# CHAPTER 1: CONCEPTUALISATION OF THE STUDY

*This chapter introduces the study on investigating learning difficulties involving volumes of solids of revolution amongst engineering students at two colleges for further education and training in South Africa. Section 1.1 introduces the setting: the country, its educational system, the structure of the colleges for further education and training and their entry requirements. Section 1.2 presents the motivation for this study in relation to my involvement in teaching and learning. Section 1.3 presents the problem description in relation to learning about volumes of solids of revolution leading to the formulation of the research questions in Section 1.4. Section 1.5 presents the significance of this study for the colleges relating to the learning, teaching and assessment of volumes of solids of revolution. Section 1.6 presents the conclusion to summarise the important ideas and arguments discussed in this chapter and finally the overview of the chapters is presented in Section 1.7.*

## 1.1 SETTING

### 1.1.1 The country

This study was conducted in South Africa, which comprises nine provinces: Gauteng, Free State, Eastern Cape, Western Cape, North West, Northern Cape, Mpumalanga, Limpopo and Kwa-Zulu Natal. These provinces were established in 1994, when South Africa was liberated from minority rule. The focus for this study is on Gauteng province, the richest province and the industrial hub of South Africa. Gauteng is the smallest province geographically but the second largest demographically with a population of 9 525 571 in 2006 (Statistics South Africa, 2006), since many people migrate to it for employment and educational opportunities. This tiny province is contributing 33.3% to the national gross domestic product. The population density for Gauteng is 576 people per square kilometre, as shown in Figure 1.1.

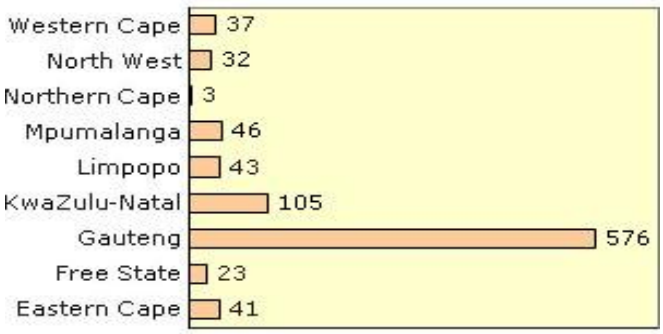



**Figure 1.1: Population density in South African provinces (Statistics South Africa, 2006)**

Most of the people moving to Gauteng are from the provinces surrounding it; Limpopo, Mpumalanga, North-West and Free State as displayed on the map in Figure 1.2.


**Figure 1.2: South African Map (www.southafrica.to/provinces/provinces.htm)**

### 1.1.2 The education system

The system of education in South Africa is classified into three bands in line with the National Qualification Framework (NQF) established in the early 1990s as shown in Table 1.1.

**Table1.1: NQF (www.saqa.org.za/show.asp?include=focus/ld.htm)**

<table>
<tr><th>NQF LEVEL</th><th>BAND</th><th colspan="2">QUALIFICATION TYPE</th></tr>
<tr><td>8</td><td rowspan="4">HIGHER EDUCATION AND TRAINING</td><td colspan="2">• Post-doctoral research degrees<br>• Doctorates<br>• Masters degrees</td></tr>
<tr><td>7</td><td colspan="2">• Professional Qualifications<br>• Honours degrees</td></tr>
<tr><td>6</td><td colspan="2">• National first degrees<br>• Higher diplomas</td></tr>
<tr><td>5</td><td colspan="2">• National diplomas<br>• National certificates</td></tr>
<tr><td colspan="4">FURTHER EDUCATION AND TRAINING CERTIFICATE</td></tr>
<tr><td>4</td><td rowspan="3">FURTHER EDUCATION AND TRAINING</td><td colspan="2" rowspan="3">• National certificates: Grade 10-12</td></tr>
<tr><td>3</td></tr>
<tr><td>2</td></tr>
<tr><td colspan="4">GENERAL EDUCATION AND TRAINING CERTIFICATE</td></tr>
<tr><td rowspan="2">1</td><td rowspan="2">GENERAL EDUCATION AND TRAINING</td><td>Grades 0 - 9</td><td>ABET Level 4</td></tr>
<tr><td colspan="2">• National certificates</td></tr>
</table>

The three bands within the NQF are as follows:

(a) *General Education and Training* (GET) at NQF level 1. GET caters for students from Grade 0 to Grade 9, aged between 6 and 15 years as well as the *Adult Basic Education and Training* (ABET) qualification catering for adults. In South Africa education is compulsory from age 7 to 15 years (McGrath, 1998).

(b) *Further Education and Training* (FET) is at NQF levels 2 to 4. FET caters for students from Grades 10 to 12 in schools (technical and normal), students mainly aged between 16 and 18 as well as students who left school after Grade 9 or Grade 12 to join FET colleges for vocational training.

(c) *Higher Education* (HE) offered by universities of technology (previously technikons) and comprehensive or academic universities at NQF level 5 to 8.

It was asserted that

> the NQF was introduced to South Africans through the education policies and debates within the trade union movement, and more broadly within the broader liberation movement. Partly because of this, many have seen the NQF as driven by a strongly egalitarian social project (Allais, 2003, p. 306).

This was done to transform the education system that was bound by apartheid law through the Congress of South African Trade Unions, and its affiliates who engaged in intensive discussions on education and training in South Africa. Lomofsky and Lazarus (2001, p. 303) point out that the previous education system in South Africa, under central government control led to discriminatory practices and that educational institutions (schools, colleges and universities) were segregated along racial lines.

The education and training policy which emerged was designed with the intention of doing away with the racial segregation and preventing workers from getting stuck in unskilled or semi-skilled jobs (Allais, 2003, p. 306). Finally, the recommendation for a national vocational qualifications system fully integrated with formal academic qualifications, thus integrating education and training to prepare students for the work environment (Ensor, 2003; McGrath, 1998; NECC, 1992), was made. In 1995 the South African Qualifications Authority (SAQA) Act, the first education and training legislation of the new democratic parliament, was passed. It brought the NQF legally into being, with SAQA as the body responsible for developing and implementing it (Allais, 2003, p. 309). Young (2003, p. 230) highlights that the NQF offered opportunities to employers to have a bigger say in the kind of skills and knowledge that 16 to 18-year-olds were expected to acquire.

This study focuses on FET colleges that prepare its students for different vocations. The terms used variously in different countries to refer to vocationally oriented education that takes place in vocational colleges/institutions are ‘Further Education’, ‘Further Education and Training’ (FET) and ‘Technical and Vocational Education and Training (Papier, 2008, p .6). In South Africa such vocational colleges are called FET colleges. During 1999-2002, the enrolments of students after school were highest in numbers at universities, followed by technikons (universities of technology) and then FET colleges (DoE & DoL, 2001; Powell & Hall, 2002, Powell & Hall, 2004).

### 1.1.3 Structure of FET colleges

In the past the FET colleges were called technical colleges, which were initiated in response to the mineral discoveries like gold in the nineteenth century and apprentices were trained (Behr, 1988; Akoojee, McGrath & Visser, 2008; Malherbe, 1977). The first technical colleges, the Cape and the Durban technical colleges were established in 1923 in line with the Apprenticeship Act of 1922 (Malherbe, 1977, p. 170). The technical colleges were mainly training artisans as apprentices from companies, not students coming straight from schools with no experience from the industry. The large technical colleges were transformed into universities of technology during the 1960s, to do more advanced technical training than what the technical colleges were doing (Fisher, Jaff, Powell & Hall, 2003).

During the 1960s to 1980s, technical colleges were obtaining their students from industry. Students were sent by the employers for the ten week ‘block release’ course (Malherbe, 1977) to do theory. These ten week courses, which were completed in a trimester, were called ‘N’ courses. That means that students came to the colleges with work experience, hence they would incorporate theory into the practical component of their work to complete an N course. The courses were done in six block releases, hence called N1, N2, N3, N4, N5 and N6. This system was beneficial to the companies as the then government was giving the employers a tax incentive for sending their employees to the colleges. With the scrapping of this arrangement, employers were no longer sending their employees to the colleges. Most employees would take the courses part-time in the evenings. FET colleges in South Africa came into existence after a merger of 152 technical colleges into 50 FET colleges in 2002, each with two or more campuses (college sites) in terms of the FET Act, No 98 of 1998 (Akoojee, 2008; Akoojee & McGrath, 2008; Fisher, Jaff, Powell & Hall, 2003; McGrath, 2004, McGrath, & Akoojee, 2009; Papier, 2008; Powell & Hall, 2004), with the initiative from the National Committee on Further Education, which presented their draft report highlighting the lack of identity of an FET level (McGrath, 2004).

Both the Department of Education (DoE) and the Department of Labour (DoL) were engaged in a legislative process (McGrath, 2000), in which skills development was regarded as important in FET colleges. Akoojee (2008, p. 297) argues that the national attention on the role of skills development has focused on the role of FET colleges in providing intermediate-level education and training necessary to meet the South African national development challenge. In this regard attention has been focused on the reorganisation and rationalisation of college structures through the merger. The reasons for the merger were to help the weaker colleges financially, and in terms of resources, to share with the stronger colleges. Presently, the FET colleges are the responsibility of the Department of Higher Education and Training in terms of salary payments for the staff but governed and controlled by the college councils which also control students' fees. The role of college councils is critical to the success of the devolved staffing arrangement (Akoojee, 2008, p.308).

After the merger in 2002, a survey was made of South African FET college lecturers' qualifications. The qualifications of the 7088 teaching staff were as follows: 15% had higher degrees, 28% had degrees or higher diplomas, 43% had diplomas and 7% were unqualified or under-qualified (Powell & Hall, 2004). The presented qualifications revealed that the highest percentages of the lecturers have diplomas, recruited from universities of technology and the FET colleges themselves and that some lecturers are under qualified or unqualified. This analysis raised serious concerns about the quality of teaching at the colleges. A concern by Akoojee (2008, p. 311) is that increasing student numbers and diversifying programme offerings needed to be matched by improvements in the quantity and quality of teaching staff.

College lecturers in technical fields have in the past been recruited from industry and usually possessed technical qualifications and wide experience and knowledge from the industry, which is not the case presently, where the majority of the lecturers are recruited from the universities of technologies, with diplomas in engineering. According to Papier (2008, p. 7), many lecturers in academic subjects such as language, mathematics or science entered colleges with school teaching qualifications but little industry experience. Papier further attests that the national Ministry of Education is currently designing a framework of recognised qualifications for lecturers in FET colleges, which will usher in a new era of curriculum development for those higher education institutions that wish to offer them. The quality of FET college lecturers has been a matter of concern, as highlighted by Papier that

> College lecturers in the old dispensation were not required to have specific teaching qualifications. Their technical qualifications and years of experience were given equivalence for remuneration purposes, using pay-scales applicable to school-teachers. Where provincial departments of education made it a requirement for lecturers to obtain a teaching qualification, a few higher education institutions offered diploma programmes which have since become outdated. The national Department of Education indicated in 2007 that it would shortly publish a new framework of qualifications recognised for teaching in FET colleges (Papier, 2008, p. 7).

### 1.1.4 FET colleges and entry requirements

The FET colleges in South Africa are mainly located in the peri-urban areas (townships), and industrial (commercial) areas, scattered all over the nine provinces. Only black students attend colleges in the peri-urban and rural areas. The majority of students in the industrial areas are predominately black even though the colleges are non-racial. The FET colleges are seen to boost the economic mobility of the country. The economy needs artisans in engineering fields and specialists in fields that require mathematics as a basic subject. As a result mathematics has become compulsory in many study fields in South Africa, especially in engineering.

The curriculum in the FET colleges is vocationally inclined, preparing students for industry. FET colleges predominately offer qualifications for engineering studies and business studies, which comprises 90% of the total enrolment (Powell & Hall, 2004). The predominant courses in the engineering studies (the focus for this study) are mechanical engineering and electrical engineering where students take mathematics (which is compulsory) and three other subjects and civil engineering. The passing mark for subjects in the FET colleges is 40%. The courses for the engineering studies are taken at N1 to N6 levels.

Many students at the colleges do not normally reach the N6 level as companies start to recruit them with N3, N4 or N5 certificates. N1 to N6 levels are classified in terms of the trimester system (studying three times a year). For example, a student at an FET college could complete N1, N2 and N3 in one year spending ten weeks on each level, with external examinations written in April, August and November and N4, N5 and N6 in the following year. When a student completes a level, a certificate is awarded by the national DoE in Pretoria. The N1 to N6 courses are called national technical education programmes, examined by the national DoE (Akoojee, McGrath & Visser, 2008, p. 263). The examinations for N1 to N6 courses are moderated, marked and the results processed nationally, with Umalusi (Council for General and Further Education and Training Quality Assurance) and experts in specific disciplines as quality assurers for N1 to N3 and N4 to N6, respectively (DoE, 2006, p.15). The national pass rate at the FET colleges is around 58%, with pass rates higher at N4 to N6 level than that at N1 to N3 level (Fisher, Jaff, Powell, & Hall, 2003, p. 338).

N1 to N3 fall under NQF level 2 to 4, while N4 to N6 fall under the NQF level 5 within the Higher Education and training level (Powell & Hall, 2004; Akoojee, McGrath & Visser, 2008). On completion of the N6 level a student receives a National Certificate that converts to a National Diploma (awarded by the national DoE) after completion of the necessary practical component for 18 months with an approved employer. After completion of the practical component, students may choose to join industry or a university of technology where they are normally given credit for some first semester courses. The universities do not recognise any qualifications from the FET colleges.

In 2007, a new qualification, the National Certificate (Vocational) abbreviated NC(V) offered at NQF level 2, 3 and 4 was introduced in order to gradually phase out the N1, N2 and N3 courses (DoE, 2006), which are believed to be outdated and obsolete (Sonn, 2008, p. 191). Unfortunately this new curriculum is being criticised as it is not recognised by universities (academic and comprehensive), universities of technology or the industry. According to Van Rooyen (2009, p. 1) “the National Certificate (Vocational) is too academic and pitched at a very high academic level, making it almost impossible for those on NQF level 2 and 3 to master”. As a result she argues that a decision was taken that the level 2 intake for NC (V) would be restricted to Grade 12 students. This decision implies that a student has to do NQF level 2 to 4 twice, with no academic benefit. In 2010 qualifications of those students who have completed NC (V) level 4 were not recognised by universities, universities of technology or the industry. These students cannot be accommodated in N4 to N6 courses either due to the curriculum mismatch from the NC (V). With the introduction of the NC (V), it is possible that the N4 to N6 courses may fade away, or may still continue with some students from the technical high schools or from industry.

The quality of students enrolled at the FET colleges varies. Some students in FET colleges spend a lot of time repeating N level subjects, even carrying subjects from lower levels. For example, you could find that a student is enrolled for N6 but still doing Mathematics N4 or/and Power Machines N5 (from mechanical engineering). In that case some students spend more than two years at the FET colleges. These problems where students do not complete their studies on time at the FET colleges can be attributed to the fact that in many instances, students struggled to complete Grade 12 or joined the FET colleges at Grade 9 level because they had difficulties with the normal school subjects (particularly mathematics and science), and hence find the mathematics content at the college difficult. The other reason in some instances may be the superficial knowledge base of lecturers in teaching the content as some of them may not be adequately qualified and also lack experience from industry. Another important aspect may be the fact that students do not have a practical component from industry, as most of their courses require preknowledge from the industry.

Students join FET colleges for different reasons and in different ways. We find students who join FET colleges after having completed Grade 12, who could not get access to universities and universities of technology as they did not acquire the required points (no endorsement in the Senior Certificate); especially in engineering and science fields that require a minimum of 50% for Grade 12 mathematics. As a result these students end up taking courses at the FET colleges, either to upgrade their marks by enrolling for mathematics N3 and/or engineering science N3 so that they can enrol at the universities of technology if they obtain a minimum of 60% for both mathematics and engineering science, or simply joining the FET college to do the full curriculum offered by the college. Students who come from technical high schools are normally allowed to continue with the N3 courses or N4 courses depending on what they have passed, because the technical schools curriculum offers courses similar to N1, N2 and N3. The students who come from normal schools are required to start from N1 since the Grade 10 to Grade 12 curricula do not offer theoretical subjects that are vocationally inclined.

Other students who join the FET colleges enter with a minimum of a Grade 9 pass and take N1 to N6 courses in their chosen fields of study. In this case students who normally join the colleges after completing Grade 9 are better off, since they realised their potential for joining the vocational fields in time. These students would be admitted for N1 straight away. The important point to be noted is that students who normally join the FET colleges, whether after Grade 12 or after Grade 9, normally have problems with mathematics and science. Anecdotal evidence suggests that these low achieving students join the FET colleges probably hoping that the mathematics and science offered at the colleges are easier than that offered in schools, perhaps also thinking that they will only be learning skills where they use their hands. Parents who bring their children to the colleges would often say: "I think my child can work well with his/her hands since he/she is struggling with the school subjects". With this misconception, the colleges attract students who normally performed badly at schools and did not excel in mathematics and science. However, there are those students who enrol at the colleges because of low fee structure and accelerated qualification times, as well as employment opportunities from industry that normally recruits students even at lower levels for example an N3 or an N4 certificate.

This study focuses on two colleges in the Gauteng province where there are eight colleges. One campus from College A with three campuses and one campus from College B with four campuses were sampled for this study. The campuses selected were the biggest engineering campuses from the two colleges used. In 2006 (during the pilot for this study), the number of FET engineering mathematics N6 students for all colleges in South Africa was 1771, 1778 and 2125 respectively for the April, August and November examinations (DoE, 2006, p. 23).

## 1.2 MOTIVATION FOR THE STUDY

### 1.2.1 My involvement

My experience of six years with the school mathematics curriculum and five years with the college mathematics curriculum (from 1993-2003) is twofold. Firstly, when comparing mathematics from Grade 10 to Grade 12, which I was fully involved with in normal schools before joining the FET colleges, I could attest to the fact that the mathematics offered at N1 to N3 levels is on a lower level than that offered in normal schools (both normal and technical). Many topics are omitted in N levels, such as euclidean geometry, linear programming, calculus applications (involving areas and volumes of right prisms). I believe that these difficult topics (that are omitted in N levels) help students in their critical and logical thinking as the main emphasis in these topics is more on conceptual thinking than procedural thinking. Secondly, I realised that a huge conceptual jump exists from the N3 to the N4 and N5 curricula (which prepare students for N6). The students are suddenly exposed to abstract mathematics in *algebra* at N4 level and abstract *calculus* at N5 and N6 level, without a proper foundation from N1 to N3 levels in terms of developing their conceptual understanding. These students might not be in a position to meet the challenges in the calculus content.

### 1.2.2 Teaching experience

During five years of lecturing at College A and as a marker of the N6 national examinations for three years during the five-year period, I encountered problems in that a large number of students from all nine provinces struggled with the section of calculating areas and volumes of three-dimensional rotational objects. I experienced the same problem at the university of technology, with the engineering mathematics students as a lecturer for three years on VSOR. At both institutions you would hear students who are repeating the course saying: "here comes a problem", when you start teaching the topic on areas and volumes of solids of revolution and they become very attentive.

I wanted to investigate where these difficulties emanate from. In order to do that, I studied the section on areas and volumes of solids of revolution (VSOR) in-depth. I decided to observe my own teaching and that of other lecturers, analyse the assessment methods used for this section as well as analyse the examination scripts of college students and to give students different tasks that focus on areas and volumes to assess their in-depth knowledge.

### 1.2.3 Criteria for selecting this topic

In learning about VSOR, the students are expected to calculate the area and volume generated and also to extend this idea to finding the *centroid* and the *distance of centre of gravity* from a certain axis as well as finding the *second moment of area* and the *moment of inertia* (refer to Appendix 1A). Finally the idea of area moments would be extended to calculating the depth of fluid pressure. The concepts above, though done in mathematics, are also applicable in subjects such as Fluid Mechanics, Thermodynamics and Strength of Materials in the fields of mechanical and civil engineering, where they deal with channel flow of fluids, heat transfer and beams applicable to industry.

When learning VSOR, the students are expected to draw graphs (in the XY- plane), shade the region bounded by the graphs, interpret the drawn graphs (in terms of points of intersection and limits of integration) and translate from the graphical representation (drawing) to the algebraic representation (formula) in order to come up with numerical representation (calculation) of areas and volumes using integration techniques. In my experience while teaching VSOR and marking students examination scripts, all these processes involve procedural and conceptual understanding with conceptual understanding as the main focus for development of cognitive skills.

### 1.2.4 Calculating the area bounded by graphs

In this study problems relating to VSOR are being considered, involving the Fundamental Theorem of Calculus (FTC) originating from Leibniz's and Newton's work. The FTC involves using Riemann sums to find areas of regions enclosed by graphs of continuous functions defined on an interval [a, b]. When using Riemann sums the area bounded by the drawn graphs is approximated by partitioning it into thin rectangular strips (horizontal or vertical) of equal width that are joined to each other. The areas of these thin rectangular strips are added to approximate the area bounded by graphs between the limits. Increasing the number of strips improves the approximation. The area therefore becomes

$$\mathbf{A} = \lim_{n\to\infty}\left[ f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots f(x_{n-1})\Delta x + f(x_n)\Delta x \right]$$

The area of each thin rectangular strip is calculated from the formula of area of a rectangle, $A = L \times B = f(x_i)\,\Delta x$, where $f(x_i)$ represents the height of the rectangle and $\Delta x = x_r - x_l$ represents the breadth of the rectangle. $x_r$ is the *x*-value on the right of the rectangle while $x_l$ is the *x*-value on the left of the rectangle. Figure 1.3 displays five vertical strips that approximate the area bounded by the graph of $f(x) = -x^2 + 5$, $x = 0$, $x = 2$ and $y = 0$.

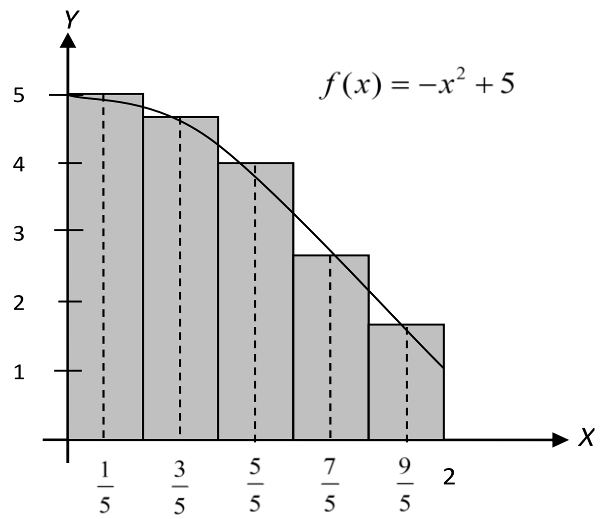



**Figure 1.3: Approximating the area**

According to the FTC, this area can be represented as

A $= \int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is an anti-derivative of $f(x)$, if a vertical rectangular strip is used.

### 1.2.5 Generating the volume of a solid of revolution

In order to understand VSOR students must be able to draw different types of graphs, identify correctly the area of the region bounded by those graphs, draw one rectangular strip that will be rotated, perform the necessary rotation and identify the correct formula for volume based on the rotated strip. In order to come up with the correct formula for volume students must have proper knowledge of figures such as a circle, an annulus and a cylinder (and their different orientations), be able to identify and draw two-dimensional and three-dimensional objects as well as rotations in general, be able to use imaginative skills and be able to do applications and calculations based on the definite integral.

In solving VSOR problems, the area bounded by drawn graphs in the XY- plane is rotated about the *x*-axis or about the *y*-axis, called an axis of revolution to form a solid (called solid of revolution) from the rotated figure. In order to form a solid, each point of the figure is rotated in a circle. If you slice the solid perpendicular to the axis of rotation, you will see a cross-section (the area revealed by many thin slices) that either resembles a coin (a full disc) or a washer (where the area between two circles in the bounded area is at a certain distance from the axis of rotation).

When learning about VSOR the emphasis would be on instruction that improves visual learning skills and development of the three formulae (disc/washer/shell). These formulae are derived from the idea of the volume of the solid as the integral of the area bounded by the given graphs. The volume in this case is found from the idea that volume = area $\times$ height, so $V = \int_a^b A(x)\,dx$ or $V = \int_c^d A(y)\,dy$. In the given formulae, $A(x)$ or $A(y)$ is the area rotated which gives rise to a disc, a washer or a shell (cylinder) after rotation, depending on the selected rectangular strip. $\Delta x$ (or d$x$) in the integral can be seen as the thickness of the selected vertical rectangular strip while $\Delta y$ (or d$y$) can be seen as the thickness of the selected horizontal rectangular strip. In these formulae, $A(x)$ or $A(y)$ represents the area of a circle for the disc method, the area of a circle with a hole in the centre for the washer method and the surface area of a cylinder for the shell method. In all calculations for volume, the volume generated with different rotations (about the *x*-axis or about the *y*-axis) is different and gives different answers since the solids generated are different. The rotations form a revolution of 360°.

#### 1.2.5.1 The disc method

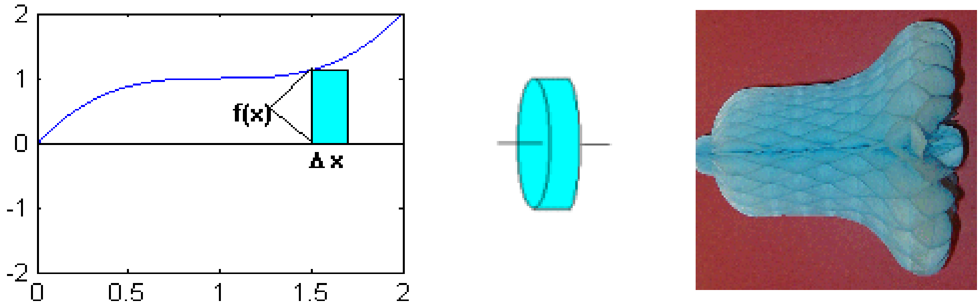



**Figure 1.4: The disc method**

The example given in Figure 1.4 is for a vertical strip (Δ *x*), rotated about the *x*-axis. If a rectangular strip chosen to be *perpendicular* to and *touching* the *x*-axis is rotated about the *x*-axis, a circular three-dimensional diagram is formed. The circular three-dimensional diagram formulated resembles a flat cylinder with radius $y = f(x)$ and height Δ*x*; hence it is termed the *disc* method. Since the rotated diagram is circular the formula for the area of a circle $(\pi r^2)$ is used. The volume in this case is given by the formula $V = \pi\int_a^b [f(x)]^2 dx$ or $V = \pi\int_a^b y^2 dx$, and *a* (the lowest *x*-value) and *b* (the highest *x*-value) are the limits of integration for the bounded area.

If a rectangular strip chosen to be *perpendicular* to and *touching* the *y*-axis is rotated about the *y*-axis, a circular three-dimensional diagram is formed. The circular three-dimensional diagram formulated resembles a flat cylinder with radius $x = f(y)$ and height Δ*y*. The volume in this case is given by the formula $V = \pi\int_c^d [f(y)]^2 dy$ or $V = \pi\int_c^d x^2 dy$, and *c* (the lowest *y*-value) and *d* (the highest *y*-value) are the limits of integration for the bounded area.

**1.2.5.2 The washer method**

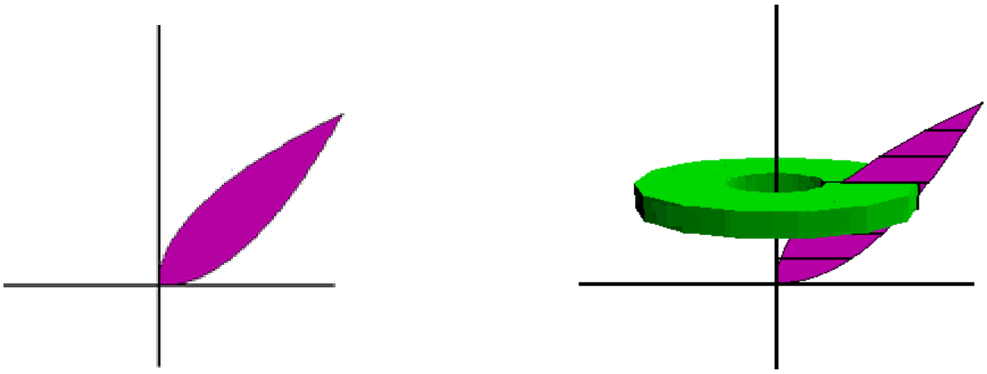


**Figure 1.5: The washer method**

The example given in Figure 1.5 is for a horizontal strip ($\Delta y$), rotated about the *y*-axis. If a rectangular strip chosen to be *perpendicular* to the *y*-axis and *not touching* it is rotated about the *y*-axis, a circular three-dimensional diagram with a *circular hole* from the *y*-axis is formed. The hollow circular three-dimensional diagram formulated resembles an annulus; hence it is termed the *washer* method. Since the rotated diagram is circular, with a hole in the middle, the formula for area of such a circle is $\pi[(r_o)^2 - (r_i)^2]$ , where $r_o$ is the radius of the outer circle and $r_i$ is the radius of the inner circle.

Since different graphs are used $V = \pi \int_c^d [f(y)]^2\, dy$ or $V = \pi \int_c^d x^2 dy = V = \pi \int_c^d (x_1^2 - x_2^2)\, dy$, where $x_1$ represents the graph on the right and $x_2$ represents the graph on the left and *c* (the lowest *y*-value) and *d* (the highest *y*- value) are the limits of integration for the bounded area.

If a rectangular strip chosen to be *perpendicular* to the *x*-axis and *not touching* it is rotated about the *x*-axis, a circular three-dimensional diagram with a *circular hole* from the *x*-axis is formed. Since different graphs are used V = $\int_a^b y^2\, dx$ can be used as $V = \pi \int_a^b (y_1^2 - y_2^2)\, dx$, where $y_1$ represents the top graph and $y_2$ represents the bottom graph, and *a* (the lowest *x*-value) and *b* (the highest *x*-value) are the limits of integration for the bounded area.

### 1.2.5.3 The shell method


**Figure 1.6: The shell method**

The example given in Figure 1.6 is for a vertical strip ($\Delta x$), rotated about the *y*-axis. If a rectangular strip chosen to be *parallel* to the *y*-axis is rotated about the *y*-axis, a cylindrical three-dimensional diagram is formed. The cylindrical three-dimensional diagram formulated resembles a shell; and thus it is termed the *shell* method. Since the rotated diagram is cylindrical, the formula for surface area of a cylinder ($2\pi rh$) is used, where *r* is the radius of the cylinder and *h* is the height of the cylinder.

Since different graphs are used $V = 2\pi \int_a^b x\, f(x)\, dx$ or $V = 2\pi \int_a^b x\, y\, dx$ can be used as

$V = 2\pi \int_a^b x(y_1 - y_2)\, dx$, where $y_1$ represents the top graph and $y_2$ represents the bottom graph, and *a* (the lowest *x*-value) and *b* (the highest *x*-value) are the limits of integration for the bounded area.

If the rectangular strip chosen is parallel to the *x*-axis and rotated about the *x* -axis, a cylindrical three-dimensional diagram is again formed. In this case the volume is given by $V = 2\pi \int_c^d y\, f(y)\, dy$ or $V = 2\pi \int_c^d y(x_1 - x_2)\, dy$ where $x_1$ represents the graph on the right and $x_2$ represents the graph on the left and *c* (the lowest *y*-value) and *d* (the highest *y*-value) are the limits of integration for the bounded area.

The formulae (disc/washer/shell) respectively if the radius is *y* and thickness is $\Delta x$ are as follows: $V = \pi \int_a^b y^2\, dx$; $V = \pi \int_a^b (y_1^2 - y_2^2)\, dx$ and $2\pi \int_a^b xy\, dx$

## 1.3 THE PROBLEM DESCRIPTION

This study is on difficulties experienced by N6 engineering students when solving problems relating to VSOR, involving the FTC and its application to integration in calculating areas and volumes. The concept of the integral in VSOR is an important part of undergraduate mathematics and the FET college curriculum in South Africa. The integral concept, along with the derivative constitutes a mathematical domain that is a language, a device, and a useful tool that is very important for other fields: physics, engineering, economy, and statistics (Kouropatov, 2008, p. 1). In order to understand it better, it is advisable to study it through accumulation, such as in Riemann sums. The concept of accumulation is central to the idea of integration, and therefore is at the core of understanding many ideas and applications in calculus (Thompson & Silverman, 2008).

Students at the FET colleges are introduced to Riemann sums in N4 level focusing on vertical rectangles only and VSOR in N5 to N6 level, focusing on both vertical and horizontal rectangles. At N5 level the students use differentiation and integration techniques including calculating areas and volumes (only with *disc* and *washer* methods) using vertical and horizontal rectangles from the Riemann sums. The section on areas and volumes constitutes about ±12% of the final N5 examination paper. At N6 level a student comes with prior knowledge from N1 to N4 on drawing graphs such as straight lines, parabolas, circles, ellipses, cubic functions, exponential functions, logarithmic functions, trigonometric functions, hyperbolas and rectangular hyperbolas, which are applied in calculus. When students enrol for N6, one can assume that they come equipped with the necessary basic knowledge from previous levels for learning VSOR. In N6 the section on application of areas and volumes (including the *shell* method) constitutes 40 % of the examination paper.

In this study, I investigated the problems that the students encounter when learning VSOR. In order to do that, I explored how students draw graphs, how they use the Riemann sum on the drawn graphs, how they interpret those graphs, how they interpret questions that require procedural and conceptual understanding, how they perform calculations in evaluating areas and volumes (using the disc, washer or shell methods), and also how they translate from area (two-dimensions) to volume (three-dimensions). I was interested in investigating the difficulties that students come across when solving problems related to the definite integral, where the area or volume to be calculated is restricted within certain $x$-values or $y$-values. In particular I wanted to know what students produce in writing (both procedural and conceptual), what they think and how they defend their mathematical content knowledge. I investigated the nature of the content learnt in VSOR and how the nature of the content learnt impacts on learning. I wanted to know what is actually happening in the classrooms in imparting this content to the students. What kind of teaching and assessments are these students exposed to? Are they taught and assessed properly? How do the teaching styles and the nature of assessment impact on the learning of VSOR? Are textbooks available for learning VSOR? If so, how useful are these textbooks in enabling them to learn a section of VSOR in terms of presentation of the content? The problem to be investigated in this study is why the students have difficulty in learning VSOR and where these difficulties emanate from.

Another aspect that may affect learning of VSOR but is not the focus of this study involves the use of language. Many students have learning difficulties caused by the use of language in mathematics (Amoah & Laridon, 2004; Howie & Pieterse, 2001; Howie 2002; Maharaj, 2005; Setati, 2008; White & Mitchelmore, 1996). Studies such as the Third International Mathematics and Science Study (TIMSS) revealed that South African students are performing badly in mathematics (Howie & Pieterse, 2001). Part of the results show that most South African students lacked the basic mathematical skills and failed to solve word problems, reason being, having to deal with English which is not the mother tongue for many students.

From the motivation for this study and the problem description discussed above, the following research question based on students' difficulties with VSOR was formulated.

## 1.4 RESEARCH QUESTION

The major research question for this study is:
**Why do students have difficulty when learning about volumes of solids of revolution?**

In order to address the major research question, seven subquestions were established, relating to what type of knowledge students display when solving problems on VSOR, revealing what they have learnt in relation to VSOR, how it is taught and how it is assessed. In selecting the sub-questions, the researcher identified aspects from literature, as well as from own experience, that can make the learning of the VSOR content successful or that could be problematic.

The subquestions are as follows:

**Subquestion 1:** *How competent are students in graphing skills? How competent are students in translating between visual graphs and algebraic equations/expressions in 2D and 3D?*

VSOR requires students to be in a position to draw correct graphs. Lack of students' competency in drawing graphs could therefore be a source of a problem when learning VSOR. After drawing the graphs, the students must be able to interpret them. They must be able to locate their points of intersection, their intercepts with the axis, identify the area bounded by these graphs by shading it as well as identifying the limits of integration and using them in the correct formula for area and volume. This is influenced by the way in which students visualise and is evident from what they produce in writing (graphically or in a form of equations/expressions) or what they verbalise. VSOR requires that students be able to visualise and use their imaginative skills, in order to translate from visual to algebraic and back.

**Subquestion 2:** *How competent are students in translating between two-dimensional and three-dimensional diagrams?*

The drawn graphs representing area are in two-dimensions and in rotating them, three-dimensional diagrams representing volume are formed. Learning difficulties arise if students cannot translate between the different dimensions, in terms of rotating properly about the given axis ($x$ or $y$). The difficulties may also emanate from the selected rectangle, whether vertical or horizontal. If an incorrect rectangle is used, an incorrect formula to calculate area in two-dimensions and volume in three-dimensions will be selected. The inability to imagine rotations may also hinder or impact on correct translations and the formulae used.

**Subquestion 3:** *How competent are students in translating between continuous and discrete representation visually and algebraically in 2D and in 3D?*

This requires the application of the Riemann sums in terms of slicing the area bounded by the drawn graphs into thin rectangular strips (vertical or horizontal) which are summed to give the approximation of the area bounded and to calculate the volume generated when this area is rotated about either the $x$-axis or the $y$-axis. If the rectangles are not correctly selected, the area and the volume to be calculated will also be incorrect.

**Subquestion 4:** *How competent are students in general manipulation skills?*

Students may encounter difficulties in VSOR if they lack general manipulation skills. When given an integral, students should be able to calculate/evaluate it, or even to do calculations required in drawing graphs, including calculating the point of intersection of the given graphs.

**Subquestion 5:** *How competent are students in dealing with the consolidation and general level of cognitive demands of the tasks?*

The VSOR content is abstract. The way in which students interpret and internalise this content may result in difficulties in learning VSOR. What is the nature of abstraction with this content? Are the students able to meet the challenges to internalise this content? How do students interpret the symbols and notations used in VSOR? This may be evident from what students produce in writing or during their discussions and how familiar students are with the terminology used in VSOR.

**Subquestion 6:** *How do students perform in tasks that require conceptual understanding and those that require procedural understanding?*

I believe that VSOR requires both conceptual and procedural understanding with conceptual knowledge as the basis for proper learning of VSOR. I want to investigate how students conceptualise when learning VSOR and how they perform in tasks that require imagination when rotating. Conceptual understanding focuses on in-depth learning of VSOR and how students engage meaningfully with it. That will be evident from students' deliberations, the questions they ask and what they produce in writing and verbally as they construct knowledge.

**Subquestion 7:** *How is VSOR taught and assessed and how does that impact on learning?*

The way in which students are being taught and assessed may result in problems in learning VSOR. How are the six subquestions above integrated in class? What kinds of methods are used in teaching and assessing VSOR? VSOR requires that students be able to visualise and imagine, and as a result methods used in teaching might require the *visual/animated exposure* in order to enhance conceptual understanding as a basis for procedural understanding. The aim is finding out whether learning difficulties are as a result of students' inadequate exposure to visual aids during the learning process or not. It is also necessary to investigate the way in which the students are being assessed, as well as finding out if perhaps there are any other learning obstacles involved.

## 1.5 SIGNIFICANCE OF THE STUDY

The significance of this study is threefold. It is significant in that it aims to impact on the nature of the content; how it is learnt, taught, and assessed and how it could be improved.

Firstly, the difficult aspects in VSOR are made public and simplified to both the lecturer and the student. Suggestions are given as to what the content of VSOR entails, in a way of simplifying the content to both the lecturer and the student from an expert perspective. The importance and the relevance of the Riemann sums and all other subparts that are crucial in understanding the VSOR is made public. The implications and the importance of the procedural and the conceptual concepts (the roots of VSOR) are differentiated for the student and the lecturer.

Secondly, as a mathematics lecturer, I am interested in improving the standard of learning, teaching and assessment in our country, in particular at the FET colleges. This study is significant in that it may lead to exposure of different methods of learning, teaching and assessment of VSOR that could be beneficial to both students and lecturers. Consequently, there is an attempt to make the section on VSOR accessible and relevant. Lecturers need to improve on their *pedagogical content knowledge,* which is the kind of knowledge that "goes beyond knowledge of the subject matter per se to the dimension of subject matter knowledge for teaching" (Shulman, 1986, p. 9). Shulman argues that teachers need to know the content they teach in-depth, understand what makes learning of specific topics easy or difficult and understand it better than others since teaching entails transformation of knowledge into a form that students can comprehend. If lecturers' pedagogical content knowledge is improved, their confidence in dealing with a section like VSOR may be improved and proper learning will take place.

This study may lead to the improvement of teaching where lecturers use appropriate methods that build the students' conceptual understanding as a basis for procedural understanding. Appropriate learning strategies used may assist in equipping students with lifelong learning that could benefit them in their working environments. The industry and the economic status as a whole could be improved by such changes, as they might receive students with a good conceptual understanding and critical thinking skills. Students may benefit from what is learnt at the colleges and its application to industry, if its relevance to the industry is made explicit in the classroom. This could lead to high awareness of the curriculum status at the colleges as it is popularised as being of high quality and not to be undermined.

Lastly, if lecturers' pedagogical content knowledge is improved, they will also have confidence in applying different methods of assessment. If lecturers know what to teach and how to teach it, proper ways of learning, teaching and assessment of VSOR which constitutes ±12% (at N5 level) and 40% (at N6 level) of the examination paper will take place. I anticipate that assessment procedures laid by the national DoE on assessment of VSOR may change. This may lead to curriculum change in that students will be assessed in better ways that are beneficiary to both the students and the lecturers and education in general. Improvements in N6 mathematics will be made and may be enjoyed by all the stakeholders. Lecturers' pedagogical content knowledge may be improved if teacher training courses are implemented at the colleges, mainly on addressing the content to be learnt and different ways of assessing it from the expert position.

When investigating the development of secondary mathematics teachers in Auckland Barton and Paterson (2009) reported that increasing the depth of understanding of mathematical knowledge may promote effective teaching of secondary mathematics. Another study conducted by Akkoç, Yeşildere and Özmantar (2007) on prospective teachers' pedagogical knowledge revealed that the teachers had difficulty in applying the concept of the limit process when teaching the definite integral. These teachers were not able to use the limit process to address how increasing the number of strips improves the approximation for area under a curve, due to their lack of pedagogical content knowledge.

This study can shed light on what is happening in classrooms when students are taught and assessed, and what could be done in order to improve it, the aim being to help all parties involved to benefit from the system, and to be empowered.

## 1.6 CONCLUSION

In Chapter 1 the setting of the country and its education system was discussed and extended to the FET colleges. The structures of the FET colleges, where they are located as well as the entry requirements were also discussed. My involvement in teaching in schools, FET colleges and university of technology were discussed. The problem description for this study was presented, in particular, how Riemann sums affect learning of VSOR as well as its implications to two-dimensional and three-dimensional representations for areas and volumes. In the motivation for the study, I highlighted the important factors involving learning, teaching and assessment of VSOR from my experience. In the problem statement, I pointed out the crucial aspects that will be investigated in VSOR. That involved how students draw graphs, identify the rectangular strip, interpret the drawn strip, rotate the area bounded by the graphs from two-dimensions to three-dimensions, how they translate from the drawn graph to algebraic equations, as well as how the students apply the disc, washer or shell methods. The role of language and its effect on VSOR were also discussed. From the problem description, the research question and its subquestions were established. The research question for this study will be expanded further in the literature to find out what was done to date in relation to the subquestions established. The significance of this study was also discussed in terms of curriculum innovation, relating to the VSOR content, how it could be learnt, taught and assessed, in a way of improving it.

## 1.7 OVERVIEW OF THE CHAPTERS

Chapter 1 presents the context of the study in relation to the education system in South Africa and the way the FET colleges operates. It also presents the motivation for the study, the problem statement, the research questions as well as the significance of the study. Chapter 2 presents the literature review for the study in order to know what has been done or not done in relation to the VSOR topic. Chapter 3 presents the conceptual framework that locates this study. Chapter 4 presents the research design and methods including the instrument for data collection. In Chapter 5 the results from the preliminary and the pilot studies are presented and analysed. In Chapter 6 the results from the first and the second runs of the questionnaire and the August 2007 examination results are presented and analysed in terms of the five skill factors. Chapter 7 presents the correlations from the questionnaire runs and the examination analysis discussed in Chapter 6. Chapter 8 presents the summaries and narratives from the classrooms observations and interviews. Chapter 9 presents the interpretations and conclusions for the whole study as well as the limitations and the recommendations.

# CHAPTER 2: LITERATURE REVIEW

*In this chapter, the literature related to the way in which students learn mathematics in general and concepts related to VSOR, how they are taught and how they are assessed are discussed. In particular, the way in which the students draw graphs and diagrams (based on their external representations) and interpret them (based on the internal representations) are discussed, hence relating to the cognitive obstacles they come across. In the interpretation of the graphs, literature related to the way in which the students translate between the visual graphs and algebraic equation from the Riemann's sums and the rotations formulated (from 2D to 3D) are discussed, revealing the effect of visual and algebraic approach to learning and the concept images formulated. The discussion also extends to how the students solve problems that are conceptual and procedural in nature. Contextual factors that affect the learning of VSOR, and learning mathematics in general are discussed in order to strengthen the focus of this study. The discussion is done under the following headings:*

- *Graphing skills and translation between visual graphs and algebraic equations/expressions in 2D and 3D.*
- *Translation between two-dimensional and three-dimensional diagrams.*
- *Translation between continuous and discrete representations.*
- *General manipulation skills.*
- *Consolidation and general level of cognitive development.*

*Contextual factors that affect learning are discussed under the following headings*

- *Writing to learn mathematics and effect of language.*
- *Scaffolding learning.*
- *Teaching approach.*
- *Curriculum level and assessment.*
- *Use of technology.*

## 2.1 GRAPHING SKILLS AND TRANSLATION BETWEEN VISUAL GRAPHS AND ALGEBRAIC EQUATIONS/EXPRESSIONS

In the learning of VSOR students draw graphs and interpret them, visually or algebraically. In translating between visual graphs and algebraic equations/expressions, students' visual ability and algebraic abilities are involved. The way in which students visualise the graphs affect the way in which they translate to equations and the way in which the students manipulate algebraic equations affect the way in which they translate those equations to visual graphs. When performing these forms of translations, algebraic equations/expressions are justified visually using diagrams. Visual justification in mathematics refers to the understanding and application of mathematical concepts using visually based representations and processes presented in diagrams, computer graphics programs and physical models (Rahim & Siddo, 2009, p. 496).

### 2.1.1 Visual learning and symbols

According to Duval (1999, p. 13) "visualization refers to a cognitive activity, that is intrinsically semiotic, that is, neither mental nor physical". Also such expressions as 'mental image', 'mental representation', 'mental imagery', are equivocal". A mental image in terms of Gutiérrez (1996, p. 9) is "any kind of cognitive representation of a mathematical concept or property by means of visual or special elements". Gutiérrez (1996, p. 9) considers visualisation as "a kind of mathematical reasoning activity based on the use of spatial or visual elements, either mental or physical performed to solve problems and or prove properties" and to mean the same thing as spatial thinking (Gutiérrez, 1996, p. 4). Haciomeroglu, Aspinwall and Presmeg, (2010, p. 159) and Jones (2001, p. 55) regard visualisation as a process involved in forming and manipulating images whether with pencil and paper or computers in order to understand the mathematical relations. Jones (2001, p. 55) further attests that "visualisation is essential to problem-solving and spatial reasoning as it enables people to use concrete means to grapple with abstract images". Visualisation is

> the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings (Arcavi, 2003, p. 217).

Spatial visualisation ability is a skill and a necessity in the engineering related areas since diagrams are used quite often. Menchaca-Brandan, Liu, Oman, and Natapoff (2007, p. 272) defined spatial ability as the "ability to generate, visualize, memorize, remember and transform any kind of visual information such as pictures, maps, 3D images, etc". Spatial ability according to Wikipedia free encyclopaedia refers to **spatial visualisation ability** or **visual spatial ability** which is the ability to mentally manipulate two-dimensional and three-dimensional figures. Menchaca-Brandan et al. (2007) use subcomponents on spatial ability as perspective-taking and mental rotations. They argue that perspective-taking (also known as spatial orientation) is the ability to imagine how an object or scene looks from different perspectives and that mental rotation (also known as *spatial relations*) refer to the ability to mentally manipulate an array of objects. Spatial visualisation is also seen by others as "the ability to imagine the rotation of a depicted object, to visualize its configuration, to transform it into a different form and to manipulate it in one's imagination." (Ryu, Chong & Song, 2007, p. 140). According to Deliyianni, Monoyiou, Elia, Georgiou and Zannettou (2009, p. 98) "students' competence in generating pictures in mathematical tasks appears to be related to their spatial ability". Such skills are concerned with manipulating, reorganising, or interpreting relationships visually (Tartre, 1990, p. 216). It is highlighted that "children who have a strong spatial sense do better at mathematics" (Clements, 2004, p. 278). It is also emphasised that spatial visualisation abilities are important for individuals who are developing and designing the three-dimensional environment and for those working in the field of engineering (Leopold, Gorska, & Sorby, 2001, p. 82).

Kozhevnikov, Hegarty and Mayer (2002), refer to those students who use visualisation as visualisers who process visual-spatial information, in a form of visual imagery and spatial imagery. Kozhevnikov et al. (2002, p. 48) argue that "*Visual imagery* refers to a representation of the visual appearance of an object, such as its shape, size, color, or brightness, whereas *spatial imagery* refers to a representation of the spatial relations between parts of an object, the location of objects in space, and their movements". They further assert that some individuals may construct vivid, concrete, and detailed images of individual objects in a situation (the iconic type: visualisers with low spatial ability), whereas others create images that represent the spatial relations between objects that facilitate the imagination of spatial transformations such as mental rotation (the spatial type: visualisers with high spatial ability). The results of their study reveal that when dealing with graphs of motion, the iconic types tend to generate images by looking for a pattern with the closest match to the stimulus, for an example the downward motion, while the spatial types visualise overall motion by breaking the graph down into intervals and visualising changes in the object's velocity from one interval to another successively (Kozhevnikov et al., 2002, p. 64). They characterised high-spatial visualisers (spatial type) as those who engage the spatial-schematic imagery system in solving problems, and low-spatial visualisers (iconic type) as those who engage the visual-pictorial imagery system in solving problems (Kozhevnikov et al., 2002, p. 69). It is argued that

> Visual-spatial representations were classified as being either primarily schematic, representations that encoded the spatial relations described within the problem, or primarily pictorial representations that encoded objects or persons described in the problem. Schematic representation was positively correlated with success in mathematical problem solving, whereas pictorial representation was negatively related to success in mathematical problem solving (Van Garderen, 2003, p. 252).

Chmela-Jones, Buys and Gaede (2007, p. 630) believe that visual learning is an approach to "helping learners communicate with imagery". They further affirm that the visual learning style is useful for learners who prefer the visual modality of learning in order to better recall what has been observed or read. This kind of learning is important in VSOR since it involves imagination as its foundation. When students engage in visual learning, the external representation and the internal representations are involved in the learning process. External representation refers to mathematics that can be visualised in terms of concrete objects while the internal representation refers to how the external representation is interpreted in the student's mind as a mental image. Dreyfus (1995) discusses the external representations in terms of diagrams and internal representations in terms of mental images. He discusses how external images and discourse interact with students' approaches to solving mathematics problems with the aid of diagrams.

Dreyfus (1995, p. 3) believes that one cannot think without mental images. He believes that for mathematics in particular, the most important type of visual information is diagrammatic (static or dynamic) which is the external representation and it plays a role in making meaning, understanding and mathematical reasoning. The mental images that students construct enable them to succeed or fail to succeed in learning mathematics. In solving mathematical problems, Dreyfus (1995, p. 13) believes that students avoid using diagrams and diagrammatic reasoning because of cognitive obstacles related to the diagrams. He points out that in one high school classroom, diagrams played a central part of students' activities and lecturer explanations, but students were observed drawing diagrams only when they were explicitly directed to do so. In that case students did not see the significance of using diagrams to express what they were thinking. Dreyfus argues that in problem-solving, students connect the external representation with the internal representation that corresponds with the diagrams. This internal representation, which he calls the *visual imagery*, is not possible to access, as it can only be made public by the individuals themselves as they experience it, unless they talk it out by writing down or drawing. According to Dreyfus (1991, p. 32), "to be successful in mathematics, it is desirable to have rich mental representations of a concept" that would enable students to interpret the external representations (diagrams) appropriately.

The impact of visual imagery was evident in the study conducted by Duval (1999), which revealed that students were able to draw graphs when given equations and to read coordinates, but could not discriminate between the drawn graphs of $y = x + 2$ and $y = 2x$.

These students were able to translate from algebraic equations/expressions to visual graphs, but failed to translate from visual to algebraic. In this case there was a mismatch between the representations.

Dettori and Lemut (1995) discuss the role of external representation in arithmetic problem-solving. Their study was aimed at the acquisition of arithmetic concepts of number and elementary operations. Their students were working in a pen-and-paper environment and a computer hypermedia environment. They also experienced that students lacked the external representation (use of diagrams), and they attributed that to some blockages (referred to as *cognitive obstacles* by Dreyfus). Students in this case were found not to be in a position to solve problems or even to relate them to what was previously learnt. They used the computer to develop representations that could assist them as a cognitive help in arithmetic problem-solving (Dettori & Lemut, 1995, p. 29). According to Dettori and Lemut (1995) a computer is a powerful means for manipulating verbal, symbolic and pictorial representations. It can also provide a rich interactive source of possible imagery, both visual and computational (Tall 1995, p. 52). The external and internal representations discussed above are significant in learning mathematics, but different students have their own preferences in terms of representations.

In their study on understanding the concept of area and the definite integral, Camacho and Depool (2003) used Calculus I students after learning with the computer software programme DERIVE. Analysing the results obtained, the study revealed that DERIVE allows students to progress slightly in their use of graphic and numerical aspects of the concept of definite integral. However, one of the students interviewed was seen to prefer to work more with algebraic than with graphic representations, while the other student was able to work in both representations. In general from the questions given graphically, the students were not fully successful in interpreting the graphs as well as translating them to algebraic equations. In another study, Maull and Berry (2000) designed a questionnaire to test engineering students on differentiation, integration, differential equations and their application to simple physical cases. The results of the study reveal that engineering students prefer verbal representations and that their individual visualisations are idiosyncratic and do not coincide with the diagrams presented to them (Maull & Berry, 2000, p. 914).

An example from Haciomeroglu et al. (2010) study showed that when presented with a derivative graph in the interviews, sometimes students sketched its antiderivative graph on paper while describing how it changed. Sometimes they described how they transformed the derivative graph into an antiderivative graph before sketching it on paper. From the students' behaviour, the authors believe that students form visual mental images guiding their thinking and employed imagery as they transform the derivative graph on paper or in their minds. Their study revealed that understanding of mathematics is strongly related to the ability to use visual and analytic thinking, as it was evident from the students' performance. They considered students' solutions as analytic (equation-based) or visual (image-based) when they translate into symbolic representations or graphic representations respectively. The results of Haciomeroglu et al. (2010) study regarding graphical tasks reveal that the one student demonstrated a strong preference for analytic thinking and relied on symbolic representations without the use of the visual representation. The results of the other two students showed that both students were more comfortable when using the $y$ values on the derivative graphs to visualise the changing slopes at various points and used these to draw the antiderivative graphs. They both used visualisation as the primary method in their work. For one of these students it was concluded that dynamic visual images, without significant support from analytic thinking, prevented his complete understanding. The other student was able to draw precise sketches with the help of analytic support of his visual images. The conclusion for their study is that the ability to synthesise analytic and visual thinking is vital in the complete understanding of differentiation and integration.

According to Aspinwall and Shaw (2002), reform efforts promote an understanding of both the analytic and the graphic representations of functions. They claim that analytic representation is in the form of symbols and is easier to manipulate, analyse or transform, whereas graphic representation conveys mathematical information visually. Tall (1991) and Habre and Abboud (2006) in their study showed that students' understanding of functions in calculus is rather algebraic (analytic) than visual. Aspinwall, Shaw and Presmeg (1997) argue that calculus courses in colleges in USA are designed to put more emphasis on the graphical approach on learning calculus as students tend to prefer the algebraic approaches, at times using graphing technology. They further argue that even if it is believed that graphs improve students' conceptual understanding, they may also serve as a source of barriers to constructing meaning through mental imagery.

Observation made by Neria and Amit (2004) from a written test (on optimisation, rate of change and area and circumference of a rectangle) based on students' solutions on mathematical problems regarding different mode of representations, indicated that students who preferred the algebraic mode achieved higher scores in the test than those who preferred the graphical mode. The results of another study by Nilklad (2004), who investigated 24 college algebra students' understanding, solution strategies, and algebraic thinking and reasoning used as they solved mathematical function problems, revealed that algebraic thinking and reasoning are lacking in students' problem-solving strategies. The study also revealed that in some instances none of the students used diagrams or pictures to clarify their examples when solving problems. However, from five students who were interviewed it was evident that the symbolic and graphical representations were used more often than any other representations while these students solved problems. The students were able to change from one representation to another, such as changing a verbal to a graphical representation or to a symbolic representation, moving from the external representation to the internal representation. An external representation, according to Gutiérrez (1996, p. 9) can be a verbal or graphical representation in a form of pictures, diagrams or drawings that helps to create or transform mental images and to reason visually. The internal representation (concepts images) is what is in the mind of the learner. Students' concept images (which are enriched by the students' ability to visualise mathematical concepts) are often based on prior knowledge which they acquire through different experiences, including their daily experiences (Harel, Selden & Selden, 2006).

For her PhD study Rouhani (2004) conducted a case study of students' knowledge of functions using technology. Her study revealed that students were more knowledgeable about the recognition of functions than its interpretation and translation. The students had more difficulty with the interpretation of functions in algebraic form than in graphical representation. However, when coming to translations, the translation of functions from a numerical to a graphical representation was easy for all the participants. On the other hand, the translation of functions from a symbolic to a graphical representation was less frequent (Rouhani, 2004, p. 120). Farmaki, Klaoudatos and Verikios (2004) argue that a function is a central concept around which school algebra can be meaningfully organised.

When translating between visual graphs and algebraic equations/expressions, symbols are used. In the study of VSOR the symbols used include the *integral sign*, $\Delta x$ and $\Delta y$. White and Mitchelmore (1996) point out that in some cases students operate with symbols without relating to their possible contextual meanings. If students do not contextualise what they are learning, they end up learning without proper understanding. According to Maharaj (2005) many students perform poorly in mathematics because they are unable to handle information given in symbolic form adequately.

Maharaj (2008) conducted a study focusing on the outcomes and implications of research on (a) use of symbols in mathematics, (b) algebraic/trigonometric expressions, (c) solving equations, and (d) dealing with functions and calculus. He argues that it is important that attention be focused on establishing the meaning of symbols when teaching mathematics as they appear in different contexts. He points out that

> Mathematics makes use of symbolic notation, which serves a dual role as an instrument of communication and thought. This special language makes it possible to represent in coded form mathematical concepts, structures and relationships (Maharaj, 2008, p. 411).

He argues that students should be encouraged to seek meaning when dealing with symbolic notation representing algebraic expressions, equations, and functions as well as get involved in verbalisation, visualisation, and appropriate mathematical questions which all contribute to sense-making (Maharaj, 2008, p.412). Another study by Samo (2009, p. 11) gives evidence that students' difficulties in algebra could be related to their difficulties and misinterpretation of symbolic notations as well as translating word problems to equations.

In her PhD study Montiel (2005) observed a Calculus II class and interviewed four students. Her study is strongly oriented towards the understanding of how learning, when applied to integral calculus, is affected by the dual nature of the integral symbol. She relates to the integral symbol as an instruction to carry out an operational process, as well as the embodiment of a specific object which is produced by that process, representing the mathematical concept of accumulation (Montiel, 2005, p. 3). Among other questions, students were given equations of graphs and were expected to set up the integrals that would permit them to find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the indicated lines. In this case they were expected to translate from algebraic to visual (by drawing the graphs of the given equations) and then expected to translate from visual to algebraic as they set up integrals that would enable them to calculate the volume (involving disk, washer and shell methods). They were also given integrals (for area and volume) and asked to sketch them. In that way they were translating from algebraic to visual where the use of symbols in integration was tested. Most students in Montiel's study were able to draw graphs, but there were cases where students had problems, for example, two of the interviewed students were seen to express a certain function algebraically in terms of *y*, but drew it in terms of *x*. In another study, Yasin and Enver (2007, p. 23) found that students had difficulty in drawing the graphs of functions except of polynomial type.

The symbolic language used in calculus was also explored where students were presented with volumes and areas (given below) expressed in terms of integrals, and were asked to make sketches $V = \pi\int_0^4 (\sqrt{x})^2\,dx$, $A = \pi\int_0^4 (\sqrt{x})^2\,dx$ (Montiel, 2005, p. 103). Two students identified the disc method (by the $\pi$) from the formula for volume, and used the boundaries correctly to sketch the two graphs. The other two students struggled with the area concept, confused by the π on the formula for area. They failed to see it as a constant. The findings were that a “cognitive obstacle” prevailed as students translated from equations to areas and solids of revolutions and back. Some students were seen to confuse the methods for calculating areas and volumes, and using incorrect rectangles for approximation of the area, as well as translating to volume. These problems that were evident in Montiel’s study were investigated in my study with more than a 100 students who solved 23 questions (classified in five categories) and another group more than a hundred, who wrote the final N6 examinations. In this study I analysed qualitatively in-depth the students’ thinking processes in written form. I did not interfere with their thinking processes. In a way of getting involved in their thinking processes, I used scaffolding for a group of eight students who were observed for five days while learning VSOR.

### 2.1.2 Transferring between mathematics and applications

In some instances students are translating from graphs to other contexts. Ubuz (2007) conducted a study where students were interpreting graphs and constructing derivatives. In my study students were translating from graphs to graphs or to the integral formula. For example students translated from area to volumes graphically and algebraically. Hjalmarson, Wage and Buck (2008) conducted a study with electrical engineering students who had completed advanced calculus or differential equations. They believe that graphical representations play a significant role in conceptual understanding within upper-level applied mathematics and that students need to be able to interpret and generate graphs as part of their mathematical reasoning. The challenge for instructors in this class was helping students learn to transfer knowledge from their mathematics class to applications in signals and systems. Students did not always connect their mathematics knowledge with the signals and systems problems. There were also representational challenges in two forms: the symbols unique to signals and systems used for representing equations and a heavy use of graphical representations (Hjalmarson et al., 2008, p. 1). Their students were asked to make an interpretation from a graph (or graphs), for periodic functions and then to select another graph based on their interpretation.

Students did periodic functions and Fourier transformations in their mathematics course, but had problems transferring that knowledge to the electrical component of the course. Students were seen to struggle to balance their conceptual and procedural knowledge. Some of the students believed that they could do the mathematics but they did not understand it. Some students expressed fear, confusion or dislike of the Fourier transformations. In a few cases, students had trouble beginning to reason through a problem because of the association with the Fourier transform. While the majority of the students interviewed could successfully interpret and analyse the graphical representations associated with the Fourier transform, it still presented a conceptual challenge to them in that they felt a bit frustrated because of their lack of confidence with the concept associated with Fourier transformations. Their discomfort with the Fourier transform is particularly notable as no computations or manipulations of equations were required in order to successfully complete the problems (Hjalmarson et al., 2008, p. 13-14).

In their study Rösken and Rolhka (2006) report on some research into what students do know with a special focus on visual aspects of the integral in relation to mental representations. In one of the questions students were asked to illustrate the geometric definition of an integral (from words to visual). The results of the study reveal that 77% of the students' illustrations were restricted to the 1st quadrant only, which represent a positive area as shown in Figure 2.1.

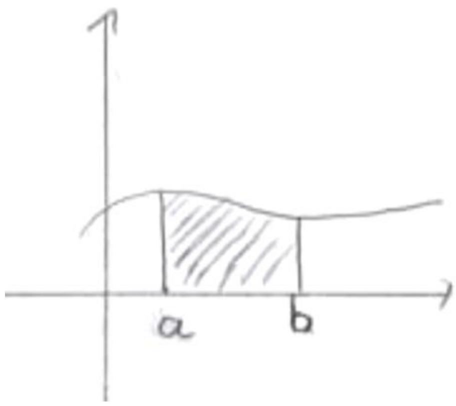



**Figure 2.1: Students' visualisation of an integral (Rösken and Rolhka, 2006, p. 459)**

In other instances the students in the Rösken and Rolhka (2006) study were unable to name the limits of integration and to use visualisation approaches. They preferred algorithmic approaches even if a visual approach was necessary for some problems. In their interpretation Rösken and Rolhka (2006, p. 463) believe that the students in their study are "cognitively fixed on algorithms and procedures instead of recognizing the advantages of visualizing".

In another study Haciomeroglu et al. (2010) aimed to analyse the thinking processes of high-achieving calculus students as they attempted to sketch antiderivative graphs when presented with derivative graphs. They developed graphical tasks to evoke imagery and probe students' thinking. Their study focussed on understanding of students' difficulties and strengths associated with visualisation as well as the types of mathematical imagery utilised by students while interpreting the derivative graphs. They assert that individuals can create different internal representations of a concept that is presented as an external instructional representation such as a diagram or graph. As it is the case in my study, students translate from graphs to other contexts within mathematics itself. They transfer what they have learnt in areas to volumes (graphically) and do the same using symbols. In my study the transfer is within, that is from mathematics to mathematics. As it is the case in Hjalmarson et al. (2008) study, where students struggled to balance their conceptual and procedural knowledge, I also investigate the relationship of students' conceptual and procedural knowledge as they transfer from areas to volumes. That is in a form of how they visualise and translate from visual to algebraic as well as how they do calculations of areas and volumes. Visualisation plays an important role in the development of algebraic skills discussed below.

Students' algebraic skills in functions were also evident in the following two similar studies done by Knuth (2000) and Santos (2000) on the *Cartesian Connection*[1], relating to how students translate from algebraic to graphical representations and vice versa. Knuth (2000) conducted a study with 178 students enrolled for calculus. The students' understandings of the connection between algebraic and graphical representation of functions were explored. He examined the abilities of students to employ, select and move between the different representations. He did that by examining how the students used a particular aspect of the Cartesian Connection, where the coordinates of any point on a line would satisfy the equation of the line (Knuth, 2000, p. 2). Students were given six tasks to work on, where both the

[1] Cartesian Connection is the way in which students are able to see how points plotted on a Cartesian plane can be joined to resemble a particular graph, as well as drawing graphs on the Cartesian plane when points are given.

algebraic representation of a function and a corresponding graphical representation were indicated. All the problems required the use of the Cartesian Connection in determining the solution. Knuth (2000, p. 3) hypothesised that if students understood the Cartesian Connection, for these problems in particular, they would then tend to use the graphical solution method (using graphs), as it was the most suitable one in responding to these problems. However, students used the algebraic solution method (using equations). The students in Knuth's study were not aware that when asked to find a solution to an equation, the answer might be represented graphically.

The findings from Knuth's study revealed that many students were able to connect between the algebraic and the graphical representations of functions when dealing with familiar routine tasks, where a table of values is used to satisfy a given equation, thereafter plotting the values on a coordinate graph, but failed to use this connection to move from a graph to an equation (Knuth, 2000, p. 4). However Knuth argues that

> students' reliance on algebraic-solution methods is due to their failure to recognise the points used in calculating a slope as solutions to an equation-recognition of which should make a graphical-solution method a viable option rather than to a perceived need for precision (Knuth, 2000, p. 4).

Students failed to connect when they were unable to realise that the selection of any point on the graph of a line would not be a solution to the line (Knuth, 2000, p. 4). Knuth (2000) further suggests that in making the connections, students normally experience problems as a result of the interactions between their internal representations with the external representations, which were discussed above. Knuth (2000, p. 4) points out that the students' preferences might be due to the curricular and instructional emphasis, which is dominated by a focus on algebraic representations and their manipulations.

Santos (2000) on the other hand conducted a study where 40 Grade 12 high school students worked on mathematical tasks, where they were asked to examine the connections between graphical and symbolic representations, on *variation*, *approximation* and *optimisation*, through graphic, table and algebraic representations. The students were using dynamic software (Cabri-Geometry), which enabled them to examine variation among main parameters attached to the problem they were engaged with as well as visualisation of mathematical relationships. Students were asked to justify and explain their arguments to support their responses relating to the different representations in written form, which were presented to the whole class later to defend their arguments. They were tape-recorded throughout the tasks. The students' work was assessed in four episodes.

In the first episode students were asked to graph on the Cartesian system, $y=-2x+8$ in the first quadrant and to examine the relationships between co-ordinates and the algebraic representations. Initially students in this study did not recognise that they could substitute the corresponding value of *x* in the expression $y=-2x+8$ to determine the value of *y*, but they finally recognised this (Santos, 2000, p. 1999). In the second episode the students were expected to make use of a table, in which they showed calculations in terms of length of the sides of the rectangle, as well as the corresponding areas and perimeters. They were able to represent that if one side of the rectangle (the one that rests on the *x*-axis) had the value *x*, then the other side b could be expressed as $b=-2x+8$. In this exercise the table became a powerful tool in enabling the students to observe variation in terms of area and perimeter (Santos, 2000, p. 203).

In the third episode the students were connecting three registers of representations, the *graphic*, *algebraic* and *numeric* with regard to the initial task. Students were asked to draw a graph which corresponds to the expression of area A(*x*), as well as to describe the behaviour of the graph in terms of the side and area of the rectangle (Santos, 2000, p. 203). Cabri-Geometry was used in graphing the area. The students were asked to reflect on connections or relationships between the graph of the area and the type of the rectangle. They were asked to observe what the expression A(*x*) becomes if *x* =1, which is A(1) = 6. They were also asked to calculate the dimensions of the rectangle whose area is 8, and to justify if there was a rectangle with an area of 10 square units. In the fourth episode students were asked to find a rectangle of maximum area with fixed perimeter.

In Santos's study, students were given the opportunity to conceptualise, ask questions, argue and defend their arguments to develop mathematical understanding. The use of technology (Cabri-Geometry) provided a learning environment where students were able to analyse main parameters associated with the tasks (Santos, 2000, p. 210). Santos argues that even if the task was routine, it could be used as a platform to discuss and introduce the use of different representations, hence transforming the task to non-routine.

Much can be learnt from the above two studies by Knuth and Santos. The studies reveal that the task given to students might help them to work in different levels of representation even if it is a routine one. It is important that the task must be designed in such a way that it engages the students in critical thinking. The students in Santos's study were more successful in working in different levels of representation than those in Knuth's study, possibly because they used dynamic software (Cabri-Geometry) to help them visualise the graphical relationships. The design and development of these two studies is used as a starting point in designing the tasks for my study, focussing on different levels of representations.

## 2.2 TRANSLATION BETWEEN 2D AND 3D DIAGRAMS

Translation between 2D and 3D diagrams requires a special kind of visualisation that involves imagery. Cube construction tasks, engineering drawing and mental rotation tasks were used to test whether manipulation and sketching activities could influence spatial visualisation ability in civil engineering students from Malaysian polytechnics (Alias, Black & Gray, 2002). The results of the study revealed that implicit teaching of mental rotation skills could be the cause for the lack of gain in mental rotation and that spatial activities (to be emphasised during teaching) enhance students' spatial visualisation ability (Alias et al., 2002). A study that focused on examining deaf and hearing students' ability to see, generate, and use relationships in mathematical problem-solving was conducted by Blatto-Vallee, Kelly, Gaustad, Porter, J and Fonzi (2007). According to Blatto-Vallee et al. (2007, p. 444), hearing students across the board generally utilised visual-spatial schematic representations to a greater degree than the deaf students in mathematical problem-solving, whereas deaf students used visual - spatial pictorial representations to a greater degree than hearing students. For that reason, hearing students were more successful in problem-solving than the deaf students.

A task designed by Ryu et al. (2007) for seven mathematically gifted students which could be solved by mentally manipulating, rotating or changing the direction of depicted objects in involving spatial visualisation abilities yielded the following results.

> Though 2 out of the 7 subjects displayed characteristic spatial visualization ability carrying out all the tasks in this research, most of the other 5 students had some difficulty in mentally manipulating an object depicted in a plane as a spatial object. The spatial visualization abilities mainly found in the students' problem-solving process are the ability to mentally rotate a 3-dimensional solid figure depicted in 2-dimensional representation and thus change the positions of its constituents, to transform a depicted object into a different form by mentally cutting it or adding to it, to see a partial configuration of the whole that is useful to solve the problem, and to mentally arrange or manipulate a 3-dimensional object depicted in 2-dimensions (Ryu et al., 2007, p. 143).

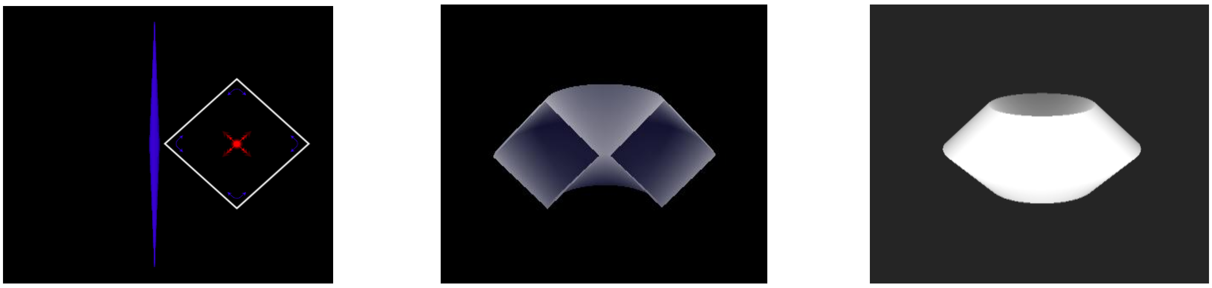
The majority of the students from Ryu et al. (2007) study, though mathematically gifted, had difficulty in imagining rotations. Leonhard Euler, the great Swiss mathematician of the eighteenth century (1707-1783) was known for his power of imagining things through which he continued to do complicated mathematics calculations in his head even when he was blind.

This power of imagining things can be useful in imagining rotations in VSOR even including the diagrams that students are confronted with for the first time.

Duval (2006, p. 119) point out the fact that there are persistent difficulties that students encounter with figures as misunderstanding of the mathematics represented. That is as a result of the fact that what one sees in a figure depends on factors of visual organisation, that is the recognition of certain one-, two- and three-dimensional forms in a figure. According to Duval, seeing in geometry requires that a student is able to recognise these dimensions. In VSOR students do not only have to see, they should be in a position to look at different orientations of these figures, for an example, by rotating them.

One topic in geometry where students deal with different orientation of figures is transformation geometry. In transformation geometry, a given figure in 2D or 3D can be reflected, translated or rotated. In order to perform these reflections, translations and rotations of given figures, imagination becomes the main skill. In their study on transformational geometry problems, Boulter and Kirby (1994) classified students' problem-solving approaches as being *holistic* (involving mental rotations of shapes) or *analytic* (involving analysing an assembling shapes). They refer to transformations in their study as *slides* (translations), *flips* (reflections) and *turns* (rotations). Most of the students succeeded in solving a question involving translating a trapezoid and a question involving rotating an arrowhead using analytic strategies and succeeded in solving a question describing the transformations from a new shape to the original shape using holistic strategies. A similar study was done by Clements, Battista, Sarama and Swaminathan (1997) using Computer Algebra Systems (CAS). In one question students were given two shapes and asked whether the first shape would have to be flipped and rotated or just rotated to be superimposed on the second shape to the other side of the given vertical line. The use of CAS increased students' awareness and conceptualisation of slides, flips, and turns. Before the use of CAS some students were seen to use trial and error to argue about the slides, flips and turns.

An example to demonstrate how a solid of revolution (Potter's Wheel, shown in Figure 2.2) could be formed was given by Christou, Jones, Pitta, Pittalis, Mousoulides, and Boytchev (2008/9, p. 5-6). In generating a Potter's Wheel, a 2D object (a square) is rotated in 3D around a vertical axis to generate a 3D rotational object. Figure 2.2 (b) can be given as an intermediated step to help students 'perceptually construct' the solid, whereby they imagine the rotation. This intermediated step enhances students to visualise the revolution procedure by identifying the fundamental 2D shape. They argue that alternatively, this didactical situation can be the other way round, where students can be given a constructed solid, Figure 2.2 (c) and asked to figure out the 2D object used. In this case students are expected to translate from 2D to 3D and from 3D to 2D.



(a) (b) (c)

**Figure 2.2: Potter Wheel construction (adapted from Christou et al., 2008, p. 6)**

Gorgorió (1998), focused on spatial rotations, where students were given geometric tasks using 2D representations of 3D objects. The results reveal that there were difficulties and errors that obstructed or hindered the students' solving processes. Some students' difficulties and errors were observed relating to their interpretation of 2D representations of 3D objects, to the use of 2D drawings to represent 3D objects, and to the use of verbal codes which refer to spatial facts. For example when talking about 3D objects, they talked about sides instead of faces. Gorgorió (1998, p. 227) attests that the "individual's spatial orientation ability depends on his/her capacity to make successful use of structuring, processing and approaching strategies". It is highlighted that the ability of individuals to visualise and manipulate mental images has been recognised as an important cognitive ability (Güven, 2008, p. 100).

Montiel (2005, p. 91) reported that one student could not visually relate the perpendicular rectangles to the slender cylinders (disks), and the parallel rectangles to the 3D shells when required to calculate the volume. This lack of mathematical fluency relates to the lack of sufficient solid schemas, which would have permitted this student to formulate the mental models of 'disks' and 'shells'. For another student, it was noted that while saying 'rotating', the student actually performed a written rotation about the line $x = y$, as is done in geometry. This, I suppose, was assisting this student to imagine the rotation about a particular axis.

Learning about 2D) and 3D objects begins at the elementary level. The investigations of Kotzé (2007, p. 33) on Grade 10 learners and the teachers enrolled for the Advanced Certificate in Education (ACE) programme indicated that space and shape were problematic areas for both teachers and learners. The following were some of the problems experienced in space and shape in geometry classes.

• Respondents had difficulty in representing characteristics of and relationships between 2D and 3D objects. 3D activities were specifically experienced as more problematic.
• If geometric objects were placed in different orientations and positions, respondents experienced problems in analysing and solving problems, especially related to viewing objects from different angles.
• In-depth knowledge into volume and surface area needed attention: respondents did not perform well in analysing a 3D problem and being able to calculate its surface area correctly.

VSOR use space and shape as well as formulae for areas of 2D and 3D objects as prior knowledge. The circle relates to the disc method which relates to the equation for the area of a circle, the washer that related to two circles, the small circle inside the big circle and the shell that relates to formula for area of a cylinder. If students lack knowledge of space and shape, especially involving 2D and 3D objects, it may become difficult to deal fully with problems related to volumes of solids of revolutions.

## 2.3 TRANSLATION BETWEEN CONTINUOUS AND DISCRETE REPRESENTATIONS

This translation normally occurs when students learn about areas bounded by continuous graphs and volume generated. Orton (1983, p. 4) discusses the results of calculus students who were given a number of items to solve. The students had great difficulty with the explanations required in the item that involved integration of sums and volume of revolution and in other comparable items concerned with areas. The results revealed and suggested that most students had little idea of the procedure of dissecting an area or volume into narrow sections, summing the areas or volumes of the sections, and obtaining an exact answer for the area or volume by narrowing the sections and increasing their number, making use of a limiting process. In translating between continuous and discrete, the Riemann sum is used. The bounded area is partitioned using vertical or horizontal rectangles, which will be used to set up the formulae for area and volume if the bounded area is rotated. One student from Montiel’s (2005) study was trying to memorise the disk and shell methods according to the formulas (especially the ‘$\pi$’ or ‘$2\pi$’ attached to one or the other) and the rectangles being parallel or perpendicular. This attempt at rote memorisation caused this student to mention statements that hint at her lack of fluency in basic geometric concepts such as height, radius and the difference between parallel and perpendicular (Montiel, 2005, p. 91).

Gerson and Walter (2008) conducted a study in which students were given a task aimed at engaging them with calculus concepts (interpreting rates, antiderivatives, concavity, extremas, points of inflection, area between curves, and average rate of change) that they had not yet learnt, providing that require high-level thinking. They were interested in studying how students collaboratively built connected understanding of the quantity of water in a reservoir. In particular they were interested in studying students' development over time of the fundamental theorem of calculus. After having drawn graphs to represent the amount of water in the tank, students made different interpretations of their results. Some students were seen to determine the quantity of water discretely as they compared quantities at different levels. Other students compared areas between curves to generate a comparison of quantities. In that way they were operating with continuous graphs, and did not see discrete entities.

The studies by Orton (1983), Montiel (2005) and Gerson and Walter (2008) above relate to my study on VSOR in that students are calculating the area under the curve and translating that to volume. In my study students are expected to start by representing area under the curve by using a number of rectangles (discrete) and to later represent it using one strip that accommodates the whole area (continuous) in terms of integration. In another study Santos (2000) points out that the use of table or numeric representations was useful to explore the *discrete behaviour* of a particular property, while the graphic representation became a visual tool and the algebraic representation allowed students to examine general cases and *continuous behaviour*. Students in Santos's study discussed in Section 2.1.2 were able to move from discrete to continuous when performing the Cartesian Connection, which did not happen in Knuths's (2000) study.

The difference in performance in the above studies might be as a result of the fact that the students in Santos's study, though at a lower level (Grade 12) used dynamic software (Cabri-Geometry), while those in Knuths's study learnt traditionally. The implication of these results to my study may be that the lack of CAS, which enabled the students in Santos's study to operate in different levels of representations, may hamper students' understanding of concepts involving translation from discrete to continuous as it was the case in Knuths's study, despite the fact that the students were at a higher level.

Farmaki and Paschos (2007) reports on the case study of Peter (with excellent results in mathematics), involving the study of motion via graphic representation of velocity versus time in a Cartesian axes system. He used the visual representation of velocity (drawing a straight line $y = 2x$ continuous in a closed interval $x \in [a, b]$ and used step functions in a form of vertical rectangles of equal heights by partitioning of the time interval [0,1] in equal subintervals) as a step to abstract mathematical thought and to full mathematical justification (Farmaki & Paschos, 2007, p. 361). The partitioning used in (Farmaki & Paschos, 2007, p. 361), may be regarded in one way where Peter was moving from continuous to discrete without being aware of it. Looking back at the origins of calculus it is asserted that

> from a modeling perspective, we see a development of calculus that starts with modeling problems about velocity and distance. Initially these problems are tackled with discrete approximations, inscribed by discrete graphs. Later, similar graphs - initially discrete and later continuous - form the basis for more formal calculus (Gravemeijer & Doorman, 1999, p. 122).

A discrete approximation of a constant changing velocity is shown in Figure 2.3.

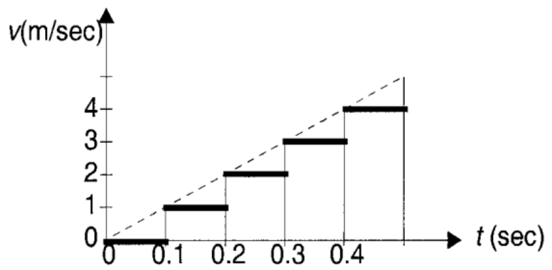



**Figure 2.3: Discrete approximation of velocity (Gravemeijer & Doorman, 1999, p. 125)**

According to Gravemeijer, and Doorman (1999) a central problem is when students use the small rectangles above through the coordination of the height and the width of the bars when they visualise a discrete approximation of a movement, as well as extending the idea to investigating the 'area of the graph', and the total distance covered over a longer period of time and extending that to integration. In another study by Camacho and Depool (2003), some students were seen to be in a position to draw a certain number of rectangles to approximate the area. The problem was that maybe some of them did perhaps not realise that the more the number of rectangles one uses, the better the approximation of area gets, since some were seen to use two rectangles while other used seven or eight rectangles. In some instances students were drawing rectangles to approximate the area for a region that is not bounded.

In their study on students' understanding of limits and integrals, Pettersson and Scheja (2008, p. 776) interviewed a student who stated that "integral is the area … and when you compute the area you often … get an approximation … the smaller strips [the better]". This student's understanding was that the strips are just an explanation of how to find the area and that in order to find the area, you compute an integral by using the antiderivative. This approximation involved a Riemann sum. Another study on Riemann sums was conducted by Seally (2006), who investigated the use of Riemann sums to calculate area under a curve as well as the definite integrals. The conclusion reached in this study was that not all functions have an antiderivative that can be expressed in terms of elementary functions. For example, the antiderivative of $f(x) = e^{x^2}$ cannot be expressed in terms of elementary functions. With such functions, the FTC cannot be applied, and other methods for evaluating the definite integral, such as Riemann sums would be needed (Seally, 2006, p. 46).

## 2.4 GENERAL MANIPULATION SKILLS

A study by Cui, Rebello, Fletcher and Bennett (2006), on transfer of learning from college calculus to Physics II courses with engineering students, revealed that students had difficulties which they also acknowledged in setting up calculus-based physics problems. They could not decide on the appropriate variable and limits of integration and in most cases tend to avoid using calculus but used oversimplified algebraic relationships in problem-solving. This occurs as a result of students' shallow knowledge of calculus and graphs in particular. In other instances the reason might be that what needs to be learnt is presented above students' cognitive abilities. In their study Yasin and Enver (2007, p. 23) found that students had difficulty in calculating the area bounded by curves. From the written responses given, students only indicated the shaded area and did not draw the rectangular strip. Most of the students had problems calculating area especially if the area was below the $x$-axis and at times used incorrect limits.

Huntley, Marcus, Kahan and Miller (2007) investigated what high school mathematics students would use to solve three linear equations. They assert that the dominant strategy used by students is symbol manipulation (while checking their solutions by substituting back in the original equation) with graphical solution being the less dominant strategy. They used symbol manipulation even when given parallel lines which could be more easily solved graphically. The problem with general manipulation was also evident in Montiel's (2005) study. Some students had problems expressing the functions 'in terms of $x$' or 'in terms of $y$', as well as confusing the line $x = -1$ with the line $y = -1$.

Some of the students also did not as well know when to use $\Delta x$ and when to use $\Delta y$. This is what one student said:

> **"**Mm, I have a question. In the disk method, the *x* ... is what we're revolving about, if it's *x* it's d*x*, if it's *y*, it's d*y* in the disk and washer method. Now the shell method, if it is revolving about the *x* it is d*y*, and about *y* is d*x* (Montiel, 2005, p. 101).

This student also did not know which method to use for trigonometric functions such as $y = \sin x$.

Gonza 'Lez-marti'n and Camacho (2004) designed a teaching sequence for improper integrals using a computer algebra system. Their study identifies difficulties, obstacles and errors experienced by first-year mathematics students in Spain while they were learning integration. Students responded to questions from a questionnaire relating to algebraic and graphic representations. The authors focused on difficulties that students have when carrying out non-routine tasks related to improper integrals in order to discover the level of students' understanding (Gonza 'Lez-marti'n and Camacho, 2004, p. 74). They highlight that students' difficulties arise from errors that students make when doing conversions between algebraic and graphic registers. From their analysis one can conclude that there are students who have difficulty in articulating the different systems of representation, and have problems in connecting and relating this knowledge as a generalisation of previous concepts. Gonza 'Lez-marti'n and Camacho (2004) assert that even 'simple' calculation of integrals causes problems for the students. This may be as a result of students seeing integration as cognitively demanding and they develop a negative attitude even for the simple exercises.

Students in the study by Camacho and Depool (2003) were in some instances unable to translate from visual to algebraic, but they performed well in calculating arithmetic and manipulating the integrals they were working with. A study done by Neria and Amit (2004) indicated that the vast majority of their students preferred verbal mode (44%) and numerical mode (37%). When solving algebraic problems, students in the study by Pugalee (2004, p. 37) actually used guess and check most frequently followed by logical reasoning and diagrams/tables/other visuals. An analysis of students' written responses revealed that the majority of students' errors were procedural (66.2% of all errors) followed by computation (23%) and algebraic (10.8%).

## 2. 5 GENERAL LEVEL OF COGNITIVE DEVELOPMENT

Learning concepts that are above the students' cognitive level are often regarded as abstract but sometimes possible if the students are given enough time to deal with such concepts. Eisenberg (1991) argues that the abstraction of the new mathematical knowledge and the pace with which it is presented often becomes the downfall of many students. When learning abstract mathematical concepts, both conceptual knowledge and procedural knowledge are involved. In learning of VSOR, both procedural and conceptual knowledge feature predominately. There are many definitions of procedural knowledge and conceptual knowledge. According to Hiebert and Lefevre (1986), procedural knowledge involves symbols, rules, algorithms, syntax of mathematics while conceptual knowledge involves individual pieces of information and their relationships. According to Haapasalo and Kadijevich (2000, p. 141), procedural knowledge often calls for unconscious steps, while conceptual knowledge requires conscious thinking.

In this study I used questions that are procedural, as they could be answered by simplistic rehearsal of a rule method as well as conceptual questions as they require the use of some thought and rules or methods committed to memory (Berry, Johnson, Maull and Monaghan 1999, p. 110). The questions are structured in such a way that the students have an opportunity to reason since reasoning skills are necessary to advance from a procedural to a conceptual approach (Kotzé, 2007, p. 23).

It has been argued that,

> conceptual knowledge has been described as being particularly rich in relationships and can be thought of in terms of a connected web of knowledge. Procedural knowledge has been defined in terms of knowledge of rules or procedures for solving mathematical problems (Pettersson & Scheja , 2008, p. 768).

According to Star (2005) conceptual knowledge involves 'knowledge of concepts' and procedural knowledge involves 'knowledge of procedures'. In my own interpretations, if one knows procedures only, I refer to that as procedural knowledge but if one knows procedures with reasoning, I refer to that as conceptual understanding (deep understanding of procedures and concepts behind that). According to Skemp (1976), that kind of understanding is instrumental understanding (knowing procedures) and relational understanding (knowing why certain procedures are done).

One of the reasons why students have difficulty in learning a subject like calculus is the deficiency in conceptual understanding (Mahir, 2009, p. 201). As it is the case in most calculus classrooms from my previous students, conceptual learning requires serious mental activity, and to avoid this, students prefer to memorise procedural rules and algorithms (Mahir, 2009, p. 201-202). Research has shown that success amongst students who memorise procedural knowledge without proper understanding of the underlying concepts is not possible (Mahir, 2009). Procedural and conceptual learning can involve routine and non-routine problems. A non-routine problem can become routine if an individual solves the same problem more than once. This is supported by Engelbrecht, Harding and Potgieter (2005) when stating that a problem that is conceptual in nature becomes procedural if it is done repeatedly.

Cognitive skills include visual skills since “our perceptions are conceptually driven” (Arcavi, 2003, p. 234). In addition to that Arcavi (2003, p. 235) comments that “visualization is no longer related to the illustrative purposes only, but is also being recognized as a key component of reasoning (deeply engaging with the conceptual and not the merely perceptual), problem-solving, and even proving”. According to Sabella and Redish (1996), studies involving students’ understanding of calculus reveals that they have superficial and incomplete understanding of many of the basic concepts of calculus. Garner and Garner (2001) suggest that when teaching calculus, instructors should focus more on conceptual teaching, since calculators and computers can be used to perform mathematical calculations.

A study by Pettersson and Scheja (2008) explores the nature of 20 engineering students’ conceptual understanding of calculus on the concepts of limit and integral. The results revealed that students’ understanding of the limit and integral concepts was algorithmic, emphasising procedures and techniques for problem-solving, rather than pointing at conceptual connections between concepts (Pettersson and Scheja, 2008, p. 781). In that case, it means that students did not have a thorough understanding of the concepts of limit and integral, hence lacked conceptual understanding. These concepts, limit and integral are regarded by Pettersson and Scheja (2008, p. 768) as ‘threshold concepts’ since they are ‘conceptual gateways’ or ‘portals’ that lead to a new way of thinking about a particular subject area. They further argue that threshold concepts have the potential to open up understanding of a topic in important ways, even though students may initially find them difficult to grasp.

In my study, threshold concepts include the '*rectangular strip'*, '*boundaries'*, '*solids of revolution*', the '*disc method'*, the '*washer method* 'and the '*shell method'*. I focus on how students deal with these concepts, with the hope that students must have grasped the notion of an integral as limit of a sum. According to Pettersson and Scheja (2008, p. 770), students may experience difficulties in understanding the relationship between a definite integral and area under a curve and sometimes seen integration just as a rule, as antidifferentiation, thus students find integration difficult (Pettersson & Scheja., 2008 & Yost, 2008). Students, according to Pettersson and Scheja (2008) seemed to be thinking about limits and integrals within an algorithmic context, emphasising procedures and techniques for problem-solving, rather than pointing at conceptual connections between concepts.

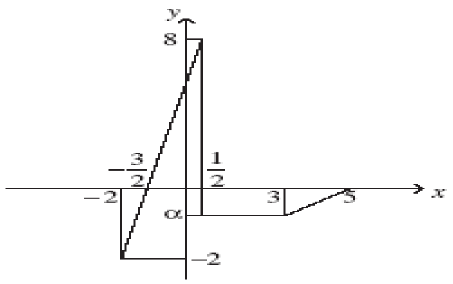
Mahir (2009) conducted a study with a sample of 62 first-year calculus students on topics including functions, limits and continuity, differentiation, transcendental functions, and some applications of differentiation including sketching graphs of functions covered in the 1st semester and topics including integration, techniques of integration, application of integration, sequences, series and power series covered in the 2nd semester. The questions given in Figure 2.4 used in Mahir's (2009) study are to a certain extent related to my study.

(1) Evaluate $\int_2^3 \frac{x+3}{x-1}\mathrm{d}x$.

(2) Evaluate $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\mathrm{d}x$.

(3) Evaluate $\int_{\sqrt{2}}^{2\sqrt{2}} (\sqrt{6}-\sqrt{8-x^2})\mathrm{d}x$.

(4) The graph of $f$ is sketched below. Given that $\int_{-2}^{5} f(x)\mathrm{d}x = \frac{39}{8}$, find the value of $\alpha$.




(5) The graph of $f'$, the derivative of $f$, is sketched below. The areas of the regions A, B, and C are 20, 8, and 5 square units, respectively. Given that $f(0) = -5$, find the value of $f(6)$.

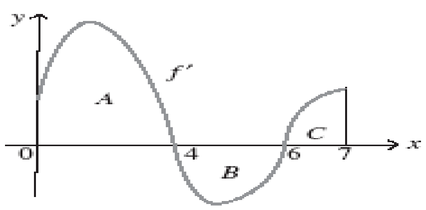



**Figure 2.4: Questions on evaluating an integral (Mahir 2009, p. 203)**

From the five questions given, the first and the second questions were procedural since they could be solved by using integral formulas and integration techniques. The third and fourth questions were both procedural and conceptual. They could be solved either by using the integral-area relation (indicating the presence of conceptual knowledge) or by using integral techniques (indicating the presence of procedural knowledge), while the fifth question was conceptual since it incorporates many concepts including the fundamental theorem of calculus, integral-area relation and the fact that the integral of a function is the algebraic sum of areas. For this reason, the ability to solve the fifth question strongly indicates the presence of conceptual knowledge (Mahir, 2009, p. 204).

The students' interpretations were interesting. Mahir (2009, p. 204-207) reported that the first and second questions were correctly solved by 92% and 74% of the students, respectively, which implies that students possess procedural knowledge of integration. For the third question, it was found that 73% of the participants tried to solve this question by using trigonometric substitution, whereas only 8% tried to make use of the integral-area relation and 19% of the students did not respond to this question at all. It was found that all the students who followed the conceptual approach correctly, solved this question by using a few simple calculations, whereas only 11% of the students who applied trigonometric substitution managed to obtain the correct solution. The remaining 89%, although successfully applying trigonometric substitution, delved into lengthy and complicated calculations and could not obtain the correct answer.

The results for the fourth question are not different from those of the third question. What is important is that the number of students that used procedural knowledge was significantly higher than the students who used conceptual knowledge. As in the third question, the number of students that used procedural knowledge is significantly higher than those who used conceptual knowledge. Most importantly only 16% of the students that used procedural knowledge were able to obtain the correct answer, whereas 71% of the students that used conceptual knowledge obtained the correct answer. This reveals that students who have a thorough understanding (conceptual knowledge) of content are better equipped to solve problems than those who have a shallow understanding (procedural knowledge) of concepts. As for the fifth question, 40% of the students did not respond, since they did not have the conceptual understanding of the fundamental theorem of calculus. Moreover, 36% of the students responded incorrectly. What was found to be problematic in most cases was that for the area below the $x$-axis the students took it as positive and did not subtract it.

In the above study, to evaluate the integral, one student calculated the area of an incorrect region. Although this student was aware of the integral-area relation, he did not know that the 'area' refers to the area of the region between the graph of the function and the *x*-axis. What was revealed again in Mahir (2009, p. 209) was that students were seen to successfully apply the fundamental theorem of calculus when the integrand is explicitly given to them as in Questions 1 and 2. On the other hand, if the integrand is not explicit, as in Question 5, they mostly failed. In his conclusion, Mahir (2009, p. 210) acknowledged that their students did not have satisfactory conceptual understanding of the integral and integral-area relation, that the integral of a function is algebraic sum of areas and of the fundamental theorem of calculus. They finally recommended among other reasons that in order to improve conceptual understanding, various graphical, algebraic and real-life examples should be given when a new concept is taught in class. Another aspect that is important in mathematics learning, especially calculus, is preknowledge. For example, the concepts of the Riemann sum must be understood well to succeed in integration as that enables a student to move from discrete to continuous.

Mahir (2009:202) is of the opinion that "one cannot understand differentiation without knowing limits and one cannot understand integration without knowing differentiation". In FET colleges, before doing integration (at N5 and N6 level), students are expected to have preknowledge of differentiation from N3 and N4 levels and from Grade 12. However, there are different beliefs of what should be taught first (differentiation or integration) or whether they should be taught simultaneously. For example Harman 2003 and Parrott 1999 believe that teaching of integration must precede teaching of differentiation.

Other studies on conceptual and procedural knowledge were done on interpretations of functions (Evangelidou, Spyrou, Elia, & Gagatsis, 2004; Hähkiöniemi, 2006; Juter, 2006; Sierpinska, 1992; Tall, 2000). Among them, Juter (2006) reports on a study involving the limits of functions with 112 first-year university students. She asserts that students encounter difficulty with definitions when they learn limits. Most students did not improve, despite the fact that their teacher displayed graphical proofs of the limits and followed the textbook. She argues that the students struggled because they learnt limits as facts (Juter, 2006, p. 426). She further pointed out that the students had an algebraic approach to limits, where they used unknown and unsuitable procedures to compute limits. The study exposes the most crucial aspects of rote learning and lack of conceptual understanding. The fact that students were seen to use unknown and unsuitable procedures, justifies that they lacked proper understanding of the concepts learnt.

In learning functions at undergraduate level, it is argued that if algebraic and procedural methods were closer connected to conceptual learning, students would be better equipped to apply their algebraic techniques appropriately in solving novel problems and tasks (Oehrtman, Carlson, & Thompson, 2008, p. 151). As students move through their school and undergraduate mathematics curricula, they are frequently asked to manipulate algebraic equations and compute answers to specific types of questions. This strong emphasis on procedures without accompanying activities to develop deep understanding of the concept has not been effective for building students' foundational function conceptions that allow for meaningful interpretation and use of functions in various representational and novel settings.

Cognitive obstacles experienced in mathematics understanding may be a result of the abstract nature of mathematics. Eraslan (2008) shows how the notion of reducing abstraction can be used for analysing mental processes of students studying quadratic function in high school mathematics. The results reveal that when students solve problems related to quadratic functions, they tend to change the given form from a less familiar form to a more familiar and manageable one, hence reducing abstraction (Eraslan, 2008, p. 1055). In most cases, when students try to reduce abstraction, they tend to change the whole meaning of the question and end up not getting the solution correct. Abstraction in most cases is possible in problems that are conceptual in nature, and that pose a challenge to both learners and their teachers. It is important that teachers assist learners to link new knowledge to existing knowledge and develop instructional techniques that would facilitate cognitive growth and change (Kotzé, 2007) in a way of promoting conceptual understanding.

In Montiel's (2005) study, cognitive obstacles were encountered in some questions. For example, when asked to sketch the region and the volume generated by this region of a particular set, say, $V = \pi\int_0^4 (\sqrt{x})^2\,dx$, students would evaluate the integral. This relates to procedural knowledge of the integral. Another student was seen to multiply by $\pi$ when asked to calculate area, not knowing exactly what the $\pi$ was for, also indicating procedural knowledge of the concepts learnt. In this case the symbol becomes the cognitive obstacle. In some instances, it was difficult for the students to picture revolving about, say, the vertical axis, and setting up the integral in terms of the horizontal axis. Students were also seen to use the disc method even if the question required the use of the shell method, pointing to the lack of conceptual understanding.

A study by Joffrion (2005) involving seventh grade teachers and their learners, revealed that

> The students of the teacher who delivered conceptual instruction improved their algebra skills from the beginning of the year to the end. The students who received more procedural instruction without the support of the conceptual network showed little improvement over the course of the year. Their knowledge stood alone as individual pieces and they were not able to apply it in new situations. These students were not well equipped to solve problems or apply algebraic reasoning. The students of the more conceptual teacher, on the other hand, were significantly better prepared to answer questions requiring algebraic reasoning (Joffrion, 2005, p. 57-58).

The above studies all point to the important fact that knowing why certain procedures should be performed when solving mathematical problems is the foundation for conceptual understanding. These concepts are relevant to my study as success in learning VSOR requires use of procedures after reasoning at a conceptual level. In that way students will be able to solve problems that are abstract in nature. In relation to other studies discussed above, concurrent validity will be ensured where the results of the tests administered in my study concur with the results of the other tests or instruments that were testing the same construct (Cohen, Manion & Morrison, 2001, p. 132), in this case learning difficulties with VSOR.

## 2.6 CONTEXTUAL FACTORS AFFECTING LEARNING

### 2.6.1 Writing to learn mathematics and effect of language

According to Duval (2006), mathematics register is produced verbally or symbolically in written form. The special symbols used in mathematics are a way of communicating it. It is through this verbal or written form that success in mathematics can be measured. It is highlighted that "research about the learning of mathematics and its difficulties must be based on what students do really by themselves, on their productions, on their voices" (Duval, 2006, p. 104). In this research, the focus is on the students' productions through their written work and in some instances their verbal interpretations.

In VSOR the issues of writing and language are relevant as students are expected to write and at the same time be confronted with the language of assessment. What students have learnt is evident from their written responses and what they verbalise individually or as they work in groups. Writing to learn mathematics was related to conceptual understanding and procedural ability of students in an introductory calculus course by Porter and Masingila (1995).

Students were seen using incorrect procedures, for example when asked to find the derivative of a function, they would then find the limit of the function. This shows that in this case language and interpretation of concepts is a problem. Students were seen to make procedural, conceptual and indeterminate errors. Students were at times asked to reflect in writing on their study habits and performance in the course. They were also asked to explain the following concepts in writing: function, derivative, Rolle's Theorem, the Mean-value Theorem and so on. Such explanations in written form improve conceptual understanding and not only knowledge of procedures (Porter and Masingila, 1995)

Writing has benefits for both the learner and the teacher. A study by Kågesten and Engelbrecht (2006) with engineering students in technical universities in Sweden reveals some interesting results. They argue that the students in their study and in many other countries tend to treat mathematics as a mechanical subject in which you do calculations and manipulations with very little explanation. They forced the students in their study to reflect and re-think the answers they gave in writing (during examinations) and comments from their teachers to clarify their explanations. Students were forced not to limit themselves to methods of calculations and manipulations when solving problems, but to reflect on the actual concepts involved. It was decided:

> Students would not be given full marks for a question (if they do not fully explain why they used the manipulations they did, or if there are any deficiencies in the linguistics of the explanation. After the marked script has been given back to the student, (s)he takes the script home and revises and appends it according to the comments and questions listed by the teacher that marked the script (Kågesten and Engelbrecht , 2006, p. 709).

This practice forced each and every student to re-construct what he/she did. During the interview that they conducted, some students admitted that they could not explain their calculations, arguing that at the time of the examination they did not understand what they were doing to a 'sufficient depth', while other students did not want to expose their understanding too much, probably because of a lack of confidence about their understanding of the concepts learnt (Kågesten & Engelbrecht, 2006, p. 711). What was significant in this study is that almost all students interviewed, emphasised the importance of having to reflect on their work. They commented that "the additional time that they spent at home, attempting to address the comments from the teacher that marked the test, contributed largely to deeper understanding of the particular concept" (Kågesten & Engelbrecht, 2006, p. 712). The students interviewed also acknowledged that in having to think about the relevant concept again, they discovered their previous misconceptions. The students reported that they were used to using mathematical symbols and avoiding verbal language.

These comments clearly allude to the view that writing improves critical thinking, and it is important to improve on one's conceptual understanding. Smith (2010) concurs with Kågesten and Engelbrecht (2006) above when they assert that computations without any writing or explanation *contain no mathematics.* In their study on science literacy, students of McDermott and Hand (2010, p. 55) clearly indicated that the writing tasks they participated, in encouraged a type of ''learning'' that was in-depth, personal, and went beyond mere memorisation or recall. Writing can also be a tool for supporting a metacognitive framework (Pugalee, 2004).

In my study, writing is used in order to evaluate students at FET Colleges, in the classroom and during examinations. If during the class assessments, students are given the opportunity to reflect on their writing (Kågesten & Engelbrecht, 2006), critical thinking and better understanding of the concepts involved in VSOR may be reached. It is for the teacher to create an environment that stimulates explanations of what is written by the students (Kågesten & Engelbrecht, 2006, p. 713). At the university undergraduate level, writing is the dominant, if not exclusive language mode through which learning is evaluated (Gambell, 1991). Writing is seen as one way to encourage critical thinking (Indris 2009; Kieft, Rijlaarsdam & Van den Bergh, 2008; Klein, Piacente-Cimini & Williams, 2007; Zohar & Peled, 2008) and reflection and evaluation of understanding in students (Indris, 2009, p. 36). Indris (2009) conducted a study, where students were given an opportunity to use writing activities to explore calculus materials, concepts and ideas freely, to assist them to develop their own intuitive ideas. In learning mathematics the meanings of some concepts need to be understood for proper learning, especially problems that are conceptual in nature. Writing helps students to gain conceptual understanding of such scientific topics (Gunel, Hand & McDermott, 2009).

Another aspect that affects learning is the language used. It may be language of instruction or language of assessment. The problems that involve language include word problems. In a number of studies, language was a barrier for proper learning of mathematics (Bell, 1995; Eiselen, Strauss & Jonck, 2007; Howie, 2002; Inoue, 2008; Pettersson et al., 2008; Setati, 2008). In the study of Pettersson et al. (2008), students' understanding of the concept of limit seemed to be deeply intertwined with everyday language use, where the everyday meaning of the word 'limit' induces conceptions of the limit as a barrier or as the last term of a process. As it was the case in Montiel's (2005) study, when solving problems that are given in words, students tend not to relate the problem to its meaning. They rather tend to focus on how to do calculations, without understanding what is being asked. When asked to sketch the region and the volume generated by this region $V = \pi\int_0^4 (\sqrt{x})^2\,dx$, the students were seen to evaluate the given integral. Studies including TIMSS have shown that English seems to be a problem especially with learners whose home language is not English. It was evident from the TIMSS study that the poor performance of most South African learners was a result of their English language proficiency (Howie, 2002).

Some terminology in integration was seen to be problematic. According to Montiel (2005, p. 89), students understood terminology such as ‘bounds’, ‘boundaries’ and, ‘region’ incorrectly. Montiel (2005) believes that spoken mathematics is the most direct way to detect metaphors that are used by students. She believes that in mathematics, unlike foreign or native language (where students do creative writing), metaphors do not usually appear as such in students’ writing, although they are present in their mental structures. She argues that the actual names of the disk and shell methods correspond to extra-mathematical metaphors. It was evident from her study that the “disk” metaphor was much more helpful for the majority of students in the class than the “shell” metaphor.

According to Duval (2006, p. 121) students encounter problems with simple “translation” of the terms of a word problem into symbolic expressions. In Rouhani’s (2004, p. 120) study, the task of translating functions from a verbal representation to an algebraic description was the most difficult task for all but one participant. The results of a study conducted by Swangrojn (2003), indicate that unsuccessful problem solvers had difficulty translating and representing word problems into equations using variables and symbols. While lecturing to first-year students at the University of KwaZulu-Natal, Maharaj (2008; 411) found that a significant number of students were unable to interpret the structures of mathematical objects, and to solve word problems. He suggests that there should be a focus on formulating the problem statement and transforming it into the relevant equation in order to give students a deeper insight into the structural features of equations, and the need for transforming them into equivalent equations.

The above studies are relevant to my study of VSOR where students are assessed in writing and the language they use will be evident in their written responses and discussions. The language used in VSOR is the everyday and the mathematical language. In this study I focus on how students interpret the language used in questions given in VSOR.

### 2.6.2 Scaffolding learning

Scaffolding takes place when a teacher or one in possession of knowledge assists learners to attain knowledge through explanations and clarifications of concepts. While teachers teach

> … pupils make sense of teachers' instructions in their own ways, sometimes very different from those of the teacher. With *cognitive structuring* teachers assist pupils to organise their own experience either by providing explanations or by suggesting meta-level strategies to help pupils organize their work (Bliss, Askew & Macrae,1996, p. 41).

In a learning environment the classroom should be considered as a social environment which involves complex exchanges that support learning (Anghileri, 2006, p. 35). Anghileri found that teachers are most effective if they can scaffold pupils' learning by employing a range of teaching approaches in their classrooms in an environment that encourages active involvement working in groups. Scaffolding has been found to be very important in mediating learning (Anghileri, 2006; Bliss et al., 1996). During scaffolding, the *Zone of Proximal Development* (ZPD) is enabled. The ZPD is the "the distance between the actual development as determined by independent problem-solving and level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86).

Bliss et al. (1996) refer to two types of scaffolds in a way of cueing. They are the *Alpine guide: step-by-step* or *foothold scaffolds;* and *hints and slots scaffolds.* With the former, arguments in teaching are sometimes a little difficult. One way to keep going is to lead step by step in a series of questions. Each step in the argument is turned into a question, and each question expects an answer which in turn, will permit the next question. The latter type of scaffold refers to those occasions when it is difficult to ask open-ended questions. Questions such as 'What is ...?' often lead to one specific answer (Bliss et al., 1996, p. 47).

In mathematics classrooms scaffolding can take place when students learn individually or cooperatively. Various studies focus on individual learning (Brijlall & Maharaj, 2009; Ebert & Mwerinde, 2002; Gagatsis & Patronis, 1990) as well as cooperative learning where students share knowledge during their interactions (Brijlall & Maharaj, 2010; Brijlall & Maharaj, 2009; Chmela-Jones et al., 2007; Ebert & Mwerinde, 2002; Juter, 2006; Walter, Barros & Gerson, 2008). When students learn individually, they are able to use their imaginative skills in order to accomplish the necessary learning. With cooperative learning, there is a set of processes or step-by-step methods that help students interact with each other in order to accomplish a task (Chmela-Jones et al., 2007, p. 631). When students learn mathematics, they require engagement in conscious reflection (a metacognitive skill), on their own mental processes (Gagatsis & Patronis, 1990), which may be activated when students work cooperatively. When students work together in a group, collective understanding is possible (Martin, Towers & Pirie, 2006), as they share meaning. These authors argue that group work enables students to make, hold, and extend particular images in growing their mathematical understanding about a particular concept. The students in this case act together; they are involved in *coacting*, which is

> a process through which mathematical ideas and actions, initially stemming from an individual learner, become taken up, built on, reworked, and elaborated by others, and thus emerge as shared understanding for and across the group, rather than remaining located within any one individual (Martin et al., 2006; 156).

During group interactions, students construct individual mental representations. It is argued that, "as children model and represent their strategies, and as they develop generalized mental models of the part/whole relations for situations and operations, they construct *mental maps* that can eventually become tools to think with" (Fosnot & Dolk, 2003, p. 14). They believe that learning requires assimilation, accommodation and reflective abstraction. They further highlight that if the problems given to students promote progressive schematisation, the development of big ideas and the construction of models, learning will occur provided the pedagogic strategies are aligned with the process of learning rather than with transmission and or activity (Fosnot & Dolk, 2003, p. 14).

Zimbardo, Butler and Wolfe (2003) looked at reasons why teams arrive at better answers than individuals. It was argued that it is possible that team members may stimulate and encourage each other through their discussion. Additionally error correction procedures may occur in groups to effectively help the student get rid of incorrect answers. Through active participation such as verbalising a reason for one's answer, a student's misconception of the content learnt may be clarified by fellow students with more knowledge. It is possible that by using a group testing approach instructors are structuring their courses so that students assist each other in mastering the course content as the collaborate as peers. During group interaction the student's misunderstanding of questions may be corrected by others through scaffolding. When group members support each other (positive affect) their motivation may be boosted, thus foster learning (Desrochers, Fink, Thomas, Kimmerling, & Tung, 2007, p. 294).

When a student is working alone it is possible that such a student carelessly reads an item and thus misinterprets it, resulting in failure to answer the question correctly. It should however be noted that group work is only important to enable students during the learning process to pick on their errors and misconceptions and to develop confidence. In the end every student should be able to work independently, since examinations are not written in groups. In one study, as students worked cooperatively, some students demonstrated the ability to apply symbols, language, and mental images to construct internal processes as a way of making sense of the concepts of monotonicity and boundedness of sequences (Brijlall et al., 2010, p. 61).

The above studies have shown how scaffolding and cooperative learning are significant in the development of critical thinking in students, since students discuss (and are guided) and come to know what they do not know, from their peers or their teachers. This aspect is not of great importance in my study but it cannot be ignored since if students learn a topic like VSOR cooperatively, there are more opportunities that they might come to know what they do not know and may succeed in counteracting the cognitive conflicts that they may have. As students learn cooperatively it is important that the teacher also interact with them in a way of scaffolding to help students deal with their cognitive conflicts.

### 2.6.3 Teaching approach

According to Maharaj (2008, p. 411) teaching should focus on and emphasise the structural features of mathematical objects such as expressions, equations and functions. Maharaj (2008) argues that teaching should not neglect the role of ordinary English in developing the symbolic notation, and suggests that word problems should be used to introduce linear, quadratic and possibly cubic equations. He pointed out that instruction should not ignore the links between arithmetic and algebra, algebra and geometry, and the teaching implications from research studies in mathematics. He believes that

> An educator who functions at the structural level, and ignores the fact that concepts in mathematics are first conceived operationally, is unlikely to meaningfully develop in learners an understanding of mathematical concepts. Furthermore, the educator is unlikely to appreciate the cognitive obstacles experienced by learners with regard to the formation of concepts and the achieving of understanding (Maharaj, 2008, p. 411).

In exploring children's algebraic thinking and generalisation through instruction that involves visual/spatial representation of a geometric growing pattern made of square tiles, it was asserted that

> children's full understanding of and ability to engage in mathematical generalization may in fact rely on a critical integration of more than one form of representation of a mathematical idea. This may more specifically be described as involving children's ability to move fluidly and fluently back and forth across multiple representations in both interpreting and applying a mathematical generalization (McNab, 2006).

Adler (2002) argues that educators must relearn mathematics to develop conceptual understanding, in order to be better equipped to develop learners' conceptual understanding. Such teachers will be able to teach for understanding (Mwakapenda, 2004). Vaughn, Klinger, and Hughes (2000, p. 169) believe that teachers must have "deep knowledge about a practice" in order to sustain their use of that practice. The question that one might ask is: "Do we have teachers who have good conceptual understanding of mathematics?" According to Setati (2008, p. 114) procedural teaching is dominant in South African classrooms and it is seen as being a function of the teachers' lack of or limited knowledge of mathematics.

Procedural teaching will continue to dilute down the mathematical knowledge if teachers do not encourage deep learning for achieving high levels of reasoning and thinking (Kasonga & Corbett, 2008). Some official conceptions of mathematics teaching according to Hoz and Weizman (2008, p. 908) are that "mathematics teaching stresses the learners' construction of mathematical knowledge … and that mathematical teaching must emphasise conceptual understanding". Bossé and Bahr (2008) on the other hand suggest that in teaching, there must be a balance between conceptual understanding and procedural knowledge. Another aspect that should not be ignored in teaching is students' abilities to solve problems. According to Clark, James, and Montelle (2009, p. 59), instructors may not take for granted what academically-able students have acquired in terms of employing their different methods with regard to problem-solving. Students' different methods must be taken into consideration for proper learning to take place, as long as they are mathematically correct.

Fricke, Horak, Meyer and Van Lingen (2008, p. 75) believe that there should be teacher development programmes on-site, focusing on individual teacher needs. The programmes should encompass both content knowledge and teaching strategies and should entail regular follow-up to ensure that there has been successful implementation of new strategies. In teaching mathematical knowledge, Rasmussen and Marrongelle (2006, p. 389) argue that teachers use a pedagogical content tool such as a graph, diagram, equation, or verbal statement intentionally to connect to students thinking while moving the mathematical agenda forward. They further attest that the use of pedagogical content tool requires blending of specific content knowledge, general pedagogical expertise and knowledge of subject matter for teaching.

A study by Bingolbali, Monaghan and Ropers (2007) on the understanding of the derivative involving a group of first-year students in Turkey reveals that mechanical engineering students considered the derivative in terms of rate of change while mathematics students considered the derivative in terms of tangent. This was influenced by the way in which they were taught. In introducing the derivative concept, on the one hand the mechanical engineering calculus lecturer was seen to spend about ten minutes on 'tangent', 'slope of the tangent line' and 'equation of the tangent line to the curve at a particular point' without solving any tangent examples, while spending 133 minutes on rate of change aspects of the derivative followed by nine examples, focusing more on practical mathematics. This lecturer introduced the idea of rate of change through velocity, distance and acceleration. On the other hand, the mathematics course lecturer used tangent ideas to introduce the derivative, attending to the 'slope of the line', 'equation of line' and 'tangent line and secant line'. This lecturer spent eleven minutes on rate of change ideas and 85 minutes on tangents followed by seven examples on tangents, focussing more on theoretical mathematics. The rate of change was only mentioned when he talked about the physical meaning of the derivative and later mentioned rate of change when he attended to acceleration with regard to the second derivative. This lecturer did not solve any examples on rate of change (Bingolbali et al., 2007, p. 771-772). According to Bingolbali and Monachan (2008, p. 31), you get what you teach.

The way in which students were taught in the above study, shaped their developing concept images of the derivative. Bingolbali and Monaghan (2008, p. 23) investigated first-year mechanical engineering and mathematics students' conceptual development of the derivative with particular reference to rate of change and tangent aspects through tests (pre-, post- and delayed post-tests), questionnaires, interviews observations and discussions. The test questions addressed 'rate of change' and 'tangent' aspects of the derivative in graphic, algebraic and application formats. The results for the tests revealed that students' concept images of the derivative changed as they progressed from entry to the end of the first year, where mechanical engineering students' concept images of the derivative developed in the direction of rate of change orientations and mathematics students' concept images developed in the direction of tangent orientations (Bingolbali and Monaghan, 2008, p. 30). The results also revealed that students' developing concept images and the way they build relationships with its particular forms are closely related to teaching practices and the department they come from (Bingolbali and Monaghan, 2008, p. 32).

According to McCormick (1997, p. 148) the schemata, which is the knowledge structures that exist in memory that the individual constructs from experience and instruction, need to be taken into account by teachers when they want students to learn a new concept or theory. Bjuland (2007, p. 27) suggests that “teacher education must stimulate metacognitive training in combination with cooperative learning among the students in order to develop problem-solving skills”.

Allowing students to develop visual skills and to be able to translate to different representations in learning may also affect the way they learn. Kreminski (2009) proposes a visual approach that helps students with the chain rule formulae, by drawing functions of functions and showing them also how these formulae generalise. In this way the symbolic representation of the chain rule was shown graphically for better understanding. Aspinwall and Shaw (2002) encourage teachers to create a learning environment where students become fluent with a variety of representations. They attest that “teachers can enhance students’ understanding by continuing to demonstrate how different representations of the same mathematical concept provide additional information” (Aspinwall & Shaw, 2002, p. 439). Neria and Amit (2004, p. 414) believe that the use of algebraic representation should be integrated into the teaching of algebra from the first stage, and students should gain experience in using algebra for argumentation and justification. In teaching functions, Nilklad (2004) noticed that the instructor did not provide examples that used more than two mathematical representations to display the same data, as well as spending time translating one representation into another.

The ways in which students are taught have an influence on how they learn. Cai (2004) conducted a study on how teaching, the teacher’s beliefs, and curriculum emphases influence the way students solve problems in algebra. The results of the study reveal that the way in which students solved problems was influenced by the way in which they were taught, the teacher’s beliefs, and curriculum emphases. It was found that Chinese students rarely used visual representation whereas the United States (US) students did. That was due to the fact that

> U.S. and Chinese teachers not only hold different learning goals, but also place different emphases on their teaching of problem solving. In particular, U.S. teachers hold a much higher value for responses involving concrete strategies and visual representations than do Chinese teachers (Cai, 2004, p. 158).

Woolner (2004) reports on a survey of the thinking styles of 36 students in Year 7 (aged 11 to 12) in a verbally taught class and a visually taught class where the verbal lessons and the visual lessons covered the same content area, also ensuring that the same questions, investigations and identical teaching materials were used. The intervention lessons were taught once a week for ten weeks. Woolner categorised students into those who prefer to be given a formula (taught verbally) and work without a diagram and those who prefer to use a diagram (taught visually) to conceptualise. The students worked on mathematics questions requiring literacy skills.

The study revealed that students who were taught verbally scored significantly higher than students who were taught visually, with a good correlation ($r = 0.669$) between the pre and the post intervention scores. Woolner postulated that the contrasting results (those taught visually scoring less) could be as a result of a "mismatch between their preferred learning style and the predominance of verbal teaching and assessment" (Woolner 2004, p. 450). Even though Woolner's study is on small kids, it highlights important aspects that if learners' preferences are contradicted in the learning process, the mismatch can lead to poor performance. Haciomeroglu, Aspinwall and Presmeg (2009) argue that in learning, calculus concepts should be represented numerically, algebraically, graphically, and verbally in order that students develop a deeper understanding of the concepts.

### 2.6.4 Curriculum level and assessment

There is no research done on mathematics assessment at the FET colleges, where students' written responses of content learnt was explored. Only examination policies and reports from the national examination are available. The reports hint on the general performance for different subjects in relation to the national average.

De Villiers (2004, p. 706) uses Van Hiele's theory to argue that the reason for failure of the geometry curriculum in high schools is that the curriculum is presented at a higher level than that of the students. In geometry it is essential to engage students at some stage in the process of defining of geometric concepts (De Villiers, 2004, p. 708). The levels are presented below.

Level 1 (Visualisation) Students represent figures by appearance only, often by comparing them to a known prototype. The properties of a figure are not perceived. At this level, students make decisions based on perception, not reasoning.

Levels 2 (Analysis) Students see figures as collections of properties of geometric figures, but they do not see relationships between these properties. When describing an object, a student operating at this level might list all the properties the student knows, but not discern which properties are necessary and which are sufficient to describe the object.

Level 3 (Abstraction) Students perceive relationships between properties and between figures. At this level students can create meaningful definitions and use informal arguments to justify their reasoning. Logical implication and class inclusions, such as squares being a type of a rectangle are understood. The role and significance of formal deductions, however, are not understood.

Level 4 (Deduction) Students can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. At this level, students should be able to construct proofs such as those typically found in a high school geometry class.

Level 5 (Rigour) Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. Students at this level can understand and use indirect proof and proof by contrapositive, and can understand non-Euclidean systems.

The levels above are crucial not only to geometry, but they can be used in other areas. Levels 1, 2 and 3 fit well in the learning of VSOR and are useful in that regard. The levels focus on what is happening in mathematic classrooms in a real sense and can be evident from classroom discussions and from students’ writings. The use of circle, washer and shell stems from these levels as they are geometric figures.

In order to ensure that the curriculum was implemented correctly, students must be assessed. Assessment can be used to verify if proper teaching and learning took place. According to Vandeyar and Killen (2007, p. 102), educators who view assessment as a useful means of gathering data upon which to base decisions about learning and their own teaching will attempt to make assessment an integral part of teaching, by emphasising formative rather than summative assessment. In my study students were assessed from classroom exercises, tests, examination and the designed 23-item instrument which includes both formative and summative assessments. Beets (2007, p. 578) argue that *conventional assessment* is an approach in which assessment normally follows teaching. Beets (2007) believes that assessment in higher education is still dominated by summative assessment practices. In the FET colleges, students are assessed by summative assessments, which may have an impact on their success. Kasonga & Corbett (2008, p. 603) argue that the quality of assessment tasks is a very important determinant of the students’ learning approach (surface or deep learning), except for those students who are intrinsically motivated by the subject.

### 2.6.5 Use of technology

Technology has been found to assist in the development of concepts (Berger, 2007; Pierce & Stacey, 2008; Smith & Shotsberger, 2001; Tall, 1991), especially those technologies that are conceptual in nature and aid visualisation. It is important to note that not only is CAS important to improve on students' conceptual understanding. Advice from various researchers is that shallow learning can be overcome if blended learning (mixing online and face to face learning) is used. The results of Groen and Carmody (2006) indicate that first-year mathematics students benefited positively from blended learning as deep learning was encouraged. The blended learning method introduced students to diverse environments. The results indicated a positive correlation between the average score on deep learning and the average score for blending.

It is believed that "CAS offers pedagogical opportunities for teaching mathematics better and for learning mathematics better" (Pierce & Stacey, 2008, p. 6) in that in some cases it creates possibilities of access to different mathematical representations such as numeric, symbolic and graphic. Berger (2007) and Tall (2000) share the sentiments of Pierce and Stacey (2008) about different mathematical representations with the use of CAS. Computer algebra systems such as Derive, Maple, Mathematica, Micromedia's Flash and others have recently been utilised to enhance the learning of mathematics in the form of animations. Animations have been found to be a successful tool in the learning and modelling of graphical data (Bakhoum, 2008).

Poohkay and Szabo (1995) conducted a study with 147 undergraduate education major students in a mathematics teaching methods course. They compared animations, still graphics and text only for their effects on the acquisition of the mathematics skill of using a compass to create triangles. They found that students, who studied through animations, performed better than students who used still graphics who in turn performed better than the students who used text only (Poohkay & Szabo, 1995, p. 4). In this study, the effects of such animations will be explored, focussing on calculus instruction with the engineering students, where learning through visualisation is an important aspect to be researched.

The birth of new mathematical software, CAS, was realised around the 90s. Among others: Maple (1990), Mathematica (Wolfram 1995), Matlab (Moler 1995) or Derive (1994) were realeased. According to (Balacheff & Kaput, 1996, p. 6) CAS "enable students to define, combine, transform, compare, visualize and otherwise manipulate functions and relations".

They further emphasise the importance of computers, that of providing ways of doing and experiencing mathematics that was not possible before through ‘chalk and talk’. It is important to ensure that while all that is possible, conceptual learning should not be hampered whereby the students, who end up being dependent to the computer, see everything that could be learnt using the computer as being procedural.

A study by Palmiter (1991) involving 78 subjects was carried out in order to investigate whether there was a significant difference between students who have been taught calculus using a CAS (MACSYMA) to compute limits, derivatives and integrals and students who used standard paper-and-pencil procedures focussing on knowledge of calculus concepts; knowledge of calculus procedures as well as grades in subsequent calculus course (Palmiter 1991, p. 151). The control group and the experimental group were each assigned one lecturer and one teaching assistant, who collaborated regularly to ensure that the topics presented were overlapping; the same examples were used and the same materials presented. The techniques for integration were not presented to the MACSYMA group; they had access to MACSYMA for both homework and examinations to compute integrals. Most of the work was covered in a form commanding MACSYMA to compute a limit, sum, derivative, integral, solving and equation or plotting a graph. The rest of the work was done on paper. In the end both groups were given the same conceptual and computational exams created by both lecturers. The results of this study reveal that the score on conceptual knowledge and computational examinations of the students who were taught calculus using a computer algebra system was higher than that of those who were taught using paper and pencil computations.

Rochowicz (1996) administered and analysed 89 questionnaires from calculus instructors and innovators from engineering and pre-engineering schools pertaining their perception on the impact of using computers and calculators on calculus instruction based on ‘calculus, student motivation, student learning, and the role of the lecturer’. He reported that there appeared to be a shift in the focus of learning, “from symbolic algebraic and skills to more interpretation, approximation, graphing, and modelling of realistic situations” (Rochowicz ,1996, p. 390), which is an important aspect for deep learning to be possible. The impact on student motivation was uncertain. In relation to student learning, it was revealed that learning improves as a more active environment is created with the use of technology and that visualisation enhances learning (Rochowicz, 1996, p. 392), even though in relation to the impact on the lecturer, the use of the computer requires more time from the instructor which is more creative and meaningful (Rochowicz 1996, p. 3).

In his study, Meagher (2005) conducted qualitative case studies focusing on college students learning calculus using Mathematica. The results of his study reveals that the most significant representations in mathematics, numerical, graphical, and algebraic can now be instrumentalised with technology, and pedagogy aiming to use technology can take advantage of this instrumentalisation (Meagher, 2005, p. 177). However, some of the students felt that the computer was doing the mathematics for them since they were giving it instructions and they sometimes felt that they want to do the mathematics on their own without the computer. They complained that instead of learning calculus, they learnt how to use Mathematica which was making the calculations easier. Students felt that too much time was spent on Mathematica not mathematics and that too many of their questions were technical (about Mathematica) rather than conceptual (about mathematics) (Meagher, 2005, p. 182).

In contrast to this study some advantages are reported on using technology. Nilklad (2004, p. 212) highlights that “the incorporation of graphing tools in the curriculum does support students’ visualisation of functions because the graphing tools help them understand more abstract views of functions”. Noinang, Wiwatanapataphee and Wu (2008) also assert that the use of CAS help in developing students’ logical/analytical reasoning by visually supporting calculus concepts like integration to be learnt with graphics. Clements et al. (1997) point out that learning by CAS creates an environment that is motivating and meaningful to the students as well as allowing students of different abilities to use a variety of approaches to solve problems. CAS also improves spatial skills (Kaufmann & Schmalstieg, 2003).

Bressoud (2001), talks about the debates that took place (around 1980) in order to come up with innovative approaches to calculus instruction in undergraduate mathematics. One of the suggestions was that when learning using a computer, key ideas should be treated graphically, numerically and symbolically and that writing should be used to foster *critical thinking* (Bressoud, 2001, p. 579). Presently Autograph and Geogebra (available online) are used during mathematics lessons to enhance visualisation including drawing different graphs and demonstrations of the Riemann sum. A comment from a teacher was that Autograph enabled learners to visualise how a “2D shape can rotate to generate a 3D shape” (McMahon, 2012, p. 3). Autograph’s unique 3D interface was also found to be useful in aiding students aged 16 -19 to visualise a volume of revolution (Barton, 2009).

## 2.7 CONCLUSION

Since the focus of this study is on students' learning difficulties involving VSOR, the importance of visual learning was discussed. The literature survey done was based on the five categories and the contextual factors affecting learning of VSOR. What one can gather from the debates and studies above and reflecting on my study is that in learning a topic such as VSOR, students have to develop critical thinking. Diagrams are seen as a starting point to aid students to learn visually. However, most students tend to avoid using diagrams and prefer to use algebraic representation where they tend to calculate even if the solution to the problem given does not require calculation or can be interpreted visually. The majority of students were seen to perform better in problems that require procedural skills, and failed when they had to visualise, requiring conceptual skills. With VSOR the students are expected to draw graphs, interpret the graphs (from the Riemann sums and after rotating them). The use of the Riemann sum and the translation from 2D to 3D were difficult aspects for most students. The threshold concepts in integration were also seen as problematic to most students. In some studies, students struggled to understand how different rotations give rise to different methods for calculating volume as well as drawing the 3D diagrams formulated.

The importance of writing and language used was also shown. In this study in particular, students are assessed in writing in order to investigate their thinking processes. The importance of the use of CAS has also been discussed to show its merit in visual learning. The debates and studies above emphasise the importance of making mathematics real and accessible, using different levels of representation, also highlighting a shift from *verbal*, *symbolic* and *numerical* representation towards visual learning, especially when learning using the CAS. Visual instruction was seen to provide room to engage students with meanings, which are not always possible when they learn symbolically and verbally. In my study the use of CAS can be useful, but there is reluctance in using it due to shortage of resources at most colleges. Even though some of the studies discussed in this section involve younger children, their mathematical foundation might affect performance at higher levels.

In sections that follow, the conceptual framework, the methods of data collection and analysis are discussed, followed by the presentation, analysis and interpretation of the results in order to answer the research question of this study. Finally the conclusions and recommendations are made.