An application of an entropy principle to short term interest rate modelling

by

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Abstract

This dissertation is based on the papers written by Platen and Rebolledo (1996), and Platen (1999). The papers focuses on modeling the short term interest rate by optimizing relative entropy of two probability measures Q and P.

The derivation of the model is done by applying the three principles of market clearing, exclusion of arbitrage and minimization of increase of arbitrage information on a simple financial market model. The last principle is equivalent to minimization of the distance between the risk neutral and the real world probability measures. We test the model on historical data from two countries, United States and South Africa from different time frames. The results are then compared to the findings of Platen (1999).

Preface

Please note that in this dissertation too much repetition from Honours level is avoided, we do not introduce concepts such as probability space, Wiener process, Itô formula, etc. However enough background needed in this dissertation is covered.

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Chapter 1

Introduction

There are different types of models that exist that specify the Q-dynamics of the interest rate, Vasiček, Cox-Ingersoll-Ross (CIR) and Dothan, etc. But Platen and Rebolledo [15] states that there seems to be no model in the financial market that clearly explains the relationship between the dynamics of these processes, the pricing of the contingent claim and some of the major economic factors. It is still a challenge to find the model that not only is a good fit to historical data, but also explains the relationship of these processes.

In this work we follow the approach and assumptions of Platen and Rebolledo. In which they model the short term interest rate using the three principles of market clearing, exclusion of instantaneous arbitrage and minimization of increase of arbitrage information. Björk [2] considers this a different approach since the dynamics of the short rate are derived as a consequence of optimizing entropy. The dynamics leads to a diffusion process that has a mean reverting property.

The first principle, *Market clearing condition* is a topic in economics which refers to the principle that, the quantity of assets bought is equal to the quantity of assets sold, and the market clearing price, depends on the actual demand and supply. The second principle, *Exclusion of arbitrage opportunities* which relates to the existence of an equivalent martingale measure, is a topic that is well treated in Finance and being discussed by a few authors like Delbaen and Schachermayer in their paper

[5]. It should be noted that the authors in their paper [15], refers to this principle as the exclusion of instantaneous arbitrage opportunities, however it seems that all the results in this work can be derived with the standard exclusion of arbitrage opportunities which is used to obtain the equivalent martingale measure. Therefore the difference between the principle of instantaneous arbitrage opportunities and arbitrage opportunities will thus not be discussed in this work. The third principle requires the *minimization of increase of arbitrage information*. As stated in their paper [15]: "Arbitrage information also represents the negative relative entropy of the market system, where maximization of negative relative entropy is equivalent to minimization of the sum of the squared market price for risk processes. This principle is thus understood to minimize the market prices of risk to be paid."

We also study the short term interest rate model by Platen [14] and show how it follows from the principle of minimizing entropy, then we look at the differences between this model and the Vasiček model. Finally, we apply this model to both American and South African historical data, for different time frames and compare the fit with results obtained by Platen where he had compared this model with historical data from three countries, United States, Australia and Germany.

Chapter 2

Preliminaries

In this whole dissertation we consider the probability space $(\Omega, \mathcal{F}, P)$,where $\mathcal{F}$ is a $\sigma - algebra$ and P is a probability measure.

2.1 Martingale Measures

In order to give the definition of a martingale measure, we first give the definition of a martingale.

Definition 2.1.1. ([1], page 443) Let $\mathcal{F}_t, t \geq 0$ be a filtration. A stochastic process $X_t, t \geq 0$ is an $\mathcal{F}_t$**-martingale** if

1. X is adapted to $\mathcal{F}_t$, i.e X_t is $\mathcal{F}_t$ measurable for each t.

2. $X_t \in L^1$ for each t

3. For every s and t with $0 \leq s \leq t$ it holds that

$$X_s = E[X_t|\mathcal{F}_s], \qquad P - a.s.$$

Definition 2.1.2. ([1], page 416) Consider a probability space$(\Omega, \mathcal{F})$ on which there are defined two probability measures Q and P

- If, for all $A \in \mathcal{F}$, it holds that

$$P(A) = 0 \to Q(A) = 0 \tag{2.1}$$

then P is said to be absolutely continuous with respect to P on $\mathcal{F}$ and we write this $Q << P$.

- If we have both $Q << P$ and $P << Q$, then Q and P are said to be equivalent and we write this as $Q \sim P$.

Definition 2.1.3. ([1], page 136) A probability measure Q on $\mathcal{F}_T$ is called an equivalent martingale measure for a market model given by X_t on $[0, T$ if it has the following properties:

- $Q \sim P$ on $\mathcal{F}_T$
- All the price processes $X_0, X_1, \ldots\ldots X_N$ are martingales under Q

Theorem 2.1.4. *([1],Theorem A.52) (The Radon-Nikodym Theorem) Consider the measure space $(\Omega, \mathcal{F}, P)$, where we assume that $P(\Omega) < \infty$. Assume that there exists a measure Q on $(\Omega, \mathcal{F})$ such that $Q << P$ on $\mathcal{F}$. Then there exists a nonnegative function $\Phi : \Omega \to \mathcal{R}$ such that Φ is $\mathcal{F}$-measurable,*

$$\int_\Omega \Phi(\omega) dP(\omega) < \infty,$$

$$Q(A) = \int_A \Phi(\omega) dP(\omega),$$

for all $A \in \mathcal{F}$ and $\omega \in \Omega$.

The function Φ is called the Radon-Nikodym derivative of Q w.r.t. P. It is uniquely determined by Q-a.e. and we write

$$\Phi(\omega) = \frac{dQ(\omega)}{dP(\omega)},$$

or alternatively,

$$dQ(\omega) = \Phi(\omega) dP(\omega)$$

2.2 Complete and Arbitrage Free Market

Definition 2.2.1. ([1], page 84)

- A contingent claim is any random variable X, defined on Ω.
- The value process V^h corresponding to the portfolio h is given by

$$V_t^h = \sum_{i=1}^{N} h_i(t) S_i(t),$$

  where $S_i(t)$ is the i-th price process of the stock at time t.

- A portfolio is said to be self-financing, if

$$dV_t^h = h_t dS_t,$$

  where dS_t P-dynamics of S.

- A given contingent claim X is said to be **hedgeable**, if there exists a self-financing portfolio h such that the corresponding value process have the property that

$$V_T^h = X, \qquad P - a.s. \tag{2.2}$$

  In this case we say that h is a hedge against X, where V_T is the value of the portfolio h at time T. If every contingent claim can be hedged we say that the market is **complete**.

2.2.1 Arbitrage Free Market

Definition 2.2.2. ([1], page 16) An **arbitrage** possibility on a financial market is a self-financing portfolio h such that

$$V_0^h = 0$$

$$P(V_T^h \geq 0) = 1$$

$$P(V_T^h > 0) > 0.$$

We say that the market is **arbitrage free** if there are no arbitrage possibilities.

The next theorem is considered to be the First Fundamental theorem the market model consisting of the asset price processes $S_0, S_1, \ldots, S_N$ on the time interval $[0, T]$. S_0 is assumed to be strictly positive.

Theorem 2.2.3. *([1], page 150) The market model is free of arbitrage if and only if there exists an* ***equivalent martingale measure****, i.e a measure $Q \sim P$ such that the processes*

$$\frac{B_t}{B_t}, \frac{S_1(t)}{B_t}, \ldots, \frac{S_N(t)}{B_t} \tag{2.3}$$

are martingales under Q.

From Theorem 2.2.3, we see that $\frac{S_t}{B_t}$ is a martingale under Q. In particular, if the bank price process is given by:

$$B_t = e^{-\int_0^t r(s)ds},$$

for $0 < s < t$ where r is the short rate process, then by the General Pricing Formula ([1] p. 148) we have the following theorem:

Theorem 2.2.4. *([1], page 151) The arbitrage free price of the claim X is given by:*

$$\prod(t; X) = E_Q[e^{-\int_t^T r(s)ds} X | \mathcal{F}_t], \tag{2.4}$$

where $\prod(t; X)$ is the price of the contingent claim X.

2.3 The Market Price of Risk

Assumption 2.3.1. *We assume that the market consist of*

- *Under the objective probability measure P, the S-dynamics are given by*

$$dS_i(t) = S_i(t)\mu_i(t)dt + S_i(t)\sigma_i(t)d\bar{W}(t)$$

for $i = 1, \ldots, n$. Here $\bar{W}_1, \ldots, \bar{W}_n$ are independent P-Wiener processes.

- *The coefficients μ_i and σ_i above are assumed to be known constants.*

- *A risk free asset (money account)with the dynamics*

$$dB_t = rB_t dt,$$

where r is the deterministic short rate of interest.

Consider two fixed T-claims F and G, of the form

$$F = \Phi(S(T)),$$

$$G = \Gamma(S(T)),$$

real where Φ and Γ are given deterministic real valued functions.

A model that is free of arbitrage possibilities, implies the existence of a martingale measure, by Theorem 2.2.3. Such a measure is specified by the market price of risk process given by the following formula:

$$\sigma_F(t)\lambda_t = \bar{\mu}_F(t) - r, \tag{2.5}$$

with r being the risk-free rate, μ_t is the expected rate of return of the stock S_t, σ_t the volatility and ψ_t is the market price of risk.

The following result is typical, and illustrates a property of market price of risk.

Proposition 2.3.2. *([1], page 210) Assume that the market for derivatives is free of arbitrage. Then there exist a process λ_t s.t.*

$$\lambda(t) = \frac{\bar{\mu}_F(t) - r}{\sigma_F(t)} \tag{2.6}$$

with probability 1, for all t regardless of the specific choice of the derivative F.

α_F is the expected return on the claim F.

2.4 Relative Entropy

The concept of entropy will be used in the third principle of this dissertation, which is the minimization of relative entropy. Before we define relative entropy, let us define entropy and the properties thereof.

Entropy is a measure of uncertainty of a random variable. The formal definition for discrete case is as follows:

Definition 2.4.1. ([17]) Let $p = (p_1 \ldots, p_n)$ be a finite discrete probability distribution. Then, the entropy of p is

$$H(p) = -\sum_{i=1}^{n} p_i \ln p_i = \sum_{i=1}^{n} p_i \ln \frac{1}{p_i}, \tag{2.7}$$

where $0 \ln 0 = 0$ and p_i is the probability of the i-th outcome.

Lemma 2.4.2. *Properties of Entropy* *Let $p = (p_1, \ldots, p_n)$*

1. $H(p) \geq 0$
2. $H(p) = 0$ *when all the $p_i = 0$, except for one probability that will be equal to* 1.
3. *Entropy is a maximum if $p_i = \frac{1}{n}$ for each i, $1 \leq i \leq n$*

For fixed n, entropy is zero when there is certainty that one outcome will be true, otherwise it is always positive. Furthermore, when the events are equally likely to occur, the entropy will be maximal. This is the maximum uncertainty of what the outcome would be in a certain state of affairs. As an illustration of entropy for discrete random variables, lets look at the following example of tossing two coins at the same time.

Example 2.4.3. *Let X be number of heads that appear in the toss*

$$X = \begin{cases} 2 & \text{with probability } \frac{1}{4}. \\ 1 & \text{with probability } \frac{1}{2}. \\ 0 & \text{with probability } \frac{1}{4}. \end{cases}$$

Then the entropy of $H(X)$ is

$$H(X) = -[2(\frac{1}{4}\ln\frac{1}{4}) + \frac{1}{2}\ln\frac{1}{2}] = \frac{3}{2}\ln 2 \approx 1.03972 \approx 1$$

The above entropy is calculated for a situation when using fair coins, where the entropy of the outcome is maximal, that is 1, since all events have an equal chance of appearance. If the coins were so unfair that the probability of landing on heads is 1, then the entropy would be zero.

For continuous random variables with density distribution function $f(x)$, the entropy is defined by

$$H(f) = -\int_{-\infty}^{\infty} f(x)\ln f(x)dx. \tag{2.8}$$

Shannon in his paper [17]pp. 38 states that, entropy in the continuous case can be considered as a measure of randomness relative to assumed standard.

For Relative Entropy or Kullback-Leibler divergence, which is a measure of the difference between two probability distributions, we have the following definition:

Definition 2.4.4. [7] Define the Relative Entropy or Kullback-Leibler divergence between two probability measures Q and P to be

$$K(P,Q) = \begin{cases} \mathbf{E}_Q[\frac{dP}{dQ}\ln\frac{dP}{dQ}] & \text{if } P << Q \\ \infty & \text{otherwise} \end{cases}$$

The Kullback-Leibler divergence can also be expressed as

$$K(P,Q) = \int \frac{dP}{dQ}\ln\frac{dP}{dQ}dQ = \int \ln\frac{dP}{dQ}dP, \tag{2.9}$$

which is a consequence of the following theorem in measure theory:

Theorem 2.4.5. *([8], page 134) If λ and μ are totally σ-finite measures such that $\mu \ll \lambda$, and if f is a finite valued measurable function for which $\int f d\mu$ is defined, then $\int f d\mu = \int f \frac{d\mu}{d\lambda} d\lambda$.*

2.5 The Girsanov Theorem

Theorem 2.5.1. *([1], Theorem 11.3) On a probability space $(\Omega, \mathcal{F}, P)$ consider ψ to be any N-dimensional adapted column vector process and consider an N-dimensional P-Wiener process denoted by W^P. Fix $t \in [0, T]$ and define the process Φ on $[0, T]$ by*

$$d\Phi_t = \psi \Phi_t dW_t^P, \tag{2.10}$$

$$\Phi_0 = 1 \tag{2.11}$$

i.e.

$$\Phi_t = \exp\{\int_0^t \psi dW_s^P - \frac{1}{2}\int_0^t |\psi_s|^2 ds\}.$$

Assume that

$$E^P[\Phi_T] = 1, \tag{2.12}$$

and define the new probability measure Q on $\mathcal{F}_T$ by

$$\Phi_T = \frac{dQ}{dP}, \quad on \quad \mathcal{F}_T. \tag{2.13}$$

Then

$$dW_t^P = \psi_t dt + dW_t^Q, \tag{2.14}$$

where W^Q is a Q-Wiener process.

Remark 2.5.2. *Note that the $\Phi = \frac{dP}{dQ}$ given in Theorem 2.1.4 is the same function as in Theorem 2.5.1. Where Φ in both Theorems is called the Radon-Nikodym derivative.*

Chapter 3

Market clearing: The first principle

The market clearing principle ensures that, at any time and for each contingent claim, the cumulative number of shares sold has to be equal to the cumulative number of shares bought, where the traded amount depends on the actual demand and supply. This principle is used to derive the logarithmic contingent claim price dynamics.

In this chapter, a simple market model is chosen to illustrate the three principle used in the dissertation. Demand and supply of shares on a contingent claim is first modeled, which allows us to describe the dynamics of the cumulative amount of shares bought and sold respectively. The models that describe the behavior of investors based on the increase or decrease in the log-price are chosen to be linear. These models describe the reaction of investors towards buying and selling of assets when there is an increase or decrease in the demand and the log-price of an asset. This approach of modeling demand and supply is used by the authors Platen and Rebolledo as they believe it is less complex than having to define utility functions.

To illustrate this principle of market clearing, let us first model supply and demand of the contingent claim.

On a probability space $(\Omega, \mathcal{F}, P)$, consider N risky assets, $S = \{S_1, \ldots, S_N\}$ and $\{W_1, \ldots, W_N\}$ are the N independent Wiener Processes. The market information available at time $t \geq 0$ is expressed by the σ-algebra $\mathcal{F}_t$ generated by the independent Wiener process up to this time. The dynamics of the ith asset price process is described by the following stochastic differential equation:

$$dS_t^i = \mu_t^i S_t^i dt + S_t^i \sum_{j=1}^{N} \sigma_t^{i,j} dW_t^j, \tag{3.1}$$

The number of risky assets and noise sources are chosen to be equal to end up with complete market. Also in the model we have the risk free asset process

$$B = \left\{ \mathrm{B}_t : 0 \leq t \leq T \right\}$$

that satisfies the following differential equation

$$dB_t = r_t B_t dt \tag{3.2}$$

with initial value $B_0 = 1$, where r_t is the short term interest rate of return.

Let X_t^i, $i \in 1, \ldots, N$ denote the price process of the ith contingent claim and $L_t^i, i \in \{1 \ldots N\}$ be the logarithmic price process, given by

$$L_t^i = \ln X_t^i. \tag{3.3}$$

Demand is the measure of the amount of shares that are bought by investors at a specified price for a specific time. Then if we sum the total demand of a contingent claim for a period of time, we call it cumulative demand.

Denote ρ_t^i, $0 \leq t < \infty$ to be the cumulative demand process of investors for buying shares of the ith contingent claim until time t. Assume that the stochastic differential equation for ρ_t^i is given by

$$d\rho_t^i = n(\bar{L}_t^i - L_t^i)dt + \sum_{j=1}^{N} b_t^{i,j} dW_t^j \tag{3.4}$$

$0 \leq t$ and $i \in 1, \ldots, N$, where $\bar{L}_t^i$ is the risk neutral log-price process which is given, and calculated using equation (2.4). Equation (3.4) has the expected drift rate that is proportional to the difference between the risk neutral log-price $\bar{L}_t^i$ and the actual log-price L_t^i. The constant n is positive, this is to reflect the realistic investment strategy of profit making by most investors of wanting to "buy low" and "sell high". The diffusion coefficients $b_t^{i,j}$ represent deterministic function of time.

Let

$$D(t, L_t^i, \rho_t^i) = at - lL_t^i + p\rho_t^i \tag{3.5}$$

denote the cumulative amount of shares of the ith contigent claim bought until time t. The $N \times 1$ dimensional L is defined by

$$L = \left(\begin{array}{c} L_1^1 \ldots L_N^N \end{array} \right),$$

and the $N \times 1$-dimensional ρ by

$$\rho = \left(\begin{array}{c} \rho_1^1 \ldots \rho_N^N \end{array} \right),$$

where equation (3.5) can now be written as

$$D(t, L, \rho) = at - lL + p\rho. \tag{3.6}$$

The constants in equation (3.6) are all chosen to be positive, where a denotes the number of shares bought per unit of time independently of the changes in the log-price and the cumulative demand. The constant l stands for the proportional decrease in the number of shares bought per currency if the price increases, this is so because naturally investors would buy less shares if the price is high. If the price decreases the term lL will be smaller hence, this will indicate more shares being purchased because the term $p\rho$ will be high to indicate high cumulative demand. Lastly, p is the intensity per currency with which the buyers react to an increase in the cumulative demand and hence it is also assumed to be positive since higher cumulative demand will encourage investors to buy larger number of shares.

In a similar way, describe the cumulative amount of shares sold by

$$S(t, L) = at + fL, \tag{3.7}$$

where the constants on this equation are chosen again as in equation (3.6) to be positive. a is the same constant as in equation (3.6), here it represents the number of shares sold per unit of time independently of the log-price and cumulative demand. The constant $f > 0$ denotes the proportional increase per currency of the number of shares sold if the log-price increases. Since it is natural that the final decision on an agreement is made by the buyer, S is chosen in a way that it does not depend on cumulative supply. This is to keep the example simple, hence cumulative supply is not modeled.

Equation (3.6) and (3.7) are chosen to be linear in order to keep the structure of the model simple, and to avoid technical difficulties.

To satisfy the market clearing condition we then have

$$D(t, L_i^t, \rho_t^i) = S(t, L_t^i) \tag{3.8}$$

for all $0 \leq t$ and $i \in \{1, \ldots, N\}$. Substituting equation (3.5) and equation (3.7) into equation (3.8) and applying the Itô formula,

$$dS(t, L_t^i) - dD(t, L_i^t, \rho_t^i) = 0$$

$$adt + fdL_t^i - adt + ldL_t^i - pd\rho_t^i = 0,$$

solving for dL_t^i above, results in

$$dL_t^i = \frac{p}{f+l} d\rho_t^i, \tag{3.9}$$

substituting equation (3.4) into (3.9), then the stochastic differential equation for log-price process is of the form

$$dL_t^i = \frac{pn}{f+l}(\bar{L}_t^i - L_t^i)dt + \frac{p}{f+l}\sum_{j=1}^{d} b_t^{i,j} dW_t^j \tag{3.10}$$

for all $0 \leq t$ and $i \in \{1, \ldots, N\}$.

We observe from equation (3.9), that the relationship between the log-price and the cumulative demand is directly proportional. The log-price will increase with increasing cumulative demand. In equation (3.10), based on our choice of the cumulative demand dynamics, we end up with the stochastic differential equation being the Ornstein-Uhlenbeck type process. The drift term is described by

$$\mu_t^i = c(\bar{L}_t^i - L_t^i), \tag{3.11}$$

and

$$c = \frac{pn}{f + l}$$

is the mean reverting rate. The mean, that is the risk neutral log-price $\bar{L}_t^i$, attracts the actual log-price L_t^i to itself. If $L_t^i < \bar{L}_t^i$ we have $\mu > 0$, but if $L_t^i > \bar{L}_t^i$ then $\mu < 0$. The value L_t^i reverts to the risk neutral log-price $\bar{L}_t^i$ exponentially at a rate c, with the value that is directly proportional to the difference between the risk neutral log-price and the actual log-price. This can be seen if we ignore the dW_t term, and consider the ordinary differential equation

$$dL_t^i = c(\bar{L}_t^i - L_t^i)dt \tag{3.12}$$

with solution

$$L_t^i = \bar{L}_t^i + (L_0^i - \bar{L}_0^i)\exp(-ct). \tag{3.13}$$

The differential equation of the log-price, is an Ornstein-Uhlenbeck process as mentioned before, that is also known as mean reverting process, where the drift term depends on the current value of the process.

Denote by $\sigma_t^{i,j}, t \geq 0 i, j \in 1, \ldots, N$ to be the volatility of the contingent claim as

$$\sigma_t^{i,j} = \frac{p}{f + l} b_t^{i,j} \tag{3.14}$$

for all $0 \leq t$ and $j \in \{1, \ldots, N\}$, then equation (3.10) becomes

$$dL_t^i = \mu_t^i dt + \sum_{j=1}^{N} \sigma_t^{i,j} dW_t^j. \tag{3.15}$$

We observe from equation (3.14) that, the volatility of the log price process is directly proportional to the volatility of the cumulative demand. This means that the fluctuations of the cumulative demand are directly transferred to those of the log-price process.

Chapter 4

Exclusion of Arbitrage: The second principle

The main use of this principle in this chapter is to introduce the market price of risk. Where we also show that the discounted price process is a martingale. The stochastic differential equation for the price process is derived in this chapter. And we show the dependency of ψ_t the market price of risk on the log-price L_t^i.

4.1 Deriving the dynamics of the price process for assets

From equation (3.3) solving for X_t^i, we see that

$$X_t^i = e^{L_t^i}. \tag{4.1}$$

If we let $F(t, L_t^i) = e^{L_t^i}$ then by applying Itô's formula to the function F we obtain

$$dF = F_t dt + F_L dL + \frac{1}{2} F_{LL}(dL)^2 \tag{4.2}$$

$$= e^{L_t^i} dL_t^i + \frac{1}{2} e^{L_t^i} (dL_t^i)^2 \tag{4.3}$$

$$= X_t^i dL_t^i + \frac{1}{2} X_t^i (dL_t^i)^2 \tag{4.4}$$

and substituting equation (3.15), we have the following stochastic differential equation for the price process

$$dX_t^i = \big[\mu_t^i + \frac{1}{2}\sum_{j=1}^{N}(\sigma_t^{i,j})^2\big] X_t^i dt + X_t^i \sum_{j=1}^{N} \sigma_t^{i,j} dW_t^j, \tag{4.5}$$

for all $0 \leq t$ and $i \in \{1, \ldots, N\}$. The integral form of equation (3.15) is given by

$$L_t^i = L_0^i + \int_0^t \mu_s^i ds + \int_0^t \sum_{j=1}^{N} \sigma_s^{i,j} dW_s^j, \tag{4.6}$$

and substituting into equation (4.1) for L_t^i

$$X_t^i = \exp\big[L_0^i + \int_0^t \mu_s^i ds + \sum_{j=1}^{N} \int_0^t \sigma_s^{i,j} dW_s^j\big], \tag{4.7}$$

which becomes

$$X_t^i = X_0^i \cdot \exp\big[\int_0^t \mu_s^i ds + \sum_{j=1}^{N} \int_0^t \sigma_s^{i,j} dW_s^j\big], \tag{4.8}$$

for all $0 \leq t$ and $i \in \{1, \ldots, N\}$. Equation (4.8) is the solution to (4.5). The above derivation is in line with the theory studied in Hull ([9]) or most theoretical work done on stochastic differential equations which shows that when the price of a contingent claim follows a Geometric Brownian Motion of the form equation (4.5), with solution (4.8), then the log-price will have the stochastic differential of the form (3.15).

4.2 Discounted price process

As we assumed in our market model that the market is free of arbitrage opportunities. Then by Theorem (2.2.3) this implies the existence of a martingale measure Q, under which the discounted contingent claim price processes are martingales.

The martingale measure Q will be specified by introducing the market price of risk process of the form $\lambda_t^j, 0 \leq t < \infty, j \in 1, \ldots, N$ that is defined by

$$\sum_{j=1}^{N} \sigma_t^{i,j} \lambda_t^j = \bar{\mu}_t^i - r_t. \tag{4.9}$$

Since

$$\psi_t^j = -\lambda_t^j \tag{4.10}$$

for all $t > 0$, then

$$\sum_{j=1}^{N} \sigma_t^{i,j} \psi_t^j = r_t - \bar{\mu}_t^i, \tag{4.11}$$

with

$$\bar{\mu}_t^i = \mu_t^i + \frac{1}{2} \sum_{j=1}^{N} (\sigma_t^{i,j})^2. \tag{4.12}$$

Under the measure Q the processes $\tilde{W}_t^j = \{\tilde{W}_t^j; 0 \leq t < \infty\}$ with

$$\tilde{W}_t^j = W_t^j - \int_0^t \psi_s^i ds, \tag{4.13}$$

$i \in \{1, \ldots, N\}$, are Wiener processes by the Girsanov Theorem (2.5.1). The differential equation is given by

$$d\tilde{W}_t^j = dW_t^j - \psi_t^i dt. \tag{4.14}$$

Let

$$\tilde{X}_t^i = \frac{X_t^i}{B_t} \tag{4.15}$$

be the discounted contingent claim price, which can also be written as

$$X_t^i = \tilde{X}_t^i B_t, \tag{4.16}$$

where B_t is the price of the risk free asset. Applying Itô on equation (4.16) we get

$$dX_t^i = \tilde{X}_t^i dB_t + B_t d\tilde{X}_t^i + d\tilde{X}_t^i dB_t.$$

The term $d\tilde{X}_t^i dB_t$ will fall off this is because $(dt)^2 = 0$ and also $dt \times dW = 0$ since we know that $dB_t = B_t r_t dt$, and we assume that $d\tilde{X}_t^i$ will contains the drift term and the diffusion term since it will be a stochastic differential equation. Solving for $d\tilde{X}_t^i$ we get

$$d\tilde{X}_t^i = \frac{1}{B_t} \times (dX_t^i - \tilde{X}_t^i dB_t).$$

Substituting equations (4.5), (4.15), (4.12) and (3.2) we get

$$d\tilde{X}_t^i = \frac{1}{B_t} \times (\bar{\mu}_t^i X_t^i dt + X_t^i \sum_{j=1}^{N} \sigma_t^{i,j} dW_t^j - X_t^i r_t dt),$$

and substituting (4.14) for dW and for ψ_t^i from equation (4.11) we obtain

$$d\tilde{X}_t^i = \frac{1}{B_t} \times \{X_t^i(\bar{\mu}_t^i - r_t)dt + X_t^i \sum_{j=1}^{N} \sigma_t^{i,j} d\tilde{W}_t^j - X_t^i(\bar{\mu}_t^i - r_t)dt\},$$

which becomes

$$d\tilde{X}_t^i = \tilde{X}_t^i \sum_{j=1}^{N} \sigma_t^{i,j} d\tilde{W}_t^j \tag{4.17}$$

for $t \geq 0$, $i \in \{1, \ldots, N\}$. The discounted price process is a martingale, that is $\tilde{X}_t^i = \mathbf{E}[\tilde{X}_s^i | \mathcal{F}_t]$, for $0 \leq t \leq s < \infty 0$, $i \in \{1, \ldots, N\}$since (4.17) contains only of the diffusion term.

4.3 Relation between market price of risk and the Log-price

From equation (4.11) substituting equation (4.12) and solving for r_t, we can derive the short-term interest rate process r_t as:

$$r_t = \mu_t^i + \frac{1}{2}\sum_{j=1}^{N}(\sigma_t^{i,j})^2 + \sum_{j=1}^{N}\sigma_t^{i,j}\psi_t^j \tag{4.18}$$

for all $j \in \{1, \ldots, N\}$ and $0 \le t < \infty$.

Writing (4.11) in vector form we let the vector be denoted by ψ_t as

$$\psi_t = \begin{pmatrix} \psi_t^1 \\ \vdots \\ \psi_t^N \end{pmatrix},$$

and the $N \times 1$-dimensional vector matrix of $\bar{\mu}_t$ by

$$\bar{\mu}_t = \begin{pmatrix} \bar{\mu}_t^1 \\ \vdots \\ \bar{\mu}_t^N \end{pmatrix},$$

and we assume that the volatility matrix is invertible.

Assume also that the demands are uncorrelated, i.e. $b_t^{i,j} > 0$ for all $i = j$ and $b_t^{i,j} = 0$ for $j \neq i$ and $0 \le t$. Then the volatility of the log-price hence the price process will also be uncorrelated. Then equation (4.11) can be written as

$$\psi_t = \sigma_t^{-1}(r_t\mathbf{1} - \bar{\mu}_t), \tag{4.19}$$

for $t \in [0, T]$ where $\mathbf{1}$ is the n-dimensional column vector 1_n given by

$$\mathbf{1} = \begin{pmatrix} 1 \ldots 1 \end{pmatrix}^T.$$

From (3.11), (4.12) and (4.19), for our Ornstein-Uhlenbeck-type log-price example we obtain in the special case the market price of risk in the form

$$\psi_t^i(r_t) = \frac{1}{\sigma_t^{i,i}}[r_t - c(\bar{L}_t^i - L_t^i) - \frac{1}{2}(\sigma_t^{i,i})^2]. \tag{4.20}$$

Note that $\psi_t^i(r_t)$ is large for larger values of the log-price L_t^i. If we square $\psi_t^i(r_t)$ we get

$$(\psi_t^i(r_t))^2 = \frac{1}{(\sigma_t^{i,i})^2}[r_t - c(\bar{L}_t^i - L_t^i) - \frac{1}{2}(\sigma_t^{i,i})^2]^2. \tag{4.21}$$

Taking the partial derivative of $(\psi_t^i)^2(r_t)$ with respect to L_t^i we obtain

$$\frac{\partial \psi_s^2}{\partial L_t^i} = \frac{2c}{(\sigma_t^{i,i})^2}[r_t - c(\bar{L}_t^i - L_t^i) - \frac{1}{2}(\sigma_t^{i,i})^2], \tag{4.22}$$

equate to zero

$$0 = \frac{2c}{(\sigma_t^{i,i})^2}[r_t - c(\bar{L}_t^i - L_t^i) - \frac{1}{2}(\sigma_t^{i,i})^2], \tag{4.23}$$

and solve for L_t^i we obtain

$$L_t^i = \bar{L}_t^i - \frac{1}{c}(r_t - \frac{(\sigma_t^{i,i})^2}{2}). \tag{4.24}$$

The above shows that the square of $\psi_t^i(r_t)$ is minimal for L_t^i close to the value given by equation (4.24) which is the optimal value. This observation will be helpful for the interpretation of our result in chapter 5. As we will see that the third principle chapter 5 minimizes the sum of squares of the market price of risk which will turn out to be equivalent to the minimization of increase of arbitrage information.

Chapter 5

Minimization of arbitrage information: The third principle

The main application of the third principle in this work, is to derive the short term interest rate. This will be done by first introducing the Kullback-Leibler information process denoted by h_t, then minimizing the rate of change of the information. Platen and Rebolledo states that this approach of minimizing the increase of the arbitrage information, is a different approach that substitute the principle of maximizing utility functions. The rate of change, is described by the conditional expectation of the sum of squares of the market price of risk. From this we end up with the short term interest rate model which is derived in section (5.1).

In chapter 4 we have formulated ψ for which this specifies a Radon-Nikodym derivative $\Phi = \frac{dQ}{dP}$ of Q w.r.t. P(Φ will be introduced in this chapter) and in turn defines a martingale measure Q. The martingale measure, uniquely defines Φ, by the Radon-Nikodym theorem. In this chapter, by minimizing the increase of the arbitrage information, we are minimizing the market price of risk where we try to find the right martingale measure that is closest to the 'real world' measure P. By so doing we are fixing r the short rate.

5.1 The third principle

Define the Kullback-Leibler information process by $h = \{h_t : 0 \leq t < \infty\}$, with

$$h_t = \Phi_t^{-1} \log \Phi_t^{-1} \tag{5.1}$$

where Φ_t is the Radon-Nikodym derivative. The Kullback-Leibler divergence is obtained by taking the $\mathbf{E}(h_t|\mathcal{F}_0)$.

The dynamics of Φ_t, are given by:

$$d\Phi_t = \psi_t^j \Phi_t dW_t^j \tag{5.2}$$

$$\Phi_0 = 1 \tag{5.3}$$

for all $\{0 \leq t < \infty\}$. Consider $F(\Phi_t, t) = \ln \Phi_t$ applying Itô's lemma on $F(\Phi_t, t)$, then Φ_t can be explicitly described by

$$\Phi_t = exp\big\{ -\frac{1}{2}\sum_{j=1}^{N}\int_0^t |\psi_s^j|^2 ds + \sum_{j=1}^{N}\int_0^t \psi_s^j dW_s^j\big\} \tag{5.4}$$

for $t \in [0, T]$.

Define the arbitrage information for the measure P with respect to $Q_{\psi(r)}$, where $\hat{\mathbf{E}}$ is the expectation with respect to $Q_{\psi(r)}$ given by,

$$I_t(P, Q_{\psi(r)}) = \hat{\mathbf{E}}(h_t|\mathcal{F}_0). \tag{5.5}$$

Substituting h_t we can write equation (5.5) as

$$I_t(P, Q_{\psi(r)}) = \hat{\mathbf{E}}((\Phi_t^{-1} \log \Phi_t^{-1})|\mathcal{F}_0). \tag{5.6}$$

If the conditional expectation in equation (5.6) does not exist, we set $I_t(P, Q) = \infty$. From Theorem 2.4.5 and (2.9) we see that equation (5.6) can now be expressed as

$$I_t(P, Q_{\psi(r)}) = \mathbf{E}((-\log \Phi_t)|\mathcal{F}_0), \tag{5.7}$$

for **E** being the expectation with respect to P. Equation (5.7) is the arbitrage information up to t at time $t = 0$. Since $\mathcal{F}_0$ is a trivial σ-field, we can rewrite equation (5.7) as

$$I_t(P, Q_{\psi(r)}) = \mathbf{E}(-\log \Phi_t), \tag{5.8}$$

Note that the arbitrage information is equivalent to the negative relative entropy, this will be shown below, where we will start with the arbitrage information and end up with the negative relative entropy:

Let $Q \sim P$ and the Radon-Nikodym derivative is given by $\Phi = \frac{dP}{dQ}$ and $\Phi^{-1} = \frac{dQ}{dP}$

.

$$I_t(P, Q_) = \hat{\mathbf{E}}((\Phi_t^{-1} \log \Phi_t^{-1})|\mathcal{F}_0) \tag{5.9}$$

$$= \int \frac{dQ}{dP} \ln \frac{dQ}{dP} dQ \tag{5.10}$$

$$= \int \ln \frac{dQ}{dP} dP \tag{5.11}$$

$$= \int \ln \Phi^{-1} dP \tag{5.12}$$

$$= -\int \ln \Phi dP \tag{5.13}$$

$$= -\int \ln \frac{dP}{dQ} dP \tag{5.14}$$

$$= -\int \frac{dP}{dQ} \ln \frac{dP}{dQ} dQ \tag{5.15}$$

$$= -K(P, Q) \tag{5.16}$$

Substituting equation (5.4) into (5.7) it follows that

$$
\begin{aligned}
I_t(P, Q_{\psi(r)}) =& \mathbf{E}((-\log(exp\{ -\frac{1}{2}\sum_{j=1}^{N}\int_0^t |\psi_s^j|^2 ds + \sum_{j=1}^{N}\int_0^t \psi_s^j dW_s^j\}))|\mathcal{F}_0) && (5.17)\\
=& \mathbf{E}((\frac{1}{2}\sum_{j=1}^{N}\int_0^t |\psi_s^j|^2 ds - \sum_{j=1}^{N}\int_0^t \psi_s^j dW_s^j)|\mathcal{F}_0) && (5.18)\\
=& \frac{1}{2}\mathbf{E}((\sum_{j=1}^{N}\int_0^t |\psi_s^j|^2) ds|\mathcal{F}_0). && (5.19)
\end{aligned}
$$

That is

$$
I_t(P, Q_{\psi(r)}) = \frac{1}{2}\sum_{j=1}^{N}\int_0^t \mathbf{E}((|\psi_s^j|^2)|\mathcal{F}_0) ds \tag{5.20}
$$

since $\mathbf{E}\{\int_0^t \psi_s^j dW_s^j\} = 0$.

Then the rate of change of equation (5.20) is given by

$$
\frac{\partial}{\partial t} I_t(P, Q_{\psi(r)}) = \frac{1}{2}\sum_{j=1}^{N} \mathbf{E}(|\psi_s^j|^2|\mathcal{F}_0). \tag{5.21}
$$

Note that the arbitrage information equation (5.20) is non-negative, where it would be zero if the P and $Q_\psi(r)$ were the same, that is, $I_t(P, Q_\psi(r)) = 0$ if and only if $Q_\psi(r) = P$. Rényi [16] proved this property on page 554 for $I_t(P, Q) = 0$ with the base of 2. The property is still true for base e since $\log_2 b = \frac{\ln b}{\ln 2}$. If $I_t(P, Q_\psi(r)) = 0$ it will affect the market price of risk to be zero $\psi_t(r) = 0$, and this means that $\alpha_F(t) = r(t)$ i.e. the expected rate of return of investing in a risky asset is the same as holding a risk free asset. This is not the case in general, but true in risk neutral pricing theory where the market price of risk is zero.

The author's [15] remark in this case is that: " Arbitrage information equation (5.7) represents negative relative entropy equation (2.4.4) that measures free energy in the system. Free energy gives rise to fluctuations of contingent claims which creates temporary over and underpricing. Investors exploit this phenomenon to generate profit by buying underpriced contingent claims and selling them at times when these are overpriced. It turns out that information about such over or underpricing is indicated by arbitrage information."

5.2 The Derivation of the short rate model: General Case

In this section we derive the short rate model, by minimizing the rate of change of the arbitrage information. We observe that to minimize the rate of increase of the difference between the 'real world' measure P and the martingale measure Q equation (5.21), we need to minimize the quadratic form of the market price of risk given by,

$$\psi_s^2(\omega) = (\sigma_s^{-1} r_s \mathbf{1})^T (\sigma_s^{-1} r_s \mathbf{1}) - 2(\sigma_s^{-1} r_s \mathbf{1})^T (\sigma_s^{-1} \bar{\mu}_s(\omega)) + (\sigma_s^{-1} r_s \bar{\mu}_s(\omega))^T (\sigma_s^{-1} \bar{\mu}_s(\omega)),$$

take the partial derivative with respect to r_t of the quadratic form of the market price of risk. Equate this to zero, then solve for r_t to find the equation of the short rate, where equation (5.21)

$$\frac{\partial}{\partial t} I_t(P, Q) = \frac{1}{2} \sum_{j=1}^{N} \mathbf{E}[|\psi_s^j|^2 | \mathcal{F}_\prime]$$

would be minimum when r_t is given by this value. Then consider the short rate using the specific case, that is, the rate of return being given by

$$\bar{\mu} = \mu_t^i + \frac{1}{2} \sum_{j=1}^{N} (\sigma_t^{i,j})^2.$$

The rate of change of the arbitrage information equation (5.21), becomes minimal if for all $s \geq 0$ and $\omega \in \Omega$ the expression

$$\psi_s^2(\omega) = \sum_{i=1}^{N} |\psi_{s,\omega}^i (r_s(\omega))|^2 \tag{5.22}$$

is minimized.

The quadratic form of equation (4.19) is obtained by multiplying the vector by its transpose, is given by

$$\psi_s^2(\omega) = (\sigma_s^{-1} r_s \mathbf{1})^T(\sigma_s^{-1} r_s \mathbf{1}) - 2(\sigma_s^{-1} r_s \mathbf{1})^T(\sigma_s^{-1}\bar{\mu}_s(\omega)) + (\sigma_s^{-1} r_s \bar{\mu}_s(\omega))^T(\sigma_s^{-1}\bar{\mu}_s(\omega)), \tag{5.23}$$

for all $s \geq 0$ and $\omega \in \Omega$. Taking the partial derivative of (5.23) with respect to r_s we have,

$$\frac{\partial \psi_s^2}{\partial r_s} = (\sigma_s^{-1}\mathbf{1})^T(\sigma_s^{-1} r_s \mathbf{1}) + (\sigma_s^{-1} r_s \mathbf{1})^T(\sigma_s^{-1}\mathbf{1}) - 2(\sigma_s^{-1}\mathbf{1})^T(\sigma_s^{-1}\bar{\mu}_s(\omega)) \tag{5.24}$$

$$= r_s(\sigma_s^{-1}\mathbf{1})^T(\sigma_s^{-1}\mathbf{1}) + r_s(\sigma_s^{-1}\mathbf{1})^T(\sigma_s^{-1}\mathbf{1}) - 2(\sigma_s^{-1}\mathbf{1})^T(\sigma_s^{-1}\bar{\mu}_s(\omega)) \tag{5.25}$$

$$= 2r_s(\sigma_s^{-1}\mathbf{1})^T(\sigma_s^{-1}\mathbf{1}) - 2(\sigma_s^{-1}\mathbf{1})^T(\sigma_s^{-1}\bar{\mu}_s(\omega)), \tag{5.26}$$

for $s \geq 0$ and $\omega \in \Omega$, where A^T denotes the transpose of matrix A.

Taking the second partial derivative of the above equation with respect to r_s we get

$$\frac{\partial^2 \psi_s^2}{\partial r_s^2} = 2(\sigma_s^{-1}\mathbf{1})^T(\sigma_s^{-1}\mathbf{1}), \tag{5.27}$$

where we see that it is positive. This means that the first derivative $\frac{\partial \psi_s^2}{\partial r_s}$ is increasing, hence it implies that $\frac{\partial \psi_s^2}{\partial r_s}$ has a minimum. Equate equation (5.26) to zero and solve to find the minimal r_s, equation (5.24) becomes

$$r_s(\omega) = \frac{(\sigma_s^{-1}\mathbf{1})^T(\sigma_s^{-1}\bar{\mu}_s(\omega))}{(\sigma_s^{-1}\mathbf{1})^T(\sigma_s^{-1}\mathbf{1})}, \tag{5.28}$$

for $s \geq 0$ and $\omega \in \Omega$, where A^T denotes the transpose of matrix A. So equation (5.21) will be minimum if r_s is defined by equation (5.28). Therefore the short rate is given by

$$r_t = \frac{(\sigma_t^{-1}\mathbf{1})^T(\sigma_t^{-1}\bar{\mu}_t)}{(\sigma_t^{-1}\mathbf{1})^T(\sigma_t^{-1}\mathbf{1})}, \tag{5.29}$$

for $t \geq 0$, which minimizes the increase of arbitrage information. We note that r_t is an average over the components of $\bar{\mu}_t$, the expected rate of return.

Let $R_{0,t} = \mathbf{E}(r_t|\mathcal{F}_0)$ denote the conditional expected short rate. To derive the formula for $R_{0,t}$, we first need to derive the equation for the conditional expected rate of return, denoted by $\hat{\mu} = \mathbf{E}(\bar{\mu}_t^i|\mathcal{F}_0)$. Substitute equation (3.11) into equation (4.12), to obtain

$$\bar{\mu}_t^i = \mu_t^i + \frac{1}{2}\sum_{j=1}^{N}(\sigma_t^{i,j})^2 \tag{5.30}$$

$$= c(\bar{L}_t^i - L_t^i) + \frac{1}{2}\sum_{j=1}^{N}(\sigma_t^{i,j})^2. \tag{5.31}$$

Then

$$\hat{\mu}_{0,t}^i = \mathbf{E}(\bar{\mu}_t^i|\mathcal{F}_0) = c\mathbf{E}((\bar{L}_t^i - L_t^i)|\mathcal{F}_0) + \frac{1}{2}\sum_{j=1}^{N}(\sigma_t^{i,j})^2, \tag{5.32}$$

where $\bar{\mu}_t = (\bar{\mu}_t^1, \ldots, \bar{\mu}_t^N)^T$. Then from equation (5.29), the expected short rate is then given by

$$\mathbf{E}(r_t|\mathcal{F}_0) = \frac{(\sigma_t^{-1}\mathbf{1})^T(\sigma_t^{-1})}{(\sigma_t^{-1}\mathbf{1})^T(\sigma_t^{-1}\mathbf{1})}\mathbf{E}(\bar{\mu}_t)|\mathcal{F}_0). \tag{5.33}$$

For uncorrelated demand processes ρ^i with $b_t^{i,i} = b^{i,i} > 0$ and $b_t^{i,j} = 0$ for $i \neq j; j \in \{1, ..., N\}, t \geq 0$, equation (4.21) becomes

$$r_t = \frac{1}{2}\left[\frac{1}{N}\sum_{i=1}^{N}(\sigma^{i,i})^{-2}\right]^{-1}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{2c}{(\sigma^{i,i})^2}(\bar{L}_t^i - L_t^i) + 1\right], \tag{5.34}$$

that is

$$r_t = \bar{r}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{2c}{(\sigma^{i,i})^2}(\bar{L}_t^i - L_t^i) + 1\right), \tag{5.35}$$

with

$$\bar{r} = \frac{1}{2}\left(\frac{1}{N}\sum_{i=1}^{N}(\sigma^{i,i})^{-2}\right)^{-1}. \tag{5.36}$$

The expected short rate is then given by:

$$\mathbf{E}(r_t|\mathcal{F}_0) = \bar{r}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{2c}{(\sigma^{i,i})^2}\mathbf{E}(\bar{L}_t^i - L_t^i|\mathcal{F}_0) + 1\right). \tag{5.37}$$

If we assume that the expected drift rate is zero, that is $\mathbf{E}[(\bar{L}_t^i - L_t^i)|\mathcal{F}_0] = 0$, then the expected short rate is represented by $\bar{r}$ that is

$$\mathbf{E}(r_t|\mathcal{F}_0) = \bar{r}$$

equation (5.36). If the volatilities are the same, that is $\sigma^{1,1}$, this then results in the expression

$$\bar{r} = \frac{(\sigma^{1,1})^2}{2}. \tag{5.38}$$

Platen and Rebolledo [15] compare equation (5.38) with findings from a paper by Finnerty and Leistikow (1993). They state that an average short rate of 6.12% from the US Treasury Bank Bill Market for the period 1958-1989 was estimated. Finnerty and Leistikow report an average inflation of 4.82% and an average equity volatility of 16.3% for the same period. Platen and Rebolledo [15] compute from equation (5.38) the value to be $\bar{r} = 0.013$. They then state that this value corresponds almost exactly with the value of the observed real short rate (without the inflation rate) of 1.3%. This will be discussed again when we look at model testing chapter 8 later in our dissertation.

Chapter 6

Platen's Model

In this chapter we discuss how Platen's model was derived. We then test the model on historical data from the United States and South Africa. We will first discuss the results obtained by Platen [14].

In deriving the dynamics, Platen first assumes that the stochastic differential equations for the market variance σ^2, the inflation rate I and the market net growth rate α are Itô processes. He also assume that the sum of the inflation rate and the market growth remains constant, this assumption is used only in this section to simplify the derivation of the short rate dynamics. We test the model on historical data from the US and again SA historical data from different time intervals.

6.1 Interest rate dynamics

From the model derived in chapter 5 above, assume that the volatilities are the same for each asset price dynamics, with $\bar{\mu}_t^j = \frac{1}{2}\sigma_t^2 + \mu_t^j$. Then equation (4.19) now becomes

$$\psi_t^j = \frac{r_t - \frac{1}{2}\sigma_t^2 - \mu_t^j}{\sigma_t} \tag{6.1}$$

for all $t \in [0, T]$, $j \in \{1, ..., N\}$. Then the quadratic form is given by

$$\psi_t^2 = \frac{1}{(\sigma_t)^2}[r_t^2 - 2r_t(\frac{1}{2}\sigma_t^2 + \mu_t^j) + (\frac{1}{2}\sigma_t^2 + \mu_t^j)^2] \tag{6.2}$$

If we let μ_t^j represents the trend of the jth asset price , defined by

$$\mu_t^j = \eta_t^j + I_t + \alpha_t. \tag{6.3}$$

Substituting equation (6.3) into (6.2) for μ_t^j and assuming that $\sum_{j=1}^N \eta_t^j = 0$, then equation (6.2) now becomes

$$\psi_t^2 = \frac{1}{\sigma_t^2}[r_t^2 - 2r_t(\frac{1}{2}\sigma_t^2 + I_t + \alpha_t) + (\frac{1}{2}\sigma_t^2 + I_t + \alpha_t)^2] \tag{6.4}$$

for $t \in [0, T]$.

Following the same procedure as in the first section, by taking the first partial derivative of equation (6.4) w.r.t r_t, and $\frac{\partial \psi_t^2}{\partial r} = 0$, then r_t is the given by

$$r_t = \frac{\sigma_t^2}{2} + I_t + \alpha_t. \tag{6.5}$$

Therefore the model equation (6.5) minimizes the rate of increase of the difference between the two probability measures mentioned in this work.

To derive the dynamics of equation (6.5), let us first assume that the stochastic differential equations for the market variance σ^2, the inflation rate I and the market net growth rate α are given. Then the dynamics of the short rate are given by

$$dr_t = \frac{1}{2}d\sigma_t^2 + dI_t + d\alpha_t, \tag{6.6}$$

for $t \in [0, T]$.

To simplify the model let us also assume that the sum of the inflation rate and the market net growth rate remains constant. This leaves us with defining only the dynamics of the market variance (squared volatility) process $\sigma^2 = \{\sigma_t^2 : 0 \leq t \leq T\}$, then (6.6) becomes

$$dr_t = \frac{1}{2} d\sigma_t^2 \tag{6.7}$$

for $t \in [0, T]$.

For the choice of the volatility dynamics, the author followed the same optimality property he used for the short rate model. He then ends up with the stochastic differential equation for the market variance of the form:

$$d\sigma_t^2 = c\sigma_t^2(\nu_t(\tau_t + 1) - \sigma_t^2 p)dt - \varrho\sigma_t^3 d\tilde{W}_t \tag{6.8}$$

for $t \in [0, T]$, where $\tilde{W}$ are the Q-Wiener process. ν_t is the average market variance, with dynamics assumed to be given by

$$d(\ln \nu_t) = c\tau_t dt. \tag{6.9}$$

The parameter τ being the $\mathcal{F}$-adapted volatility trend process, corresponding initial values $\nu_0 > 0$, $\sigma_0^2 > 0$ and τ_0 which are deterministic. The constants c, p and ϱ are all assumed to be greater than zero. By substituting (6.8) into (6.7), that is

$$dr_t = \frac{1}{2}\left[c\sigma_t^2(\nu_t(\tau_t + 1) - \sigma_t^2 p)dt - \varrho\sigma_t^3 d\tilde{W}_t\right], \tag{6.10}$$

and for simplicity by setting $p = 1$ and $\sigma_t^2 = 2(r_t - I_t - \alpha_t)$ we obtain that

$$dr_t = \frac{1}{2}\left[c2(r_t - I_t - \alpha_t)(\nu_t(\tau_t + 1) - 2(r_t - I_t - \alpha_t))dt - \varrho\left(\sqrt{2(r_t - I_t - \alpha_t)}\right)^3 d\tilde{W}_t\right]. \tag{6.11}$$

After some manipulation the stochastic differential equation for the short rate is now given by

$$dr_t = 2c(r_t - I_t - \alpha_t)(\frac{1}{2}\nu_t(\tau_t + 1) + I_t + \alpha_t - r_t))dt - \sqrt{2}\varrho(r_t - I_t - \alpha_t)^{\frac{3}{2}} d\tilde{W}_t \tag{6.12}$$

which is written as

$$dr_t = b_t[\hat{r}_t - r_t]dt - s_t d\tilde{W}_t, \tag{6.13}$$

for $t \in [0, T]$.

Note that $b_t = 2c(r_t - I_t - \alpha_t)$, $s_t = \sqrt{2}\varrho(r_t - I_t - \alpha_t)^{\frac{3}{2}}$ and $\hat{r}_t = \frac{1}{2}\nu_t(\tau_t + 1) + I_t + \alpha_t$ which is called the average interest rate. The model equation (6.12) has a mean reverting property, where the drift term depends on the current value of the process. Again we observe that equation (6.8), with $\tau_t = 0$ and $p = 1$ has a mean reverting property, with the mean variance given by ν_t.

6.2 Model Testing

Some of the most known short rate dynamics like, Vasiček model, Cox-Ingersoll-Ross models, to mention a few, have a mean reverting property. So is the Platen model (1996), given by

$$dr_t = 2c(r_t - I_t - \alpha_t)(\frac{1}{2}\nu_t(\tau_t + 1) + I_t + \alpha_t - r_t))dt - \sqrt{2}\varrho(r_t - I_t - \alpha_t)^{\frac{3}{2}}d\tilde{W}_t. \quad (6.14)$$

Platen [14] mentions that it is important that in the long run the theoretical model correctly reflects major movements of the empirical short rate, which was his goal with this model. He compares his model with historical data from three countries. The countries are United States, Germany and Australia, where the values for the inflation rate, average variance and the net market growth are taken from a study done by Finnerty and Leistikow (1993).

6.2.1 Results by Platen

The author's finding were as follows:

Set $\hat{r}_t$ the average interest rate to be described by the following model

$$\hat{r}_t = \frac{1}{2}\nu_t(1 + \tau_t) + I_t + \alpha_t \quad (6.15)$$

for $t \geq 0$. Historical values are chosen for the inflation rate I_t, a constant average value for $\hat{\alpha}$ for the market net growth rate, a constant average $\hat{\nu}$ for the average variance ν_t and a given cyclical volatility trend τ_t, which will be studied in the average interest rate $\hat{r}_t$, equation (6.15).

The author compares the "theoretical value" $\hat{r}_t$, with the historical data from the United States, Australia and Germany. In his model equation (6.15), he states that the equation (6.15) suggests a low volatility trend, since the if $\tau = 1$ the average variance contributes strongly to the average interest rate and if $\tau = -1$, the average variance does not contribute at all to the average short rate.

For his model he chose for the United States the market net growth $\hat{\alpha} = 0$, which reflects the fact that the US markets is the largest financial market in the world. From the study done by Finnerty and Leistikow [6] the following values were taken, where the study covers a period 1958 to 1989. For $\hat{\nu} = 0.027$ and $\hat{I} = 0.0482$ for the 30 years average US inflation rate and the estimated average short rate was $\tilde{r} = 0.0612$ in the same study. With these same values it was noted that for his model equation (6.15) he gets a value that is close to the one given by [6], which is

$$\hat{r} = \frac{1}{2}\hat{\nu} + \hat{I} = 0.0617 \approx \tilde{r} = 0.0612.$$

From the States again, one month's US treasury bank bill rate is taken as the empirical short rate, where this was compared to the theoretical model which produced a good fit.

For the second market Platen considered the Australian data: Based on an average market variance of $\hat{\nu} = 0.068$ estimated from 25 leading stocks over the period of 1987-95, and a market net growth of $\hat{\alpha} = 0.03$, he compares the average short rate computed by his model equation (6.15) with empirical interest rate, represented by the three months treasury bill bank rate from 1987 until 1995. The author chooses the Australian inflation rate and volatility trend. There he also assume the volatility trend to follow sinusoidal oscillations, again with a period of seven years but delayed by six months from the corresponding US volatility trend.

Finally for Germany, the three months bank bill rate is taken as proxy for the empirical interest rate. The average variance was estimated from 30 leading stocks over the period 1987-1995 with a value of $\hat{\nu} = 0.0254$, the market net growth rate was set to $\hat{\alpha} = 0.03$ and the cycle length in the volatility trend was again seven years but with eighteen months delay against the US market.

With these comparisons, the author indicates that the formula (6.15) represents a reasonable model to explain the average of the empirical forward rate dynamics, for major financial markets as his model resulted to be a good fit with the empirical data from these three countries.

6.3 Historical Data Testing

In this section we test the model against US data for the same time period (1987-1996) as with Platen [14], and also for (1958-1989) to see if we can obtain the same values from the study by Finnerty and Leistikow [6]. We use the 3 months Treasury bill instead of the 1 month Treasury bill and also test the model using historical data from (2001-2009). We also test the model on South African historical data for the times (1991-2009) and (2001-2009). The aim of these tests is to investigate if we can come to the same conclusion as the author.

For both countries, we take the stock price observed at a fixed interval (daily) and we define the following symbols, and derivations below with reference from [9]:

- $n + 1$: the number of observations
- S_i: Stock price at the end of the ith interval ($i = 0, 1, \dots, n$)
- $t_i - t_{i-1}$: length of time interval in years
- u_i: continuously compounded return at t_i for $i = 0, 1, \dots n$
- s: the standard deviation of the u_i
- σ^*: estimated volatility
- ν: variance .

If the stock follows a geometric Brownian motion in discrete time:

$$S_i - S_{i-1} = \mu S_{i-1} \cdot [t_i - t_{i-1}] + \sigma S_{i-1}[W_i - W_{i-1}]$$

then

$$ln[\frac{S_i - S_{i-1}}{S_{i-1}}] = (\mu - \frac{\sigma_2}{2})t_i - t_{i-1} + \sigma[W_i - W_{i-1}].$$

This means that

$$ln[\frac{S_i - S_{i-1}}{S_{i-1}}] \sim N\big[(\mu - \frac{\sigma_2}{2})(t_i - t_{i-1}), \sigma\sqrt{t_i - t_{i-1}}\big] \tag{6.16}$$

Define the continuously compounded return between $t_i - t_{i-1}$ by u_i. Then

$$S_i = S_{i-1}e^{u_i(t_i - t_{i-1})},$$

where

$$u_i = ln(\frac{S_i}{S_{i-1}}) \cdot \frac{1}{t_i - t_{i-1}}.$$

Therefore, it follows from (6.16) that

$$u_i \sim N\big[\mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{t_i - t_{i-1}}}\big]. \tag{6.17}$$

Let the standard deviation of u_i be denoted by s, then from (6.17) we have that

$$s = \frac{\sigma^*}{\sqrt{t_i - t_{i-1}}}$$

which implies that

$$\sigma^* = s \cdot \sqrt{t_i - t_{i-1}},$$

where $t_i - t_{i-1}$ is taken as the length of trading days in a year. And s is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} u_i^2 - \frac{1}{n(n-1)}\big(\sum_{i=1}^{n} u_i\big)^2}.$$

6.3.1 US Empirical and Approximate short term rate

The following results were obtained in our test. For the US, a 3 months Treasury bill was used as the historical short rate between the years (1958-1989),(1987-1996) and (2001-2009). The historical inflation rate during 1987-1996 is shown in figure 6.4 and 2001-2009 in figure 6.5. The average inflation values are calculated to be $\hat{I} = 0.037$, $\hat{I} = 0.025$ for 1987-1996 and 2001-2009 respectively. The average variance $\hat{\nu}$ is calculated using the S & P 500 stock index as a constant, with values 0.019, 0.025 and 0.051 for (1958-1989),(1987-1996) and (2001-2009) respectively. We assume that the net growth rate $\hat{\alpha}$ to be zero. As stated by Platen [14] that, this reflects the fact that the US market is the largest financial market in the world. Figure 6.1, 6.2 and 6.3 shows the average rate $\hat{r}$, as computed using equation (6.15) together with historical US interest rate $\tilde{r}$. The market volatility trend τ, have been represented as sinusoidal oscillation, with the average taken to be zero.

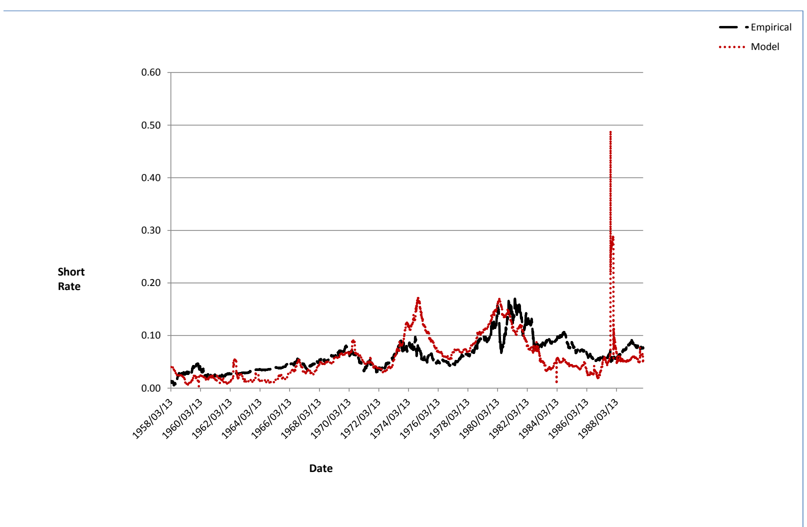

Figure 6.1: US empirical and Platen's model interest rate, $1958 - 1989$.

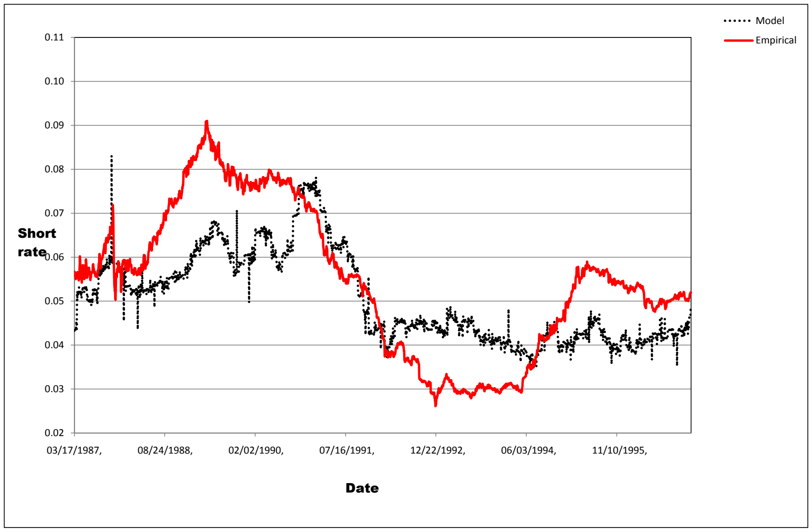

Figure 6.2: US empirical and Platen's model interest rate, 1987 – 1996.

Table 6.1 below shows different estimated and calculated values of the short rate for each time period.

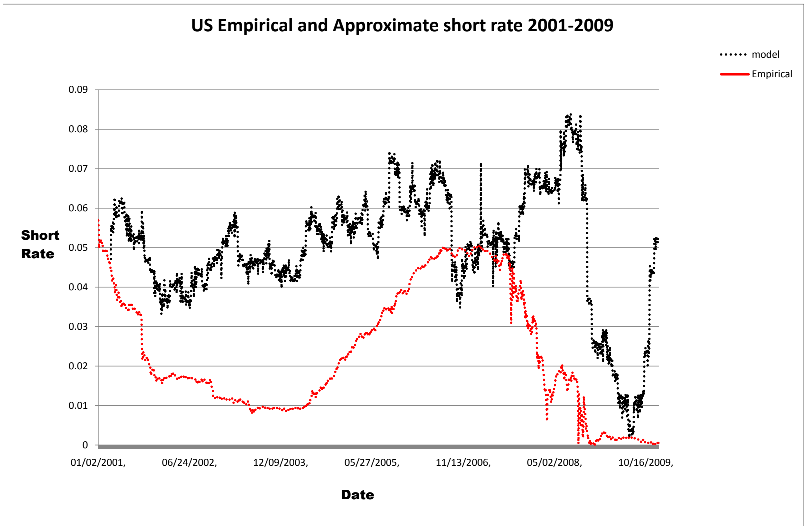

Figure 6.3: US empirical and Platen's model interest rate, 2001 – 2009.

Time Period	$\hat{I}$	$\hat{\nu}$	$\hat{\alpha}$	$\hat{\tau}$	$\hat{r}$	$\tilde{r}$
1958-1989	0.0480	0.0194	0	0	0.0577	0.0613
1987-1996	0.0368	0.0255	0	0	0.0496	0.0547
2001-2009	0.0248	0.0506	0	0	0.0528	0.0235

Table 6.1: Summary of US empirical and Platen's model for the short rate

We compare the values that we obtained with those of the authors. We observe that the values of the average variance from our test and that from the study by Finnerty and Leistikow during 1958-1989 are not the same. We obtained $\hat{\nu} = 0.0194$, while that form [6] is $\hat{\nu} = 0.027$. It should be noted that, for both studies the historical S & P 500 was used for that period. The average inflation values are almost equal, with values of $\hat{I} = 0.0480$ and $\hat{I} = 0.0482$. In our test we use the historical 3 months Treasury Bill as an empirical short rate, and [6] used a historical 1 month Treasury Bill. The empirical average values from both studies, were surprisingly the same. We obtained a $\tilde{r} = 0.0613$, while that [6] obtained $\tilde{r} = 0.0612$. However, because of the different values we obtained of the average variance, when substituting in equation (6.15) to estimate $\hat{r}$, the values from our test and that from [14] are different. Platen

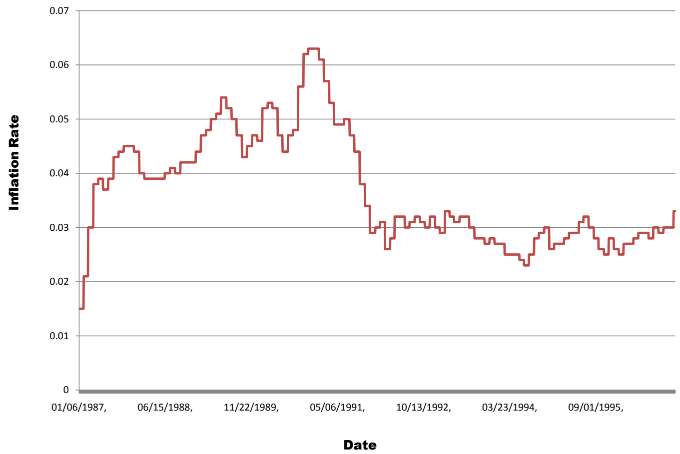

Figure 6.4: US inflation rate, 1987 − 1996.

obtained an estimate $\hat{r} = 0.0617$, and in our study, we obtained $\hat{r} = 0.0577$. Table 6.1 shows our calculated values using equation (6.15). The graphs of figure 6.1 and 6.2, shows that Platen's model is a good fit to the average empirical short rate. While the graph of figure 6.3 shows the model is not a good representation for the average empirical short rate, this is during the 2001-2009.

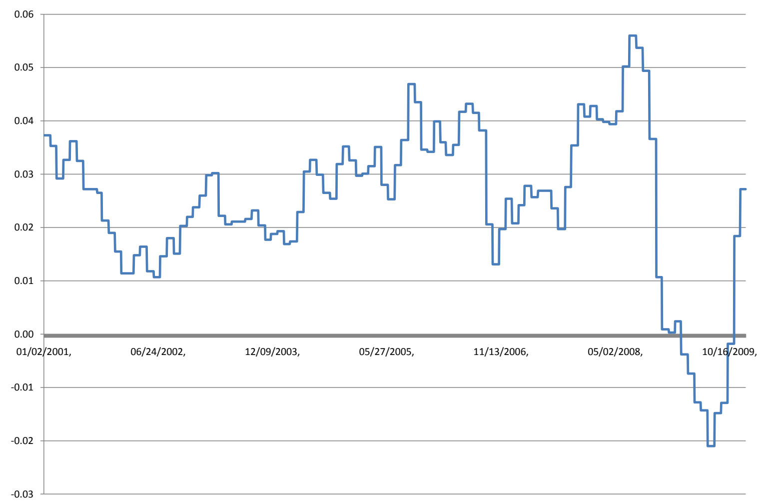

Figure 6.5: US inflation rate, 2001 − 2009.

6.3.2 SA Empirical and Platen's model short term rate

In this section, we test the model equation (6.15) on South African historical data. We test the model on a longer period (1991-2009), then on subperiods (1991-2000) and (2001-2009). A 91 days Treasury bill was used as an empirical short rate. Figure 6.6 and 6.8 shows the South African inflation rate with the average values of $\hat{I} =$ 0.0.0974 and $\hat{I} = 0.0623$. The average variance from the JSE Alshare Index for the period (1991-2000) and (2001-2009) was calculated to be $\hat{\nu} = 0.0274$ and $\hat{\nu} = 0.0469$, respectively. Market net growth with a values of $\hat{\alpha} = 0.03$ and 0.0021 for 1991-2000 and 2001-2009 respectively. Figure 6.7 and 6.9, shows the empirical short rate and the model using the equation (6.15). The market volatility trend was calculated to represent a sinusoidal oscillations, this is to reflect the impact of the business cycle on the market, with average set to be zero. Using these values in the model the following was obtained: $\hat{r} = 0.1411$ for (1991-2000) and $\hat{r} = 0.0879$ for (2001-2009). Comparing these values with the estimated average historical short rate, we obtain $\tilde{r} = 0.01328$ for (1991-2000) and $\tilde{r} = 0.0901$ for (2001-2009). See table 6.2 for reference. The

Time Period	$\hat{I}$	$\hat{\nu}$	$\hat{\alpha}$	$\hat{\tau}$	$\hat{r}$	$\tilde{r}$
1991-2009	0.0730	0.0370	0.01	0	0.1015	0.1114
1991-2000	0.0974	0.0274	0.03	0	0.1411	0.1328
2001-2009	0.0623	0.0496	0.0021	0	0.0879	0.0901

Table 6.2: Summary of SA empirical and Platen's model for the short rate

graphs of the short rate for the empirical and the model, figure 6.7 and 6.9, shows that equation (6.15) seems to be a good fit to the empirical short rate. The test we did for the longer interval period, that is 1991-2009, was to see if there would be any effect in longer time span compared to a shorter time span. With our findings, there seems to be no difference in the values we obtain, see table 6.2.

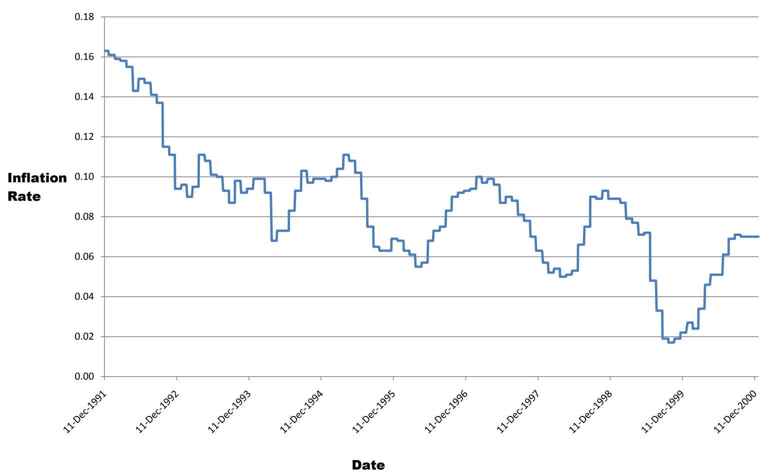

Figure 6.6: SA inflation rate, $1991 - 2000$.

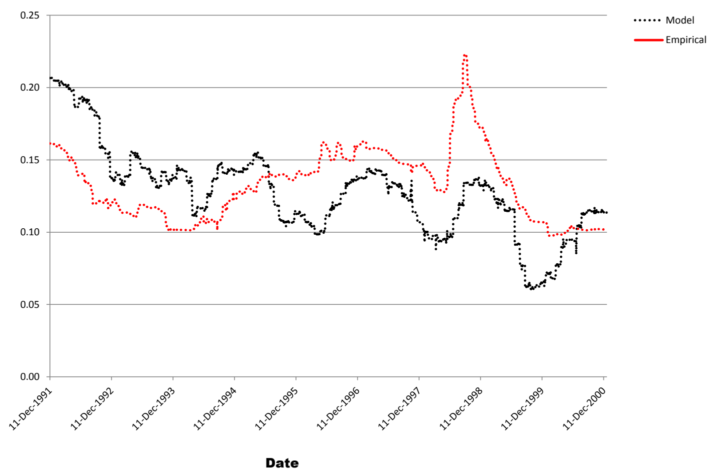

Figure 6.7: SA empirical and Platen's model interest rate, 1991 – 2000.

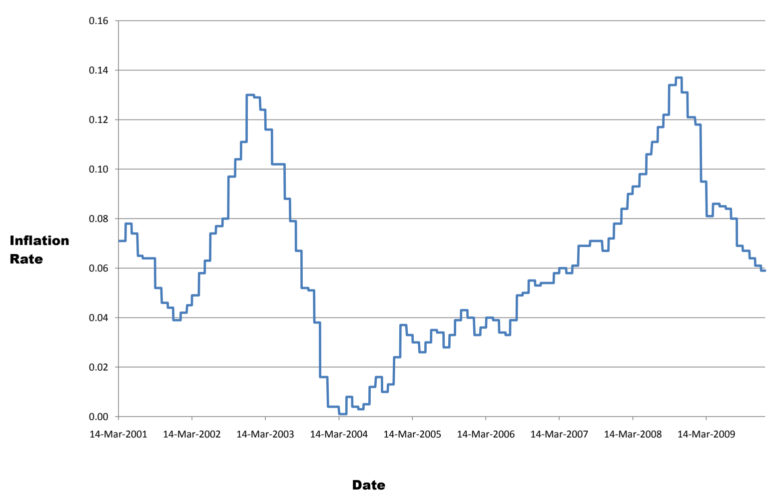

Figure 6.8: SA inflation rate, 2001 – 2009.

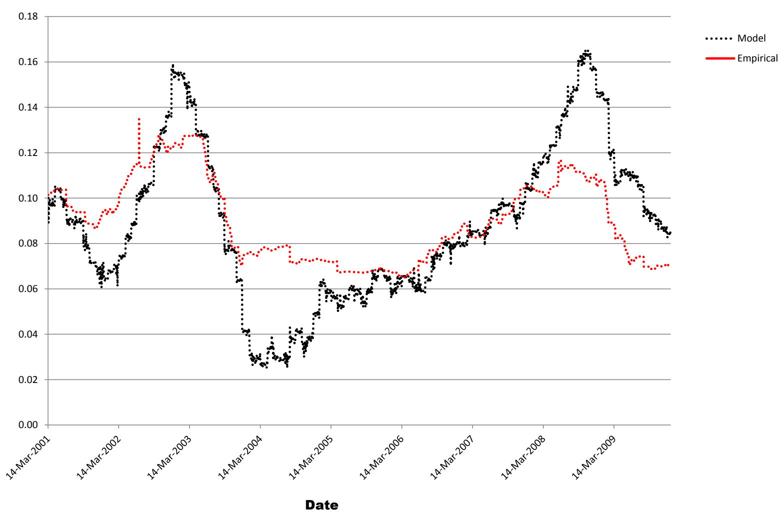

Figure 6.9: SA empirical and Platen's model interest rate, 2001 − 2009.

Chapter 7

Differences between the Vasiček and the Platen Model

Bond prices, under the martingale measure are generated by equation

$$p(t,T) = \mathbf{E}^{Q}\big[\int_t^T r(s)ds|\mathcal{F}_t\big], \tag{7.1}$$

from [3]. Björk [1] states that, it is possible to derive the above equations for models that describe the short rate using a linear SDE, this includes the Vasiček, Ho-Lee and Hull-White models.

The Vasiček model is given by the following dynamics:

$$dr_t = (b - ar_t)dt + \sigma d\tilde{W}_t, \tag{7.2}$$

for $a\&b$ being positive constants. Such equations, are said to be linear SDE's by Björk [1], he continues to state that such r-processes can be shown to be normally distributed, where the normal property is inherited by the integral $\int_t^T r(s)ds$.

In this chapter, we use the Euler approximation with different value parameters, to see if the r in Platen's model is normally distributed or not. We also compare Platen's model to Vasiček, the comparison is not to determine which is the best model, but to see if there are any similarities or how different is the Platen model to the already existing models.

From equation (6.12), the dynamics of the Platen model are given by

$$dr_t = 2c(r_t - I_t - \alpha_t)(\frac{1}{2}\nu_t(\tau_t + 1) + I_t + \alpha_t - r_t))dt - \sqrt{2}\varrho(r_t - I_t - \alpha_t)^{\frac{3}{2}}d\tilde{W}_t. \quad (7.3)$$

Rewriting Platen's model in a simple form, we have:

$$dr_t = b_t[\bar{r}_t - r_t]dt - s_t d\tilde{W}_t. \quad (7.4)$$

The parameters of the model are time dependent whiles those of Vasiček are constant. The similarities are that both models have the mean reverting property, while the difference is that, the Vasiček model was assumed to be Gaussian and that of Platen, was derived by means of an optimality property.

The dynamics of the Vasiček model are Gaussian, where the disadvantage for such models is that r_t can be negative for $t > 0$. Negative interest rate might bring rise to arbitrage opportunities.

Using the Euler approximation and different values for parameters we investigate the distribution of Platen's model.

From equation (7.3), if we let $I_t + \alpha_t = M_t$ and for simplicity, $c = \varrho = 1$ and average of the volatility trend $\tau = 0$. Then the Euler's approximation for equation (6.13) is

$$R_{t+h} = R_t + 2(R_t - M_t)(\frac{1}{2}\nu_t + M_t - R_t)\Delta t - \sqrt{2(R_t - M_t)^3}(W_{t+h} - W_t), \quad (7.5)$$

where $(W_{t+h} - W_t) \sim N(0, 1)$. We test different values of M_t where M_t is set to be constant and ν_t being a sine function. We test the model for the following values: $M = r_0$, $M > r_0$ and also for $M < r_0$. For $M = r_0$ we found the value of the R_{t+h} to be constant with values lying between -0.5 and 0.5. Platen's model is always positive and this can also be observed by looking at the diffusion property in equation (7.3), where it is not possible to take the square root of any negative number in the real number system.

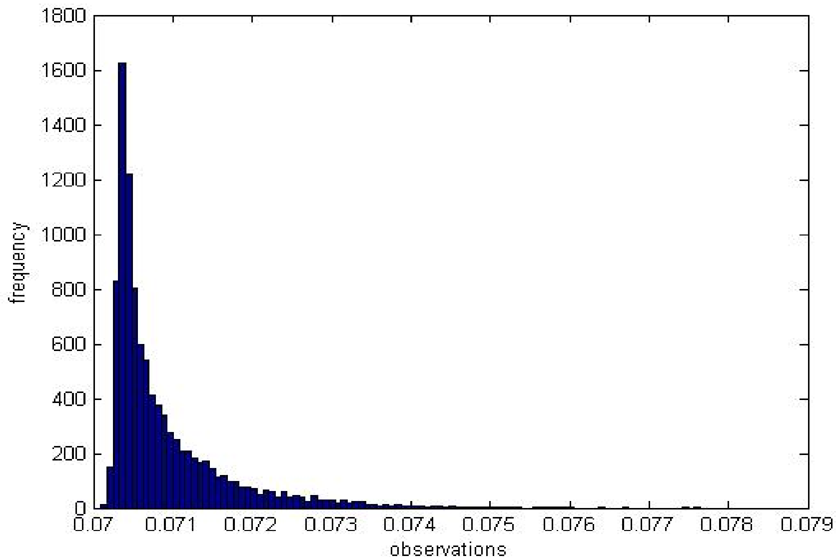

Figure 7.1: Euler's approximation of the short rate for $M > R$.

From figures (7.1) and (7.2) we can conclude that, the Platen's model is not a Gaussian distribution. The figures shows a tail to the right. Hence, it might be a heavy task, if not impossible, do derive the equation of the bond prices for this model.

The following sketches below figure (7.3) and (7.4)shows the distributions of SA and US empirical data respectively, none of which is closer to be a Gaussian distribution.

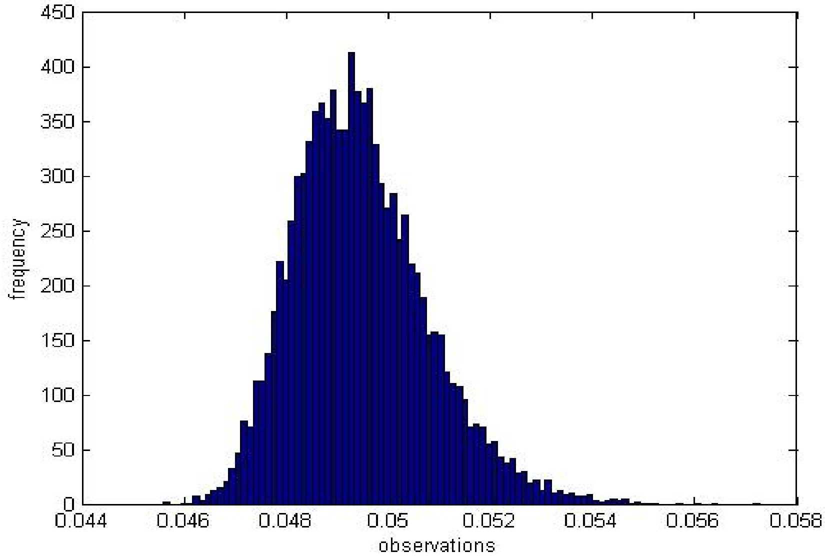

Figure 7.2: Euler's approximation of the short rate for $M < R$.

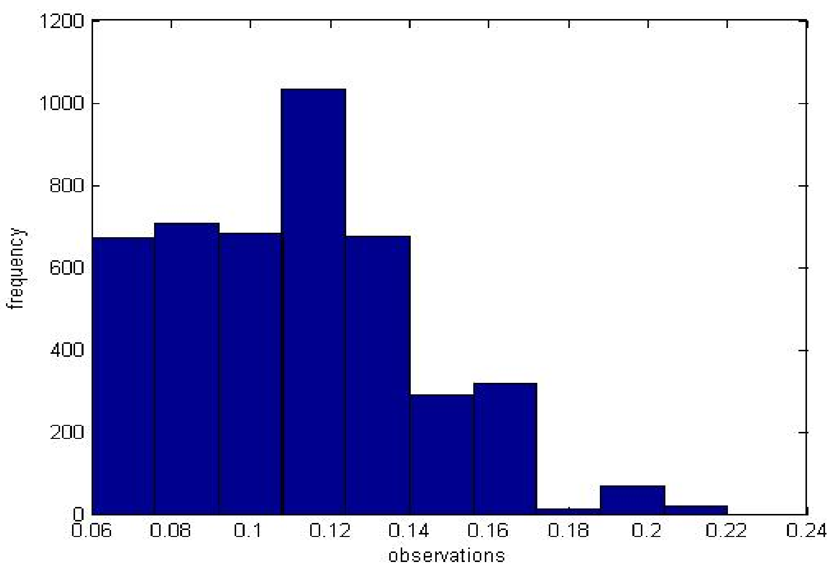

Figure 7.3: SA short rate empirical data.

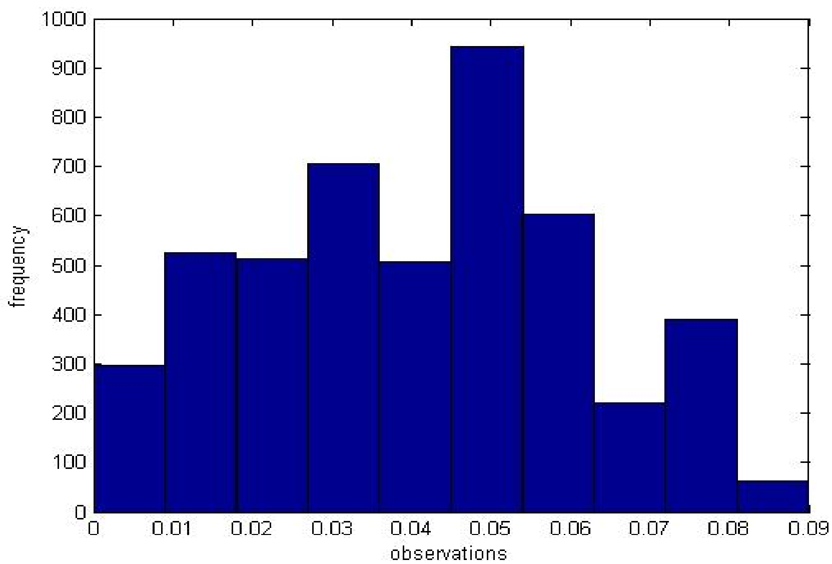

Figure 7.4: US short rate empirical data.

Chapter 8

Conclusion

The aim of this work was to present the completely different approach and possibly more realistic short rate model introduced by Platen and Rebolledo [15], and Platen [14]. This model links the short rate with other major economic factors such as inflation, market variance, market net growth etc. We also investigated the relation that this model might have with already well known models. We studied the tests done by Platen [14]. And we did our own test to investigate if we could obtain the same results as the author.

As already mentioned, Platen and Rebolledo [15] suggested a different approach of deriving the short rate dynamics. Their approach was to use an optimality property involving relative entropy to derive the dynamics of the short rate. This was done by applying three principles to a simple financial market model.

The first principle considered in their work, was that of *market clearing*, where demand and supply was modelled to derive the dynamics of the log-price process of a contingent claim. These dynamics were in turn used to derive the dynamics of the price process of a contingent claim on an underlying asset.

By using the second principle *exclusion of arbitrage opportunities*, we formulated the market price of risk for which it specifies a Radon-Nikodym derivative of the martingale measure w.r.t the 'real world' measure. This ensures the existence of a martingale measure.

Time Period	$\hat{r}$	$\tilde{r}$	difference
1958-1989	0.0577	0.0613	0.0036
1987-1996	0.0496	0.0547	0.0051
2001-2009	0.0528	0.0235	0.0293

Table 8.1: Difference of the US empirical and Platen's model for the short rate

Time Period	$\hat{r}$	$\tilde{r}$	difference
1991-2009	0.1015	0.1114	0.0099
1991-2000	0.1411	0.1328	0.0083
2001-2009	0.0879	0.0901	0.0022

Table 8.2: Difference of the SA empirical and Platen's model for the short rate

We use the third principle, *Minimizing the increase of arbitrage information*, to minimize the market price of risk, where we try to find a martingale measure that is closest to the 'real world' measure. This process then fixes the short rate r_t.

The above procedure resulted in an equation for r_t, which was used to derive the dynamics of the short rate dr_t. The derived SDE for r_t has a mean reverting property. The mean is used in this case to test against historical data, since the value of the short rate will always fluctuate towards the mean, in a mean reverting model. Platen found [14] his model to be a good fit for the US, Germany and Australian historical data. It should be noted that his conclusion was not from a statistical analysis, but from a mere observations of the graphs maintaining the same movements for same time period. It should also be noted that, for the calculation with the average values in equation (6.15), Platen used values from a study over 30 years time interval period by Finnerty and Leistikow. In our own investigations using the US and SA historical data, we observed that the model is a good fit for 1958-1989 and 1987-1996 and not for the years 2001-2009 for the US. The test on the SA data showed a good fit for both interval periods.

Tables 8.1 and 8.2 show the difference obtained between the average empirical short rate and the average short rate of the model. Platen's values had a difference of 0.0005 between his model and the empirical short rate for the US, where he concluded that the model is a good fit. Looking at the values in the above tables 8.1 and 8.2, we can also conclude that these values seem to indicate that the model is a good fit to the historical short rate.

Platen's model has a mean reverting property like the Vasiček model, however unlike the Vasiček model it is not Gaussian. We could not conclude at this stage if it is possible to derive an equation for bond prices using this model. Further research using this approach, might result in obtaining a model that will have a link with other major economic factors, for financial situations which might be relevant to describe the economic down turn like the one in 2008.

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