Generalized statistics and the formation of a quark-gluon plasma

by

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Submitted in partial fulfillment of the requirements for the degree

Magister Scientiae

in the Faculty of Natural and Agricultural Sciences
University of Pretoria
Pretoria
May 2003

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Degree: Magister Scientiae

Abstract

Substantial theoretical research has been carried out to study the phase transition between hadronic matter and a quark-gluon plasma (QGP). When calculating the QGP signatures in relativistic nuclear collisions, the distribution functions of quarks and gluons are traditionally described by Boltzmann-Gibbs (BG) statistics. Here we investigate the effect of both extensive and non-extensive forms of statistical mechanics on the formation of the QGP. We suggest to represent the dominant part of the *long-range* interactions among the constituents in the QGP by a change in the statistics of the system in this phase, and we study the relevance of this statistics for the phase transition. The results show that for small deviations ($\approx 10\%$) from BG statistics in the QGP phase, the critical temperature for the formation of a QGP does not change substantially for a large variation of the chemical potential. This can be interpreted as the formation of a QGP occurs at a critical temperature

which is almost independent of the total number of baryons participating in heavy ion collision. The resulting insensitivity of the critical temperature to the total number of baryons presents a clear experimental signature for the existence of fractal statistics for the constituents of the QGP.

Acknowledgments

My sincere gratitude extends to Professors H.G. Miller and R. Tegen for guiding me through my thesis. I really appreciate for their valuable assistance and constructive remarks.

I would also like to thank R.Q. Odendaal for the help he offered me in fixing computer related problems.

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Chapter 1

Introduction

Quarks are usually bound in hadronic states. However, lattice calculations of Quantum Chromodynamics (QCD) [3, 4, 5, 6, 7] predict that at high temperature and pressure, the hadrons essentially melt and the quarks and gluons become asymptotically free. Such a state is called a quark-gluon plasma (QGP). One of the primary objectives of colliding heavy ions at very high energies is to study this new phase of matter, the QGP. Collisions of nuclei with highly relativistic speeds are expected to produce small volumes of matter in which the quarks and gluons, ordinarily confined to protons and neutrons, interact freely with each other[1].

When the density of quarks and antiquarks in a system is low, the quarks are confined in individual hadrons, surrounded by *normal* vacuum. However, as the density is raised, by increasing temperature or baryon density, the hadrons begin to overlap and matter is expected eventually to undergo a transition to the QGP phase, in which the quarks and gluons are no longer locally confined, but are free to roam over the entire system. If one imagines hadrons as surrounded by little islands of *perturbative* vacuum, as in bag

[1] For a brief introduction of quarks and gluons see Appendix A.

models, then at sufficiently high density, the inbetween regions of *normal* vacuum are squeezed out, and the space becomes filled with *perturbative* vacuum [8].

In the absence of a complete solution of QCD, one can describe confinement and possible deconfinement of quarks to a first approximation in terms of a picture of vacuum having two possible phases. The first, the *normal* vacuum outside hadrons, is that in the absence of physical quarks and color fields. Quarks and gluons modify the vacuum in their neighborhood, transforming it into a second, high energy state, the *perturbative* vacuum, the form of the vacuum inside hadrons. The crucial difference between between these two states is that the *normal* vacuum excludes physical quark and gluon fields, while they can propagate freely throughout the *perturbative* vacuum. In terms of quark masses, one would say that in the *normal* vacuum the mass of an isolated quark is infinite (provided confinement is exact), while in the *perturbative* vacuum the quarks have the current mass values (see Appendix A) [8].

In addition to man-made production of the QGP in heavy ion collisions where heavy ions are accelerated to relativistic energies and made to collide, the QGP can be found in cosmic rays, supernovae, neutron stars and the early universe [8].

Substantial theoretical research has been carried out to study the phase transition between hadronic matter and the QGP [1, 2, 9, 10, 11, 12]. When calculating the QGP signatures in relativistic nuclear collisions, the distribution functions of quarks and gluons are traditionally described by Boltzmann-Gibbs (BG) statistics. Here we investigate the effect of the non-extensive form of statistical mechanics proposed by Tsallis [13] on the formation of a QGP [14]. The crucial difference between the hadronic and QGP phase is the relative importance of *short-range* and *long-range* interactions among the constituents on either side of the anticipated phase transition. The hadronic phase is characterized by a dominant *short-range* interaction among hadrons (which lends itself to BG statistics) while the QGP phase has a greatly reduced *short-range* interaction (due to "asymptotic freedom") and consequently a dominant *long-range* interaction. We suggest to account for the effects of the dominant part of this *long-range* interaction by a change in statistics for the constituents in the QGP phase.

Since hadron-hadron interactions are of *short-range*, the BG statistics is successful in describing particle production ratios seen in relativistic heavy ion collisions below the phase transition [15, 16, 17, 18, 19, 20]. Our motivation for the use of generalized statistics in the QGP phase lies in the necessity to include the *long-range* interactions on the QGP side. Recently Hagedorn's [21] statistical theory of the momentum spectra produced in heavy ion collisions has been generalized using Tsallis statistics to provide a good description of e^+e^- annihilation experiments [22, 23]. Furthermore, Walton and Rafelski [24] studied a Fokker-Planck equation describing charmed quarks in a thermal quark-gluon plasma and showed that Tsallis statistics were relevant. These results suggest that perhaps BG statistics may not be adequate in the quark-gluon phase.

It has been demonstrated [25, 26] that the non-extensive statistics can be considered as the natural generalization of the extensive BG statistics in the presence of *long-range* interactions, *long-range* microscopic memory, or *fractal space-time* constraints. It was suggested in [27, 28] that the extreme conditions of high density and temperature in ultra-relativistic heavy ion collisions can lead to memory effects and *long-range* color interactions. We deem the non-dominant part of this *long-range* interaction negligible for the purpose of the phase diagram which we study here in detail. This latter view is supported by the empirical insensitivity of the phase diagram to *details* of the interaction among the constituents on either side of the phase transition. Therefore, we use the generalized statistics of Tsallis to describe the QGP phase while maintaining the usual BG statistics in the hadron phase (as we shall see this may also be regarded as choosing Tsallis statistics in the hadron phase with the Tsallis parameter $q=1$). In chapter two, we will discuss the generalized statistics proposed by Tsallis and formulate the distribution functions of fermions and bosons from maximum entropy principle. In chapter three, we will apply the distribution functions formulated in chapter two to the constituents of the QGP and study the effect of the generalized statistics on the formation of a QGP.

Chapter 2

Generalized Statistics

New developments in statistical mechanics have shown that in the presence of *long-range* forces and/or in irreversible processes related to microscopic long-time memory effects, the extensive thermodynamics, based on the conventional BG thermostatistics, may not be correct and, consequently, the equilibrium particle distribution functions can show different shapes from the conventional well-known distributions [27, 28].

An interesting generalization of the conventional BG statistics has been proposed by Tsallis [13] and proves to be able to overcome the shortcomings of the conventional statistical mechanics in many physical problems, where the presence of *long-range* interactions, *long-range* microscopic memory, or *fractal space-time* constraints hinders the usual statistical assumptions.

In the past few years the non-extensive form of statistical mechanics proposed by Tsallis has found applications in astrophysical self-gravitating systems [29], solar neutrinos [30, 31], high energy nuclear collisions [27, 28], cosmic microwave back ground radiation [32], high temperature superconductivity [33, 34] and many others. In these cases a small deviation of the Tsallis parameter q ($\approx 10\%$) from one (BG statistics) reduces the discrepancies between experimental data and theoretical models.

The generalized entropy proposed by Tsallis [13] takes the form:

$$S_q = \mathcal{K}\left(\frac{1-\sum_{i=1}^{W} p_i^q}{q-1}\right) \qquad (q \in \Re), \tag{2.1}$$

where $\mathcal{K}$ is a positive constant (from now on set equal to 1), W is the total number of microstates in the system, p_i are the associated probabilities with $\sum_{i=1}^{W} p_i = 1$, and the Tsallis parameter (q) is a real number.

The new entropy has the usual properties of positivity, equiprobability, concavity and irreversibility, preserves the whole mathematical structure of thermodynamics (Legendre transformations) and reduces to the conventional BG logarithmic entropy, $S = -\sum_{i=1}^{W} p_i \ln p_i$, in the limit $q \to 1$. Only in this limit is the ensuing statistical mechanics extensive [13, 26, 35]. For general values of q, the measure S_q is non-extensive. That is, the entropy of a composite system $A \oplus B$ consisting of two subsystems A and B, which are statistically independent in the sense that $p_{i,j}^{(A\oplus B)} = p_i^{(A)} p_j^{(B)}$, is not equal to the sum of the individual entropies associated with each subsystem. Instead, the entropy of the composite system is given by Tsallis' q-additive relation [13],

$$S_q(A \oplus B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)\,. \tag{2.2}$$

The quantity $|1-q|$ can be regarded as a measure of the degree of non-extensivity exhibited by S_q.

Suppose that the set of W microstates is arbitrarily separated into two subsets having W_L and W_M microstates ($W_L + W_M = W$) and define their corresponding probabilities as $p_L \equiv \sum_{i=1}^{W_L} p_i$ and $p_M \equiv \sum_{i=W_L+1}^{W} p_i$ with $p_L + p_M = 1$. It can be shown that [36]

$$S_q(\{p_i\}) = S_q(p_L, p_M) + p_L^q\, S_q(\{p_i/p_L\}) + p_M^q\, S_q(\{p_i/p_M\}), \tag{2.3}$$

where the sets $\{p_i/p_L\}$ and $\{p_i/p_M\}$ are the conditional probabilities. This is a generalization of the famous Shanon's property except for the appearance of p_L^q and p_M^q instead of p_L and p_M in the second and third terms of the right hand side of (2.3). Since the probabilities $\{p_i\}$ are normalized, $p_i^q > p_i$ for $q < 1$ and $p_i^q < p_i$ for $q > 1$. As a consequence the values $q < 1$ $(q > 1)$ will favor rare (frequent) events, respectively [27, 28].

Starting from the one parameter deformation of the exponential function $\exp_{\{\kappa\}}(x) = (\sqrt{1+\kappa^2x^2} + \kappa x)^{\frac{1}{\kappa}}$, a generalized statistical mechanics has been recently constructed by Kaniadakis [37, 38], which reduces to the ordinary BG statistical mechanics as the deformation parameter κ approaches to zero. The difference between Tsallis and Kaniadakis statistics is that: Tsallis statistics is non-extensive and reduces to BG statistics (extensive) as the Tsallis parameter q tends to one. On the other hand, Kaniadakis statistics is extensive and tends to BG statistics as the deformation parameter κ tends to zero. The distribution functions for fermions and bosons can be derived from maximum entropy principle [37, 39]. The κ-entropy is linked to Tsallis entropy $S_q^{(T)}$ through the following relationship [37]:

$$S_\kappa = \frac{1}{2}\frac{\alpha^\kappa}{1+\kappa}S_{1+\kappa}^{(T)} + \frac{1}{2}\frac{\alpha^{-\kappa}}{1-\kappa}S_{1-\kappa}^{(T)} + const. \tag{2.4}$$

where α is a real positive constant. Here we consider the generalized statistics proposed by the Tsallis to represent the dominant part of the *long-range* interactions among the constituents in the QGP[1].

The standard quantum mechanical distributions can be obtained from a maximum entropy principle based on the entropic measure [40, 41],

$$S = -\sum_i [\bar{n}_i \ln \bar{n}_i \mp (1 \pm \bar{n}_i)\ln(1 \pm \bar{n}_i)], \tag{2.5}$$

[1]The phase diagram for Kaniadakis statistics is shown in fig. 3.8.

where the upper and lower signs correspond to bosons and fermions, respectively, and $\bar{n}_i$ denotes the number of particles in the i^{th} energy level with energy ϵ_i. The extremization of the above measure under the constraints imposed by the total number of particles,

$$\sum_i \bar{n}_i = N \tag{2.6}$$

and the total energy of the system,

$$\sum_i \bar{n}_i \epsilon_i = E, \tag{2.7}$$

leads to the standard quantum distributions (see Appendix B),

$$\bar{n}_i = \frac{1}{\exp \beta(\epsilon_i - \mu) \mp 1}, \tag{2.8}$$

where $\beta = \frac{1}{T}$, μ is the chemical potential which is associated with the number of particles and the upper and lower signs correspond to the Bose-Einstein and Fermi-Dirac distributions, respectively.

To deal with non-extensive scenarios (characterized by $q \neq 1$), the extended measure of entropy for fermions proposed in [33, 39] is:

$$S_q^{(F)} = \sum_i \left[\frac{\bar{n}_i - \bar{n}_i^q}{q-1} + \frac{(1-\bar{n}_i) - (1-\bar{n}_i)^q}{q-1} \right], \tag{2.9}$$

which for $q \to 1$ reduces to the entropic functional (2.5) (with lower signs).

The constraints

$$\sum_i \bar{n}_i^q = N \tag{2.10}$$

and

$$\sum_i \bar{n}_i^q \epsilon_i = E \tag{2.11}$$

lead to (see Appendix B)

$$\bar{n}_i = \frac{1}{[1 + (q-1)\beta(\epsilon_i - \mu)]^{\frac{1}{q-1}} + 1}. \tag{2.12}$$

In the limit $q \to 1$ one recovers the usual Fermi-Dirac distribution (2.8) (with lower sign).

Similarly,

$$\bar{n}_i = \frac{1}{[1 + (q-1)\beta(\epsilon_i - \mu)]^{\frac{1}{q-1}} - 1} \tag{2.13}$$

for bosons.

Chapter 3

The Formation of a QGP

3.1 Deconfinement in Heavy Ion Collisions

Under given conditions of temperature and density the hadronic matter undergoes a phase transition towards a plasma of deconfined quarks and gluons. The possibility of obtaining energy densities which are large enough to cause deconfinement in ultra-relativistic heavy ion collisions has acted as one of the main stimuli to interest in such collisions, from both the experimental and theoretical points of view.

Historically, much of the early interest in the deconfinement transition came from the cosmological community, where the emphasis was on hadronization in the early universe. The resulting calculations tended to be performed at zero net baryon number, as befits the early universe scenario. More recently, the high energy physics community, spurred by the development of more powerful particle accelerators, have considered the possibility of obtaining the reverse process, i.e. deconfinement, in the laboratory from ultra-relativistic heavy ion collisions. These collisions result in systems which certainly have non-zero baryon number; as a result, much effort has been devoted to determine full phase diagrams for strongly interacting systems. The major theoretical thrusts in this direction have centered on two approaches, namely lattice QCD and phenomenological models. While lattice QCD [3, 4, 5, 6, 7] is the preferred method of studying the behavior of strongly interacting systems, the enormous computational requirements of such calculations have led to an appreciable amount of work being directed towards phenomenological descriptions of deconfinement [9].

In phenomenological calculations the relevant quantities are the densities of the thermodynamic variables. Anticipating the formation of a QGP during a heavy ion collision, it is at present by no means clear whether there is sufficient time available for the hadron gas and the QGP to equilibrate, and therefore whether traditional equilibrium thermodynamics should be applied to these systems. In what follows, we will assume that the system is in fact in equilibrium. However, the limitations inherent in this assumption should be borne in mind when using the results of our calculations in analysis of heavy ion collisions [9]. We now turn to the description of the system in the QGP phase.

3.2 The QGP Phase

In this approach, the quarks and gluons are treated as forming an ideal gas, apart from the non-perturbative corrections to the pressure and energy density resulting from the bag model [2]. We initially (3.2.1) describe the system by BG statistics in order to define our notations and general numerical procedure. In (3.2.2) we lay out the differences due to the non-extensive statistics. We emphasize that only (3.2.2) can incorporate the anticipated *long-range* forces in the QGP phase.

3.2.1 Boltzmann-Gibbs (BG) Statistics

According to the BG statistics the energy density, pressure, and baryon number density for a QGP consisting of massless up and down quarks and antiquarks, each with degeneracy factor $d_Q = 12$ (2 (spin) x 3 (color) x 2 (quarks and antiquarks)), and gluons, with degeneracy $d_G = 16$ (2 (spin) x 8 (color)), at a temperature T and baryon chemical potential μ are given by (see Appendix C for details)

$$u_{QGP} = \frac{d_Q}{2\,\pi^2}\int_0^\infty dk\, k^3\,(\bar{n}_Q + \bar{n}_{\bar{Q}}) + \frac{d_G}{2\,\pi^2}\int_0^\infty dk\, k^3\,\bar{n}_G + B \tag{3.1}$$

$$P_{QGP} = \frac{d_Q\,T}{2\,\pi^2}\int_0^\infty dk\, k^2\left\{\ln\left[1+\exp\frac{1}{T}(\mu_Q - k)\right] + \ln\left[1+\exp\frac{-1}{T}(\mu_Q + k)\right]\right\}$$

$$-\frac{d_G\,T}{2\,\pi^2}\int_0^\infty dk\, k^2\,\ln\left[1-\exp\left(\frac{-k}{T}\right)\right] - B \tag{3.2}$$

and

$$n_{QGP} = \frac{d_Q}{6\,\pi^2}\int_0^\infty dk\, k^2\,(\bar{n}_Q - \bar{n}_{\bar{Q}})\,, \tag{3.3}$$

where

$$\bar{n}_{Q(\bar{Q})} = \frac{1}{\exp\frac{1}{T}(k \mp \mu_Q) + 1}\,, \tag{3.4}$$

$$\bar{n}_G = \frac{1}{\exp\left(\frac{k}{T}\right) - 1} \tag{3.5}$$

and $\mu_Q = \frac{\mu}{3}$.

Integration by parts of (3.2) yields

$$P_{QGP} = \frac{1}{3}\,(u_{QGP} - 4B)\,. \tag{3.6}$$

Evaluating the integrals in (3.1-3.3) yields (see Appendix C)

$$u_{QGP} = \frac{\pi^2}{30}\left(d_G + \frac{7}{4}d_Q\right)T^4 + \frac{d_Q\,\mu^2\,T^2}{36} + \frac{d_Q\,\mu^4}{648\,\pi^2} + B\,, \tag{3.7}$$

$$P_{QGP} = \frac{\pi^2}{90}\left(d_G + \frac{7}{4}d_Q\right)T^4 + \frac{d_Q\,\mu^2\,T^2}{108} + \frac{d_Q\,\mu^4}{1944\,\pi^2} - B \tag{3.8}$$

and

$$n_{QGP} = d_Q \left(\frac{\mu T^2}{54} + \frac{\mu^3}{486\,\pi^2} \right) , \tag{3.9}$$

where B is the bag constant which is taken here as $(210 \text{ MeV})^4$ [1] with an uncertainty of $\approx 15\%$.

3.2.2 Generalized Statistics

The quantum mechanical distribution function proposed by Buyukklic and Demirhan (BD) [42] is given by

$$\bar{n}_i = \frac{1}{[1 + (q-1)\beta(\epsilon_i - \mu)]^{\frac{1}{q-1}} \mp 1}, \tag{3.10}$$

where the upper and lower signs correspond to bosons and fermions, respectively.

Extremizing the entropic functional,

$$S_q^{(B)} = \sum_i \left[\frac{\bar{n}_i - \bar{n}_i^q}{q-1} - \frac{(1+\bar{n}_i) - (1+\bar{n}_i)^q}{q-1} \right], \tag{3.11}$$

under the constraints imposed by (2.10) and (2.11) yields the q-generalized BD Bose-Einstein distribution, which is given by (3.10) with the minus sign. A "probabilistic" interpretation of the entropic measure (3.11) and the associated variational procedure leading to BD approach distribution for bosons (3.10) is somewhat problematic [39]. If we use the generalized statistics to describe the entropic measure of the whole system, the distribution function can not , in general, be reduced to a finite, closed, analytical expression [39, 43, 44, 45, 46, 47]. For this reason, we use generalized statistics to describe the entropies of the individual particles, rather than of the system as a whole. Even in this case, we are unable to obtain the "probabilistic" interpretation for bosons (see Appendix B).

The single particle distribution functions of quarks, antiquarks and gluons are given by

$$\bar{n}_{Q(\bar{Q})} = \frac{1}{\left[1+\frac{1}{T}(q-1)(k\mp\mu_Q)\right]^{\frac{1}{q-1}}+1} \tag{3.12}$$

and

$$\bar{n}_G = \frac{1}{\left[1+\frac{1}{T}(q-1)k\right]^{\frac{1}{q-1}}-1}, \tag{3.13}$$

respectively. In the limit $q \to 1$, (3.12) and (3.13) reduce to (3.4) and (3.5) respectively.

The expression for the pressure is given by

$$P_{QGP} = \frac{d_Q\, T}{2\,\pi^2}\int_0^\infty dk\, k^2 \left(\frac{f_Q^{q-1}-1}{q-1}+\frac{f_{\bar{Q}}^{q-1}-1}{q-1}\right) - \frac{d_G\, T}{2\,\pi^2}\int_0^\infty dk\, k^2 \left(\frac{f_G^{q-1}-1}{q-1}\right) - B\,, \tag{3.14}$$

where

$$f_Q = 1+\left[1+\frac{1}{T}(q-1)(k-\mu_Q)\right]^{\frac{1}{1-q}}, \tag{3.15}$$

$$f_{\bar{Q}} = 1+\left[1+\frac{1}{T}(q-1)(k+\mu_Q)\right]^{\frac{1}{1-q}} \tag{3.16}$$

and

$$f_G = 1-\left[1+\frac{1}{T}(q-1)k\right]^{\frac{1}{1-q}}, \tag{3.17}$$

which in the limit $q \to 1$ reduces to (3.2).

Since the integrals in (3.1-3.3) are not integrable analytically, one has to calculate these integrals numerically. For $q > 1$, the quantity $[1+\frac{1}{T}(q-1)(k-\mu_Q)]$ becomes negative if $\mu_Q > k$. To avoid this problem we use [48],

$$f_Q = 1+\left[1+\frac{1}{T}(q-1)(k-\mu_Q)\right]^{\frac{1}{1-q}}, \qquad k \geq \mu_Q \tag{3.18}$$

and

$$f_Q = 1+\left[1+\frac{1}{T}(1-q)(k-\mu_Q)\right]^{\frac{1}{q-1}}, \qquad k < \mu_Q\,. \tag{3.19}$$

In the limit $q \rightarrow 1$ one recovers, of course, the appropriate Fermi-Dirac distribution in both cases. We now turn to the hadronic phase of the system which is more readily accessible to experiment.

3.3 The Hadron Phase

The hadron phase is taken to contain only interacting nucleons and antinucleons and an ideal gas of massless pions motivated by the findings in [49, 50]. The interactions between nucleons can be treated either by means of an excluded volume approximation or by a mean field approximation.

3.3.1 The Excluded Volume Approximation

In this approximation, the short range repulsive hadron-hadron interactions are taken into account by a Van der Waals-type method. The assumption is that a hadron is deformable, but has an intrinsic hard-core volume V_o which prevents compression of the hadron gas beyond a close-packing density $\frac{1}{V_o}$. One of the problems of the excluded volume is this close-packing density; the physics in such a limit is by no means clear. The hard-core volumes of hadrons are taken into account in calculations by reducing the total volume available to the system from V to $V - \sum_i N^i V_o^i$, where N^i is the number of hadrons of species i and V_o^i is the hard-core volume of these species [9].

We consider a nucleon which consists of a meson cloud and a hard-core volume where the quarks reside. The relationship between the hard-core volume V_0 and the hard-core radius R_0 is $V_0 = \frac{4}{3}\pi R_0^3$, where the hard-core radius R_0 is in the range between 0.5 and 1 fm [51, 52]. In the simplest approximation, one can assume that all baryons have the same hard-core volume, and use the relevant value of V_o (e.g. the nucleon volume) for all baryons, and ignore the meson hard-cores altogether. Alternatively, one can assume a parameterization of the hard-core volume in terms of the hadron mass. One such parameterization, which draws its inspiration from the MIT bag model [53], is

$$V_o^i = \frac{m^i}{m_N} V_o^N, \tag{3.20}$$

where m_N is the nucleon mass and V_o^N is the nucleon hard-core volume.

Once the form of the hard-core volume has been determined, the particle number density n^i, the energy density u^i and the pressure P^i are given by [54]

$$n^i = \frac{n_{id}^i}{1 + \sum_j V_o^j n_{id}^j}, \tag{3.21}$$

$$u^i = \frac{u_{id}^i}{1 + \sum_j V_o^j n_{id}^j} \tag{3.22}$$

and

$$P^i = \frac{P_{id}^i}{1 + \sum_j V_o^j n_{id}^j}, \tag{3.23}$$

where the quantities with subscript *id* refer to the relevant quantities calculated for an ideal gas of point-like particles.

While the excluded volume approximation (3.21-3.23) with various volume corrections has been widely used as a method of including hadron-hadron interactions in phenomenological hadron gas models, all such formulations suffer from two main and severe deficiencies [54, 55, 56, 57, 58]. Firstly, the equations of state (EOS) in these models are not *thermodynamically consistent* because we do not have a well defined partition function or thermodynamic potential Ω such that the baryon number N, energy E, pressure P and entropy S can be obtained directly from it (i.e. $N \neq -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V}$, $E \neq \left(\frac{\partial}{\partial\beta}\{\beta\Omega\}\right)_{V,\mu} + \mu N$, $P \neq -\left(\frac{\partial\Omega}{\partial V}\right)_{T,\mu}$ and $S \neq -\left(\frac{\partial\Omega}{\partial T}\right)_{V,\mu}$). The second and more crucial deficiency is that these models violate *causality* at high densities, i.e. information travels at a speed larger than the speed of light. Several proposals appeared in the literature which removed the *inconsistency* problem [59, 60, 61, 62, 63, 64] but they suffer from the *causality* problem.

It is possible to construct a *thermodynamically consistent* model using an excluded volume approximation by formulating the problem in the pressure ensemble [59, 60, 61, 62, 63, 64]. For simplicity, let's consider one particle species with eigenvolume v. The pressure P is related to the grand partition function $\mathcal{Z}$ according to

$$P(T,\mu) = T \lim_{V\to\infty} \frac{\ln \mathcal{Z}(T,\mu,V)}{V}. \qquad (3.24)$$

where μ, T and V are the chemical potential, temperature and volume of the system respectively.

The grand partition function is defined as

$$\mathcal{Z}(T,\mu,V) = \sum_{N=0}^{\infty} \exp\left(\frac{\mu N}{T}\right) Z(T,N,V). \qquad (3.25)$$

To introduce the excluded volume, it is necessary to substitute the canonical partition function Z in (3.25) by [63]

$$Z^{excl}(T,N,V) = Z(T,N,V - vN)\,\theta(V - vN). \qquad (3.26)$$

This ansatz is motivated by considering N particles with eigenvolume v in a volume V as N point-like particles in the "available volume", $V - vN$. Substituting (3.26) in (3.25), one obtains

$$\mathcal{Z}^{excl}(T,\mu,V) = \sum_{N=0}^{\infty} \exp\left(\frac{\mu N}{T}\right) Z(T,N,V - vN)\,\theta(V - vN). \qquad (3.27)$$

The main problem in evaluating (3.27) is the dependence of the available volume on the varying number of particles N. To overcome this difficulty one has to perform a Laplace transformation of (3.27). This method of the "isobaric partition function" [65] was successfully used [60, 61, 66] to investigate the excluded volume effect in a gas of quark-gluon bags. In doing so, one obtains [63]

$$\hat{\mathcal{Z}}^{excl}(T,\mu,x) \equiv \int_0^\infty dV\, \exp(-x\,V)\,\mathcal{Z}^{excl}(T,\mu,V)$$

$$= \int_0^\infty d\hat{V}\, exp(-x\,\hat{V})\,\mathcal{Z}(T,\hat{\mu},\hat{V})\,, \tag{3.28}$$

where $\hat{\mu} \equiv \mu - v\,T\,x$ and $\hat{V} \equiv V - v\,N$.

From the definition of the pressure function one concludes that the grand canonical partition function of the system, in the thermodynamical limit, approaches

$$\mathcal{Z}^{excl}(T,\mu,V)|_{V\to\infty} \sim \exp\left[\frac{P^{excl}(T,\mu)\,V}{T}\right].$$

From the first equality in (3.28) one sees that this exponentially increasing part of $\mathcal{Z}^{excl}(T,\mu,V)$ generates an extreme right singularity in the function $\hat{\mathcal{Z}}^{excl}(T,\mu,x)$ at some point x^*. For $x < P^{excl}/T$ the integration over V for $\hat{\mathcal{Z}}^{excl}(T,\mu,x)$ diverges at its upper limit. Therefore, the extreme right singularity of $\hat{\mathcal{Z}}^{excl}(T,\mu,x)$ at $x^*(T,\mu)$ gives a pressure [63],

$$P^{excl}(T,\mu) \equiv T \lim_{V\to\infty} \frac{\ln \mathcal{Z}^{excl}(T,\mu,V)}{V} = T\,x^*(T,\mu). \tag{3.29}$$

The direct connection of the extreme right x-singularity of $\hat{\mathcal{Z}}^{excl}$ to the asymptotic behavior $V \to \infty$ of $\mathcal{Z}^{excl}$ is a general mathematical property of the Laplace transform. Using the above equations one obtains

$$x^*(T,\mu) = \lim_{\hat{V}\to\infty} \frac{\ln \mathcal{Z}(T,\tilde{\mu},\hat{V})}{\hat{V}}\ ;\ \tilde{\mu} = \mu - v\,T\,x^*(T,\mu).$$

Applying (3.24) for $\mathcal{Z}(T,\tilde{\mu},\hat{V})$ and making use of (3.29) to eliminate x^* one gets

$$P^{excl}(T,\mu) = P(T,\tilde{\mu})\ ;\ \tilde{\mu} = \mu - v\,P^{excl}(T,\mu). \tag{3.30}$$

Therefore, we have an implicit equation for $P^{excl}(T,\mu)$ if P is a known function of its arguments.

If we consider the ideal gas case, we have in (3.28)

$$\mathcal{Z}_{id}(T,\tilde{\mu},\hat{V}) = \exp[\hat{V}\,F(T,\tilde{\mu})]\,, \tag{3.31}$$

where

$$F(T,\mu) = \frac{1}{a}\,\frac{d}{(2\pi)^3}\int d^3k\,\ln\left[1 + a\exp\left(\frac{-\sqrt{k^2+m^2}+\mu}{T}\right)\right], \tag{3.32}$$

where $a = \pm 1$ refers for fermions/bosons and $a \to 0$ for classical (Boltzmann) approximation. Performing the integral in (3.28) yields

$$\hat{\mathcal{Z}}^{excl}_{id}(T,\mu,x) = \frac{1}{x - F(T,\hat{\mu})}\,,$$

which gives

$$P^{excl}_{id}(T,\mu) = T\,F(T,\tilde{\mu}) = P_{id}\left(T,\mu - v\,P^{excl}_{id}(T,\mu)\right). \tag{3.33}$$

This is a special case of (3.30) for the ideal gas [62, 67]. The expression (3.30) is valid also for more general cases.

The particle number density, the entropy density and the energy density can be found from (3.33) using the relations [63]

$$n^{excl}_{id}(T,\mu) \equiv \left(\frac{\partial P^{excl}_{id}}{\partial \mu}\right)_T = \frac{n_{id}(T,\tilde{\mu})}{1 + v\,n_{id}(T,\tilde{\mu})}\,, \tag{3.34}$$

$$s^{excl}_{id}(T,\mu) \equiv \left(\frac{\partial P^{excl}_{id}}{\partial T}\right)_\mu = \frac{s_{id}(T,\tilde{\mu})}{1 + v\,n_{id}(T,\tilde{\mu})} \tag{3.35}$$

and

$$u^{excl}_{id}(T,\mu) \equiv T\,s^{excl}_{id} - P^{excl}_{id} + \mu\,n^{excl}_{id} = \frac{u_{id}(T,\tilde{\mu})}{1 + v\,n_{id}(T,\tilde{\mu})}\,, \tag{3.36}$$

where n_{id}, s_{id} and u_{id} are the well-known expressions for an ideal gas of point-like particles. The relations (3.33-3.36) are *thermodynamically consistent*, i.e. fundamental thermodynamical relations are fulfilled.

Since the excluded volume approximation suffers from the *causality* problem even if the *consistency* problem is addressed, we use the relativistic mean field or Hartree approximation proposed by Walecka [68, 69], which is *thermodynamically self-consistent*, to treat the interactions.

3.3.2 The Mean Field Approximation

In this approximation, the interaction between nucleons is described by the *scalar-isoscalar* ϕ and *vector-isoscalar* V^μ mesonic fields with baryon-meson interaction terms in the Lagrangian: $g_s\bar{\psi}\psi\phi$ and $g_v\bar{\psi}\gamma_\mu V^\mu\psi$. For nuclear matter in thermodynamical equilibrium these mesonic fields (ϕ and V^μ) are considered to be *constant* classical quantities. The *scalar* field ϕ describes the attraction between nucleons and lowers the nucleon (antinucleon) mass M to $M^* = M - g_s < \phi >$. The nucleon-nucleon repulsion is described by the *vector* field V^μ which changes the nucleon (antinucleon) energy by ($\pm U(n)$) [1].

The Lagrangian density in the in the Walecka model is given by [70]

$$\mathcal{L} = \bar{\psi}\left[\gamma_\mu\left(i\partial^\mu - g_v V^\mu\right) - (M - g_s\phi)\right]\psi + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2\right)$$
$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_v^2 V_\mu V^\mu + \delta\mathcal{L}, \quad (3.37)$$

where $F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$ and $\delta\mathcal{L}$ contains renormalization counterterms required for quantum field theory. The parameters M, g_s, g_v, m_s and m_v are phenomenological constants that can be determined from experimental measurements.

The Euler-Lagrange equations [70]

$$\frac{\partial}{\partial x^\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial q_i/\partial x^\mu)}\right] - \frac{\partial\mathcal{L}}{\partial q_i} = 0, \quad (3.38)$$

[1] The odd G-parity of the ω-meson is responsible for the attractive ω-exchange in $N\bar{N}$ scattering as compared to the repulsive ω-exchange in $NN(\bar{N}\bar{N})$ scattering.

where q_i is one of the generalized coordinates (ψ, ϕ and V^μ), yield the field equations

$$\left(\partial_\mu \partial^\mu + m_s^2\right)\phi = g_s \bar{\psi}\psi\,, \tag{3.39}$$

$$\partial_\mu F^{\mu\nu} + m_v^2 V^\nu = g_v \bar{\psi}\gamma^\nu \psi \tag{3.40}$$

and

$$[\gamma^\mu (i\partial_\mu - g_v V_\mu) - (M - g_s\phi)]\psi = 0\,. \tag{3.41}$$

Equation (3.39) is simply the Klein-Gordon equation with a scalar source, Equation (3.40) looks like massive quantum electrodynamics (QED) with the conserved baryon current

$$B^\mu = \bar{\psi}\gamma^\mu\psi\,; \quad \partial_\mu B^\mu = 0 \tag{3.42}$$

rather than the (conserved) electromagnetic current as source and equation (3.41) is the Dirac equation with the *scalar* and *vector* fields introduced in a minimal fashion. These field equations also imply that the canonical energy-momentum tensor

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - \frac{1}{2}\left(\partial_\sigma\phi\partial^\sigma\phi - m_s^2\phi^2\right)g^{\mu\nu} + \partial^\mu\phi\partial^\nu\phi$$
$$+\frac{1}{2}\left(\partial_\sigma V_\lambda \partial^\sigma V^\lambda - m_v^2 V_\sigma V^\sigma\right)g^{\mu\nu} - \partial^\mu V_\lambda \partial^\nu V^\lambda \tag{3.43}$$

is conserved ($\partial_\mu T^{\mu\nu} = \partial_\nu T^{\mu\nu} = 0$).

Equations (3.39-3.41) are *nonlinear quantum field equations*, and their exact solutions are very complicated. In particular, they describe mesons and baryons that are not *point like particles*, but rather objects with intrinsic structure due to the implied (virtual) mesonic and baryon-antibaryon loops. When the source terms are large, the meson field operators can be replaced by their expectation values, which are classical fields [70]:

$$\phi \rightarrow <\phi> \equiv \phi_0\,; \quad V^\mu \rightarrow <V^\mu> \equiv \delta^{\mu 0} V_0 \tag{3.44}$$

For a static, uniform system, the quantities ϕ_0 and V_0 are *constants* that are independent of space and time. Rotational invariance implies that the expectation value of the three-vector piece of V^μ vanishes.

The meson field equations (3.39) and (3.40) can be solved for constants ϕ_0 and V_0 to give

$$\phi_0 = \frac{g_s}{m_s^2} < \bar{\psi}\psi > \tag{3.45}$$

and

$$V_0 = \frac{g_v}{m_v^2} < \psi^\dagger \psi > . \tag{3.46}$$

When the meson fields in (3.41) are approximated by the classical fields of (3.44), the Dirac equation becomes *linear*,

$$[i\gamma_\mu \partial^\mu - g_v \gamma^0 V_0 - (M - g_s \phi_0)]\psi = 0 \tag{3.47}$$

and can be solved exactly. It is this *linearization* of the full field equation (3.41) that allows the baryons to be interpreted as *point particles*. As for free particles, the stationary state solutions for a uniform system have the form

$$\psi = \psi(\mathbf{k}, \lambda) e^{i\mathbf{k}\cdot\mathbf{x} - i\epsilon(k)t} \tag{3.48}$$

where $\psi(\mathbf{k}, \lambda)$ is a four-component Dirac spinor and λ denotes the spin index, which corresponds to one of the two orthogonal polarizations chosen in the particle's rest frame [70]. The Dirac equation then becomes

$$(\boldsymbol{\alpha} \cdot \mathbf{k} + \beta M^*)\psi(\mathbf{k}, \lambda) = [\epsilon(k) - g_v V_0]\, \psi(\mathbf{k}, \lambda)\,, \tag{3.49}$$

where $\boldsymbol{\alpha}$ and β are the four matrices:

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.50}$$

The *effective* mass M^* is defined by

$$M^* = M - g_s \phi_0\,. \tag{3.51}$$

The condensed *scalar* field ϕ_0 thus serves to shift the mass of the baryons. Evidently, the condensed *vector* field V_0 shifts the frequency (or energy) of the solutions. The square of equation (3.49) and the properties of the Dirac matrices yield the eigenvalue equations [70]

$$\epsilon^{(\pm)}(k) = g_v V_0 \pm \sqrt{k^2 + M^{*2}} \equiv g_v V_0 \pm E^*(k)\,. \tag{3.52}$$

The presence of both positive and negative square roots is the characteristic of the Dirac equation. These solutions can be used to define quantum field operators and the Hamiltonian density for the system can be constructed in the canonical fashion (for the details of these procedures, see ref. [70]). The result is

$$\hat{H} = \hat{H}_{MFT} + \delta H\,, \tag{3.53}$$

$$\hat{H}_{MFT} = g_v V_0 \hat{B} + \sum_{\mathbf{k}\lambda} E^*(k)\left(A^\dagger_{\mathbf{k}\lambda} A_{\mathbf{k}\lambda} + B^\dagger_{\mathbf{k}\lambda} B_{\mathbf{k}\lambda}\right) + \frac{1}{2} V\left(m_s^2 \phi^2 - m_v^2 V_0^2\right), \tag{3.54}$$

$$\hat{B} = \sum_{\mathbf{k}\lambda}\left(A^\dagger_{\mathbf{k}\lambda} A_{\mathbf{k}\lambda} - B^\dagger_{\mathbf{k}\lambda} B_{\mathbf{k}\lambda}\right) \tag{3.55}$$

and

$$\delta H = -\sum_{\mathbf{k}\lambda}\left(\sqrt{k^2 + M^{*2}} - \sqrt{k^2 + M^2}\right). \tag{3.56}$$

Here $A^\dagger_{\mathbf{k}\lambda}$, $B^\dagger_{\mathbf{k}\lambda}$, $A_{\mathbf{k}\lambda}$ and $B_{\mathbf{k}\lambda}$ are creation and destruction operators for baryons and antibaryons with shifted mass and energy. The properties of these operators are completely determined by the anticommutation relations:

$$\left\{A_{\mathbf{k}\lambda}, A^\dagger_{\mathbf{k}'\lambda'}\right\} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\lambda\lambda'} = \left\{B_{\mathbf{k}\lambda}, B^\dagger_{\mathbf{k}'\lambda'}\right\}$$

$$\left\{A_{\mathbf{k}\lambda}, B_{\mathbf{k}'\lambda'}\right\} = 0 = \left\{A^\dagger_{\mathbf{k}\lambda}, B^\dagger_{\mathbf{k}'\lambda'}\right\}\,,\ etc. \tag{3.57}$$

$\hat{B}$ is the baryon number operator, which clearly counts the number of baryons minus the number of antibaryons, the index λ denotes both spin and isospin projections and the correction term δH arises from placing the operators in $\hat{H}_{MFT}$ in normal order [70].

The thermodynamic potential $\Omega(\mu, V, T)$ can be computed using the standard expression from statistical mechanics:

$$\Omega(\mu, V, T) = -T \ln \left\{ Tr \left[\exp\left(-\frac{\hat{H} - \mu \hat{B}}{T} \right) \right] \right\}, \tag{3.58}$$

where $\hat{H}$ is the Hamiltonian of the mean field theory and μ is the chemical potential of baryons.

The general form of the *thermodynamically self-consistent* EOS for nuclear matter which includes the mean field theory and pure phenomenological models as special cases is [71, 72]:

$$P(T, \mu) = \frac{\gamma_N}{3} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\sqrt{k^2 + M^{*2}}} (\bar{n}_N + \bar{n}_{\bar{N}}) + n\, U(n) - \int_0^n dn'\, U(n') + p(M^*), \tag{3.59}$$

$$\bar{n}_{N(\bar{N})} = \left[\exp\left(\frac{\sqrt{k^2 + M^{*2}} \mp \mu \pm U(n)}{T} \right) + 1 \right]^{-1}, \tag{3.60}$$

$$\left(\frac{\delta p}{\delta M^*} \right)_{T,\mu} \equiv \frac{dp(M^*)}{dM^*} - \gamma_N \int \frac{d^3k}{(2\pi)^3} \frac{M^*}{\sqrt{k^2 + M^{*2}}} (\bar{n}_N + \bar{n}_{\bar{N}}) = 0, \tag{3.61}$$

$$n(T, \mu) = \gamma_N \int \frac{d^3k}{(2\pi)^3} (\bar{n}_N - \bar{n}_{\bar{N}}) \tag{3.62}$$

and

$$u(T, \mu) = \gamma_N \int \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + M^{*2}}\, (\bar{n}_N + \bar{n}_{\bar{N}}) + \int_0^n dn'\, U(n') - p(M^*), \tag{3.63}$$

where P, n and u are the pressure, baryon number density and energy density respectively, $\bar{n}_{N(\bar{N})}$ is the distribution function of nucleons (antinucleons), μ is the baryon chemical potential and γ_N is the spin-isospin degeneracy of the nucleon which is 4 for symmetric nuclear matter. Equation (3.61) describes the dependence of the effective nuclear mass M^* on T and μ which is defined by extremizing the thermodynamical potential (maximizing the pressure).

If we choose [68, 69],

$$p(M^*) = -\frac{1}{2\,C_s^2}(M - M^*)^2, \qquad U(n) = C_v^2\, n \tag{3.64}$$

where $C_s \equiv g_s\left(\frac{M}{m_s}\right)$ and $C_v \equiv g_v\left(\frac{M}{m_v}\right)$. The parameters M, M^*, g_s, g_v, m_s and m_v are taken here to be M= 0.940 GeV, M^*= 0.543M, $g_s \approx 11$ GeV^{-1}, $g_v \approx 14$ GeV^{-1}, m_s= 0.520 GeV and m_v= 0.783 GeV, which reproduce data (see ref. [70, 72] for details) in the hadronic phase.

The energy density, pressure and baryon number density for the hadron gas taken to contain nucleons, antinucleons and an ideal gas of massless pions are:

$$u_H(T,\mu) = \frac{\gamma_N}{2\,\pi^2}\int_0^\infty dk\, k^2\sqrt{k^2 + M^{*2}}\,(\bar{n}_N + \bar{n}_{\bar{N}}) + \frac{1}{2}C_v^2\, n^2 + \frac{1}{2\,C_s^2}(M - M^*)^2 + \frac{1}{10}\pi^2\, T^4\,, \tag{3.65}$$

$$P_H(T,\mu) = \frac{\gamma_N}{6\,\pi^2}\int_0^\infty \frac{dk\, k^4}{\sqrt{k^2 + M^{*2}}}\,(\bar{n}_N + \bar{n}_{\bar{N}}) + \frac{1}{2}C_v^2\, n^2 - \frac{1}{2\,C_s^2}(M - M^*)^2 + \frac{1}{30}\pi^2\, T^4 \tag{3.66}$$

and

$$n_H(T,\mu) = \frac{\gamma_N}{2\,\pi^2}\int_0^\infty dk\, k^2\,(\bar{n}_N - \bar{n}_{\bar{N}})\,, \tag{3.67}$$

where $\bar{n}_{N(\bar{N})}$ as in (3.60) with $U(n)$ as in (3.64).

Finally, we address the phase transition from the hadronic to the QGP phase, within our model.

3.4 Phase Transition

Assuming a first order phase transition between hadronic matter and QGP, one matches an EOS for the hadronic system and the QGP via Gibbs conditions for phase equilibrium:

$$P_H = P_{QGP}, \qquad T_H = T_{QGP}, \qquad \mu_H = \mu_{QGP} \tag{3.68}$$

With these conditions the pertinent regions of temperature T and baryon chemical potential μ are shown in figures (3.1-3.3) for $q=$ 1, 1.1 (0.9) and 1.25 (0.75). At low values of μ and T the nuclear matter is composed of confined hadrons, but as the energy density is raised, with increasing T or μ, or both, the hadronic matter undergoes a phase transition towards a plasma of deconfined quarks and gluons. The critical temperature at $\mu = 0$ for $q=$ 1, 1.1 (0.9) and 1.25 (0.75) are found to be 122 MeV, 101 MeV and 66 MeV with B=(180 MeV)^4; 148 MeV, 122 MeV and 79 MeV with B=(210 MeV)^4 and 180 MeV, 148 MeV and 96 MeV with B=(250 MeV)^4, respectively. As the non-extensive parameter deviates from $q=$ 1 to 1.25 (0.75), the critical temperature becomes almost independent of the baryon chemical potential which is associated with the number of particles. Variation of the bag constant B between (180 MeV)^4 and (250 MeV)^4 (fig. 3.1-3.3) and excluding interactions among the constituents in the hadron phase (fig. 3.4) do not alter our findings significantly. One still observes a flattening of the $T(\mu)$ curves as $|1-q|$ increases. The only effect is a shift of the value of the maximal μ and $T(0)$. The agreement between BG and the generalized statistics as the critical temperature approaches to zero is evident from equations (3.2) and (3.14). Figures (3.5-3.7) show the dependence of the critical temperature on the non-extensive parameter at $\mu = 0$, 250 MeV and 1000 MeV. The dependence is almost linear for small values of $|1-q|$

with a slope $|\frac{\Delta T}{\Delta q}| \approx 190$ MeV , $|\frac{\Delta T}{\Delta q}| \approx 240$ MeV, and $|\frac{\Delta T}{\Delta q}| \approx 302$ MeV for B=(180 MeV)^4, B=(210 MeV)^4 and B=(250 MeV)^4, respectively . This can be interpreted as a new type of universality condition [33] which suggests that the formation of a QGP occurs (almost) independent of the total number of baryons participating in heavy ion collisions.

We also consider the extensive Kaniadakis statistics [37, 38] to represent the constituents of the QGP [73]. Figure (3.8) shows the $T(\mu)$ curves for κ=0, κ=0.23 and κ=0.29. Since the two statistics are fractal in nature, we observe a similar flattening of the $T(\mu)$ curves as the deformation κ increases. For κ=0.23 (see fig. 3.9), we obtain essentially the same phase diagram as in the case of Tsallis statistics with q=1.1.

3.5 Conclusion

We have studied the phase transition from a system in the hadronic phase to the QGP phase. The phase diagrams shown in figures (3.1-3.3, 3.8) are for interacting hadrons and non-interacting QGP. Excluding interactions among the constituents in the hadron phase (fig. 3.4) does not change the flattening of the $T(\mu)$ curves. The empirical insensitivity of the phase diagram to *details* of the interactions in the hadron phase shows that the *short-range* interactions among the constituents in the hadron phase are unimportant for the phase transition. On the QGP side the *short-range* interactions have died out (due to “asymptotic freedom”) and have made room for the only remaining *long-range* interactions among the constituents. Although we do not explicitly consider the interactions, we account for the dominant part of this interaction by a change in the statistics of the system in the QGP phase.

In the present work, we consider two entropic measures (Tsallis and Kaniadakis) to represent the constituents of the QGP. Both are fractal in nature but differ in that: Tsallis statistics is non-extensive and reduces to BG statistics (extensive) as the Tsallis parameter q tends to one. On the other hand, Kaniadakis statistics is extensive and tends to BG statistics as the deformation parameter κ tends to zero. Due to lack of experimental data, we choose different values of q and κ to determine the phase diagram in both cases. For κ=0.23 (see fig. 3.9), we obtain essentially the same phase diagram as in the case of Tsallis statistics with q=1.1. This agreement suggests that the flattening in the phase diagram is due to the fractal nature of both Tsallis and Kaniadakis statistics.

We present here testable consequences of using extensive and non-extensive form of generalized statistical mechanics on the formation a QGP. The resulting insensitivity of the critical temperature to the total number of baryons presents a clear experimental signature for the existence of generalized statistics for the constituents of the QGP.

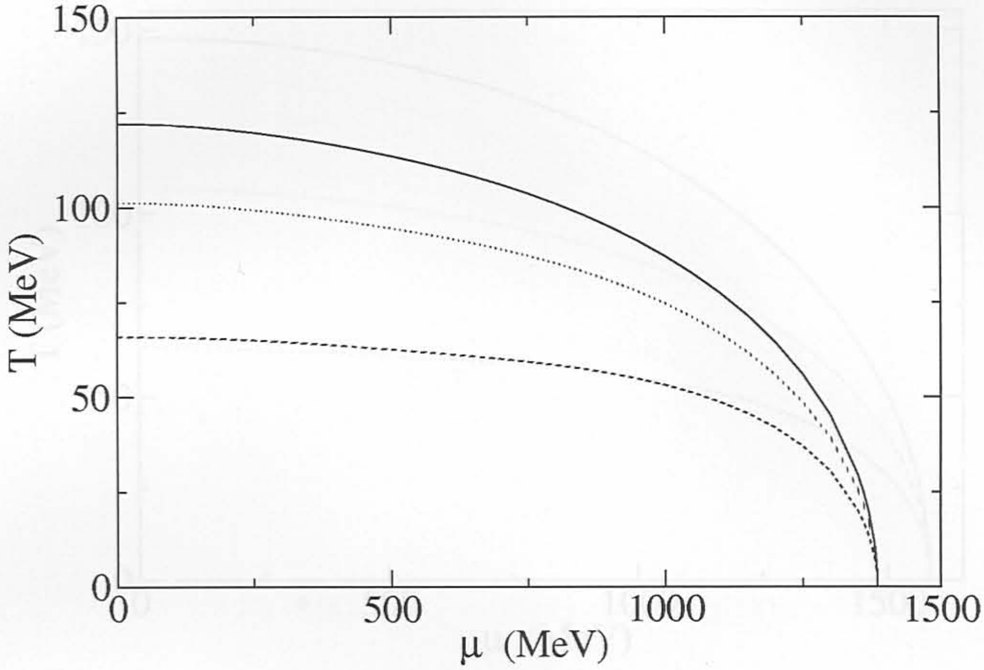

Figure 3.1: Phase transition curves between the hadronic matter and QGP for q=1 (solid line), q=1.1 (0.9) (dotted line) and q=1.25 (0.75) (dashed line) with B=(180 MeV)^4.

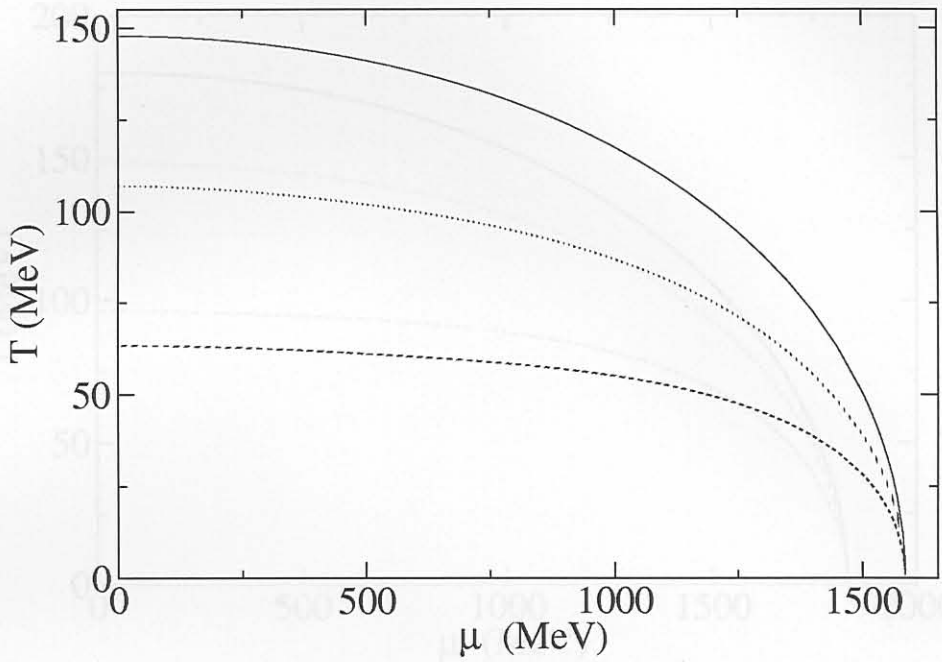

Figure 3.2: Phase transition curves between the hadronic matter and QGP for q=1 (solid line) [1], q=1.1 (0.9) (dotted line) and q=1.25 (0.75) (dashed line) with B=(210 MeV)^4.

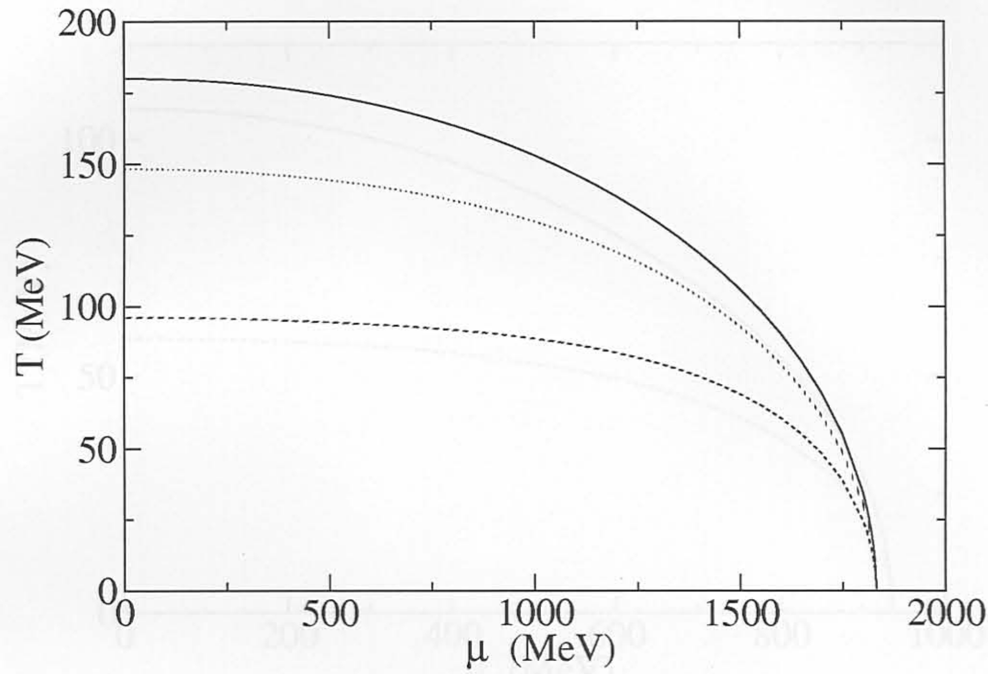

Figure 3.3: Phase transition curves between the hadronic matter and QGP for q=1 (solid line), q=1.1 (0.9) (dotted line) and q=1.25 (0.75) (dashed line) with B=(250 MeV)^4.

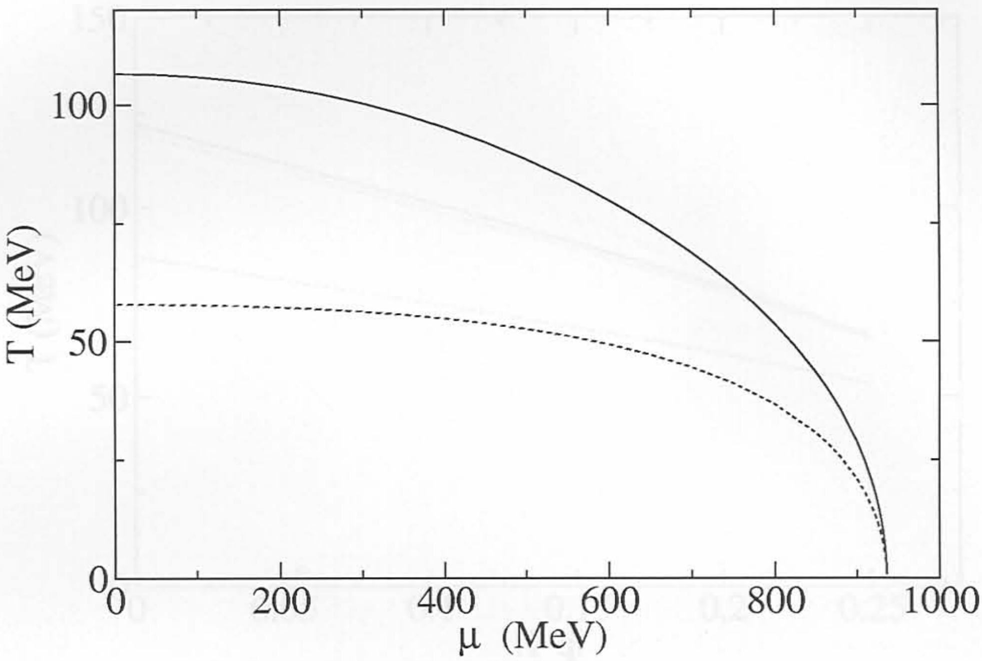

Figure 3.4: Phase transition curves between the hadronic matter and QGP for q=1 (solid line) [2] and q=1.25 (0.75) (dashed line) with B=(148 MeV)^4.

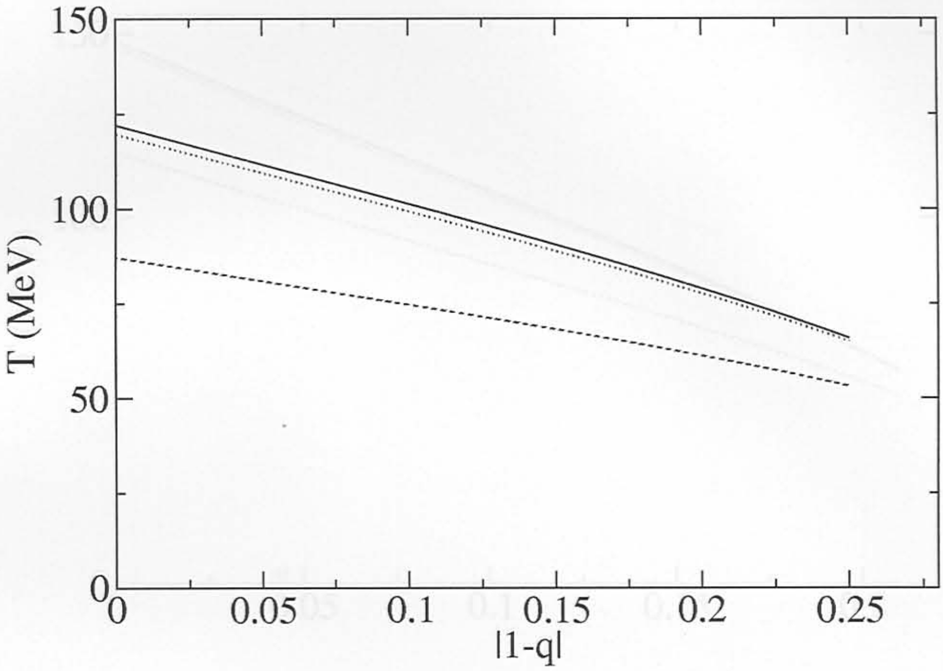

Figure 3.5: The dependence of T on $|1-q|$ at μ= 0 (solid line), μ= 250 MeV (dotted line) and μ= 1000 MeV (dashed line) with B=(180 MeV)^4.

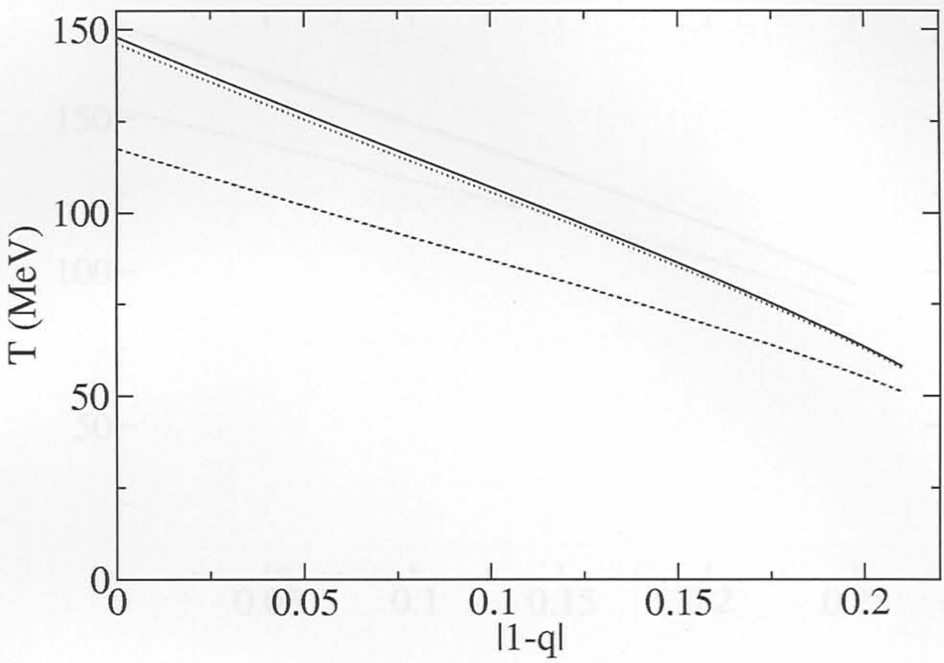

Figure 3.6: The dependence of T on $|1-q|$ at μ= 0 (solid line), μ= 250 MeV (dotted line) and μ= 1000 MeV (dashed line) with B=(210 MeV)^4.

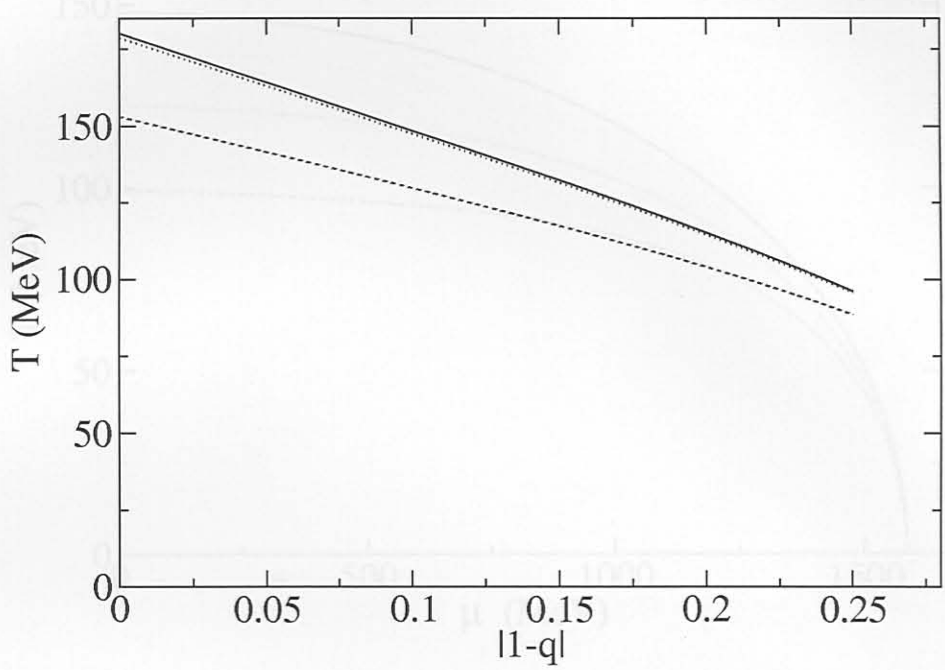

Figure 3.7: The dependence of T on $|1-q|$ at μ= 0 (solid line), μ= 250 MeV (dotted line) and μ= 1000 MeV (dashed line) with B=(250 MeV)^4.

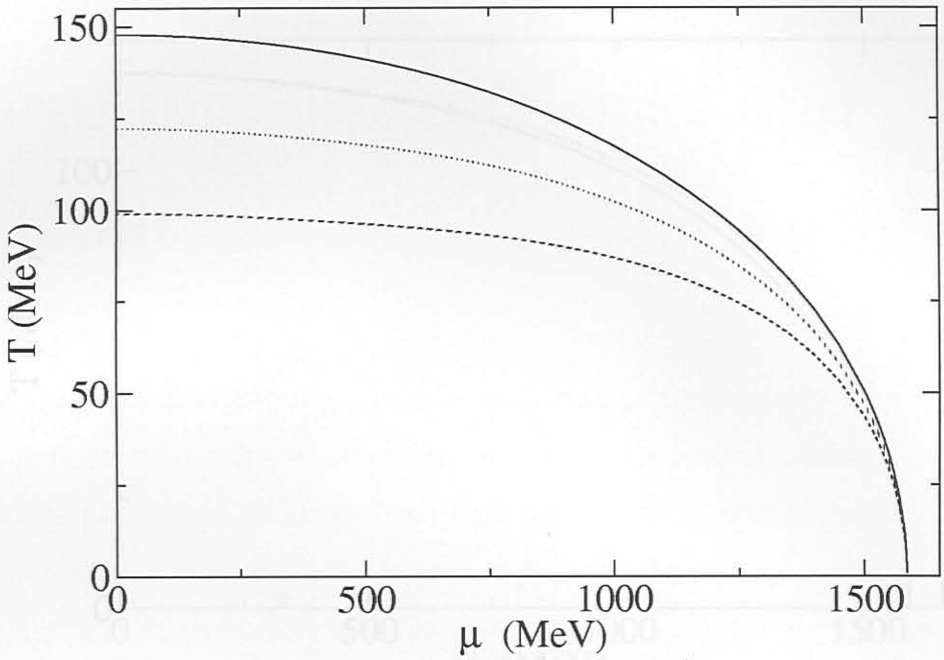

Figure 3.8: Phase transition curves between the hadronic matter and QGP for κ=0 (solid line) [1], κ=0.23 (dotted line) and κ=0.29 (dashed line) with B=(210 MeV)^4.

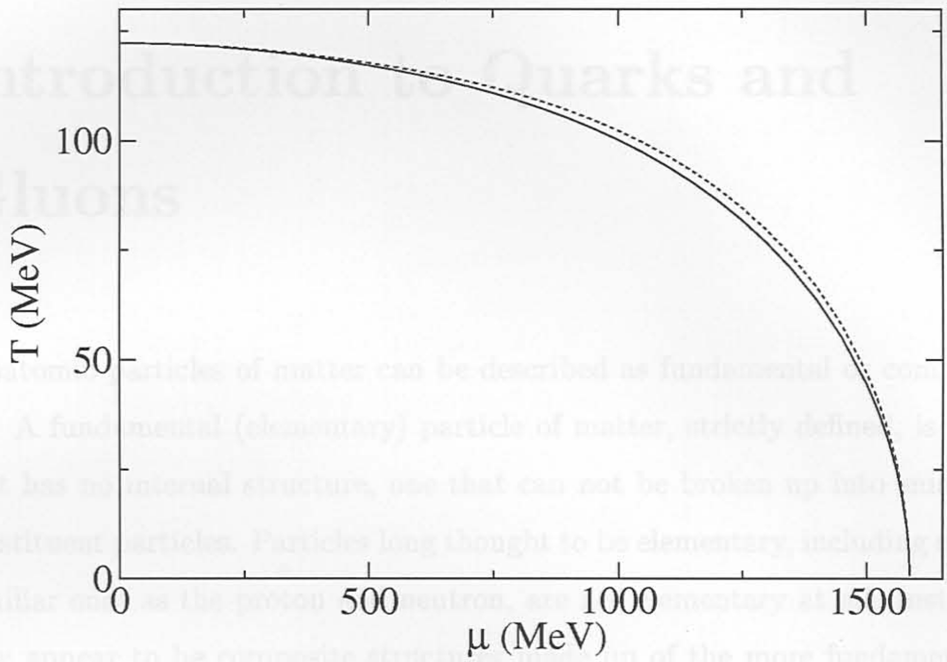

Figure 3.9: Phase transition curves between the hadronic matter and QGP for κ=0.23 (dashed line) and q=1.1 (solid line) with B=(210 MeV)^4.

Appendix A

Introduction to Quarks and Gluons

Subatomic particles of matter can be described as fundamental or composite. A fundamental (elementary) particle of matter, strictly defined, is one that has no internal structure, one that can not be broken up into smaller constituent particles. Particles long thought to be elementary, including such familiar ones as the proton and neutron, are not elementary at all. Instead they appear to be composite structures made up of the more fundamental entities named quarks and gluons, in much the same way that an atom is made up of a nucleus and electrons [74].

The fundamental particles of matter are called quarks and leptons. There are six flavors of quarks and six flavors of leptons (three pairs). The six flavors of quarks and leptons can be summarized as follows:

Quarks

Flavor	Mass	Electric Charge
Up	3±2 MeV	$+\frac{2}{3}$
Down	6±3 MeV	$-\frac{1}{3}$
Charm	$1.3^{+0.05}_{-0.15}$ GeV	$+\frac{2}{3}$
Strange	100^{+70}_{-25} MeV	$-\frac{1}{3}$
Top	174.3±5.1 GeV	$+\frac{2}{3}$
Bottom	$4.3^{+0.1}_{-0.3}$ GeV	$-\frac{1}{3}$

Leptons

Flavor	Mass (MeV)	Electric Charge
Electron	0.510998902±0.000000021	-1
Electron Neutrino	$< 3\text{x}10^{-6}$	0
Muon	105.658357±0.000005	-1
Muon Neutrino	<0.19	0
Tauon	$1777.03^{+0.30}_{-0.26}$	-1
Tauon Neutrino	<18.2	0

All particles have spin (intrinsic angular momentum), which is either odd-$\frac{1}{2}$-integral-spin ($\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, ...) or integral-spin (0, 1, 2, ...). For both force-carrier and fundamental particles, spin determines the energy distribution function, which can be either Bose-Einstein (bosons) or Fermi-Dirac (fermions). Particles with odd-$\frac{1}{2}$-integral-spin (fermions) obey the Pauli Exclusion Principle, whereas particles with integral-spin (bosons) do not. In sum:

Quantum Energy Distribution Functions

Particles	Spin	Statistics	Pauli Exclusion Principle
Fermions	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$	Fermi-Dirac	Obey
Bosons	0, 1, 2, ...	Bose-Einstein	Do not obey

The particles thought to be made up of quarks are those called hadrons; they are distinguished by the fact that they interact with each other through a strong nuclear force, the force that binds together the particles in the atomic nucleus. ("Hadron" is derived from the Greek hadros, meaning stout or strong). Leptons and photons do not respond to this strong force [74].

The hadrons are divided into two large subgroups named the baryons (fermions) and the mesons (bosons). These two kinds of particle differ in many of their properties, and indeed they play different roles in the structure of matter, but the distinction between them can be made most clearly in the context of a simple quark model. All baryons consist of three quarks, and there are also antibaryons consisting of three antiquarks. The least massive and the most familiar of the baryons are the proton and the neutron [74]. Since these baryons are stable against decay through strong interactions, they are composed of the lightest and most stable quarks: the up-quarks and the down-quarks. A proton is composed of two up-quarks and one down-quark, whereas a neutron is composed of two down-quarks and one up-quark. Mesons have a different structure: they consist of quarks bound to antiquarks. The least massive and the most long-lived meson against decay through strong interactions is the positively charged pi-meson (pion, composed of an up-quark and a down-antiquark, mass of 0.14 GeV), which has an average lifetime measured in nanoseconds [75].

Gluons are the exchange particles for the color force between quarks, analogous to the exchange of photons in the electromagnetic force between two charged particles. The photon does not carry electric charge with it, while gluons do carry the "color" charge. Analogous to electrical charge associated with electromagnetic force there is a "color" charge associated with quarks and gluons. The colors of this charge are called red, green and blue, not visual colors, but a kind of charge based on an analogy to colors. Just as combining electrical positive and negative charge results in a neutral electrical charge, combining red, green and blue color charge gives a neutral color charge (analogy to color being that mixing the red, green and blue primary colors gives neutral white) [75].

All quarks and gluons have color charge, but all the hadrons (protons, neutrons, mesons) comprised of quarks, antiquarks and gluons have neutral color charge (analogous to most atoms having a neutral electrical charge). A quark can change color by emitting or absorbing gluons. If a red quark becomes a green quark, it must have emitted a gluon carrying the colors red and antigreen. Like the electric charge, color charge is always conserved [75].

For every matter particle there corresponds an antimatter particle. Antimatter particles correspond to matter particles in every respect except that they have opposite charge, spin and chemical potential. An antielectron (positron) has the same mass as an electron, but it is electrically positive. Antiquarks have electrical charges $-\frac{2}{3}$ and $+\frac{1}{3}$. Associated with the antiquarks, however, there are anticolor charges: antired, antigreen and antiblue. An antiproton is composed of two up-antiquarks and one down-antiquark [75].

Appendix B

Quantum Distributions from Maximum Entropy Principle

I. The Standard Maximum Entropy Principle

In standard quantum mechanical statistics, the entropic measure is given by [40, 41]

$$S = -\sum_i [\bar{n}_i \ln \bar{n}_i \mp (1 \pm \bar{n}_i) \ln(1 \pm \bar{n}_i)], \tag{B.1}$$

where the upper and lower signs correspond to bosons and fermions, respectively, and $\bar{n}_i$ denotes the number of particles in the i^{th} energy level with energy ϵ_i. The extremization of the above measure under the constraints imposed by the total number of particles,

$$\sum_i \bar{n}_i = N \tag{B.2}$$

and the total energy of the system,

$$\sum_i \bar{n}_i \, \epsilon_i = E, \tag{B.3}$$

leads to a variational problem

$$\frac{\delta}{\delta \bar{n}_i} \left[S + \alpha \left(N - \sum_i \bar{n}_i \right) + \beta \left(E - \sum_i \epsilon_i \, \bar{n}_i \right) \right] = 0 \, . \tag{B.4}$$

Substituting (B.1) in (B.4), one obtains

$$[\ln(1-\bar{n}_i) - \ln\bar{n}_i - (\alpha + \beta\,\epsilon_i)] = 0 \tag{B.5}$$

for fermions and

$$[\ln(1+\bar{n}_i) - \ln\bar{n}_i - (\alpha + \beta\,\epsilon_i)] = 0 \tag{B.6}$$

for bosons.

Rearranging the terms in (B.5) and (B.6) and introducing a chemical potential

$$\mu = -\frac{\alpha}{\beta}, \tag{B.7}$$

yields

$$\bar{n}_i = \frac{1}{\exp\beta(\epsilon_i - \mu) \mp 1}, \tag{B.8}$$

where the upper and lower signs correspond to the Bose-Einstein and Fermi-Dirac distributions, respectively.

II. The Non-extensive Maximum Entropy Principle for Fermions

The extended measure of entropy for fermions is given by [33, 39]

$$S_q^{(F)} = \sum_i \left[\frac{\bar{n}_i - \bar{n}_i^q}{q-1} + \frac{(1-\bar{n}_i) - (1-\bar{n}_i)^q}{q-1}\right], \tag{B.9}$$

which for $q \to 1$ reduces to the entropic functional (B.1) (with lower signs).

Maximizing the extended measure of entropy in (B.9) subject to the constraints

$$\sum_i \bar{n}_i^q = N \tag{B.10}$$

and

$$\sum_i \bar{n}_i^q \, \epsilon_i = E \tag{B.11}$$

leads to the variational problem

$$\frac{\delta}{\delta \bar{n}_i} \left[S_q^{(F)} + \alpha \left(N - \sum_i \bar{n}_i^q \right) + \beta \left(E - \sum_i \epsilon_i \, \bar{n}_i^q \right) \right] = 0 \, . \tag{B.12}$$

Following the same procedure as in (I) yields

$$\bar{n}_i = \frac{1}{[1 + (q-1)\beta(\epsilon_i - \mu)]^{\frac{1}{q-1}} + 1} . \tag{B.13}$$

In the limit $q \to 1$ one recovers the usual Fermi-Dirac distribution (B.8) (with lower sign).

III. The Bosonic Problem

The quantum mechanical distribution function proposed by Buyukklic and Demirhan (BD) [42] is given by

$$\bar{n}_i = \frac{1}{[1 + (q-1)\beta(\epsilon_i - \mu)]^{\frac{1}{q-1}} \mp 1} , \tag{B.14}$$

where the upper and lower signs correspond to bosons and fermions, respectively. Ever since the BD proposal, the two cases suggested in (B.14) (that is, the fermionic and bosonic case) were regarded as sharing the same degree of validity. On the basis of [39], the fermionic and bosonic BD distributions do not stand on an equal footing. First of all, each term in the entropic functional (B.9) gives the correct expression, within the context of Tsallis non-extensive thermostatistics, for the entropy of one single fermionic oscillator (in thermal equilibrium) in terms of its average occupation number $\bar{n}_i$. Secondly, the entropic functional (B.9) admits a reasonable "probabilistic" interpretation (see Sommerfeld [41] for a discussion of this point in the standard q=1 case). $\bar{n}_i$ can be regarded as the probability of the i^{th} state being occupied and $(1-\bar{n}_i)$ as the probability of that state being empty.

Extremizing the entropic functional,

$$S_q^{(B)} = \sum_i \left[\frac{\bar{n}_i - \bar{n}_i^q}{q-1} - \frac{(1+\bar{n}_i) - (1+\bar{n}_i)^q}{q-1} \right], \tag{B.15}$$

under the constraints imposed by (B.10) and (B.11) yields the q-generalized BD Bose-Einstein distribution, which is given by (B.14) with the minus sign. A "probabilistic" interpretation of the entropic measure (B.15) and the associated variational procedure leading to BD approach distribution for bosons (B.14) is somewhat problematic [39]. Let us denote by $f_{i,n}$ the probability of having n bosons in the state i with energy ϵ_i. If we try to follow the steps of Sommerfeld's probabilistic approach for bosons [41], we should start by extremizing the functional

$$S_q = \sum_{i,n} \frac{f_{i,n} - f_{i,n}^q}{q-1}, \tag{B.16}$$

under the set of constraints

$$\sum_n f_{i,n} = 1, \qquad \forall i\,,$$

$$\sum_{n,i} f_{i,n}^q n = N$$

and

$$\sum_{n,i} f_{i,n}^q \, n \, \epsilon_i = E\,. \tag{B.17}$$

For the sake of simplicity we use unnormalized q-constraints here, but the same conclusions would obtain if normalized q-constraints are used instead (notice that the quantities $f_{i,n}$ are not occupation numbers. They are true probabilities and, strictly speaking, the associated mean values should be normalized). The set in (B.17) does not consist of just three constraints. There is one "normalization" constraint for each i. So, in principle, we are dealing with an infinite set of constraints. Accordingly, we have to introduce the Lagrange multipliers,

$$\alpha,\ \beta,\ \lambda_i\,, \tag{B.18}$$

where there is one λ_i for each state i. The variational principle then reads,

$$\frac{\delta}{\delta f_{i,n}} \sum_{i,n} \left(\frac{f_{i,n} - f_{i,n}^q}{q-1} - \lambda_i\, f_{i,n} - \alpha\, n\, f_{i,n}^q - \beta\, n\, \epsilon_i\, f_{i,n}^q \right) = 0. \tag{B.19}$$

The above variational problem leads to

$$\frac{1 - q\, f_{i,n}^{q-1}}{q-1} - \lambda_i - q\, \alpha\, n\, f_{i,n}^{q-1} - q\, \beta\, n\, \epsilon_i\, f_{i,n}^{q-1} = 0\,, \tag{B.20}$$

which can be solved for $f_{i,n}$, yielding

$$f_{i,n} = \left[\frac{q}{1 + \lambda_i\,(1-q)} \right]^{\frac{1}{1-q}} \left[1 - (1-q)(\alpha\, n + \beta\, n\, \epsilon_i) \right]^{\frac{1}{1-q}}. \tag{B.21}$$

The mean occupation number of the state i would then be given by

$$\bar{n}_i = \sum_{n=0}^{\infty} n\, f_{i,n}^q\,, \tag{B.22}$$

where $f_{i,n}$ is given by (B.21). Now, the difficulty is that the sum appearing in (B.22) is not equal to the BD expression for the q-generalized Bose-Einstein distribution. The sum in (B.22) cannot, in general, be reduced to a finite, closed, analytical expression. The alluded to summation can be done analytically only for q=1, and it leads to the standard Bose-Einstein distribution [41].

Appendix C

Evaluation of Integrals

In BG statistics, the energy density, pressure and baryon number density of fermions and bosons can be derived as follows[1]:

Fermions

The energy of fermions is given by

$$E = d \int\int \frac{d^3q\, d^3p\, \epsilon}{\exp \frac{1}{T}(\epsilon - \mu) + 1}, \tag{C.1}$$

where

- ϵ is relativistic energy
- d is degeneracy factor
- μ is chemical potential
- T is temperature, and
- $d^3q\, d^3p$ is the element of phase space.

[1]For the sake of simplicity the Boltzmann's constant k_B, the reduced Planck's constant $\hbar$ and the speed of light c are set equal to one.

The integral over the phase space yields

$$E = \frac{V\,d}{2\,\pi^2} \int_0^\infty \frac{dp\,p^2\,\epsilon}{\exp\frac{1}{T}(\epsilon-\mu)+1}\,, \tag{C.2}$$

where

- V is volume and
- $\boldsymbol{p}$ is momentum.

The relativistic energy is given by

$$\epsilon^2 = k^2 + m^2\,, \tag{C.3}$$

where $\boldsymbol{k}$ is wave vector and m is rest mass.

For relativistic fermions with m=0,

$$\epsilon = k\,. \tag{C.4}$$

Substituting ϵ by k and $dp\,p^2$ by $dk\,k^2$ in (C.2), one obtains

$$E = \frac{V\,d}{2\,\pi^2} \int_0^\infty \frac{dk\,k^3}{\exp\frac{1}{T}(k-\mu)+1}\,. \tag{C.5}$$

Therefore, the energy density, that is energy per unit volume, is given by

$$u = \frac{d}{2\,\pi^2} \int_0^\infty \frac{dk\,k^3}{\exp\frac{1}{T}(k-\mu)+1}\,. \tag{C.6}$$

The expressions for the pressure and baryon number density can be derived in a similar manner and are given by

$$P = \frac{d\,T}{2\,\pi^2} \int_0^\infty dk\,k^2 \left\{\ln\left[1+\exp\frac{1}{T}(\mu-k)\right]\right\} \tag{C.7}$$

and

$$n = \frac{d}{2\,\pi^2} \int_0^\infty \frac{dk\,k^2}{\exp\frac{1}{T}(k-\mu)+1}\,. \tag{C.8}$$

Integration by parts of (C.7) yields

$$P = \frac{1}{3}\,u\,. \tag{C.9}$$

Bosons

The energy of bosons is given by

$$E = d \int\int \frac{d^3q\, d^3p\, \epsilon}{\exp\left(\frac{\epsilon}{T}\right) - 1}. \tag{C.10}$$

The integral over the phase space yields

$$E = \frac{V\, d}{2\,\pi^2} \int_0^\infty \frac{dp\, p^2\, \epsilon}{\exp\left(\frac{\epsilon}{T}\right) - 1}. \tag{C.11}$$

For relativistic bosons with m=0, the energy density is given by

$$u = \frac{d}{2\,\pi^2} \int_0^\infty \frac{dk\, k^3}{\exp\left(\frac{k}{T}\right) - 1}. \tag{C.12}$$

Letting $x = \frac{k}{T}$, one obtains

$$u = \frac{d\, T^4}{2\,\pi^2} \int_0^\infty \frac{dx\, x^3}{\exp(x) - 1}. \tag{C.13}$$

Evaluating the integral yields

$$u = \frac{1}{30}\, d\, \pi^2\, T^4\,. \tag{C.14}$$

The pressure, which is $P = \frac{1}{3}\, u$, is given by

$$P = \frac{1}{90}\, d\, \pi^2\, T^4\,. \tag{C.15}$$

The energy density residing in the quarks alone, or the antiquarks alone, can not be calculated analytically in the general case, $\mu, T \neq 0$. However, the sum of both yields a simple analytical formula as can be derived in the following way:

The energy density carried by quarks is given by

$$u_Q = \frac{d_Q}{2\,\pi^2} \int_0^\infty \frac{dk\, k^3}{\exp\beta(k-\mu) + 1}. \tag{C.16}$$

Letting $x = \beta(k - \mu)$, one obtains

$$u_Q = \frac{d_Q}{2\,\pi^2\,\beta^4}\int_{-\beta\,\mu_Q}^{\infty}\frac{dx\,(x+\beta\,\mu_Q)^3}{\exp(x)+1}. \tag{C.17}$$

The only change occurring in the same expression for antiquarks is the replacement of μ by (-μ). Up on letting $x = \beta(k + \mu)$, one obtains

$$u_{\bar{Q}} = \frac{d_Q}{2\,\pi^2\,\beta^4}\int_{\beta\,\mu_Q}^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1}. \tag{C.18}$$

The integral in (C.18) can be split up in the following way:

$$\int_{\beta\,\mu_Q}^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} = \int_0^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} - \int_0^{\beta\,\mu_Q}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1}. \tag{C.19}$$

Substituting x by (-x) in the second integral and using the relation

$$\frac{1}{\exp(-x)+1} = 1 - \frac{1}{\exp(x)+1} \tag{C.20}$$

yields

$$\int_{\beta\,\mu_Q}^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} = \int_0^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} + \int_0^{-\beta\,\mu_Q}\frac{dx\,(x+\beta\,\mu_Q)^3}{\exp(x)+1} - \int_0^{-\beta\,\mu_Q} dx\,(x+\beta\,\mu_Q)^3. \tag{C.21}$$

Therefore, the energy density of the QGP, which is the sum of the energy density of quarks, antiquarks and gluons is given by

$$u_{QGP} = \frac{d_Q}{2\,\pi^2\,\beta^4}\left\{\int_{-\beta\,\mu_Q}^{\infty}\frac{dx\,(x+\beta\,\mu_Q)^3}{\exp(x)+1} + \int_0^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} + \int_0^{-\beta\,\mu_Q}\frac{dx\,(x+\beta\,\mu_Q)^3}{\exp(x)+1} + \int_{-\beta\,\mu_Q}^{0} dx\,(x+\beta\,\mu_Q)^3\right\} + \frac{d_G}{2\,\pi^2\,\beta^4}\int_0^{\infty}\frac{dx\,x^3}{\exp(x)-1} + B. \tag{C.22}$$

Combining the first and the third integrals yields

$$u_{QGP} = \frac{d_Q}{2\,\pi^2\,\beta^4}\left\{\int_0^{\infty}\frac{dx\,(x+\beta\,\mu_Q)^3}{\exp(x)+1} + \int_0^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} +\right.$$

$$\int_{-\beta\,\mu_Q}^{0} dx\,(x+\beta\,\mu_Q)^3\bigg\} + \frac{d_G}{2\,\pi^2\,\beta^4}\int_0^\infty \frac{dx\,x^3}{\exp(x)-1} + B\,. \tag{C.23}$$

Expanding and rearranging these integrals, one obtains

$$u_{QGP} = \frac{d_Q}{2\,\pi^2\,\beta^4}\bigg\{2\int_0^\infty \frac{dx\,x^3}{\exp(x)+1} + 6\,\beta^2\,\mu_Q^2\int_0^\infty \frac{dx\,x}{\exp(x)+1} +$$

$$\int_{-\beta\,\mu_Q}^{0} dx\,(x+\beta\,\mu_Q)^3\bigg\} + \frac{d_G}{2\,\pi^2\,\beta^4}\int_0^\infty \frac{dx\,x^3}{\exp(x)-1} + B\,. \tag{C.24}$$

Evaluation of these integrals yields

$$u_{QGP} = \frac{\pi^2}{30}\left(d_G + \frac{7}{4}\,d_Q\right)T^4 + \frac{d_Q\,\mu^2\,T^2}{36} + \frac{d_Q\,\mu^4}{648\,\pi^2} + B\,. \tag{C.25}$$

The pressure, which is $\frac{1}{3}\,(u_{QGP} - 4B)$, is given by

$$P_{QGP} = \frac{\pi^2}{90}\left(d_G + \frac{7}{4}\,d_Q\right)T^4 + \frac{d_Q\,\mu^2\,T^2}{108} + \frac{d_Q\,\mu^4}{1944\,\pi^2} - B\,. \tag{C.26}$$

The baryon number density is given by

$$n_{QGP} = \frac{1}{3}\,(n_Q - n_{\bar{Q}})\,, \tag{C.27}$$

where

$$n_Q = \frac{d_Q}{2\,\pi^2}\int_0^\infty \frac{dk\,k^2}{\exp\beta(k-\mu)+1}$$

and

$$n_{\bar{Q}} = \frac{d_Q}{2\,\pi^2}\int_0^\infty \frac{dk\,k^2}{\exp\beta(k+\mu)+1}\,.$$

Using the same procedure as the energy density, it can be shown that

$$n_{QGP} = d_Q\left(\frac{\mu\,T^2}{54} + \frac{\mu^3}{486\,\pi^2}\right). \tag{C.28}$$

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