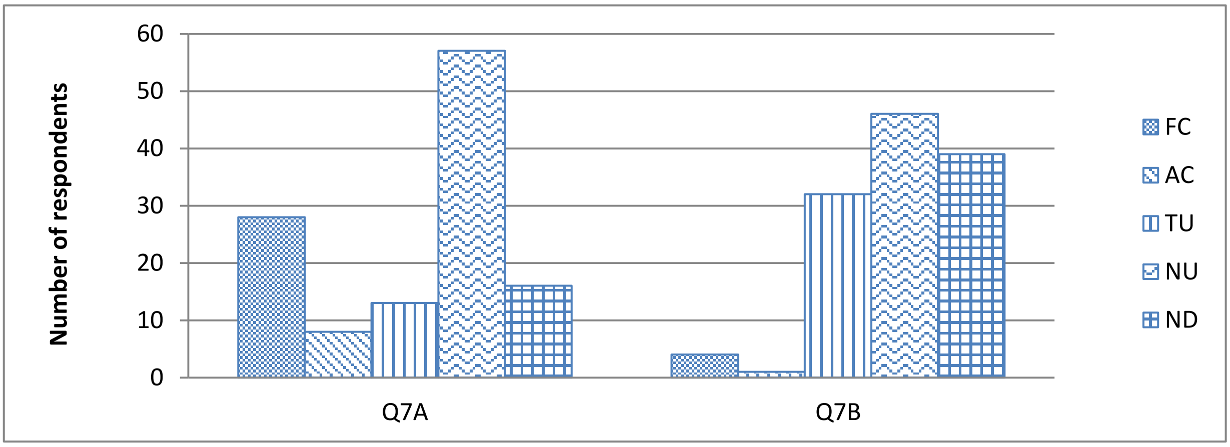


Figure 6.29: Questionnaire 2nd run for Question 7

From Tables 6.13, 6.14, 6.15 and 6.16 as well as Figures 6.26, 6.27, 6.26 and 6.29 the performance was different for the two questionnaire runs. For Question 6A a large proportion of responses in the Questionnaire 1st run, 40.5% (15), reveal that the students understood the question producing fully correct responses, even though there was also a higher percentage (29.7%) of responses showing that the students did not respond to this question with 21.6% (8) of the responses showing no understanding, evident from the bi-modal graph. In contrast, for the Questionnaire 2nd run, a large proportion of responses, 54.1% (66), reveal that the students did not understand the question with no fully correct responses, evident from the negatively skewed graph. In Question 6A, the students were expected to draw a 3D solid from a given 2D diagram after rotation of a region bounded by straight line graphs. In overall, 43.2% (16) of the responses for the Questionnaire 1st run were acceptably correct, regarded as satisfactory performance with only 12.3% (15) of the acceptably correct for the Questionnaire 2nd run, regarded as poor performance.

For Question 6B, a large proportion of responses in the Questionnaire 1st run, 51.4% (19), reveal that the students did not understand the question. In the case of the Questionnaire 2nd run the same proportion of responses, 33.3% (18) showed that the students had traces of understanding and no understanding respectively. In Question 6B a torus was an outcome after rotation of the given circle that was a certain distance from the axis. Only 13.5% (5) of the responses were acceptably correct from the Questionnaire 1st run, regarded as poor performance, evident from the negatively skewed graph. For the Questionnaire 2nd run, 31.5% (17) of the responses were acceptably correct, regarded as performance that is not satisfactory, evident from the negatively skewed graph. In both runs of the questionnaire, students performed better in Question 6A compared to Question 6B.

In Question 7A the students were expected to translate from 3D to 2D by drawing a graph that generates a sphere when rotated about the *y*-axis. The results from the Questionnaire 1st run reveal that a higher proportion of responses, 56.8% (21) were fully correct. For the Questionnaire 2nd run, a higher proportion of responses, 46.7% (57) were for the responses showing no understanding, with only 23% (28) of the responses being fully correct. Based on 67.6% (25) of the acceptably correct responses in the Questionnaire 1st run, evident from the positively skewed graph in Figure 6.26, the performance was regarded as being good. In the Questionnaire 2nd run, with only 29.6% of the responses being acceptably correct, evident from the negatively skewed graph as in Figure 6.29, the performance was regarded as not satisfactory.

In Question 7B the students were expected to sketch graphs of the given the solid of revolution. In the Questionnaire 1st run the same highest proportion of responses (37.8%) showed traces of understanding and those that were not done (37.8%). In the Questionnaire 2nd run, a higher proportion of the responses, 37.7% (46) was for no understanding and 32% (39) for traces of understanding. For both runs of the questionnaire, the performance was poor, with 13.5% (5) and 4.1 % (5) of the acceptably correct responses for the Questionnaire 1st run and for the Questionnaire 2nd run respectively. As in Figures 6.26 and 6.29, both graphs were negatively skewed.

In both runs of the questionnaire, performance in Question 7A was better than in Question 7B. In Questions 6 and 7, the overall impression is that for the Questionnaire 2nd run, there were few acceptably correct responses, evident from all four negatively skewed graphs in Figures 6.27, 6.28 and 6.29, compared to the Questionnaire 1st run.

Below some examples of what students actually did in Question 6A and 6B are given.

Question 6A: Typical students' errors when drawing a 3D solid resembling a cone

Traces of understanding: Students drew

- Only a disc from a Δx strip showing rotation about the x – axis
- A rotated graph about the y-axis with a Δx strip rotated about the x – axis

No understanding: Students drew

- Only a Δx strip on the drawn graph
- A graph of $y = x$ and $x = c$ with a Δy strip in the 1st quadrant
- A Δy strip on the drawn graph

The examples above reveal that students are not familiar with drawing 3D diagrams, representing solids of revolution. The rotation was only shown with a rotated strip, not a solid of revolution. The students were rotating the graphs, but in most cases the 3D shapes that arise as a result of rotation remained the original graphs. In Figures 6.30 and 6.31, examples of actual written responses are given showing *no understanding*.

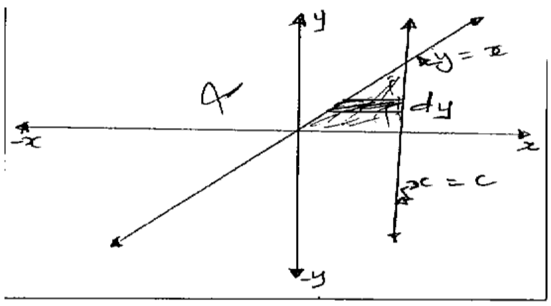


Figure 6.30: The graph of $y = x$ and the Δy strip

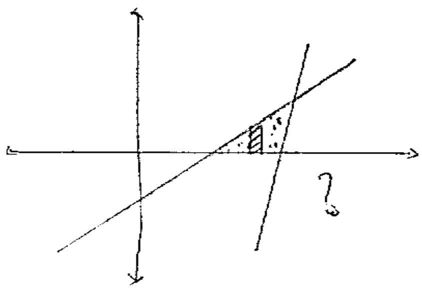


Figure 6.31: The same graph with Δx strip

Question 6B: Typical students' errors when drawing a 3D solid resembling a torus

Almost correct: Students drew

- A torus even if it was not an accurate one

Traces of understanding: Students drew

- A torus, but rotated about the x – axis
- A horizontal cylindrical pipe away from the x- axis, in the 1st and 2nd quadrant
- A horizontal cylindrical pipe away from the x- axis in the 1st quadrant

No understanding: Students drew

- A Δy strip on the given diagram
- A circle in the 1st quadrant away from the origin, with a Δx and a Δy strip
- A semicircle in the 1st and the 4th quadrant with a Δx strip
- A semicircle in the 1st and the 2nd quadrant with a Δy strip
- The same given diagram in the 4th quadrant with a Δx strip
- Three quarter circles in quadrants 1, 2 and 3
- A quarter circle in quadrant 1
- Something like a leaf on the y – axis
- Two big circles in quadrant 1 and 4
- A rectangular hyperbola rotated about the y – axis

Very few students managed to draw a torus. Students drew cylinders and other nonsensical diagrams. Students did not see the significance of the distance of this circle from both axes and did not realise that such a distance could give rise to a hole after rotation.

In Figures 6.32 and 6.33 examples of actual written responses are given showing *no understanding* and *traces of understanding*.

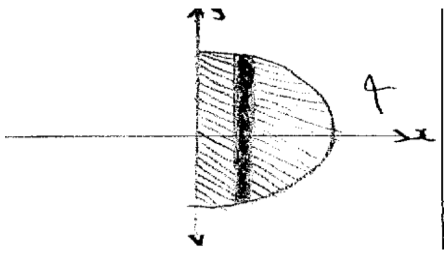


Figure 6.32: A hemisphere about the x axis

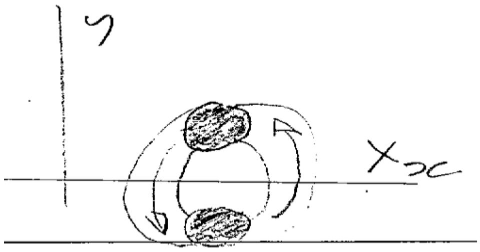


Figure 6.33: Rotation about the x-axis

Discussion on Element 6

In the Questionnaire 1st run, students performed satisfactorily (43.2%), in Question 6A where the region bounded by straight lines was rotated, and poorly (13.5%) in Question 6B involving rotation of a circle that was a certain distance from the origin, probably because it was unfamiliar. The majority of the students drew vertical or horizontal rectangles and rotated them, instead of the whole diagram as a solid of revolution. In the Questionnaire 2nd run the students' performance was not satisfactory in Question 6B with 31.5% of acceptably correct responses and poor in Question 6A with only 12.3% of acceptably correct responses.

Below some examples of what students actually did in Question 7A and 7B are given.

Question 7A: Typical students' errors when drawing a graph that could generate a hemisphere after rotation about the y - axis

Almost correct: Students drew

- Half a circle in quadrant 1 and 2, with a Δy strip in quadrant 1
- Half a circle in quadrant 1 and 4 with a Δx strip in both quadrants, in other cases without a strip.

Traces of understanding: Students drew

- A graph of $y = 1 - x^2$

No understanding: Students drew

- Different graphs, for example the graph of $y = e^x$, in other cases different strips Δx strip or Δy
- Different diagrams including a cylinder and a parabola facing down without the x intercepts and other different parabolas including a horizontal parabola $x = y^2$

The different graphs drawn reveal that some of the students did not know what a sphere was.

Figures 6.34 and 6.35 display actual written responses showing *no understanding*.

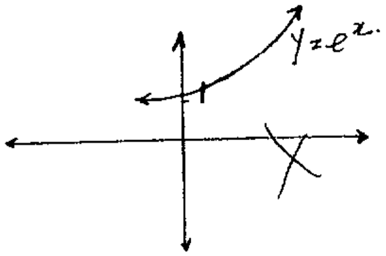


Figure 6.34: An exponential function

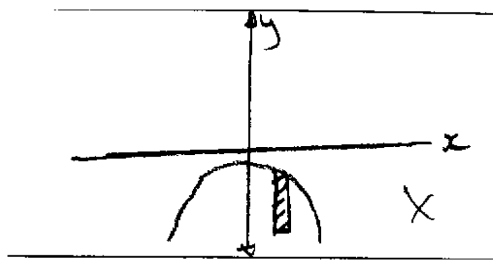


Figure 6.35: The parabolic diagram

Question 7B: Typical students' errors when drawing a graph that could generate a sphere with a cylindrical hole in the centre

Almost correct: Students drew

- Half a cylinder was drawn to show a hole in the 2^{nd} quadrant

Traces of understanding: Students drew

- A circle and two rectangles crossing at the origin, one vertical and one horizontal
- Half a circle on the Cartesian plane with a hole
- A circle on a Cartesian plane
- A circle on a Cartesian plane with a hole like a pipe
- A circle with a Δy strip in quadrants 1 and 2
- A line $x = c$ and rotated about the y – axis, showing a cylinder
- A circle with intercepts $\pm r$; a circle with a Δy strip in quadrants 1 and 2 like a cylinder
- A cylinder with the x – axis and the y – axis intersecting at its center
- A circle and a cylinder to show a hole

No understanding: Students drew

- A cone on the Cartesian plane
- An exponential graph with a Δy strip
- A rectangular hyperbola rotated about the y – axis.

In Figures 6.36 and 6.37, examples showing *traces of understanding* are given.

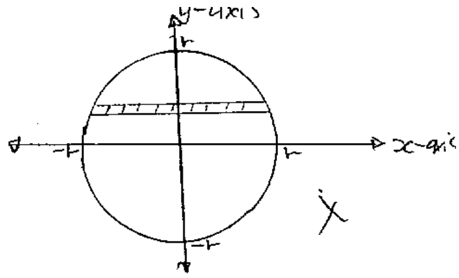


Figure 6.36: A circular shape

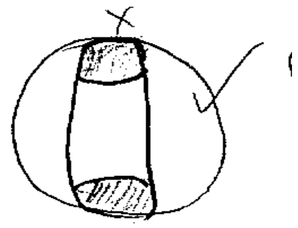


Figure 6.37: A circle and a rod

Discussion on Skill factor II

In the Questionnaire 1st run, students' performance was good when translating from 3D to 2D, when drawing a graph that could generate half a sphere after rotation and performed satisfactorily when translating from 2D to 3D, where the given straight line results in a cone after rotation. Students struggled mainly with the questions that require more imaginative skills at a higher level, when translating between 2D and 3D. Most students failed to comprehend a question where a given 2D diagram resulted in a torus after rotation (translation from 2D to 3D), regarded as poor performance. Most students did not respond to a question when they were expected to draw the graphs that could give rise to a sphere with a cylindrical hole in the centre (translation from 3D to 2D). This was also regarded as poor performance. The student partially managed to work in 2D and in 3D. Therefore translation from 2D to 3D and from 3D to 2D was partially achieved only for simple diagrams, such as a straight line that gave rise to a cone and a semicircle that gives rise to half a sphere, mainly in the Questionnaire 1st run. Most students failed when the diagrams involved more imaginative skills at a higher level of conceptualising.

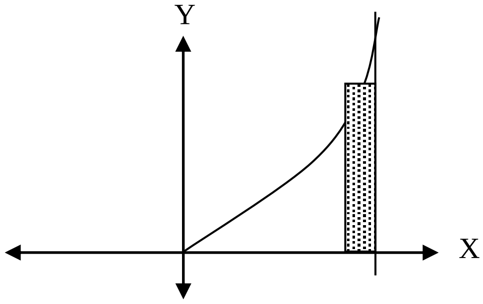
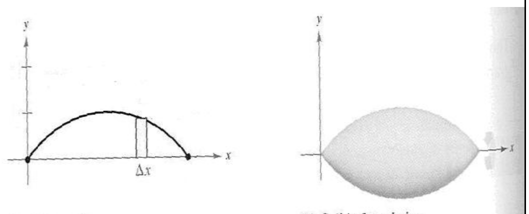
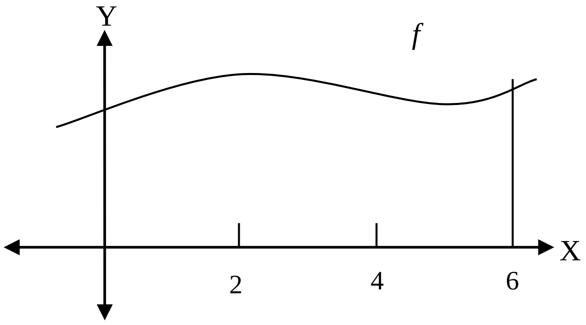
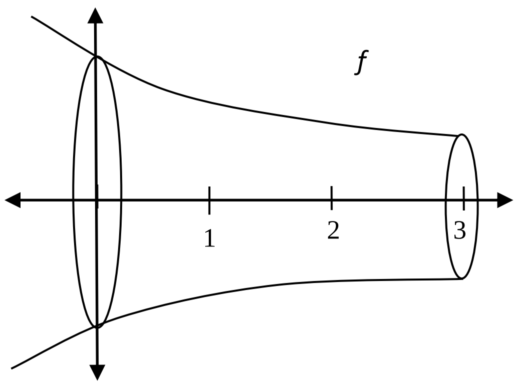
The result from the Questionnaire 2nd run revealed that many students struggled with most of the questions in this element. The performance was poor when the students were expected to draw the graphs that could give rise to a sphere with a cylindrical hole in the centre (translation from 3D to 2D) and when translating from 2D to 3D, where the given straight line results in a cone after rotation. The performance was not satisfactory in the remaining two questions. For the one question, the students solved a problem where a given 2D diagram resulted in a torus after rotation (translation from 2D to 3D), while for the other question the students were expected to draw a graph that could generate half a sphere after rotation.

6.1.3 Skill factor III: Moving between discrete and continuous

- **Element 8: Continuous to discrete (visual) in 2D and 3D and**
  Element 9: Discrete to continuous and continuous to discrete (algebraic) in 2D and 3D

The questions for Element 8 and 9 were as follows:

Table 6.17: Element 8 and 9 questions

8A: Sketch three additional rectangular strips (similar to the given rectangle) so that the total area of the rectangles approximates the area under the graph.  	8B: When the plane region (a) on the left is rotated, the 3-dimensional solid of revolution (b) on the right is generated. Show using diagrams how you would cut the solid of revolution (b) in appropriate shapes (discs, shells or washers) to approximate its volume.  (a) Plane region (b) Solid of revolution
9 A: Show in terms of rectangles what the following represent with a sketch: $2f(0)+2f(2)+2f(4)$  	9 B: If the volume of the given solid of revolution is approximated by discs, sketch the discs that would give the volume: $\pi\left(f(0)\right)^2+\pi\left(f(1)\right)^2+\pi\left(f(2)\right)^2$ 

Questionnaire 1st run

Table 6.18: Element 8 and 9 for the Questionnaire 1st run as Question 8 and Question 9

RESPONSES	Q8A	%	Q8B	%	Q9A	%	Q9B	%
Fully correct	8	21.6	2	5.4	1	2.7	0	0
Almost correct	3	8.1	1	2.7	2	5.4	0	0
Traces of understanding	15	40.5	12	32.4	2	5.4	17	46
No understanding	10	27	6	16.2	12	32.4	9	24.3
Not done	1	2.7	16	43.2	20	54.1	11	29.7
% (FC + AC)		**29.7**		**8.1**		**8.1**		**0**

N = 37

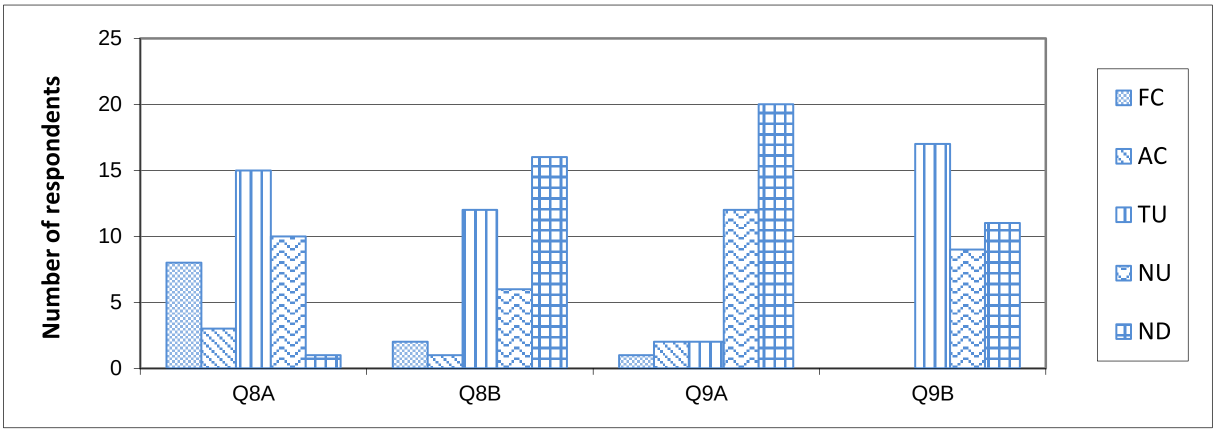


Figure 6.38: Questionnaire 1st run for Question 8 and Question 9

Questionnaire 2nd run

Table 6.19: Element 8 and 9 for the Questionnaire 2nd run as Question 8B and Question 9A

RESPONSES	Q8B	%	Q9A	%
Fully correct	0	0	0	0
Almost correct	0	0	15	12.3
Traces of understanding	14	11.5	19	15.6
No understanding	47	38.5	33	27.1
Not done	61	50	55	45.1
% (FC + AC)		**0**		**12.3**

N = 122

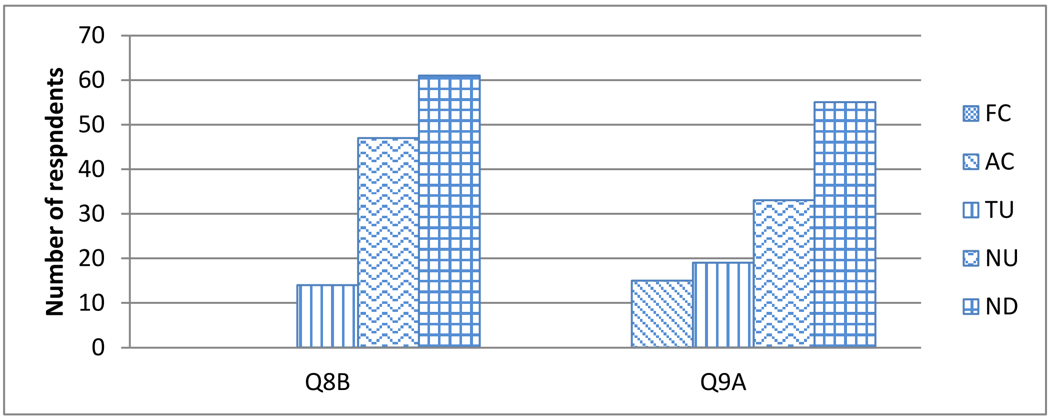


Figure 6.39: Questionnaire 2nd run for Question 8B and Question 9A

Table 6.20: Element 8 and 9 for the Questionnaire 2nd run as Question 8A and Question 9B

RESPONSES	Q8A	%	Q9B	%
Fully correct	4	7.4	0	0
Almost correct	2	3.7	0	0
Traces of understanding	38	70.4	44	81.5
No understanding	10	18.5	6	11.1
Not done	0	0	4	7.4
% (FC + AC)		11.1		0

N = 54

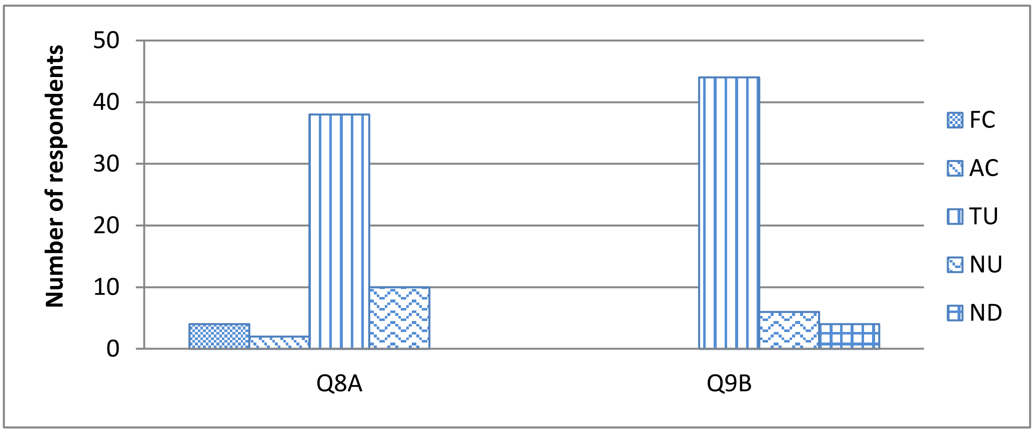


Figure 6.40: Questionnaire 2nd run for Question 8A and Question 9B

The performance in this element was poor for all questions. For both runs of the questionnaire, the highest responses in Questions 8A and 9B revealed traces of understanding, 40.5% (15) and 46% (17) respectively for the Questionnaire 1st run and 70.4% (38) and 81.5% (44) for Questionnaire 2nd run. The proportion of acceptably correct responses in Question 8A, 29.7% (11) for Questionnaire 1st run reveals that the performance was not satisfactory. In the Questionnaire 2nd run, the performance in this question was poor, with only 11.1% (6) of the responses being acceptably correct. The graphs representing these questions (as in Figures 6.38 and 6.40) are negatively skewed. None of the students obtained fully correct or almost correct responses in Question 9B, represented by the first and the longest bar for traces of understanding as in Figures 6.38 and 6.40. In Question 9B the students were expected to represent the three discs from the given formula on the given diagram. In Question 8A the region bounded by graphs was to be approximated using three additional rectangles.

Most of the students did not respond to Questions 8B and 9A, 43.2% (16) and 54.1% (20) respectively for Questionnaire 1st run and 50% (61) and 45.1% (55) respectively for the Questionnaire 2nd run. There were only 8.1% (3) acceptably correct responses for Questions

8B and 9A for the Questionnaire 1st run and 0% and 12.3% (15) for the Questionnaire 2nd run, all regarded as poor performance. The graphs representing these questions (as in Figures 6.38, and 6.39) are negatively skewed. Question 8B involved the approximation of the rotated region bounded by graphs using discs and Question 9A involved a representation of three rectangles from the given formula on the given diagram.

Question 8 involved operating visually in 2D and 3D and approximating area and volume by slicing based on the concepts of the Riemann sums, whereas Question 9 involved approximating area and volume algebraically in 2D and in 3D.

Below some examples of what students actually did in Question 8A and 8B are given.

Question 8A: Typical students' errors when approximating area using rectangular strips

Almost correct: Students drew

- Three additional rectangles correctly, but not well on scale.

Traces of understanding: Students drew

- Two additional rectangles correctly
- Three additional rectangles but separated them
- Seven additional rectangles
- Two additional separated rectangles, not joint

No understanding: Students drew

- An image of the rotated given graph about the *x*-axis
- The image of the rotated given graph about the *y*-axis using a Δx strip
- Some vertical lines separately
- Mirror images of the original graph when reflected about the *x* and the *y*-axis separately
- A rectangle along the *x*-axis

Question 8A required that the students must approximate the area bounded by a curve using rectangles that are joined to each other, originating from Riemann sums. In Figures 6.41 and 6.42 examples of actual written responses for students showing *no understanding* and *traces of understanding* are given.

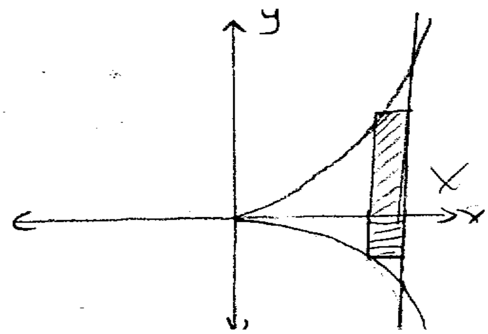


Figure 6.41: One rectangle

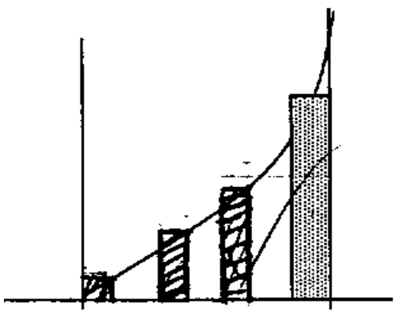


Figure 6.42: Four rectangles

Question 8B: Typical students' errors when approximating volume using discs

Almost correct: Students drew

- Two discs

Traces of understanding: Students drew

- A vertical disc, but labelled it Δy
- One disc in the middle of the diagram
- One strip in quadrants 1 and 4 in the middle of the diagram

No understanding: Students drew

- A Δy strip on the x – axis in the middle of the diagram; cut it in the middle leaving an open disc; a rectangular hyperbola rotated about the x – axis.

In Question 8B students were required to approximate the volume of a given solid using discs. Figures 6.43 and 6.44 give examples of actual written responses showing *traces of understanding*.

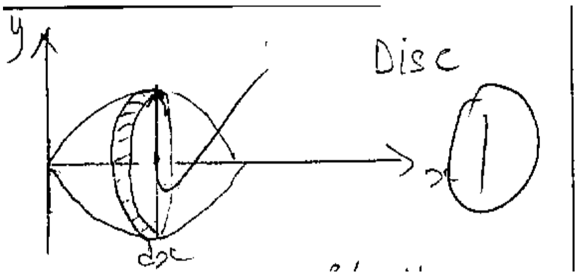


Figure 6.43: The first ring

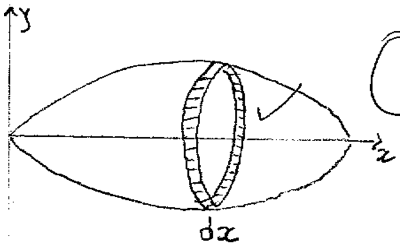


Figure 6.44: The second ring

Below some examples of what students actually did in Question 9A and 9B are given.

Question 9A: Typical students' errors when representing area from the given equation

Almost correct: Students drew

- Rectangles, where the middle rectangle, $f(2)$ not properly done

Traces of understanding: Students drew

- 4 vertical lines without joining the top
- 3 rectangles not corresponding to the given function
- Rectangles drawn, without drawing the function

No understanding: Students drew

- A rectangle of length 4 and breadth 2 separately
- A rectangle of unspecified breadth and length 6 separately
- A rectangle of unspecified length and breadth 2 separately
- A rectangle of unspecified breadth and length 4 separately
- A line $y = 2$ from the y – axis
- A rectangle of length 6 and breadth 2 separately
- A rectangle of length 6 and breadth 12 separately
- A rectangle of length $2f(4)$ and breadth $2f(2)$ separately
- A trapezium

The performance in Question 9A was disappointing. Students could not interpret the given expressions and could not relate it to the given function. They did not recognise that the width (relating to the x values) of all the rectangles is 2 and that $f(0)$, $f(2)$ and $f(4)$ represented the length (relating to the y values) of the rectangles, hence the length times the width represented by, $2f(0)$, $2f(2)$, and $2f(4)$ respectively which need to be summed, as in Riemann sums. Despite the fact that the question stated explicitly that the rectangles should be drawn, the students had no idea where the rectangles should be located on the given diagram as an approximation for area. In Figures 6.45 and 6.46 examples actual written responses display *no understanding*.

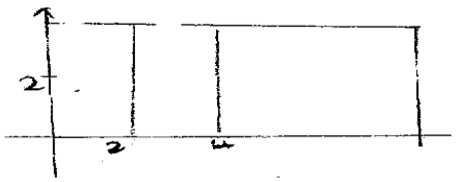


Figure 6.45: Unequal rectangles

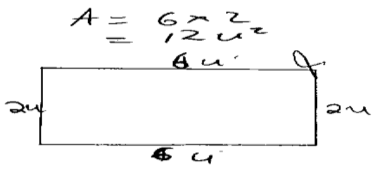


Figure 6.46:A rectangle of area 12

Question 9B: Typical students' errors when representing volume from the given equation

Traces of understanding: Students drew

- Vertical then discs at the given x-intercepts of 1and 2
- 2 vertical half discs at the given x-intercepts of 1and 2
- One disc in the middle of the diagram
- 2 discs anywhere

No understanding: Students drew

- A horizontal disc around the y – axis
- The same given diagram upside down
- A cylinder
- The top half of the given graph
- A semicircle
- The same graph reduced from 0 to 2 for the x values

In Question 9B rather than using rectangles, students were asked for an approximation of volume using discs of the same thickness, to approximate the volume on the given diagram. Even though it was explicitly asked that the volume should be represented by discs, also in the formula πr^2, from the given formula where in each case r is represented as $f(0)$, $f(1)$ or $f(2)$ respectively, the majority of the students struggled to use the formula to translate to the appropriate discs. None of the students responded correctly to this question. The students simply failed to interpret the question, even though there were traces of understanding in most instances. The students drew discs which did not have the same thickness, as it was required from the given x- intercepts on the diagram where three discs of thickness 1 could be drawn.

In Figures 6.47 and 6.48, examples of actual written responses showing *traces of understanding* are given.

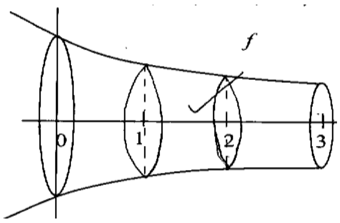


Figure 6.47: The thin circles

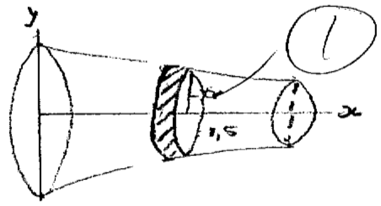


Figure 6.48: A circle of radius 1.5

Discussion on Skill factor III

In Skill factor III, performance was very low for both runs of the questionnaire. Students approximated the given area or volume, for other questions by using disjoint rectangles of the same width or slices of the same thickness, not showing any continuity of points on a continuous function. It was evident that students were not familiar with the concept of Riemann sums. Students were certainly not able to translate from continuous to discrete representations and from discrete to continuous representations in 2D and in 3D, where the given equations represented rectangles and discs.

6.1. 4 Skill factor IV: General manipulation skills

Skill factor IV comprises Element 10 only.

The questions for Element 10 were as follows:

10A: Calculate the point of intersection of $4x^2 + 9y^2 = 36$ and $2x + 3y = 6$

10B: Calculate $\int_0^1 \pi (1-x^2)^2 \, dx$

10C: Calculate $\int_0^1 2\pi \, x (1-\sin x) \, dx$

<u>Questionnaire 1st run</u>

Table 6.21: Element 10 for Questionnaire 1st run as Questions 10

RESPONSES	Q10A	%	Q10B	%	Q10C	%
Fully correct	10	27.1	15	40.5	2	5.4
Almost correct	3	8.1	7	18.9	22	59.5
Traces of understanding	1	2.7	2	5.4	9	24.3
No understanding	22	59.5	4	10.8	3	8.1
Not done	1	2.7	9	24.3	1	2.7
% (FC + AC)		**35.2**		**59.4**		**64.9**

N = 37

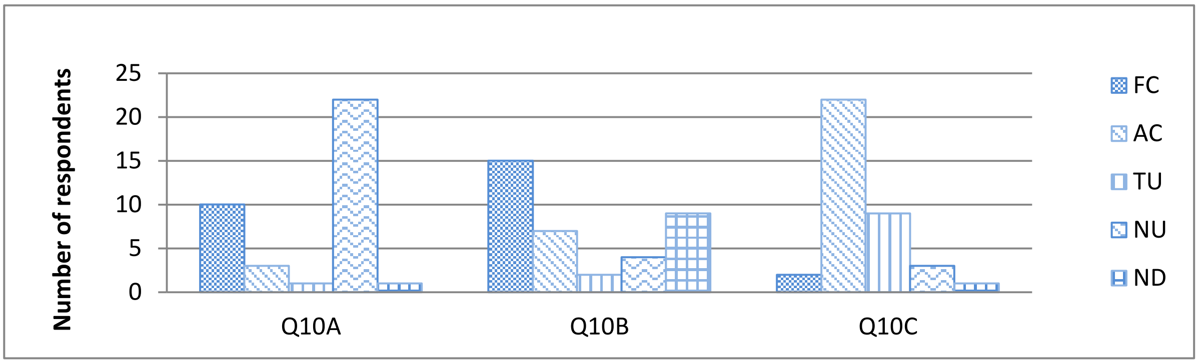


Figure 6.49: Questionnaire 1st run for Questions 10

<u>Questionnaire 2nd run</u>

Table 6.22: Element 10 for Questionnaire 2nd run as Questions 10

RESPONSES	Q10A	%	Q10B	%	Q10C	%
Fully correct	8	6.6	55	45.1	4	3.3
Almost correct	5	4.1	25	20.5	28	23
Traces of understanding	7	5.7	15	12.3	32	26.2
No understanding	97	79.5	12	9.8	58	47.5
Not done	5	4.1	15	12.3	0	0
% (FC + AC)		10.7		65.6		26.3

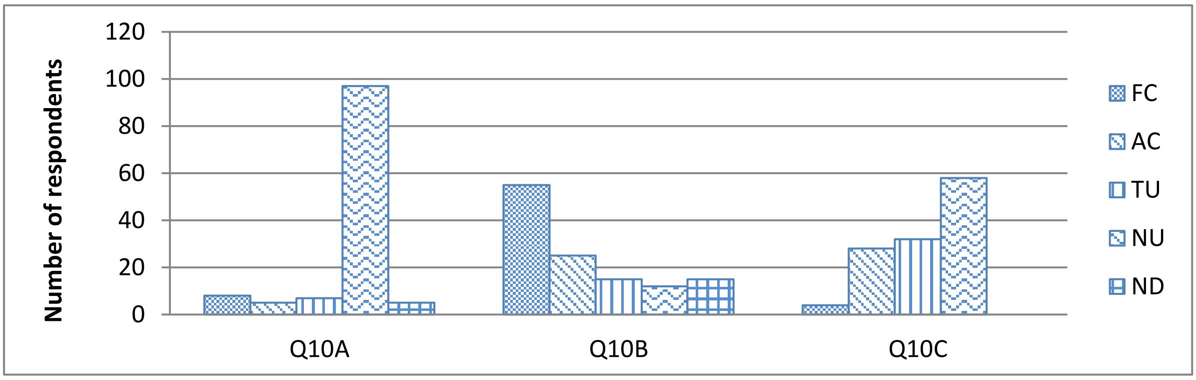


Figure 6.50: Questionnaire 2nd run for Questions 10

From Tables 6.21 and 6.22 as well as Figures 6.49 and 6.50, for both questionnaire runs, the performance was low in Question 10A, where students were expected to calculate the point of intersection of $4x^2+9y^2=36$ and $2x+3y=6$. A higher proportion of responses, 59.5% (22) in the Questionnaire 1st run and large proportion of responses 79.5% (97), for Questionnaire 2nd run showed that most of the students were unable to calculate the point of intersection. It is also evident from the bar graphs (Figures 6.49 and 6.50) for Question 10 A, which are negatively skewed, that few responses were acceptably correct, 35.2% (13) in the Questionnaire 1st run and 10.7% (13) in the Questionnaire 2nd run. The performance was not satisfactory (35.2%) for the Questionnaire 1st run and poor (10.7%) in the Questionnaire 2nd run.

In relation to the other questions, the performance was better in Question 10B for both Questionnaire 1st run and Questionnaire 2nd run, where most of the responses were fully correct, 40.5 % (15) for Questionnaire 1st run and 45.1% (55) for Questionnaire 2nd run. In Question 10B, the students were expected to evaluate a definite integral involving basic power rules, with all students in the Questionnaire 1st run responding to this question. Performance in this question was satisfactory in Questionnaire 1st run with 59.4% (22) of the responses being acceptably correct and good in the Questionnaire 2nd run with 65.6% (80) of the responses being acceptably correct, also shown in Figures 6.49 and 6.50 which are positively skewed for Question 10B.

In Question 10C a higher proportion of responses, 59.5% (22) in the Questionnaire 1st run were almost correct responses, and 24.3% (9), showed some traces of understanding. In the Questionnaire 2nd run, there was a higher proportion of non-responses, 47.5% (58) with only 3.3% (4) of fully correct responses. In the Questionnaire 1st run, as evident from the positively skewed graph (Figure 6.49), there were a large proportion of acceptably correct responses (64.9%). In contrast, as evident from the negatively skewed graph (Figure 6.50) in Questionnaire 2nd run, there were few acceptably correct responses (26.3%). In this question, students were expected to evaluate a definite integral involving integration by parts. Even though there was an indication of some manipulation skills, some of the mistakes that students made were because they used incorrect algorithms.

The results reveal that from both questionnaire runs, students were less successful in solving Question 10A, where they had to calculate the point of intersection, which seemed simpler than the other two questions, where they had to evaluate the definite integral.

Below some examples of what students actually did in Question 10A, 10B and 10C are given.

Question 10A: Typical students' errors when calculating the point of intersection
Almost correct: Students

- Substituted correctly but manipulated incorrectly

Traces of understanding: Students

- Substituted correctly but made a mathematical error when simplifying the roots, since they did not square them

No understanding: Students

- Solved for the x-intercept and the y-intercept for each equation
- Equated some graphs in an incorrect way
- Only made y the subject of the formula in the linear equation and did not proceed
- Took square roots incorrectly like $\sqrt{36-4x^2}$ as $6-2x$
- Differentiated the two equations representing the two graphs

Some of the students were able to solve problems in general manipulation skills, even though there were substantial mathematical errors in some cases. Such errors occurred mostly in Question 10A, for example when some students were finding the *x*- and the *y*-intercepts for the graphs, some equated the equations representing the two graphs incorrectly while others took square roots incorrectly: $\sqrt{36-4x^2} = 6-2x$. Figures 6.51 and 6.52 present examples of actual written responses *showing no understanding.* In Figure 6.51 the solution is incomplete and in Figure 6.52 the solution is completely incorrect.

$$y = 2 - \frac{2x}{3} \quad \ldots \quad (1)$$
$$y = 2 - \frac{2x}{3} \quad \ldots \quad (2)$$

Figure 6.51: Incorrect solution 1

$$y = y$$
$$-\tfrac{2}{3}x^2 + 2 = 2 - \tfrac{2}{3}x$$
$$-\tfrac{2}{3}x^2 + \tfrac{2}{3}x = 0$$
$$-\tfrac{2}{3}x(x-1) = 0$$
$$x = 0 \quad \text{or} \quad x = 1$$
$$(0, 2) \quad \& \quad (1, \tfrac{4}{3})$$

Figure 6.52: Incorrect solution 2

Question 10B: Typical students' errors when evaluating the integral

Almost correct: Students

- Calculated incorrectly after correct integration and substitution

Traces of understanding: Students

- Integrated incorrectly

No understanding: Students simplified $(1-x^2)^2$ as follows

- $(1-x^2)^2 = 1-x^4$
- $(1-x^2)^2 = 1+x^4$

Question 10C: Typical students' errors when using the integration by parts

Almost correct: Students

- Used integration by parts, but made mistakes with signs
- Integrated by parts correctly, but made mistakes at the end of the calculation

Traces of understanding: Students

- Used integration by parts, but lost the other part
- Did not use integration by parts correctly

No understanding: Students

- Made a mathematical error by multiplying $x \sin x$ to be $\sin^2 x$
- Made a mathematical error by multiplying $x \sin x$ to be $\sin x^2$

For questions such as in Question 10C, although some students made mathematical errors such as shown above, most students were able to solve the integral even if they were making errors with the signs.

Conclusions from Skill factor IV

Students are fairly proficient with general manipulation skills, especially in the Questionnaire 1st run. They mainly make mistakes in applying the integration techniques, like the integration by parts. In other instances they take square roots incorrectly or make mathematical errors. However, as they continue to calculate, they showed proficiency in the general manipulation skills. It can be argued that students were reasonably successful in this element.

6.1.5 Skill factor V: Consolidation and general level of cognitive development

Skill factor V comprises Element 11 only.

In this element there were 2 questions. The aim of the questions was to test whether students can do an entire problem correctly, in this way consolidating the individual skills tested from five different elements: graphing skills, general manipulation skills, moving from continuous to discrete (visual 2D and 3D), translation from visual to algebraic in 2D and translation from visual to algebraic 3D combined, from Skill factors I, II, III and IV.

The questions for Element 11 were as follows:

11A: Given: $y = \sin x$ and $y = 1$, where $x \in \left[0\,,\frac{\pi}{2}\right]$

(i) Sketch the graphs and shade the area bounded by the graphs and $x = 0$.
(ii) Show the rotated area about the *y*-axis and the representative strip to be used to calculate the volume generated.
(iii) Calculate the volume generated when this area is rotated about the *y*-axis.

11B: Use integration methods to show that the volume of a cone of radius *r* and height *h is* given by

$$V = \frac{1}{3}\pi r^2 h.$$

<u>Questionnaire 1st run</u>

Table 6.23: Element 11 for the Questionnaire 1st run as Question 11

RESPONSES	Q11A	%	Q11B	%
Fully correct	0	0	1	2.7
Almost correct	6	16.2	4	10.8
Traces of understanding	24	64.9	2	5.4
No understanding	4	10.8	17	46
Not done	3	8.1	13	35.1
% (FC + AC)		**16.2**		**13.5**

N = 37

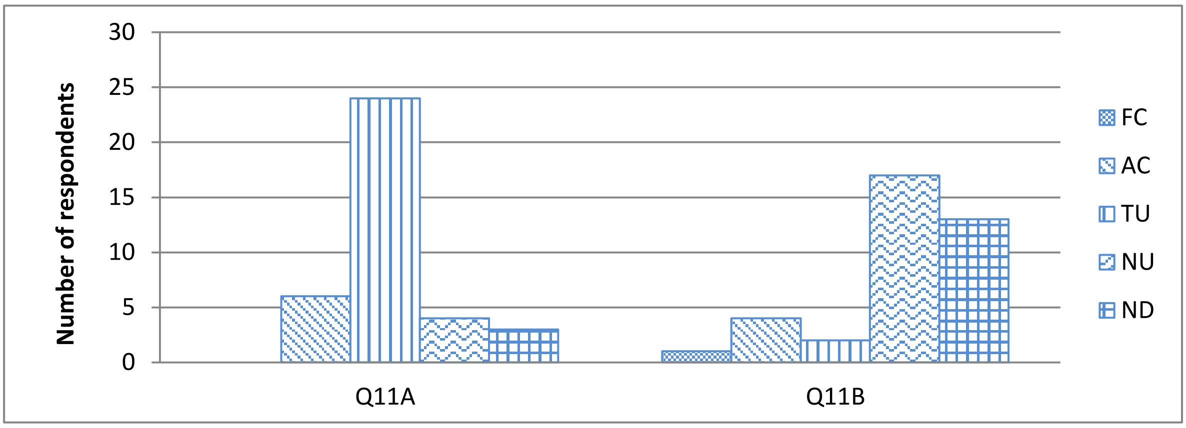


Figure 6.53: Questionnaire 1st run as Question 11

Questionnaire 2nd run

Table 6.24: Element 11 for the Questionnaire 2nd run as Question 11A

RESPONSES	Q11A	%
Fully correct	0	0
Almost correct	4	3.3
Traces of understanding	63	51.6
No understanding	52	42.6
Not done	3	2.5
% (FC + AC)		3.3

N = 122

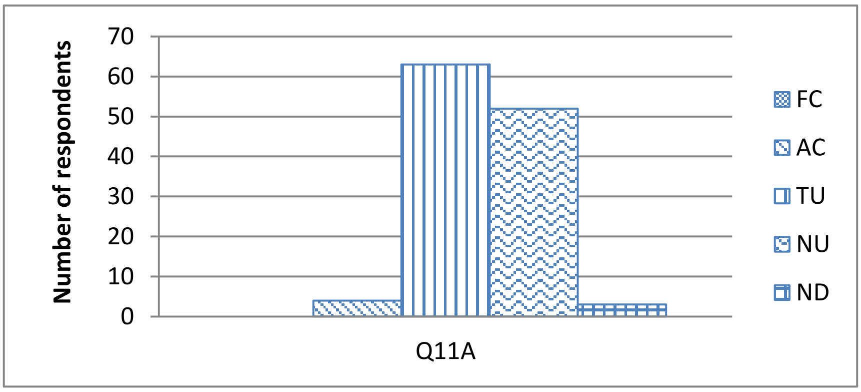


Figure 6.54: Questionnaire 2nd run for Question 11A

Table 6.25: Element 11 for the Questionnaire 2nd run as Question 11B

RESPONSES	Q11B	%
Fully correct	0	0
Almost correct	0	0
Traces of understanding	2	3.7
No understanding	39	72.2
Not done	13	24.1
% (FC + AC)		0

N = 54

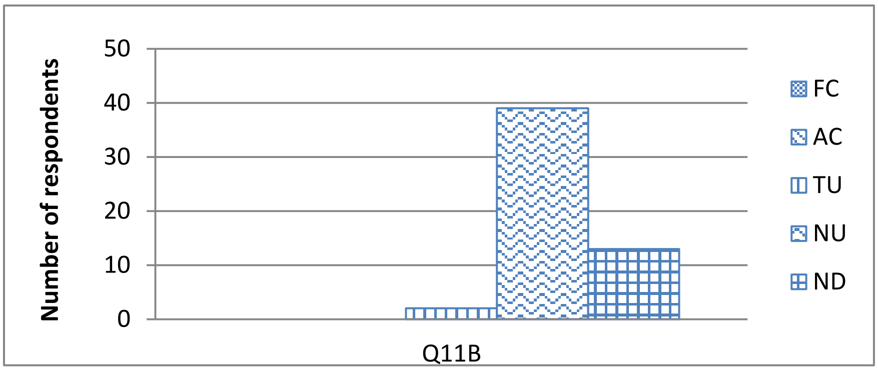


Figure 6.55: Questionnaire 2nd run for Question 11B

From Tables 6.23, 6.24 and 6.25 and Figures 6.53, 6.54 and 6.55 it is evident that the students did not do well in this element. In the Questionnaire 1st run, none of the students produced fully correct responses in Question 11A, with a large proportion of responses 64.9% (24), showing traces of understanding, as displayed from Figure 6.53. The responses showing traces of understanding reveal that the students had an idea on how to approach the question but were confused at times. In Question 11A, 16.2% (6) of the responses were acceptably correct, regarded as poor performance. In the Questionnaire 2nd run, a large proportion of responses 51.6% (63) and 42.6% (52) were for traces of understanding and no understanding respectively, as shown in Figure 6.54. As it was in the Questionnaire 1st run, there were no fully correct responses. Only 3.3% (4) of the responses were acceptably correct.

In Question11B, from the Questionnaire 1st run, the highest proportion of responses, 46% (17), was an indication of no understanding, with 35.1% (17) of non-responses and only 2.7% (1) of the responses being fully correct. The proportion of acceptably correct responses in this question is 13.5% (5), revealing that there were few acceptably correct responses, as indicated in Figure 6.63, which is negatively skewed. The performance for Question 11B was worse than for Question 11A, with the highest number of responses showing no understanding or not done at all. For the Questionnaire 2nd run, the performance was the lowest in Question 11B with no fully correct or almost correct responses. Most of the responses, 72.2% (39) was an indication of no understanding. There were no acceptably correct responses, as shown in Figure 6.55.

The performance in both questionnaire runs in Question 11 was poor (less that 20% of acceptably correct responses). This question, structured in the same way as the final N6 examinations portrays the students' level of cognitive development as very low or not developed.

Below are some examples of what students actually did in Question 11A and 11B.

Question 11A: Typical students' errors in drawing the graphs, shading the bounded region, selecting the strip and selecting the formula for volume

Traces of understanding: Students

- Drew correct graphs, shaded correctly, drew a disc using a Δy strip, but used formulas for disc and shell in same equation used $2\pi \int_0^1 y(\sin^{-1} y)^2\, dy$
- Drew a Δx strip but used a washer method when substituting, which was incorrect
- Drew two Δx strips on separate graphs, but labelled them as Δy, one incorrectly (i) then (ii) correctly
- Drew a Δy strip and used the formula $x = \int \sin^{-1} y\, dy$ or $\pi \int_0^{\frac{\pi}{2}} \left(\sin^2 x\right) dx$ or $\pi \int_0^{\frac{\pi}{2}} (\sin^2 x - 1)\, dx$
- Drew a Δy strip and used an incorrect formula as $\pi \int_0^1 x\,(\sin x)\, dx$, after substituting in the correct formula
- Used a Δy strip and shaded below, used formula, even if a Δy strip was drawn
- Selected a Δx strip incorrectly under the sine graph, drew a disc as if a Δy strip was rotated about the *y*-axis and used an incorrect formula as $\pi \int_0^{90} x\,(\sin x)\, dx$, after substituting in the correct formula

No understanding: Students

- Drew an incorrect sine graph and selected a Δx strip below it, using a the formula $\int_0^{\frac{\pi}{2}} x^2\, (\sin x) dx$
- Drew a correct sine graph and selected a Δy strip below it

The variety of different attempts in this question reveals how confused the students were, and how failure in one facet of the problem can make them fail in the rest of the problem. A vast number of students were seen to use different formulae to calculate volume despite the fact that they might have chosen the correct strip and perhaps also rotated it correctly. None of the responses were correct. All the students struggled to come up with the correct formula for a shell as $2\pi \int_0^{\frac{\pi}{2}} x(1 - \sin x)\, dx,$ despite the fact that some of them managed to draw the correct graph and the correct strip. It seems as if students do not have a clue about how different strips results in different solids leading to a disc, a washer or a shell method. A significant number of students were seen to use the boundaries as 1 and 0, instead of seeing the graph of $y = 1$ as the top graph and the graph of $y = \sin x$ as the bottom graph. The results also reveal that students avoid the shell method; they tend to use the disc method or the washer method.

In Figures 6.56 and 6.57, examples of actual written responses are given, showing *traces of understanding.*

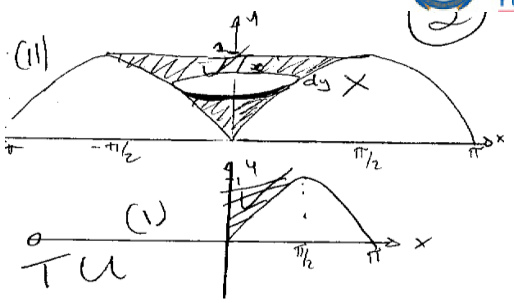


Figure 6.56: Correct graph with a Δy strip

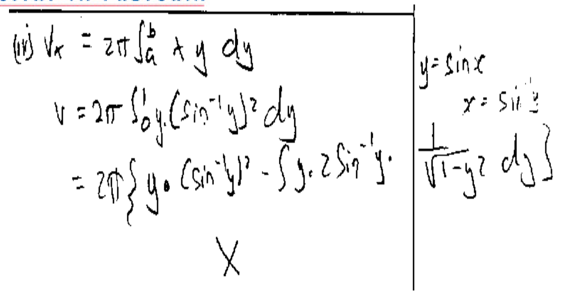


Figure 6.57: Inverse function manipulation

Question 11B: Typical students' errors when deriving the formula for volume of a cone

Almost correct: Students

- Used $y=\frac{r}{h}x+c$ and integrated correctly with the shell method, but lost *h* on the way.
- Drew a line similar to $y=x$, labelled it $y=mx+c$ and rotated about the *y*-axis formulating a cone with radius *r* and height *h*, and used the formula $V_y=\pi\int_0^r\left(\frac{h}{r}x\right)^2 dy$, but erased the square on *h* and *r* in the next step, working towards the formula

Traces of understanding: Students

- Drew a correct straight line in the 2nd quadrant, but did not give its equation nor integrate
- Wrote the formula $y=\frac{r}{h}+c$, without the graph and integrated using the formula $V=2\pi\int_0^r(h-x)y\,dy$

No understanding: Students

- Did not draw graphs, the equation $\frac{1}{3}\pi h$ and $\frac{1}{3}\pi r^2$ were integrated with no limits of integration
- Calculated volume as $\frac{dV}{dr}$, also $\frac{dV}{dh}$ same as in rate of change
- Integrated $\int_a^b\frac{1}{3}\pi r^2 dx$ but still failed, some without limits
- Integrated $\frac{1}{3}\pi r^2 h$ to be, $\frac{1}{3}\pi x\frac{r^3}{3}\frac{h^2}{2}$ or $\frac{1}{3}\pi\frac{r^3}{3}\frac{h^2}{2}$, or $\frac{1}{9}\pi r^2 h$ and $\frac{1}{6}\pi r^2 h^2$ as in rate of change

Discussion on Skill factor V

The majority of the students lacked the broader cognitive skills required to solve Question 11A. Students in this case did not display the possibility of cognitive development. They were actually performing poorly. Students were in most cases seen to use a Δ*x* strip and a disc or a washer method. At times a disc or washer method was used with a Δy strip. Those who tried to use the shell method struggled when having to substitute using the two graphs or from the correct limits. For Question 11B, most of the students were seen to integrate the given expression. It seems as if most of the students do not know that the given expression can be derived from the 1st principles as formula for volume of a cone which is formulated when a straight line with a positive or negative gradient is rotated about the *x*-axis or the *y*-axis, depending on how the line is drawn and how the strip is selected.

6.1.6 Overall responses per question, per element and per skill factor for the questionnaire runs

Table 6.26 and the Figures 6.58 and 6.59 display the sum of individual rank scores (from Appendix 4A) of the responses from the Questionnaire 1st run under each question. Overall performance is presented, to indicate performance in terms of the proportion of fully correct and almost acceptably correct responses in questions, elements and skill factors.

Table 6.26: All 11 elements for the Questionnaire 1st run

RESPONSES	GR		AV2D		VA2D		AV3D		VA3D		2D-3D		3D-2D		CD(V)		DC-CD(A)		GMNP			CGLCD	
	1A	1B	2A	2B	3A	3B	4A	4B	5A	5B	6A	6B	7A	7B	8A	8B	9A	9B	10A	10B	10C	11A	11B
FC	4	9	3	13	31	16	4	4	23	4	15	3	21	3	8	2	1	0	10	15	2	0	1
AC	6	10	28	8	2	4	8	18	6	6	1	2	4	2	3	1	2	0	3	7	22	6	4
TU	0	8	4	4	3	8	3	10	5	0	2	10	2	14	15	12	2	17	1	2	9	24	2
NU	17	10	1	10	1	8	11	2	3	18	8	19	7	4	10	6	12	9	22	4	3	4	17
ND	10	0	1	2	0	1	11	3	0	9	11	3	3	14	1	16	20	11	1	9	1	3	13
(FC+AC)	**10**	**19**	**31**	**21**	**33**	**20**	**12**	**22**	**29**	**10**	**16**	**5**	**25**	**5**	**11**	**3**	**3**	**0**	**13**	**22**	**24**	**6**	**5**
Questions %	27.0	51.4	**83.8**	56.8	**89.2**	54.1	32.4	59.5	**78.4**	27.0	43.2	**13.5**	67.6	**13.5**	29.7	**8.1**	**8.1**	**0.0**	35.1	59.5	64.9	**16.2**	**13.5**
ELM%(FC+AC)	39.2%		**70.3%**		**71.6%**		45.9%		52.7%		28.4%		40.5%		**18.9%**		**4.1%**		53.2%			**14.9%**	
SKF %(FC+AC)	55.9%										34.5%				**11.5%**				53.2%			**14.9%**	
OVERALL%	**40.5%**																						

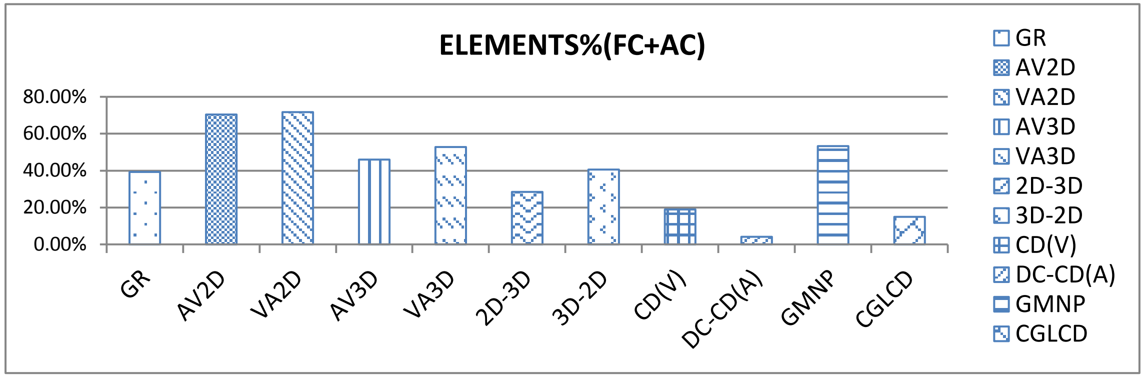


Figure 6.58: Comparing the 11 elements for the Questionnaire 1st run

For the tables in this section, the following acronyms are used
GR: Graphing skills
AV2D: Translation from algebraic to visual in 2D
VA2D: Translation from visual to algebraic in 2D
AV3D: Translation from algebraic to visual in 3D
VA3D: Translation from visual to algebraic in 3D
2D-3D: Translation from 2D to 3D
3D-2D: Translation from 3D to 2D
CD(V): Translation from continuous to discrete (visually)
DC-CD(A): Translation from discrete to continuous and continuous to discrete (algebraic)
GMNP: General manipulation skills
CGLCD: Consolidation and general level of cognitive development

From Table 6.26, it is evident that the performance was poor in 7 questions falling under 5 elements (translation from 2D to 3D, translation from 3D to 2D, translation from continuous to discrete (visually), translation from discrete to continuous and continuous to discrete (algebraic) and consolidation and general level of cognitive development), not satisfactory in 5 questions falling under 5 elements (graphing skills, translation from algebraic to visual in 3d, translation from visual to algebraic in 3D, translation from continuous to discrete (visually) and general manipulation skills), satisfactory in 6 questions falling under 6 elements (graphing skills, translation from algebraic to visual in 2D, translation from visual to algebraic in 2d, translation from algebraic to visual in 3D, translation from 2D to 3D and general manipulation skills), good in 2 questions falling under 2 elements (translation from 3D to 2D and general manipulation skills), and excellent in 3 questions falling under 3 elements (translation from algebraic to visual in 2D, translation from visual to algebraic in 2D and translation from visual to algebraic in 3D). The discussion that follows is based on the difficulties that the students have from the 11 elements regarding VSOR (refer to Table 6.26 and Figure 6.58) in terms of performance level from the average of the questions under each element.

Students' performance was poor (less than 20%) in three elements (Elements 8, 9 and 11), evident from the shortest bars. The proportion of acceptably correct responses under these three elements were 4.1% for the translating from discrete to continuous and continuous to discrete, where the students were expected to represent the equations using the rectangular strips and discs on the given diagram; 14.9% for consolidation and general level of cognitive development, where the students were expected to do skills required in the five different elements as one question and 18.9% for translation from continuous to discrete (visually), where the students were expected to represent the rectangular strips and discs on the given diagram. These reveal that the students have major difficulties in these three elements.

For the two elements (Elements 8 and 9), translating from discrete to continuous and continuous to discrete (algebraic) and for translation from continuous to discrete (visually), using the representative strip, stemming from the Riemann sum, is the main aspect. If students lack knowledge of the main concepts of the Riemann sum, selection and interpretation of the representative strip becomes problematic, even if a correct graph might be drawn. In regard to Element 11, consolidation and general level of cognitive development, the students were required to consolidate what was done in five elements (graphing skills; translation from visual to algebraic in 2D; translation from visual to algebraic in 3D; translation from continuous to discrete (visually) and general manipulation skills) as one question. All these elements consolidated, result in skills that require a certain level of cognitive development. The students are expected to first draw graphs. The correctness of the graphs that are drawn depends on the general manipulation skills, from the calculations for the important points for the graph. The students are then asked to select the representative strip (translation from continuous to discrete (visually)), representing the region bounded by the drawn graphs, which is also affected by the correctness of the graph. Based on the drawn graph and the representative strip selected, the students are then required to interpret the drawn graph so as to come up with the formula for area, (translation from visual to algebraic in 2D) or for volume, (translation from visual to algebraic in 3D) after rotation of the bounded region, where the strip has been selected. Students are then expected to calculate the area or volume from the selected strip which again requires general manipulation skills. It is clear from the above results, based on the proportion of the acceptably correct responses (14.9%), that most of the students did not reach the required cognitive level when solving VSOR problems.

Students' performance was not satisfactory in Elements 1, involving graphing skills and Element 6 involving translation from 2D to 3D, with 39.2% and 28.4% respectively as the proportion of acceptably correct responses. This reveals that even though the students have difficulties in drawing graphs as well as rotating the 2D diagrams to 3D diagrams (solids of revolution), the difficulties are not major as compared with the selection of the strip and the questions that require consolidation and general cognitive development.

However, despite the difficulties, the students' performance was satisfactory in four elements (translation from algebraic to visual in 3D, translation from visual to algebraic in 3D, translation from 3D to 2D, general manipulation skills), and good in only two elements, translation from algebraic to visual in 2D involving representation of area in a form of a diagram, with 70.3% of acceptably correct responses and translation from visual to algebraic in 2D, involving representation of area in a form of an equation with 71.6% of acceptably correct responses. The results reveal that even if students' performance was excellent in individual questions (three questions only) from different elements (translation from algebraic to visual in 2D, translation from visual to algebraic in 2D and translation from visual to algebraic in 3D), overall there were no elements where the average performance in all questions was excellent. This is an indication that generally VSOR is difficult for the students.

When grouping the 11 elements into five skill factors, Table 6.26 and Figure 6.59 are used to display students' performance from the Questionnaire 1st run.

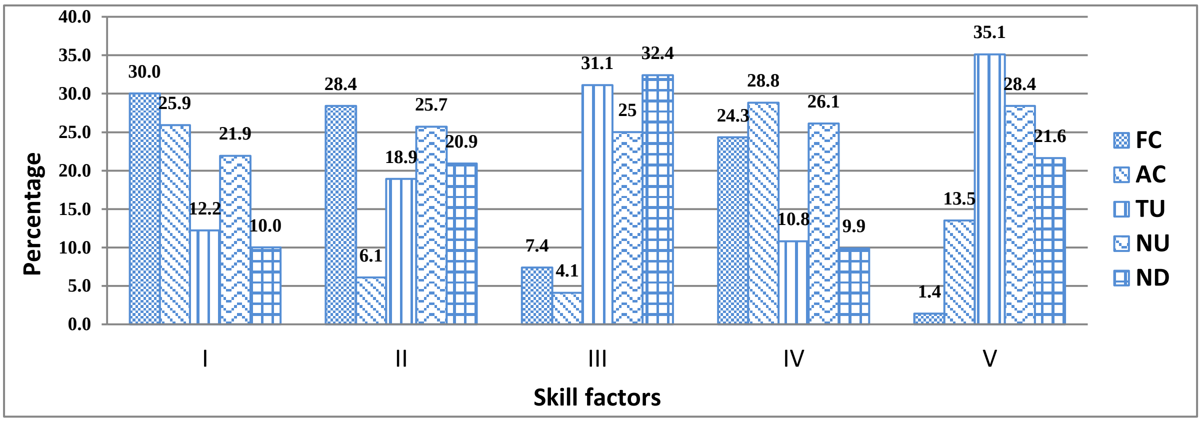


Figure 6.59: The five skill factors compared

As indicated in Table 6.26 and Figure 6.59 it is clear that most of the students are performing satisfactorily (55.9%) in the skill factor where students are drawing graphs and translating between the visual graphs and algebraic equations/expressions both in 2D and in 3D (Skill factor I). The performance in skill factor involving questions where students' general manipulation skills were tested was also satisfactory, where 53.2% of acceptably correct responses were produced (Skill factor IV).

In the other three skill factors, a large proportion of the responses were incorrect. Most of the students struggled mainly with the skill factor involving moving between discrete and continuous representations (Skill factor III). Only 11.5 % of the responses were acceptably correct, regarded as poor performance, with a large proportion (32.4%) of responses being where the students did not respond to the question. This skill factor has the highest proportion of non-responses in relation to the other four skill factors. In this skill factor, the selection of the representative strip which approximate the region bounded by the drawn graphs and the Riemann sum are the main concepts.

Another skill factor where the performance was poor, with 14.9% of acceptably correct responses, is the skill factor involving questions that require that students be at a certain level of cognitive development (Skill factor V), namely consolidation and general level of cognitive development. With general level of cognitive development, students use manipulation skills to draw graphs, select the representative strip, rotate it in terms of volume, translate the drawn graph to represent formula for area or volume and perform manipulation skills based on the selected formula. With this skill factor, students are expected to possess the necessary skill from all the other four skill factors. Only 14.9 % of the responses were acceptably correct, with the lowest of 1.4 % fully correct responses compared to the other skill factors, also with higher proportion (21.6%) of non-responses.

Finally only 34.5 % (performance that was not satisfactory) of the responses were acceptably correct for the skill factor with questions of a conceptual nature, where students were translating between two-dimensional and three-dimensional diagrams (Skill factor II), where solids of revolution are drawn. This low performance in the three skill factors (II, III, and V) discussed above reveal that students are having difficulties with VSOR.

In Table 6.27 (results for Test 1 and 2 from one group of students) and Table 6.28 (results for Test 3 from a different group of students), the responses from the Questionnaire 2nd run are indicated and discussed. Individual questions and mainly elements, where all questions were written are compared.

Table 6.27: All 4 elements for the Test 1 and 2; and other questions from the Questionnaire 2nd run

RESPONSES	GR		AV2D	VA2D	AV3D		VA3D	2D-3D	3D-2D		CD(V)	DC-CD(A)	GMNP			CGLCD
	1A	1B	2B	3B	4A	4B	5B	6A	7A	7B	8B	9A	10A	10B	10C	11A
FC	8	54	9	43	0	2	18	0	28	4	0	0	8	55	4	0
AC	17	16	28	11	5	16	23	15	8	1	0	15	5	25	28	4
TU	32	19	30	44	25	15	14	19	13	32	14	19	7	15	32	63
NU	58	31	48	23	61	75	57	66	57	46	47	33	97	12	58	52
ND	7	2	7	1	31	14	10	22	16	39	61	55	5	15	0	3
	122	122	122	122	122	122	122	122	122	122	122	122	122	122	122	122
FC+AC	**25**	**70**	**37**	**54**	**5**	**18**	**41**	**15**	**36**	**5**	**0**	**15**	**13**	**80**	**32**	**4**
Questions %	20.5	57.4	30.3	44.3	**4.1**	**14.8**	33.6	**12.3**	29.5	**4.1**	**0.0**	**12.3**	**10.7**	65.6	26.2	**3.3**
ELM(FC+AC)	38.9%				9.4%				16.8%				34.2%			

Table 6.28: Responses for Test 3 from the Questionnaire 2nd run

RESPONSES	AV2D	VA2D	VA3D	2D-3D	CD(V)	DC-CD(A)	CGLCD
	2A	**3A**	**5A**	**6B**	**8A**	**9B**	**11B**
FC	0	30	32	9	4	0	0
AC	29	17	4	8	2	0	0
TU	18	0	9	18	38	44	2
NU	5	5	9	18	10	6	39
ND	2	2	0	1	0	4	13
	54	**54**	**54**	**54**	**54**	**54**	**54**
FC+AC	29	47	36	17	6	0	0
Questions %	53.7%	**87.1%**	66.7%	31.5%	**11.1%**	**0.0%**	**0.0**

From Tables 6.27 and 6.28 the results reveal that less than 20% (regarded as poor performance) of the students got acceptably correct responses in 10 questions, 7 in Test 1 and Test 2, under seven elements (translation from algebraic to visual in 3D; translation from 2D to 3D; translation from 3D to 2D; translation from continuous to discrete (visually); translation from discrete to continuous and continuous to discrete (algebraic); general manipulation skills and consolidation and general level of cognitive development) and three questions in Test 3 under three elements (translation from continuous to discrete (visually); translation from discrete to continuous and continuous to discrete (algebraic) and consolidation and general level of cognitive development). From these 10 questions, where performance was poor, there were three questions, where none of the students answered the questions. One of these three questions is Question 8B from Test 1 and 2, requiring an approximation of the given 3D diagram using discs where translation is from continuous to discrete (visual). The other two questions, both from Test 3 are Questions 9B and 11B, respectively requiring the use of a given formula for volume to represent it on a given 3D diagram to represent discs where translation is from discrete to continuous and continuous to discrete (algebraic) and the derivation of the formula for volume of a cone from the first principles, possible if one has competency in the skills for the consolidation and general level of cognitive development.

Similar to the results of the Questionnaire 1st run, poor performance in the Questionnaire 2nd run was mainly for the same three elements, namely translation from continuous to discrete (visually), translation from discrete to continuous and continuous to discrete (algebraic) and the consolidation and general level of cognitive development. In the Questionnaire 1st run, the average response percentage for the questions under these three elements were calculated since both questions were given to the same group of students, but were not calculated in the Questionnaire 2nd run since the questions were given to two different groups of students, thus resulting in two separate response percentages.

The proportion for the acceptably correct responses under the three elements are given, respectively for the Questionnaire 1st run and for the Questionnaire 2nd run as 18.9%, 0% and 11.1% under translation from continuous to discrete (visually), 4.1%, 12.3% and 0% under translation from discrete to continuous and continuous to discrete (algebraic) and 14.9%, 3.3% and 0% under the consolidation and general level of cognitive development. The performance was poor in Questionnaire 1st run for the element where translation is from discrete to continuous and continuous to discrete with 4.1%, while for the Questionnaire 2nd run, performance was poor for the element requiring consolidation and general level of cognitive development with 3.3% and 0% from both groups. As argued above, for these three elements, the representative strip, stemming from the Riemann sum, is the main aspect.

Looking at the four elements (graphing skills, translation from algebraic to visual in 3D, translation from 3D to 2D and general manipulation skills) in the Questionnaire 2nd run where the same group of students wrote all questions, different results were found. In contrast, to the results of the Questionnaire 1st run, it is evident that the performance in Test 1 and 2 (Questionnaire 2nd run) was poor in the elements, translating from algebraic to visual in 3D and translating from 3D to 2D with the proportion of acceptably correct responses as 9.4% and 16.8% respectively, while for the Questionnaire 1st run, the response percentages were 45.9% and 40.5% respectively. However, for the graphing skills, the performance in both Questionnaire 1st run and Questionnaire 2nd run were not satisfactory, with the proportion of acceptably correct responses as 38.9% and 39.2% respectively nearly equal.

6.1.7 Total responses for all categories

In Table 6.29 and Figure 6.60, the overall responses are given for all 23 questions from the 37 responses. The total number of fully correct responses, almost correct responses, responses showing traces of understanding, no understanding and not done are given for the 37 students for the 23 questions, resulting in $37 \times 23 = 851$ responses in total.

Table 6.29: The responses for all questions

RESPONSES	Total	%	Total %
Fully correct	192	22.6	40.5
Almost correct	153	18	
Traces of understanding	157	18.4	59.5
No understanding	206	24.2	
Not done	143	16.8	
Σ	851		

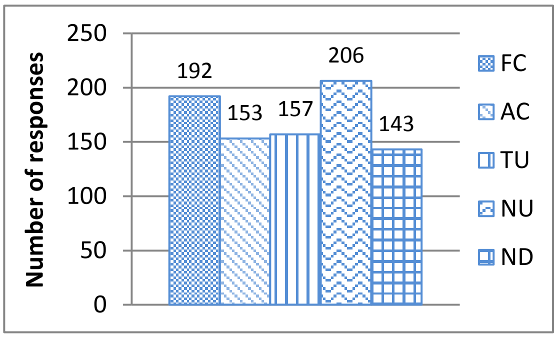


Figure 6.60: All responses represented

There were 192 (22.6%) fully correct responses; 153 (18%) showing almost correct responses; 157 (18.4%) responses showing traces of understanding; 206 (24.2%) responses showing no understanding and finally 143 (16.8%) where the students did not attempt to answer the questions, categorised as not done, shown in Figure 6.60, which is bi-modal. It can be argued that in general the students had problems with VSOR, with the highest percentage (24.2%), revealing that the students showed no understanding of the questions given. Even though the fully correct responses also show a higher percentage (22.6%), it is mainly because most of the problems in the instrument involved general manipulation skills. Overall, students struggled with the problems that required higher order thinking skills, where general manipulation skills were not used. Overall the performance was satisfactory (40.5%), yet nearly below 40%.

6.1.8 Performance in the five skill factors classified in terms of procedural and/or conceptual knowledge

In Table 6.30 (adapted from Appendix 4D), the five skill factors are classified as either requiring the use of procedural and conceptual skills, conceptual skills or procedural skills.

Table 6.30: Procedural and conceptual skills from the Questionnaire 1st run

<table>
<tr><th></th><th>Procedural and conceptual Skills %
Skill factors I and V</th><th>Conceptual skills %
Skill factor II and III</th><th>Procedural Skills %
Skill factor IV</th></tr>
<tr><td>FC</td><td rowspan="2">49.1</td><td rowspan="2">23</td><td rowspan="2">53.1</td></tr>
<tr><td>AC</td></tr>
<tr><td>TU</td><td rowspan="3">50.9</td><td rowspan="3">77</td><td rowspan="3">46.9</td></tr>
<tr><td>NU</td></tr>
<tr><td>ND</td></tr>
</table>

Questions that are procedural and conceptual in nature are found in Skill factor I, where students draw graphs and solve problems where they translate between the visual graphs and algebraic equations/expressions and in Skill factor V, where students perform skills required after consolidation of five elements from Skill factors I, II, III and IV, requiring a certain level of cognitive development. The performance with questions that were procedural and conceptual in nature was satisfactory (49.1% of the acceptably correct responses).

Questions that are conceptual in nature are found in Skill factor II, where students were translating between two-dimensional and three-dimensional diagrams and Skill factor III involving moving between discrete and continuous representations. In these questions, the students' performance was poor (23% of the acceptably correct responses). This performance was the lowest compared to the other skills.

The performance in Skill factor IV, involving questions that are more procedural in nature where students' general manipulation skills were tested, was also satisfactory (53.1% of the acceptably correct responses). In general the students were partially competent in skill factor IV that involved procedural knowledge, and not fully competent in the skill factors that involve both procedural and conceptual skills and not competent in the skill factors that involve conceptual skills only.

In the section that follows, general observations are made based on the five skill factors.

6.1.9 General observations for the five skill factors from the questionnaire runs

Skill factor I: *Graphing skills and translating between visual graphs and algebraic equations/ expressions in 2D and 3D*

The results of the Questionnaire 1st run and the Questionnaire 2nd run reveal that students' performance in drawing graphs was not satisfactory. Students were in most cases able to draw graphs that they were familiar with or completely failed to draw the graphs that they had not seen before or worded differently. Students had difficulty mainly in answering a question requiring that a line with a positive gradient passing through the origin on a given interval be drawn. In most cases the interval was interpreted as coordinates, where $x = 0$ and $y = 3$. By contrast, in drawing the graphs of $x^2 - y^2 = 9$ and $x = 5$ that were expected to be more difficult to draw than a line with a positive gradient, the performance was good. It seems as if students had difficulty in interpreting this problem since it was presented as a word problem, they did not know what "a line with a positive gradient" is, but could easily draw a line $y = x$ as was seen in other questions. The students struggled to interpret the symbolic notation representing an interval.

When translating from algebraic representation to visual representation, where an integral formula for area was to be translated to a diagram, most students in the Questionnaire 2nd run were not successful. In the Questionnaire 1st run, the performance was good when translating from algebraic to visual, involving the integral formula for area. In both questionnaire runs, more students succeeded in translating from visual graphs to algebraic equations in 2D, especially if a Δx strip was appropriate, compared to having to draw the graphs.

When students solved the problems that relate to translation between algebraic and visual representations in 3D, the performance in the Questionnaire 1st run was satisfactory while not satisfactory in the Questionnaire 2nd run. Even if the performance in the Questionnaire 1st run was satisfactory, students in both runs of the questionnaire were seen to struggle with the formula that involves the translation from algebraic to visual when the formula for a disc was used, compared to when a shell was used. The disc in the disc method was not interpreted correctly. Most students used it as if it related to a parabola that was to be drawn and not to a straight line that was squared as a result of the formula for a disc (refer to Question 4A). For the other formula, probably the presence of 2π may have triggered some form of awareness to them that it represents a shell. Many students were seen to draw the parabolas as they appear in 2D without drawing the 3D solids from the 2D rotations.

The translation from visual graphs to algebraic equations in 3D was seen to be simpler only when a disc method was appropriate. A fair number of students managed to come up with the correct formula for volume for a disc. They failed when it resulted as a shell. Most students ignored the Δx strip drawn and used a Δy in the formula that represented a disc method or a washer method for the cosine graph. For those students who tried to use a shell method, errors were found as the students used incorrect limits of integration.

Overall the results from the Questionnaire 1st run reveal that the students' performance was satisfactory (55.9%) in graphing skills and translating between visual graphs and algebraic equations/expressions (composed of five elements).

Skill factor II: *Three-dimensional thinking*

In both questionnaire runs students find it easier to translate when simple diagrams like a straight line were rotated to formulate 3D solids, involving translation from 2D to 3D. They found it difficult to rotate when given diagrams that required more imaginative skills at a higher level of conceptualising like when a torus had to be formed after rotation of a circle that was a certain distance away from the *x*-axis and from the *y*-axis. Generally, students had difficulty in drawing solids of revolution, involving translation from 2D to 3D, where 2D diagrams were given. Instead of drawing solids of revolution, 2D diagrams were drawn. The students' performance in the Questionnaire 1st run, when translating from 2D to 3D was not satisfactory.

When a 3D solid or its description was to be represented as a 2D diagram, where translation from is 3D to 2D, students also encountered difficulties. The performance in the Questionnaire 1st run was satisfactory, while poor in the Questionnaire 2nd run.

Overall, the performance in the Questionnaire 1st run in three-dimensional thinking (composed of two elements) was not satisfactory (34.5%).

Skill factor III: *Moving between discrete and continuous representations*

The questions involved the approximation of area and volume from rectangles and discs that represented a bounded region. The results revealed that many students struggled mainly with the translation from the algebraic expressions to the approximation of area or volume on the given diagram, involving translation from discrete to continuous and continuous to discrete (algebraic). It was quite clear that most of the students were not familiar with the concept of Riemann sums algebraically as representing the area of rectangles for a 2D diagram and the area of the circle for a 3D diagram on a bounded region. The students also struggled to approximate the area of the region bounded by graphs using rectangles and discs involving translation from continuous to discrete (visually).

The results of both Questionnaire runs reveal that students were not competent in approximating the area of the region bounded by graphs and in translating from the algebraic expressions to the approximation of area or volume on the given diagram, using rectangles and discs. Evidence from the students' responses points to the students' lack of in-depth knowledge about the Riemann sums. Overall, the performance in the Questionnaire 1st run in moving between discrete and continuous representations was poor (11.5%) and very low compare to the other four skill factors. The results reveal that the students do not have in-depth knowledge about the significance of the representative strip. This calls for serious interventions so that the learning of VSOR becomes meaningful to students who will not just want a formula to substitute in, but think carefully about the selection of the representative strip, which is where the formulae come from. In that way conceptual understanding may be enforced.

Skill factor IV: *General manipulation skills*

The results from the Questionnaire 1st run revealed that most of the students were fairly fluent with regard to general manipulation skills. In some cases errors were made as students solved problems that involved the evaluation of an integral involving integration by parts. Mathematical errors were also made when students calculated the point of intersection of graphs. Many students were seen to make y the subject of the formula from the ellipse by seeing $\sqrt{36-4x^2}$ as $6-2x$, hence making the whole solution incorrect after substitution. The results from the Questionnaire 1st run reveal that students' performance was satisfactory (53.2%) with general manipulation skills. In regard to the Questionnaire 2nd run, students' performance was not satisfactory (34.2%).

Skill factor V: *Consolidation and general level of cognitive development*

In the Questionnaire 1st run it was found that the majority of the students lack the general cognitive skills required to solve problems that involve five elements from Skill factors I, II, III and IV, when consolidated, since they lack in-depth understanding of VSOR. Students' partial competency in drawing graphs; failure in identifying the strip correctly and drawing the 3D diagram represented by the rotated strip after rotation, impacts heavily on the consolidation and general level of cognitive development. This leads to difficulty in learning about VSOR, which leads to poor performance. If the students' performance in the five elements from Skill factors I to IV is so low, how do they then manage to solve problems that address these skill factors all at once, when consolidated? The poor performance in Question 11A and 11B give evidence of that. In some instances students drew the graphs correctly, but gave incorrect limits for integration, meaning that they failed to translate from a visual graph to an algebraic equation. It was also found that the students struggled to use integration techniques. The students' performance in Skill factor V was also poor (14.9%), same as that from skill factor III (11.5%).

In the Questionnaire 2nd run the results were similar to those from the Questionnaire 1st run. The students also partially managed to draw the graphs and failed to interpret them correctly. From the responses given by students, it seems as if tasks similar to those in Skill factor V are cognitively demanding for these students. Cognitive obstacles were encountered when dealing with such tasks, as the ones under Skill factor V. The majority of these students lacked skills in the general level of development. They could not meet the cognitive demands of the tasks. The students' cognitive abilities are not at a level that enables them to solve the tasks that involve the consolidation and general level of cognitive development.

6.1.10 Discussion and conclusion

From the discussion on the five skill factors above, it seems as if students struggled mostly with Skill factor III, involving questions that are conceptual in nature where students were translating between continuous and discrete, which relates to the use of the Riemann sums. The results reveal that students are not competent in the concept of the Riemann sums. Students do not know how to approximate the bounded region or the volume generated from the rectangles or the discs from the Riemann sums. What emerges from these results is that even if students manage to draw correct graphs, or graphs are given, most of them struggle to locate the rectangular strip that approximates the area or the volume after rotation of the bounded region.

The general conclusion after these investigations is that although students perform better in some of the elements, their overall performance (40.5%), which is very close to being not satisfactory (below 40%) indicates that this section of the syllabus involving VSOR, is perhaps cognitively more demanding than other topics (those that constitute the other 60% of the paper). Students are seen to rely on types of problems that they have been exposed to before, so they fail if the problems in examinations differ from what they have seen before. The same applies to the 23-item instrument where most of the questions were somewhat different from the format of the examination. However performance seems to be higher in certain questions, especially those questions where a graph is given to students. It was also found that the students cannot draw properly where graphs are not given. At times they tend to abandon the drawn graphs when they do the translations, especially if the graph is a bit complicated. The skill factors that the students seem to be partially competent in are the general manipulation skills as well as graphing skills and translating between visual graphs and algebraic equations/expressions, where the students' performance was satisfactory.

Overall, from both runs of the questionnaire, even though different students were used at different times and different results were obtained, some similar trends were evident. More attention, in terms of areas where learning should be improved needs to be on the three skill factors of knowledge where the students are not competent in and where performance is not satisfactory and poor namely:

- Moving between discrete and continuous representations (poor performance).
- Three-dimensional thinking (performance that is not satisfactory).
- Consolidation and general level of cognitive development (poor performance).

6.2 EXAMINATION ANALYSIS AND THE DETAILED WRITTEN EXAMINATION RESPONSES

The discussion of the results includes a quantitative analysis of the examination and a qualitative analysis of the detailed examination responses from seven students. Before presenting the results, the analysis of the questions that students responded to, is presented.

6.2.1 Examination analysis

In the section that follows, Question 5 from the August 2007 examination paper, contributing 40% of the whole examination is analysed in terms of the five skill factors.

6.2.1.1 Analysis of the examination scripts of 151 students

Below the subquestions of Question 5 are given (refer to a detailed marking memorandum of the Question 5 in Appendix 6A). Each subquestion is discussed in relation to the given classified elements from the skill factors.

<u>Question 5.1</u>

5.1.1 Calculate the points of intersection of: $y = -4x^2 + 4$ and $y = -x^2 + 4$
Sketch the TWO graphs and show the representative strip/element that you will use to calculate the volume of the solid generated when the area bounded by the graphs is rotated about the *y*-axis. (3)

5.1.2 Calculate the volume described in QUESTION 5.1.1 by means of integration. (4)

5.1.3 Calculate the volume moment of the solid about the *x*-axis as well as the *y*-ordinate of the centre of gravity of the solid. (5)

In Question 5.1.1 three subsections were evident. Students were required to

- Firstly, calculate the point of intersection of two graphs, which involves *general manipulation skills* (Skill factor IV).
- Secondly, sketch the two graphs, which involve *graphing skills* (Skill factor I).
- Thirdly, show the representative strip/element to be used in calculation of volume, which involves *translation from continuous representation to discrete representation* (Skill factor III).

In Question 5.1.2 students were required to

- Calculate the volume generated by means of integration, which involves the *translation between 2D and 3D* (Skill factor II); *the translation between visual graphs and* *algebraic equations* (Skill factor I) as well as the *general manipulation skills* (Skill factor IV), as the calculation is performed. In the marking memorandum, there is no mention as to how the drawn graph in 5.1.1 is *translated between 2D and 3D* (Skill factor II), in order to show the new solid generated as well as the resulting *disc, washer* or *shell*, whichever is applicable. It was anticipated that in calculating the volume the visual representation would also be shown.

In Question 5.1.3 students were required to

- Calculate the volume moment as well as the *y*-ordinate of the centre of gravity, which also involves the *translation between the visual graphs and algebraic equations for volume* (Skill factor I) as well as the *general manipulation skills* (Skill factor IV), when calculating volume.

<u>Question 5.2</u>

5.2.1	A vertical sluice gate in the form of a trapezium is 7 m high. The longest horizontal side is 8 m in length and in the water level. The shorter side is 4m in length and 7 m below the water surface. Make a neat sketch of the sluice gate and calculate the relationship between the two variables x and y.	(3)
5.2.2	Calculate the first moment of area of the sluice gate about the water level.	(4)
5.2.3	Calculate the second moment of area of the sluice gate about the water level as well as the depth of the centre of pressure of the sluice gate by means of integration.	(5)

In Question 5.2.1 two subsections were evident. Here students were required to

- Firstly, sketch the sluice gate, which involves the *graphical skills* (Skill factor I).
- Secondly, calculate the relationship between x and y, which involve *general manipulation skills* (Skill factor IV), but *translating between visual graphs and algebraic equations* (Skill factor I).

In Question 5.2.2, the students were required to

- Calculate the first moment of area which involves the *translation between visual graphs and algebraic equations* (Skill factor I), from the solution in 5.2.1 as well as *general manipulation skills* (Skill factor I).

In Question 5.2.3 they are required to

- Calculate the second moment of area as well as the depth of the centre of pressure which involves the *general manipulation skills* (Skill factor I) and incorporating the solution in 5.2.2.

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Question 5.3

5.3.1 Make a neat sketch of the curve $y = 3\cos x$ and show the area bounded by the curve and the lines $x = 0$ and $y = 0$. Show the representative strip/element that you will use to calculate the volume, by using the SHELL METHOD only, if the area bounded is rotated about the *y*-axis. (2)

5.3.2 Calculate the volume described in QUESTION 5.3.1.
Use the SHELL METHOD only. (5)

In Question 5.3.1 two subsections were evident. Here students were required to

- Firstly, sketch the graph and show the area bounded, which involves the *graphical skills* (Skill factor I) as well as the integration of the *general manipulation* (Skill factor IV) *and visual skills* (Skill factor I).
- Secondly, show the representative strip/element using SHELL METHOD only that will be used in the calculation of volume, which involves *translation between continuous and discrete representations* (Skill factor III).

In Question 5.3.2 the students were asked to

- Calculate volume, which involves *translation between visual and algebraic* (Skill factor I) as well as the *general manipulation skills* (Skill factor IV). In the marking memorandum, there is no mention as to how the drawn graph in 5.3.1 is *translated between 2D and 3D* (Skill factor II), in order to show the new solid generated as well as the resulting *shell*. It was anticipated that in calculating the volume the visual representation would also be shown, in a form of a solid of revolution.

Question 5.4

5.4.1 Calculate the coordinates of the points of intersection of: $y - 2x = 0$ and $x = \frac{1}{4}y^2$.

Sketch the graphs and show the representative strip/element that you will use to calculate the area bounded by the graphs. (3)

5.4.2 Calculate the area described in Question 5.4.1 (3)

5.4.3 Calculate the second moment of area described in Question 5.4.1 with respect to the *y*-axis. (3)

In Question 5.4.1 three subsections were evident. Here students were required to

- Firstly, calculate the point of intersection of two graphs, which involves the *general manipulation skills* (Skill factor I).
- Secondly, sketch the two graphs, which involve *graphing skills* (Skill factor I).
- Thirdly, show the representative strip/element to be used in calculation of the area, which involves part of *translation between continuous and discrete representations* (Skill factor III). In the marking memorandum the 2D diagram is shown.

In Question 5.4.2 students were required to

- Calculate the area generated by means of integration, which involves the *translation between visual graphs and algebraic equations* (Skill factor I), as well as *general manipulation skills* (Skill factor IV) as the calculation is performed.

In Question 5.4.3 students were required to

- Calculate the second moment of area from Question 4.1 with respect to the *y-axis*, which also involves *general manipulation skills* (Skill factor IV) and *translation between visual graphs and algebraic equations general manipulation skills* (Skill factor I).

Total Marks for the section [40]

In the examination paper, only five elements were assessed explicitly, the other six elements were assessed implicitly. In this section, the discussion will centre only on these five elements. The other six elements were indirectly embedded in the assessment.

The five elements are

- General manipulation skills (Skill factor I)
- Graphing skills (Skill factor IV)
- Translation from continuous to discrete (visually) (Skill factor III)
- Translation from visual to algebraic in 2D (Skill factor I)
- Translation from visual to algebraic in 3D (Skill factor I)

In all the tables and figures, abbreviations were used as follows, GMNP: general manipulation skills, GR: graphing skills, CD(V): translation from continuous to discrete representation (visually) from the selected strip, VA2D: translation from visual to algebraic in 2D and VA3D: translation from visual to algebraic in 3D.

6.2.1.2 Quantitative analysis of five elements that were tested directly from the question paper

- **The response for Question 5**

In this section performance of 151 students is discussed based on Table 6.31 and Figure 6.61 from the 17 subquestions. The responses were coded as follows: FC if the answer is fully correct; AC if the answer is almost correct; TU if there were some traces of understanding; NU if there was no indication of understanding and ND if there was no attempt in answering the question, drawn from Appendix 6B.

Table 6.31: Students' responses in five elements

	GMNP	**GR**	**CD(V)**	**VA3D**	**VA3D**	**GR**	**VA2D**	**VA2D**	**VA2D**	**GR**	**CD(V)**	**VA3D**	**GMNP**	**GR**	**CD(V)**	**VA2D**	**VA2D**
	1.1	1.1	1.1	1.2	1.3	2.1	2.1	2.2	2.3	3.1	3.1	3.2	4.1	4.1	4.1	4.2	4.3
FC	70	63	32	17	10	38	33	16	9	89	31	21	38	50	52	45	6
AC	13	26	19	21	10	15	12	31	38	25	2	41	37	12	4	28	8
TU	22	8	21	18	14	66	65	56	51	6	3	29	24	23	23	20	6
NU	28	29	33	67	57	12	19	17	17	9	81	29	17	27	20	18	74
ND	18	25	46	28	60	20	22	31	36	22	34	31	35	39	52	40	57
% FC+AC	54.9	58.9	33.8	25.2	13.2	35.1	29.8	31.1	31.1	**75.5**	21.9	41.1	49.7	41.1	37.1	48.3	9.3
% FC+AC	**GMNP: 52.3%**			**GR: 52.6%**				**CD(V): 30.9%**				**VA2D: 29.9%**			**VA3D: 26.5%**		
% FC+AC	**OVERALL: 37.5%**																

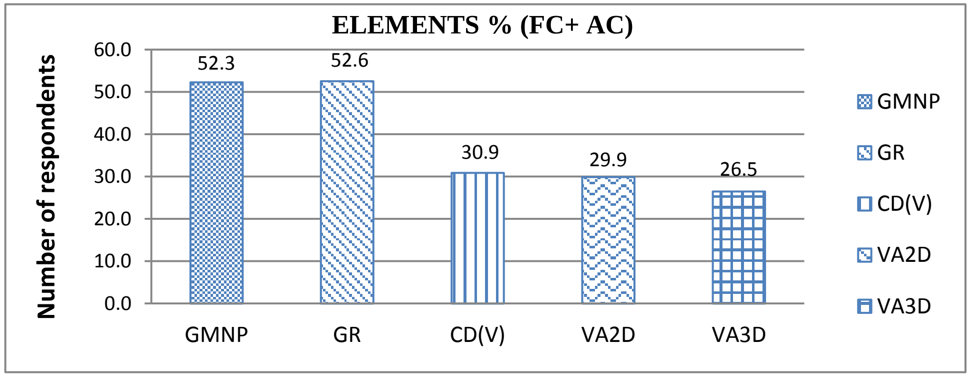


Figure 6.61: Comparing the five elements

For the tables in this section, the following acronyms are used
GMNP: General manipulation skills
GR: Graphing skills
CD(V): Translation from continuous to discrete (visually)
VA2D: Translation from visual to algebraic in 2D
VA3D: Translation from visual to algebraic in 3D

Questions 5.1.1 to 5.4.3 were classified under the five elements, depending on whether they involve graphing skills, general manipulation skills (only for calculation of points of intersection or intercept points or any other points necessary for drawing the graphs), moving from continuous to discrete (visual 2D and 3D), translation from visual to algebraic in 2D or translation from visual to algebraic 3D, leading to 17 subsections in line with the five elements. In the presentation and discussion of the results the general manipulations involving integration (evaluation of area, volume, centroid, centre of gravity and so on) that occurred after translation from visual to algebraic in 2D or in 3D are excluded. The reason for excluding them is that there were many different solutions based on the previous parts on the questions since the students were expected to start by drawing graphs, selecting the representative strip and translating from visual to algebraic in 2D or in 3D and errors made would affect the final general manipulations required.

In Table 6.31 and Figure 6.61 the data revealed that, out of the 17 questions, the question in which students performed well in involved graphing skills, where a graph of the curve $y = 3\cos x$, bounded by the lines $x = 0$ and $y = 0$ (graphing skills 3.1) was to be drawn, where 75.5% of the responses were correct and regarded as excellent performance. Question 3.1 indicated some interesting trends. The majority of the students did well in this question with 89 fully correct responses. Surprisingly most of the students were unable to draw the correct strip, a huge jump to only 31 fully correct responses, which means that the interpretation for the question that follows might be incorrect. In this question, it was evident that even if the students were able to draw the proper graph, the idea of translating the area of the graph in accordance with the Riemann sum, to show the strip that approximates the area correctly, was not well understood. These results are also confirmed from a huge jump in Question 1.1 with 63 fully correct responses for graphing skills 1.1, followed by 32 fully correct responses for a question involving translation from continuous to discrete (visually) 1.1. Clearly students have huge problems in selecting the correct strip, whether it should be a Δx strip or a Δy strip.

The results also indicate that in most cases, students' responses deteriorated from the first question to its subquestions. If one considers Question 5.1 for example, one will observe some trends for the fully correct responses. For the general manipulation skills 1.1, there were 70 fully correct responses, followed by 63 fully correct for graphing skills 1.1, followed by 32 fully correct responses for translation from continuous to discrete (visually) 1.1 followed by 17 fully correct responses for translation from visual to algebraic in 3D 1.2 and finally 10 fully correct responses for translation from visual to algebraic in 3D 1.3. That may imply that even if students get the first answer correct, they may fail to interpret it correctly to get to the next question. It may also mean that the incorrect response from the first question may affect the rest of the questions in such a way that the performance deteriorates.

The above analysis shows that there were more or less same patterns for Questions 5.1.1 and Questions 5.3.1. The question that may be asked is, if these questions are similar in some ways. Most students were able to draw the parabolas and a cosine graph asked in these two questions respectively. However, on the one hand a large number of students who chose a Δx strip in Question 5.1.1 indicated such a preference, even if it does not approximate the chosen area correctly. On the other hand, in Question 5.3.1, only a few students drew the correct Δx strip, which was different than in Question 5.1.1. The choice of a strip in this case was not well justified.

There was no question where the students' performance was good. The performance was satisfactory in six questions. The questions that need serious attention are the eight questions (Translation from continuous to discrete (visually),1.1; Translation from visual to algebraic in 3D, 1.2; Graphing skills, 2.1; Translation from visual to algebraic in 2D, 2.1; Translation from visual to algebraic in 2D, 2.2; Translation from visual to algebraic in 2D, 2.3; Translation from continuous to discrete (visually),3.1 and Translation from continuous to discrete (visually), 4.1) where performance was not satisfactory and two questions (Translation from visual to algebraic in 3D, 1.3 and Translation from visual to algebraic in 2D, 4.3) where performance was poor. Most of the questions where the students are having difficulty involve the selection of the representative strip and the translation from a visual graph to an algebraic equation in 2D and in 3D. The results also reveal that overall students' performance in general manipulation skills and in some of the graphing skills is satisfactory. Overall, from Table 6.34, the performance in Skill factor V was not satisfactory, where only 37.5% of the responses were acceptably correct.

It was found that even if the students were able to draw the correct graphs and sometimes the correct strip, the problem arose when the students had to translate from the graph to the algebraic formula for area or volume. The chosen strip was in most cases not considered when writing down the algebraic formula. The average mark that students obtained for Question 5 is 15.4 out of 40 and for the whole examination paper, it is 45.5 out of 100, which is satisfactory performance (refer to Appendix 6B).

In the next section written responses for seven students in Question 5 are presented.

6.2.2 Detailed written examination responses

6.2.2.1 Actual written responses from the seven students

Some students from College A (who were part of the group that was observed for 5 days) were given Question 5.1, 5.3 and 5.4 (as well as some other questions) to respond to as a test. Only seven scripts from this group were collected and analysed qualitatively. The percentages of acceptably correct responses (given in Figure 6.62) are calculated for each of the five elements so as to know how the seven students performed overall (Refer to Appendix 6D), in each element, for an example in graphing skills and in other elements.

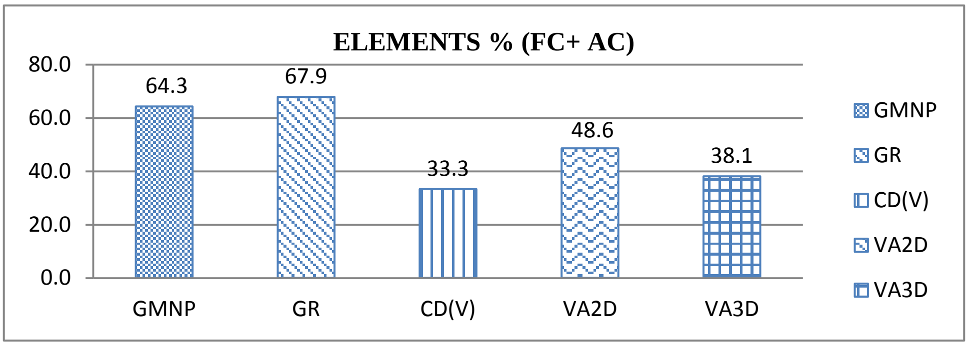


Figure 6.62: The performance from the seven students

The results from Figure 6.62 reveal that students' performance was good in graphing skills (67.9%) and in general manipulation skills (64.3%). Students' performance was satisfactory (48.6%) in translation from visual to algebraic in 2D. The performance was not satisfactory in translation from visual to algebraic in 3D (38.1%) and translation from continuous to discrete representations (33.3%). Overall, from the average of the 5 elements (50.2%), the performance in Skill factor V was satisfactory. However, even if the overall performance was satisfactory, the results reveal that most students had difficulty when selecting the strip and when interpreting the rotated strip so as to come up with the formula for volume.

Some of the examples from students' written responses are presented below.

- **Responses for Question 5.1**

In Question 5.1 students were asked to

5.1.1 Calculate the points of intersection of $y = -4x^2 + 4$ and $y = -x^2 + 4$.
Sketch the TWO graphs and show the representative strip/element that you will use to calculate the volume of the solid generated when the area bounded by the graphs is rotated about the *y*-axis. (3)

5.1.2 Calculate the volume described in QUESTION 5.1.1 by means of integration. (4)

5.1.3 Calculate the volume moment of the solid about the *x*-axis as well as the *y*-ordinate of the centre of gravity of the solid. (5)

Of the seven responses, five students were able to draw the correct graphs, but were unable to draw the proper Δy strip, hence failed to solve the problem correctly. The sixth student drew straight lines and not parabolas. The seventh student drew the graph correctly and drew the correct Δy strip for rotation. In Figure 6.63, written responses for one of the five students are given.

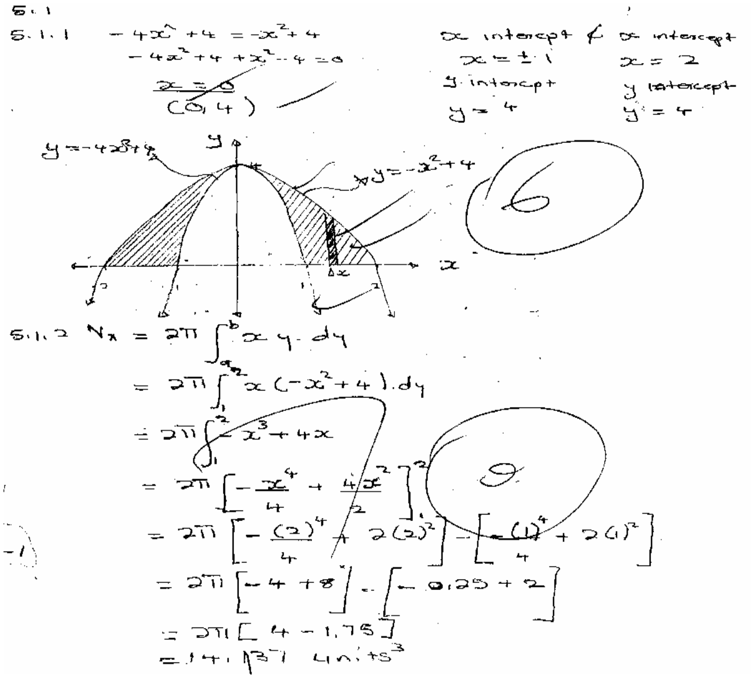


Figure 6.63: The incorrect approximation with a Δx strip

The results reveal that this student was able to draw the graphs correctly. The student drew a strip that does not accommodate all of the bounded area. This raises questions about the in-depth knowledge about the Riemann sum and the idea of slicing the area to touch all graphs used. What was interesting from the above example was that this student, compared to the other students used a Δx strip but in solving the problem referred to it as a Δy strip. The other four students also used a Δx strip and carried on solving the problem using the Δx strip, but did not get the solution correct since instead of the *'washer method'*, they used the '*shell method'*. Even if the first step was incorrect, based on the formula used, the general manipulation skills that were required from this student were correct. The student carried on to simplify and integrate correctly, until the final step. What is disappointing is that despite the other steps being correct, the student forfeits all the marks because the steps that follow are not in accordance with the marking memorandum, which is followed when marking this question. The reason behind this is that the formula used for the first step was incorrect. The wrong interpretation of the drawn graph raises questions as to whether the students refer to their drawn diagrams when they select the equations. This student also did not relate to the correct point of intersection that was calculated, which relates to the Δy strip.

The written responses given in Figure 6.64 are for the seventh student who drew the graph correctly, drew the correct strip, but failed to substitute in the formula correctly.

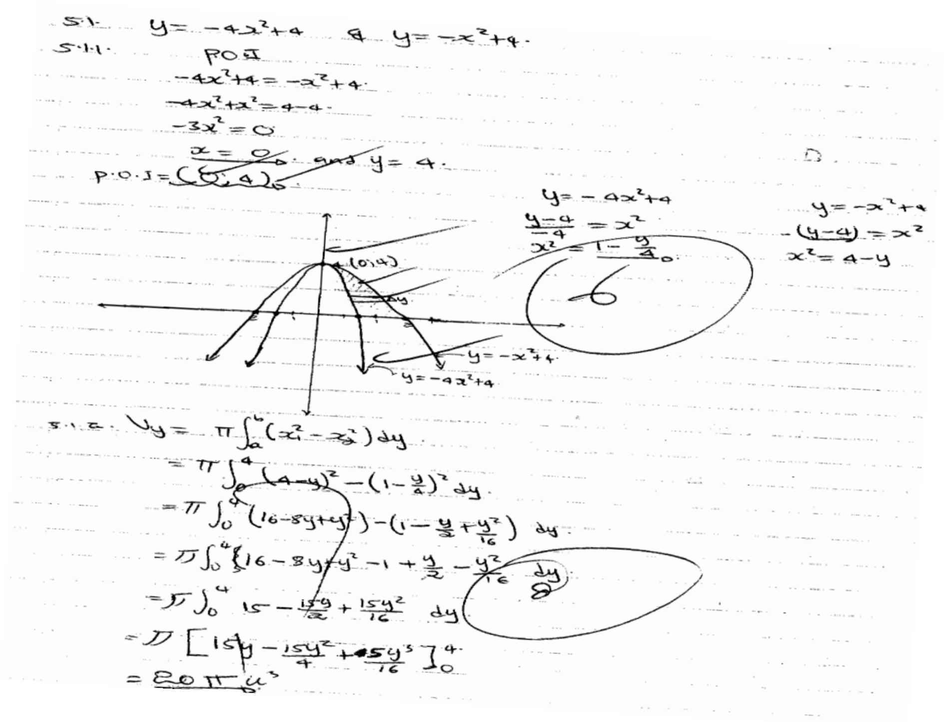


Figure 6.64: Incorrect substitution in the equation for volume

What is evident from the above answer is that this particular student was able to calculate the correct points of intersection, to draw the correct graph and to select the correct strip. This student could not substitute correctly, hence failed to transfer. The student used the correct formula to integrate volume as given in Figure 6.64 but failed to substitute correctly. The equations used to calculate the point of intersection were correctly given as $x^2=1-\frac{y}{4}$ and $x^2=4-y$. When substituting these equations in the formula for a washer as x_1^2 and x_2^2, the student failed to use them. The student continued with the squares again as if *x* was not squared already. The student wrote $\pi\int_0^4(4-y)^2-\left(1-\frac{y}{4}\right)^2 dy$ instead of $\pi\int_0^4(4-y)\,dy-\left(1-\frac{y}{4}\right)dy$.

From the responses given above, this student was able to draw the proper graph and the proper strip. The problem was the second step of the substitution where the squares were not necessary in the equation. The steps that followed were mathematically correct but the correct volume was not found. Of the seven scripts, five were able to draw the correct graphs, but were unable to draw the proper Δy strip, and thus failed to solve the problem correctly. The sixth student drew straight lines and not parabolas. The seventh student drew the graph correctly and drew the correct Δy strip for rotation.

- **Responses for Question 5.3**

  In Question 5.3 students were asked to

5.3.1 Make a neat sketch of the curve $y = 3\cos x$ and show the area bounded by the curve and the lines $x = 0$ and $y = 0$. Show the representative strip/element that you will use to calculate the volume, by using the SHELL METHOD only, if the area bounded is rotated about the *y*-axis. (2)

5.3.2 Calculate the volume described in QUESTION 5.3.1. Use the SHELL METHOD only (5)

For this question most of the students managed to draw the correct graphs, and the correct strip, but struggled to use the formula for the shell method correctly. There was one student who nearly got the correct answer for the volume, but did not use radians when evaluating the definite integral. Overall, most students were unable to use integration by parts when integrating $x\cos x$. Students where seen to use incorrect rules for integration including adding 1 and writing $\cos x$ as $\frac{\cos x^2}{2}$ even if they used the correct formula for the shell.

In Figure 6.65, an example is given of a student who drew the graph correctly, without drawing a strip (the strip on the diagram was drawn by the lecturer when he was marking). In the interpretation of the graph, this student used a Δy strip when substituting in the formula for volume where incorrect limits were used as 0 and 3 instead of being 0 and $\frac{\pi}{2}$, hence the student failed to translate from visual to algebraic in 3D. The volume to be calculated was therefore incorrect. This student was unable to calculate the volume correctly because of the incorrect formula for integration and incorrect integration techniques as shown in Figure 6.65.

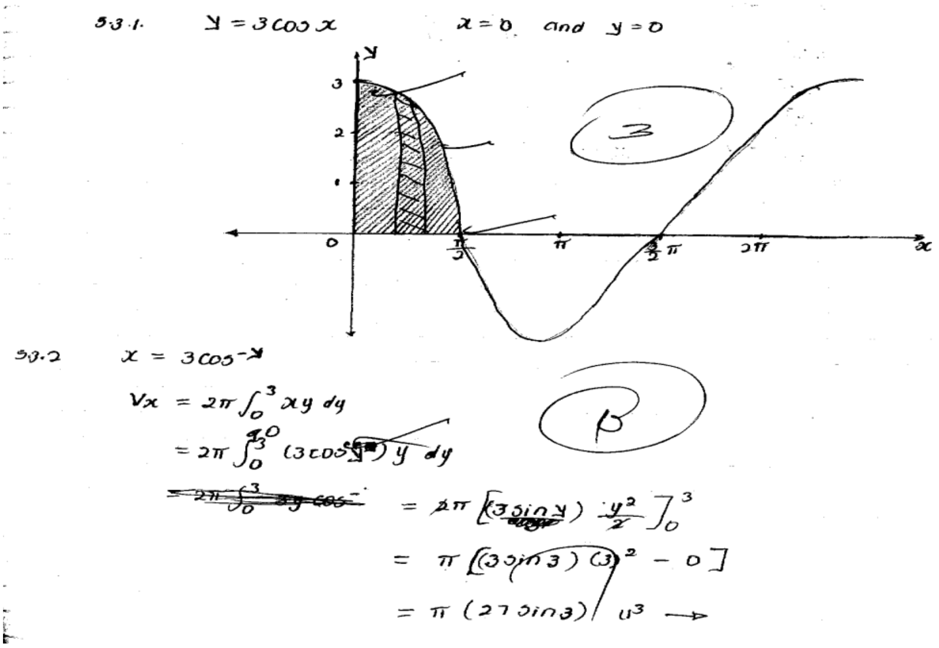


Figure 6. 65: A cosine graph without the strip

- **Responses for Question 5.4**

In Question 5.4, students were asked to

5.4.1 Calculate the coordinates of the points of intersection of $y-2x=0$ and $x=\frac{1}{4}y^2$.
Sketch the graphs and show the representative strip/element that you will use to calculate the area bounded by the graphs. (3)
5.4.2 Calculate the area described in Question 5.4.1 (3)
5.4.3 Calculate the second moment of area described in Question 5.4.1 with respect to the *y*-axis. (3)

In Figure 6.66 the student was trying to calculate the points of intersection of the graphs before drawing them, but failed and no graph was drawn. In that case the whole question was never answered. This student failed to manipulate at the step where cross multiplication was to be used. The student solved $x=\frac{1}{4}y^2$ incorrectly as $y^2=\frac{x}{4}$ in stead of $y^2=4x$, hence could not find the correct solution. This student failed in all five elements and as a result failed in the consolidation and general level of cognitive development.

5.4.1. $y-2x=0$ and $x=\frac{1}{4}y^2$

$y=2x$ $\qquad y^2=\frac{x}{\frac{1}{4}}$

$y^2=4x^2$ $\qquad y^2=\frac{4}{x}$

$4x^2=\frac{4}{x}$

$4x^3=4$

$x^3=\frac{4}{4}$

$x^3=1 \quad \therefore x=\pm 1$

$y-$

Figure 6.66: Incomplete manipulation

6.2.2.2 Summary for the detailed written examination responses

In this section in particular students did not fail because they got everything incorrect, they failed because the questions are accumulative in nature. In answering the questions, the solution for the first step is used in order to answer the second step. If the graph is drawn incorrectly, then the strip drawn will not be correct. Again if the strip is incorrect, the other steps as well as the rotations will be affected.

6.2.3 Discussion and conclusion

The results from the examination reveal that Question 5 may be problematic for students since the questions are asked in a hierarchical order. One must start by drawing graphs first, at times calculating the points of intersection before calculating area, volume and so on. These may create problems if the graphs drawn are not correct. In other instances students who draw correct graphs fail to interpret the drawn graphs in relation to the selection of the correct strip that approximates the area of the bounded region and rotating that drawn strip in cases where the volume is to be calculated.

6.3 SUMMARY OF THE EXAMINATION ANALYSIS

Special trends can be established from the discussions on the examination analysis in relation to the students' difficulties with VSOR. Students' performance was satisfactory in general manipulation skills and in drawing graphs, but they were unable to interpret the drawn graphs. In cases where graphs are drawn correctly, there are many instances where the students could not select the strip correctly, interpreting it to calculate the area as well as rotating it properly to calculate the volume generated. Similar to the results of the questionnaire runs, more attention needs to be on the three skill factors of knowledge where performance is not satisfactory namely:

- Moving between discrete and continuous representations.
- Three-dimensional thinking.
- Consolidation and general level of cognitive development.

Overall for the examination analysis of the 151 respondents, performance in Skill factor V was not satisfactory, revealing how challenging the VSOR content is.

In the section that follows, a model question paper for the August 2007 paper is designed in line with the five skill factors as a guide on how Question 5 analysed above could be assessed.

6.4 A MODEL QUESTION PAPER

In Figure 6.67, a proposed model on how VSOR should be assessed is presented, in an attempt to reduce the cognitive constraints brought about by the consolidation of the four skill factors as one question. The model proposes that VSOR be assessed in line with the five skill factors where most of the questions are broken down. The reason is that it is evident that students have problems with the consolidation of the four skill factors since some of the skill factors require conceptual understanding of the VSOR content which these students do not have.

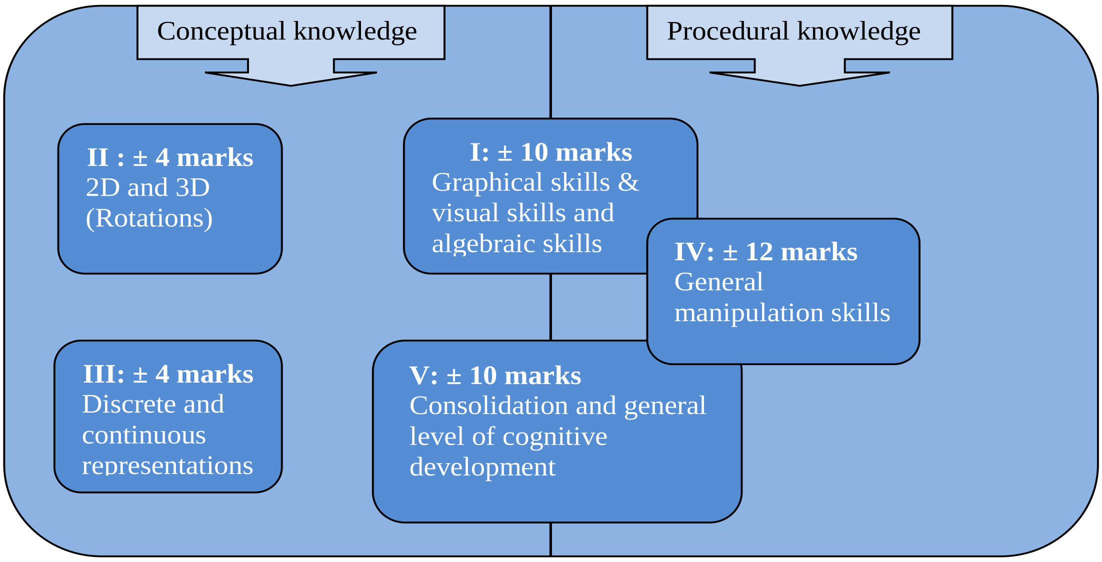


Figure 6.67: The proposed VSOR assessment model

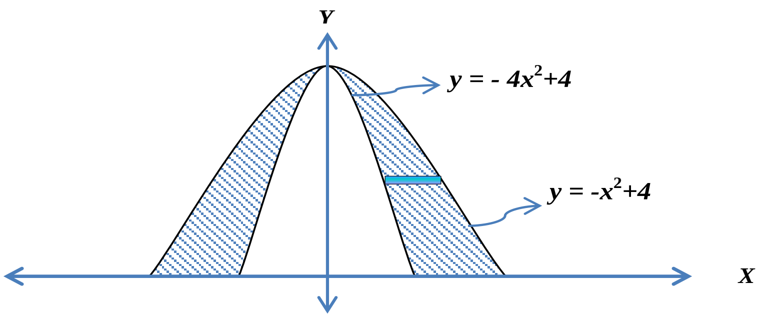
In the above model, the 40 marks of Question 5 will be separated into 15 marks of the conceptual knowledge and 25 marks of the procedural knowledge from the five skill factors. Below an example of how Question 5 should be assessed using questions from the August 2007 examination paper is designed as four separate questions.

Question 5.1 tests for both conceptual and procedural knowledge (12marks), from Skill factors I and IV. Question 5.2 tests for conceptual knowledge from Skill factor II and their applications (5marks), where Skill factor IV is required. Question 5.3 tests for conceptual knowledge and their applications from Skill factor III (11 marks), where Skill factor IV is required, while Question 5.4 tests for consolidation of all four skill factors (12 marks). The total marks allocated is still 40 but the VSOR content is assessed differently, using the same questions as it was from the August 2007 examination paper, where most of the elements are tested explicitly.

A. With this approach, 12 marks tests for both conceptual and procedural knowledge as it is with Skill factor I.

Question 5.1

5.1 Below the area bounded by the graphs of $y = -4x^2 + 4$ and $y = -x^2 + 4$ is represented. A representative strip is also indicated in first quadrant area.



5.1.1 Calculate the intercepts as well as the coordinates of the point of intersection of the graphs. (2)

5.1.2 Draw the 3D representation of the rotated strip about the *y*-axis, and the solid of revolution formulated. (2)

5.1.3 Using the selected strip, substitute the equations of the given graphs in a suitable formula to represent the volume generated when the area bounded is rotated about the *y*-axis. (2)

5.1.4 Calculate the volume generated when this area is rotated about the *y*-axis. (2)

5.1.5 Calculate the volume moment of the solid about the *y*-axis as well as the co-ordinates for the centre of gravity of the solid. (Hint: Show the position of the centre of gravity on the solid. (4)

[12]

B. The next 5 marks are for Skill factor II, when translating from 3D to 2D as well as some general manipulation skills from Skill factor I as the given integral will be evaluated.

Question 5.2

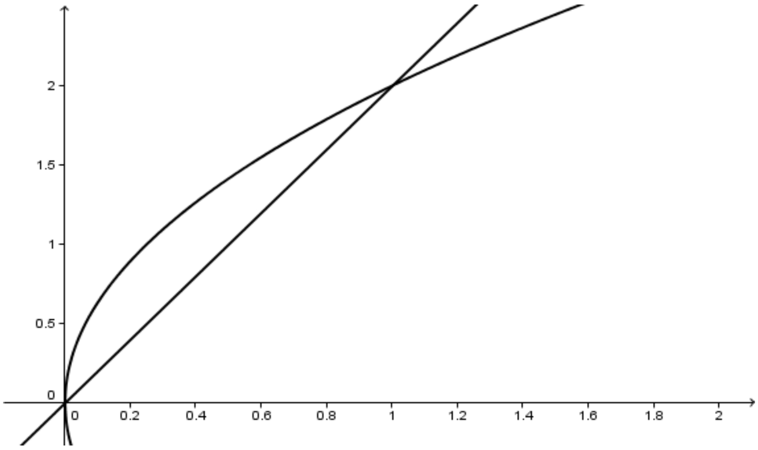
5.2.1 Draw a 2D diagram from which the volume is given by $V = 2\pi \int_0^{90^0} x(3\cos x)\,dx$ (2)

5.2.2 Evaluate the integral $V = 2\pi \int_0^{90^0} x(3\cos x)\,dx$ (3)

[5]

C. The next 11 marks focus on the questions that test for conceptual knowledge and their applications as it is with skill III as well as some from Skill factor I as students will be calculating the area.

Question 5.3



5.3.1 Above the graphs of $y-2x=0$ and $4x=y^2$ are drawn. Use five rectangles to **approximate** the area bounded by the drawn graphs.
(NB: Highlight the representative strip) (4)

5.3.2 Calculate the area bounded by the graphs using integration methods. (3)

5.3.3 Show the coordinates of the centroid of the strip and calculate them. (4)

[11]

D. The last 12 marks use questions where the next questions depend on the graph(s) drawn testing both conceptual and procedural knowledge as it is with Skill factor V. Students are expected to calculate area, volume, centroid, centre of gravity, moment of inertia and second moment of area. An example given below is for fluid pressure on sluice gates.

Question 5.4

5.4.1 A vertical sluice gate in the form of a trapezium is 7 m high. The longest horizontal side is 8 m in length and in the water level. The shorter side is 4 m in length and 7 m below the water surface. Make a neat sketch of the sluice gate and calculate the relationship between the two variables *x* and *y*. (3)

5.4.2 Calculate the first moment of area of the sluice gate about the water level. (4)

5.4.3 Calculate the second moment of area of the sluice gate about the water level as well as the depth of the centre of pressure of the sluice gate by means of integration. (5)

[12]

The general manipulation skills for Skill factor IV should not be tested separately since ±15 marks of it are tested in Skill factors I and V as students are calculating points of intersection, other important points of the graphs, the equations of the sluice gates and other necessary calculations before drawing the graphs, as well as evaluating the integrals after selecting the correct formula and calculating the centroids and others. The general manipulation skills are also tested in most of the questions that constitute the remaining 60 marks of the examination paper. It is suggested that in alternative trimesters, some of the concepts including the centroid, centre of gravity, second moment of area, the moment of inertia and the application of fluid pressure be tested in Skill factor I where the diagram and the strip are given and the students interpret them. The way of questioning in Skill factor I enables one to determine whether the students are competent in these concepts or not. The assessment in Skill factors II and III, the focus in mainly on the development of the conceptual understanding. In Skill factor V, a concept that was not tested in Skill factor I may now be tested, where the focus is now on the level of cognitive development. The table 6.32 summarises the composition of the paper.

Table 6.32: The composition of the paper

QS	GMNP	GR	CD(V)	VA3D	VA2D	AV3D	2D-3D	Marks
5.1	7			3			2	**= 12**
5.2	3					2		**= 5**
5.3	5		4		2			**= 11**
5.4	7	3			2			**CGLCD = 12**
	22	**3**	**4**	**3**	**4**	**2**	**2**	**TOTAL=40**
	25		**15**					

Procedural skills

Conceptual skills