Part II

Convergence Spaces and Generalized Functions

Chapter 6

Initial Uniform Convergence Spaces

6.1 Initial Uniform Convergence Structures

As mentioned in Section 2.4, uniform spaces, and more generally uniform convergence spaces, appear in many important applications of topology, and in particular analysis. In this regard, the concepts of completeness and completion of a uniform convergence space play a central role. Indeed, Baire's celebrated Category Theorem asserts that a *complete* metric space cannot be expressed as the union of a countable family of closed nowhere dense sets. The importance of this result is demonstrated by the fact that the Banach-Steinhauss Theorem, as well as the Closed Graph Theorem in Banach spaces follow from it.

However, in many situations one deals with a space X which is *incomplete*, and in these cases one may want to construct the *completion* of X. In this regard, the main result, see for instance [63], [64] and [161] and Section 2.4, is that every Hausdorff uniform convergence space X may be uniformly continuously embedded into a *complete*, Hausdorff uniform convergence space $X^\sharp$ in a unique way such that the image of X in $X^\sharp$ is dense. Moreover, the following *universal property* is satisfied. For every complete, Hausdorff uniform convergence space Y, and any uniformly continuous mapping

$$\varphi : X \to Y$$

the diagram

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[dr, "\iota_X"'] & & Y \\
& X^\sharp \arrow[ur, "\exists ! \varphi^\sharp"'] &
\end{tikzcd} \tag{6.1}$$

commutes, with $\varphi^\sharp$ uniformly continuous, and ι_X the canonical embedding of X into its completion $X^\sharp$.

It is often not only the completion $X^\sharp$ of a uniform convergence space X that is of interest, but also the extension $\varphi^\sharp$ of uniformly continuous mappings from X to $X^\sharp$. In this regard, we recall that one of the major applications of uniform spaces, and recently also uniform convergence spaces, is to the solutions of PDEs. Indeed, let us consider a PDE

$$Tu = f, \tag{6.2}$$

with T a possibly nonlinear partial differential operator which acts on some relatively small space X of classical functions, u the unknown function, while the right hand term f belongs to some space Y. One usually considers some uniformities, or more generally uniform convergence structures, on X and Y in such a way that the mapping

$$T : X \rightarrow Y \tag{6.3}$$

is uniformly continuous. It is well known that the equation (6.2), or typically some suitable extension of it, can have solutions of *physical interest* which, however, may fail to be *classical*, in the sense that they do not belong to X. From here, therefore, the particular interest in *generalized solutions* to (6.2). Such generalized solutions to (6.2) may be obtained by constructing the completions $X^\sharp$ and $Y^\sharp$ of X and Y, respectively. The mapping (6.3) extends uniquely to a mapping

$$T^\sharp : X^\sharp \rightarrow Y^\sharp \tag{6.4}$$

so that the diagram

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\iota_X"'] & Y \arrow[d, "\iota_Y"] \\ X^\sharp \arrow[r, "T^\sharp"] & Y^\sharp \end{tikzcd}$$

commutes, with ι_X and ι_Y the uniformly continuous embeddings associated with the completions $X^\sharp$ and $Y^\sharp$ of X and Y, respectively. One may now consider the *extended* equation

$$T^\sharp u^\sharp = f \tag{6.5}$$

where the solutions of (6.5) are interpreted as generalized solutions of (6.2). Note that the *existence* and *uniqueness* of generalized solutions depend on the properties of the mapping $T^\sharp$ and the uniform convergence structure on $X^\sharp$ and $Y^\sharp$, as opposed to the *regularity* of the generalized solutions, which may be interpreted as the extent to which a generalized solution exhibits characteristics of classical solutions, which depends on the properties of the elements of the space $X^\sharp$. It is therefore clear that not only the *completion* $X^\sharp$ of a u.c.s. X, but also the the associated *extensions* of uniformly continuous mappings, defined on X, are of interest.

The example given above indicates a particular point of interest. The uniform convergence structure $\mathcal{J}_X$ on the domain X of the PDE operator T is usually defined as the *initial* uniform convergence structure [26] with respect to some uniform convergence structure $\mathcal{J}_Y$ on Y, and a family of mappings

$$(\psi_i : X \to Y)_{i\in I} \tag{6.6}$$

In the case of PDEs, the mappings ψ_i are typically usual partial differential operators, up to a given order m. A natural question arises as to the connection between the completion of X, and the completion of Y. More generally, consider a set X, a family of mappings

$$(\psi_i : X \to X_i)_{i\in I}$$

where each X_i is a uniform convergence space. If the family $(\psi_i)_{i\in I}$ separates the points of X, then the initial uniform convergence structure on X with respect to the family of mappings (6.6) is also Hausdorff, and we may consider its completion

$X^\sharp$. It appears that the issue of the possible connections between the completion of X and that of the spaces X_i, respectively, has not yet been fully explored. We aim to clarify the possible connection between the completion $X^\sharp$ of X, and the completions $X_i^\sharp$ of the X_i.

6.2 Subspaces of Uniform Convergence Spaces

It can easily be shown that the Bourbaki completion of a uniform space X preserves subspaces. In particular, the completion $Y^\sharp$ of any subspace Y of X is isomorphic to a subspace of the completion $X^\sharp$ of X. For uniform convergence spaces in general, and the associated Wyler completion, this is not the case. In this regard, consider the following[1].

Example 35 *Consider the real line $\mathbb{R}$ equipped with the uniform convergence structure associated with the usual uniformity on $\mathbb{R}$. Also consider the set $\mathbb{Q}$ of rational numbers equiped with the subspace uniform convergence structure induced from $\mathbb{R}$. The Wyler completion $\mathbb{Q}^\sharp$ of $\mathbb{Q}$ is the set $\mathbb{R}$ equipped with a suitable uniform convergence structure. As such, the inclusion mapping $i : \mathbb{Q} \to \mathbb{R}$ extends to a uniformly continuous bijection*

$$i^\sharp : \mathbb{Q}^\sharp \to \mathbb{R} \tag{6.7}$$

Furthermore, a filter $\mathcal{F}$ on $\mathbb{Q}^\sharp$ converges to $x^\sharp$ if and only if

$$[\mathcal{V}\left(x^\sharp\right)_{|\mathbb{Q}}] \cap [x^\sharp] \subseteq \mathcal{F}$$

where $\mathcal{V}\left(x^\sharp\right)$ is the neighborhood filter in $\mathbb{R}$ at $x^\sharp$, and $\mathcal{V}\left(x^\sharp\right)_{|\mathbb{Q}}$ denotes its trace on $\mathbb{Q}$. As such, it is clear that the neighborhood filter at $x^\sharp$ does not converge in $\mathbb{Q}^\sharp$. Therefore the mapping (6.7) does not have a continuous inverse, so that it is not an embedding.

In view of Example 35, it is clear that Wyler completion does not preserve subspaces. The underlying reason for this phenomenon is is twofold. In the first place, and as mentioned in Section 2.3, the adherence operator on a convergence space is in general not idempotent. Furthermore, and perhaps more fundamentally, for a subset Y of a set X, and a filter $\mathcal{F}$ on X, we have the inclusion

$$\mathcal{F} \subseteq [\mathcal{F}_{|Y}]_X,$$

with equality only holding in case $Y \in \mathcal{F}$. In terms of the underlying set associated with the uniform convergence space completion $Y^\sharp$ of a subspace Y of a uniform convergence space X, we may still say something. In particular, we have the following.

[1]This example was communicated to the author by Prof. H. P. Butzmann

Proposition 36 *Let Y be a subspace of the uniform convergence space X. Then there is an injective, uniformly continuous mapping*

$$i^{\sharp} : Y^{\sharp} \to X^{\sharp}$$

which extends the inclusion mapping $i : Y \to X$. In particular,

$$i^{\sharp}\left(Y^{\sharp}\right) = a_{X^{\sharp}}\left(\iota_X\left(Y\right)\right).$$

Proof. In view of the fact that the inclusion mapping $i : Y \to X$ is a uniformly continuous embedding, we obtain a uniformly continuous mapping

$$i^{\sharp} : Y^{\sharp} \to X^{\sharp} \tag{6.8}$$

so that the diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "\iota_Y"'] & X \arrow[d, "\iota_X"] \\ Y^{\sharp} \arrow[r, "i^{\sharp}"] & X^{\sharp} \end{tikzcd} \tag{6.9}$$

commutes. To see that the mapping (6.8) is injective, consider any $y_0^{\sharp}, y_1^{\sharp} \in Y^{\sharp}$ and suppose that

$$i^{\sharp}\left(y_0^{\sharp}\right) = i^{\sharp}\left(y_1^{\sharp}\right) = x^{\sharp} \tag{6.10}$$

for some $x^{\sharp} \in X^{\sharp}$. Since $\iota_Y\left(Y\right)$ is dense in $Y^{\sharp}$ there exists Cauchy filters $\mathcal{F}$ and $\mathcal{G}$ on Y such that $\iota_Y\left(\mathcal{F}\right)$ converges to $y_0^{\sharp}$ and $\iota_Y\left(\mathcal{G}\right)$ converges to $y_1^{\sharp}$. From the diagram above it follows that $\iota_X\left(i\left(\mathcal{F}\right)\right)$ and $\iota_X\left(i\left(\mathcal{G}\right)\right)$ converges to $x^{\sharp}$. Therefore the filter

$$\mathcal{H} = \iota_X\left(i\left(\mathcal{F}\right)\right) \cap \iota_X\left(i\left(\mathcal{G}\right)\right)$$

converges to $x^{\sharp}$ in $X^{\sharp}$. Note that the filter

$$i^{-1}\left(\iota_X^{-1}\left(\mathcal{H}\right)\right)$$

is a Cauchy filter on Y so that $\iota_Y\left(i^{-1}\left(\iota_X^{-1}\left(\mathcal{H}\right)\right)\right)$ must converge in $Y^{\sharp}$ to some $y^{\sharp}$. But $\iota_Y\left(i^{-1}\left(\iota_X^{-1}\left(\mathcal{H}\right)\right)\right) \subseteq \iota_Y\left(\mathcal{F}\right)$ and $\iota_Y\left(i^{-1}\left(\iota_X^{-1}\left(\mathcal{H}\right)\right)\right) \subseteq \iota_Y\left(\mathcal{G}\right)$ so that $\iota_Y\left(\mathcal{F}\right)$ and $\iota_Y\left(\mathcal{G}\right)$ must converge to $y^{\sharp}$ as well. Since $Y^{\sharp}$ is Hausdorff it follows by (6.10) that

$y_0^\sharp = y_1^\sharp = y^\sharp$. Therefore $i^\sharp$ is injective.
Clearly $i^\sharp\left(Y^\sharp\right) \subseteq a_{X^\sharp}\left(\iota_X\left(Y\right)\right)$. To verify the reverse inclusion, consider any $x^\sharp \in a_{X^\sharp}\left(\iota_X\left(Y\right)\right)$. Then

$$\begin{array}{ll} \exists & \mathcal{F} \text{ a filter on } \iota_X\left(Y\right) : \\ & [\mathcal{F}]_{X^\sharp} \text{ converges to } x^\sharp \text{ in } X^\sharp \end{array}.$$

Then there is a Cauchy filter $\mathcal{G}$ on X so that

$$\iota_X\left(\mathcal{G}\right) \cap [x^\sharp] \subseteq [\mathcal{F}]_{X^\sharp}$$

This implies that the Cauchy filter $\mathcal{G}$ has a trace $\mathcal{H} = \mathcal{G}_{|Y}$ on Y, which is a Cauchy filter on Y. The result now follows by the commutative diagram (6.9). ∎

The following is an immediate consequence of Proposition 36.

Corollary 37 *Let X and Y be uniform convergence spaces, and $\varphi : X \to Y$ a uniformly continuous embedding. Then there exists an injective uniformly continuous mapping $\varphi^\sharp : X^\sharp \to Y^\sharp$, where $X^\sharp$ and $Y^\sharp$ are the completions of X and Y respectively, which extends F.*

It should be noted that Wyler completion is minimal, with respect to inclusion, among complete, Hausdorff uniform convergence on the set $X^\sharp$, as demonstrated in the following proposition. We may obtain this as an easy conseqeunce of the universal property (6.1) and Corollary 37.

Proposition 38 *Consider a Hausdorff uniform convergence space X. For any complete, Hausdorff uniform convergence space $X_0^\sharp$ that contains X as a dense subspace, there is a bijective and uniformly continuous mapping*

$$\iota_{X,0}^\sharp : X^\sharp \to X_0^\sharp.$$

Proof. Let $X_0^\sharp$ be a complete, Hausdorff uniform convergence space that contains X as a dense subspace, so that the inclusion mapping

$$i : X \ni x \mapsto x \in X_0^\sharp \tag{6.11}$$

is a uniformly continuous embedding. It follows from Corollary 37 that the mapping (6.11) extends to an injective uniformly continuous mapping

$$i : X^\sharp \ni x^\sharp \mapsto i^\sharp\left(x^\sharp\right) \in X_0^\sharp. \tag{6.12}$$

It remains to verify that the mapping (6.12) is surjective. In this regard, consider any $x_0^\sharp \in X_0^\sharp$. Since X is dense in $X_0^\sharp$, there is a Cauchy filter $\mathcal{F}$ on X so that $[\mathcal{F}]_{X_0^\sharp}$ converges to $x_0^\sharp$ in $X_0^\sharp$. As such, there exists $x^\sharp \in X^\sharp$ so that $[\mathcal{F}]_{X^\sharp}$ converges to $x^\sharp$. Therefore $i^\sharp\left([\mathcal{F}]_{X^\sharp}\right)$ converges to $x_0^\sharp$ in $X_0^\sharp$ so that $i^\sharp\left(x^\sharp\right) = x_0^\sharp$. This completes the proof. ∎

For a subspace Y of a Hausdorff uniform convergence space X, this leads to the following.

Corollary 39 *Let Y be a subspace of the Hausdorff uniform convergence space X. The uniform convergence structure on the Wyler completion $Y^\sharp$ of Y is the finest complete, Hausdorff uniform convergence structure on $a_{X^\sharp}(Y)$ so that Y is contained in it as a dense subspace.*

Remark 40 *It should be noted, and as mentioned in Section 2.4, that the completion of a convergence vector space [65], the completion of a convergence group [61], and the Wyler completion [161] of a uniform convergence space are in general all different. Indeed, the Wyler completion is typically not compatible with the algebraic structure of a convergence group or convergence vector space [26], [65], while the convergence group completion of a convergence vector space does in general not induce a vector space convergence structure [21].*

6.3 Products of Uniform Convergence Spaces

In this section we consider the completion of the product of a family of uniform convergence spaces. In contradistinction with subspaces of a uniform convergence space, products of uniform convergence spaces are well behaved with respect to the Wyler completion. In particular, it is well known [161] that the product of complete, Hausdorff uniform convergence structures are complete and Hausdorff. Furthermore, we obtain the following result.

Theorem 41 *Let $(X_i)_{i\in I}$ be a family of Hausdorff uniform convergence spaces, and let X denote their Cartesian product equipped with the product uniform convergence structure. Then the completion $X^\sharp$ of X is the product of the completions $X_i^\sharp$ of the X_i.*

Proof. First note that $\prod_{i\in I} X_i^\sharp$ is complete. For every i, let $\iota_{X_i} : X_i \rightarrow X_i^\sharp$ be the uniformly continuous embedding associated with the completion $X_i^\sharp$ of X_i. Define the mapping $\iota_X : X \rightarrow \prod X_i^\sharp$ through

$$\iota_X : x = (x_i)_{i\in I} \mapsto (\iota_{X_i}(x_i))_{i\in I}$$

For each i, let $\pi_i : X \rightarrow X_i$ be the projection. Since each ι_{X_i} is injective, so is ι_X. Moreover, we have

$$\begin{aligned}
\mathcal{U} \in \mathcal{J}_X \quad &\Rightarrow (\pi_i \times \pi_i)(\mathcal{U}) \in \mathcal{J}_{X_i} \\
&\Rightarrow (\iota_{X_i} \times \iota_{X_i})((\pi_i \times \pi_i)(\mathcal{U})) \in \mathcal{J}^\sharp_{X_i} \\
&\Rightarrow \prod_{i\in I} (\iota_{X_i} \times \iota_{X_i})((\pi_i \times \pi_i)(\mathcal{U})) \in \mathcal{J}^\sharp_{\prod} \\
&\Rightarrow (\iota_X \times \iota_X)(\mathcal{U}) \in \mathcal{J}^\sharp_{\prod}
\end{aligned}$$

where $\mathcal{J}^\sharp_{\prod}$ denotes the product uniform convergence structure on $\prod_{i\in I} X_i^\sharp$. Hence ι_X is uniformly continuous. Similarly, if the filter $\mathcal{V}$ on $\iota_X(X) \times \iota_X(X)$ belongs to the subspace uniform convergence structure, then

$$\begin{aligned}(\pi_i \times \pi_i)(\mathcal{V}) \in \mathcal{J}^\sharp_{X_i} &\Rightarrow \left(\iota^{-1}_{X_i} \times \iota^{-1}_{X_i}\right)((\pi_i \times \pi_i)(\mathcal{V})) \in \mathcal{J}_{X_i} \\ &\Rightarrow \prod_{i \in I} \left(\iota^{-1}_{X_i} \times \iota^{-1}_{X_i}\right)((\pi_i \times \pi_i)(\mathcal{V})) \in \mathcal{J}_X \\ &\Rightarrow \left(\iota^{-1}_X \times \iota^{-1}_X\right)(\mathcal{V}) \in \mathcal{J}_X\end{aligned}$$

so that ι^{-1}_X is uniformly continuous. Hence ι_X is a uniformly continuous embedding. That $\iota_X(X)$ is dense in $\prod_{i \in I} X^\sharp_i$ follows by the denseness of $\iota_{X_i}(X_i)$ in $X^\sharp_i$, for each $i \in I$. The extension property of uniformly continuous mappings into a complete u.c.s. follows in the standard way. ■

6.4 Completion of Initial Uniform Convergence Structures

In view of the fact that the Wyler completion of uniform convergence spaces do not, in general, preserve subspace, initial structures are not invariant under the formation of completions. That is, if X carries the initial uniform convergence structure with respect to a family of mappings

$$(\psi_i : X \to X_i)_{i \in I}$$

into u.c.s.s X_i, then the completion $X^\sharp$ of X does not necessarily carry the initial uniform convergence structure with respect to

$$\left(\psi^\sharp_i : X^\sharp \to X^\sharp_i\right)_{i \in I}$$

where $\psi^\sharp_i$ denotes the uniformly continuous extension of $\psi^\sharp_i$ to $X^\sharp$. In this regard, one can only obtain a generalization of Proposition 36. The first, and in fact straight forward, result in this regard is the following.

Proposition 42 *Suppose that X is equipped with the initial uniform convergence structure with respect to a family of mappings*

$$(\varphi_i : X \to X_i)_{i \in I}, \tag{6.13}$$

where each uniform convergence space X_i is Hausdorff, and the family of mappings (8.22) separates the points on X. Then each mapping φ_i extends uniquely to a uniformly continuous mapping

$$\varphi^\sharp_i : X^\sharp \to X^\sharp_i \tag{6.14}$$

and the uniform convergence structure on $X^\sharp$ is finer than the initial uniform convergence structure with respect to the mappings (6.14).

Proof. It follows by the universal property (6.1) that each of the mappings (8.22) extend to a uniformly continuous mapping (6.14). From the continuity of the mappings (6.14) it follows that the uniform convergence structure on $X^\sharp$ is finer than the initial uniform convergence structure with respect to the mapping of mappings (6.14). ■

In connection with the actual uniform convergence structure on the set $X^\sharp$, we cannot in general make a stronger claim. However, it is possible to describe the structure of the set $X^\sharp$ itself in terms of the completions of the X_i. In this regard, we first note that the uniform convergence structure on X may be described in terms of the product uniform convergence structure on $\prod_{i \in I} X_i$.

Proposition 43 *For each $i \in I$, let X_i be a Hausdorff uniform convergence space, with uniform convergence structure $\mathcal{J}_{X_i}$. Let the uniform convergence space X carry the initial uniform convergence structure $\mathcal{J}_X$ with respect to the family of mappings*

$$(\psi_i : X \to X_i)_{i \in I}$$

Assume that $(\psi_i)_{i \in I}$ separates the points of X. Then there exists a unique uniformly continuous embedding

$$\Psi : X \to \prod_{i \in I} X_i \tag{6.15}$$

such that, for each $i \in I$, the diagram

$$\begin{tikzcd} X \arrow[rr, "\psi_i"] \arrow[dr, "\Psi"'] & & X_i \\ & \prod X_i \arrow[ur, "\pi_i"'] & \end{tikzcd}$$

commutes, with π_i the projection.

Proof. Define the mapping Ψ as

$$\Psi : X \ni x \mapsto (\psi_i(x))_{i \in I} \in \prod_{i \in I} X_i \tag{6.16}$$

Since the family $(\varphi_i)_{i\in I}$ seperates the points of X, the mapping (6.16) is injective. Furthermore, the diagram

$$\begin{tikzcd} X \arrow[rr, "\psi_i"] \arrow[dr, "\Psi"'] & & X_i \\ & \prod X_i \arrow[ur, "\pi_i"'] & \end{tikzcd} \tag{6.17}$$

commutes for every $i \in I$. Suppose that $\mathcal{U} \in \mathcal{J}_X$. Then

$$\begin{array}{l} \forall \quad i \in I : \\ \quad (\psi_i \times \psi_i)(\mathcal{U}) \in \mathcal{J}_{X_i} : \end{array}$$

and hence

$$\begin{array}{l} \forall \quad i \in I : \\ \quad (\pi_i \times \pi_i)(\Psi \times \Psi)(\mathcal{U}) \in \mathcal{J}_{X_i} : \end{array} .$$

Therefore $(\Psi \times \Psi)(\mathcal{U}) \in \mathcal{J}_{\prod}$, which is the product uniform convergence structure, so that Ψ is uniformly continuous.
Let $\mathcal{V} \in \mathcal{J}_{\prod}$ be a filter on $\prod_{i\in I} X_i \times \prod_{i\in I} X_i$ with a trace on $\Psi(X) \times \Psi(X)$. Then

$$\begin{array}{l} \forall \quad i \in I : \\ \quad \text{a) } (\pi_i \times \pi_i)(\mathcal{V}) \in \mathcal{J}_{X_i} \\ \quad \text{b) } W \in (\pi_i \times \pi_i)(\mathcal{V}) \Rightarrow W \cap (\psi_i(X) \times \psi_i(X)) \neq \emptyset \end{array}$$

so that

$$\begin{array}{l} \forall \quad i \in I : \\ \quad (\psi_i \times \psi_i)((\Psi^{-1} \times \Psi^{-1})(\mathcal{V})) \supseteq (\pi_i \times \pi_i)(\mathcal{V}) \end{array}$$

Form the definition of an initial uniform convergence structure, and in particular the product uniform convergence structure, it follows that $(\Psi^{-1} \times \Psi^{-1})(\mathcal{V}) \in \mathcal{J}_X$. Hence Ψ is a uniformly continuous embedding. The uniqueness of the mapping Ψ is obvious from the construction of Ψ. ∎

The following now follows as an immediate consequence of Proposition 43.

Theorem 44 *For each $i \in I$, let X_i be a Hausdorff uniform convergence space, with uniform convergence structure $\mathcal{J}_{X_i}$. Let the uniform convergence space X carry the initial uniform convergence structure $\mathcal{J}_X$ with respect to the family of mappings*

$$(\psi_i : X \to X_i)_{i\in I}$$

Assume that $(\psi_i)_{i\in I}$ separates the points of X. Then there exists a unique injective, uniformly continuous mapping

$$\Psi^\sharp : X^\sharp \to \prod_{i\in I} X_i^\sharp \tag{6.18}$$

such that, for each $i \in I$, the diagram

$$\begin{tikzcd}
X^\sharp \arrow[rr, "\psi_i^\sharp"] \arrow[dr, "\Psi^\sharp"'] & & X_i^\sharp \\
& \prod X_i^\sharp \arrow[ur, "\pi_i"'] &
\end{tikzcd}$$

commutes, with π_i the projection, and $\psi_i^\sharp$ the unique extension of ψ_i to $X^\sharp$.

Proof. The result follows by Proposition 36, Theorem 41 and Proposition 43. ∎

Within the context of nonlinear PDEs, as explained in Section 6.1, Theorem 44 may be interpreted as a regularity result. Indeed, consider some space $X \subseteq \mathcal{C}^\infty(\Omega)$ of classical, smooth functions on an open, nonempty subset Ω of $\mathbb{R}^n$. Equip X with the initial uniform convergence structure $\mathcal{J}_X$ with respect to the family of mappings

$$D^\alpha : X \to Y,\ \alpha \in \mathbb{N}^n \tag{6.19}$$

where Y is some space of functions on Ω that contains $D^\alpha(X)$ for each $\alpha \in \mathbb{N}^n$. In view of Theorem 44, the mapping

$$\mathbf{D} : X \ni u \to (D^\alpha u) \in Y^{\mathbb{N}} \tag{6.20}$$

is a uniformly continuous embedding, and as such (6.20) extends to an injective uniformly continuous mapping

$$\mathbf{D}^\sharp : X^\sharp \ni u \to (D^\alpha u) \in Y^{\sharp\mathbb{N}} \tag{6.21}$$

so that the diagram

$$
\begin{tikzcd}
X^{\sharp} \arrow[rr, "D^{\alpha\sharp}"] \arrow[dr, "\mathbf{D}^{\sharp}"'] & & Y^{\sharp} \\
& Y^{\sharp\mathbb{N}} \arrow[ur, "\pi_i"'] &
\end{tikzcd}
\tag{6.22}
$$

commutes. Here

$$D^{\alpha\sharp} : X^{\sharp} \to Y^{\sharp},\ \alpha \in \mathbb{N}^n$$

are the uniformly continuous extension of the mappings (6.19). As such, each *generalized function* $u^{\sharp} \in X^{\sharp}$ may be identified with $\mathbf{D}^{\sharp}u^{\sharp} \in Y^{\sharp\mathbb{N}}$.

The above interpretation of the completion of a uniform convergence space which is equipped with an initial structure is central to the theory of the solutions of nonlinear PDEs presented in the chapters to follow. In particular, we employ exactly the construction (6.22) to obtain our first and basic regularity properties for the solutions of such systems of equations.

Chapter 7

Order Convergence on $\mathcal{ML}(X)$

7.1 Order Convergence and the Order Completion Method

We may recall from Section 1.4 that our approach to the enrichment of the Order Completion Method [119] is motivated by the fact that the process of taking the supremum of a subset A of a partially ordered set X is essentially a process of approximation. Such approximation-type statements, and in particular the process of forming the Dedekind completion of a partially ordered set, may be reformulated in terms of topological type structures, which may turn out to be more general than the usual Hausdoff-Kuratowski-Bourkabi concept of topology.

In this regard, and as mentioned in Chapter 4, there are several useful modes of convergence on a partially ordered set which are defined in terms of the partial order, see for instance [29], [101] and [124]. A particularly relevant concept is that of the order convergence of sequences defined on a partially ordered set through (4.6). In general, and as mentioned in Section 4.2, there is no topology on a partially ordered set X that induces the order convergence of sequences. That is, for a partially ordered set X there is in general no topology τ on X such that the τ-convergent sequences are exactly the order convergent sequences. However, the more general context of convergence structures and convergence spaces provides an adequate setting within which to describe the order convergence of sequences. Namely, if X is a σ-distributive lattice, then the convergence structure (4.8) induces the order convergence of sequences.

In particular, and as is discussed in Section 4.2, every Archimedean vector lattice is fully distributive, and hence σ-distributive. In this case the convergence structure (4.8) is a vector space convergence structure, and as such it is induced by a uniform convergence structure [26]. In this case, the Cauchy filters may be defined through

$$\mathcal{F} \text{ a Cauchy filter on } X \;\Leftrightarrow \mathcal{F} - \mathcal{F} \in \lambda_o(0)\,.$$

Furthermore, the convergence vector space completion of an Archimedean vector lattice X, equipped with the order convergence structure λ_o, may be constructed as the Dedekind σ-completion $X^\sharp$ of X, equipped with the order convergence structure. If X is order separable, the completion of X is in fact its Dedekind completion. In the particular case when $X = \mathcal{C}(Y)$, with Y a metric space, the convergence vector space completion is the set $\mathbb{H}_{ft}(X)$ of finite Hausdorff continuous functions on Y, which is the Dedekind completion of $\mathcal{C}(Y)$.

Let us now consider the possibility of applying the above results to the problem of solving nonlinear PDEs through the Order Completion Method. In this regard, consider a nonlinear PDE of the form (1.100), and the associated mapping

$$T : \mathcal{M}^m(\Omega) \to \mathcal{M}^0(\Omega)$$

The Order Completion Method is based on the abundance of *approximate solutions* to (1.100), which are elements of $\mathcal{M}^m(\Omega)$, and in general one cannot expect these approximations to be continuous, let alone *sufficiently smooth*, on the whole of Ω. Moreover, the space $\mathbb{H}_{ft}(\Omega)$ does not contain the space $\mathcal{M}^0(\Omega)$.

On the other hand, the space $\mathcal{M}^0(\Omega)$ is an order separable Archimedean vector lattice [119], and therefore one may equip it with the order convergence structure. The completion of this space will be its Dedekind completion $\mathcal{M}^0(\Omega)^\sharp$, as desired. However, there are several obstacles to applying the theory of the order convergence structure to the Order Completion Method. If one equips $\mathcal{M}^m(\Omega)$ with the subspace convergence structure, then the nonlinear mapping T is not necessarily continuous. Moreover, the quotient space $\mathcal{M}^m_T(\Omega)$ is *not* a linear space, so that the completion process for convergence vector spaces does not apply. It is therefore necessary to develop a *nonlinear* convergence theoretic model for the Dedekind completion of $\mathcal{M}(\Omega)$.

7.2 Spaces of Lower Semi-Continuous Functions

We may recall from Section 3.1 that the notion of a normal lower semi-continuous function, respectively normal upper semi-continuous function, was introduced by Dilworth [47] in connection with the Dedekind completion of spaces of continuous functions. Dilworth introduced the concept for *bounded*, real valued functions. Subsequently the definition was extended to *locally bounded* functions [6]. The definition extends in a straightforward way to extended real valued functions. In particular, a function $u : X \to \overline{\mathbb{R}}$, with X a topological space, is normal lower semi-continuous at $x \in X$ whenever

$$(I \circ S)(u)(x) = u(x) \tag{7.1}$$

It is called normal lower semi-continuous on X if it is normal lower semi-continuous at every $x \in X$. Here I and S are the Lower- and Upper Baire Operators defined through (3.9) and (3.10), respectively. Note that if a function u is real valued and continuous at a point $x \in X$, then it is also normal lower semi-continuous at x.

In analogy with H-continuous interval valued functions, we call a normal lower semi-continuous function u *nearly finite* whenever the set

$$\{x \in X \, : \, u(x) \in \mathbb{R}\}$$

is open and dense in X. We denote the space of all nearly finite normal lower semi-continuous functions by $\mathcal{NL}(X)$. The space $\mathcal{NL}(X)$ is ordered in a pointwise way through

$$\begin{array}{ll} \forall & u, v \in \mathcal{NL}(X) \, : \\ & u \leq v \Leftrightarrow \left(\begin{array}{ll} \forall & x \in X : \\ & u(x) \leq v(x) \end{array} \right) \end{array} \tag{7.2}$$

The space $\mathcal{NL}(X)$ is the fundamental space upon which a convergence theoretic approach to nonlinear PDEs will be constructed. In this regard, the following basic order theoric properties of this space are fundamental.

Theorem 45 *The space $\mathcal{NL}(X)$ is Dedekind complete. Moreover, if $A \subseteq \mathcal{NL}(X)$ is bounded from above, and $B \subseteq \mathcal{NL}(X)$ is bounded from below, then*

$$\sup A = (I \circ S)(\phi)$$

$$\inf B = (I \circ S \circ I)(\varphi)$$

where

$$\phi : X \ni x \mapsto \sup\{u(x) \ : u \in A\}$$

and

$$\varphi : X \ni x \mapsto \inf\{u(x) \ : u \in B\}$$

Proof. Consider a set $A \subset \mathcal{NL}(X)$ which is bounded from above. Then it follows by (3.17) and (3.15) that the function $u_0 = (I \circ S)(\varphi)$ is nearly finite and normal lower semi-continuous. Furthermore, u_0 is an upper bound for A, that is,

$$\begin{array}{ll} \forall & u \in A : \\ & u \leq u_0 \end{array} .$$

Now suppose that u_0 is not the least upper bound of A. That is, we assume

$$\begin{array}{ll} \exists & w \in \mathcal{NL}(X) \, : \\ \forall & u \in A : \\ & u \leq w < u_0 \end{array} . \tag{7.3}$$

Then it follows that

$$\varphi(x) \leq w(x) \, , \tag{7.4}$$

so that (3.15) and (3.17) imply

$$u_0 \leq w$$

which contradicts (7.3).
The existence of a greatest lower bound follows in the same way. ■

We now proceed to establish further properties of the space $\mathcal{NL}(X)$ concerning the pointwise order (7.2). In this regard, the following result generalizes the well known property of continuous functions. If D is a dense subset of X, then

$$\begin{array}{l} \forall \quad u, v \in \mathcal{C}(X) : \\ \left(\begin{array}{ll} \forall & x \in D : \\ & u(x) \leq v(x) \end{array} \right) \Rightarrow u \leq v \end{array} .$$

Proposition 46 *Consider any* $u \in \mathcal{NL}(X)$*. Then there is a set* $R \subseteq X$ *such that* $X \setminus R$ *is of First Baire Category and* u *is continuous at every* $x \in R$*. If* $v \in \mathcal{NL}(X)$ *and* $D \subseteq X$ *is dense in* X*, then*

$$\left(\begin{array}{ll} \forall & x \in D : \\ & u(x) \leq v(x) \end{array} \right) \Rightarrow u \leq v$$

Proof. Consider any $u \in \mathcal{NL}(X)$. Then u is lower semi-continuous on X, and real valued on some open and dense subset D of X. Fix $\epsilon > 0$. We claim

$$\begin{array}{ll} \exists & \Gamma_\epsilon \subset D \text{ closed nowhere dense :} \\ & 0 < S(u)(x) - u(x) < \epsilon,\ x \in D \setminus \Gamma_\epsilon \end{array} . \tag{7.5}$$

In this regard, suppose that there is a nonempty, open subset V of D such that

$$S(u)(x) \geq u(x) + \epsilon,\ x \in V.$$

Since u is lower semi-continuous, so is the function $u + \epsilon$. As such, it follows by (3.11) and (3.20) that

$$u(x) \geq u(x) + \epsilon,\ x \in V,$$

which is a contradiction. As such, the set of points

$$\{x \in D : 0 < S(u)(x) - u(x) < \epsilon\}$$

is dense in D. That it is open follows by the semi-continuity of the functions u and $S(u)$. Then we have

$$u(x) = S(u)(x),\ x \in R = D \setminus \left(\bigcup_{n \in \mathbb{N}} \Gamma_{\frac{1}{n}} \right).$$

As such, and in view of (3.12) and (3.14) it follows that u is upper semi-continuous at every point of R. Since u is both lower semi-continuous and upper semi-continuous on R, it is continuous on R.
Consider now any dense subset D of X, and any $u, v \in \mathcal{NL}(X)$ so that

$$u(x) \leq v(x), \, x \in D.$$

Take any $x \in X$ arbitrary but fixed, and neighborhoods V_1 and V_2 of x. Since D is dense in X there is some $z_0 \in V_1 \cap V_2 \cap D$ so that

$$\inf\{u(y) \,:\, y \in V_1\} \leq u(z_0) \leq v(z_0) \leq \sup\{v(y) \,:\, y \in V_2\}.$$

Since V_1 and V_2 are chosen independent of each other, and that x is arbitrary, we have

$$I(u)(x) \leq S(v)(x), \, x \in X.$$

From (3.13), (3.15) and (3.17) it follows that

$$u = I(I(u)) \leq I(S(v)) = v$$

which completes the proof. ■

Recall from Section 4.2 that the order convergence structure may be defined on an arbitrary lattice. However, this convergence structure induces the order convergence of sequences only on σ-distributive lattices. As such, the following property is essential.

Proposition 47 *The space* $\mathcal{NL}(X)$ *is a fully distributive lattice.*

Proof. Consider any $u, v \in \mathcal{NL}(X)$, and the normal lower semi-continuous function

$$w = (I \circ S)(\varphi)$$

where $\varphi : X \rightarrow \overline{\mathbb{R}}$ is the pointwise supremum of u and v, namely,

$$\varphi : X \ni x \mapsto \sup\{u(x), v(x)\}.$$

Since both u and v are nearly finite, there is some open and dense subset D of X such that φ is finite on D. Note that both u and v must be locally bounded on D. As such, it follows that

$$\begin{array}{ll} \forall & x \in D : \\ \exists & V \in \mathcal{V}_x : \\ \exists & M > 0 : \\ & -M < \varphi(y) < M, \, y \in V \end{array}.$$

Therefore we must have

$$-M \leq w(x) \leq M,\ x \in V$$

so that w is nearly finite. It now follows by Theorem 45 that $w = \sup\{u, v\}$. The existence of $\inf\{u, v\}$ follows in the same way.
Now let us show that $\mathcal{NL}(X)$ is distributive. Consider a set $A \subset \mathcal{NL}(X)$ such that

$$\sup A = u_0$$

For $v \in \mathcal{NL}(X)$ we must show

$$u_0 \wedge v = \sup\{u \wedge v : u \in A\} \tag{7.6}$$

Suppose that (7.6) fails for some $A \subset \mathcal{NL}(X)$ and some $v \in \mathcal{NL}(X)$. That is,

$$\begin{array}{ll} \exists & w \in \mathcal{NL}(X) : \\ & u \in A \Rightarrow u \wedge v \leq w < u_0 \wedge v \end{array} \tag{7.7}$$

Clearly, $u_0, v > w$ so that there is some $u \in A$ such that w is not larger than u. In view of Proposition 46

$$\begin{array}{ll} \exists & V \subseteq X \text{ nonempty, open} : \\ & x \in V \Rightarrow w(x) < u(x) \end{array} \tag{7.8}$$

From (3.13), (3.14), (3.15) and Proposition 45 it follows that

$$(v \wedge u)(x) > w(x),\ x \in V.$$

Hence (7.7) cannot hold. This completes the proof. ■

It is a well known fact that a pointwise bounded subset of $\mathcal{C}(X)$ may fail to be uniformly bounded, even when X is compact. Furthermore, such a pointwise bounded set may not even be bounded with respect to the pointwise order on $\mathcal{C}(X)$. In this regard, consider the following.

Example 48 *Consider the sequence (u_n) of continuous, real valued functions on $\mathbb{R}$, defined through*

$$u_n(x) = \begin{cases} n - n^2|x - \frac{1}{n}| & if \quad |x - \frac{1}{n}| < \frac{1}{n} \\ 0 & if \quad |x - \frac{1}{n}| \geq \frac{1}{n} \end{cases}$$

Clearly the sequence (u_n) is pointwise bounded on $\mathbb{R}$. Indeed,

$$\begin{array}{ll} \forall & x \in \mathbb{R} : \\ \exists & N_x \in \mathbb{N} : \\ & n \geq N_x \Rightarrow u_n(x) = 0 \end{array}$$

which validates our claim. However, in view of

$$u_n\left(\frac{1}{n}\right) = n,\ n \in \mathbb{N}$$

it follows that there cannot be a continuous, real valued function u *on* $\mathbb{R}$ *so that* $u_n \leq u$ *for each* $n \in \mathbb{N}$.

Within the more general setting of spaces of normal lower semi-continuous functions there is quite a strong relationship between pointwise bounded sets and order bounded sets. In particular, we have the following.

Proposition 49 *Consider a set* $\mathcal{A} \subset \mathcal{NL}(X)$ *that satisfies*

$$\begin{array}{ll} \exists & R \subseteq X \text{ a residual set :} \\ \forall & x \in R : \\ & \sup\{u(x) \ : u \in \mathcal{A}\} < +\infty \end{array} . \tag{7.9}$$

If X *is a Baire space, then*

$$\begin{array}{ll} \exists & \mu \in \mathcal{NL}(\Omega) \ : \\ & u \in \mathcal{A} \Rightarrow u(x) \leq \mu(x),\ x \in X \end{array} . \tag{7.10}$$

If X *is a metric space, then*

$$\begin{array}{ll} \exists & \Gamma \subset X \text{ closed nowhere dense :} \\ \exists & \mu \in \mathcal{NL}(X) \ : \\ & \quad 1) \quad \mu \in \mathcal{C}(X \setminus \Gamma) \\ & \quad 2) \quad u \in \mathcal{A} \Rightarrow u \leq \mu \end{array} \tag{7.11}$$

The corresponding result for sets bounded from below is also true.

Proof. Consider the function $\varphi : X \to \overline{\mathbb{R}}$ defined through

$$\varphi(x) = \sup\{u(x) \ : \ u \in \mathcal{A}\},\ x \in X.$$

Since each $u \in \mathcal{A}$ is lower semi-continuous, it follows that φ is lower semi-continuous on X. Moreover, φ is finite on the residual set R. Set

$$\mu(x) = (I \circ S)(\varphi)(x).$$

In view of the fact that $I \circ S$ is idempotent it follows that μ is normal lower semi-continuous and $u \leq \mu$ for every $u \in \mathcal{A}$. We claim that μ is nearly finite. Suppose this were not the case, so that

$$\begin{array}{ll} \exists & V \subset X \text{ nonempty, open :} \\ & x \in V \Rightarrow \mu(x) = +\infty \end{array} . \tag{7.12}$$

Then it follows by the inequality

$$I(S(\varphi)) \leq S(\varphi)$$

that

$$\begin{array}{ll} \forall & x \in V : \\ & S(\varphi)(x) = +\infty \end{array}$$

Then, in view of (3.10), we have

$$\begin{array}{ll} \forall & M > 0 : \\ \forall & x_0 \in V : \\ \forall & W \in \mathcal{V}_{x_0} : \\ \exists & x_M \in V \cap W : \\ & \varphi(x_M) > M \end{array}$$

Since φ is lower semi-continuous, we must have

$$\begin{array}{ll} \exists & D_M \subseteq V \text{ open and dense in } V : \\ & x \in D_M \Rightarrow \varphi(x) > M \end{array}$$

Therefore

$$\varphi(x) = +\infty,\ x \in R' = \bigcap_{M \in \mathbb{N}} D_M$$

Since φ is finite on R, it follows that

$$R \cap V \subseteq V \setminus R'$$

Since X is a Baire space, V is a Baire space in the subspace topology, and $R \cap V$ is residual in V. But R' is clearly also residual in V so that $R \cap V$ is of first Baire category, which is a contradiction. Therefore (7.12) cannot hold. Therefore μ is nearly finite, and we have proven (7.10).
The validity of (7.11) follows by (3.42). $\blacksquare$

The following related result provides a useful connection between pointwise convergence and order convergence in $\mathcal{NL}(X)$.

Proposition 50 *Let X be a Baire space. Consider a decreasing sequence (u_n) in $\mathcal{NL}(X)$ which is bounded from below. Let*

$$u = \inf\{u_n : n \in \mathbb{N}\} \in \mathcal{NL}(\Omega).$$

Then the following holds:

$$\begin{array}{ll} \forall & \epsilon > 0 : \\ \exists & \Gamma_\epsilon \subseteq \Omega \textit{ closed nowhere dense} : \\ & x \in \Omega \setminus \Gamma_\epsilon \Rightarrow \left(\begin{array}{ll} \exists & N_\epsilon \in \mathbb{N} : \\ & u_n(x) - u(x) < \epsilon,\ n \geq N_\epsilon \end{array} \right) \end{array}$$

The corresponding statement for increasing sequences is also true.

Proof. Take $\epsilon > 0$ arbitrary but fixed. We start with the set

$$C = \left\{ x \in X \;\middle|\; \begin{array}{ll} \forall & n \in \mathbb{N} : \\ & u_n,\, u \text{ continuous at } x \end{array} \right\},$$

the complement of which is a set of first Baire category. Hence C is dense. In view of Proposition 46, the set of points

$$C_\epsilon = \left\{ x \in C \;\middle|\; \begin{array}{ll} \exists & N_\epsilon \in \mathbb{N} : \\ & u_n(x) - u(x) < \epsilon,\, n \geq N_\epsilon \end{array} \right\}$$

must be dense in C. From the continuity of u and the u_n on C it follows that

$$\begin{array}{ll} \forall & x_0 \in C_\epsilon : \\ \exists & \delta_{x_0} > 0 : \\ & x \in C,\, \|x - x_0\| < \delta_{x_0} \Rightarrow x \in C_\epsilon \end{array}$$

Since C is dense in X, the result follows. ■

The set $\mathcal{C}_{nd}(X)$ of all functions $u : X \to \mathbb{R}$ that are continuous everywhere except on some closed nowhere dense subset of X, that is,

$$u \in \mathcal{C}_{nd}(X) \Leftrightarrow \left(\begin{array}{ll} \exists & \Gamma_u \subset X \text{ closed nowhere dense} : \\ & u \in \mathcal{C}(X \setminus \Gamma_u) \end{array} \right) \tag{7.13}$$

plays a fundamental role in the theory of Order Completion [119], as discussed in Section 1.4. In particular, one considers the quotient space $\mathcal{M}(X) = \mathcal{C}_{nd}(X)/\sim$, where the equivalence relation $\sim$ on $\mathcal{C}_{nd}(X)$ is defined by

$$u \sim v \Leftrightarrow \left(\begin{array}{lll} \exists & \Gamma \subset X \text{ closed nowhere dense} : & \\ & 1) & x \in X \setminus \Gamma \Rightarrow u(x) = v(x) \\ & 2) & u, v \in \mathcal{C}(X \setminus \Gamma) \end{array} \right) \tag{7.14}$$

The canonical partial order on $\mathcal{M}(X)$ is defined as

$$U \leq V \Leftrightarrow \left(\begin{array}{lll} \forall & u \in U,\, v \in V : & \\ \exists & \Gamma \subset X \text{ closed nowhere dense} : & \\ & 1) & u, v \in \mathcal{C}(X \setminus \Gamma) \\ & 2) & u(x) \leq v(x),\, x \in X \setminus \Gamma \end{array} \right).$$

An order isomorphic representation of the space $\mathcal{M}(X)$, consisting of normal lower semi-continuous functions, is obtained by considering the set

$$\mathcal{ML}(X) = \left\{ u \in \mathcal{NL}(X) \;\middle|\; \begin{array}{ll} \exists & \Gamma \subset X \text{ closed nowhere dense} : \\ & u \in \mathcal{C}(X \setminus \Gamma) \end{array} \right\}$$

The advantage of considering the space $\mathcal{ML}(X)$ rather than $\mathcal{M}(X)$ is that the elements of $\mathcal{ML}(X)$ are actual point valued functions on X, in contradistinction with the elements of $\mathcal{M}(X)$ which are equivalence classes of functions. In particular, the singularity set Γ associated with a function $u \in \mathcal{ML}(X)$, as well as the values of u on Γ are fully specified. Hence the value $u(x)$ of $u \in \mathcal{ML}(X)$ are completely determined. That is, for each $x \in X$, and every $u \in \mathcal{ML}(X)$, the value $u(x)$ of u at x is a well defined element of $\overline{\mathbb{R}}$, which is not the case for an equivalence class in $\mathcal{M}(X)$. Indeed, for every $U \in \mathcal{M}(X)$, and each $x \in X$ one may find $u_1, u_2 \in U$ so that

$$u_1(x) \neq u_2(x).$$

Proposition 51 *The mapping*

$$I_S : \mathcal{M}(X) \ni U \mapsto (I \circ S)(u) \in \mathcal{ML}(X),\ u \in U \tag{7.15}$$

is a well defined order isomorphism.

Proof. First note that, in view of (3.17) and (7.13), the mapping I_S does indeed take values in $\mathcal{ML}(X)$. Now we show that the mapping I_S is well defined. That is, we show that $I_S(U)$ does not depend on the particular representation $u \in U$ that is used in (7.15). In this regard, consider some $U \in \mathcal{M}(X)$ and any $u, v \in U$. Let $\Gamma \subset X$ be the closed nowhere dense set associated with u and v through (7.14). Since Γ is closed, it follows by (3.9), (3.10) and (7.13) that

$$(I \circ S)(u)(x) = (I \circ S)(v)(x),\ x \in X \setminus \Gamma \tag{7.16}$$

Since $X \setminus \Gamma$ is dense in X, it follows by Proposition 46 that equality holds on the whole of X.
It is obvious that the mapping I_S is surjective. Indeed, each element $u \in \mathcal{ML}(X)$ generates an equivalence class U in $\mathcal{ML}(X)$, so that (7.15) and (3.17) implies that $I_S(U) = u$. To see that it is injective, consider any $U, V \in \mathcal{M}(X)$. From (7.14) it follows that

$$\begin{array}{ll} \forall & u \in U,\ v \in V: \\ \exists & A \subseteq X \text{ nonempty, open}: \\ \exists & \epsilon > 0: \\ & \quad 1)\quad x \in A \Rightarrow u(x) < v(x) - \epsilon \\ & \quad 2)\quad u, v \in \mathcal{C}(A) \end{array}$$

so that

$$I_S(U)(x) < I_S(V)(x) - \epsilon,\ x \in A$$

It remains to verify

$$\begin{array}{ll} \forall & U, V \in \mathcal{M}(X): \\ & U \leq V \Leftrightarrow I_S(U) \leq I_S(V) \end{array}$$

The implication

$$U \leq V \Rightarrow I_S(U) \leq I_S(V)$$

follows by (3.20), Proposition 46 and (7.13). Conversely, suppose that $I_S(U) \leq I_S(V)$ for some $U, V \in \mathcal{M}(X)$. The result now follows in the same way as the injectivity of I_S. This completes the proof. ■

The following is immediate.

Corollary 52 *The space $\mathcal{ML}(X)$ is a fully distributive lattice.*

7.3 The Uniform Order Convergence Structure on $\mathcal{ML}(X)$

As a consequence of Corollary 52 one may define the order convergence structure λ_o on the space $\mathcal{ML}(X)$. The order convergence structure induces the order convergence of sequences on $\mathcal{ML}(X)$ and is Hausdorff, regular and first countable. In order to define a uniform convergence structure on $\mathcal{ML}(X)$ that induces the order convergence structure, we introduce the following notation. For any open subset V of X, and any subset F of $\mathcal{ML}(X)$, we denote by $F_{|V}$ the restriction of F to V. That is,

$$F_{|V} = \left\{ v \in \mathcal{ML}(V) \left| \begin{array}{ll} \exists & w \in F : \\ & x \in V \Rightarrow w(x) = v(x) \end{array} \right. \right\}$$

Definition 53 *Let Σ consist of all nonempty order intervals in $\mathcal{ML}(X)$. Let $\mathcal{J}_o$ denote the family of filters on $\mathcal{ML}(X) \times \mathcal{ML}(X)$ that satisfy the following: There exists $k \in \mathbb{N}$ such that*

$$\begin{array}{ll} \forall & i = 1, ..., k : \\ \exists & \Sigma_i = (I_n^i) \subseteq \Sigma : \\ & 1) \quad I_{n+1}^i \subseteq I_n^i,\ n \in \mathbb{N} \\ & 2) \quad ([\Sigma_1] \times [\Sigma_1]) \cap ... \cap ([\Sigma_k] \times [\Sigma_k]) \subseteq \mathcal{U} \end{array} \tag{7.17}$$

where $[\Sigma_i] = [\{I \,:\, I \in \Sigma_i\}]$. Moreover, for every $i = 1, ..., k$ and $V \in \tau_X$ one has

$$\begin{array}{ll} \exists & u_i \in \mathcal{ML}(X) : \\ & \bigcap_{n \in \mathbb{N}} I_{n|V}^i = \{u_i\}_{|V} \end{array} \quad or \quad \bigcap_{n \in \mathbb{N}} I_{n|V}^i = \emptyset \tag{7.18}$$

Before we proceed to establish that the family $\mathcal{J}_o$ of filters on $\mathcal{ML}(X) \times \mathcal{ML}(X)$ does indeed constitute a uniform convergence structure, let us recall the following useful technical lemma.

Lemma 54 **[26] Let X be a set.*

(i) *Consider filters $\mathcal{U}_1, ..\mathcal{U}_n$ and $\mathcal{V}_1, ..., \mathcal{V}_m$ on $X \times X$. Then the filter*

$$(\mathcal{U}_1 \cap ... \cap \mathcal{U}_n) \circ (\mathcal{V}_1 \cap ... \cap \mathcal{V}_m)$$

exists if and only if $\mathcal{U}_i \circ \mathcal{V}_j$ exists for some $i = 1, ..., n$ and $j = 1, ..., m$. In this case, we have

$$(\mathcal{U}_1 \cap ... \cap \mathcal{U}_n) \circ (\mathcal{V}_1 \cap ... \cap \mathcal{V}_m) = \bigcap \{\mathcal{U}_i \circ \mathcal{V}_j \ : \mathcal{U}_i \circ \mathcal{V}_j \ \textit{exists}\}.$$

(ii) *Consider filters $\mathcal{F}_1, \mathcal{F}_2, \mathcal{G}_1$ and $\mathcal{G}_2$ on X. Then $(\mathcal{F}_1 \times \mathcal{F}_2) \circ (\mathcal{F}_G \times \mathcal{G}_2)$ exists if and only if $\mathcal{F}_1 \vee \mathcal{G}_2$ exists. If this is true, then*

$$(\mathcal{F}_1 \times \mathcal{F}_2) \circ (\mathcal{F}_G \times \mathcal{G}_2) = \mathcal{G}_1 \times \mathcal{F}_2.$$

Theorem 55 *The family $\mathcal{J}_o$ of filters on $\mathcal{ML}(X) \times \mathcal{ML}(X)$ constitutes a uniform convergence structure.*

Proof. The first four axioms of Definition 21 are trivially fulfilled, so it remains to verify

$$\begin{array}{ll} \forall & \mathcal{U}, \mathcal{V} \in \mathcal{J}_o : \\ & \mathcal{U} \circ \mathcal{V} \text{ exists } \Rightarrow \mathcal{U} \circ \mathcal{V} \in \mathcal{J}_o \end{array} \tag{7.19}$$

In this regard, take any $\mathcal{U}, \mathcal{V} \in \mathcal{J}_o$ such that $\mathcal{U} \circ \mathcal{V}$ exists, and let $\Sigma_1, ..., \Sigma_k$ and $\Sigma'_1, ..., \Sigma'_l$ be the collections of order intervals associated with $\mathcal{U}$ and $\mathcal{V}$, respectively, through Definition 53. Set

$$\Phi = \{(i, j) \ : \ [\Sigma_i] \circ [\Sigma'_j] \text{ exists}\}$$

Then, by Lemma 54 (i) it follows that

$$\mathcal{U} \circ \mathcal{V} \supseteq \bigcap \{([\Sigma_i] \times [\Sigma_i]) \circ ([\Sigma_j] \times [\Sigma_j]) \ : \ (i, j) \in \Phi\}. \tag{7.20}$$

Now $(i, j) \in \Phi$ if and only if

$$\begin{array}{ll} \forall & m, n \in \mathbb{N} : \\ & I^i_m \cap I^j_n \neq \emptyset \end{array}$$

For any $(i, j) \in \Phi$, set $\Sigma_{i,j} = (I^{i,j}_n)$ where, for each $n \in \mathbb{N}$

$$I^{i,j}_n = \left[\inf\left(I^i_n\right) \wedge \inf\left(I^j_n\right), \sup\left(I^i_n\right) \vee \sup\left(I^j_n\right)\right]$$

Now, using (7.20), we find

$$\mathcal{U} \circ \mathcal{V} \supseteq \bigcap \{[\Sigma_i] \times [\Sigma_j] : \ (i, j) \in \Phi\} \supseteq \bigcap \{[\Sigma_{i,j}] \times [\Sigma_{i,j}] : \ (i, j) \in \Phi\}$$

Clearly each $\Sigma_{i,j}$ satisfies 1) of (7.17). Since $\mathcal{NL}(X)$ is fully distributive, see Proposition 47, (7.18) also holds. This completes the proof. ■

An important fact to note is that the uniform order convergence structure $\mathcal{J}_o$ is defined solely in terms of the order on $\mathcal{ML}(X)$ and the topology on X. This is unusual for a uniform convergence structure on a function space. Indeed, for a space of functions $F(X,Y)$, defined on some set X, and taking values in Y, one defines the uniform convergence structure either in terms of the uniform convergence structure on Y, or in terms of a convergence structure on $F(X,Y)$ which is suitably compatible with the algebraic structure of the space. Indeed, a convergence vector space carries a natural uniform convergence structure, where the Cauchy filters are determined by the linear structure. That is,

$$\mathcal{F} \text{ a Cauchy filter} \Leftrightarrow \mathcal{F} - \mathcal{F} \to 0 \tag{7.21}$$

The motivation for introducing a uniform convergence structure that does not depend on the algebraic structure of the set $\mathcal{ML}(X)$ comes from nonlinear PDEs, and in particular the Order Completion Method [119]. As mentioned in Chapter 1, as well as in Section 7.1, such linear topological structures are inappropriate when it comes to the highly nonlinear phenomena inherent in the study of nonlinear PDEs.

Recall from Section 2.4 that every uniform convergence structure induces a convergence structure through (2.69). In the case of the uniform order convergence structure, this induced convergence structure on $\mathcal{ML}(X)$ may be characterized as follows.

Theorem 56 *A filter $\mathcal{F}$ on $\mathcal{ML}(X)$ belongs to $\lambda_{\mathcal{J}_o}(u)$, for some $u \in \mathcal{ML}(X)$, if and only if there exists a family $\Sigma_{\mathcal{F}} = (I_n)$ of nonempty order intervals on $\mathcal{ML}(X)$ such that*

$$\begin{array}{ll} 1) & I_{n+1} \subseteq I_n,\ n \in \mathbb{N} \\ 2) & \begin{array}{l} \forall \quad V \in \tau : \\ \quad \bigcap_{n\in\mathbb{N}} I_{n|V} = \{u\}_{|V} \end{array} \end{array}$$

and $[\Sigma_{\mathcal{F}}] \subseteq \mathcal{F}$.

Proof. Let the filter $\mathcal{F}$ converge to $u \in \mathcal{ML}(X)$. Then, by (2.71), $[u] \times \mathcal{F} \in \mathcal{J}_o$. Hence by Definition 53 there exist $k \in \mathbb{N}$ and $\Sigma_i \subseteq \Sigma$ for $i = 1, ..., k$ such that (7.17) through (7.18) are satisfied. Set $\Psi = \{i : [\Sigma_i] \subset [u]\}$. We claim

$$\mathcal{F} \supset \bigcap_{i\in\Psi} [\Sigma_i] \tag{7.22}$$

Take a set $A \in \cap_{i\in\Psi}[\Sigma_i]$. Then for each $i \in \Psi$ there is a set $A_i \in [\Sigma_i]$ such that $A \supset \cup_{i\in\Psi} A_i$. For each $i \in \{1, ..., k\} \setminus \Psi$ choose a set $A_i \in [\Sigma_i]$ with $u \notin \mathcal{ML}(X) \setminus A_i$. Then

$$(A_1 \times A_1) \cup ... \cup (A_k \times A_k) \in ([\Sigma_1] \times [\Sigma_1]) \cap ... \cap ([\Sigma_k] \times [\Sigma_k]) \subset \mathcal{F} \times [u]$$

and so there is a set $B \in \mathcal{F}$ such that

$$B \times \{u\} \subset (A_1 \times A_1) \cup ... \cup (A_k \times A_k)$$

If $w \in B$ then $(u, w) \in A_i \times A_i$ for some i. Since $u \in A_i$, we get $i \in \Psi$ and so $w \in \cup_{i \in \Psi} A_i$. This gives $B \subseteq \cup_{i \in \Psi} A_i \subseteq A$ and so $A \in \mathcal{F}$ so that (7.22) holds. Clearly, for each $i \in \Psi$, we have

$$\begin{array}{l} \forall \quad V \in \tau : \\ \quad \cap_{n \in \mathbb{N}} I^i_{n|V} = \{u\}_{|V} \end{array} \tag{7.23}$$

Writing each $I^i_n \in \Sigma_i$ in the form $I^i_n = [\lambda^i_n, \mu^i_n]$, we claim

$$\sup\{\lambda^i_n : n \in \mathbb{N}\} = u = \inf\{\mu^i_n : n \in \mathbb{N}\}$$

Suppose this were not the case. Then there exists $v, w \in \mathcal{ML}(X)$ such that

$$\lambda_n \leq v < w \leq \mu_n,\ n \in \mathbb{N}$$

Then, in view of Proposition 46, there is some nonempty $V \in \tau$ such that

$$v(x) < w(x),\ x \in V$$

which contradicts (7.18). Since $\mathcal{ML}(X)$ is fully distributive, the result follows upon setting

$$\Sigma_{\mathcal{F}} = \left\{ [\lambda_n, \mu_n] \,\middle|\, \begin{array}{ll} 1) & \lambda_n = \inf\{\lambda^i_n : i \in \Psi\} \\ 2) & \mu_n = \sup\{\mu^i_n : i \in \Psi\} \end{array} \right\}$$

The converse is trivial. ■

The following is now immediate

Corollary 57 *Consider a filter $\mathcal{F}$ on $\mathcal{ML}(X)$. Then $\mathcal{F} \in \lambda_{\mathcal{J}_o}(u)$ if and only if $\mathcal{F} \in \lambda_o(u)$. Therefore $\mathcal{ML}(X)$ is a uniformly Hausdorff uniform convergence space. In particular, a sequence (u_n) on $\mathcal{ML}(X)$ converges to u if and only if (u_n) order converges to u.*

7.4 The Completion of $\mathcal{ML}(X)$

This section is concerned with the construction of the completion of the uniform convergence space $\mathcal{ML}(X)$. In this regard, recall that the completion of the convergence vector space $\mathcal{C}(X)$, equipped with the order convergence structure, is the set of finite Hausdorff continuous functions on X, see Section 4.3 and [10]. This space is order isomorphic to the set of all *finite* normal lower semi-continuous functions. Note, however, that functions $u \in \mathcal{ML}(X)$ need not be finite everywhere, but may, in contradistinction with functions in $\mathcal{C}(X)$, assume the values $\pm\infty$ on any closed nowhere dense subset of X. Hence we consider the space $\mathcal{NL}(X)$ of nearly finite normal lower semi-continuous functions on X. Following the results in Section 7.3, we introduce the following uniform convergence structure on $\mathcal{NL}(X)$.

Definition 58 *A filter $\mathcal{U}$ on $\mathcal{NL}(\Omega) \times \mathcal{NL}(\Omega)$ belongs to the family $\mathcal{J}_o^\sharp$ whenever, for some positive integer k, we have the following:*

$$\begin{array}{l} \forall \quad i = 1, ..., k \ : \\ \exists \quad (\lambda_n^i)\,,\ (\mu_n^i) \subset \mathcal{ML}^0(\Omega) \ : \\ \exists \quad u^i \in \mathcal{NL}(\Omega) \ : \\ \quad 1) \quad \lambda_n^i \leq \lambda_{n+1}^i \leq \mu_{n+1}^i \leq \mu_n^i,\ n \in \mathbb{N} \\ \quad 2) \quad \sup\{\lambda_n^i \ : \ n \in \mathbb{N}\} = u^i = \inf\{\mu_n^i \ : \ n \in \mathbb{N}\} \\ \quad 3) \quad \bigcap_{i=1}^k (([\Sigma^i] \times [\Sigma^i]) \cap ([u^i] \times [u^i])) \subseteq \mathcal{U} \end{array} \tag{7.24}$$

Here $\Sigma^i = \{I_n^i \ : \ n \in \mathbb{N}\}$ with $I_n^i = \{u \in \mathcal{ML}^0 \ : \ \lambda_n^i \leq u \leq \mu_n^i\}$.

The following now results by the same arguments and techniques used in Section 7.3, notably those employed in the proof of Theorems 55 and 56.

Theorem 59 *The family $\mathcal{J}_o^\sharp$ of filters on $\mathcal{NL}(X) \times \mathcal{NL}(X)$ is a Hausdorff uniform convergence structure.*

Theorem 60 *A filter $\mathcal{F}$ on $\mathcal{NL}(X)$ belongs to $\lambda_{\mathcal{J}_o^\sharp}$ if and only if*

$$\begin{array}{l} \exists \quad (\lambda_n)\,,\ (\mu_n) \subset \mathcal{ML}(X) \ : \\ \quad 1) \quad \lambda_n \leq \lambda_{n+1} \leq \mu_{n+1} \leq \mu_n,\ n \in \mathbb{N} \\ \quad 2) \quad \sup\{\lambda_n \ : \ n \in \mathbb{N}\} = u = \inf\{\mu_n \ : \ n \in \mathbb{N}\} \\ \quad 3) \quad [\{I_n \ : \ n \in \mathbb{N}\}] \subseteq \mathcal{F} \end{array} ,$$

where $I_n = \{v \in \mathcal{ML}(X) \ : \ \lambda_n \leq v \leq \mu_n\}$.

We now proceed to show that $\mathcal{NL}(X)$ is the completion of $\mathcal{ML}(X)$. That is, we show that the following three conditions are satisfied:

- The uniform convergence space $\mathcal{NL}(X)$ is complete
- $\mathcal{ML}(X)$ is uniformly isomorphic to a dense subspace of $\mathcal{NL}(X)$
- Any uniformly continuous mapping φ on $\mathcal{ML}(X)$ into a complete, Hausdorff uniform convergence space Y extends uniquely to a uniformly continuous mapping $\varphi^\sharp$ from $\mathcal{NL}(X)$ into Y.

Proposition 61 *The uniform convergence space $\mathcal{NL}(X)$ is complete.*

Proof. Clearly $\mathcal{J}_o^\sharp$ is simply the uniform convergence structure associated with the convergence structure described in Theorem 60. Therefore it is complete. ■

Theorem 62 *Let X be a metric space. Then the space $\mathcal{NL}(X)$ is the uniform convergence space completion of $\mathcal{ML}(X)$.*

Proof. First we show that $\iota(\mathcal{ML}(X))$ is dense in $\mathcal{NL}(X)$, where $\iota : \mathcal{ML}(X) \to \mathcal{NL}(X)$ is the inclusion mapping. To see this, consider any $u \in \mathcal{NL}(X)$, and set

$$D_u = \{x \in X \,:\, u(x) \in \mathbb{R}\}$$

Since D_u is open, it follows that u restricted to D_u is normal lower semi-continuous. Since u is also finite on D_u it follows by (3.42) that there exists a sequence (u_n) of continuous functions on D_u such that

$$u(x) = \sup\{u_n(x) \,:\, n \in \mathbb{N}\},\ x \in D_u \tag{7.25}$$

Consider now the sequence $(v_n) = ((I \circ S)(u_n^0))$ where

$$u_n^0(x) = \begin{cases} u_n(x) & \text{if} \quad x \in D_u \\ \\ 0 & \text{if} \quad x \notin D_u \end{cases}$$

Clearly $v_n(x) = u_n(x)$ for every $x \in D_u$. We claim

$$u = \sup\{v_n \,:\, n \in \mathbb{N}\} \tag{7.26}$$

If (7.26) does not hold, then

$$\begin{array}{ll} \exists & v \in \mathcal{NL}(X) \,: \\ & n \in \mathbb{N} \Rightarrow v_n \leq v < u \end{array}$$

But then, in view of Proposition 46, and the fact that D_u is open and dense, there exists an open and nonempty set $W \subseteq D_u$ such that

$$\begin{array}{ll} \forall & x \in W \,: \\ & n \in \mathbb{N} \Rightarrow u_n(x) \leq v(x) < u(x) \end{array}$$

which contradicts (7.25). Therefore (7.26) must hold. The sequence $(\iota(v_n))$ is clearly a convergent sequence in $\mathcal{NL}(X)$ so that $\iota(\mathcal{ML}(X))$ is dense in $\mathcal{NL}(X)$.
Now let us show that the inclusion mapping is a uniformly continuous embedding. In this regard, it is sufficient to consider a filter $[\Sigma_{\mathcal{F}}]$ where

$$\Sigma_{\mathcal{F}} = \{I_n = [\lambda_n, \mu_n] \,:\, n \in \mathbb{N}\}$$

is a family of nonempty order intervals in $\mathcal{ML}(X)$ that satisfies 1) of (7.17) as well as (7.18). We claim

$$\begin{array}{ll} \exists & u \in \mathcal{NL}(X) \,: \\ & \sup\{\lambda_n \,:\, n \in \mathbb{N}\} = u = \inf\{\mu_n \,:\, n \in \mathbb{N}\} \end{array}. \tag{7.27}$$

Since the sequence (λ_n) is bounded from above, and the sequence (μ_n) is bounded from below, it follows from the Dedekind completeness of $\mathcal{NL}(X)$, Theorem 45, that

$$\begin{array}{ll} \exists & u, v \in \mathcal{NL}(X) \,: \\ & \sup\{\lambda_n \,:\, n \in \mathbb{N}\} = u \leq v = \inf\{\mu_n \,:\, n \in \mathbb{N}\} \end{array}. \tag{7.28}$$

To see that (7.27) holds, we proceed by contradiction. Suppose that $u \neq v$. Then, by Proposition 46, we have

$$\exists \quad \begin{array}{l} W \subset X \text{ nonempty and open}: \\ x \in W \Rightarrow u(x) < v(x) \end{array} . \tag{7.29}$$

We may assume that both u and v are finite on W. Since v is lower semi-continuous,

$$\begin{array}{l} \forall \quad x \in W: \\ \quad v(x) = \sup \left\{ \varphi(x) \left| \begin{array}{ll} 1) & \varphi \in \mathcal{C}(W) \\ 2) & \varphi(x) \leq v(x),\, x \in W \end{array} \right. \right\} \end{array}$$

Clearly, there is a function $\varphi \in \mathcal{C}(W)$, and a nonempty open set $A \subseteq W$ such that

$$u(x) < \varphi(x) < v(x),\, x \in A$$

Applying the Katětov-Tong Theorem to the continuous function φ and the lower semi-continuous function v, one finds a function $\psi \in \mathcal{C}(A)$ such that

$$u(x) < \varphi(x) < \psi(x) < v(x),\, x \in A$$

which contradicts (7.17). Therefore ι is uniformly continuous.
That ι^{-1} is uniformly continuous follows immediately from (7.24).
The extension property for uniformly continuous mappings on $\mathcal{ML}(X)$ follows in the standard way. ■

Note that in the above proof, we actually showed that $\mathcal{NL}(X)$ is the Dedekind completion of $\mathcal{ML}(X)$. Hence the uniform order convergence structure provides a nonlinear topological model for the process of taking the Dedekind completion of $\mathcal{ML}(X)$. In view of Proposition 51, this extends a previous result of Anguelov and Rosinger [9] on the Dedekind completion of $\mathcal{M}(X)$.

However, it should be noted that Theorem 62 is in fact *more general* than the result in [9]. Indeed, along with the uniform convergence space completion of $\mathcal{ML}(X)$ we obtain a class of mappings, namely, uniformly continuous mappings into *any* Hausdorff uniform convergence space Y, that can be extended uniquely to the completion of $\mathcal{ML}(X)$. In contradistinction with the uniform convergence space completion constructed in Theorem 62, the Dedekind completion result in [9] allows only for the extension of order isomorphic embeddings into partially ordered sets, see Section 1.4 and [119, Appendix A].

Chapter 8

Spaces of Generalized Functions

8.1 The Spaces $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$

The aim of the current investigation is to enrich the basic theory of Order Completion for systems of nonlinear PDEs. In this regard we have two objectives, namely, to obtain a better understanding of the possible *structure of generalized solutions*, and to determine to what extent we may obtain stronger regularity properties of such generalized solutions. A first step in this direction is to *recast* the basic existence, uniqueness and regularity results in the Order Completion Method within the context of uniform convergence spaces.

Such a reformulation of the basic results of the Order Completion Method in terms of uniform convergence spaces allows for the application of tools from the theory of convergence spaces to questions related to the structure and regularity of generalized solutions. Such convergence theoretic techniques may turn out to be more suited to address these issues than the basic order theoretic techniques upon which the Order Completion Method is based.

In particular, our first efforts go towards the construction of the spaces of generalized functions as the completion of suitable uniform convergence spaces, rather than the Dedekind order completion of appropriate partially ordered sets as discussed in Section 1.4. Such a reformulation of the theory of Order Completion in topological terms is motivated by the difficulties, such as those mentioned at the end of Section 1.4, involved in going beyond the basic results presented in [119] in purely order theoretic terms.

A key feature of the Order Completion Method is that, with the particular nonlinear partial differential operator that defines the equation, one associates a space of generalized functions. In particular, the partial order (1.123) on the space $\mathcal{M}^m_T(\Omega)$ is defined in exactly such a way as to make the nonlinear partial differential operator *compatible* with the given order structures on its domain and range. It is exactly this idea of defining the structure on the domain of the operator, in this case a uniform convergence structure, in such a way as to ensure a certain compatibility with the particular nonlinear mapping involved which we exploit.

Consider now a system of K possibly nonlinear PDEs, each of order at most m, of the form

$$\mathbf{T}(x,D)\,\mathbf{u}(x)=\mathbf{f}(x)\,,\ x\in\Omega, \tag{8.1}$$

where $\Omega\subseteq\mathbb{R}^n$ is nonempty and open. The righthand term $\mathbf{f}$ is assumed to be a continuous mapping $\mathbf{f}:\Omega\to\mathbb{R}^K$, with components $f_1,...,f_K$. The partial differential operator $\mathbf{T}(x,D)$ is supposed to be defined by a jointly continuous mapping

$$\mathbf{F}:\Omega\times\mathbb{R}^M\to\mathbb{R}^K \tag{8.2}$$

through

$$\mathbf{T}(x,D)\,\mathbf{u}(x)=\mathbf{F}\left(x,...,u_i(x)\,,...,D^\alpha u_i(x)\,,...\right),\ |\alpha|\leq m;\ i=1,...,K \tag{8.3}$$

where each component $u_1,...,u_K$ of the unknown $\mathbf{u}$ belongs to $\mathcal{C}^m(\Omega)$. In view of the continuity of the mapping (8.2), we may associate with the nonlinear operator $\mathbf{T}(x,D)$ the mapping

$$\mathbf{T}:\mathcal{C}^m(\Omega)^K\to\mathcal{C}^0(\Omega)^K \tag{8.4}$$

defined through

$$\mathbf{Tu}:\Omega\ni x\mapsto\mathbf{T}(x,D)\,\mathbf{u}(x)\in\mathbb{R}^K$$

for each $\mathbf{u}\in\mathcal{C}^m(\Omega)^K$.

The mapping (8.4) associated with the system of equations (8.1) extends in a canonical way to a mapping between suitable spaces of normal lower semi-continuous functions. In this regard, we introduce, for an integer $l\geq 0$, the following space of nearly finite normal lower semi-continuous functions

$$\mathcal{ML}^l(\Omega)=\left\{u\in\mathcal{ML}(\Omega)\left|\begin{array}{ll}\exists & \Gamma\subset\Omega\text{ closed nowhere dense :}\\ & u\in\mathcal{C}^l(\Omega\setminus\Gamma)\end{array}\right.\right\}. \tag{8.5}$$

Clearly, in case $l=0$, we have recovered simply the space $\mathcal{ML}(\Omega)$. We may also note that, in contradistinction with the space $\mathcal{C}^l(\Omega)$, for $l\geq 1$, of smooth functions, each of the spaces $\mathcal{ML}^l(\Omega)$ is a fully distributive lattice with respect to the pointwise order (7.2).

Proposition 63 *For each $l\geq 0$, the space $\mathcal{ML}^l(\Omega)$ is a fully distributive lattice with respect to the pointwise order (7.2).*

Proof. Consider any $u,v\in\mathcal{ML}^l(\Omega)$. Then there is a closed and nowhere dense subset Γ of Ω such that $u,v\in\mathcal{C}^l(\Omega\setminus\Gamma)$. Define open subsets U, V and W of $\Omega\setminus\Gamma$ through

$$U=\{x\in\Omega\setminus\Gamma:\ u(x)<v(x)\},$$

$$V = \{x \in \Omega \setminus \Gamma : v(x) < u(x)\}$$

and

$$W = \operatorname{int}\{x \in \Omega \setminus \Gamma : u(x) = v(x)\},$$

respectively. It is clear that the function

$$\varphi : \Omega \ni x \mapsto \sup\{u(x), v(x)\} \in \overline{\mathbb{R}}$$

is $\mathcal{C}^l$-smooth on $U \cup V \cup W$. Clearly the set $U \cup V \cup W$ is dense in $\Omega \setminus \Gamma$. As such, it follows by Theorem 45 that $\sup\{u, v\}$ belongs to $\mathcal{ML}^l(\Omega)$.
The existence of the infimum of $u, v \in \mathcal{ML}^l(\Omega)$ in $\mathcal{ML}^l(\Omega)$ follows in the same way. The distributivity of $\mathcal{ML}^l(\Omega)$ now follows by Proposition 47. $\blacksquare$

The usual partial differential operators

$$D^\alpha : \mathcal{C}^l(\Omega) \to \mathcal{C}^0(\Omega), \; |\alpha| \leq l \tag{8.6}$$

may be extended in a straightforward way to the larger space $\mathcal{ML}^l(\Omega)$. Indeed, in view of (8.5), it is clear that, for each $u \in \mathcal{ML}^l(\Omega)$, we have

$$\begin{array}{ll} \exists & \Gamma \subset \Omega \text{ closed nowhere dense}: \\ \forall & |\alpha| \leq l: \\ & D^\alpha\left(u_{|\Omega\setminus\Gamma}\right) \in \mathcal{C}^0(\Omega \setminus \Gamma) \end{array} \tag{8.7}$$

which allows for an extension of the mapping (8.6) to a mapping

$$\mathcal{D}^\alpha : \mathcal{ML}^l(\Omega) \mapsto \mathcal{ML}^0(\Omega) \tag{8.8}$$

through

$$\mathcal{D}^\alpha : u \mapsto (I \circ S)(D^\alpha u). \tag{8.9}$$

Indeed, in view of (8.7) and (7.1), the function $\mathcal{D}^\alpha u$ is nearly finite and normal lower semi-continuous for every $|\alpha| \leq l$. Furthermore, each partial derivatives $\mathcal{D}^\alpha u$ belongs to $\mathcal{ML}^l(\Omega)$. In particular,

$$\mathcal{D}^\alpha u(x) = D^\alpha u(x), \; x \in \Omega \setminus \Gamma,$$

where Γ is the closed nowhere dense subset of Ω associated with u through (8.5).

In order to now extend the mapping (8.4) to a mapping

$$\mathbf{T} : \mathcal{ML}^m(\Omega)^K \to \mathcal{ML}^0(\Omega)^K, \tag{8.10}$$

we express (8.4) componentwise as

$$T_j : \mathcal{C}^m(\Omega)^K \ni \mathbf{u} \mapsto F_j(\cdot, ..., u_i, ..., D^\alpha u_i, ...) \in \mathcal{C}^0(\Omega) \tag{8.11}$$

where $F_1, ..., F_K : \Omega \times \mathbb{R}^M \to \mathbb{R}$ are the components of the mapping (8.2). The components (8.11) extend in a straight forward way to mappings

$$T_j : \mathcal{ML}^m(\Omega)^K \to \mathcal{ML}^0(\Omega)$$

which are defined as

$$T_j : \mathcal{ML}^m(\Omega)^K \ni \mathbf{u} \mapsto (I \circ S)(F_j(\cdot, ..., u_i, ..., \mathcal{D}^\alpha u_i, ...)) \in \mathcal{ML}^0(\Omega). \quad (8.12)$$

In view of (8.7), it follows by (7.1) and the continuity of each of the components $F_1, ..., F_K$ of the mapping (8.2) that the mapping (8.12) is well defined for each $j = 1, ..., K$. As such, we may define the extension (8.10) of the mapping (8.4) componentwise, with components defined in (8.12). That is,

$$\mathbf{T} : \mathcal{ML}^m(\Omega)^K \ni \mathbf{u} \mapsto (T_j\mathbf{u})_{j\leq K} \in \mathcal{ML}^0(\Omega)^K.$$

The mapping (8.10) extends the mapping (8.4) associated with the nonlinear partial differential operator (8.3). Therefore we may formulate the system of nonlinear PDEs (8.1) in the significantly more general framework of the spaces of normal lower semi-continuous functions $\mathcal{ML}^m(\Omega)^K$ and $\mathcal{ML}^0(\Omega)^K$. In particular, we formulate the generalized equation

$$\mathbf{Tu} = \mathbf{f} \quad (8.13)$$

where the unknown $\mathbf{u}$ ranges over $\mathcal{ML}^m(\Omega)^K$. It should be noted that this extended formulation of the problem allows for functions with singularities on arbitrary closed nowhere dense subsets of the domain of definition Ω to act as *global* solutions of the system of nonlinear PDEs (8.1). This should be compared with the global version of the Cauchy-Kovalevskaia Theorem [141] which is also mentioned in Section 1.3. Furthermore, such a solution will in general not belong to any of the customary spaces of generalized functions, such as the Sobolev spaces $H^{2,m}(\Omega)$, or the space $\mathcal{D}'(\Omega)$ of distributions on Ω. Indeed, a function $u \in \mathcal{ML}^m(\Omega)$ will in general fail to be locally integrable on Ω, since it does not satisfy any growth conditions near the closed nowhere dense singularity set Γ associated with u through (8.5).

Throughout this section, the space $\mathcal{ML}^0(\Omega)$ is equipped with the uniform order convergence structure $\mathcal{J}_o$, while the product space $\mathcal{ML}^0(\Omega)^K$ will carry the product uniform convergence structure $\mathcal{J}_o^K$ with respect to $\mathcal{J}_o$. That is,

$$\mathcal{U} \in \mathcal{J}_o^K \Leftrightarrow \left(\begin{array}{ll} \forall & i = 1, ..., K : \\ & (\pi_i \times \pi_i)(\mathcal{U}) \in \mathcal{J}_o \end{array} \right) \quad (8.14)$$

where π_i denotes the projection

$$\pi_i : \mathcal{ML}^0(\Omega)^K \ni \mathbf{u} = (u_i)_{i\leq K} \mapsto u_i \in \mathcal{ML}^0(\Omega).$$

The basic properties of the space $\mathcal{ML}^0(\Omega)^K$ that are relevant to this investigation are summarized in the following proposition.

Proposition 64 *The uniform convergence space $\mathcal{ML}^0(\Omega)^K$ is first countable and Hausdorff. Furthermore, its completion is the space $\mathcal{NL}(\Omega)^K$ equipped with the product uniform convergence structure with respect to the uniform convergence structure $\mathcal{J}_o^\sharp$.*

Proof. The assertions of the proposition follow immediately from Proposition 41, Corollary 57 and Theorem 62, respectively. ■

Within the context of the nonlinear mapping associated with a given system of nonlinear PDEs introduced in this section, and in particular the extended mapping (8.10), the most simple way in which to define a suitable uniform convergence structure on $\mathcal{ML}^m(\Omega)^K$ is to introduce the initial uniform convergence structure on $\mathcal{ML}^m(\Omega)^K$ with respect to the mapping (8.10). However, the completion results for uniform convergence spaces discussed in Sections 2.4 and 6.1 apply to Hausdorff uniform convergence spaces only, while the initial uniform convergence structure on $\mathcal{ML}^m(\Omega)^K$ with respect to the mapping (8.10) is Hausdorff if and only if the mapping (8.10) is *injective*, which is typically not the case.

This difficulty can be overcome if we associate with the mapping (8.10) an equivalence relation on $\mathcal{ML}^m(\Omega)^K$ through

$$\mathbf{u} \sim_{\mathbf{T}} \mathbf{v} \Leftrightarrow \mathbf{Tu} = \mathbf{Tv}. \tag{8.15}$$

The mapping (8.10) induces an *injective* mapping

$$\widehat{\mathbf{T}} : \mathcal{ML}^m_{\mathbf{T}}(\Omega) \to \mathcal{ML}^0(\Omega)^K, \tag{8.16}$$

where $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$ denotes the quotient space $\mathcal{ML}^m(\Omega)^K / \sim_{\mathbf{T}}$, such that the diagram

$$\begin{tikzcd}
\mathcal{ML}^m(\Omega)^K \arrow[rr, "\mathbf{T}"] \arrow[dr, "q_{\mathbf{T}}"'] & & \mathcal{ML}^0(\Omega)^K \\
& \mathcal{ML}^m_{\mathbf{T}}(\Omega) \arrow[ur, "\widehat{\mathbf{T}}"'] &
\end{tikzcd} \tag{8.17}$$

commutes, where $q_{\mathbf{T}}$ is the canonical quotient mapping associated with the equivalence relation (8.15). The diagram (8.17) amounts simply to a *representation* of the mapping $\mathbf{T}$. In particular, the equation (8.13) is, in a certain precise sense, *equivalent* to the equation

$$\widehat{\mathbf{T}}\mathbf{U} = \mathbf{f}, \tag{8.18}$$

with the unknown $\mathbf{U}$ ranging over $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$. Indeed, in view of the diagram (8.17) and the surjectivity of $q_{\mathbf{T}}$, it follows that

$$\begin{array}{ll} \forall & \mathbf{u} \in \mathcal{ML}^m(\Omega)^K : \\ & \mathbf{T}\mathbf{u} = \mathbf{f} \Leftrightarrow \widehat{\mathbf{T}}\left(q_{\mathbf{T}}\mathbf{u}\right) = \mathbf{f} \end{array}$$

and

$$\begin{array}{ll} \forall & \mathbf{U} \in \mathcal{ML}^m_{\mathbf{T}}(\Omega) : \\ & \widehat{\mathbf{T}}\mathbf{U} = \mathbf{f} \Leftrightarrow \mathbf{T}\mathbf{u} = \mathbf{f},\, \mathbf{u} \in q^{-1}_{\mathbf{T}}(\mathbf{U}) \end{array}$$

Since the mapping (8.16) is *injective* it follows that the initial uniform convergence structure $\mathcal{J}_{\mathbf{T}}$ on $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$ with respect to (8.16) is Hausdorff. In particular,

$$\mathcal{U} \in \mathcal{J}_{\mathbf{T}} \Leftrightarrow \left(\widehat{\mathbf{T}} \times \widehat{\mathbf{T}}\right)(\mathcal{U}) \in \mathcal{J}_o^K \tag{8.19}$$

so that $\widehat{\mathbf{T}}$ is in fact a *uniformly continuous embedding*. As such, and in view of Proposition 36, the uniform convergence space completion $\mathcal{NL}_{\mathbf{T}}(\Omega)$ of $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$ may be identified with a subspace of the completion $\mathcal{NL}^0(\Omega)^K$ of $\mathcal{ML}^0(\Omega)^K$. In particular, the mapping (8.16) extends to an injective uniformly continuous mapping

$$\widehat{\mathbf{T}}^{\sharp} : \mathcal{NL}_{\mathbf{T}}(\Omega) \to \mathcal{NL}^0(\Omega)^K. \tag{8.20}$$

Within the context of the construction (8.15) to (8.20), we may formulate the *generalized equation*

$$\widehat{\mathbf{T}}^{\sharp}\mathbf{U}^{\sharp} = \mathbf{f} \tag{8.21}$$

corresponding to the equation (8.18), where the unknown $\mathbf{U}^{\sharp}$ ranges over $\mathcal{NL}_{\mathbf{T}}(\Omega)^K$. In view of the equivalence of the equations (8.13) and (8.18), we will interpret any solution to (8.21) as a generalized solution of the system of nonlinear PDEs (8.1). The question of *existence* of solutions to (8.21) will be addressed in Section 9.2.

8.2 Sobolev Type Spaces of Generalized Functions

The Order Completion Method [119] involves a construction of spaces of generalized functions which are associated with the particular nonlinear partial differential operator which defines the equation. The spaces of generalized functions constructed in Section 8.1 employ essentially the same technique, with the key difference that the spaces of generalized functions are obtained not through the process of order completion, but rather through the more general topological process of completion of a uniform convergence space.

As mentioned, the spaces of generalized functions constructed in Section 8.1 are constructed with a particular nonlinear partial differential operator in mind. As such, they may depend to a large extent on this operator. Furthermore, there is no concept of derivative of generalized functions. In this section we construct, in the original spirit of Sobolev [148] and [149], spaces of generalized functions which are *independent* of any particular nonlinear partial differential operator. Moreover, these spaces are equipped in a natural and canonical way with partial differential operators that extend the classical operators on spaces of smooth functions. Furthermore, and as we will show in Section 8.3, these spaces are, in a certain precise sense, compatible with the spaces constructed in Section 8.2.

Recall that the Sobolev space $H^{2,l}(\Omega)$ may be constructed as the completion of $\mathcal{C}^l(\Omega)$ equipped with the initial vector space topology induced by the family of mappings

$$\left(D^\alpha : \mathcal{C}^l(\Omega) \to L_2(\Omega)\right)_{|\alpha|\leq l}$$

where $L_2(\Omega)$ is the Hilbert space of square integrable functions on Ω. We follow a similar approach in constructing spaces of generalized functions. In this regard, we equip the space $\mathcal{ML}^l(\Omega)$, where $l \geq 1$, with the initial uniform convergence structure $\mathcal{J}_D$ with respect to the family of mappings

$$\left(\mathcal{D}^\alpha : \mathcal{ML}^l(\Omega) \to \mathcal{ML}^0(\Omega)\right)_{|\alpha|\leq l} \tag{8.22}$$

That is, for any filter $\mathcal{U}$ on $\mathcal{ML}^l(\Omega) \times \mathcal{ML}^l(\Omega)$, we have

$$\mathcal{U} \in \mathcal{J}_D \Leftrightarrow \left(\begin{array}{ll} \forall & |\alpha| \leq l \\ & (\mathcal{D}^\alpha \times \mathcal{D}^\alpha)(\mathcal{U}) \in \mathcal{J}_o \end{array}\right) \tag{8.23}$$

Since the family of mappings (8.22) separates the elements of $\mathcal{ML}^l(\Omega)$, that is,

$$\begin{array}{ll} \forall & u, v \in \mathcal{ML}^l(\Omega), u \neq v : \\ \exists & |\alpha| \leq l : \\ & \mathcal{D}^\alpha u \neq \mathcal{D}^\alpha v \end{array},$$

it follows that $\mathcal{J}_D$ is uniformly Hausdorff. A filter $\mathcal{F}$ on $\mathcal{ML}^l(\Omega)$ is a Cauchy filter if and only if

$$\begin{array}{ll} \forall & |\alpha| \leq l : \\ & \mathcal{D}^\alpha(\mathcal{F}) \text{ is a Cauchy filter in } \mathcal{ML}^0(\Omega) \end{array} \tag{8.24}$$

In particular, a filter $\mathcal{F}$ on $\mathcal{ML}^l(\Omega)$ converges to $u \in \mathcal{ML}^l(\Omega)$ if and only if

$$\begin{array}{ll} \forall & |\alpha| \leq l : \\ & \mathcal{D}^\alpha(\mathcal{F}) \in \lambda_o(\mathcal{D}^\alpha u) \end{array} \tag{8.25}$$

In view of the results in Chapter 6 on the completion of uniform convergence spaces, the completion of $\mathcal{ML}^l(\Omega)$ is realized as a subspace of $\mathcal{NL}(\Omega)^M$, for an appropriate $M \in \mathbb{N}$. In this regard, we note, see Proposition 43, that the mapping

$$\mathbf{D} : \mathcal{ML}^l(\Omega) \ni u \mapsto (\mathcal{D}^\alpha u)_{|\alpha|\leq l} \in \mathcal{ML}^0(\Omega)^M \tag{8.26}$$

is a uniformly continuous embedding. In particular, for each $|\alpha| \leq l$, the diagram

$$\begin{tikzcd} \mathcal{ML}^{l}(\Omega) \arrow[rr, "\mathbf{D}"] \arrow[dr, "\mathcal{D}^{\alpha}"'] & & \mathcal{ML}^{0}(\Omega)^{M} \arrow[dl, "\pi_{\alpha}"] \\ & \mathcal{ML}^{0}(\Omega) & \end{tikzcd} \tag{8.27}$$

commutes, with π_α the projection. This diagram amounts to a *decomposition* of $u \in \mathcal{ML}^{l}(\Omega)$ into its partial derivatives. In view of the uniform continuity of the mapping $\mathbf{D}$ and its inverse, it follows by Theorem 44 that $\mathbf{D}$ extends to an injective uniformly continuous mapping

$$\mathbf{D}^{\sharp} : \mathcal{NL}^{l}(\Omega) \rightarrow \mathcal{NL}(\Omega)^{M} \tag{8.28}$$

where $\mathcal{NL}^{l}(\Omega)$ denotes the uniform convergence space completion of $\mathcal{ML}^{l}(\Omega)$. Moreover, since each mapping $\mathcal{D}^{\alpha}$ is uniformly continuous, one obtains the commutative diagram

$$\begin{tikzcd} \mathcal{NL}^{l}(\Omega) \arrow[rr, "\mathbf{D}^{\sharp}"] \arrow[dr, "\mathcal{D}^{\alpha\sharp}"'] & & \mathcal{NL}^{0}(\Omega)^{M} \arrow[dl, "\pi_{\alpha}^{\sharp}"] \\ & \mathcal{NL}^{0}(\Omega) & \end{tikzcd} \tag{8.29}$$

where

$$\mathcal{D}^{\alpha\sharp} : \mathcal{NL}^{l}(\Omega) \rightarrow \mathcal{NL}^{0}(\Omega) \tag{8.30}$$

is the extension through uniform continuity of the partial differential operator $\mathcal{D}^{\alpha}$. Since the mapping $\mathbf{D}^{\sharp}$ is injective and uniformly continuous, and in view of the commutative diagram (8.29) above, each *generalized function* $u^\sharp \in \mathcal{NL}^l(\Omega)$ may be uniquely represented by its generalized partial derivatives

$$u^\sharp \mapsto \mathbf{D}^\sharp u^\sharp = \left(\mathcal{D}^{\alpha\sharp}\mathbf{u}^\sharp\right)_{|\alpha|\leq l} \tag{8.31}$$

Each generalized partial derivative $\mathcal{D}^{\alpha\sharp}u^\sharp$ of $u^\sharp$ is a nearly finite normal lower semi-continuous function. We note, therefore, that the set of singular points of each $u^\sharp \in \mathcal{NL}^l(\Omega)$, that is, the set

$$\left\{ x \in \Omega \;\middle|\; \begin{array}{l} \exists \quad |\alpha| \leq l : \\ \mathcal{D}^{\alpha\sharp}u^\sharp \text{ not continuous at } x \end{array} \right\}$$

is at most a set of first Baire category, that is, it is a topologically small set. However, this set may be dense in Ω. Furthermore, such a set may have arbitrarily large positive Lebesgue measure [121]. Highly singular objects, such as the generalized functions that are the elements of $\mathcal{ML}^l(\Omega)$ may turn out to model highly relevant real world situations, like turbulence or other chaotic phenomena.

8.3 Nonlinear Partial Differential Operators

This section deals with the general class of nonlinear partial differential operators associated with systems of nonlinear PDEs of the form (8.1) to (8.3). In this regard, we investigate the properties of such operators in the context of the Sobolev type spaces of generalized functions introduced in Section 8.2, and in particular the extent to which such operators are compatible with the topological structures of these spaces. Furthermore, the extent to which the Sobolev type spaces are compatible with the 'pull back' spaces of generalized functions introduced in Section 8.1 are demonstrated.

The first part of this section concerns the general class of nonlinear partial differential operators introduced in Section 8.1. It is shown that the mapping (8.10) induced by such an operator is uniformly continuous with respect to the Sobolev type uniform convergence structure on $\mathcal{ML}^m(\Omega)$, and the uniform order convergence structure on $\mathcal{ML}^0(\Omega)$. It is also shown that the Sobolev type spaces of generalized functions are *compatible* with the pull back spaces. In the second part of this section we introduce additional smoothness properties on the nonlinear partial differential operators, and some basic properties of these operators are discussed.

The approach to generalized solutions of nonlinear PDEs pursued in this work is based on extending nonlinear partial differential operators to the completion of a suitable uniform convergence space. As is mentioned in Section 1.2, some care must be taken in constructing such extensions. In particular, it is essential that the mapping associated with such a nonlinear operator is compatible with the relevant uniform convergence structures, namely, it must be uniformly continuous.

In this regard, consider a system of nonlinear PDEs of the form (8.1) through (8.3), and the mapping (8.10) associated with the system of equations, that is, the mapping

$$\mathbf{T} : \mathcal{ML}^m(\Omega)^K \to \mathcal{ML}^0(\Omega)^K$$

The Cartesian product $\mathcal{ML}^m(\Omega)^K$ will throughout be equipped with the product uniform convergence structure $\mathcal{J}_D^K$ with respect to the uniform convergence structure $\mathcal{J}_D$ on $\mathcal{ML}^m(\Omega)$, that is,

$$\mathcal{U} \in \mathcal{J}_D^K \Leftrightarrow \left(\begin{array}{ll} \forall & i = 1, ..., K : \\ & (\pi_i \times \pi_i)(\mathcal{U}) \in \mathcal{J}_o \end{array}\right). \tag{8.32}$$

Since $\mathcal{ML}^m(\Omega)$ is Hausdorff, so is the product. Furthermore, in view of Theorem 41, the completion of $\mathcal{ML}^m(\Omega)^K$ is $\mathcal{NL}^m(\Omega)^K$. Within the context of the Sobolev type uniform convergence structure (8.23) on $\mathcal{ML}^m(\Omega)$, and the uniform order convergence structure on $\mathcal{ML}^0(\Omega)$, the basic result concerning the mapping (8.10) is the following.

Theorem 65 *Consider a mapping*

$$\boldsymbol{T} : \mathcal{ML}^m(\Omega)^K \to \mathcal{ML}^0(\Omega)^K$$

defined through a jointly continuous mapping (8.2) as in (8.12). Then this mapping is uniformly continuous.

Proof. The mapping $\mathbf{T}$ may be represented through the diagram

$$\begin{tikzcd} \mathcal{ML}^m(\Omega)^K \arrow[rr, "\mathbf{T}"] \arrow[dr, "\overline{\mathbf{D}}"'] & & \mathcal{ML}^0(\Omega)^K \\ & \mathcal{ML}^0(\Omega)^{M\times K} \arrow[ur, "\overline{\mathbf{F}}"'] & \end{tikzcd} \tag{8.33}$$

where $\overline{\mathbf{F}} = \left(\overline{F}_i\right)_{i\leq K}$ is defined componentwise through

$$\overline{F}_i : \mathcal{ML}^0(\Omega)^{M\times K} \ni \mathbf{u} \mapsto (I \circ S)(F_i(\cdot, u_1, ..., u_M)) \in \mathcal{ML}^0(\Omega) \tag{8.34}$$

and $\overline{\mathbf{D}}$ is defined as

$$\overline{\mathbf{D}} : \mathcal{ML}^m(\Omega)^K \ni \mathbf{u} \mapsto (\mathcal{D}^\alpha u_i)_{i\leq K}^{|\alpha|\leq m} \in \mathcal{ML}^0(\Omega)^{M\times K}$$

Clearly $\mathbf{D}$ is uniformly continuous, so in view of the diagram (8.33) it suffices to show that $\overline{\mathbf{F}}$ is uniformly continuous with respect to the product uniform convergence structure on $\mathcal{ML}^0(\Omega)^{M\times K}$.
In this regard, we consider sequences of order intervals (I_n^i) in $\mathcal{ML}^0(\Omega)$, which, for $i=1,...,M\times K$, satisfies condition 1) of (7.17) and (7.18). We claim

$$\begin{array}{ll} \forall & n\in\mathbb{N}: \\ \exists & \text{Order intervals } J_n^1,...,J_n^K\subseteq\mathcal{ML}^0(\Omega) : \\ & F_j\left(\prod_{i=1}^{M\times K} I_n^i\right)\subseteq J_n^j,\ j=1,...,K \end{array} \tag{8.35}$$

To verify (8.35), observe that there is a closed nowhere dense set $\Gamma_n\subseteq\Omega$ so that

$$\begin{array}{ll} \forall & x\in\Omega\setminus\Gamma : \\ \exists & a(x)>0 : \\ \forall & i=1,...,M\times K : \\ & u\in I_n^i\Rightarrow |u(x)|\le a(x) \end{array} \tag{8.36}$$

Since $F_j:\Omega\times\mathbb{R}^M\to\mathbb{R}$ is continuous, it follows from (8.36) that

$$\begin{array}{ll} \forall & x\in\Omega\setminus\Gamma : \\ \exists & b(x)>0 : \\ & \left(\begin{array}{ll} \forall & i=1,...,M\times K : \\ & u_i\in I_n^i \end{array}\right)\Rightarrow |F_j(x,u_1(x),...,u_M(x))|\le b(x) \end{array} \tag{8.37}$$

Therefore, in view of Proposition 49, our claim (8.35) holds. In particular, since $\mathcal{NL}(\Omega)$ is Dedekind complete by Theorem 45 , we may set

$$J_n^j=[\lambda_n^j,\mu_n^j]$$

where, for each $n\in\mathbb{N}$ and each $j=1,...,K$

$$\lambda_n^j=\inf\{\overline{F}_j\mathbf{u}:\mathbf{u}\in\prod_{i=1}^{M\times K} I_n^i\}$$

and

$$\mu_n^j=\sup\{\overline{F}_j\mathbf{u}:\mathbf{u}\in\prod_{i=1}^{M\times K} I_n^i\}$$

The sequence (λ_n^j) and (μ_n^j) are increasing and decreasing, respectively. For each $j=1,...,K$ we may consider

$$\sup\{\lambda_n^j:n\in\mathbb{N}\}=u^j\le v^j=\inf\{\mu_n^j:n\in\mathbb{N}\}$$

We claim that $u^j=v^j$. To see this, we note that for each $i=1,...,M\times K$ there is some $w^i\in\mathcal{NL}(\Omega)$ so that

$$\sup\{l_n^i:n\in\mathbb{N}\}=w^i=\inf\{u_n^i:n\in\mathbb{N}\}$$

where $I_n^i = [l_n^i, u_n^i]$. Applying Proposition 50 and the continuity of F_j our claim is verified. Applying the same technique as in the proof of Theorem 62, as well as Proposition 34 we obtain a sequence $\left(\overline{I}_n^j\right)$ of order intervals in $\mathcal{ML}^0(\Omega)$ that satisfies 1) of (7.17), (7.18) and

$$\overline{F}_j\left(\prod_{i=1}^{M} I_n^i\right) \subseteq \overline{I}_n^j$$

This completes the proof. ■

Since the mapping (8.10) is uniformly continuous, it extends in a unique way to a uniformly continuous mapping

$$\mathbf{T}^\sharp : \mathcal{NL}^m(\Omega)^K \rightarrow \mathcal{NL}(\Omega)^K . \tag{8.38}$$

Therefore, one may formulate a generalized equation corresponding to (8.13) as

$$\mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{f} \tag{8.39}$$

where the unknown $\mathbf{u}^\sharp$ ranges over $\mathcal{NL}^m(\Omega)$. In view of the fact that the mapping (8.10) is the *unique uniformly continuous extension* of (8.4), we interpret any solution to (8.39) as a generalized solution to the system of nonlinear PDEs (8.1).

Recall also that the mapping (8.16) is a uniformly continuous embedding. As such, and in view of the commutative diagram (8.17), the canonical quotient mapping

$$q_{\mathbf{T}} : \mathcal{ML}^m(\Omega)^K \rightarrow \mathcal{ML}^m_{\mathbf{T}}(\Omega)$$

associated with the equivalence relation (8.15) is uniformly continuous, and extends in a unique way to a uniformly continuous mapping

$$q^\sharp_{\mathbf{T}} : \mathcal{NL}^m(\Omega)^K \rightarrow \mathcal{NL}_{\mathbf{T}}(\Omega) \tag{8.40}$$

In particular, the mapping (8.20) may be interpreted as a *representation* for the mapping (8.38) through the commutative diagram

$$\begin{tikzcd}
\mathcal{NL}^m(\Omega)^K \arrow[rr, "\mathbf{T}^\sharp"] \arrow[dr, "q^\sharp_{\mathbf{T}}"'] & & \mathcal{NL}(\Omega)^K \\
& \mathcal{NL}_{\mathbf{T}}(\Omega) \arrow[ur, "\widehat{\mathbf{T}}^\sharp"'] &
\end{tikzcd} \tag{8.41}$$

which is nothing but an extension of the diagram (8.17). Indeed, since the mapping (8.20) is an injective uniformly continuous mapping, it follows that

$$\begin{array}{ll} \forall & \mathbf{u}^\sharp, \mathbf{v}^\sharp \in \mathcal{NL}^m(\Omega) \ : \\ & \mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{T}^\sharp \mathbf{v}^\sharp \Leftrightarrow q^\sharp_{\mathbf{T}} \mathbf{u}^\sharp = q^\sharp_{\mathbf{T}} \mathbf{v}^\sharp \end{array} . \tag{8.42}$$

In particular, $q^\sharp_{\mathbf{T}}$ is nothing but the canonical quotient map associated with the equivalence relation

$$\begin{array}{ll} \forall & \mathbf{u}^\sharp, \mathbf{v}^\sharp \in \mathcal{NL}^m(\Omega) \ : \\ & \mathbf{u}^\sharp \sim_{\mathbf{T}^\sharp} \mathbf{v}^\sharp \Leftrightarrow \mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{T}^\sharp \mathbf{v}^\sharp \end{array} . \tag{8.43}$$

The meaning of (8.41) to (8.43) is clear. Indeed, any solution to the generalized equation (8.39) corresponds to a solution to (8.21). In particular, any generalized function

$$\mathbf{U}^\sharp \in q^\sharp_{\mathbf{T}} \left(\mathcal{NL}^m(\Omega)^K \right) \subseteq \mathcal{NL}_{\mathbf{T}}(\Omega)$$

may be interpreted a $\sim_{\mathbf{T}^\sharp}$-equivalence class of generalized functions in $\mathcal{NL}^m(\Omega)^K$. This may be interpreted as a regularity result for the generalized functions in $\mathcal{NL}_{\mathbf{T}}(\Omega)$. However, from the diagram (8.41) we only obtain the *inclusion*

$$q^\sharp_{\mathbf{T}} \left(\mathcal{NL}^m(\Omega)^K \right) \subseteq \mathcal{NL}_{\mathbf{T}}(\Omega), \tag{8.44}$$

and equality in (8.44) may not hold for all nonlinear partial differential operators $\mathbf{T}$. In this regard, we will present sufficient conditions for equality to hold in (8.44) in Section 9.3.

We have so far considered nonlinear partial differential operators which satisfy minimal assumptions on smoothness of the mapping (8.2). In particular, it is only assumed that the mapping (8.2) is *continuous*. However, it most often happens in practice that (8.2) satisfies additional smoothness conditions, namely, that it is continuously differentiable up to a given order. Such additional smoothness conditions will be exploited in Chapter 10 to obtain dramatic regularity results for the solutions of a large class of systems of nonlinear PDEs.

In this regard, we consider now the case of a system of nonlinear PDEs of the form (8.1) to (8.3) where the mapping $\mathbf{F} : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}^K$ which defines the nonlinear operator through (8.3), is assumed to be not only continuous, but also $\mathcal{C}^k$-smooth, that is, $\mathbf{F} \in \mathcal{C}^k\left(\Omega \times \mathbb{R}^M, \mathbb{R}^K\right)$ for some $k \in \mathbb{N} \cup \{\infty\}$. Since $\mathcal{C}^{m+k}(\Omega) \subset \mathcal{C}^m(\Omega)$, we may compute $\mathbf{Tu}$ for each $\mathbf{u} \in \mathcal{C}^{m+k}(\Omega)^K$. In this case, in view of the chain rule of differentiation, it is clear that $\mathbf{Tu} \in \mathcal{C}^k(\Omega)^K$, that is,

$$\mathbf{T} : \mathcal{C}^{m+k}(\Omega)^K \rightarrow \mathcal{C}^k(\Omega)^K . \tag{8.45}$$

More generally, given any $\mathbf{u} \in \mathcal{ML}^{m+k}(\Omega)^K$, applying the mapping (8.10) we obtain $\mathbf{Tu} \in \mathcal{ML}^k(\Omega)^K$. That is, restricting (8.10) to $\mathcal{ML}^{m+k}(\Omega)^K$ yields a mapping

$$\mathbf{T} : \mathcal{ML}^{m+k}(\Omega)^K \to \mathbf{u} \in \mathcal{ML}^k(\Omega)^K . \tag{8.46}$$

Indeed, in view of (8.5) we have, for each $\mathbf{u} \in \mathcal{ML}^{m+k}(\Omega)^K$,

$$\begin{array}{ll} \exists & \Gamma \subset \Omega \text{ closed nowhere dense :} \\ \forall & i = 1, ..., K : \\ \forall & |\alpha| \leq m : \\ & \mathcal{D}^\alpha u_i \in \mathcal{C}^k(\Omega \setminus \Gamma) \end{array}$$

From the smoothness of the mapping (8.2) and the chain rule, it follows that

$$\mathbf{Tu} \in \mathcal{C}^k(\Omega \setminus \Gamma)^K \tag{8.47}$$

which verifies (8.46).

In case the nonlinear partial differential operator satisfies sufficient smoothness conditions, such as those introduced in (8.45) to (8.47), we may introduce a suitable notion of *derivative* of the partial differential operator $\mathbf{T}$. Indeed, for each $\mathbf{u} \in \mathcal{ML}^{m+k}(\Omega)^K$, we may calculate the partial derivatives

$$\mathcal{D}^\beta T_j \mathbf{u} \in \mathcal{ML}^0(\Omega),\ |\beta| \leq k,\ j = 1, ..., K$$

where the T_j, for $j \leq K$, are the components (8.12) of the mapping (8.46). In this regard, we may define a mapping

$$\mathbf{T}^k : \mathcal{ML}^{m+k}(\Omega)^K \to \mathcal{ML}^0(\Omega)^N , \tag{8.48}$$

for a suitable choice of $N \in \mathbb{N} \cup \{\infty\}$, so that the diagram

$$\begin{tikzcd} \mathcal{ML}^{m+k}(\Omega)^K \arrow[rr, "\mathbf{T}^k"] \arrow[dr, "\mathbf{T}"'] & & \mathcal{ML}^0(\Omega)^N \\ & \mathcal{ML}^k(\Omega)^K \arrow[ur, "\mathbf{D}"'] & \end{tikzcd} \tag{8.49}$$

commutes, with the mapping $\mathbf{D}$ defined through

$$\mathbf{D} : \mathcal{ML}^k(\Omega)^K \ni \mathbf{v} \mapsto \left(\mathcal{D}^\beta v_i\right)_{i\leq K}^{|\beta|\leq k} \in \mathcal{ML}^0(\Omega)^N \tag{8.50}$$

Applying the chain rule, we can obtain an explicit expression for the mapping (8.48) in terms of the mapping (8.2), which defines the partial differential operator (8.46), and its derivatives. Such a formula, however, is typically rather involved. As such, we will rather express it in terms of a suitable jointly continuous mapping

$$\mathbf{F}^k : \Omega \times \mathbb{R}^L \to \mathbb{R}^N, \tag{8.51}$$

for a suitable integer L. In particular, we may define the components $T_{j,\beta}^k$ of the mapping (8.48) through

$$T_{j,\beta}^k \mathbf{u} = (I \circ S)\left(F_{j,\beta}^k(\cdot, ..., u_i, ..., D^\alpha u_i, ...)\right),\ |\alpha| \leq m+k;\ i = 1, ..., K \tag{8.52}$$

where the $F_{j,\beta}^k$ are components of the mapping (8.51). The main result concerning the mapping (8.46) is the following.

Theorem 66 *Let k be finite. Then the mapping (8.46) is uniformly continuous with respect to the Sobolev uniform convergence structures on $\mathcal{ML}^{m+k}(\Omega)^K$ and $\mathcal{ML}^k(\Omega)^K$.*

Proof. The uniform continuity of the mapping (8.48) defined through (8.52) follows by the same arguments used in the proof of Theorem 65. Furthermore, the mapping (8.50) is clearly a uniformly continuous embedding. The uniform continuity of (8.46) now follows from the commutative diagram (8.49). ■

In view of Theorem 66, the mapping (8.46) extends uniquely to a uniformly continuous mapping

$$\mathbf{T}^\sharp : \mathcal{NL}^{m+k}(\Omega)^K \to \mathcal{NL}^k(\Omega)^K \tag{8.53}$$

Furthermore, both the mappings (8.48) and (8.50) are uniformly continuous, so that these mappings may be uniquely extended to uniformly continuous mappings

$$\mathbf{T}^{k\sharp} : \mathcal{NL}^{m+k}(\Omega)^K \to \mathcal{NL}^0(\Omega)^N, \tag{8.54}$$

and

$$\mathbf{D}^\sharp : \mathcal{NL}^k(\Omega)^K \ni \mathbf{v} \mapsto \left(\mathcal{D}^{\beta\sharp} v_i\right)_{i\leq K}^{|\beta|\leq k} \in \mathcal{NL}^0(\Omega)^N. \tag{8.55}$$

As such, the diagram (8.49) extends to the commutative diagram

$$\begin{tikzcd}
\mathcal{NL}^{m+k}(\Omega)^K \arrow[rr, "\mathbf{T}^{k\sharp}"] \arrow[dr, "\mathbf{T}^{\sharp}"'] & & \mathcal{NL}^{0}(\Omega)^N \\
& \mathcal{NL}^{k}(\Omega)^K \arrow[ur, "\mathbf{D}^{\sharp}"'] &
\end{tikzcd} \tag{8.56}$$

Note that, in case both the nonlinear partial differential operator and the right-hand term in the system of nonlinear PDEs (8.1) are $\mathcal{C}^k$-smooth, the extended equation (8.13) is equivalent to

$$\mathbf{T}^k\mathbf{u} = \mathbf{D}\mathbf{f}. \tag{8.57}$$

In view of the extensions (8.53) and (8.55) of the smooth nonlinear partial differential operator, and the uniformly continuous embedding (8.50), respectively, we may formulate the equation corresponding to (8.57) as

$$\mathbf{T}^{k\sharp}\mathbf{u}^{\sharp} = \mathbf{D}^{\sharp}\mathbf{f}. \tag{8.58}$$

It should be noted that the generalized equation (8.39) corresponding to (8.13) is no longer equivalent to the equation (8.58). Indeed, a solution $\mathbf{u}^{\sharp} \in \mathcal{NL}^m(\Omega)^K$ may not have generalized derivatives up to order $m+k$, which is required of any solution to (8.58).

Such additional, and in fact rather minimal, smoothness conditions on the nonlinear partial differential operator turn out to be sufficient for particularly strong regularity properties of generalized solutions to large classes of systems of nonlinear PDEs. As will be shown in Section 10.2, only very basic assumptions of a simple topological nature are involved in the relevant regularity properties of generalized solutions of (8.1).

Chapter 9

Existence of Generalized Solutions

9.1 Approximation Results

In this section we obtain the basic approximation results used to prove the existence of solutions to the generalized equations (8.20) and (8.39). We also show that functions in $\mathcal{ML}^m(\Omega)$ may be suitably approximated by sequences of smooth functions. In particular, we show that $\mathcal{C}^m(\Omega)$ is dense in $\mathcal{ML}^m(\Omega)$.

The first and basic approximation results are essentially multi dimensional versions of the fundamental approximation results (1.108) and (1.110) underlying the Order Completion Method. These results allow for the existence of generalized solutions to (8.1) in the space $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$, that is, a solution to (8.21). Further specializations of these basic results will also be presented. In particular, under certain mild assumptions on the nonlinear partial differential operator (8.3) we obtain bounds for such approximate solutions. These bounds will be used to obtain generalized solutions to (8.1) in $\mathcal{NL}^m(\Omega)^K$, that is, solutions to (8.39). Similar approximation results are also proved for equations that satisfy additional smoothness assumptions, namely, assumptions such as those introduced in Section 8.3. These approximation results for such equations that satisfy additional smoothness conditions result in a strong regularity property for solutions in the Sobolev type spaces of generalized functions. Finally, we investigate the extent to which functions in $\mathcal{ML}^m(\Omega)$ may be approximated by $\mathcal{C}^m$-smooth functions.

We now again consider a system of K nonlinear PDEs of the form (8.1) through (8.3). Recall that the Order Completion Method, as discussed in Section 1.4, for single nonlinear PDEs of the form (1.100) through (1.102) is based on the simple approximation result (1.110). In this section we extend this result to the general K-dimensional case, for $K \geq 1$ arbitrary but given, see [119] for a particular case of such an extension.

A natural assumption on the function $\mathbf{F} : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}^K$, and hence the PDE-operator $\mathbf{T}(x, D)$, and the righthand term $\mathbf{f}$ is that, for every $x \in \Omega$

$$\mathbf{f}(x) \in \operatorname{int}\{\mathbf{F}(x, \xi_{1\alpha}, ..., \xi_{i\alpha}, ...) \, : \, \xi_{i\alpha} \in \mathbb{R}, \, i = 1, ..., K, \, |\alpha| \leq m\}, \tag{9.1}$$

which is a multidimensional version of (1.107). The condition (9.1) is noting but a *sufficient* condition for the system of nonlinear PDEs (8.1) to have usual classical solution on Ω. Note that (9.1) is of a technical nature, and hardly a restriction on the class of PDEs considered. In fact, every linear PDE, and also most nonlinear PDEs of applicable interest satisfy (9.1). It is in fact, as discussed in Section 1.4, a necessary condition for the existence of a classical solution to (8.3) in a neighborhood of x. Assuming that the condition (9.1) holds, we obtain the following basic result.

Theorem 67 *Consider a system of PDEs of the form (8.1) through (8.3) that also satisfies (9.1). For every $\epsilon > 0$ there exists a closed nowhere dense set $\Gamma_\epsilon \subset \Omega$ with zero Lebesgue measure, and a function $\boldsymbol{U}_\epsilon \in \mathcal{C}^m(\Omega \setminus \Gamma_\epsilon)^K$ with components $U_{\epsilon,1}, ..., U_{\epsilon,K}$ such that*

$$f_i(x) - \epsilon \leq T_i(x, D)\,\boldsymbol{U}_\epsilon(x) \leq f_i(x)\,,\; x \in \Omega \setminus \Gamma_\epsilon \tag{9.2}$$

Proof. Let

$$\Omega = \bigcup_{\nu \in \mathbb{N}} C_\nu \tag{9.3}$$

where, for $\nu \in \mathbb{N}$, the compact sets C_ν are n-dimensional intervals

$$C_\nu = [a_\nu, b_\nu] \tag{9.4}$$

with $a_\nu = (a_{\nu,1}, ..., a_{\nu,n})$, $b_\nu = (b_{\nu,1}, ..., b_{\nu,n}) \in \mathbb{R}^n$ and $a_{\nu,i} \leq b_{\nu,i}$ for every $i = 1, ..., n$. We also assume that C_ν, with $\nu \in \mathbb{N}$ are locally finite, that is,

$$\begin{array}{ll} \forall & x \in \Omega : \\ \exists & V_x \subseteq \Omega \text{ a neighborhood of } x : \\ & \{\nu \in \mathbb{N} : C_\nu \cap V_x \neq \emptyset\} \text{ is finite} \end{array} \tag{9.5}$$

We also assume that the interiors of C_ν, with $\nu \in \mathbb{N}$, are pairwise disjoint. We note that such C_ν exist, see [58].
Let us now take $\epsilon > 0$ given arbitrary but fixed. Let us take $\nu \in \mathbb{N}$ and apply Proposition 68 to each $x_0 \in C_\nu$. Then we obtain $\delta_{x_0} > 0$ and $P_{x_0,1}, ..., P_{x_0,K}$ polynomial in $x \in \mathbb{R}^n$ such that

$$f_i(x) - \epsilon \leq T_i(x, D)\,\mathbf{P}_{x_0}(x) \leq f(x)\,,\; x \in \Omega \cap \overline{B}(x_0, \delta_{x_0}) \text{ and } i = 1, ..., K \tag{9.6}$$

where $\mathbf{P}_{x_0} : \mathbb{R}^n \to \mathbb{R}^K$ is the K-dimensional vector valued function with components $P_{x_0,1}, ..., P_{x_0,K}$. Since C_ν is compact, it follows that

$$\begin{array}{ll} \exists & \delta > 0 : \\ \forall & x_0 \in C_\nu : \\ \exists & P_{x_0,1}, ..., P_{x_0,K} \text{ polynomial in } x \in \mathbb{R}^n : \\ & \|x - x_0\| \leq \delta \Rightarrow f_i(x) - \epsilon \leq T_i(x, D)\,\mathbf{P}_{x_0}(x) \leq f(x)\,,\; x \in \overline{B}(x_0, \delta) \cap C_\nu \end{array} \tag{9.7}$$

where $i = 1, ..., K$. Subdivide C_ν into n-dimensional intervals $I_{\nu,1}, ..., I_{\nu,\mu}$ with diameter not exceeding δ such that their interiors are pairwise disjoint. If a_j with

$j = 1, ..., \mu$ is the center of the interval $I_{\nu,j}$ then by (9.7) there exists $P_{a_j,1}, ..., P_{a_j,K}$ polynomial in $x \in \mathbb{R}^n$ such that

$$f_i(x) - \epsilon \leq T_i(x, D)\,\mathbf{P}_{a_j}(x) \leq f_i(x),\ x \in I_{\nu,j} \tag{9.8}$$

where $i = 1, ..., K$. Now set

$$\Gamma_{\nu,\epsilon} = C_\nu \setminus \left(\left(\bigcup_{j=1}^{\mu} \mathrm{int} I_{\nu,j} \right) \cup \mathrm{int} C_\nu \right) \tag{9.9}$$

that is, $\Gamma_{\nu,\epsilon}$ is a rectangular grid generated as a finite union of hyperplanes. Furthermore, using (9.8), we find

$$\mathbf{U}_{\nu,\epsilon} \in \mathcal{C}^m\left(C_\nu \setminus \Gamma_{\nu,\epsilon}\right) \tag{9.10}$$

such that

$$f_i(x) - \epsilon \leq T_i(x, D)\,\mathbf{U}_{\nu,\epsilon}(x) \leq f_i(x),\ x \in C_\nu \setminus \Gamma_{\nu,\epsilon} \tag{9.11}$$

In view of (9.5) it follows that

$$\Gamma_\epsilon = \bigcup_{\nu \in \mathbb{N}} \Gamma_{\nu,\epsilon} \text{ is closed nowhere dense and } \mathrm{mes}(\Gamma_\epsilon) = 0 \tag{9.12}$$

From (9.3), (9.10) and (9.11) we obtain (9.2). ■

The above proof relies on the following proposition which is in fact the basic approximation result.

Proposition 68 *Consider a system of PDEs of the form (8.1) through (8.3) that also satisfies (9.1). Then*

$$\begin{array}{ll} \forall & x_0 \in \Omega : \\ \forall & \epsilon > 0 : \\ \exists & \delta > 0,\ P_1, ..., P_K \text{ polynomial in } x \in \mathbb{R}^n : \\ & x \in B(x_0, \delta) \cap \Omega,\ 1 \leq i \leq k \Rightarrow f_i(x) - \epsilon \leq T_i(x, D)\,\boldsymbol{P}(x) \leq f_i(x) \end{array} \tag{9.13}$$

Here $\boldsymbol{P}$ *is the* K*-dimensional vector valued function with components* $P_1, ..., P_K$*.*

Proof. For any $x_0 \in \Omega$ and $\epsilon > 0$ small enough it follows by (9.1) that there exist

$$\xi_{i\alpha} \in \mathbb{R} \text{ with } i = 1, ..., K \text{ and } |\alpha| \leq m \tag{9.14}$$

such that

$$F_i(x_0, ..., \xi_{i\alpha}, ...) = f_i(x_0) - \frac{\epsilon}{2} \tag{9.15}$$

Now take $P_1, ..., P_K$ polynomials in $x \in \mathbb{R}^n$ that satisfy

$$D^\alpha P_i(x_0) = \xi_{i\alpha} \text{ for } i = 1, ..., K \text{ and } |\alpha| \leq m \tag{9.16}$$

Then it is clear that

$$T_i(x,D)\,\mathbf{P}(x_0) - f_i(x_0) = -\frac{\epsilon}{2} \tag{9.17}$$

where $\mathbf{P}$ is the K-dimensional vector valued function on $\mathbb{R}^n$ with components $P_1, ..., P_K$. Hence (9.13) follows by the continuity of the f_i and the F_i. ∎

It should be observed that, in contradistinction with the usual functional analytic methods, the local *lower solution* in Proposition 68 is constructed in a particularly simple way. Indeed, it is obtained by nothing but a straightforward application of the continuity of the mapping $\mathbf{F}$. Using exactly these same techniques, one may prove the existence of the corresponding approximate *upper solutions*.

Proposition 69 *Consider a system of PDEs of the form (8.1) through (8.3) that also satisfies (9.1). Then*

$$\begin{array}{ll} \forall & x_0 \in \Omega : \\ \forall & \epsilon > 0 : \\ \exists & \delta > 0,\ P_1, ..., P_K \text{ polynomial in } x \in \mathbb{R}^n : \\ & x \in B(x_0,\delta) \cap \Omega,\ 1 \leq i \leq k \Rightarrow f_i(x) < T_i(x,D)\,\boldsymbol{P}(x) < f_i(x) + \epsilon \end{array} \tag{9.18}$$

Here $\boldsymbol{P}$ *is the* K*-dimensional vector valued function with components* $P_1, ..., P_K$*.*

In connection with the *global* approximation result presented in Theorem 67, and as was mentioned in connection with Proposition 68, the approximation result above is based *solely* on the existence of a compact tiling of open subsets of $\mathbb{R}^n$, the properties of compact subsets of $\mathbb{R}^n$ and the continuity of usual real valued functions. Hence it makes no use of so called *advanced mathematics*. In particular, techniques from functional analysis are not used at all. Instead, the relevant techniques belong rather to the classical theory of real functions.

Note that Theorem 67 makes no claim concerning the convergence, or otherwise, of the sequence $(\mathbf{u}_n)$ in $\mathcal{ML}^m(\Omega)^K$. Indeed, assuming only that (9.1) is satisfied, it is typically possible to construct a sequence $(\mathbf{U}_n)$ that satisfies Theorem 67, and is unbounded on every neighborhood of every point of Ω. This follows easily from the fact that, in general, for a fixed $x_0 \in \Omega$, the sets

$$\{\xi \in \mathbb{R}^M : \mathbf{F}(x_0,\xi) = \mathbf{f}(x_0)\}$$

may be unbounded.

In view of the above remarks, it appears that a stronger assumption than (9.1) may be required in order to construct generalized solutions to (8.1) in $\mathcal{NL}^m(\Omega)^K$. When formulating such an appropriate condition on the system of PDEs (8.1), one should keep in mind that the Order Completion Method [119], and in particular the pseudo-topological version of the theory developed in this work, is based on some basic topological processes, namely, the completion of uniform convergence spaces, and the simple condition (9.1), which is formulated entirely in terms of the usual real mappings $\mathbf{F}$ and $\mathbf{f}$. In particular, (9.1) does not involve any topological structures on function spaces, or mappings on such spaces. Furthermore, other than the mere continuity of the mapping $\mathbf{F}$, (9.1) places no restriction on the *type* of equation treated. As such, it is then clear that any further assumptions that we may wish to impose on the system of equations (8.1) in order to obtain generalized solutions in $\mathcal{NL}^m(\Omega)^K$ should involve only basic topological properties of the mapping $\mathbf{F}$, and should not involve any restrictions on the type of equations.

In formulating such a condition on the system of PDEs (8.1) that will ensure the existence of a generalized solution in $\mathcal{NL}^m(\Omega)^K$, it is helpful to first understand more completely the role of the condition (9.1) in the proof of the local approximation result Proposition 68. In particular, and as is clear from the proof of Proposition 68, the condition (9.1) relates to the continuity of the mapping $\mathbf{F}$. Furthermore, and as has already been mentioned, the approximations constructed in Theorem 67 and Proposition 68 concern only convergence in the range space of the operator $\mathbf{T}$ associated with (8.1). Our interest here lies in constructing suitable approximations in the domain of $\mathbf{T}$, and as such, properties of the inverse of the mapping $\mathbf{F}$ may prove to be particularly useful. In view of these remarks, we introduce the following condition.

$$\begin{array}{ll} \forall & x_0 \in \Omega : \\ \exists & \xi(x_0) \in \mathbb{R}^M,\ \mathbf{F}(x_0, \xi(x_0)) = \mathbf{f}(x_0) \ : \\ \exists & V \in \mathcal{V}_{x_0},\ W \in \mathcal{V}_{\xi(x_0)} : \\ & \mathbf{F} : V \times W \to \mathbb{R}^K \text{ open} \end{array} \tag{9.19}$$

Note that the condition (9.19) above, although more restrictive than (9.1), allows for the treatment of a large class of equations. In particular, each equation of the form

$$D_t\mathbf{u}(x,t) + \mathbf{G}(x, t, \mathbf{u}(x,t), ..., D_x^\alpha \mathbf{u}(x,t), ...) = \mathbf{f}(x,t)$$

with the mapping $\mathbf{G}$ merely continuous, satisfies (9.19). Indeed, in this case the mapping $\mathbf{F} : \Omega \times \mathbb{R}^M \to \mathbb{R}^K$ that defines the equation through (8.3) is both open and surjective. Indeed, each component F_j of $\mathbf{F}$ is *linear* in ξ_j, where $\xi = (\xi_1, ..., \xi_j, ..., \xi_M)$ belongs to $\mathbb{R}^M$, from which our assertion follows immediately. Other classes of equations that satisfy (9.19) can be easily exhibited by using, for instance, various open mapping theorems, see for instance [19, 41.7]. The following is a specialization of the global approximation result Theorem 67.

Theorem 70 *Consider a system of nonlinear PDEs of the form (8.1) through (8.3) that also satisfies (9.19). Then there is a sequence (Γ_n) of closed nowhere dense set $\Gamma_n \subset \Omega$, which is increasing with respect to inclusion, and a sequence of function $(\boldsymbol{V}_n)$ such that $\boldsymbol{V}_n \in \mathcal{C}^m(\Omega \setminus \Gamma_n)$ and*

$$\begin{array}{l} \forall \quad j = 1, ..., K : \\ \quad f_j(x) - \frac{1}{n} \le T_j(x, D)\, \boldsymbol{V}_n(x) \le f_j(x),\ x \in \Omega \setminus \Gamma_n \end{array}.$$

Furthermore, for each $|\alpha| \leq m$ *and every* $i = 1, ..., K$ *there are sequences* $\left(\lambda_{n,i}^{\alpha}\right)$ *and* $\left(\mu_{n,i}^{\alpha}\right)$ *such that* $\lambda_{n,i}^{\alpha}, \mu_{n,i}^{\alpha} \in \mathcal{C}^0(\Omega \setminus \Gamma_n)$ *which sequences satisfy*

$$\begin{array}{ll} \forall & n \in \mathbb{N} : \\ \forall & |\alpha| \leq m : \\ \forall & i = 1, ..., K : \\ & 1) \quad \lambda_{n,i}^{\alpha}(x) < D^{\alpha} V_{n,i}(x) < \mu_{n,i}^{\alpha}(x), \ x \in \Omega \setminus \Gamma_n \\ & 2) \quad \lambda_{n,i}^{\alpha}(x) < \lambda_{n+1,i}^{\alpha}(x) < \mu_{n+1,i}^{\alpha}(x) < \mu_{n,i}^{\alpha}(x), \ x \in \Omega \setminus \Gamma_{n+1} \end{array}$$

and

$$\begin{array}{ll} \forall & x \in \Omega \setminus \left(\bigcup_{n \in \mathbb{N}} \Gamma_n\right) : \\ \forall & |\alpha| \leq m : \\ \forall & i = 1, ..., K : \\ & \sup\{\lambda_{n,i}^{\alpha}(x) \ : n \in \mathbb{N}\} = \inf\{\mu_{n,i}^{\alpha}(x) \ : n \in \mathbb{N}\} \end{array}$$

Proof. Set

$$\Omega = \bigcup_{\nu \in \mathbb{N}} C_{\nu} \tag{9.20}$$

where, for $\nu \in \mathbb{N}$, the compact set C_{ν} is an n-dimensional intervals

$$C_{\nu} = [a_{\nu}, b_{\nu}] \tag{9.21}$$

with $a_{\nu} = (a_{\nu,1}, ..., a_{\nu,n})$, $b_{\nu} = (b_{\nu,1}, ..., b_{\nu,n}) \in \mathbb{R}^n$ and $a_{\nu,j} \leq b_{\nu,j}$ for every $j = 1, ..., n$. We also assume that the collection of sets $\{C_{\nu} : \nu \in \mathbb{N}\}$ is locally finite, that is,

$$\begin{array}{ll} \forall & x \in \Omega : \\ \exists & V \subseteq \Omega \text{ a neighborhood of } x : \\ & \{\nu \in \mathbb{N} : C_{\nu} \cap V \neq \emptyset\} \text{ is finite} \end{array} \tag{9.22}$$

Furthermore, let the interiors of the C_{ν}, with $\nu \in \mathbb{N}$, be pairwise disjoint.
Let C_{ν} be arbitrary but fixed. In view of (9.19) and the continuity of $\mathbf{f}$, we have

$$\begin{array}{ll} \forall & x_0 \in C_{\nu} : \\ \exists & \xi(x_0) \in \mathbb{R}^M, \ \mathbf{F}(x_0, \xi(x_0)) = \mathbf{f}(x_0) \ : \\ \exists & \delta, \epsilon > 0 : \\ & 1) \quad \{(x, \mathbf{f}(x)) \ : \ \|x - x_0\| < \delta\} \subset \operatorname{int}\left\{(x, \mathbf{F}(x, \xi)) \left| \begin{array}{l} \|x - x_0\| < \delta \\ \|\xi - \xi(x_0)\| < \epsilon \end{array} \right.\right\} \\ & 2) \quad \mathbf{F} : B_{\delta}(x_0) \times B_{2\epsilon}(\xi(x_0)) \rightarrow \mathbb{R}^K \text{ open} \end{array} \tag{9.23}$$

For each $x_0 \in C_{\nu}$, fix $\xi(x_0) \in \mathbb{R}^M$ in (9.23). Since C_{ν} is compact, it follows from (9.23) that

$$\begin{array}{ll} \exists & \delta > 0 : \\ \forall & x_0 \in C_{\nu} : \\ \exists & \epsilon_{x_0} > 0 : \\ & 1) \quad \{(x, \mathbf{f}(x)) \ : \ \|x - x_0\| < \delta\} \subset \operatorname{int}\left\{(x, \mathbf{F}(x, \xi)) \left| \begin{array}{l} \|x - x_0\| < \delta \\ \|\xi - \xi(x_0)\| < \epsilon_{x_0} \end{array} \right.\right\} \\ & 2) \quad \mathbf{F} : B_{\delta}(x_0) \times B_{2\epsilon_{x_0}}(\xi(x_0)) \rightarrow \mathbb{R}^K \text{ open} \end{array} \tag{9.24}$$

Subdivide C_ν into n-dimensional intervals $I_{\nu,1},...,I_{\nu,\mu_\nu}$ with diameter not exceeding δ such that their interiors are pairwise disjoint. If $a_{\nu,j}$ with $j=1,...,\mu_\nu$ is the center of the interval $I_{\nu,j}$ then by (9.24) we have

$$\begin{array}{ll} \forall & j=1,...,\mu_\nu : \\ \exists & \epsilon_{\nu,j}>0 : \\ & \begin{array}{ll} 1) & \{(x,\mathbf{f}(x)) \ : \ x\in I_{\nu,j}\} \subset \operatorname{int}\left\{(x,\mathbf{F}(x,\xi)) \left| \begin{array}{l} x\in I_{\nu,j} \\ \|\xi-\xi(a_{\nu,j})\|<\epsilon_{\nu,j} \end{array}\right.\right\} \\ 2) & \mathbf{F}: I_{\nu,j}\times B_{2\epsilon_{\nu,j}}(\xi(a_{\nu,j}))\rightarrow \mathbb{R}^K \text{ open} \end{array} \end{array} \tag{9.25}$$

Take $0<\gamma<1$ arbitrary but fixed. In view of Proposition 68 and (9.25), we have

$$\begin{array}{ll} \forall & x_0\in I_{\nu,j} : \\ \exists & \mathbf{U}_{x_0}=\mathbf{U}\in\mathcal{C}^m(\mathbb{R}^n)^K \ : \\ \exists & \delta=\delta_{x_0}>0 : \\ & x\in B_\delta(x_0)\cap I_{\nu,j}\Rightarrow\left(\begin{array}{ll} 1) & (D^\alpha U_i(x))_{i\leq K}^{|\alpha|\leq m}\in B_{\epsilon_{\nu,j}}(\xi(a_{\nu,j})) \\ 2) & i\leq K\Rightarrow f_i(x)-\gamma<T_i(x,D)\mathbf{U}(x)<f_i(x) \end{array}\right) \end{array}$$

As above, we may subdivide $I_{\nu,j}$ into pairwise disjoint, n-dimensional intervals $J_{\nu,j,1},...,J_{\nu,j,\mu_{\nu,j}}$ so that for $k=1,...,\mu_{\nu,j}$ we have

$$\begin{array}{ll} \exists & \mathbf{U}^{\nu,j,k}=\mathbf{U}\in\mathcal{C}^m(\mathbb{R}^n)^K \ : \\ \forall & x\in J_{\nu,j,k} : \\ & \begin{array}{ll} 1) & \left(D^\alpha U_i(x)_{i\leq K}^{|\alpha|\leq m}\right)\in B_{\epsilon_{\nu,j}}(\xi(a_{\nu,j})) \, , \, |\alpha|\leq m \\ 2) & i\leq K\Rightarrow f_i(x)-\gamma<T_i(x,D)\mathbf{U}(x)<f_i(x) \end{array} \end{array} \tag{9.26}$$

Set

$$\Gamma_1=\Omega\setminus\left(\bigcup_{\nu\in\mathbb{N}}\left(\bigcup_{j=1}^{\mu_\nu}\left(\bigcup_{k=1}^{\mu_{\nu,j}}\operatorname{int}J_{\nu,j,k}\right)\right)\right).$$

and

$$\mathbf{V}_1=\sum_{\nu\in\mathbb{N}}\left(\sum_{j=1}^{\mu_\nu}\left(\sum_{k=1}^{\mu_{\nu,j}}\chi_{J_{\nu,j,k}}\mathbf{U}_{\nu,j,k}\right)\right)$$

where $\chi_{J_{\nu,j,k}}$ is the characteristic function of $J_{\nu,j,k}$. Then Γ_1 is closed nowhere dense, and $\mathbf{V}_1\in\mathcal{C}^m(\Omega\setminus\Gamma_1)^K$. In view of (9.26) we have, for each $i=1,...,K$

$$f_i(x)-\gamma<T_i(x,D)\mathbf{V}_1(x)<f_i(x)\, , \, x\in\Omega\setminus\Gamma_1$$

Furthermore, for each $\nu\in\mathbb{N}$, for each $j=1,...,\mu_\nu$, each $k=1,...,\mu_{\nu,j}$, each $|\alpha|\leq m$ and every $i=1,...,K$ we have

$$x\in\operatorname{int}J_{\nu,j,k}\Rightarrow\xi_i^\alpha(a_{\nu,j})-\epsilon<D^\alpha V_{1,i}(x)<\xi_i^\alpha(a_{\nu,j})+\epsilon$$

Therefore the functions $\lambda_{1,i}^{\alpha}, \mu_{1,i}^{\alpha} \in \mathcal{C}^0(\Omega \setminus \Gamma_1)$ defined as

$$\lambda_{1,i}^{\alpha}(x) = \xi_i^{\alpha}(a_j) - 2\epsilon_{\nu,j} \text{ if } x \in \mathrm{int}I_{\nu,j}$$

and

$$\mu_{1,i}^{\alpha}(x) = \xi_i^{\alpha}(a_j) + 2\epsilon_{\nu,j} \text{ if } x \in \mathrm{int}I_{\nu,j},$$

respectively, satisfy

$$\lambda_{1,i}^{\alpha}(x) < D^{\alpha}V_{1,i}(x) < \mu_{1,i}^{\alpha}(x), \ x \in \Omega \setminus \Gamma_1$$

and

$$\mu_{1,i}^{\alpha}(x) - \lambda_{1,i}^{\alpha}(x) < 4\epsilon_{\nu,j}, \ x \in \mathrm{int}I_{\nu,j}$$

Applying (9.25) restricted to $\Omega \setminus \Gamma_1$, and proceeding in a fashion similar as above, we may construct, for each $n \in \mathbb{N}$ such that $n > 1$, a closed nowhere dense set $\Gamma_n \subset \Omega$, so that $\Gamma_n \subseteq \Gamma_{n+1}$, a function $\mathbf{V}_n \in \mathcal{C}^m(\Omega \setminus \Gamma_n)^K$ and functions $\lambda_{n,i}^{\alpha}, \mu_{n,i}^{\alpha} \in \mathcal{C}^0(\Omega \setminus \Gamma_n)$ so that, for each $i = 1, ..., K$

$$f_i(x) - \frac{\gamma}{n} < T_i(x, D)\mathbf{V}_n(x) < f_i(x), \ x \in \Omega \setminus \Gamma_n. \tag{9.27}$$

and for every $|\alpha| \leq m$

$$\lambda_{n-1,i}^{\alpha}(x) < \lambda_{n,i}^{\alpha}(x) < D^{\alpha}V_{n,i}(x) < \mu_{n,i}^{\alpha}(x) < \mu_{n-1,i}^{\alpha}(x), \ x \in \Omega \setminus \Gamma_n \tag{9.28}$$

and

$$\mu_{n,i}^{\alpha}(x) - \lambda_{n,i}^{\alpha}(x) < \frac{4\epsilon_{\nu,j}}{n}, \ x \in (\mathrm{int}I_{\nu,j}) \cap (\Omega \setminus \Gamma_n). \tag{9.29}$$

This completes the proof. ∎

At this point we proceed to establish an approximation result for equations that satisfy addition smoothness conditions such as those introduced in Section 8.3. In particular, we will establish a version of Theorem 70 that incorporates also the derivatives of $\mathbf{Tu}$, for a sufficiently smooth function $\mathbf{u}$. Owing to the representation (8.49) of the operator (8.46), this result follows by the same elementary arguments that lead to Theorem 70.

In this regard, we consider a system of nonlinear PDEs of the form (8.1) to (8.3) so that the mapping (8.2) is $\mathcal{C}^k$-smooth, for some $k \in \mathbb{N}$. In view of the representation (8.49), the condition (9.19) on the mapping (8.2) is replaced with a suitable assumption on the mapping (8.51), namely, we assume

$$\begin{array}{ll} \forall & x_0 \in \Omega : \\ \exists & \xi(x_0) \in \mathbb{R}^L, \ \mathbf{F}^k(x_0, \xi(x_0)) = (D^{\alpha}f_i(x_0))_{i \leq K}^{|\alpha| \leq m} : \\ \exists & V \in \mathcal{V}_{x_0}, \ W \in \mathcal{V}_{\xi(x_0)} : \\ & \mathbf{F}^k : V \times W \to \mathbb{R}^N \text{ open} \end{array}. \tag{9.30}$$

The following now follows by the representation (8.49) and the same arguments used in the proof of Theorem 70.

Theorem 71 *Consider a system of nonlinear PDEs of the form (8.1) through (8.3) with both the righthand term and the mapping (8.2) $\mathcal{C}^k$-smooth. Also assume that (9.30) holds. Then there is an increasing sequence (Γ_n) of closed nowhere dense sets $\Gamma_n \subset \Omega$ and a sequence of function $(\boldsymbol{V}_n)$ such that $\boldsymbol{V}_n \in \mathcal{C}^{m+k}(\Omega \setminus \Gamma_n)$ and*

$$\begin{array}{ll} \forall & i = 1, ..., K : \\ \forall & |\beta| \leq k : \\ & D^\beta f_i(x) - \frac{1}{n} \leq D^\beta T_i(x, D)\, \boldsymbol{V}_n(x) \leq D^\beta f_i(x),\ x \in \Omega \setminus \Gamma_n \end{array}.$$

Furthermore, for each $|\alpha| \leq m + k$ and every $i = 1, ..., K$ there are sequences $\left(\lambda_{n,i}^\alpha\right)$ and $\left(\mu_{n,i}^\alpha\right)$ so that $\lambda_{n,i}^\alpha, \mu_{n,i}^\alpha \in \mathcal{C}^0(\Omega \setminus \Gamma_n)$ which satisfy

$$\begin{array}{lll} \forall & n \in \mathbb{N} : & \\ \forall & |\alpha| \leq m + k : & \\ \forall & i = 1, ..., K : & \\ & 1) & \lambda_{n,i}^\alpha(x) < D^\alpha V_{n,i}(x) < \mu_{n,i}^\alpha(x),\ x \in \Omega \setminus \Gamma_n \\ & 2) & \lambda_{n,i}^\alpha(x) < \lambda_{n+1,i}^\alpha(x) < \mu_{n+1,i}^\alpha(x) < \mu_{n,i}^\alpha(x),\ x \in \Omega \setminus \Gamma_{n+1} \end{array}$$

and and

$$\begin{array}{ll} \forall & x \in \Omega \setminus \left(\bigcup_{n \in \mathbb{N}} \Gamma_n\right) : \\ \forall & |\alpha| \leq m + k : \\ \forall & i = 1, ..., K : \\ & \sup\{\lambda_{n,i}^\alpha(x) \ : n \in \mathbb{N}\} = \inf\{\mu_{n,i}^\alpha(x) \ : n \in \mathbb{N}\} \end{array}.$$

By employing the representation (8.49), we may verify Theorem 71 by using exactly the same techniques and arguments as in the proof of Theorem 70. As such we omit it.

Remark 72 *It should be noted that Theorem 67 may be reproduced for nonlinear partial differential operators that satisfy additional smoothness conditions. In particular, if we assume that the mapping (8.2) as well as the righthand term in (8.1) are both $\mathcal{C}^k$-smooth, for some $k \in \mathbb{N}$, then we may obtain version of Theorem 67 that also incorporates the derivatives of $\boldsymbol{Tu}$ up to order k. This is not a significant improvement, as it does not lead to a more general or powerful existence or regularity result than are already possible using only Theorem 67.*

We now turn to the final result of the section, namely, we show that each function $u \in \mathcal{ML}^m(\Omega)$ may be suitably approximated by functions in $\mathcal{C}^m(\Omega)$. Together with certain basic compactness results to be presented in Section 10.1, this results in a significant improvements on the regularity of the generalized solutions to a large class of equations.

The result we present now is based on the well known principle of *Partition of Unity*. In this regard, we may recall the following version of this principle.

Theorem 73 **[150] Let $\mathcal{O}$ be a locally finite open cover of a smooth manifold M. Then there is a collection*

$$\{\varphi_U : M \to [0,1] : U \in \mathcal{O}\}$$

of $\mathcal{C}^\infty$-smooth mappings φ_U such that the following hold:

i) *For each $U \in \mathcal{O}$, the support of φ_U is contained in U.*

ii) *For each $x \in M$, we have $\sum_{U \in \mathcal{O}} \varphi_U(x) = 1$.*

A useful consequence of Theorem 73 concerns the separation of disjoint, closed sets by $\mathcal{C}^\infty$-smooth, real valued mappings. In this regard, consider a nonempty, open set $\Omega \subseteq \mathbb{R}^n$. Let S and T be disjoint, nonempty, closed subsets of Ω. Then it follows from Theorem 73 that

$$\begin{array}{ll} \exists & \varphi \in \mathcal{C}^\infty(\Omega, [0,1]) : \\ & 1) \quad x \in A \Rightarrow \varphi(x) = 1 \\ & 2) \quad x \in B \Rightarrow \varphi(x) = 0 \end{array} . \tag{9.31}$$

This leads to the following simple approximation result.

Theorem 74 *For any $u \in \mathcal{ML}^m(\Omega)$, denote by $\Gamma_u \subset \Omega$ the smallest closed nowhere dense set such that $u \in \mathcal{C}^m(\Omega \setminus \Gamma_u)$. Then there exists a sequence (u_n) in $\mathcal{C}^m(\Omega)$ such that*

$$\begin{array}{ll} \forall & A \subset \Omega \setminus \Gamma_u \textit{ compact :} \\ \forall & |\alpha| \leq m : \\ & (D^\alpha u_n) \textit{ converges uniformly to } \mathcal{D}^\alpha u \textit{ on } A \end{array} .$$

Proof. For each $n \in \mathbb{N}$, we consider the set $\overline{B}_{\frac{1}{n}}(\Gamma)$, which is the closure of the set

$$\left\{ x \in \Omega \,\middle|\, \begin{array}{ll} \exists & x_0 \in \Gamma : \\ & \|x - x_0\| \leq \frac{1}{2n} \end{array} \right\}$$

and the set

$$\overline{C}_{\frac{1}{n}}(\Gamma) = \left\{ x \in \Omega \,\middle|\, \begin{array}{ll} \forall & x_0 \in \Gamma : \\ & \|x - x_0\| \geq \frac{1}{n} \end{array} \right\}$$

Clearly, each of the sets $\overline{B}_{\frac{1}{n}}(\Gamma)$ and $\overline{C}_{\frac{1}{n}}(\Gamma)$ is closed, and for each $n \in \mathbb{N}$, $\overline{B}_{\frac{1}{n}}(\Gamma)$ and $\overline{C}_{\frac{1}{n}}(\Gamma)$ are disjoint. As such, by (9.31), there exists a function $\varphi_n \in \mathcal{C}^\infty(\Omega, [0,1])$ so that

$$\varphi_n(x) = \begin{cases} 0 & if \quad x \in \overline{B}_{\frac{1}{n}}(\Gamma) \\ & \\ 1 & if \quad x \in \overline{C}_{\frac{1}{n}}(\Gamma) \end{cases}$$

Clearly, each of the functions $u_n = \varphi_n u$ is $\mathcal{C}^m$-smooth and satisfies

$$u_n(x) = \begin{cases} 0 & if \quad x \in \overline{B}_{\frac{1}{n}}(\Gamma) \\ u(x) & if \quad x \in \overline{C}_{\frac{1}{n}}(\Gamma) \end{cases}$$

Furthermore,

$$\bigcap_{n \in \mathbb{N}} \overline{B}_{\frac{1}{n}}(\Gamma) = \Gamma$$

and

$$\bigcup_{n \in \mathbb{N}} \overline{C}_{\frac{1}{n}}(\Gamma) = \Omega \setminus \Gamma$$

which completes the proof. ∎

Remark 75 *It should be noted that the approximations constructed in Theorems 67, 70 and 71 are in fact $\mathcal{C}^\infty$-smooth everywhere except on a closed nowhere dense set. Indeed, each approximating functions is obtained by arranging, in an appropriate way, suitable functions obtained through Proposition 68, which are polynomials in $x \in \mathbb{R}^n$.*

The approximation results presented in this section are fundamental to our approach to constructing generalized solutions to large classes of nonlinear PDEs. In this regard, and as we have mentioned already, it should be noted that none of the results are based on so called ‘advanced mathematics’. Indeed, functional analysis and topology are not used at all. Rather, the techniques used belong to the classical theory of real functions.

9.2 Solutions in Pullback Uniform Convergence Spaces

In this section we present the first and basic existence result within the context of the spaces of generalized functions introduced in Chapter 8. In particular, we prove that every system of nonlinear PDEs of the form (8.1) to (8.3) that also satisfies the natural and rather minimal condition (9.1) will have a solution in the pullback type space of generalized functions associated with the particular nonlinear operator (8.10). As a consequence of the way in which the space of generalized functions is constructed, one also obtains immediately the uniqueness of a generalized solution to (8.1). This result amounts to a reformulation of the main existence and uniqueness result obtained through the Order Completion Method [119] in terms of uniform convergence spaces and their completions. Furthermore, and as mentioned in Section

8.1, such a recasting allows for the application of convergence theoretic techniques to questions related to the structure and regularity of generalized solutions. Such methods may prove to be more suitable to these problems than the order theoretic tools involved in the Order Completion Method.

Recall that the space $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$ associated with the mapping (8.10) consists of equivalence classes of functions in $\mathcal{ML}^m(\Omega)^K$ under the equivalence relation (8.15). With the mapping (8.10) we associate in a canonical way the *injective* mapping (8.16). In view of the commutative diagram (8.17), the equations (8.13) and (8.18) are *equivalent*. Since the mapping (8.16) is injective, the initial uniform convergence structure (8.19) on $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$ with respect to (8.16) is Hausdorff. As such, we may construct its completion $\mathcal{NL}_{\mathbf{T}}(\Omega)$. In particular, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{ML}^m_{\mathbf{T}}(\Omega) \arrow[r, "\widehat{\mathbf{T}}"] \arrow[d, "\phi"'] & \mathcal{ML}^0(\Omega)^K \arrow[d, "\varphi"] \\ \mathcal{NL}_{\mathbf{T}}(\Omega) \arrow[r, "\widehat{\mathbf{T}}^\sharp"] & \mathcal{NL}(\Omega)^K \end{tikzcd} \tag{9.32}$$

where ϕ and φ are the canonical uniformly continuous embeddings associated with the completions of $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$ and $\mathcal{ML}^0(\Omega)^K$, respectively, and $\widehat{\mathbf{T}}^\sharp$ is the unique extension of the mapping $\widehat{\mathbf{T}}$ through uniform continuity. Note that, in view of the injectivity of the mapping (8.16), it is in fact a uniformly continuous embedding. As such, and as an immediate consequence of Corollary 37, it follows that the mapping $\widehat{\mathbf{T}}^\sharp$ is injective. The existence and uniqueness result we present now follows by the basic approximation result Theorem 67, and the diagram (9.32).

Theorem 76 *For every $\boldsymbol{f} \in \mathcal{C}^0(\Omega)^K$ that satisfies (9.1), there exists a unique $\boldsymbol{U}^\sharp \in \mathcal{NL}_{\boldsymbol{T}}(\Omega)$ such that*

$$\widehat{\boldsymbol{T}}^\sharp \boldsymbol{U}^\sharp = \boldsymbol{f} \tag{9.33}$$

Proof. First let us show existence. For every $n \in \mathbb{N}$, Theorem 67 yields a closed nowhere dense set $\Gamma_n \subset \Omega$ and a function $\mathbf{u}_n \in \mathcal{C}^m(\Omega \setminus \Gamma_n)$ that satisfies

$$x \in \Omega \setminus \Gamma_n \Rightarrow f_i(x) - \frac{1}{n} \leq T_i(x, D)\,\mathbf{u}_n(x) \leq f_i(x)\,,\ i = 1, ..., K \tag{9.34}$$

Since Γ_n is closed nowhere dense we associate $\mathbf{u}_n$ with a function $\mathbf{v}_n \in \mathcal{ML}^m(\Omega)$ in a unique way. Indeed, consider for instance the function

$$\mathbf{w}_n : x \mapsto \begin{cases} \mathbf{u}_n(x) & \text{if } x \in \Omega \setminus \Gamma \\ 0 & \text{if } x \in \Gamma \end{cases}$$

Now let $\mathbf{v}_n$ be the K-dimensional vector valued function with components $v_n^i = (I \circ S)(w_n^i)$.
Denote by $\mathbf{V}_n$ the equivalence class generated by $\mathbf{v}_n$ under the equivalence relation (8.15). In view of the the fact that each term in the sequence $(\mathbf{u}_n)$ satisfies

$$\mathbf{u}_n \in \mathcal{C}^m (\Omega \setminus \Gamma_n)^K ,$$

it follows by (3.17), (8.9), (8.11), Proposition 46 and the continuity of the mapping (8.2) that

$$\begin{array}{ll} \forall & i = 1, ..., K : \\ & f_i - \frac{1}{n} \leq T_i \mathbf{v}_n \leq f_i \end{array} .$$

As such, and in view of the diagram (8.17), it is clear that for each $i = 1, ..., K$, the sequence $\left(\widehat{\mathbf{T}}\mathbf{V}_{n,i}\right)$ order converges to f_i in $\mathcal{ML}^0(\Omega)$. Hence the sequence $\left(\widehat{\mathbf{T}}(\mathbf{V}_n)\right)$ converges to $\mathbf{f}$ in $\mathcal{ML}^0(\Omega)^K$. It now follows that $(\mathbf{V}_n)$ is a Cauchy sequence in $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$ so that there exists $\mathbf{U}^\sharp \in \mathcal{NL}_{\mathbf{T}}(\Omega)$ that satisfies (9.33).
Since the mapping $\widehat{\mathbf{T}} : \mathcal{ML}^m_{\mathbf{T}}(\Omega) \to \mathcal{ML}^0(\Omega)^K$ is a uniformly continuous embedding, the uniqueness of the solution $\mathbf{U}^\sharp$ found above now follows by Corollary 37. ■

The relative simplicity, and lack of *technical* difficulty, of the proof of Theorem 76 should be compared to the highly involved techniques used to prove the existence of generalized solutions of a *single* equation in the context of the usual functional analytic approach, including those involving weak solutions or distributions. Indeed, the existence result presented in Theorem 76 applies to what may be considered as all nonlinear partial differential equations. Furthermore, in contradistinction with the customary functional analytic methods, the nonlinearity of the partial differential operator does not give rise to any additional difficulties. Indeed, the Order Completion Method [119], as well as the theory presented here, do not make any distinction between linear and nonlinear equations, this being one of the main strengths of this approach.

Let us now consider the structure of the space $\mathcal{NL}_{\mathbf{T}}(\Omega)$. In this regard, we recall the construction of the completion of a Hausdorff uniform convergence space X [161]. One considers the set X_C of all Cauchy filters on X, and an equivalence relation $\sim_C$ on X_C, defined as

$$\mathcal{F} \sim_C \mathcal{G} \Leftrightarrow \left(\begin{array}{ll} \exists & \mathcal{H} \in X_C : \\ & \mathcal{H} \subseteq \mathcal{F} \cap \mathcal{G} \end{array} \right) \tag{9.35}$$

The space X is embedded in $X^\sharp = X_C / \sim_C$ through

$$X \ni x \mapsto \lambda(x) \in X^\sharp$$

where λ is the induced convergence structure on X. The uniform convergence structure $\mathcal{J}^\sharp$ on $X^\sharp$ is defined as

$$\mathcal{U} \in \mathcal{J}_X^\sharp \Leftrightarrow \left(\begin{array}{ll} \exists & \mathcal{V} \in \mathcal{J}_X : \\ & [\mathcal{V}]_{X^\sharp} \subseteq \mathcal{U} \end{array} \right)$$

In view of the above construction, the space $\mathcal{NL}_{\mathbf{T}}(\Omega)$ consists of all filters $\mathcal{F}$ on $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$ such that the filter $\widehat{\mathbf{T}}(\mathcal{F})$ is a Cauchy filter in $\mathcal{ML}^0(\Omega)^K$, under the equivalence relation (9.35). In particular, the unique generalized solution $\mathbf{U}^\sharp$ to (8.1) may be represented as the set

$$\mathbf{U}^\sharp\{\mathcal{F} \text{ a filter on } \mathcal{ML}^m_{\mathbf{T}}(\Omega) : \widehat{\mathbf{T}}(\mathcal{F}) \text{ converges to } \mathbf{f} \text{ in } \mathcal{NL}(\Omega)^K\} \tag{9.36}$$

Note that each *classical solution* $\mathbf{u} \in \mathcal{C}^m(\Omega)^K$ to (8.1), and also each nonclassical $\mathbf{u} \in \mathcal{ML}^m(\Omega)^K$, generates the Cauchy filter

$$[\mathbf{U}] = \{F \subseteq \mathcal{ML}^m_{\mathbf{T}}(\Omega) \; : \; \mathbf{U} = q_{\mathbf{T}}\mathbf{u} \in F\}$$

on $\mathcal{NL}_{\widetilde{\mathbf{T}}}(\Omega)$, which belongs to the set (9.36). Hence our concept of generalized solution is *consistent* with the usual classical and nonclassical solutions in $\mathcal{ML}^m(\Omega)^K$. Moreover, the generalized solution to (8.3) may be assimilated with usual, nearly finite normal lower semi-continuous functions on Ω, in the sense that there is an injective uniformly continuous mapping

$$\widehat{\mathbf{T}}^\sharp : \mathcal{NL}_{\mathbf{T}}(\Omega) \to \mathcal{NL}(\Omega)^K$$

In this regard, we have a *blanket regularity* for the solutions of a rather large class of systems of nonlinear PDEs. It should be noted that this does not mean that the solution obtained in Theorem 76 is in fact a normal lower semi-continuous function, but rather that it may be *constructed* using such functions. In particular, since the mapping (8.20) is *injective*, the space $\mathcal{NL}_{\mathbf{T}}(\Omega)$ of generalized functions may be considered as the subset

$$\widehat{\mathbf{T}}^\sharp(\mathcal{NL}_{\mathbf{T}}(\Omega))$$

of the set $\mathcal{NL}(\Omega)^K$ of K-tuples of normal lower semi-continuous functions, equipped with a suitable uniform convergence structure.

In view of the above remarks concerning the structure of the unique generalized solution of (8.1), the uniqueness of the solution may be interpreted as follows. As mentioned, each classical solution $\mathbf{u} \in \mathcal{C}^m(\Omega)^K$ of (8.1), as well as each generalized solution $\mathbf{u} \in \mathcal{ML}^m(\Omega)^K$ to the extended equation (8.13), generates a Cauchy filter in $\mathcal{ML}^m_{\mathbf{T}}(\Omega)$. Such a Cauchy filter would then belong to the equivalence class (9.36), which is the *representation* of the generalized solution to (8.1). This class of Cauchy filters will also include other, more general filters. In particular, and in view of the commutative diagrams (8.17) and (9.49) we have

$$\begin{array}{ll} \forall & \mathcal{F} \text{ a Cauchy filter on } \mathcal{ML}^m(\Omega)^K : \\ & \mathbf{T}(\mathcal{F}) \to \mathbf{f} \text{ in } \mathcal{ML}^0(\Omega)^K \Rightarrow q_{\mathbf{T}}(\mathcal{F}) \in \mathbf{U}^\sharp \end{array}$$

so that $\mathbf{U}^\sharp$ also contains every generalized solution of (8.1) in the Sobolev type space of generalized functions $\mathcal{NL}^m(\Omega)^K$. Therefore, we may interpret the unique generalized solution $\mathbf{U}^\sharp \in \mathcal{NL}_{\mathbf{T}}(\Omega)$ of (8.1) as the set of *all* solutions of (8.1) in the context of the spaces of generalized solutions associated with the theory of PDEs presented here.

9.3 How Far Can Pullback Go?

In Section 9.2 we presented the first and basic existence, uniqueness and regularity result for the solutions of a large class of systems of nonlinear PDEs within the setting of the so called pullback spaces of generalized functions. This result essentially amounts to a reformulation of the fundamental results in the Order Completion Method [119] within the context of uniform convergence spaces. However, the underlying approach to constructing generalized solutions to systems of nonlinear PDEs presented in Sections 8.1 and 9.2 can result in significant improvements in the regularity of generalized solutions of (8.1). In this section we address the issue of improving upon the regularity of the generalized solutions obtained in Section 9.2 within that general and type independent setting. This is done by imposing rather minimal conditions on the smoothness of the nonlinear operator (8.10).

In this regard, we consider a system of nonlinear PDEs of the form (8.1) to (8.3), with both the right hand term $\mathbf{f}$ in (8.1) as well as the mapping (8.2) are $\mathcal{C}^k$-smooth, for some $k \in \mathbb{N} \cup \{\infty\}$. Recall that, in this case, we obtain the mapping (8.46) with domain $\mathcal{ML}^{m+k}(\Omega)^K$ and range contained in $\mathcal{ML}^k(\Omega)^K$, rather than the mapping (8.4) with domain $\mathcal{ML}^m(\Omega)^K$ and range contained in $\mathcal{ML}^0(\Omega)^K$. In this case, we may reproduce the construction (8.15) through (8.17) as follows. We introduce an equivalence relation on $\mathcal{ML}^{m+k}(\Omega)^K$ through

$$\mathbf{u} \sim_{\mathbf{T},k} \mathbf{v} \Leftrightarrow \mathbf{Tu} = \mathbf{Tv}. \tag{9.37}$$

Exactly as in Section 8.1, we may associate with the mapping (8.46) an *injective* mapping

$$\widehat{\mathbf{T}}_k : \mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega) \to \mathcal{ML}^k(\Omega)^K \tag{9.38}$$

in a canonical way so as to produce the commutative diagram

$$\begin{tikzcd}[column sep=small]
\mathcal{ML}^{m+k}(\Omega)^K \arrow[rr, "\mathbf{T}"] \arrow[dr, "q_{\mathbf{T},k}"'] & & \mathcal{ML}^{k}(\Omega)^K \\
& \mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega) \arrow[ur, "\widehat{\mathbf{T}}_k"'] &
\end{tikzcd} \tag{9.39}$$

Here $q_{\mathbf{T},k}$ is the canonical quotient mapping associated with the equivalence relation (9.37), and $\mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega)$ is the quotient space $\mathcal{ML}^{m+k}(\Omega)^K / \sim_{\mathbf{T},k}$.

In introducing a suitable uniform convergence structure on $\mathcal{ML}^k(\Omega)$, and by implication also on $\mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega)$, it should be noted that the Cauchy sequence $(\mathbf{V}_n)$ constructed in Theorem 76 actually satisfies

$$\left(\widehat{\mathbf{T}}_k \mathbf{V}_n\right) \text{ converges to } \mathbf{f} \text{ in } \mathcal{ML}^0(\Omega)^K \tag{9.40}$$

As such, there is in fact no need to go beyond the space $\mathcal{ML}^0(\Omega)^K$ when constructing the generalized solution of (8.1).

Furthermore, we may observe that, as shown in Proposition 63, the space $\mathcal{ML}^k(\Omega)$ equipped with the usual pointwise order (7.2) is a sublattice of $\mathcal{ML}^0(\Omega)$. As such, the order convergence structure (4.8) is a well defined convergence structure which induces the order convergence of sequences (2.35). Moreover, recall from Section 2.4 that every reciprocal convergence structure, and in particular every Hausdorff convergence structure, is induced by the complete uniform convergence structure (2.70) through (2.69).

In view of (9.40), we equip the space $\mathcal{ML}^k(\Omega)^K$ with the uniform convergence structure (2.70) associated with product convergence structure with respect to the order convergence structure λ_o on each copy of $\mathcal{ML}^k(\Omega)$. That is,

$$\mathcal{U} \in \mathcal{J}^K_{\lambda_o} \Leftrightarrow \left(\begin{array}{ll} \exists & \mathbf{u}_1, ..., \mathbf{u}_k \in \mathcal{ML}^k(\Omega)^K \ : \\ \exists & \mathcal{F}_1, ..., \mathcal{F}_k \text{ filters on } \mathcal{ML}^k(\Omega)^K \ : \\ & 1) \quad \mathcal{F}_i \text{ converges to } \mathbf{u}_i,\ i = 1, ..., k \\ & 2) \quad (\mathcal{F}_1 \times \mathcal{F}_1) \cap ... \cap (\mathcal{F}_k \times \mathcal{F}_k) \subseteq \mathcal{U} \end{array}\right). \tag{9.41}$$

The space $\mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega)$ is equipped with the initial uniform convergence structure $\mathcal{J}_{\mathbf{T},k}$ with respect to the mapping (9.38). That is,

$$\mathcal{U} \in \mathcal{J}_{\mathbf{T},k} \Leftrightarrow \left(\widehat{\mathbf{T}}_k \times \widehat{\mathbf{T}}_k\right)(\mathcal{U}) \in \mathcal{J}^K_{\lambda_o}. \tag{9.42}$$

Since the mapping (9.38) is injective, it is a uniformly continuous embedding, and the uniform convergence structure (9.42) is Hausdorff. As such, we may construct the completion of $\mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega)$, which we denote by $\mathcal{NL}_{\mathbf{T},k}(\Omega)$, and a unique uniformly continuous mapping

$$\widehat{\mathbf{T}}^\sharp_k : \mathcal{NL}_{\mathbf{T},k}(\Omega) \to \mathcal{ML}^k(\Omega)^K \tag{9.43}$$

so that the diagram

$$\begin{tikzcd}
\mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega) \arrow[rr, "\widehat{\mathbf{T}}_k"] \arrow[dr, "\phi"'] & & \mathcal{ML}^k(\Omega)^K \\
& \mathcal{NL}_{\mathbf{T},k}(\Omega) \arrow[ur, "\widehat{\mathbf{T}}^\sharp_k"'] &
\end{tikzcd} \tag{9.44}$$

commutes, with ϕ the canonical uniformly continuous embedding associated with the completion $\mathcal{NL}_{\mathbf{T},k}(\Omega)$ of $\mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega)$. In particular, in view of Corollary 37, the mapping (9.43) is *injective*. As in Sections 8.2 and 9.1, and in view of the diagram (9.39), we consider any solution $\mathbf{U}^\sharp \in \mathcal{NL}_{\mathbf{T},k}(\Omega)$ of the equation

$$\widehat{\mathbf{T}}^\sharp_k \mathbf{U}^\sharp = \mathbf{f} \tag{9.45}$$

as a generalized solution of (8.1). The main result of this section is now the following.

Theorem 77 *Consider a system of nonlinear PDEs of the form (8.1) through (8.2) that also satisfies (9.1). If both the righthand term* $\boldsymbol{f}$ *in (8.1) and the mapping (8.2) are* $\mathcal{C}^k$*-smooth, then there is a unique* $\boldsymbol{U}^\sharp \in \mathcal{NL}_{\boldsymbol{T},k}(\Omega)$ *so that*

$$\widehat{\boldsymbol{T}}^\sharp_k \boldsymbol{U}^\sharp = \boldsymbol{f}$$

Proof. Let us first show existence. By Theorem 67, see also Remark 75, there is exists a sequence (Γ_n) of closed nowhere dense subsets of Ω, and functions $\mathbf{U}_n \in \mathcal{C}^{m+k}(\Omega \setminus \Gamma_n)^K$ so that

$$\begin{array}{ll} \forall & i = 1, ..., K : \\ \forall & x \in \Omega \setminus \Gamma_n : \\ & f_i(x) - \frac{1}{n} \leq T_i(x, D)\mathbf{U}_n(x) \leq f_i(x) \end{array}$$

In view of (3.20), Proposition 46 and (8.12) it follows that

$$\begin{array}{ll} \forall & i = 1, ..., K : \\ & f_i - \frac{1}{n} \leq T_i \mathbf{v}_n \leq f_i \end{array}$$

where $\mathbf{v}_n \in \mathcal{ML}^{m+k}(\Omega)^K$ is the function with components $v_{n,i}$ defined through

$$v_{n,i} = (I \circ S)(U_{n,i}).$$

Clearly each sequence $(T_i \mathbf{v}_n)$ converges to f_i in $\mathcal{ML}^k(\Omega)$ so that the sequence $(\mathbf{Tv}_n)$ converges to $\mathbf{f}$ in $\mathcal{ML}^k(\Omega)^K$. As such, the sequence $(\mathbf{V}_n)$ associated with $(\mathbf{v}_n)$ through (9.44) is a Cauchy sequence in $\mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega)$. The existence of a solution now follows by the uniform continuity of the mapping (9.38).
Since the mapping (9.38) is a uniformly continuous embedding, the uniqueness of the solution follows by Corollary 37. ■

The structure of the generalized solution obtained in Theorem 77 may be explained in terms of the structure of the completion of a uniform convergence space. In particular, each element $\mathbf{U}^\sharp$ of the completion $\mathcal{NL}_{\mathbf{T},k}(\Omega)$ of $\mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega)$ may be interpreted as consisting of the equivalence class of Cauchy filters

$$\left\{ \mathcal{F} \text{ a Cauchy filter on } \mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega) \ : \ \widehat{\mathbf{T}}(\mathcal{F}) \text{ converges to } \mathbf{f} \right\} \tag{9.46}$$

under the equivalence relation (9.35). In view of (9.46), the unique generalized solution $\mathbf{U}^\sharp \in \mathcal{NL}_{\mathbf{T},k}(\Omega)$ contains all possible sufficiently smooth solutions of (8.1) within the context of the Order Completion Method. In particular, each solution $\mathbf{u} \in \mathcal{ML}^{m+k}(\Omega)^K$ of the equation (8.13) generates a Cauchy sequence in $\mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega)$ which belongs to the equivalence class (9.46). As such, this notion of solution is consistent with solutions in $\mathbf{u} \in \mathcal{ML}^{m+k}(\Omega)^K$, which includes also all classical solutions of (8.1).

Furthermore, since the mapping (9.38) is a uniformly continuous embedding, it follows by Corollary 37 that the extended mapping (9.43) associated with (9.38) is an injection. This may be interpreted as a regularity result for the unique generalized solution obtained in Theorem 77, in the sense that each generalized function in $\mathcal{NL}_{\mathbf{T},k}(\Omega)$ may be assimilated with usual functions in $\mathcal{ML}^k(\Omega)^K$.

It should be noted that the generalized solution of (8.1) constructed in Theorem 76 contains also the solution obtained in Theorem 77. Indeed, since the uniform convergence structure (9.41) is *finer* than the subspace uniform convergence structure induced from $\mathcal{ML}^0(\Omega)^K$, the inclusion mapping

$$i : \mathcal{ML}^k(\Omega)^K \ni \mathbf{u} \mapsto \mathbf{u} \in \mathcal{ML}^0(\Omega)^K$$

is uniformly continuous. Combining the diagrams (8.17) and (9.44), we obtain an injective uniformly continuous mapping

$$\widehat{i} : \mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega) \to \mathcal{ML}^m_{\mathbf{T}}(\Omega) \tag{9.47}$$

so that the diagram

$$\begin{tikzcd}
\mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega) \arrow[r, "\widehat{\mathbf{T}}_k"] \arrow[d, "\widehat{i}"'] & \mathcal{ML}^k(\Omega)^K \arrow[d, "i"] \\
\mathcal{ML}^m_{\mathbf{T}}(\Omega) \arrow[r, "\widehat{\mathbf{T}}"] & \mathcal{ML}^0(\Omega)^K
\end{tikzcd} \tag{9.48}$$

commutes. Upon extension of the uniformly continuous mappings (8.20), (9.43) and (9.47) to the completions of their respective domains, one obtains the commutative diagram

$$\begin{tikzcd}
\mathcal{NL}_{\mathbf{T},k}(\Omega) \arrow[r, "\widehat{\mathbf{T}}^\sharp_k"] \arrow[d, "\widehat{i}^\sharp"'] & \mathcal{ML}^k(\Omega)^K \arrow[d, "i"] \\
\mathcal{NL}_{\mathbf{T}}(\Omega) \arrow[r, "\widehat{\mathbf{T}}^\sharp"] & \mathcal{NL}(\Omega)^K
\end{tikzcd} \tag{9.49}$$

corresponding to (9.48). Since the mappings $\widehat{T}^\sharp$, $\widehat{T}^\sharp_k$ and i are all *injective* by Corollary 37, it follows by the diagram (9.49) that the mapping

$$\widehat{i}^\sharp : \mathcal{NL}_{\mathbf{T},k}(\Omega) \to \mathcal{NL}_{\mathbf{T}}(\Omega) \tag{9.50}$$

must also be injective. In particular, if $\mathbf{U}^\sharp \in \mathcal{NL}_{\mathbf{T},k}(\Omega)$ is a solution of (9.45), then $\widehat{i}^\sharp \mathbf{U}^\sharp \in \mathcal{NL}_{\mathbf{T}}(\Omega)$ is a solution of (8.21).

The results on existence, uniqueness and regularity of generalized solutions of (8.1) obtained in this section are, to a certain extent, maximal with respect to the regularity of solutions within the framework of the so called pullback spaces of generalized functions. In this regard, let us now present the construction of generalized solution in an abstract framework. Consider spaces X and Y of functions $\mathbf{g} : \Omega \to \mathbb{R}^K$ such that $\mathbf{f} \in Y$, and the nonlinear partial differential operator $\mathbf{T}$ associated with (8.1) acts as

$$\mathbf{T} : X \to Y. \tag{9.51}$$

Also suppose that Y is equipped with a complete and Hausdorff uniform convergence structure $\mathcal{J}_Y$ which is first countable. Proceeding in the same way as is done in this section, we introduce an equivalence relation on X through

$$\mathbf{u} \sim_{\mathbf{T}} \mathbf{v} \Leftrightarrow \mathbf{T}\mathbf{u} = \mathbf{T}\mathbf{v},$$

and associate with the mapping (9.51) the injective mapping

$$\widehat{\mathbf{T}} : X_{\mathbf{T}} \to Y, \tag{9.52}$$

where $X_{\mathbf{T}}$ is the quotient space $X/\sim_{\mathbf{T}}$. In particular, the mapping (9.52) is supposed to satisfy

$$\begin{array}{ll} \forall & \mathbf{U} \in X_{\mathbf{T}} : \\ \forall & \mathbf{u} \in \mathbf{U} : \\ & \mathbf{T}\mathbf{u} = \widehat{\mathbf{T}}\mathbf{U} \end{array}.$$

If we equip $X_{\mathbf{T}}$ with the initial uniform convergence structure $\mathcal{J}_{\mathbf{T}}$ with respect to the mapping (9.52), then $\mathcal{J}_{\mathbf{T}}$ is Hausdorff and first countable. In particular, the mapping (9.52) is a uniformly continuous embedding, and extends uniquely to a injective uniformly continuous mapping

$$\widehat{\mathbf{T}}^\sharp : X_{\mathbf{T}}^\sharp \to Y, \tag{9.53}$$

where $X_{\mathbf{T}}^\sharp$ is the completion of $X_{\mathbf{T}}$. A generalized solution of (8.1) in this context is any solution $\mathbf{U}^\sharp \in X_{\mathbf{T}}^\sharp$ of the equation

$$\widehat{\mathbf{T}}^\sharp \mathbf{U}^\sharp = \mathbf{f}. \tag{9.54}$$

Note that, in view of the fact that the mapping (9.52) is a uniformly continuous embedding, and (9.53) therefore an injection, the equation (9.54) can have at most one solution.

Now, in order to obtain the existence of a solution of (9.54), we must construct a sequence $(\mathbf{u}_n)$ in X so that $(\mathbf{T}\mathbf{u}_n)$ converges to $\mathbf{f}$ in Y. In this regard, the most general such result is given by Theorem 67. As such, within such a general context as considered here, it follows that, if the mapping (8.2) is $\mathcal{C}^k$-smooth, we should have

$$X \supseteq \mathcal{ML}^{m+k}(\Omega)^K . \tag{9.55}$$

It now follows by (9.51) and (9.55) that

$$Y \supseteq \mathcal{ML}^{k}(\Omega)^K . \tag{9.56}$$

This may be summarized in the following commutative diagram.

$$\begin{tikzcd} \mathcal{ML}^{m+k}(\Omega)^K \arrow[r, "\mathbf{T}"] \arrow[d, "\subset"'] & \mathcal{ML}^{k}(\Omega)^K \arrow[d, "\subset"] \\ X \arrow[r, "\mathbf{T}"] & Y \end{tikzcd} \tag{9.57}$$

Combining the diagram (9.57) with (9.39) and

$$\begin{tikzcd} X \arrow[rr, "\mathbf{T}"] \arrow[dr, "q_{\mathbf{T}}"'] & & Y \\ & X_{\mathbf{T}} \arrow[ur, "\widehat{\mathbf{T}}"'] & \end{tikzcd} \tag{9.58}$$

we obtain an injective mapping

$$\iota_{\mathbf{T}} : \mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega) \rightarrow X_{\mathbf{T}} \tag{9.59}$$

so that the diagram

$$\begin{tikzcd} \mathcal{ML}^{m+k}_{\mathbf{T},k}(\Omega) \arrow[r, "\widehat{\mathbf{T}}_k"] \arrow[d, "\iota_{\mathbf{T}}"'] & \mathcal{ML}^k(\Omega)^K \arrow[d, "\subset"] \\ X_{\mathbf{T}} \arrow[r, "\widehat{\mathbf{T}}"] & Y \end{tikzcd} \tag{9.60}$$

commutes. In particular, if the subspace convergence structure induced on $\mathcal{ML}^k(\Omega)^K$ from Y is coarser than the order convergence structure, then the mapping (9.59) is uniformly continuous. Furthermore, in this case the mapping (9.59) extends to an injective uniformly continuous mapping

$$\iota^\sharp_{\mathbf{T}} : \mathcal{NL}_{\mathbf{T},k}(\Omega) \to X^\sharp_{\mathbf{T}} \tag{9.61}$$

so that the extended diagram

$$\begin{tikzcd} \mathcal{NL}_{\mathbf{T},k}(\Omega) \arrow[r, "\widehat{\mathbf{T}}^\sharp_k"] \arrow[d, "\iota^\sharp_{\mathbf{T}}"'] & \mathcal{ML}^k(\Omega)^K \arrow[d, "\subset"] \\ X^\sharp_{\mathbf{T}} \arrow[r, "\widehat{\mathbf{T}}^\sharp"] & Y \end{tikzcd} \tag{9.62}$$

commutes. The existence of the injective mapping (9.61) may be interpreted as follows. Any pullback type space of generalized functions $X^\sharp_{\mathbf{T}}$ which is constructed as above, and subject to the condition of *generality* of the nonlinear partial differential operator $\mathbf{T}$ must contain the space $\mathcal{NL}_{\mathbf{T},k}(\Omega)$. As such, within the context of *general*, continuous systems of nonlinear PDEs, the generalized functions in $\mathcal{NL}_{\mathbf{T},k}(\Omega)$ may be considered to be 'more regular' than those in any other space of generalized functions constructed in this way.

9.4 Existence of Solutions in Sobolev Type Spaces

In the previous two section we obtained existence, uniqueness and regularity results for the generalized solutions of large classes of systems of nonlinear PDEs in the context of the so called pullback spaces of generalized functions. However, and as explained at the end of Section 9.3, it is not possible, in the general case of arbitrary systems of continuous nonlinear PDEs, to go beyond the basic regularity properties of such generalized solutions within the framework of the mentioned pullback type spaces of generalized functions.

In this regard, there are two obstacles. In particular, the spaces of generalized functions $\mathcal{NL}_{\mathbf{T}}(\Omega)$ are constructed with a given nonlinear operator $\mathbf{T}$ in mind. As such, both the generalized functions $\mathbf{U}^{\sharp} \in \mathcal{NL}_{\mathbf{T}}(\Omega)$, as well as the uniform convergence structure on $\mathcal{NL}_{\mathbf{T}}(\Omega)$, may depend on this nonlinear mapping. A second difficulty, and connected with the first, is that there is no concept of generalized derivative on $\mathcal{NL}_{\mathbf{T}}(\Omega)$. In fact, it is not clear how one should define the derivatives of the generalized functions in $\mathcal{NL}_{\mathbf{T}}(\Omega)$.

Within the context of the Sobolev type spaces of generalized functions introduced in Section 8.2, the difficulties discussed above are resolved. In particular, these spaces are constructed independent of any given nonlinear partial differential operator $\mathbf{T}$. Furthermore, the usual partial differential operators

$$\mathcal{D}^{\alpha} : \mathcal{ML}^{m}(\Omega) \rightarrow \mathcal{ML}^{0}(\Omega)$$

extend uniquely to uniformly continuous mappings

$$\mathcal{D}^{\alpha\sharp} : \mathcal{NL}^{m}(\Omega) \rightarrow \mathcal{NL}(\Omega)$$

so that we may associate with each generalized function $u^{\sharp} \in \mathcal{NL}^{m}(\Omega)$ the vector of generalized derivatives

$$\mathbf{D}^{\sharp}u = \left(\mathcal{D}^{\alpha\sharp}u\right)_{|\alpha|\leq m} \in \mathcal{NL}(\Omega)^{M}.$$

Note also that, in view of the commutative diagram (8.41), the space $\mathcal{NL}^{m}(\Omega)$ provides also an additional clarification of the structure of generalized functions in the pullback type spaces of generalized functions, in case the generalized equation (8.39) admits a solution.

In this section we investigate the existence of solutions to the generalized equation (8.39). In this regard, the main result is that a large class of systems of nonlinear PDEs have generalized solutions in the Sobolev type spaces of generalized functions $\mathcal{NL}^{m}(\Omega)$. We also consider systems of equations that satisfy additional smoothness conditions, such as those introduced in Section 8.3, over and above the mere continuity of the mapping (8.2). Such equations turn out to admit solutions in the Sobolev type spaces of generalized functions $\mathcal{NL}^{m+k}(\Omega)^{K}$, the elements of which have generalized partial derivatives up to order $m+k$. Here m is the order of the system of equations (8.1) and k is the degree of smoothness of the righthand term $\mathbf{f}$ and the mapping (8.2).

As mentioned, the central result of this section concerns the existence of solutions to (8.39) for a large class of nonlinear partial differential operators. This result, and as is also the case for the existence results presented in Sections 9.2 and 9.3, uses only rather basic topological processes associated with the completion of uniform convergence spaces, and the approximation results presented in Section 9.1, most notably Theorem 70.

Theorem 78 *Consider a system of nonlinear PDEs of the form (8.1) through (8.3) that satisfies (9.19). Then there is some* $\boldsymbol{u}^\sharp \in \mathcal{NL}^m(\Omega)^K$ *such that*

$$\boldsymbol{T}^\sharp \boldsymbol{u}^\sharp = \boldsymbol{f}.$$

Proof. We may apply Theorem 70 to obtain a sequence (Γ_n) of closed nowhere dense sets such that

$$\begin{array}{ll} \forall & n \in \mathbb{N} : \\ & \Gamma_n \subseteq \Gamma_{n+1} \end{array},$$

and a sequence of functions $(\mathbf{V}_n)$ such that

$$\begin{array}{ll} \forall & n \in \mathbb{N} : \\ & \mathbf{V}_n \in \mathcal{C}^m(\Omega \setminus \Gamma_n)^K \end{array}.$$

The sequence $(\mathbf{V}_n)$ satisfies

$$\begin{array}{ll} \forall & j = 1, ..., K : \\ & f_j(x) - \frac{1}{n} \leq T_j(x, D)\mathbf{V}_n(x) \leq f_j(x), \; x \in \Omega \setminus \Gamma_n \end{array}. \tag{9.63}$$

Furthermore, for each $|\alpha| \leq m$ and every $i = 1, ..., K$ there are sequences $\left(\lambda_{n,i}^\alpha\right)$ and $\left(\mu_{n,i}^\alpha\right)$ so that $\lambda_{n,i}^\alpha, \mu_{n,i}^\alpha \in \mathcal{C}^0(\Omega \setminus \Gamma_n)$, and both

$$\begin{array}{ll} \forall & n \in \mathbb{N} : \\ \forall & |\alpha| \leq m : \\ \forall & i = 1, ..., K : \\ & 1) \;\; \lambda_{n,i}^\alpha(x) < D^\alpha V_{n,i}(x) < \mu_{n,i}^\alpha(x), \; x \in \Omega \setminus \Gamma_n \\ & 2) \;\; \lambda_{n,i}^\alpha(x) < \lambda_{n+1,i}^\alpha(x) < \mu_{n+1,i}^\alpha(x) < \mu_{n,i}^\alpha(x), \; x \in \Omega \setminus \Gamma_{n+1} \end{array} \tag{9.64}$$

and

$$\begin{array}{ll} \forall & x \in \Omega \setminus \left(\bigcup_{n \in \mathbb{N}} \Gamma_n\right) : \\ \forall & |\alpha| \leq m : \\ \forall & i = 1, ..., K : \\ & \sup\{\lambda_{n,i}^\alpha(x) \; : \; n \in \mathbb{N}\} = \inf\{\mu_{n,i}^\alpha(x) \; : \; n \in \mathbb{N}\} \end{array} \tag{9.65}$$

are satisfied. Consider the sequence of functions $(\mathbf{u}_n)$ in $\mathcal{ML}^m(\Omega)^K$, the components of which are defined through

$$u_{n,i} = (I \circ S)(V_{n,i}), \; i = 1, ..., K.$$

In view of (9.63) it is clear that the sequence $(\mathbf{Tu}_n)$ converges to $\mathbf{f} \in \mathcal{ML}^0(\Omega)^K$. Now define, for each $i = 1, ..., K$ and every $|\alpha| \leq m$, the sequences $\left(\overline{\lambda}^{\alpha}_{n,i}\right)$ and $\left(\overline{\mu}^{\alpha}_{n,i}\right)$ in $\mathcal{ML}^0(\Omega)$ as

$$\overline{\lambda}^{\alpha}_{n,i} = (I \circ S)\left(\lambda^{\alpha}_{n,i}\right)$$

and

$$\overline{\mu}^{\alpha}_{n,i} = (I \circ S)\left(\mu^{\alpha}_{n,i}\right)$$

Applying (3.17), (3.20) and Propositions 46 to (9.64) it follows that, for each $n \in \mathbb{N}$,

$$\overline{\lambda}^{\alpha}_{n,i} \leq \overline{\lambda}^{\alpha}_{n+1,i} \leq \mathcal{D}^{\alpha} u_{n,i} \leq \overline{\mu}^{\alpha}_{n,i} \leq \overline{\mu}^{\alpha}_{n,i}.$$

Furthermore, from (3.20), Definition 53 and (9.65) it follows that each of the filters

$$[\{[\overline{\lambda}^{\alpha}_{n,i}, \overline{\mu}^{\alpha}_{n,i}] \,:\, n \in \mathbb{N}\}]$$

is a Cauchy filter in $\mathcal{ML}^0(\Omega)$. As such, each of the sequences $(\mathcal{D}^{\alpha} u_{n,i})$ is a Cauchy sequence in $\mathcal{ML}^0(\Omega)$ so that the sequence $(\mathbf{u}_n)$ is a Cauchy sequence in $\mathcal{ML}^m(\Omega)^K$. The result now follows by Theorem 65. ∎

Theorem 78 states that the generalized equation (8.39) corresponding to the system of nonlinear PDEs (8.1) has a solution in $\mathcal{NL}^m(\Omega)^K$. Since the mapping (8.38) which defines the left hand side of the equation (8.39) is the unique uniformly continuous extension of the mapping (8.10), the solution $\mathbf{u}^{\sharp} \in \mathcal{NL}^m(\Omega)^K$ to (8.39) is interpreted as a generalized solution to the system of PDEs (8.1).

Furthermore, each of the partial differential operators (8.8) extends uniquely to uniformly continuous mapping (8.30) which represent the generalized derivatives of the generalized functions $u^{\sharp} \in \mathcal{NL}^m(\Omega)$. In particular, and in view of the definition of the uniform convergence structure $\mathcal{J}_D$ on $\mathcal{ML}^m(\Omega)$ as the initial uniform convergence structure with respect to the family of mappings (8.22), the mapping (8.26) is a uniformly continuous embedding of $\mathcal{ML}^m(\Omega)$ into $\mathcal{ML}^0(\Omega)^M$. As such, and in view of Corollary 37, the mapping (8.26) extends uniquely to the injective uniformly continuous mapping (8.28). Thus, the commutative diagram (8.29) amounts to a representation of the generalized functions that are the elements of $\mathcal{NL}^m(\Omega)$ in terms of their generalized derivatives $\mathcal{D}^{\alpha\sharp} u^{\sharp} \in \mathcal{NL}(\Omega)$.

The representation of a generalized function $u^{\sharp} \in \mathcal{NL}^m(\Omega)$ in terms of its generalized derivatives may be interpreted as a regularity result for the generalized solutions to (8.1) obtained in Theorem 78. Indeed, each generalized derivative $\mathcal{D}^{\alpha\sharp} u^{\sharp}_i$

of a component $u_i^\sharp$ of the solution $\mathbf{u}^\sharp$ to (8.39) is a nearly finite normal lower semi-continuous functions. As such, we have

$$\begin{array}{ll} \exists & B \subset \Omega \text{ of first Baire category :} \\ \forall & i = 1, ..., K : \\ \forall & |\alpha| \leq m : \\ \forall & x \in \Omega \setminus B : \\ & \mathcal{D}^\alpha u_i^\sharp \text{ continuous at } x \end{array}$$

That is, each generalized solution $\mathbf{u}^\sharp \in \mathcal{NL}^m(\Omega)^K$ of (8.1) may be represented as a K-tuple of usual nearly finite normal lower semi-continuous functions which, in view of Proposition 46, are continuous and real valued on a residual subset of Ω.

The existence of generalized solutions of (8.1) in the Sobolev type space of generalized functions $\mathcal{NL}^m(\Omega)^K$ also provides some insight into the structure of the generalized solutions in the pullback type spaces of generalized functions. In this regard, consider now a system of nonlinear PDEs of the form (8.1) such that the mapping (8.2) is both *open* and *surjective*. In that case, it follows by Theorems 76 and 78 that

$$\begin{array}{ll} \forall & \mathbf{f} \in \mathcal{C}^0(\Omega)^K \ : \\ \exists! & \mathbf{U}^\sharp \in \mathcal{NL}_{\mathbf{T}}(\Omega) \ : \\ & \widehat{\mathbf{T}}^\sharp \mathbf{U}^\sharp = \mathbf{f} \end{array}$$

and

$$\begin{array}{ll} \forall & \mathbf{f} \in \mathcal{C}^0(\Omega)^K \ : \\ \exists & \mathbf{u}^\sharp \in \mathcal{NL}^m(\Omega) \ : \\ & \mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{f} \end{array}$$

In view of the commutative diagram (8.41) it follows that the unique generalized solution to (8.1) in $\mathcal{NL}_{\mathbf{T}}(\Omega)$ consists precisely of all generalized solutions to (8.1) in $\mathcal{NL}^m(\Omega)^K$. That is,

$$\mathbf{U}^\sharp = \left\{ \mathbf{u}^\sharp \in \mathcal{NL}^m(\Omega)^K \ : \ \mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{f} \right\}.$$

Moreover, and as is explained in Section 8.3, the mapping (8.40) is the canonical quotient mapping associated with the equivalence relation (8.43) on $\mathcal{NL}^m(\Omega)^K$.

The existence result presented in Theorem 78 applies to a general class of systems of nonlinear PDEs. In particular, it requires rather minimal assumptions on the smoothness of the both the nonlinear partial differential operator $\mathbf{T}$, as well as the righthand term $\mathbf{f}$. In this regard, it is only assumed that the righthand term $\mathbf{f}$ and the mapping (8.2) that defines the nonlinear operator $\mathbf{T}$ through (8.12) are continuous.

As is shown in Section 9.3 in connection with generalized solutions in the pullback type spaces of generalized functions, additional regularity assumptions on the operator $\mathbf{T}$ and the righthand term $\mathbf{f}$, such as those introduced in Section 8.3, may lead to significant improvements in the regularity of generalized solutions. As we shall see shortly, this is also the case for solutions constructed in the Sobolev type spaces of generalized functions.

In this regard, we now consider a system of nonlinear PDEs of the form (8.1), with the mapping (8.2) which defines the nonlinear operator through (8.12) a $\mathcal{C}^k$-smooth function, for some $k \in \mathbb{N}$. We may recall from Section 8.3 that, in this case, we obtain a uniformly continuous mapping

$$\mathbf{T} : \mathcal{ML}^{m+k}(\Omega)^K \to \mathcal{ML}^k(\Omega)^K .$$

In particular, this mapping may be represented by the uniformly continuous mappings (8.48) and (8.50) in the commutative diagram (8.49). This shows that the equation (8.57) is equivalent to (8.13). Furthermore, and in view of the uniform continuity of the mappings (8.46), (8.48) and (8.50), each of these mappings extend uniquely the uniformly continuous mappings (8.53), (8.54) and (8.55), respectively. Moreover, since the mapping (8.55) is injective, one obtains also the representation (8.56) for the extended nonlinear partial differential operator (8.53). In this regard, it follows that the generalized equation (8.58) is equivalent to

$$\mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{f}, \tag{9.66}$$

where the unknown $\mathbf{u}^\sharp$ is supposed to belong to the space $\mathcal{NL}^{m+k}(\Omega)^K$. Note, however, that, as is mentioned in Section 8.3, the equivalence with the generalized equation (8.39) breaks down, since in that case the solution is only assumed to have generalized derivatives up to order m. Under assumptions similar to those required for Theorem 78, we now obtain the existence of a solution to the generalized equation (9.66). In this regard, the approximation result Theorem 71 is the key.

Theorem 79 *Consider a system of nonlinear PDEs of the form (8.1) to (8.3) with the mapping (8.2) and the righthand term $\boldsymbol{f}$ both $\mathcal{C}^k$-smooth for some $k \in \mathbb{N}$. If the system satisfies (9.30), then there is some $\boldsymbol{u}^\sharp \in \mathcal{NL}^{m+k}(\Omega)^K$ such that*

$$\boldsymbol{T}^\sharp \boldsymbol{u}^\sharp = \boldsymbol{f}.$$

Proof. The proof of this result utilizes exactly the same techniques by which Theorem 78 is verified. Hence we do not include it here. ■

The structure of the generalized solution to (8.1) obtained in Theorem 79 may be explained by the same arguments used to describe the generalized solution constructed in Theorem 78. In particular, each solution $\mathbf{u}^\sharp \in \mathcal{NL}^{m+k}(\Omega)^K$ to (9.66) may be uniquely represented through its generalized derivatives

$$\mathcal{D}^{\alpha\sharp} u_i^\sharp,\ |\alpha| \leq m+k \text{ and } i = 1, ..., K$$

with each such generalized derivative a nearly finite normal lower semi-continuous function on Ω.

The existence results presented in Sections 9.2 and 9.3 for generalized solutions to (8.1) within the context of pullback type spaces of generalized functions apply to a large class of such systems of equations. In particular, every system of linear PDEs, and more generally every system of polynomial type nonlinear PDEs satisfy the condition (9.1), see [119], so that Theorem 76 applies to all such systems of equations. In connection with the existence results presented in this section, namely Theorems 78 and 79, a large class of equations to which these results apply will be discussed in Section 10.2.

Chapter 10

Regularity of Generalized Solutions

10.1 Compactness Theorems in Function Spaces

In chapter 9 we obtained several existence results for generalized solutions of systems of nonlinear PDEs of the form (8.1) to (8.3). In particular, solutions are constructed in the pullback type spaces of generalized functions, the elements of which may be *assimilated* with usual nearly finite normal lower semi-continuous functions. Under minimal assumptions on the smoothness of the nonlinear partial differential operator, it is shown that such solutions may in fact be assimilated even with piecewise smooth functions. As is mentioned also in Section 9.3, this is to some extent the maximal regularity for solutions in these pullback type spaces of generalized functions.

In Section 9.4 solutions to (8.1) are constructed in the Sobolev type spaces $\mathcal{NL}^m(\Omega)^K$ of generalized functions. These solutions are represented as nearly finite normal lower semi-continuous functions through the injective, uniformly continuous mapping (8.28). These solutions provide additional insight into the structure of the generalized solutions in the pullback type space $\mathcal{NL}_{\mathbf{T}}(\Omega)$ of generalized functions through the commutative diagram (8.41). In particular, the unique generalized solution $\mathbf{U}^\sharp \in \mathcal{NL}_{\mathbf{T}}(\Omega)$ may be represented as the equivalence class

$$\left\{\mathbf{u}^\sharp \in \mathcal{NL}^m(\Omega)^K \,:\, \mathbf{T}^\sharp\mathbf{u}^\sharp = \mathbf{f}\right\}$$

under the equivalence relation (8.43).

As discussed in Section 8.2, the generalized derivatives $\mathcal{D}^{\alpha\sharp}u^\sharp$ of a generalized function $u^\sharp \in \mathcal{NL}^m(\Omega)$ are normal lower semi-continuous functions. In particular, each such generalized derivative is continuous on a residual set, that is,

$$\begin{array}{ll} \exists & B \subset \Omega \text{ of first Baire category :} \\ \forall & |\alpha| \leq m : \\ & \mathcal{D}^\alpha u^\sharp \text{ is continuous at every } x \in \Omega \setminus B \end{array}$$

In general these generalized derivatives cannot be interpreted as usual derivatives of real functions. However, as we shall show in Section 10.2, under rather mild assumptions on the nonlinear partial differential operator (8.4), we can construct generalized solutions to (8.1) in $\mathcal{NL}^m(\Omega)^K$ such that

$$\begin{array}{ll} \exists & \mathbf{u} \in \mathcal{ML}^m(\Omega)^K : \\ \forall & |\alpha| \leq m : \\ \forall & i = 1, ..., K : \\ & \mathcal{D}^{\alpha\sharp} u_i^\sharp = \mathcal{D}^\alpha u_i \end{array} \tag{10.1}$$

The regularity property (10.1) for generalized solutions $\mathbf{u}^\sharp \in \mathcal{NL}^m(\Omega)^K$ is obtained as an application of suitable compactness theorems in $\mathcal{C}^k(\Omega)$, these being the subject of the present section. Some of the results presented in this section can be found in [1]. We include the proofs, as these are of independent interest.

In this regard, the notion of *equicontinuity* of sets of continuous functions is a key concept. Recall that for a topological space X, a set $\mathcal{A} \subset \mathcal{C}(X)$ is equi-continuous at $x_0 \in X$ whenever

$$\begin{array}{ll} \forall & \epsilon > 0 : \\ \exists & V \in \mathcal{V}_{x_0} : \\ \forall & u \in \mathcal{A} : \\ & x \in V \Rightarrow |u(x_0) - u(x)| < \epsilon \end{array} .$$

The set $\mathcal{A}$ is called equicontinuous on X if it is equicontinuous at each $x_0 \in X$, see for instance [81]. Equicontinuity is closely related to compactness in $\mathcal{C}(X)$. In particular, the well known theorem of Arzellà-Ascoli is the standard result.

Theorem 80 **[110] Consider a subset $\mathcal{A}$ of $\mathcal{C}(X)$. Then $\mathcal{A}$ has compact closure in the topology of uniform convergence on compacta in X whenever $\mathcal{A}$ is equicontinuous, and*

$$\mathcal{A}(x) = \{u(x) \ : u \in \mathcal{A}\}$$

has compact closure in $\mathbb{R}$ for each $x \in X$.
The converse holds whenever X is locally compact.

The special case of Theorem 80 which is relevant in the context of nonlinear PDEs is when X is a suitable subset of $\mathbb{R}^n$ and $\mathcal{A} \subset \mathcal{C}^m(X)$. In this regard, we at first consider a compact, convex subset X of $\mathbb{R}^n$ with nonempty interior. We equip the space $\mathcal{C}^m(X)$ with the norm

$$\|u\|_m = \sup\left\{ |D^\alpha u(x)| \left| \begin{array}{ll} 1) & |\alpha| \leq m \\ 2) & x \in X \end{array} \right. . \right\} \tag{10.2}$$

Theorem 81 **[1] With the norm (10.2), the space $\mathcal{C}^m(X)$ is a Banach space.*

Proof. Let (u_n) be a Cauchy sequence in $\mathcal{C}^m(X)$. Then, in view of the completeness of $\mathcal{C}^0(X)$ with respect to the uniform norm, it follows that

$$\begin{array}{ll} \forall & |\alpha| \leq m : \\ \exists & u^\alpha \in \mathcal{C}^0(X) : \\ & (D^\alpha u_n) \text{ converges uniformly to } u^\alpha \end{array}.$$

Denote by u the function u^α for $|\alpha| = 0$. We claim

$$\begin{array}{ll} \forall & |\alpha| \leq m : \\ & D^\alpha u = u^\alpha \end{array} \tag{10.3}$$

which would complete the proof. In this regard, fix some $i_0 \in \{1, ..., n\}$ and consider any $c = (c_i)_{i \leq n} \in \mathrm{int}X$. Define the nontrivial line segment $I_{i_0}(c)$ as

$$I_{i_0}(c) = \left\{ x \in X \left| \begin{array}{ll} \forall & i \neq i_0 : \\ & x_i = c_i \end{array} \right. \right\}.$$

Fix $x^0 \in I_{i_0}(c)$. By virtue of the Mean Value Theorem we have

$$\begin{array}{ll} \forall & x \in I_{i_0}(c) : \\ \forall & m, n \in \mathbb{N} : \\ \exists & y \in I_{i_0}(c) : \\ & (u_m(x) - u_n(x)) - (u_m(x^0) - u_n(x^0)) \\ & = \left(x_{i_0} - x^0_{i_0}\right)\left(\frac{\partial u_m}{\partial x_{i_0}}(y) - \frac{\partial u_n}{\partial x_{i_0}}(y)\right) \end{array}$$

From this it follows that, whenever $x^0 \neq x$, we have

$$\left| \frac{u_m(x) - u_m(x^0)}{x_{i_0} - x^0_{i_0}} - \frac{u_n(x) - u_n(x^0)}{x_{i_0} - x^0_{i_0}} \right| \leq \left\| \frac{\partial u_m}{\partial x_{i_0}} - \frac{\partial u_n}{\partial x_{i_0}} \right\|.$$

As such, and in view of the uniform convergence of the sequence of derivatives, it follows that

$$\begin{array}{ll} \forall & \epsilon > 0 : \\ \exists & M_\epsilon \in \mathbb{N} : \\ \forall & m, n \geq M_\epsilon : \\ & \left| \frac{u_m(x) - u_m(x^0)}{x_{i_0} - x^0_{i_0}} - \frac{u_n(x) - u_n(x^0)}{x_{i_0} - x^0_{i_0}} \right| < \epsilon \end{array}.$$

Therefore we have

$$\left| \frac{u(x) - u(x^0)}{x_{i_0} - x^0_{i_0}} - \frac{u_n(x) - u_n(x^0)}{x_{i_0} - x^0_{i_0}} \right| < \epsilon,\ n \geq M_\epsilon. \tag{10.4}$$

Since the sequence $\left(\frac{\partial u_n}{\partial x_{i_0}}\right)$ converges uniformly to u^α, with $\alpha = (0, ..., 0, 1, 0, ..., 0)$, it follows that

$$\begin{array}{ll} \exists & N_\epsilon \in \mathbb{N} : \\ & \left| \frac{\partial u_n}{\partial x_{i_0}}(x^0) - u^\alpha(x^0) \right| < \epsilon,\ n \geq N_\epsilon \end{array} \tag{10.5}$$

Set $K = \sup\{M_\epsilon, N_\epsilon\}$. Since $u_K \in \mathcal{C}^m(X)$ it follows that

$$\begin{array}{ll} \exists & \delta_\epsilon(K) > 0 : \\ \forall & x \in I_{i_0}(c) \\ & \left|\frac{u_K(x) - u_K(x^0)}{x_{i_0} - x_{i_0}^0} - \frac{\partial u_K}{\partial x_{i_0}}(x^0)\right| < \epsilon,\ 0 < |x_{i_0} - x_{i_0}^0| < \delta_\epsilon(K) \end{array}. \tag{10.6}$$

From the inequalities (10.4), (10.5) and (10.6) it follows that

$$\left|\frac{u(x) - u(x^0)}{x_{i_0} - x_{i_0}^0} - u^\alpha(x^0)\right| < 3\epsilon$$

whenever $0 < |x_{i_0} - x_{i_0}^0| < \delta_\epsilon(K)$. This proves that $\frac{\partial u}{\partial x_{i_0}}(x^0) = u^\alpha(x^0)$. This argument can be replicated for all $x \in X$ and all $|\alpha| \leq m$. As such, (10.3) must hold, and the proof is complete. ■

The main result of this section, in regard to the space $\mathcal{C}^m(X)$, is a useful sufficient condition for a set $\mathcal{A} \subseteq \mathcal{C}^m(X)$ to be precompact. As mentioned, equicontinuity is closely connected with compactness in spaces of continuous functions. Indeed, this concept *characterizes* the compact sets in $\mathcal{C}(X)$ through the Arzellà-Ascoli Theorem 80. In this regard, within the context of sets of smooth functions discussed here, a useful class of equicontinuous sets may be easily described.

Proposition 82 **[1] A subset $\mathcal{A}$ of $\mathcal{C}^1(X)$ is equicontinuous whenever*

$$\begin{array}{ll} \exists & C > 0 : \\ \forall & |\alpha| = 1 : \\ \forall & u \in \mathcal{A} : \\ & \|D^\alpha u\| \leq C \end{array} \tag{10.7}$$

Proof. For $u \in \mathcal{A}$, and $c \in X$, denote by $D_u(c)$ the Frechét derivative of u at x. That is, the linear functional defined through

$$D_u(c) : \mathbb{R}^n \ni x \mapsto \sum_{i=1}^n x_i \frac{\partial u}{\partial x_i}(c) \in \mathbb{R}.$$

By the Mean Value Theorem [19, 40.4], it follows that

$$\begin{array}{ll} \forall & u \in \mathcal{A} : \\ \forall & x, y \in X : \\ \exists & z \text{ on the line segment from } x \text{ to } y : \\ & D_u(z)(x - y) = u(x) - u(y) \end{array}$$

This leads to

$$|u(x) - u(y)| \leq \|D_u(z)\| \cdot \|x - y\|$$

where we take the supremum norm of $D_u(z)$. It now follows from (10.7) that

$$\begin{array}{ll} \forall & u \in \mathcal{A} : \\ \forall & x, y \in X : \\ & |u(x) - u(y)| \leq C\|x - y\| \end{array}.$$

For a fixed $x \in X$ we now have

$$\begin{array}{ll} \forall & u \in \mathcal{A} : \\ & |u(x) - u(y)| < C\delta \end{array}$$

whenever $\|x - y\| < \delta$. As such, for every $\epsilon > 0$, and if we choose $\delta < \frac{\epsilon}{M}$, it follows that

$$\begin{array}{ll} \forall & u \in \mathcal{A} : \\ \forall & x, y \in X : \\ & \|x - y\| < \delta \Rightarrow |u(x) - u(y)| < \epsilon \end{array}$$

which completes the proof. ∎

As an easy application of Proposition 82 we now obtain the following result on the compactness of sets in $\mathcal{C}^m(X)$.

Theorem 83 **[1] Consider a set $\mathcal{A} \subseteq \mathcal{C}^{m+1}(X)$. If*

$$\begin{array}{ll} \exists & C > 0 : \\ \forall & |\alpha| \leq m + 1 : \\ \forall & u \in \mathcal{A} : \\ & \|D^\alpha u\| \leq C \end{array}, \tag{10.8}$$

then $\mathcal{A}$ is precompact in $\mathcal{C}^m(X)$, with respect to the topology induced by the norm (10.2).

Proof. It is sufficient to show that $\mathcal{A}$ is sequentially precompact. In this regard, consider any sequence (u_n) in $\mathcal{A}$. From Proposition 82 it follows that, for each $|\alpha| \leq m$, the set

$$\{D^\alpha u : u \in \mathcal{A}\}$$

is equicontinuous. As such, and in view of (10.8) and Theorem 80, it follows that there exists a subsequence (u_{n_k}) of (u_n), and functions $u^\alpha \in \mathcal{C}^0(X)$, for $|\alpha| \leq m$, so that each sequence $(D^\alpha u_{n_k})$ converges to u^α. The result now follows by Theorem 81. ∎

The results obtained so far apply only to functions defined on a compact, convex subset of $\mathbb{R}^n$, the interior of which is nonempty. As such, and in particular in connection with nonlinear PDEs, the power of the respective results resides rather in the sphere of *local* properties of solutions of a systems of nonlinear PDEs, as apposed to the global properties of such a solution. More precisely, in general the domain of definition Ω of a system of nonlinear PDEs (8.1) is in general neither convex, nor compact. In particular, Ω is typically some open subset of $\mathbb{R}^n$, which may fail to be convex or bounded. In this regard, we introduce the following topology on $\mathcal{C}^m(\Omega)$, with Ω any nonempty and open subset of $\mathbb{R}^n$.

Definition 84 *Denote by τ_m the topology on $\mathcal{C}^m(\Omega)$ which is generated by the collection of subsets*

$$\left\{ S\left(A, \{U_\alpha\}_{|\alpha|\leq m}\right) \middle| \begin{array}{ll} 1) & A \subset \Omega \textit{ compact} \\ 2) & U_\alpha \subseteq \mathbb{R} \textit{ open, } |\alpha| \leq m \end{array} \right\}$$

of $\mathcal{C}^m(\Omega)$, where for $A \subset \Omega$ compact and $U_\alpha \subseteq \mathbb{R}$, $|\alpha| \leq m$, open

$$S\left(A, \{U_\alpha\}_{|\alpha|\leq m}\right) = \left\{ u \in \mathcal{C}^m(\Omega) \middle| \begin{array}{ll} \forall & |\alpha| \leq m : \\ & D^\alpha u(A) \subseteq U_\alpha \end{array} \right\}.$$

It is clear that τ_m does indeed define a topology on $\mathcal{C}^m(\Omega)$. Furthermore, a sequence (u_n) in $\mathcal{C}^m(\Omega)$ converges to $u \in \mathcal{C}^m(\Omega)$ if and only if

$$\begin{array}{ll} \forall & |\alpha| \leq m : \\ \forall & A \subset \Omega \text{ compact} : \\ & (D^\alpha u_n) \text{ converges } D^\alpha u \text{ uniformly on } A \end{array}.$$

Theorem 85 *The topology τ^m metrizable and complete.*

Proof. Let $\{A_i : i \in \mathbb{N}\}$ be a collection of compact, convex perfect subsets of Ω such that the family $\{\mathrm{int} A_i : i \in \mathbb{N}\}$ covers Ω, see for instance [58]. Then each of the sets $\mathcal{C}^m(A_i)$ is a complete metric space with respect to the metric induced through the norm (10.2). As such, the space

$$\prod_{i\in\mathbb{N}} \mathcal{C}^m(A_i)$$

is complete and metrizable in the product topology. This follows by the Urysohn Metrization Theorem, see for instance [110]. Consider the mapping

$$E : \mathcal{C}^m(\Omega) \to \prod_{i\in\mathbb{N}} \mathcal{C}^m(A_i)$$

defined through

$$E(u) = \left(u_{|A_i}\right)_{i\in\mathbb{N}}, \tag{10.9}$$

where $u_{|A_i}$ denotes the restriction of u to A_i. Clearly the mapping (10.9) is injective and continuous with a continuous inverse. As such, $\mathcal{C}^m(\Omega)$ is homeomorphic to the subspace $E(\mathcal{C}^m(\Omega))$ of $\prod_{i\in\mathbb{N}} \mathcal{C}^m(A_i)$, and hence it is metrizable. Completeness now follows by Theorem 81. ∎

Now, as mentioned, in the context of nonlinear PDEs, Theorem 83 is inappropriate, since the domain of definition of a system of nonlinear PDEs will in general fail to be compact and convex. However, the metrizable topology τ_m on $\mathcal{C}^m(\Omega)$, with Ω a nonempty and open subset of $\mathbb{R}^n$, provides a suitable framework for proving similar results in the noncompact case.

Theorem 86 *Let Ω be a nonempty and open subset of $\mathbb{R}^n$. Suppose that the set $\mathcal{A} \subset \mathcal{C}^{m+1}(\Omega)$ satisfies*

$$\begin{array}{ll} \forall & A \subset \Omega \textit{ compact :} \\ \exists & M_A > 0 \textit{ :} \\ \forall & |\alpha| \leq m+1 \textit{ :} \\ \forall & x \in A \textit{ :} \\ & u \in \mathcal{A} \Rightarrow |D^\alpha u(x)| < M_A \end{array}.$$

Then $\mathcal{A}$ is precompact in $\mathcal{C}^m(\Omega)$ with respect to the topology τ_m.

Proof. Note that, by Proposition 82, every set $D^\alpha(\mathcal{A})$, with $|\alpha| \leq m$ is equicontinuous, and hence, by Theorem 80, precompact in $\mathcal{C}^0(\Omega)$ with respect to the compact open topology. As such, each sequence (u_n) in $\mathcal{A}$ contains a subsequence (u_{n_k}) such that

$$\begin{array}{ll} \forall & |\alpha| \leq m : \\ \exists & u^\alpha \in \mathcal{C}^0(\Omega) \ : \\ \forall & A \subset \Omega \text{ compact :} \\ & (D^\alpha u_{n_k}) \text{ converges uniformly to } u^\alpha \text{ on } A \end{array}.$$

The result now follows by the same techniques used in the proof of Theorem 81. ∎

Theorems 83 and 86 provide a sufficient condition for a subset $\mathcal{A}$ of $\mathcal{C}^{m+1}(X)$, respectively $\mathcal{C}^{m+1}(\Omega)$, to be compact in $\mathcal{C}^m(X)$, respectively $\mathcal{C}^m(\Omega)$. It should be noted that, due to reasons from elementary Banach space theory, such sets need not be compact in $\mathcal{C}^{m+1}(X)$, $\mathcal{C}^{m+1}(\Omega)$ respectively. Indeed, suppose sets $\mathcal{A} \subset \mathcal{C}^{m+1}(X)$ which satisfy (10.8) are compact in $\mathcal{C}^{m+1}(\Omega)$. Then the closed unit ball is also compact, so that $\mathcal{C}^{m+1}(X)$ is finite dimensional, which is obviously not the case.

In order to obtain compactness of a set $\mathcal{A} \subset \mathcal{C}^{m+1}(X)$, one must impose additional assumptions on the set $\mathcal{A}$. In particular, in the one dimensional case when X is a compact interval in $\mathbb{R}$, the compact subsets of $\mathcal{C}^m(X)$ are characterized by the conditions

$$\begin{array}{ll} 1) & \mathcal{A} \text{ is bounded w.r.t. the norm (10.2)} \\ 2) & \{D^m u : u \in \mathcal{A}\} \text{ is equicontinuous} \end{array},$$

see for instance [49]. This characterization can be generalized to the arbitrary n dimensional case studied here. However, within the context of nonlinear PDEs, and in particular the construction of generalized solutions though approximation by smooth functions, the condition of equicontinuity of the set of highest order derivatives is rather difficult to satisfy.

Theorems 83 and 86 illustrate the phenomenon of 'loss of smoothness', which is well known in the field of partial differential equations. In this context, Theorem 83 states that, if you obtain a solution u of a PDE as the limit of a sequence (u_n) of functions that are $\mathcal{C}^m$-smooth, then u will be only $\mathcal{C}^{m-1}$-smooth. In this regard, consider some iterative method for constructing successive approximations to the solution of a given nonlinear PDE

$$T(x, D)u(x) = f(x), x \in \Omega. \tag{10.10}$$

Such an algorithm produces a sequence (u_n) of approximate solutions to (10.10). It often happens, see for instance [108] , that if $u_n \in \mathcal{C}^m(\Omega)$ for some $n \in \mathbb{N}$, then the next approximation u_{n+1} in the sequence will be *less smooth* than u_n. That is, we will typically have $u_{n+1} \in \mathcal{C}^{m-1}(\Omega) \setminus \mathcal{C}^m(\Omega)$. This has lead to the consideration of so called *smoothing operators*, which are supposed to restore the desired regularity of the approximations, see for instance [108], [114] and [117].

In this way, we may come to appreciate another novelty of the method of obtaining generalized solutions of systems of nonlinear PDEs of the form (8.1) presented here. Namely, that no such loss of smoothness of the approximating solutions occur. There is therefore no need to introduce any kind of smoothing operators. However, the approximate solutions are not smooth on the whole domain of definition of the system of equations. Indeed, each such approximate solution $\mathbf{u}_n \in \mathcal{ML}^m(\Omega)^K$ may be nonsmooth on some closed nowhere dense set $\Gamma_n \subset \Omega$, and the sets $\cup_{n \geq K}\Gamma_n$, for $K \in \mathbb{N}$, are typically dense in Ω. As such, one cannot apply the results of this section to obtain even just local regularity of generalized solutions. In the next section we shall present a way, based on Theorem 74, of going beyond these difficulties.

10.2 Global Regularity of Solutions

As is mentioned in Section 10.1, in the method for obtaining generalized solutions of systems of nonlinear PDEs presented in this work, and in particular, the construction of solutions in the Sobolev type spaces of generalized functions in Sections 8.2 and Section 9.4, there is no loss of smoothness of the approximating functions. Indeed, recall that the generalized solutions to (8.1) in $\mathcal{NL}^m(\Omega)^K$ are constructed as the limits of sequences in $\mathcal{ML}^m(\Omega)^K$. As such, there is no need to introduce any kind of smoothing operator in order to restore the regularity of successive approximations.

However, the results developed in Section 10.1, in particular Theorems 81, 83, 85 and 86 do not apply in the setting of the Sobolev type spaces of generalized functions, since the approximate solutions to (8.1) in $\mathcal{ML}^m(\Omega)$ allow singularities across arbitrary closed nowhere dense subsets of Ω. Furthermore, no suitable generalization of these results to the larger space $\mathcal{ML}^m(\Omega)$ seems possible. Indeed, the compactness results presented in Section 10.1 is based on the Arzellà-Ascoli Theorem 80, which requires pointwise *boundedness* and *equicontinuity* of the set of functions. However, note that a function $u \in \mathcal{ML}(\Omega)$, as well as its derivatives, will typically become unbounded in every neighborhood of the singularity set Γ_u associated with it through (8.5). Furthermore, if a set $\mathcal{A}$ of real valued functions on Ω is equicontinuous on Ω, then we must have

$$\mathcal{A} \subseteq \mathcal{C}^0(\Omega),$$

which is is in general not the case for subsets of $\mathcal{ML}^m(\Omega)$.

The aim of this section is to show that there exist generalized solutions to (8.1) in $\mathcal{NL}^m(\Omega)^K$ which are in fact classical solutions everywhere except on a closed nowhere dense set. This will follow as an application of Theorems 74 and 86. Note, however, that Theorem 83, and therefore Theorem 86, involves a loss of smoothness. In particular, given a sequence (u_n) of $\mathcal{C}^m$-smooth functions on a compact, convex subset X of $\mathbb{R}^n$ with nonempty interior, which is bounded with respect to the norm (10.2), we are in general only able to extract a subsequence of (u_n) which converges in $\mathcal{C}^{m-1}(\Omega)$. As such, and in view of the results presented in Section 9.4, it is clear that some additional smoothness conditions on the nonlinear partial differential operator (8.10), beyond the mere continuity of the mapping (8.2), must be imposed in order to apply Theorem 86.

In this regard, we consider a system of nonlinear PDEs of the form (8.1) such that the mapping (8.3), as well as the righthand term $\mathbf{f}$ are $\mathcal{C}^k$-smooth, for some $k \geq 1$. Theorem 79 states that such a system of equations admits a solution $\mathbf{u}^\sharp \in \mathcal{NL}^{m+k}(\Omega)^K$ whenever the condition (9.30) is satisfied. The main result of this section is a significant strengthening of Theorem 79 in terms of the regularity of the solution constructed.

Theorem 87 *Suppose that a system of nonlinear PDEs of the form (8.1) satisfies (9.30). Then there exists some* $\boldsymbol{u} \in \mathcal{ML}^{m+k-1}(\Omega)^K$ *so that*

$$\boldsymbol{Tu} = \boldsymbol{f}$$

Proof. By Theorem 79 we have

$$\begin{array}{ll} \exists & \mathbf{u}^\sharp \in \mathcal{NL}^{m+k}(\Omega)^K : \\ & \mathbf{T}^\sharp \mathbf{u}^\sharp = \mathbf{f} \end{array}.$$

In particular, there exists a Cauchy sequence $(\mathbf{u}_n) \subset \mathcal{ML}^{m+k}(\Omega)^K$ so that $(\mathbf{Tu}_n)$ converges to $\mathbf{f}$ in $\mathcal{NL}^k(\Omega)^K$. Furthermore, for each $j = 1, ..., K$ and each $|\beta| \leq k$ we have

$$\mathcal{D}^\beta f_j - \frac{1}{n} \leq \mathcal{D}^\beta T_j \mathbf{u}_n \leq \mathcal{D}^\beta f_j. \tag{10.11}$$

For each $n \in \mathbb{N}$ there is a closed nowhere dense set $\Gamma_n \subset \Omega$ such that $\mathbf{u}_n \in \mathcal{C}^{m+k}(\Omega \setminus \Gamma_n)^K$. Therefore, in view of Theorem 74, we have

$$\begin{array}{ll} \forall & n \in \mathbb{N} : \\ \exists & (\mathbf{u}_{n,r}) \subset \mathcal{C}^{m+k}(\Omega)^K : \\ \forall & |\alpha| \leq m+k : \\ \forall & i = 1, ..., K : \\ \forall & A \subset \Omega \setminus \Gamma_n \text{ compact} : \\ & \|\mathcal{D}^\alpha u_{n,r,i} - \mathcal{D}^\alpha u_{n,i}\|_A \rightarrow 0 \end{array} \tag{10.12}$$

where $\|\cdot\|_A$ denotes the uniform norm on $\mathcal{C}^0(A)$. It follows by the construction of the sequence $(\mathbf{u}_n)$ in Theorem 71 that

$$\begin{array}{ll} \forall & x \in \Omega \setminus \left(\bigcup_{n\in\mathbb{N}} \Gamma_n\right) : \\ \forall & |\alpha| \leq m+k : \\ \forall & i = 1, ..., K : \\ & |\mathcal{D}^\alpha u_{n,i}(x) - \mathcal{D}^{\alpha\sharp} u_i(x)| \to 0 \end{array} . \tag{10.13}$$

Therefore, and in view of (10.12), it follows that there is a strictly increasing sequence of integers (r_n) so that

$$\begin{array}{ll} \forall & x \in \Omega \setminus (\bigcup \Gamma_n) : \\ \forall & |\alpha| \leq m+k : \\ \forall & i = 1, ..., K : \\ & |\mathcal{D}^\alpha u_{n,r_n,i}(x) - \mathcal{D}^{\alpha\sharp} u_j(x)| \to 0 \end{array} \tag{10.14}$$

From (10.11), as well as the continuity of the mapping (8.2) and its derivatives, it follows that

$$\begin{array}{ll} \forall & x \in \Omega \setminus (\bigcup \Gamma_n) : \\ \forall & |\beta| \leq k : \\ \forall & j = 1, ..., K : \\ & |\mathcal{D}^\beta T_j \mathbf{u}_{n,r_n}(x) - \mathcal{D}^\beta f_j(x)| \to 0 \end{array} \tag{10.15}$$

In view of Proposition 49 there is a function $\mu \in \mathcal{ML}(\Omega)$ so that

$$\begin{array}{ll} \forall & n \in \mathbb{N} : \\ \forall & |\beta| \leq k : \\ \forall & j = 1, ..., K : \\ & |\mathcal{D}^\beta T_j \mathbf{u}_{n,r_n}| \leq \mu \end{array} .$$

As such, there is a closed nowhere dense set $\Gamma \subset \Omega$ so that

$$\begin{array}{ll} \forall & A \subset \Omega \setminus \Gamma \text{ compact} : \\ \exists & M_A > 0 : \\ \forall & |\beta| \leq k : \\ \forall & j = 1, ..., K : \\ & \|\mathcal{D}^\beta T_j \mathbf{u}_{n,r_n}\|_A \leq M_A,\ n \in \mathbb{N} \end{array} .$$

As an application of Theorem 86 it follows that there is a subsequence of $(\mathbf{u}_{n,r_n})$, which we dote by $(\mathbf{v}_n)$, so that $\left(\mathcal{D}^\beta \mathbf{T}\mathbf{v}_n\right)$ converges to $\mathcal{D}^\beta \mathbf{f}$ uniformly on compact subsets of $\Omega \setminus \Gamma$ for each $|\beta| \leq k-1$. Hence the sequence $(\mathbf{T}\mathbf{v}_n)$ converges to $\mathbf{f}$ in $\mathcal{ML}^{k-1}(\Omega)^K$.

By similar arguments as those used above, it may be shown that there is a closed nowhere dense set $\Gamma_0 \subset \Omega$ so that

$$\begin{array}{ll} \forall & A \subset \Omega \setminus \Gamma_0 \text{ compact} : \\ \exists & M_A^0 > 0 : \\ \forall & |\alpha| \leq m+k : \\ \forall & i = 1, ..., K : \\ & \|\mathcal{D}^\alpha v_{n,i}\|_A \leq M_A^0,\ n \in \mathbb{N} \end{array}$$

Applying Theorem 86, we find that there is a subsequence of $(\mathbf{v}_n)$, which we again denote by $(\mathbf{v}_n)$, and some $\mathbf{v} \in \mathcal{C}^{m+k-1}(\Omega \setminus \Gamma_0)^K$ so that

$$\begin{array}{ll} \forall & A \subset \Omega \setminus \Gamma_0 \text{ compact :} \\ \forall & |\alpha| \leq m+k-1 : \\ \forall & i = 1, ..., K : \\ & \|\mathcal{D}^\alpha v_{n,i} - D^\alpha v_i\|_A \rightarrow 0 \end{array}$$

Clearly the sequence $(\mathbf{u}_n)$, the components of which are defined as

$$u_{n,i} = (I \circ S)(v_{n,i}),$$

converges in $\mathcal{ML}^{m+k-1}(\Omega)^K$ to the function $\mathbf{u} \in \mathcal{ML}^{m+k-1}(\Omega)^K$, the components of which are defined as $u_i = (I \circ S)(v_i)$. The result now follows by the uniform continuity of the mapping $\mathbf{T} : \mathcal{ML}^{m+k-1}(\Omega)^K \rightarrow \mathcal{ML}^{k-1}(\Omega)^K$. ■

Theorem 87 states that every system of nonlinear PDEs of the form (8.1) such that the mapping (8.2) and the righthand term $\mathbf{f}$ are $\mathcal{C}^k$-smooth, has a generalized solution $\mathbf{u}^\sharp \in \mathcal{NL}^m(\Omega)^K$ such that $\mathbf{u}^\sharp \in \mathcal{ML}^{m+k-1}(\Omega)$, provided that the condition (9.30) is satisfied. That is,

$$\begin{array}{ll} \exists & \Gamma \subset \Omega \text{ closed nowhere dense :} \\ \exists & \mathbf{u} \in \mathcal{C}^{m+k-1}(\Omega \setminus \Gamma)^K \ : \\ & \mathbf{T}(x, D)\mathbf{u}(x) = \mathbf{f}(x), \ x \in \Omega \setminus \Gamma \end{array}$$

A highly important particular case of Theorem 87 occurs when the system of equations is $\mathcal{C}^1$-smooth, in the sense that the mapping (8.2) and the righthand term $\mathbf{f}$ in (8.1) are $\mathcal{C}^1$-smooth. In this case, Theorem 87 may be stated as

$$\begin{array}{ll} \exists & \Gamma \subset \Omega \text{ closed nowhere dense :} \\ \exists & \mathbf{u} \in \mathcal{C}^{m}(\Omega \setminus \Gamma)^K \ : \\ & \mathbf{T}(x, D) = \mathbf{f}(x), \ x \in \Omega \setminus \Gamma \end{array} \tag{10.16}$$

We may recall [129] that a property of a system is called a *strongly generic property* of this system if and only if it holds on an open and dense subset of the domain of definition of that system. Therefore, in view of (10.16) the existence of a classical solution to a system of nonlinear PDEs (8.1) that satisfies (9.30) is a strongly generic property of such a system.

A question naturally arises as to the actual scope of the result. That is, can we describe a significantly large class of systems of nonlinear PDEs to which Theorem 87 applies? To this question, the answer is affirmative. In this regard, note that the condition (9.30) is *sufficient* for the existence of classical solutions to (8.1) on an open and dense subset of the domain of definition Ω of the system of equations. As such, we need only demonstrate that this condition is satisfied. Furthermore, and as we shall shortly see, the condition (9.30) is, in many cases, rather easily verified through some standard techniques in real analysis. In particular, certain open mapping type theorems [19] are useful in this regard. We shall exhibit one considerably general class of equations to which Theorem 87 applies.

In this regard, we consider a system of K nonlinear PDEs of the form

$$D_t\mathbf{u}(x,t) + \mathbf{G}(x,t,...,D^\alpha u_i(x,t),...) = \mathbf{f}(x,t),\ i = 1,...,K \tag{10.17}$$

where $(x,t) \in \Omega \times [0,\infty)$, with $\Omega \subseteq \mathbb{R}^n$ nonempty and open, and

$$\mathbf{G} : \Omega \times [0,\infty) \times \mathbb{R}^M \to \mathbb{R}^K \tag{10.18}$$

a $\mathcal{C}^1$-smooth mapping. With the system of equations (10.17) we may associate a mapping

$$\mathbf{T} : \mathcal{ML}^m(\Omega \times [0,\infty))^K \to \mathcal{ML}^0(\Omega \times [0,\infty))^K. \tag{10.19}$$

In particular, and in view of the fact that the mapping (10.21) is $\mathcal{C}^1$-smooth, the mapping (10.19) satisfies

$$\mathbf{T} : \mathcal{ML}^{m+1}(\Omega \times [0,\infty))^K \to \mathcal{ML}^1(\Omega \times [0,\infty))^K. \tag{10.20}$$

Theorem 88 *Consider a system of nonlinear PDEs of the form (10.17). If both the mapping (10.21) and the righthand term $\boldsymbol{f}$ are $\mathcal{C}^1$-smooth, then the system of equations satisfies (9.30).*

Proof. Note that, for each $\beta \in \{0,1\}^n$, and every $j = 1,...,K$ there is a jointly continuous mapping

$$G_j^\beta : \Omega \times [0,\infty) \times \mathbb{R}^L \to \mathbb{R} \tag{10.21}$$

so that, for each $\mathbf{u} \in \mathcal{C}^{m+1}(\Omega)^K$, we have

$$D^\beta T_j\mathbf{u}(x,t) = D^\beta D_t u_j(x,t) + G_j^\beta(x,t,...,D^\alpha u_i(x,t),...),\ |\alpha| \leq m+1.$$

As such, the $K \times 2^n$ components of the mapping (8.51) may be expressed as

$$F_{j,\beta} : (x,t,\xi) \mapsto \xi_j + G_j^\beta(x,t,...,\xi_i,...),\ K \times 2^n < i \leq L. \tag{10.22}$$

From (10.22) it is clear that the mapping (8.51) is both open and surjective. As such, the condition (9.30) is satisfied. ∎

The following is now a straight forward consequence of Theorems 87 and 88.

Corollary 89 *Consider any system of nonlinear PDEs of the form (10.17). Then there is some $\boldsymbol{u} \in \mathcal{ML}^m(\Omega \times [0,\infty))^K$ such that*

$$\boldsymbol{Tu} = \boldsymbol{f}$$

The results on the existence of generalized solutions to (8.1) presented in Chapter 9, as well as the regularity properties of such solutions obtained in this chapter, do not take into account any possible initial and / or boundary conditions that may be associated with a particular system of nonlinear PDEs. In the next chapter, we shall adapt the general method developed over the course of the last three chapters so as to also incorporate such additional conditions. We shall see that, in contradistinction with with usual functional analytic methods, in particular those involving distributions, boundary and / or initial value problems are solved by, essentially, the same techniques that apply to the free problem.

Chapter 11

A Cauchy-Kovalevskaia Type Theorem

11.1 Existence of Generalized Solutions

The first general and type independent existence and regularity result for the solutions of systems of nonlinear PDEs, namely, Theorem 2, dates back to Cauchy. The first rigorous proof of this result was given by Kovalevskaia [86] more than a century ago. It should noted that, and as mentioned in Section 1.1, the original proof of Theorem 2 does not involve *any* so called 'advanced mathematics'. In particular, functional analysis is not used at all.

As is well known, ever since Sobolev [148], [149] introduced the sequential method for solving linear and nonlinear PDEs in the setting of Hilbert spaces over 70 years ago, the main, and to some extent nearly exclusive, approach to PDEs has been that of linear functional analysis. However, during the nearly eighty years of functional analysis, the mentioned Cauchy-Kovalevskaia Theorem has not been extended on its own general and type independent grounds. It was only in the 1987 monograph [139], see also [141], that, based on algebraic rather than functional analytic methods, a global version of the local existence and regularity result in Theorem 2 was obtained. The mentioned global version of Theorem 2 still requires both the equation (1.2) and the initial data (1.3) to be *analytic*. As such, this does not present a generalization of the type of equations that may be solved, but rather the domain of definition of the solution is enlarged. In fact, and in view of Lewy's impossibility result [97], see also [88], it may appear that an extension of Theorem 2 to nonanalytic equations is highly unlikely. As we shall see in the sequel, this is in fact a misunderstanding.

In this section, we present a first in the literature. Namely, we show that a system of K nonlinear PDEs of the form

$$D_t^m \mathbf{u}(t,y) = \mathbf{G}\left(t, y, ..., D_y^q D_t^p u_i(t,y), ...\right) \tag{11.1}$$

with $t \in \mathbb{R}$, $y \in \mathbb{R}^{n-1}$, $m \geq 1$, $0 \leq p < m$, $q \in \mathbb{N}^{n-1}$, $|q| + p \leq m$ and with the

Cauchy data

$$D_t^p \mathbf{u}\left(t_0, y\right) = \mathbf{g}_p\left(y\right), \; 0 \leq p < m, \; \left(t_0, y\right) \in S \tag{11.2}$$

on the noncharacteristic analytic hypersurface

$$S = \left\{\left(t_0, y\right) \; : \; y \in \mathbb{R}^{n-1}\right\}$$

admits a generalized solution $\mathbf{u}^\sharp \in \mathcal{NL}^m\left(\mathbb{R}^{n-1} \times \mathbb{R}\right)$, provided that the mapping

$$\mathbf{G} : \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^M \rightarrow \mathbb{R}^K \tag{11.3}$$

is jointly continuous, and the initial data (11.2) satisfies

$$\begin{array}{l} \forall \quad 0 \leq p < m : \\ \quad \mathbf{g}_p \in \mathcal{C}^{m-p}\left(\mathbb{R}^{n-1}\right)^K \end{array} \tag{11.4}$$

That is, we give the first extension of the Cauchy-Kovalevskaia Theorem, on its own general and type independent grounds, to equations which are not analytic.

Furthermore, if the mapping (11.3) is $\mathcal{C}^1$-smooth, and the initial condition (11.2) satisfies

$$\begin{array}{l} \forall \quad 0 \leq p < m : \\ \quad \mathbf{g}_p \in \mathcal{C}^{m-p+2}\left(\mathbb{R}^{n-1}\right)^K \end{array} \tag{11.5}$$

then the generalized solution of (11.1) through (11.2) is in fact a *classical* solution in the sense that

$$\begin{array}{l} \exists \quad \Gamma \subset \mathbb{R}^{n-1} \times \mathbb{R} \text{ closed nowhere dense :} \\ \qquad 1) \quad \Gamma \cap S \text{ closed nowhere dense in } S \\ \qquad 2) \quad \mathbf{u}^\sharp \in \mathcal{C}^m\left(\left(\mathbb{R}^{n-1} \times \mathbb{R}\right) \setminus \Gamma\right) \end{array} \tag{11.6}$$

It is clear that the existence of a solution of the system of nonlinear PDEs (11.1) is a straight forward consequence of the general existence results proved in Chapter 9. Furthermore, the regularity property (11.6) of the solution follows easily from the results in Chapter 10. In order to also incorporate the initial data (11.2), the methods presented in the mentioned chapters need only be adapted slightly. In this way, we come to appreciate yet another key feature of the solution method for systems of nonlinear PDEs presented in Chapters 6 through 10. Namely, that initial and / or boundary value problems may be solved by essentially the same techniques that apply to the free problem. This should be compared with the customary functional analytic methods, in particular those involving distributions, where such additional conditions often lead to significant complications which often require entirely new techniques.

In order to incorporate the initial condition (11.2) into our solution method, we introduce the following spaces of functions. Denote by $\mathcal{ML}^m_{\mathbf{g}}(\Omega)$ the set

$$\mathcal{ML}^m_{\mathbf{g}}(\Omega) = \left\{ \mathbf{u} \in \mathcal{ML}^m(\Omega)^K \left| \begin{array}{l} \forall \quad i = 1, ..., K : \\ \forall \quad 0 \leq p < m : \\ \forall \quad q \in \mathbb{N}^{n-1},\, 0 \leq |q| + p \leq m : \\ \quad 1) \quad \mathcal{D}^{qp}_{yt} u_i(y, t_0) = D^q g_{p,i}(y)\,,\; y \in \mathbb{R}^{n-1} \\ \quad 2) \quad \mathcal{D}^{qp}_{yt} u_i \text{ continuous at } (y, t_0) \end{array} \right. \right\}$$

where $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}$. For each $i = 1, ..., K$, every $0 \leq p < m$ and each $q \in \mathbb{N}^{n-1}$ such that $0 \leq |q| + p \leq m$, we consider the space $\mathcal{ML}^0_{i,q,p}(\Omega)$, which is defined through

$$\mathcal{ML}^0_{i,q,p}(\Omega) = \left\{ u \in \mathcal{ML}^0(\Omega) \left| \begin{array}{l} \forall \quad y \in \mathbb{R}^{n-1} : \\ \quad 1) \quad u(y, t_0) = D^q g_{p,i}(y) \\ \quad 2) \quad u \text{ continuous at } (y, t_0) \end{array} \right. \right\}.$$

Clearly, for every $0 \leq p < m$, and $p \in \mathbb{N}^{n-1}$ such that $0 \leq |q| + p \leq m$, and each $i = 1, ..., K$ we may define the partial differential operators

$$\mathcal{D}^{qp}_{i,yt} : \mathcal{ML}^m_{\mathbf{g}}(\Omega) \to \mathcal{ML}^0_{i,q,p}(\Omega)\,, \tag{11.7}$$

as in Chapter 8, through

$$\mathcal{D}^{qp}_{i,yt}\mathbf{u} = (I \circ S)\left(D^{qp}_{yt} u_i\right).$$

The partial differential operators $\mathcal{D}^m_{i,t}$, is defined in a similar way, namely, as

$$\mathcal{D}^m_{i,t} : \mathcal{ML}^m_{\mathbf{g}}(\Omega) \ni \mathbf{u} \mapsto (I \circ S)(D^m_t u_i) \in \mathcal{ML}^0(\Omega)\,. \tag{11.8}$$

The method for constructing generalized solutions to the initial value problem (11.1) to (11.2) presented here is essentially the same as that used in the case of arbitrary systems of nonlinear PDEs, which is developed in Chapters 8 and 9. In particular, generalized solutions are constructed as elements of the completion of the space $\mathcal{ML}^m_{\mathbf{g}}(\Omega)$, equipped with a suitable uniform convergence structure. In this regard, the space $\mathcal{ML}^0(\Omega)$ carries the uniform order convergence structure introduced in Chapter 7. We introduce the following uniform convergence structure on $\mathcal{ML}^0_{i,q,p}(\Omega)$.

Definition 90 *Let Σ consist of all nonempty order intervals in $\mathcal{ML}^0_{i,q,p}(\Omega)$. Let $\mathcal{J}_{i,q,p}$ denote the family of filters on $\mathcal{ML}^0_{i,q,p}(\Omega) \times \mathcal{ML}^0_{i,q,p}(\Omega)$ that satisfy the following: There exists $k \in \mathbb{N}$ such that*

$$\begin{array}{l} \forall \quad j = 1, ..., k : \\ \exists \quad \Sigma_j = (I^j_n) \subseteq \Sigma : \\ \quad 1) \quad I^j_{n+1} \subseteq I^j_n,\; n \in \mathbb{N} \\ \quad 2) \quad ([\Sigma_1] \times [\Sigma_1]) \cap ... \cap ([\Sigma_k] \times [\Sigma_k]) \subseteq \mathcal{U} \end{array} \tag{11.9}$$

where $[\Sigma_j] = [\{I : I \in \Sigma_j\}]$. *Moreover, for each* $j = 1, ..., k$ *and every open subset* V *of* Ω *one has*

$$\begin{array}{l} \exists \quad u_j \in \mathcal{ML}^0_{i,q,p}(\Omega) \ : \\ \quad \cap_{n\in\mathbb{N}} I^j_{n|V} = \{u_j\}_{|V} \end{array} \quad or \quad \cap_{n\in\mathbb{N}} I^j_{n|V} = \emptyset \tag{11.10}$$

Proposition 91 *The family of filters* $\mathcal{J}_{i,q,p}$ *on* $\mathcal{ML}^0_{i,q,p}(\Omega) \times \mathcal{ML}^0_{i,q,p}(\Omega)$ *is a Hausdorff uniform convergence structure.*
Furthermore, a filter $\mathcal{F}$ *on* $\mathcal{ML}^0_{i,q,p}(\Omega)$ *converges to* $u \in \mathcal{ML}^0_{i,q,p}(\Omega)$ *if and only if there exists a family* $\Sigma_{\mathcal{F}} = (I_n)$ *of nonempty order intervals on* $\mathcal{ML}^0_{i,q,p}(\Omega)$ *such that*

$$\begin{array}{ll} 1) & I_{n+1} \subseteq I_n,\ n \in \mathbb{N} \\ 2) & \begin{array}{l} \forall \quad V \subseteq \Omega \text{ nonempty and open :} \\ \quad \cap_{n\in\mathbb{N}} I_{n|V} = \{u\}_{|V} \end{array} \end{array}$$

and $[\Sigma_{\mathcal{F}}] \subseteq \mathcal{F}$.

Proof. The first four axioms of Definition 21 are clearly fulfilled, so it remains to verify

$$\begin{array}{l} \forall \quad \mathcal{U}, \mathcal{V} \in \mathcal{J}_o : \\ \quad \mathcal{U} \circ \mathcal{V} \text{ exists } \Rightarrow \mathcal{U} \circ \mathcal{V} \in \mathcal{J}_o \end{array} \tag{11.11}$$

In this regard, take any $\mathcal{U}, \mathcal{V} \in \mathcal{J}_o$ such that $\mathcal{U} \circ \mathcal{V}$ exists, and let $\Sigma_1, ..., \Sigma_k$ and $\Sigma'_1, ..., \Sigma'_l$ be the collections of order intervals associated with $\mathcal{U}$ and $\mathcal{V}$, respectively, through Definition 90. Set

$$\Phi = \{(l, j) \ : \ [\Sigma_l] \circ [\Sigma'_j] \text{ exists}\}$$

Then, by Lemma 54

$$\mathcal{U} \circ \mathcal{V} \supseteq \bigcap \{([\Sigma_l] \times [\Sigma_l]) \circ ([\Sigma_j] \times [\Sigma_j]) \ : \ (l, j) \in \Phi\} \tag{11.12}$$

Now $(l, j) \in \Phi$ if and only if

$$\begin{array}{l} \forall \quad m, n \in \mathbb{N} : \\ \quad I^l_m \cap I^j_n \neq \emptyset \end{array}$$

For any $(l, j) \in \Phi$, set $\Sigma_{l,j} = \left(I^{l,j}_n\right)$ where, for each $n \in \mathbb{N}$

$$I^{l,j}_n = \left[\inf\left(I^l_n\right) \wedge \inf\left(I^j_n\right), \sup\left(I^l_n\right) \vee \sup\left(I^j_n\right)\right]$$

Now, using (11.12), we find

$$\mathcal{U} \circ \mathcal{V} \supseteq \bigcap \{[\Sigma_l] \times [\Sigma_j] : \ (l, j) \in \Phi\} \supseteq \bigcap \{[\Sigma_{l,j}] \times [\Sigma_{l,j}] : \ (l, j) \in \Phi\}$$

Clearly each $\Sigma_{l,j}$ satisfies 1) of (11.9). Since $\mathcal{ML}^0(\Omega)$ is fully distributive, see Corollary 52, (11.10) follows by Lemma 92.
The second part of the proposition follows by the same arguments used in the proof of Theorem 56. ■

The proof of Proposition 91 relies on the following.

Lemma 92 *The set $\mathcal{ML}^0_{i,q,p}(\Omega)$ is a lattice with respect to the pointwise order.*

Proof. Consider any functions $u, v \in \mathcal{ML}^0_{i,q,p}(\Omega)$, and set $w = \sup\{u, v\} \in \mathcal{ML}^0(\Omega)$. In view of Theorem 45 it follows that

$$w(x) = (I \circ S)(\varphi)(x), \; x \in \Omega$$

where

$$\varphi(x) = \sup\{u(x), v(x)\}, \; x \in \Omega.$$

Assume that

$$\begin{array}{ll} \exists & y_0 \in \mathbb{R}^n : \\ \exists & a \in \mathbb{R} : \\ & w(y_0, t_0) > a > D^q g_{p,i}(y_0) \end{array} . \tag{11.13}$$

It then follows that $S(\varphi)(y_0, t_0) > a > D^q g_{p,i}(y_0)$. Therefore

$$\begin{array}{ll} \forall & \delta > 0 : \\ \exists & (y_\delta, t_\delta) \in B_\delta(y_0, t_0) \; : \\ & \varphi(y_\delta, t_\delta) > a > D^q g_{p,i}(y_0) \end{array}$$

so that we obtain a sequence (y_n, t_n) in Ω which converges to (y_0, t_0) and satisfies

$$\begin{array}{ll} \forall & n \in \mathbb{N} : \\ & u(y_n, t_n) > a > D^q g_{i,p}(y_0) = u(y_0, t_0) \end{array} \tag{11.14}$$

or

$$\begin{array}{ll} \forall & n \in \mathbb{N} : \\ & v(y_n, t_n) > a > D^q g_{i,p}(y_0) = v(y_0, t_0) \end{array} . \tag{11.15}$$

But both u and v are continuous at (y, t_0) for each $y \in \mathbb{R}^n$, which contradicts (11.14) to (11.15). Hence (11.13) cannot hold, so that $w \in \mathcal{ML}^0_{i,q,p}(\Omega)$.
The existence of the infimum of u and v follows in the same way. ∎

The completion of $\mathcal{ML}^0_{i,q,p}(\Omega)$ may be represented as a suitable space of nearly finite normal lower semi-continuous functions. In particular, we consider the space

$$\mathcal{NL}_{i,q,p}(\Omega) = \left\{ u \in \mathcal{NL}(\Omega) \left| \begin{array}{ll} \exists & \lambda, \mu \in \mathcal{ML}^0_{i,q,p}(\Omega) \; : \\ & \lambda \leq u \leq \mu \end{array} \right. \right\}.$$

Note that $\mathcal{ML}^0_{i,q,p}(\Omega) \subset \mathcal{NL}_{i,q,p}(\Omega)$. As such, in order to show that $\mathcal{NL}_{i,q,p}(\Omega)$ is the completion of $\mathcal{ML}^0_{i,q,p}(\Omega)$, we must introduce a Hausdorff uniform convergence structure $\mathcal{J}^\sharp_{i,q,p}$ on $\mathcal{NL}_{i,q,p}(\Omega)$ in such a way that the following conditions are satisfied:

1. $\mathcal{NL}_{i,q,p}(\Omega)$ is complete with respect to $\mathcal{J}^\sharp_{i,q,p}$.

2. $\mathcal{NL}_{i,q,p}(\Omega)$ contains $\mathcal{ML}^0_{i,q,p}(\Omega)$ as a dense subspace.

3. If Y is a complete, Hausdorff uniform convergence space, then any uniformly continuous mapping $\varphi : \mathcal{ML}^0_{i,q,p}(\Omega) \to Y$ extends in a unique way to a uniformly continuous mapping $\varphi^\sharp : \mathcal{NL}_{i,q,p}(\Omega) \to Y$.

In this regard, the definition of the uniform convergence structure on $\mathcal{NL}_{i,q,p}(\Omega)$ is similar to Definition 58

Definition 93 *Let $\mathcal{J}^\sharp_{i,q,p}$ denote the family of filters on $\mathcal{NL}_{i,q,p}(\Omega)\times\mathcal{NL}_{i,q,p}(\Omega)$ that satisfy the following: There exists $k \in \mathbb{N}$ such that*

$$\begin{array}{ll} \forall & j = 1, ..., k : \\ \exists & (\lambda^j_n),\ (\mu^j_n) \subseteq \mathcal{ML}^0_{i,p}(\Omega) \ : \\ \exists & u_j \in \mathcal{ML}^0_{i,p}(\Omega) \ : \\ & 1) \quad \lambda^j_n \leq \lambda^j_{n+1} \leq \mu^j_{n+1} \leq \mu^i_n,\ n \in \mathbb{N} \\ & 2) \quad \bigcap_{j=1}^k \left(([\Sigma_j] \times [\Sigma_j]) \cap ([u_j] \times [u_j]) \right) \subseteq \mathcal{U} \end{array} \tag{11.16}$$

where each $u_j \in \mathcal{NL}_{i,q,p}(\Omega)$ satisfies $u_j = \sup\{\lambda^j_n : n \in \mathbb{N}\} = \inf\{\mu^j_n : n \in \mathbb{N}\}$. Here $\Sigma_j = \{I^j_n : n \in \mathbb{N}\}$ with

$$I^j_n = \{u \in \mathcal{ML}_{i,q,p}(\Omega) \ : \ \lambda^j_n \leq u \leq \mu^j_n\}.$$

That the family of filters $\mathcal{J}^\sharp_{i,q,p}$ does indeed constitute a Hausdorff uniform convergence structure on $\mathcal{NL}_{i,q,p}(\Omega)$ can easily be seen. Indeed, $\mathcal{J}^\sharp_{i,q,p}$ is nothing but the uniform convergence structure associated with the following Hausdorff convergence structure through (2.70): A filter $\mathcal{F}$ on $\mathcal{NL}_{i,q,p}(\Omega)$ converges to $u \in \mathcal{NL}_{i,q,p}(\Omega)$ if and only if

$$\begin{array}{ll} \exists & (\lambda_n),\ (\mu_n) \subset \mathcal{ML}^0_{i,q,p}(\Omega) \ : \\ & 1) \quad \lambda_n \leq \lambda_{n+1} \leq \mu_{n+1} \leq \mu_n,\ n \in \mathbb{N} \\ & 2) \quad \cap_{n\in\mathbb{N}}[\lambda_n, \mu_n]_{|V} = \{u\}_{|V},\ V \subseteq \Omega \text{ open} \\ & 3) \quad [\{[\lambda_n, \mu_n] : n \in \mathbb{N}\}] \subseteq \mathcal{F} \end{array} .$$

Theorem 94 *The space $\mathcal{NL}_{i,q,p}(\Omega)$ equipped with the uniform convergence structure $\mathcal{J}^\sharp_{i,q,p}$ is the uniform convergence space completion of $\mathcal{ML}^0_{i,q,p}(\Omega)$.*

Proof. That $\mathcal{NL}_{i,q,p}(\Omega)$ is complete follows immediate by our above remarks. Furthermore, it is clear that the subspace uniform convergence structure on $\mathcal{ML}^0_{i,q,p}(\Omega)$ is equal to $\mathcal{J}_{i,q,p}$.
The extension property for uniformly continuous mappings follows by a straight forward argument. ∎

An important property of the uniform convergence space $\mathcal{ML}^0_{i,q,p}(\Omega)$ and its completion $\mathcal{NL}_{i,q,p}(\Omega)$ relates to the inclusion mapping

$$i : \mathcal{ML}^0_{i,q,p}(\Omega) \to \mathcal{ML}^0(\Omega) \tag{11.17}$$

and its extension through uniform continuity

$$i^{\sharp} : \mathcal{NL}_{i,q,p}(\Omega) \to \mathcal{NL}(\Omega). \tag{11.18}$$

Indeed, it is clear form Definitions 53 and 90 that the mapping (11.17) is in fact uniformly continuous. Similarly, the inclusion mapping

$$i_0 : \mathcal{NL}_{i,q,p}(\Omega) \to \mathcal{NL}(\Omega) \tag{11.19}$$

is uniformly continuous. Since the mappings (11.18) and (11.19) coincide on a dense subset of $\mathcal{NL}_{i,q,p}(\Omega)$, it follows that (11.18) is simply the inclusion mapping (11.19).

This is related to the issue of *consistency* of generalized solutions of (11.1) to (11.2) that we construct in the sequel with solutions in the space $\mathcal{NL}^m(\Omega)^K$, that is, solutions of the generalized equation (8.39). We will discuss this in some detail in what follows, after the uniform convergence structure on $\mathcal{ML}^m_{\mathbf{g}}(\Omega)$ has been introduced.

In this regard, the uniform convergence structure $\mathcal{J}_{\mathbf{g}}$ on $\mathcal{ML}^m_{\mathbf{g}}(\Omega)$ is defined as the initial uniform convergence structure with respect to the mappings (11.7) to (11.8). That is, a filter $\mathcal{U}$ on $\mathcal{ML}^m_{\mathbf{g}}(\Omega) \times \mathcal{ML}^m_{\mathbf{g}}(\Omega)$ belongs to $\mathcal{J}_{\mathbf{g}}$ if and only if

$$\begin{array}{l} \forall \quad i = 1, ..., K : \\ \quad \left(\mathcal{D}^m_{i,t} \times \mathcal{D}^m_{i,t}\right)(\mathcal{U}) \in \mathcal{J}_o \end{array},$$

and

$$\begin{array}{l} \forall \quad 0 \leq p < m : \\ \forall \quad q \in \mathbb{N}^{n-1},\ 0 < |q| + p \leq m : \\ \forall \quad i = 1, ..., K : \\ \quad \left(\mathcal{D}^{qp}_{i,yt} \times \mathcal{D}^{qp}_{i,yt}\right)(\mathcal{U} \times \mathcal{U}) \in \mathcal{J}_{i,q,p} \end{array},$$

Clearly the family consisting of the mappings (11.7) through (11.8) separates the points of $\mathcal{ML}^m_{\mathbf{g}}(\Omega)$. As such, the uniform convergence structure $\mathcal{J}_{\mathbf{g}}$ is uniformly Hausdorff. In particular, and in view of Theorem 44, the mapping

$$\mathbf{D} : \mathcal{ML}^m_{\mathbf{g}}(\Omega) \to \left(\prod \mathcal{ML}^0_{i,q,p}(\Omega)\right) \times \mathcal{ML}^0(\Omega)^K$$

which is defined through

$$\mathbf{D} : \mathbf{u} \mapsto \left(..., \mathcal{D}^{qp}_{i,yt}\mathbf{u}, ...\mathcal{D}^m_{i,t}\mathbf{u}, ...\right) \tag{11.20}$$

is a uniformly continuous embedding. As such, it follows from Theorem 37 that the mapping (11.20) extends to an injective, uniformly continuous mapping

$$\mathbf{D}^{\sharp} : \mathcal{NL}^m_{\mathbf{g}}(\Omega) \to \left(\prod \mathcal{NL}_{i,p}(\Omega)\right) \times \mathcal{NL}(\Omega)^K$$

where $\mathcal{NL}^m_{\mathbf{g}}(\Omega)$ denotes the uniform convergence space completion of $\mathcal{ML}^m_{\mathbf{g}}(\Omega)$. In particular, for each $i = 1, ..., K$ $0 \leq p < m$, and each $q \in \mathbb{N}^{n-1}$ such that $0 \leq p + |q| \leq m$ the diagrams

$$\begin{tikzcd}
\mathcal{NL}^m_{\mathbf{g}}(\Omega) \arrow[rr, "\mathbf{D}^\sharp"] \arrow[dr, "\mathcal{D}^{qp\sharp}_{i,yt}"'] & & \left(\prod \mathcal{NL}_{i,q,p}(\Omega)\right) \times \mathcal{NL}(\Omega)^K \arrow[dl, "\pi_{i,p,q}"] \\
& \mathcal{NL}_{i,q,p}(\Omega) &
\end{tikzcd} \tag{11.21}$$

and

$$\begin{tikzcd}
\mathcal{NL}^m_{\mathbf{g}}(\Omega) \arrow[rr, "\mathbf{D}^\sharp"] \arrow[dr, "\mathcal{D}^{m\sharp}_{i,t}"'] & & \left(\prod \mathcal{NL}_{i,q,p}(\Omega)\right) \times \mathcal{NL}(\Omega)^K \arrow[dl, "\pi_{i}"] \\
& \mathcal{NL}(\Omega) &
\end{tikzcd} \tag{11.22}$$

commute, with $\pi_{i,q,p}$ and π_i the projections, and $\mathcal{D}^{qp\sharp}_{i,yt}$ and $\mathcal{D}^{m\sharp}_{i,t}$ the extensions through uniform continuity of the mappings (11.7) and (11.8), respectively.

The meaning of the diagrams (11.21) and (11.22) is twofold. In the first instance, it explains the regularity of generalized functions in $\mathcal{NL}^m_{\mathbf{g}}(\Omega)$. In particular, each generalized partial derivative of a generalized function $\mathbf{u}^\sharp \in \mathcal{NL}^m_{\mathbf{g}}(\Omega)$ is a nearly finite normal lower semi-continuous function. Therefore, each such generalized function may be represented as an element of the space $\left(\prod_{i\leq K}^{0\leq p<m} \mathcal{NL}_{i,p}(\Omega)\right) \times \mathcal{NL}(\Omega)^L$. Secondly, these diagrams state that each generalized function $\mathbf{u}^\sharp \in \mathcal{NL}^m_{\mathbf{g}}(\Omega)$ satisfies

the initial condition (11.2) in the sense that

$$\begin{array}{ll} \forall & i = 1, ..., K : \\ \forall & 0 \leq p < m : \\ \forall & q \in \mathbb{N}^{n-1},\, 0 \leq p + |q| \leq m : \\ & \mathcal{D}^{p\sharp}_{i,t} \mathbf{u}^{\sharp}\left(t_0, y\right) = g_{p,i}\left(t_0, y\right),\ y \in \mathbb{R}^{n-1} \end{array} . \tag{11.23}$$

With the system of nonlinear PDEs (11.1) we may associate a mapping

$$\mathbf{T} : \mathcal{ML}^{m}_{\mathbf{g}}\left(\Omega\right) \to \mathcal{ML}^{0}\left(\Omega\right)^{K}, \tag{11.24}$$

the components of which are defined as in (8.12). Generalized solutions to the initial value problem (11.1) and (11.2) are obtained by suitably extending the mapping (11.24) to a mapping

$$\mathbf{T}^{\sharp} : \mathcal{NL}^{m}_{\mathbf{g}}\left(\Omega\right) \to \mathcal{NL}\left(\Omega\right)^{K}. \tag{11.25}$$

Such an extension is obtained through the uniform continuity of the mapping (11.24). In this regard, we have the following.

Theorem 95 *The mapping (11.24) is uniformly continuous.*

Proof. It follows from (11.17) through (11.18) that the inclusion mapping

$$i : \mathcal{ML}^{m}_{\mathbf{g}}\left(\Omega\right) \to \mathcal{ML}^{m}\left(\Omega\right)^{K}$$

is uniformly continuous. The result now follows from the commutative diagram

$$\begin{tikzcd} \mathcal{ML}^{m}_{\mathbf{g}}\left(\Omega\right) \arrow[rr, "\mathbf{T}"] \arrow[dr, "i"'] & & \mathcal{ML}^{0}\left(\Omega\right)^{K} \\ & \mathcal{ML}^{m}\left(\Omega\right)^{K} \arrow[ur, "\mathbf{T}_0"'] & \end{tikzcd} \tag{11.26}$$

and Theorem 65, with $\mathbf{T}_0$ the mapping defined on $\mathcal{ML}^{m}\left(\Omega\right)^{K}$ through the nonlinear partial differential operator. ∎

In view of Theorem 95 the mapping (11.24) extends in a unique way to a uniformly continuous mapping with domain $\mathcal{NL}^{m}_{\mathbf{g}}\left(\Omega\right)$ and range contained in $\mathcal{NL}\left(\Omega\right)^{K}$, which is the generalized nonlinear partial differential operator (11.25). As such, the generalized initial value problem corresponding to (11.1) and (11.2) is the single equation

$$\mathbf{T}^{\sharp}\mathbf{u}^{\sharp} = \mathbf{0}, \tag{11.27}$$

where $\mathbf{0}$ denotes the the element in $\mathcal{NL}(\Omega)^K$ with all components identically 0. A solution to (11.27) is interpreted as a generalized solution to (11.1) through (11.2) based on the facts that the mapping (11.25) is the unique and canonical extension of (11.24), and each solution of (11.27) satisfies the initial condition in an extended sense, as mentioned in (11.23). Furthermore, in view of (11.17) to (11.19) and the diagram (11.26) we obtain the commutative diagram

$$\begin{tikzcd}
\mathcal{NL}^m_{\mathbf{g}}(\Omega) \arrow[rr, "\mathbf{T}^{\sharp}"] \arrow[dr, "i^{\sharp}"'] & & \mathcal{NL}(\Omega)^K \\
& \mathcal{NL}^m(\Omega)^K \arrow[ur, "\mathbf{T}_0^{\sharp}"'] &
\end{tikzcd} \tag{11.28}$$

with $i^{\sharp}$ injective and $\mathbf{T}_0^{\sharp}$ the uniformly continuous extension of the mapping

$$\mathbf{T}_0 : \mathcal{ML}^m(\Omega)^K \to \mathcal{ML}^0(\Omega)^K$$

associated with the system of nonlinear PDEs (11.1). In particular, the mapping $i^{\sharp}$ is the inclusion mapping. As such, each solution $\mathbf{u}^{\sharp} \in \mathcal{NL}^m_{\mathbf{g}}(\Omega)$ of (11.27) is a solution of the system of nonlinear PDEs (11.1) in the sense of the Sobolev type spaces of generalized functions introduced in Section 8.2. In this regard, the main result of this section is the following.

Theorem 96 *For each $0 \leq p < m$, let $\boldsymbol{g}_p \in \mathcal{C}^{m-p}(\mathbb{R}^{n-1})^K$. Then there is some $\boldsymbol{u}^{\sharp} \in \mathcal{NL}^m_{\boldsymbol{g}}(\Omega)$ so that*

$$\boldsymbol{T}^{\sharp}\boldsymbol{u}^{\sharp} = \boldsymbol{0}.$$

Proof. Let us express $\Omega = \mathbb{R}^{n-1} \times \mathbb{R}$ as

$$\Omega = \bigcup_{\nu \in \mathbb{N}} C_{\nu}$$

where, for $\nu \in \mathbb{N}$, the compact sets C_ν are n-dimensional intervals

$$C_\nu = [a_\nu, b_\nu] \tag{11.29}$$

with $a_\nu = (a_{\nu,1}, ..., a_{\nu,n})$, $b_\nu = (b_{\nu,1}, ..., b_{\nu,n}) \in \mathbb{R}^n$ and $a_{\nu,j} \leq b_{\nu,j}$ for every $j = 1, ..., n$. We assume that the C_ν, with $\nu \in \mathbb{N}$, are locally finite, that is,

$$\begin{array}{ll} \forall & x \in \Omega : \\ \exists & V \subseteq \Omega \text{ a neighborhood of } x : \\ & \{\nu \in \mathbb{N} : C_\nu \cap V \neq \emptyset\} \text{ is finite} \end{array} \tag{11.30}$$

Such a partition of Ω exists, see for instance [58]. We also assume that, for each $\nu \in \mathbb{N}$,

$$\mathcal{S} \cap C_\nu = \emptyset$$

or

$$\mathcal{S} \cap \mathrm{Int} C_\nu \neq \emptyset \tag{11.31}$$

where $\mathcal{S}$ is the noncharacteristic hypersurface

$$\mathcal{S} = \{(y, t_0) \ : \ y \in \mathbb{R}^{n-1}\}$$

For the sake of convenience, let us write $x = (y, t)$ for each $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Let $\mathbf{F} : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}^K$ is the mapping that defines the nonlinear operator $\mathbf{T}$ through

$$\mathbf{T}(x, D)\, \mathbf{u}(x) = \mathbf{F}(x, ..., D^\alpha u_i(x), ...).$$

Fix $\nu \in \mathbb{N}$ such that (11.31) is satisfied. In view of the fact that the mapping $\mathbf{F}$ is both open and surjective, we have

$$\begin{array}{ll} \forall & x_1 = (y_1, t_1) \in C_\nu : \\ \exists & \xi(x_1) \in \mathbb{R}^M,\ \mathbf{F}(x_1, \xi(x_1)) = 0 : \\ \exists & \delta,\ \epsilon > 0 : \\ & \quad 1) \quad \{(x, 0) \ : \ \|x - x_1\| < \delta\} \subset \mathrm{int}\left\{ (x, \mathbf{F}(x, \xi)) \left| \begin{array}{l} \|x - x_1\| < \delta \\ \|\xi - \xi(x_1)\| < \epsilon \end{array} \right. \right\} \\ & \quad 2) \quad \mathbf{F} : B_\delta(x_1) \times B_{2\epsilon}(\xi(x_1)) \rightarrow \mathbb{R}^K \text{ open} \end{array} \tag{11.32}$$

In particular, if $t_1 = t_0$, we may take $\xi(x_1) = (\xi_i^{q,p}, \xi_i^m)$ such that

$$\begin{array}{ll} \forall & i = 1, ..., K : \\ \forall & 0 \leq p < m : \\ \forall & q \in \mathbb{N}^{n-1},\ 0 < |q| + p \leq m : \\ & \xi_i^{q,p} = D^q g_{p,i}(y_1) \end{array} \tag{11.33}$$

For each $x_1 \in C_\nu$, fix $\xi(x_1) \in \mathbb{R}^M$ in (11.32) so that (11.33) is satisfied in case $t_1 = t_0$. Since C_ν is compact, it follows from (11.32) that

$$\begin{array}{ll} \exists & \delta > 0 : \\ \forall & x_1 \in C_\nu : \\ \exists & \epsilon_{x_1} > 0 : \\ & 1) \quad \{(x,0) \,:\, \|x - x_1\| < \delta\} \subset \operatorname{int}\left\{ (x, \mathbf{F}(x,\xi)) \left| \begin{array}{l} \|x - x_1\| < \delta \\ \|\xi - \xi(x_1)\| < \epsilon_{x_1} \end{array} \right. \right\} \\ & 2) \quad \mathbf{F} : B_\delta(x_1) \times B_{2\epsilon_{x_0}}(\xi(x_1)) \rightarrow \mathbb{R}^K \text{ open} \end{array} \tag{11.34}$$

Subdivide C_ν into n-dimensional intervals $I_{\nu,1}, ..., I_{\nu,\mu_\nu}$ with diameter not exceeding δ such that their interiors are pairwise disjoint and, for each $j = 1, ..., \mu_\nu$,

$$I_{\nu,j} \cap \mathcal{S} = \emptyset \tag{11.35}$$

or

$$\operatorname{int} I_{\nu,j} \cap \mathcal{S} \neq \emptyset \tag{11.36}$$

If $a_{\nu,j}$ with $j = 1, ..., \mu_\nu$ is the center of the interval $I_{\nu,j}$ that satisfies (11.35), then by (11.34) we have

$$\begin{array}{ll} \exists & \epsilon_{\nu,j} > 0 : \\ & 1) \quad \{(x,0) \,:\, x \in I_{\nu,j}\} \subset \operatorname{int}\left\{ (x, \mathbf{F}(x,\xi)) \left| \begin{array}{l} x \in I_{\nu,j} \\ \|\xi - \xi(a_{\nu,j})\| < \epsilon_{\nu,j} \end{array} \right. \right\} \\ & 2) \quad \mathbf{F} : I_{\nu,j} \times B_{2\epsilon_{\nu,j}}(\xi(a_{\nu,j})) \rightarrow \mathbb{R}^K \text{ open} \end{array} \tag{11.37}$$

On the other hand, if $I_{\nu,j}$ satisfies (11.36), set $a_{\nu,j}$ equal to the midpoint of $\mathcal{S} \cap I_{\nu,j}$. Then we obtain (11.37) by (11.34) such that (11.33) also holds. Take $0 < \gamma < 1$ arbitrary but fixed. In view of Proposition 68 and (11.37), we have

$$\begin{array}{ll} \forall & x_1 \in I_{\nu,j} : \\ \exists & \mathbf{U}_{x_1} = \mathbf{U} \in \mathcal{C}^m(\mathbb{R}^n)^K \;: \\ \exists & \delta = \delta_{x_1} > 0 : \\ & x \in B_\delta(x_1) \cap I_{\nu,j} \Rightarrow \left(\begin{array}{ll} 1) & (D^\alpha U_i(x))_{i \leq K}^{|\alpha| \leq m} \in B_{\epsilon_{\nu,j}}(\xi(a_{\nu,j})) \\ 2) & i \leq K \Rightarrow \gamma < T_i(x,D)\mathbf{U}(x) < 0 \end{array} \right) \end{array},$$

with $\alpha = (q,p)$. Furthermore, if $I_{\nu,j}$ satisfies (11.36), then we also have

$$\begin{array}{ll} \forall & i = 1, ..., K : \\ \forall & 0 \leq p < m : \\ \forall & q \in \mathbb{N}^{n-1},\, 0 < |q| + p \leq m : \\ \forall & y \in \mathbb{R}^{n-1} : \\ & D_{yt}^{pq} U_i(y, t_0) = D^q g_{p,i}(y) \end{array}.$$

As above, we may subdivide $I_{\nu,j}$ into pairwise disjoint, n-dimensional intervals $J_{\nu,j,1}, ..., J_{\nu,j,\mu_{\nu,j}}$ so that for $k = 1, ..., \mu_{\nu,j}$ we have

$$\begin{array}{ll} \exists & \mathbf{U}^{\nu,j,k} = \mathbf{U} \in \mathcal{C}^m(\mathbb{R}^n)^K \;: \\ \forall & x \in J_{\nu,j,k} : \\ & 1) \quad \left(D^\alpha U_i(x)_{i \leq K}^{|\alpha| \leq m} \right) \in B_{\epsilon_{\nu,j}}(\xi(a_{\nu,j})),\ |\alpha| \leq m \\ & 2) \quad i \leq K \Rightarrow f_i(x) - \gamma < T_i(x,D)\mathbf{U}(x) < f_i(x) \end{array} \tag{11.38}$$

and

$$J_{\nu,j,k}\cap\mathcal{S}=\emptyset \tag{11.39}$$

or

$$\mathrm{int}I_{\nu,j,k}\cap\mathcal{S}\neq\emptyset. \tag{11.40}$$

Furthermore, whenever $J_{\nu,j,k}$ satisfies (11.40), we have

$$\begin{array}{ll}\forall & i=1,...,K:\\ \forall & 0\leq p<m:\\ \forall & q\in\mathbb{N}^{n-1},\ 0<|q|+p\leq m:\\ \forall & y\in\mathbb{R}^{n-1}:\\ & D^{qp}_{yt}U_i\left(y,t_0\right)=D^q g_{p,i}\left(y\right)\end{array}.$$

In particular, in this case we may simply set

$$U_i\left(y,t\right)=\sum_{p=0}^{m-1}\left(t-t_0\right)^p g_{p,i}\left(y\right)+w_i\left(t\right)$$

for a suitable function $w_i\in\mathcal{C}^m\left(\mathbb{R}\right)$ that satisfies

$$\begin{array}{ll}\forall & 0\leq p<m:\\ & w_i^{(p)}\left(t_0\right)=0\end{array}.$$

Set

$$\Gamma_1=\Omega\setminus\left(\bigcup_{\nu\in\mathbb{N}}\left(\bigcup_{j=1}^{\mu_\nu}\left(\bigcup_{k=1}^{\mu_{\nu,j}}\mathrm{int}J_{\nu,j,k}\right)\right)\right).$$

and

$$\mathbf{V}_1=\sum_{\nu\in\mathbb{N}}\left(\sum_{j=1}^{\mu_\nu}\left(\sum_{k=1}^{\mu_{\nu,j}}\chi_{J_{\nu,j,k}}\mathbf{U}_{\nu,j,k}\right)\right)$$

where $\chi_{J_{\nu,j,k}}$ is the characteristic function of $J_{\nu,j,k}$. Then Γ_1 is closed nowhere dense, and $\mathbf{V}_1\in\mathcal{C}^m\left(\Omega\setminus\Gamma_1\right)^K$. Furthermore, $\mathcal{S}\cap\Gamma_1$ is closed nowhere dense in $\mathcal{S}$ and

$$\begin{array}{ll}\forall & i=1,...,K:\\ \forall & 0\leq p<m:\\ \forall & q\in\mathbb{N}^{n-1},\ 0<|q|+p\leq m:\\ \forall & \left(y,t_0\right)\in\mathcal{S}\setminus\left(\mathcal{S}\cap\Gamma_1\right):\\ & D^{qp}_{yt}V_{1,i}\left(y,t_0\right)=D^q g_{p,i}\left(y\right)\end{array}.$$

In view of (11.38) we have, for each $i=1,...,K$

$$-\gamma<T_i\left(x,D\right)\mathbf{V}_1\left(x\right)<0,\ x\in\Omega\setminus\Gamma_1$$

Furthermore, for each $\nu \in \mathbb{N}$, for each $j = 1, ..., \mu_\nu$, each $k = 1, ..., \mu_{\nu,j}$, each $|\alpha| \leq m$ and every $i = 1, ..., K$ we have

$$x \in \mathrm{int} J_{\nu,j,k} \Rightarrow \xi_i^\alpha (a_{\nu,j}) - \epsilon < D^\alpha V_{1,i} (x) < \xi_i^\alpha (a_{\nu,j}) + \epsilon \tag{11.41}$$

For $0 \leq p < m$, define the functions $\lambda_{1,i}^\alpha, \mu_{1,i}^\alpha \in \mathcal{C}^0 (\Omega \setminus \Gamma_1)$, where $\alpha = (p, q)$ with $|q| = 0$, as

$$\lambda_{1,i}^\alpha (x) = \begin{cases} \xi_i^\alpha (a_{\nu,j}) - 2\epsilon_{\nu,j} & if \quad x \in \mathrm{int} I_{\nu,j,k} \text{ and } I_{\nu,j,k} \cap \mathcal{S} = \emptyset \\ D_t^p V_{1,i} (y, t) - v_{\nu,j} (t) & if \quad x \in \mathrm{int} I_{\nu,j,k} \text{ and } I_{\nu,j,k} \cap \mathcal{S} \neq \emptyset \end{cases}$$

and

$$\mu_{1,i}^\alpha (x) = \begin{cases} \xi_i^\alpha (a_{\nu,j}) + 2\epsilon_{\nu,j} & if \quad x \in \mathrm{int} I_{\nu,j,k} \text{ and } I_{\nu,j,k} \cap \mathcal{S} = \emptyset \\ D_t^p V_{1,i} (y, t) + v_{\nu,j} (t) & if \quad x \in \mathrm{int} I_{\nu,j,k} \text{ and } I_{\nu,j,k} \cap \mathcal{S} \neq \emptyset \end{cases}$$

Here $v_{\nu,j}$ is a continuous, real valued function on $\mathbb{R}$ such that

$$v_{\nu,j} (t_0) = 0 \tag{11.42}$$

and

$$0 < v_{\nu,j} (t) < 2\epsilon_{\nu,j}, \; t \in \mathbb{R} \tag{11.43}$$

For all other α, consider the functions

$$\lambda_{1,i}^\alpha (x) = \xi_i^\alpha (a_{\nu,j}) - 2\epsilon_{\nu,j} \text{ if } x \in \mathrm{int} I_{\nu,j}$$

and

$$\mu_{1,i}^\alpha (x) = \xi_i^\alpha (a_{\nu,j}) + 2\epsilon_{\nu,j} \text{ if } x \in \mathrm{int} I_{\nu,j}.$$

Then it follows by (11.41) that

$$\lambda_{1,i}^\alpha (x) < D^\alpha V_{1,i} (x) < \mu_{1,i}^\alpha (x), \; x \in \Omega \setminus \Gamma_1$$

and

$$\mu_{1,i}^\alpha (x) - \lambda_{1,i}^\alpha (x) < 4\epsilon_{\nu,j}, \; x \in \mathrm{int} I_{\nu,j}$$

Applying (11.37) restricted to $\Omega \setminus \Gamma_1$, and proceeding in a fashion similar as above, we may construct, for each $n \in \mathbb{N}$ such that $n > 1$, a closed nowhere dense set $\Gamma_n \subset \Omega$ such that

$$\Gamma_n \cap \mathcal{S} \text{ closed nowhere dense in } \mathcal{S},$$

a function $\mathbf{V}_n \in \mathcal{C}^m(\Omega \setminus \Gamma_n)^K$ and functions $\lambda^\alpha_{n,i}, \mu^\alpha_{n,i} \in \mathcal{C}^0(\Omega \setminus \Gamma_n)$ so that, for each $i = 1, ..., K$

$$-\frac{\gamma}{n} < T_i(x, D)\mathbf{V}_n(x) < 0,\ x \in \Omega \setminus \Gamma_n. \tag{11.44}$$

and for every $|\alpha| \leq m$

$$\lambda^\alpha_{n-1,i}(x) < \lambda^\alpha_{n,i}(x) < D^\alpha V_{n,i}(x) < \mu^\alpha_{n,i}(x) < \mu^\alpha_{n-1,i}(x),\ x \in \Omega \setminus \Gamma_n \tag{11.45}$$

and

$$\mu^\alpha_{n,i}(x) - \lambda^\alpha_{n,i}(x) < \frac{4\epsilon_{\nu,j}}{n},\ x \in (\mathrm{int} I_{\nu,j}) \cap (\Omega \setminus \Gamma_n). \tag{11.46}$$

Furthermore, for each $0 \leq p < m$ and $q \in \mathbb{N}^{n-1}$ so that $0 \leq |q| + p \leq m$ we have

$$D^{qp}_{yt} V_{n,i}(y, t_0) = \lambda^\alpha_{n,i}(y, t_0) = \mu^\alpha_{n,i}(y, t_0) = D^q g_{p,i}(y),\ (y, t_0) \notin \mathcal{S} \cap \Gamma_n$$

where $\alpha = (p, q)$.
Notice that the functions $\mathbf{u}_n$, the components of which are defined through

$$u_{n,i} = (I \circ S)(V_{n,i})$$

belongs to $\mathcal{ML}^m_{\mathbf{g}}(\Omega)$. In view of (11.45) it follows that the functions $\overline{\lambda}^\alpha_{n,i}, \overline{\mu}^\alpha_{n,i} \in \mathcal{ML}^0(\Omega)$, which are defined as

$$\overline{\lambda}^\alpha_{n,i} = (I \circ S)\left(\lambda^\alpha_{n,i}\right),\ \overline{\mu}^\alpha_{n,i} = (I \circ S)\left(\mu^\alpha_{n,i}\right),$$

satisfies

$$\overline{\lambda}^\alpha_{n-1,i} \leq \overline{\lambda}^\alpha_{n,i} \leq \mathcal{D}^\alpha u_{n,i} \leq \overline{\mu}^\alpha_{n,i} \leq \overline{\mu}^\alpha_{n-1,i}$$

Furthermore, in case $\alpha = (p, q)$ with $q \in \mathbb{N}^{n-1}$ such that $0 \leq p + |q| \leq m$, then $\overline{\lambda}^\alpha_{n,i}, \overline{\mu}^\alpha_{n,i} \in \mathcal{ML}^0_{i,q,p}(\Omega)$. It now follows by (11.46) that the sequence $(\mathbf{u}_n)$ is a Cauchy sequence in $\mathcal{ML}^m_{\mathbf{g}}(\Omega)$. Moreover, (11.44) implies that the sequence $(\mathbf{T}\mathbf{u}_n)$ converges to $\mathbf{0}$ in $\mathcal{ML}^0(\Omega)^K$. The result now follows from Theorem 95. ■

We have shown that the initial value problem (11.1) through (11.2) admits a generalized solution in the space $\mathcal{NL}^m_{\mathbf{g}}(\Omega)$. In particular, and in view of the commutative diagram (11.26), the generalized solution constructed in Theorem 96 is a generalized solution of the system of nonlinear PDEs (11.1) in the sense of the Sobolev type spaces of generalized functions introduced in Section 8.2. Furthermore, this solution satisfies the initial data (11.2) in the sense that

$$\begin{array}{ll} \forall & 0 \leq p < m : \\ \forall & q \in \mathbb{N}^{n-1},\ 0 \leq |q| + p \leq m : \\ \forall & y \in \mathbb{R}^{n-1} : \\ & \mathcal{D}^{qp\sharp}_{yt,i}\mathbf{u}^\sharp(y, t_0) = D^q g_{p,i}(y) \end{array}.$$

As such, it follows from Proposition 46 the singularity set

$$\left\{ (y,t) \in \Omega \;\middle|\; \begin{array}{ll} \exists & |\alpha| \leq m : \\ \exists & i = 1, ..., K : \\ & \mathcal{D}_i^{\alpha\sharp}\mathbf{u} \text{ not continuous at } (y,t) \end{array} \right\}$$

of the solution is of first Baire category.

This result is *a first in the literature*. Indeed, during the seventy years since Sobolev introduced functional analysis in the study of PDEs, the Cauchy-Kovalevskaia Theorem 2 has not been *extended*, in the context of any of the usual spaces of generalized functions, on its own general and type independent grounds. The only *improvement* upon this result which has been obtained to date is related to the domain of definition of the solutions. In particular, it has been shown [139] that the Cauchy problem (1.2) and (1.3) admits a generalized solution in a suitable algebra of generalized functions, which is defined on the whole domain of definition of the respective system of equations (1.2). Furthermore, such a solution is analytic everywhere except possibly for a closed nowhere dense set. However, the class of equations to which the result applies is the same as in the original version of the theorem, which was obtained more than a hundred years ago [86].

Theorem 96 delivers the existence of global generalized solutions of the initial value problem (11.1) and (11.2), as described above, provided only that the mapping (11.3) is continuous, and that the initial data satisfies rather obviously necessary smoothness conditions. As such, it is an extension of both the original Cauchy Kovalevskaia Theorem 2, and the global version of that result obtained in [139] in the context of the Sobolev type spaces of generalized functions introduced in Chapter 8.

11.2 Regularity of Generalized Solutions

The results presented in the previous section, in particular Theorem 96, concern only the first and basic properties regarding existence and regularity of solutions of the Cauchy problem (11.1) to (11.2). In contradistinction with Theorem 2, and the global version of that result [139], the solution cannot be interpreted as a classical solution on any part of the domain of definition of the equation. However, and as we shall see in the sequel, such additional regularity properties of the solution may be obtained with only minimal additional assumptions on the nonlinear partial differential operator (11.24). In particular, such conditions do not involve any restrictions on the type of equation, but instead involves only very mild assumptions on the smoothness of the mapping (11.3) and the initial data (11.2).

In this regard, consider now a system of nonlinear PDEs of the form (11.1) such that the mapping (11.3) is $\mathcal{C}^1$-smooth. Furthermore, we shall assume that the initial data (11.2) satisfies (11.5). In this case, and in view of the results presented in

Chapter 10, it is clear that the system of equations (11.1) admits a generalized solution $\mathbf{u}^{\sharp} \in \mathcal{NL}^{m}(\Omega)^{K}$ that satisfies

$$\begin{array}{ll} \exists & \Gamma \subset \mathbb{R}^{n-1} \times \mathbb{R} \text{ closed nowhere dense :} \\ \exists & \mathbf{U} \in \mathcal{C}^{m}(\Omega \setminus \Gamma)^{K} : \\ \forall & i = 1, ..., K : \\ \forall & |\alpha| \leq m : \\ & \mathcal{D}^{\alpha\sharp} u_{i}^{\sharp}(y,t) = D^{\alpha} U_{i}(y,t) \ , (y,t) \in \Omega \setminus \Gamma \end{array} . \tag{11.47}$$

That is, the solution $\mathbf{u}^{\sharp}$ is in fact a classical solution everywhere except for a closed nowhere dense set. Indeed, in this regard it is sufficient to show that the mapping

$$\mathbf{F} : \Omega \times \mathbb{R}^{M} \to \mathbb{R}^{K}$$

which defines the system of equations through (8.3) satisfies (9.30). This follows easily from the fact that the equation is linear in the terms $D_{t}^{m} u_{i}$. We now show that such a solution, that is, one that satisfies (11.47) may be constructed so as to also satisfy the initial condition (11.2).

The idea is to apply the techniques from Chapter 10. In particular, we will construct a suitable generalized solution of the system of nonlinear PDEs (11.1) in the space $\mathcal{NL}^{m+1}(\Omega)^{K}$. Smooth approximations are then constructed using Theorem 74. Note, however, that this approach can, in its present form, deliver only the existence of solutions in $\mathcal{ML}^{m}(\Omega)^{K}$ of the system of PDEs (11.1), solutions which may not satisfy the initial condition (11.2). Indeed, suppose that a generalized solution $\mathbf{u}^{\sharp} \in \mathcal{NL}^{m+1}(\Omega)^{K}$ of the system of equations (11.1) is constructed so as to also satisfy the initial condition (11.2) in the sense that

$$\begin{array}{ll} \forall & 0 \leq p < m : \\ \forall & q \in \mathbb{N}^{n-1}, 0 \leq p + |q| \leq m+1 : \\ \forall & i = 1, ..., K : \\ \forall & y \in \mathbb{R}^{n-1} : \\ & 1) \quad \mathcal{D}_{yt,i}^{qp\sharp} \mathbf{u}^{\sharp}(y,t_{0}) = D^{q} g_{p,i}(y) \\ & 2) \quad \mathcal{D}_{yt,i}^{qp\sharp} \mathbf{u}^{\sharp}(y,t_{0}) \text{ continuous at } (y,t_{0}) \end{array} .$$

Such a solutions is constructed as the limit of a sequence $(\mathbf{u}_{n})$ in $\mathcal{ML}^{m+1}(\Omega)^{K}$ that satisfies

$$\begin{array}{ll} \forall & 0 \leq p < m : \\ \forall & q \in \mathbb{N}^{n-1}, 0 \leq p + |q| \leq m+1 : \\ \forall & i = 1, ..., K : \\ \forall & y \in \mathbb{R}^{n-1} : \\ & 1) \quad \mathcal{D}_{yt,i}^{qp} \mathbf{u}_{n}(y,t_{0}) = D^{q} g_{p,i}(y) \\ & 2) \quad \mathcal{D}_{yt,i}^{qp} \mathbf{u}_{n}(y,t_{0}) \text{ continuous at } (y,t_{0}) \end{array} .$$

The next step is to approximate each function $\mathbf{u}_n$ by a sequence $(\mathbf{u}_{n,r}) \subset \mathcal{C}^{m+1}(\Omega)^K$, in the sense that

$$\begin{array}{ll} \forall & i = 1, ..., K : \\ \forall & |\alpha| \leq m+1 : \\ \forall & A \subset \Omega \setminus \Gamma_n \text{ compact} : \\ & \|\mathcal{D}^\alpha u_{n,i} - \mathcal{D}^\alpha u_{n,r,i}\|_A \to 0 \end{array}$$

where $\Gamma_n \subset \Omega$ is closed nowhere dense such that $\mathbf{u}_n \in \mathcal{C}^{m+1}(\Omega \setminus \Gamma_n)^K$. Using Proposition 49 and Theorem 86, one may extract a sequence $(\mathbf{u}_{n,r_n})$ which converges to some function $\mathbf{u} \in \mathcal{ML}^m(\Omega)^K$ such that $(\mathbf{Tu}_{n,r_n})$ converges to $\mathbf{0}$ in $\mathcal{ML}^0(\Omega)^K$. In particular, the sequence $(\mathbf{u}_{n,r_n})$ may be chosen in such a way that, for some closed nowhere dense set $\Gamma \subset \mathcal{S}$

$$\begin{array}{ll} \forall & 0 \leq p < m : \\ \forall & q \in \mathbb{N}^{n-1}, 0 \leq p + |q| \leq m : \\ \forall & i = 1, ..., K : \\ \forall & A \subset \mathcal{S} \text{ compact} : \\ & \|\mathcal{D}_y^q \mathcal{D}_t^p u_{n,r_n,i} - \mathcal{D}^q g_{p,i}\|_A \to 0 \end{array}.$$

However, the above construction does not imply that the solution $\mathbf{u} \in \mathcal{ML}^m(\Omega)^K$ of the system of PDEs (11.1) satisfies the initial condition (11.2). Indeed, the sequence $(\mathbf{u}_{n,r_n})$ may be unbounded on every neighborhood of every point of $\mathcal{S}$. In this regard, consider the following.

Example 97 *For each $n \in \mathbb{N}$, consider the function $u_n \in \mathcal{C}^1(\mathbb{R})$ given by*

$$u_n(t) = \begin{cases} e^{(n^2t^2-1)^4} & if \quad |t| < \frac{1}{n} \\ 0 & if \quad |t| \geq \frac{1}{n} \end{cases}$$

Clearly, $u_n(0) = e$ for every $n \in \mathbb{N}$. However, this sequence, and the sequence (u_n') converge to 0 uniformly on every compact subset of $\mathbb{R} \setminus \{0\}$, and the sequence of derivatives (u_n') is unbounded on every neighborhood of 0.

The difficulties mentioned above may be overcome by carefully constructing the original approximating sequence in $\mathcal{ML}^{m+1}(\Omega)$. As such, the method used to construct the approximations in the proof of following result is slightly different from those used in the proofs of Theorems 76 and 96.

Theorem 98 *The nonlinear Cauchy problem*

$$D_t^m \boldsymbol{u} = \boldsymbol{G}\left(y, t, ..., D_y^q D_t^p u_i(y,t), ...\right)$$

$$D_t^p \boldsymbol{u}(y, t_0) = \boldsymbol{g}^p(y),$$

with $0 \leq p < m$ and $q \in \mathbb{N}^{n-1}$ such that $0 \leq p + |q| < m$, admits a generalized solution $\boldsymbol{u}^\sharp \in \mathcal{NL}^m(\Omega)$ that also satisfies (11.47), provided that the mapping (11.3) is $\mathcal{C}^2$-smooth, and the initial data satisfies (11.5).

Proof. Write

$$\mathbb{R}^{n-1} = \bigcup_{\nu \in \mathbb{N}} J_\nu$$

where, for each $\nu \in \mathbb{N}$, J_ν is a compact $n-1$-dimensional interval $[a_\nu, b_\nu]$, with $a_{\nu,i} < b_{\nu,i}$ for each $i = 1, ..., n-1$. We also assume that the J_ν are locally finite, and have pairwise disjoint interiors.
Fix $\nu \in \mathbb{N}$ and $y_0 \in J_\nu$. Then it follows by Picard's Theorem 1 and the compactness of J_ν that there is some $\delta_\nu > 0$ such that the system of ODEs

$$\mathbf{F}^1 \left(y_0, t, ..., D^p v_i^q (t), ...\right) = \mathbf{0} \tag{11.48}$$

has a solution $\mathbf{v} = \mathbf{v}_{y_0} = \left(v_{y_0,i}^q\right)_{i \leq K}^{|q| \leq m+1} \in \mathcal{C}^{m+1} \left(t_0 - \delta_{y_0}, t_0 + \delta_\nu\right)^L$ such that

$$\begin{array}{ll} \forall & i = 1, ..., K : \\ \forall & 0 \leq p < m : \\ \forall & q \in \mathbb{N}^n, \, 0 \leq p + |q| \leq m+1 : \\ & D_t^p v_{y_0,i}^q (t_0) = D_y^q g_{p,i} (y_0) \end{array} . \tag{11.49}$$

Here $\mathbf{F}^1 : \Omega \times \mathbb{R}^L \to \mathbb{R}^P$ is the continuous mapping such that

$$\left(D^\beta \left(D_t^m \mathbf{u} + \mathbf{G}\left(y, t, ..., D_t^p D^q u_i, ...\right)\right)\right)_{|\beta| \leq 1} = \mathbf{F}^1 \left(y, t, ..., D_y^q D_t^p u_i, ...\right).$$

Also note that since the mapping $\mathbf{F}_1$, as well as the functions $D_y^p g_{p,i}$ defining the initial data are $\mathcal{C}^1$-smooth, the solutions $\mathbf{v}_{y_0}$ of (11.48) may be chosen in such a way that they depend continuously on $y_0 \in J_\nu$, see for instance [69]. That is,

$$\begin{array}{ll} \forall & \epsilon > 0 : \\ \exists & \theta_\epsilon > 0 : \\ \forall & i = 1, ..., K : \\ \forall & 0 \leq p \leq m+1 : \\ \forall & q \in \mathbb{N}^{n-1}, \, 0 \leq p + |q| \leq m+1 : \\ \forall & |t - t_0| < \delta_\nu : \\ & \|y_0 - y_1\| < \theta_\epsilon \Rightarrow |D_t^p v_{y_0,i}^q (t) - D_t^p v_{y_1,i}^q (t)| < \frac{\epsilon}{2} \end{array} . \tag{11.50}$$

Now define the functions $U_{y_0,i} \in \mathcal{C}^{m+1} \left(\mathbb{R}^{n-1} \times [t_0 - \delta_\nu, t_0 + \delta_\nu]\right)$ through

$$U_{y_0,i}(y,t) = \sum_{|q| \leq m+1} \left(\prod_{j=1}^{n-1} (y_j - y_{0,j})^{q_j} \, v_{y_0,i}^q\right)$$

where $q = (q_1, ..., q_{n-1})$. Then we have

$$\begin{array}{ll} \forall & i = 1, ..., K : \\ \forall & 0 \leq p \leq m+1 : \\ \forall & q \in \mathbb{N}^n, \, 0 \leq p + |q| \leq m+1 : \\ \forall & |t - t_0| < \delta_\nu \\ & D_t^p D_y^q U_{y_0,i}(y_0, t) = D^p v_{y_0,i}^q (t) \end{array} \tag{11.51}$$

and

$$\begin{array}{ll} \forall & i=1,...,K: \\ \forall & 0\leq p<m: \\ \forall & q\in\mathbb{N}^n,\ 0\leq p+|q|\leq m+1: \\ & D_t^p D_y^q U_{y_0,i}\left(y_0,t_0\right)=D_y^q g_{p,i}\left(y_0\right) \end{array} \tag{11.52}$$

As such, and in view of the fact that $\mathbf{v}=(v_i^q)_{i\leq K}^{|q|\leq m+1}$ satisfies the system of ODEs (11.48), it follows that

$$\begin{array}{ll} \forall & |\beta|\leq 1: \\ \forall & |t-t_0|<\delta_\nu: \\ \forall & j=1,...,K: \\ & D^\beta T_j\left(y,t,D\right)\mathbf{U}_{y_0}\left(y_0,t\right)=0 \end{array}$$

where $\mathbf{U}_{y_0}=\left(U_{y_0,i}\right)_{i\leq K}$ and the T_j are the components of the partial differential operator $\mathbf{T}\left(y,t,D\right)$. As such, and in view of the continuity of the mapping $\mathbf{F}_1$ and the function $\mathbf{U}_{y_0}$ and its derivatives, it now follows that

$$\begin{array}{ll} \forall & \epsilon>0: \\ \exists & \delta_{y_0}^\epsilon>0: \\ \forall & \|y-y_0\|<\delta_{y_0}^\epsilon: \\ \forall & |t-t_0|<\delta_\nu: \\ \forall & |\beta|\leq 1: \\ & -\epsilon<D^\beta T_j\left(y,t,D\right)\mathbf{U}_{y_0}\left(y,t\right)<\epsilon \end{array}. \tag{11.53}$$

Furthermore, from (11.50) and (11.51) it follows that

$$\begin{array}{ll} \forall & \epsilon>0: \\ \exists & \delta_{y_0}^\epsilon>0: \\ \forall & \|y-y_0\|<\delta_{y_0}^\epsilon: \\ \forall & 0\leq p\leq m+1: \\ \forall & q\in\mathbb{N}^n,\ 0\leq p+|q|\leq m+1: \\ \forall & |t-t_0|<\delta_\nu: \\ & \|y_0-y_1\|<\delta_{y_0}^\epsilon\Rightarrow\left(\begin{array}{ll} 1) & |D_{yt,i}^{qp}\mathbf{U}_{y_0}\left(y,t\right)-D_{yt,i}^{qp}\mathbf{U}_{y_1}\left(y,t\right)|<\epsilon \\ 2) & |D_{yt,i}^{qp}\mathbf{U}_{y_1}\left(y,t\right)-D_t^p v_{y_0,i}^q\left(t\right)|<\epsilon \end{array}\right) \end{array}. \tag{11.54}$$

Fix $\epsilon > 0$. Since J_ν is compact, it follows by (11.53) and (11.54) that

$$\begin{array}{ll} \exists & \delta_\nu, \delta_\epsilon > 0 : \\ \forall & y_0 \in J_\nu : \\ \exists & \mathbf{U}_{y_0} \in \mathcal{C}^{m+1}\left(J_\nu \times [t_0 - \delta_\nu, t_0 + \delta_\nu]\right)^K : \\ \forall & |\beta| \leq 1 : \\ \forall & 0 \leq p \leq m+1 : \\ \forall & q \in \mathbb{N}^n,\ 0 \leq p + |q| \leq m+1 : \\ \forall & i, j = 1, ..., K : \\ \forall & \|y - y_0\| < \delta_\epsilon : \\ \forall & |t - t_0| < \delta_\nu : \\ & 1) \quad -\epsilon < D^\beta T_j(y,t,D)\,\mathbf{U}_{y_0}(y,t) < \epsilon \\ & 2) \quad D^\beta T_j(y,t,D)\,\mathbf{U}_{y_0}(y,t) = 0 \\ & 3) \quad \|y_0 - y_1\| < \delta_\epsilon \Rightarrow \left(\begin{array}{ll} 3.1) & |D^{qp}_{yt,i}\mathbf{U}_{y_0}(y,t) - D^{qp}_{yt,i}\mathbf{U}_{y_1}(y,t)| < \epsilon \\ 3.2) & |D^{qp}_{yt,i}\mathbf{U}_{y_1}(y,t) - D^p_t v^q_{y_0,i}(t)| < \epsilon \end{array} \right) \end{array} .$$

Furthermore, (11.52) implies that

$$\begin{array}{ll} \forall & y_0 \in \mathbb{R}^{n-1} : \\ \forall & i = 1, ..., K : \\ \forall & 0 \leq p < m : \\ \forall & q \in \mathbb{N}^n,\ 0 \leq p + |q| \leq m+1 : \\ & D^{qp}_{yt,i}\mathbf{U}_{y_0}(y_0, t_0) = D^q g_{p,i}(y_0) \end{array} .$$

Subdivide J_ν into $n-1$-dimensional, compact intervals $I_{\nu,1}, ..., I_{\nu,\gamma_\nu}$ with nonempty interiors, and diagonal not exceeding δ_ϵ. In particular, the $I_{\nu,k}$ must be locally finite with pairwise disjoint interiors. Let $y_{\nu,k}$ denote the midpoint of $I_{\nu,k}$. Then

$$\begin{array}{ll} \forall & k = 1, ..., \gamma_\nu : \\ \exists & \mathbf{U}_{\nu,k} \in \mathcal{C}^{m+1}\left(I_{\nu,k} \times [t_0 - \delta_\nu, t_0 + \delta_\nu]\right)^K : \\ \forall & (y,t) \in B_{\nu,k} : \\ \forall & 0 \leq p \leq m+1 : \\ \forall & q \in \mathbb{N}^{n-1},\ 0 \leq p + |q| \leq m+1 : \\ \forall & |\beta| \leq 1 : \\ \forall & i, j = 1, ..., K : \\ & 1) \quad -\epsilon < D^\beta\left(T_j(y,t,D)\,\mathbf{U}_{\nu,k}(y,t)\right) < \epsilon \\ & 2) \quad D^\beta\left(T_j(y,t,D)\,\mathbf{U}_{\nu,k}(y_{\nu,k},t)\right) = 0 \\ & 3) \quad D^{qp}_{yt,i}\mathbf{U}_{\nu,k}(y_{\nu,k},t) = D^p_t v^q_i(t),\ |t - t_0| < \delta_\nu \\ & 4) \quad y_0 \in I_{\nu,k} \Rightarrow \left(\begin{array}{ll} 4.1) & |D^{qp}_{yt,i}\mathbf{U}_{y_0}(y,t) - D^{qp}_{yt,i}\mathbf{U}_{\nu,k}(y,t)| < \epsilon \\ 4.2) & |D^{qp}_{yt,i}\mathbf{U}_{y_0}(y,t) - D^p_t v^q_{y_{\nu,k}}(y,t)| < \epsilon \end{array} \right) \end{array} , \quad (11.55)$$

where $B_{\nu,k}$ is the set

$$B_{\nu,k} = I_{\nu,k} \times [t_0 - \delta_\nu, t_0 + \delta_\nu],$$

and

$$\begin{array}{ll} \forall & i=1,...,K: \\ \forall & 0\leq p<m: \\ \forall & q\in\mathbb{N}^n,\ 0\leq p+|q|\leq m+1: \\ & D^{qp}_{yt,i}\mathbf{U}_{\nu,k}\left(y_{\nu,k},t_0\right)=D^q g_{p,i}\left(y_{\nu,k}\right) \end{array} \tag{11.56}$$

Consider the set

$$\Omega_1=\bigcup_{\nu\in\mathbb{N}}\left(J_\nu\times[t_0-\delta_\nu,t_0+\delta_\nu]\right)$$

and the function

$$\mathbf{V}_1=\sum_{\nu\in\mathbb{N}}\left(\sum_{k=1}^{\gamma_\nu}\chi_{\nu,k}\mathbf{U}_{\nu,k}\right)$$

where each $\chi_{\nu,k}$ denotes the characteristic function of the interior of $B_{\nu,k}$. The function $\mathbf{V}_1$ is clearly $\mathcal{C}^{m+1}$-smooth everywhere on Ω_1 except for a closed nowhere dense subset Γ_1 of Ω_1 which satisfies

$$\Gamma_1\cap\mathcal{S}\text{ closed nowhere dense in }\mathcal{S}.$$

It follows from (11.55) that the function $\mathbf{V}_1$ satisfies

$$\begin{array}{ll} \forall & \nu\in\mathbb{N}: \\ \forall & k=1,...,\gamma_\nu: \\ \forall & |\beta|\leq 1: \\ \forall & (y,t)\in\mathrm{int}B_{\nu,k}: \\ \forall & i,j=1,...,K: \\ \forall & 0\leq p\leq m+1: \\ \forall & q\in\mathbb{N}^n,\ 0\leq p+|q|\leq m+1: \\ & \begin{array}{ll} 1) & -\epsilon<D^\beta\left(T_j\left(y,t,D\right)\mathbf{V}_1\left(y,t\right)\right)<\epsilon \\ 2) & D^\beta\left(T_j\left(y,t,D\right)\mathbf{V}_1\left(y_{\nu,k},t\right)\right)=0 \\ 3) & D^{qp}_{yt,i}\mathbf{V}_1\left(y_{\nu,k},t\right)=D^p_t v^q_{y_{\nu,k},i}\left(t\right) \\ 4) & y_0\in\mathrm{Int}I_{\nu,k}\Rightarrow\left(\begin{array}{ll} 4.1) & |D^{qp}_{yt,i}\mathbf{V}_1\left(y,t\right)-D^{qp}_{yt,i}\mathbf{U}_{y_0}\left(y,t\right)|<\epsilon \\ 4.2) & |D^{qp}_{yt,i}\mathbf{V}_1\left(y,t\right)-D^p_t v^q_{y_{\nu,k}}\left(y,t\right)|<\epsilon \end{array}\right) \end{array} \end{array} ,\tag{11.57}$$

and from (11.56) we obtain

$$\begin{array}{ll} \forall & \nu\in\mathbb{N}: \\ \forall & k=1,...,\gamma_\nu: \\ \forall & i=1,...,K: \\ \forall & 0\leq p<m: \\ \forall & q\in\mathbb{N}^n,\ 0\leq p+|q|\leq m+1: \\ & D^{qp}_{yt,i}\mathbf{V}_1\left(y_{\nu,k},t_0\right)=D^q g_{p,i}\left(y_{\nu,k}\right) \end{array} .$$

From (11.57) it follows that the function $\mathbf{V}_1$ satisfies

$$\begin{array}{ll} \forall & \nu \in \mathbb{N} : \\ \forall & k = 1, ..., \gamma_\nu : \\ \forall & i = 1, ..., K : \\ \forall & 0 \leq p \leq m+1 : \\ \forall & q \in \mathbb{N}^{n-1},\, 0 \leq p + |q| \leq m+1 : \\ & (y,t) \in \Omega_1 \setminus \Gamma_1 \Rightarrow \lambda_{1,i}^{p,q}(y,t) \leq \mathcal{D}_{yt,i}^{qp} \mathbf{V}_1 (y,t) \leq \mu_{1,i}^{p,q}(y,t) \end{array} \tag{11.58}$$

where $\lambda_{1,i}^{p,q}, \mu_{1,i}^{p,q} \in \mathcal{C}^0(\Omega_1 \setminus \Gamma_1)$ are the functions

$$\lambda_{1,i}^{p,q}(y,t) = D_t^p v_{y_{\nu,k},i}^q - 2\epsilon,\ (y,t) \in \mathrm{Int} B_{\nu,k} \tag{11.59}$$

and

$$\mu_{1,i}^{p,q}(y,t) = D_t^p v_{y_{\nu,k},i}^q + 2\epsilon,\ (y,t) \in \mathrm{Int} B_{\nu,k}. \tag{11.60}$$

Continuing in this way, we may construct a countable and dense subset $A = \{y_k : k \in \mathbb{N}\}$ of $\mathbb{R}^{n-1}$, a sequence (Γ_n) of closed nowhere dense subsets of Ω_1 that satisfies

$$\Gamma_n \cap \mathcal{S} \text{ closed nowhere dense in } \mathcal{S} \text{ and } (y_k, t_0) \notin \Gamma_n,$$

and functions $\mathbf{V}_n \in \mathcal{C}^{m+1}(\Omega_1 \setminus \Gamma_n)^K$ so that

$$\begin{array}{ll} \forall & |\beta| \leq 1 : \\ \forall & (y,t) \in \Omega_1 \setminus \Gamma_n : \\ \forall & j = 1, ..., K : \\ & -\frac{\epsilon}{n} < D^\beta (T_j(y,t,D) \mathbf{V}_n(y,t)) < \frac{\epsilon}{n} \end{array} . \tag{11.61}$$

Furthermore, the sequence $(\mathbf{V}_n)$ also satisfies

$$\begin{array}{ll} \forall & k \in \mathbb{N} : \\ \exists & N_k \in \mathbb{N} : \\ \forall & i = 1, ..., K : \\ & 1) \quad n \geq N_k \Rightarrow D_{yt,i}^{qp} \mathbf{V}_n(y_k, t_0) = D_y^q g_{p,i}(y_k),\ 0 \leq p < m,\ 0 \leq p + |q| \leq m+1 \\ & 2) \quad n \geq N_k \Rightarrow D_{yt,i}^{qp} \mathbf{V}_n(y_k, t) = D_t^p v_{y_k,i}^q,\ 0 \leq p \leq m+1,\ 0 \leq p + |q| \leq m+1 \end{array} \tag{11.62}$$

and

$$\begin{array}{ll} \forall & \nu \in \mathbb{N} : \\ \forall & k = 1, ..., \gamma_\nu : \\ \forall & i = 1, ..., K : \\ \forall & 0 \leq p \leq m+1 : \\ \forall & q \in \mathbb{N}^{n-1},\, 0 \leq p + |q| \leq m+1 : \\ & (y,t) \in \Omega_1 \setminus \Gamma_1 \Rightarrow \lambda_{n,i}^{p,q}(y,t) \leq D_{yt,i}^{qp} \mathbf{V}_n(y,t) \leq \mu_{n,i}^{p,q}(y,t) \end{array}$$

where $\lambda_{n,i}^{p,q}, \mu_{n,i}^{p,q} \in \mathcal{C}^0(\Omega_1 \setminus \Gamma_n)$ are functions that satisfy

$$0 < \lambda_{n,i}^{p,q}(y,t) - \mu_{n,i}^{p,q}(y,t) < \frac{4\epsilon}{n} \tag{11.63}$$

and

$$\lambda_{n+1,i}^{p,q}(y,t) < \lambda_{n,i}^{p,q}(y,t) < \mu_{n+1,i}^{p,q}(y,t) < \mu_{n+1,i}^{p,q}(y,t) \tag{11.64}$$

for each $n \in \mathbb{N}$.
Let $(\mathbf{U}_n)$ denote the sequence of approximating solutions to the system of PDEs (11.1) constructed Theorem 71. That is, for each $n \in \mathbb{N}$, we have $\mathbf{U}_n \in \mathcal{C}^{m+1}(\Omega \setminus \Gamma_n)^K$, for some closed nowhere dense set $\Gamma_n \subset \Omega$. Consider the functions

$$\mathbf{W}_n = \chi_1 \mathbf{U}_n + \mathbf{V}_n$$

where χ_1 is the characteristic function of $\Omega \setminus Omega_1$. Clearly, for each $n \in \mathbb{N}$, we have $\mathbf{W}_n \in \mathcal{C}^{m+1}(\Omega \setminus \Gamma_n')^K$ for some closed nowhere dense set $\Gamma_n' \subseteq \Omega$. In particular,

$$\Gamma_n' \cap \mathcal{S} \text{ closed nowhere dense in } \mathcal{S} \text{ and } y_k \notin \Gamma_n'.$$

Furthermore, it follows from (11.61) and the corresponding property of the functions $\mathbf{U}_n$, that the sequence $(\mathbf{W}_n)$ satisfies

$$\begin{array}{ll} \forall & |\beta| \leq 1 : \\ \forall & j = 1, ..., K : \\ \forall & (y,t) \in \Omega \setminus \Gamma_n' \\ & -\frac{1}{n} < D^\beta T_j(y,t,D)\mathbf{W}_n(y,t) < \frac{1}{n} \end{array} \tag{11.65}$$

and (11.62) implies

$$\begin{array}{ll} \forall & k \in \mathbb{N} : \\ \exists & N_k \in \mathbb{N} : \\ \forall & i = 1, ..., K : \\ \forall & 0 \leq p < m : \\ \forall & q \in \mathbb{N}^{n-1},\ 0 \leq p + |q| \leq m+1 : \\ & n \geq N_k \Rightarrow D_{yt,i}^{qp}\mathbf{W}_n(y_k, t_0) = D_y^q g_{p,i}(y_k) \end{array} . \tag{11.66}$$

Moreover, for $0 \leq p < m+1$ and $q \in \mathbb{N}^{n-1}$ such that $0 \leq p + |q| \leq m+1$ in (11.66) we have

$$n \geq N_k \Rightarrow \mathcal{D}_{yt,i}^{qp}\mathbf{W}_n(y_k, t) = D^p v_{y_k,i}^q(t),\ (y_k, t) \in \mathcal{S}_1 \setminus \Gamma_n' \tag{11.67}$$

As such, it follows from (11.65), (11.66) and (11.67) that the sequence $(\mathbf{u}_n)$ in $\mathcal{ML}^{m+1}(\Omega)^K$, the components of which are defined as

$$u_{n,i} = (I \circ S)(W_{n,i})$$

satisfies

$$\begin{array}{ll} \forall & |\beta| \leq 1 : \\ & -\frac{1}{n} \leq \mathcal{D}^{\beta} T_j \mathbf{u} < \frac{1}{n} \end{array}, \tag{11.68}$$

$$\begin{array}{ll} \forall & k \in \mathbb{N} : \\ \exists & N_k \in \mathbb{N} : \\ \forall & n \geq N_k : \\ \forall & (y_k, t) \in \mathcal{S}_1 \setminus \Gamma_n' : \\ & 1) \quad \mathcal{D}_t^p \mathcal{D}_y^q u_{n,i}(y_k, t_0) = D_y^q g_{p,i}(y_k),\ 0 \leq p < m,\ 0 \leq p + |q| \leq m+1 \\ & 2) \quad \mathcal{D}_t^p \mathcal{D}_y^q u_{n,i}(y_k, t) = D_t^p v_{y_k,i}^q,\ 0 \leq p \leq m+1,\ 0 \leq p + |q| \leq m+1 \end{array} \tag{11.69}$$

and

$$\begin{array}{ll} \forall & n \in \mathbb{N} : \\ \forall & i = 1, ..., K : \\ \forall & 0 \leq p \leq m+1 : \\ \forall & q \in \mathbb{N}^{n-1},\ 0 \leq p + |q| \leq m+1 : \\ & \overline{\lambda}_{n,i}^{q,p} \leq \mathcal{D}_{yt,i}^{qp} \mathbf{u}_n \leq \overline{\mu}_{n,i}^{q,p} \end{array}. \tag{11.70}$$

where

$$\overline{\lambda}_{n,i}^{q,p} = (I \circ S)\left(\lambda_{n,i}^{q,p}\right)$$

and

$$\overline{\mu}_{n,i}^{q,p} = (I \circ S)\left(\mu_{n,i}^{q,p}\right).$$

In particular, the sequence $(\mathbf{u}_n)$ in $\mathcal{ML}^{m+1}(\Omega)^K$ is a Cauchy sequence, while the sequence $(\mathbf{Tu}_n)$ converges to $\mathbf{0}$ in $\mathcal{ML}^1(\Omega)^K$. It now follows by exactly the same arguments used in the proof of Theorem 87 that there is a sequence $(\mathbf{v}_n)$ in $\mathcal{C}^{m+1}(\Omega)^K$, and a function $\mathbf{u} \in \mathcal{ML}^m(\Omega)^K$ such that $(\mathbf{Tv}_n)$ converges to $\mathbf{0}$ in $\mathcal{ML}^0(\Omega)^K$, and $(\mathbf{v}_n)$ converges to $\mathbf{u}$ in $\mathcal{ML}^m(\Omega)^K$. In particular, there is a closed nowhere dense set $\Gamma \subset \Omega$ such that $\mathbf{u} \in \mathcal{C}^m(\Omega \setminus \Gamma)^K$ and

$$\begin{array}{ll} \forall & A \subset \Omega \setminus \Omega \text{ compact} : \\ \forall & |\alpha| \leq m : \\ \forall & i = 1, ..., K : \\ & \|\mathcal{D}^{\alpha} v_{n,i} - \mathcal{D} u_i\|_A \rightarrow 0 \end{array}. \tag{11.71}$$

It now follows by Theorem 65 that

$$\mathbf{Tu} = \mathbf{0}.$$

We claim

$$\Gamma \cap \mathcal{S} \text{ closed nowhere dense in } \mathcal{S}. \tag{11.72}$$

In this regard, fix $\nu \in \mathbb{N}$ and consider, for each $i =, ..., K$, every $0 \leq p \leq m$ and $q \in \mathbb{N}^{n-1}$ such that $0 \leq p + |q| \leq m$ the function

$$w_{i,\nu}^{pq} : \mathrm{int}J_\nu \times [t_0 - \delta_\nu, t_0 + \delta_\nu] \ni (y,t) \mapsto D_t^p v_{y,i}^q (t)$$

It follows from (11.50) that the function $w_{i,\nu}^{pq}$ is continuous at every point $(y,t) \in J_\nu \times [t_0 - \delta_\nu, t_0 + \delta_\nu]$. Furthermore, in view of (11.69), it is clear that the sequence $(\mathbf{v}_n)$ may be constructed in such a way that

$$\begin{array}{ll} \forall & \nu \in \mathbb{N} \\ \forall & y \in A \cap \mathrm{int}J_\nu : \\ \exists & \delta_\nu > 0 : \\ \exists & N_y \in \mathbb{N} : \\ \forall & n \geq N_y : \\ \forall & 0 \leq p \leq m : \\ \forall & q \in \mathbb{N}^{n-1}, \ 0 \leq p + |q| \leq m : \\ \forall & i = 1, ..., K : \\ & D_{yt,i}^{qp} \mathbf{v}_n (y,t) = D_t^p v_{y,i}^q (t), \ |t - t_0| < \delta_y \end{array} .$$

As such, it follows that the solution $\mathbf{u}$ satisfies

$$\begin{array}{ll} \forall & \nu \in \mathbb{N} \\ \exists & \delta_\nu > 0 : \\ \forall & 0 \leq p \leq m : \\ \forall & y \in A \cap \mathrm{int}J_\nu : \\ \forall & i = 1, ..., K : \\ \forall & q \in \mathbb{N}^{n-1}, \ 0 \leq p + |q| \leq m : \\ & D_{yt,i}^{qp} \mathbf{u} (y,t) = D_t^p v_{y,i}^q (t), \ |t - t_0| < \delta_y \end{array} . \tag{11.73}$$

Since $A \cap \mathrm{int}J_\nu$ is dense in $\mathrm{int}J_\nu$, our claim (11.72) follows from Proposition 46. That $\mathbf{u}$ satisfies the initial condition on $\mathcal{S} \setminus \Gamma$ follows by (11.73) and (11.49) ■

It should also be mentioned that many of the interesting systems of PDEs that arise in applications may be written in the form (11.1). In particular, the equations of fluid mechanics typically take the form

$$D_t u (y,t) + \mathbf{G} \left(y, t, ..., D_y^q u_i (y,t), ... \right) = \mathbf{0}$$

where $\mathbf{G} : \Omega \times \mathbb{R}^M \to \mathbb{R}^K$ is often a $\mathcal{C}^\infty$-smooth mapping. These include, amongst others, the Navier-Stokes equations which have attracted a lot of attention in recent years, see for instance [36], [41] and [98]. Although such equations may not be expressed in exactly the form (11.1), and while the additional conditions, such as boundary and / or initial conditions, may in general not take the form (11.2), the techniques presented here may be applied in these cases as well.

Chapter 12

Concluding Remarks

12.1 Main Results

We have constructed a general and type independent theory for the solutions of a large class of systems of nonlinear PDEs. The spaces of generalized functions upon which the theory is based are constructed as the Wyler completions of suitable uniform convergence spaces. A significant advantage of this method, when compared with the typical spaces of generalized functions used in the customary functional analytic methods is that the generalized functions introduced here may be represented with usual nearly finite normal lower semi-continuous functions. This provides a first basic, and so far unprecedented, *blanket regularity* for the generalized solutions of systems of nonlinear PDEs that are constructed.

It should be noted that, in the basic construction of spaces of generalized functions, and the fundamental existence results for the solutions of systems of nonlinear PDEs, those functional analytic techniques that are typical in the study of PDEs, do not appear. However, this is not to say that functional analysis, or, for that matter, any mathematics, may not, and should not, be used in the study of nonlinear PDEs. Rather, the meaning of this is that such sophisticated mathematical tools should perhaps not form the basis for the study of the *existence* of solutions of nonlinear PDEs, but rather of their additional regularity properties, beyond the mentioned blanket regularity which anyhow results from the theory presented here. Indeed, perhaps the most dramatic results presented in this work, namely, the regularity results obtained for the solutions of a large class of systems of nonlinear PDEs in Chapter 10, and the Cauchy-Kovalevskaia type Theorem proved in Chapter 11, certainly make use of advanced tools from functional analysis. Namely, it is based on sufficient conditions for precompactness of sets in suitable Frechét spaces. However, these results arise as an application of the general existence and regularity theory presented in Chapters 7, 8 and 9, which is based on far simpler techniques.

Let us now summarize the main results of this work. In Chapter 6 we present some auxiliary results on the completion of uniform convergence spaces. These results are used extensively in the text, in particular in regard to the interpretation of generalized functions as nearly finite normal lower semi-continuous functions. Chapter 7 introduces suitable spaces of nearly finite normal lower semi-continuous functions. These are the fundamental spaces upon which the spaces of generalized functions studied here are constructed. It should be noted that the results obtained in Chapter 7, and especially those connected with the construction of the uniform order convergence structure and its completion, are of interest in their own right. Indeed, the uniform convergence structure $\mathcal{J}_o$ on $\mathcal{ML}(X)$ does not depend on the uniform structure on $\mathbb{R}$, or the algebraic structure of $\mathcal{ML}(X)$. This might suggest more general result on constructing the Dedekind order completion of a partially ordered set as the completion of a suitable uniform convergence structure.

Chapter 8 concerns the construction of spaces of generalized functions, and the action of nonlinear partial differential operators on the mentioned spaces of generalized functions. As mentioned, the generalized functions, which are the elements of these spaces, may be represented as usual nearly finite normal lower semi-continuous functions. This may be interpreted as a blanket regularity for these generalized functions. The development of pullback type spaces of generalized functions introduced in Section 8.1 comes down to a reformulation, in terms of uniform convergence spaces, of the construction of spaces of generalized functions in the Order Completion Method [119]. Such a recasting of the Order Completion Method in terms of uniform convergence spaces allows for the application of convergence theoretic techniques to problems related to the structure and regularity of generalized solutions. As is shown in this work, such tools turn out to be highly effective in this regard. The mentioned spaces are associated with a given nonlinear partial differential operator. In particular, one cannot, in general, define generalized derivatives of the elements in these spaces. The Sobelev type spaces of generalized functions are introduced in Section 8.2 in order to address these issues. In particular, the spaces are defined without reference to any particular nonlinear partial differential operator, which, to a certain extent, makes them universal. Furthermore, the generalized functions in these spaces may be uniquely represented through their generalized partial derivatives as nearly finite normal lower semi-continuous functions.

The issue of existence of generalized solutions of systems of nonlinear PDEs in the spaces constructed in Chapter 8 is addressed in Chapter 9. Section 9.1 contains the approximation results upon which the theory is based. These include a multi-dimensional version of (1.110), as well as relevant refinements of this result. In Section 9.2, the basic existence and regularity result obtained in the Order Completion Method [119] is recast in the setting of the so called pullback uniform convergence spaces of generalized functions, while Section 9.3 deals with additional regularity properties of these solutions. In particular, it is shown that such solutions in the pullback spaces of generalized solutions may be assimilated with functions that are $\mathcal{C}^k$-smooth, for $k \in \mathbb{N} \cup \{\infty\}$, everywhere except on a closed nowhere dense set, provided that the nonlinear operator is $\mathcal{C}^k$-smooth. It should be noted that such regularity results have so far not been obtained within the setting of the partially ordered sets within which the Order Completion Method [119] is formulated. Indeed, our result relies on the existence of a compatible complete, Hausdorff uniform convergence structure on a given Hausdorff convergence space [26]. Section 9.4 deals with the question of existence of generalized solutions in the Sobolev type spaces of generalized functions. It is shown that a large class of systems of nonlinear PDEs admit generalized solutions in this sense. This also provides additional insight into the structure of generalized solutions in the pullback type spaces of generalized functions. Indeed, each unique generalized solution in the pullback type spaces may be identified with the set of all solutions in the Sobolev type spaces of generalized functions. We also consider the effect of additional smoothness conditions on the nonlinear partial differential operator and the righthand term $\mathbf{f}$ of the system of equations on the regularity of generalized solutions. In this regard, it is shown that, under suitable conditions on the operator $\mathbf{T}$, an equation of the form (8.1) admits a generalized solution in the Sobolev type space of order $m+k$, provided that the nonlinear operator $\mathbf{T}$, and the righthand term $\mathbf{f}$ are $\mathcal{C}^k$-smooth.

As mentioned, the generalized solutions constructed in the Sobolev type spaces of generalized functions may be uniquely represented through their generalized partial derivatives, which are nearly finite normal lower semi-continuous functions. As such, there is a set $R \subseteq \Omega$ with complement a set of first Baire category such that each generalized partial derivative is continuous and real valued at each $x \in R$. However, even in case the set R has nonempty interior, the generalized derivatives cannot, in general, be interpreted as usual partial derivatives at any point of R. In Chapter 10 it is shown that a large class of systems of nonlinear PDEs admit generalized solutions, in suitable Sobolev type spaces of generalized functions, which are in fact classical solutions everywhere except possibly on a closed nowhere dense set. This result is based on a useful sufficient condition for the precompactness of subsets of a suitable Frechét space of sufficiently smooth functions. In view of the various nonexistence results for certain partial differential equations, see for instance [97], this result is counter intuitive. Indeed, these results show that, contrary to common belief, most systems of nonlinear PDEs admit generalized solutions which are in fact classical solutions everywhere except on a closed nowhere dense subset of the domain of definition of the system. That is, the existence of a classical solution to such a system of nonlinear PDEs is a *strongly generic property* of that system [129].

The solution methods for systems of nonlinear PDEs developed in Chapters 8 to 10 do not take into account any possible additional conditions, such as initial and / or boundary conditions. However, and as is shown in Chapter 11, the theory developed in Chapters 8 to 10 may be applied to problems including such additional conditions with only minimal modifications. This results in the first extension of the Cauchy-Kovalevskaia Theorem 2 to systems of equations that may not be analytic, on its own, general and type independent grounds. In particular, it is shown that any initial value problem of the form (11.1) to (11.2) admits a generalized solution in a suitably constructed Sobolev type space of generalized functions. Furthermore, if the system of equations, and the initial data satisfy suitable smoothness conditions, such a solution can be constructed so that it is a classical solution everywhere except on a closed nowhere dense set. Furthermore, this solution satisfies the initial condition in the classical sense. It should be noted that these methods may be applied to many of the equations that arise in applications. In particular, the equations of fluid mechanics, including the Navier-Stokes equations, may be treated by similar techniques.

12.2 Topics for Further Research

In this work we have initiated a general and type independent theory for the existence and regularity of generalized solutions for a large class of systems of nonlinear PDEs. The results obtained in this regard apply also to many of those equations that that have been proven to be unsolvable in the usual linear topological spaces of generalized functions, and are therefor generally believed to be unsolvable, such as the Lewy equation (1.32), see for instance [88] and [97]. As such, the issues of solvability of such systems of linear and nonlinear PDEs must be carefully reconsidered.

Systems of linear and nonlinear PDEs appear frequently in the applications of mathematics to physics, chemistry, engineering and, recently, even biology. For such applications, knowledge of the qualitative properties of the solutions of such systems, and effective numerical computation of the solutions are required. The development of analytic and numerical tools for this purpose is an important issue.

The spaces of generalized functions that we have constructed are not contained in any of the standard linear functional analytic spaces of generalized functions that are typical in the literature. In fact, even if some generalized function may be represented in, say, one of the Sobolev type spaces of generalized functions, and in one of the standard spaces, such as the $\mathcal{D}'$ distributions, they may exhibit rather different properties. Indeed, the Heaviside function

$$u(x) = \begin{cases} 1 & if \quad x \leq 0 \\ \\ 0 & if \quad x > 0 \end{cases} \tag{12.1}$$

belongs to both $\mathcal{NL}^1(\mathbb{R})$, and to $\mathcal{D}'(\mathbb{R})$. However, in $\mathcal{NL}^1(\mathbb{R})$ its derivative is $u'(x) = 0$, while in $\mathcal{D}'(\mathbb{R})$ its derivative is $u' = \delta$, the Dirac distribution, which is not the 0 function. The exact clarification of the interrelations between the new spaces of generalized functions introduced here, and those of the classical theory of PDEs, is another interesting and important open problem.