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Generalized solutions of systems of nonlinear partial differential equations

by

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Submitted in partial fulfilment of the requirements of the degree

Philosophiae Doctor

in the Department of Mathematics and Applied Mathematics
in the Faculty of Natural and Agricultural Sciences

University of Pretoria
Pretoria

February 2009

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DECLARATION

I, the undersigned, hereby declare that the thesis submitted herewith for the degree Philosophiae Doctor to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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Date: February 2009

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Degree Philosophiae Doctor

Summary

In this thesis, we establish a general and type independent theory for the existence and regularity of generalized solutions of large classes of systems of nonlinear partial differential equations (PDEs). In this regard, our point of departure is the Order Completion Method. The spaces of generalized functions to which the solutions of such systems of PDEs belong are constructed as the completions of suitable uniform convergence spaces of normal lower semi-continuous functions.

It is shown that large classes of systems of nonlinear PDEs admit generalized solutions in the mentioned spaces of generalized functions. Furthermore, the generalized solutions that we construct satisfy a blanket regularity property, in the sense that such solutions may be assimilated with usual normal lower semi-continuous functions. These fundamental existence and regularity results are obtain as applications of basic topological processes, namely, the completion of uniform convergence spaces, and elementary properties of real valued continuous functions. In particular, those techniques from functional analysis which are customary in the study of nonlinear PDEs are not used at all.

The mentioned sophisticated methods of functional analysis are used only to obtain additional regularity properties of the generalized solutions of systems of nonlinear PDEs, and are thus relegated to a secondary role. Over and above the mentioned blanket regularity of the solutions, it is shown that for a large class of equations, the generalized solutions are in fact usual classical solutions of the respective system of equations everywhere except on a closed, nowhere dense subset of the domain of definition of the system of equations. This result is obtained under minimal assumptions on the smoothness of the equations, and is an application of convenient compactness theorems for sets of sufficiently smooth functions with respect to suitable topologies on spaces of such functions. As an application of the existence and regularity results presented here, we obtain for the first time in the literature an extension of the celebrated Cauchy-Kovalevskaia Theorem, on its own general and type independent grounds, to equations that are not analytic.

Preface

For nearly four centuries, ordinary and partial differential equations have been one of the main tools by which scientists sought to describe the laws of nature in exact mathematical terms. At first, most of these equations were of a particular form, namely, linear and of second order. However, with the emergence of increasingly sophisticated scientific theories and state of the art technologies, in particular during the second half of the twentieth century, the interest of mathematicians, and scientists in general, shifted towards nonlinear equations.

It became clear rather early on that the methods developed to deal with linear equations, such as the linear theory of distributions, in particular in the case of partial differential equations, are inappropriate for nonlinear equations. In fact, it is typically believed that a convenient and general theory for the solutions of nonlinear partial differential equations is impossible, or at best highly unlikely. This perception has lead to the development of several ad hoc solution methods for nonlinear partial differential equations, each developed with but a small class of equations, if not one single equation, in mind. While such methods may prove to be highly effective in those cases to which they apply, there is no attempt at a deeper understanding of the underlying nonlinear phenomena involved.

The alternative to the mentioned ad hoc solution methods is to establish a general theory for the existence and regularity of solutions of nonlinear partial differential equations. To date there are three such general theories, namely, the theory of algebras of generalized functions introduced independently by Colombeau and Rosinger, the so called Cental Theory of partial differential equations developed by Neuberger, and the Order Completion Method developed by Oberguggenberger and Rosinger. These three theories, each based on different techniques and perspectives on partial differential equations, apply to large classes of nonlinear partial differential equations, and are not restricted to any particular type of partial differential equation.

In this work, we present a fourth such general and type independent theory for the existence and regularity of solutions of systems of nonlinear partial differential equations. Our point of departure is the mentioned Order Completion Method. As such the theory that we present here may, to a certain extent, be considered also as a regularity theory for the solutions of systems of nonlinear partial differential equations delivered through the Order Completion Method. However, we go far beyond that basic theory by introducing new spaces of generalized functions, the elements of which act as solutions of systems of nonlinear partial differential equations in a suitable extended sense.

The mentioned spaces of generalized functions are constructed as the completion of suitable spaces of usual real valued functions, equipped with appropriate uniform convergence structures. Generalizations of the usual systems of partial differential equations are obtained by extending suitable mappings associated with such a given system of equations to the mentioned spaces of generalized functions. This is done in a consistent and rigorous way by ensuring that these mappings are suitably compatible with the mentioned uniform convergence structures on the spaces of functions. The existence of generalized solutions follows as an application of certain basic approximation results.

The thesis is divided into two parts. Part I contains the introductory chapters, which include a historical overview of the subjects of nonlinear partial differential equations and topology. We also include chapters on real and interval valued functions, and the role of ordered structures in analysis and topology. These chapters contain some results and definitions that are relevant to the work presented in subsequent chapters. Part II contains our original contributions, which we now mention briefly.

- In Chapter 6 we investigate the structure of the completion of a uniform convergence space. In particular, we consider uniform convergence structures that arise as initial structures with respect to families of mappings.

- Nearly finite normal lower semi-continuous functions are introduced in Chapter 7. A uniform convergence structure is defined on a suitable space of such functions, and its completion is characterized. These spaces of normal lower semi-continuous functions are the fundamental spaces upon which the spaces of generalized functions used in this work are constructed.

- The spaces of generalized functions that we introduce here are constructed in Chapter 8. In particular, Section 8.1 concerns the so called pullback type spaces of generalized functions, while Section 8.2 introduces the new Sobolev type spaces of generalized functions. In Section 8.3 we discuss the nonlinear partial differential operators which act on these spaces, as well as the extent to which the different types of spaces are related to one another.

- Chapter 9 deals with the issues of existence of solutions of large classes of systems of nonlinear partial differential equations in the mentioned spaces of generalized functions. In Section 9.1 we give a number of approximation results for the solutions of such systems of equations. These are used in Sections 9.2 through 9.4 to prove the existence of solutions in the various spaces of generalized functions that are constructed in Chapter 8.

- We proceed in Chapter 10 to show that a large class of equations admit solutions in the mentioned Sobolev type spaces of generalized functions, which are in fact classical solutions everywhere except on a closed nowhere dense subset of the domain of definition of system of equations. This regularity result is obtained as an application of certain compactness results for sets of sufficiently smooth functions, with respect to an appropriate topology on suitable spaces of such smooth functions.

- Chapter 11 deals with the issues of boundary and / or initial conditions that may be associated with a given system of nonlinear partial differential equations. In this regard, we consider a general, nonlinear Cauchy problem. It is shown that such an initial value problem admits a solution in a suitable generalized sense. We also show that, under minimal assumptions on the smoothness of the initial data and the nonlinear partial differential operator that defines the system of equations, such an initial value problem admits a solution which is a classical solution everywhere except on a closed and nowhere dense subset of the domain of definition of the equations.

At this point, a remark on the numbering of results is appropriate. This work contains 98 definitions, propositions, lemmas, corollaries, theorems, examples and remarks. These are numbered 1 through 98. All definitions in Part I are taken from the literature, and the relevant citations are indicated. Results form the literature are always marked with an asterisk (*), followed immediately by the relevant citation.

Acknowledgements

The author would like to thank the following people, each of whom, in some way or another, had made an invaluable contribution towards the completion of this thesis. My supervisors, Prof. Roumen Anguelov and Prof. Elemer Rosinger, with whom I had many insightful discussions, and who, after reading preliminary drafts of the thesis, made many useful suggestions. I would like to thank my family, most notably may parents, Peet and Susan van der Walt, my brother Pieter van der Walt and my sister Lydia du Toit for their support and encouragement. A special thanks is due to Lizelda Botha, who's love and support has been a pilar of strength to me during the course of the past two and a half years.

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