Appendix A

Introduction to Quarks and Gluons

Subatomic particles of matter can be described as fundamental or composite. A fundamental (elementary) particle of matter, strictly defined, is one that has no internal structure, one that can not be broken up into smaller constituent particles. Particles long thought to be elementary, including such familiar ones as the proton and neutron, are not elementary at all. Instead they appear to be composite structures made up of the more fundamental entities named quarks and gluons, in much the same way that an atom is made up of a nucleus and electrons [74].

The fundamental particles of matter are called quarks and leptons. There are six flavors of quarks and six flavors of leptons (three pairs). The six flavors of quarks and leptons can be summarized as follows:

Quarks

Flavor	Mass	Electric Charge
Up	3±2 MeV	$+\frac{2}{3}$
Down	6±3 MeV	$-\frac{1}{3}$
Charm	$1.3^{+0.05}_{-0.15}$ GeV	$+\frac{2}{3}$
Strange	100^{+70}_{-25} MeV	$-\frac{1}{3}$
Top	174.3±5.1 GeV	$+\frac{2}{3}$
Bottom	$4.3^{+0.1}_{-0.3}$ GeV	$-\frac{1}{3}$

Leptons

Flavor	Mass (MeV)	Electric Charge
Electron	0.510998902±0.000000021	-1
Electron Neutrino	$< 3\text{x}10^{-6}$	0
Muon	105.658357±0.000005	-1
Muon Neutrino	<0.19	0
Tauon	$1777.03^{+0.30}_{-0.26}$	-1
Tauon Neutrino	<18.2	0

All particles have spin (intrinsic angular momentum), which is either odd-$\frac{1}{2}$-integral-spin ($\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, ...) or integral-spin (0, 1, 2, ...). For both force-carrier and fundamental particles, spin determines the energy distribution function, which can be either Bose-Einstein (bosons) or Fermi-Dirac (fermions). Particles with odd-$\frac{1}{2}$-integral-spin (fermions) obey the Pauli Exclusion Principle, whereas particles with integral-spin (bosons) do not. In sum:

Quantum Energy Distribution Functions

Particles	Spin	Statistics	Pauli Exclusion Principle
Fermions	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$	Fermi-Dirac	Obey
Bosons	0, 1, 2, ...	Bose-Einstein	Do not obey

The particles thought to be made up of quarks are those called hadrons; they are distinguished by the fact that they interact with each other through a strong nuclear force, the force that binds together the particles in the atomic nucleus. ("Hadron" is derived from the Greek hadros, meaning stout or strong). Leptons and photons do not respond to this strong force [74].

The hadrons are divided into two large subgroups named the baryons (fermions) and the mesons (bosons). These two kinds of particle differ in many of their properties, and indeed they play different roles in the structure of matter, but the distinction between them can be made most clearly in the context of a simple quark model. All baryons consist of three quarks, and there are also antibaryons consisting of three antiquarks. The least massive and the most familiar of the baryons are the proton and the neutron [74]. Since these baryons are stable against decay through strong interactions, they are composed of the lightest and most stable quarks: the up-quarks and the down-quarks. A proton is composed of two up-quarks and one down-quark, whereas a neutron is composed of two down-quarks and one up-quark. Mesons have a different structure: they consist of quarks bound to antiquarks. The least massive and the most long-lived meson against decay through strong interactions is the positively charged pi-meson (pion, composed of an up-quark and a down-antiquark, mass of 0.14 GeV), which has an average lifetime measured in nanoseconds [75].

Gluons are the exchange particles for the color force between quarks, analogous to the exchange of photons in the electromagnetic force between two charged particles. The photon does not carry electric charge with it, while gluons do carry the "color" charge. Analogous to electrical charge associated with electromagnetic force there is a "color" charge associated with quarks and gluons. The colors of this charge are called red, green and blue, not visual colors, but a kind of charge based on an analogy to colors. Just as combining electrical positive and negative charge results in a neutral electrical charge, combining red, green and blue color charge gives a neutral color charge (analogy to color being that mixing the red, green and blue primary colors gives neutral white) [75].

All quarks and gluons have color charge, but all the hadrons (protons, neutrons, mesons) comprised of quarks, antiquarks and gluons have neutral color charge (analogous to most atoms having a neutral electrical charge). A quark can change color by emitting or absorbing gluons. If a red quark becomes a green quark, it must have emitted a gluon carrying the colors red and antigreen. Like the electric charge, color charge is always conserved [75].

For every matter particle there corresponds an antimatter particle. Antimatter particles correspond to matter particles in every respect except that they have opposite charge, spin and chemical potential. An antielectron (positron) has the same mass as an electron, but it is electrically positive. Antiquarks have electrical charges $-\frac{2}{3}$ and $+\frac{1}{3}$. Associated with the antiquarks, however, there are anticolor charges: antired, antigreen and antiblue. An antiproton is composed of two up-antiquarks and one down-antiquark [75].

Appendix B

Quantum Distributions from Maximum Entropy Principle

I. The Standard Maximum Entropy Principle

In standard quantum mechanical statistics, the entropic measure is given by [40, 41]

$$S = -\sum_i [\bar{n}_i \ln \bar{n}_i \mp (1 \pm \bar{n}_i) \ln(1 \pm \bar{n}_i)], \tag{B.1}$$

where the upper and lower signs correspond to bosons and fermions, respectively, and $\bar{n}_i$ denotes the number of particles in the i^{th} energy level with energy ϵ_i. The extremization of the above measure under the constraints imposed by the total number of particles,

$$\sum_i \bar{n}_i = N \tag{B.2}$$

and the total energy of the system,

$$\sum_i \bar{n}_i \, \epsilon_i = E, \tag{B.3}$$

leads to a variational problem

$$\frac{\delta}{\delta \bar{n}_i} \left[S + \alpha \left(N - \sum_i \bar{n}_i \right) + \beta \left(E - \sum_i \epsilon_i \, \bar{n}_i \right) \right] = 0 \,. \tag{B.4}$$

Substituting (B.1) in (B.4), one obtains

$$[\ln(1-\bar{n}_i) - \ln \bar{n}_i - (\alpha + \beta\, \epsilon_i)] = 0 \tag{B.5}$$

for fermions and

$$[\ln(1+\bar{n}_i) - \ln \bar{n}_i - (\alpha + \beta\, \epsilon_i)] = 0 \tag{B.6}$$

for bosons.

Rearranging the terms in (B.5) and (B.6) and introducing a chemical potential

$$\mu = -\frac{\alpha}{\beta}, \tag{B.7}$$

yields

$$\bar{n}_i = \frac{1}{\exp \beta(\epsilon_i - \mu) \mp 1}, \tag{B.8}$$

where the upper and lower signs correspond to the Bose-Einstein and Fermi-Dirac distributions, respectively.

II. The Non-extensive Maximum Entropy Principle for Fermions

The extended measure of entropy for fermions is given by [33, 39]

$$S_q^{(F)} = \sum_i \left[\frac{\bar{n}_i - \bar{n}_i^q}{q-1} + \frac{(1-\bar{n}_i) - (1-\bar{n}_i)^q}{q-1} \right], \tag{B.9}$$

which for $q \to 1$ reduces to the entropic functional (B.1) (with lower signs).

Maximizing the extended measure of entropy in (B.9) subject to the constraints

$$\sum_i \bar{n}_i^q = N \tag{B.10}$$

and

$$\sum_i \bar{n}_i^q \, \epsilon_i = E \tag{B.11}$$

leads to the variational problem

$$\frac{\delta}{\delta \bar{n}_i} \left[S_q^{(F)} + \alpha \left(N - \sum_i \bar{n}_i^q \right) + \beta \left(E - \sum_i \epsilon_i \, \bar{n}_i^q \right) \right] = 0 \, . \tag{B.12}$$

Following the same procedure as in (I) yields

$$\bar{n}_i = \frac{1}{[1 + (q-1)\beta(\epsilon_i - \mu)]^{\frac{1}{q-1}} + 1} . \tag{B.13}$$

In the limit $q \to 1$ one recovers the usual Fermi-Dirac distribution (B.8) (with lower sign).

III. The Bosonic Problem

The quantum mechanical distribution function proposed by Buyukklic and Demirhan (BD) [42] is given by

$$\bar{n}_i = \frac{1}{[1 + (q-1)\beta(\epsilon_i - \mu)]^{\frac{1}{q-1}} \mp 1} , \tag{B.14}$$

where the upper and lower signs correspond to bosons and fermions, respectively. Ever since the BD proposal, the two cases suggested in (B.14) (that is, the fermionic and bosonic case) were regarded as sharing the same degree of validity. On the basis of [39], the fermionic and bosonic BD distributions do not stand on an equal footing. First of all, each term in the entropic functional (B.9) gives the correct expression, within the context of Tsallis non-extensive thermostatistics, for the entropy of one single fermionic oscillator (in thermal equilibrium) in terms of its average occupation number $\bar{n}_i$. Secondly, the entropic functional (B.9) admits a reasonable "probabilistic" interpretation (see Sommerfeld [41] for a discussion of this point in the standard q=1 case). $\bar{n}_i$ can be regarded as the probability of the i^{th} state being occupied and $(1-\bar{n}_i)$ as the probability of that state being empty.

Extremizing the entropic functional,

$$S_q^{(B)} = \sum_i \left[\frac{\bar{n}_i - \bar{n}_i^q}{q-1} - \frac{(1+\bar{n}_i) - (1+\bar{n}_i)^q}{q-1} \right], \tag{B.15}$$

under the constraints imposed by (B.10) and (B.11) yields the q-generalized BD Bose-Einstein distribution, which is given by (B.14) with the minus sign. A "probabilistic" interpretation of the entropic measure (B.15) and the associated variational procedure leading to BD approach distribution for bosons (B.14) is somewhat problematic [39]. Let us denote by $f_{i,n}$ the probability of having n bosons in the state i with energy ϵ_i. If we try to follow the steps of Sommerfeld's probabilistic approach for bosons [41], we should start by extremizing the functional

$$S_q = \sum_{i,n} \frac{f_{i,n} - f_{i,n}^q}{q-1}, \tag{B.16}$$

under the set of constraints

$$\sum_n f_{i,n} = 1, \qquad \forall i\,,$$

$$\sum_{n,i} f_{i,n}^q n = N$$

and

$$\sum_{n,i} f_{i,n}^q \, n \, \epsilon_i = E\,. \tag{B.17}$$

For the sake of simplicity we use unnormalized q-constraints here, but the same conclusions would obtain if normalized q-constraints are used instead (notice that the quantities $f_{i,n}$ are not occupation numbers. They are true probabilities and, strictly speaking, the associated mean values should be normalized). The set in (B.17) does not consist of just three constraints. There is one "normalization" constraint for each i. So, in principle, we are dealing with an infinite set of constraints. Accordingly, we have to introduce the Lagrange multipliers,

$$\alpha,\, \beta,\, \lambda_i\,, \tag{B.18}$$

where there is one λ_i for each state i. The variational principle then reads,

$$\frac{\delta}{\delta f_{i,n}} \sum_{i,n} \left(\frac{f_{i,n} - f_{i,n}^q}{q-1} - \lambda_i\, f_{i,n} - \alpha\, n\, f_{i,n}^q - \beta\, n\, \epsilon_i\, f_{i,n}^q \right) = 0. \tag{B.19}$$

The above variational problem leads to

$$\frac{1 - q\, f_{i,n}^{q-1}}{q-1} - \lambda_i - q\, \alpha\, n\, f_{i,n}^{q-1} - q\, \beta\, n\, \epsilon_i\, f_{i,n}^{q-1} = 0\,, \tag{B.20}$$

which can be solved for $f_{i,n}$, yielding

$$f_{i,n} = \left[\frac{q}{1 + \lambda_i\,(1-q)} \right]^{\frac{1}{1-q}} \left[1 - (1-q)(\alpha\, n + \beta\, n\, \epsilon_i)\right]^{\frac{1}{1-q}}\,. \tag{B.21}$$

The mean occupation number of the state i would then be given by

$$\bar{n}_i = \sum_{n=0}^{\infty} n\, f_{i,n}^q\,, \tag{B.22}$$

where $f_{i,n}$ is given by (B.21). Now, the difficulty is that the sum appearing in (B.22) is not equal to the BD expression for the q-generalized Bose-Einstein distribution. The sum in (B.22) cannot, in general, be reduced to a finite, closed, analytical expression. The alluded to summation can be done analytically only for q=1, and it leads to the standard Bose-Einstein distribution [41].

Appendix C

Evaluation of Integrals

In BG statistics, the energy density, pressure and baryon number density of fermions and bosons can be derived as follows[1]:

Fermions

The energy of fermions is given by

$$E = d \int\int \frac{d^3q\, d^3p\, \epsilon}{\exp \frac{1}{T}(\epsilon - \mu) + 1}, \tag{C.1}$$

where

- ϵ is relativistic energy
- d is degeneracy factor
- μ is chemical potential
- T is temperature, and
- $d^3q\, d^3p$ is the element of phase space.

[1]For the sake of simplicity the Boltzmann's constant k_B, the reduced Planck's constant $\hbar$ and the speed of light c are set equal to one.

The integral over the phase space yields

$$E = \frac{V\,d}{2\,\pi^2}\int_0^\infty \frac{dp\,p^2\,\epsilon}{\exp\frac{1}{T}(\epsilon-\mu)+1}\,, \tag{C.2}$$

where

- V is volume and
- $\boldsymbol{p}$ is momentum.

The relativistic energy is given by

$$\epsilon^2 = k^2 + m^2\,, \tag{C.3}$$

where $\boldsymbol{k}$ is wave vector and m is rest mass.

For relativistic fermions with m=0,

$$\epsilon = k\,. \tag{C.4}$$

Substituting ϵ by k and $dp\,p^2$ by $dk\,k^2$ in (C.2), one obtains

$$E = \frac{V\,d}{2\,\pi^2}\int_0^\infty \frac{dk\,k^3}{\exp\frac{1}{T}(k-\mu)+1}\,. \tag{C.5}$$

Therefore, the energy density, that is energy per unit volume, is given by

$$u = \frac{d}{2\,\pi^2}\int_0^\infty \frac{dk\,k^3}{\exp\frac{1}{T}(k-\mu)+1}\,. \tag{C.6}$$

The expressions for the pressure and baryon number density can be derived in a similar manner and are given by

$$P = \frac{d\,T}{2\,\pi^2}\int_0^\infty dk\,k^2\left\{\ln\left[1+\exp\frac{1}{T}(\mu-k)\right]\right\} \tag{C.7}$$

and

$$n = \frac{d}{2\,\pi^2}\int_0^\infty \frac{dk\,k^2}{\exp\frac{1}{T}(k-\mu)+1}\,. \tag{C.8}$$

Integration by parts of (C.7) yields

$$P = \frac{1}{3}\,u\,. \tag{C.9}$$

Bosons

The energy of bosons is given by

$$E = d \int\int \frac{d^3q\, d^3p\, \epsilon}{\exp\left(\frac{\epsilon}{T}\right) - 1}. \tag{C.10}$$

The integral over the phase space yields

$$E = \frac{V\,d}{2\,\pi^2} \int_0^\infty \frac{dp\, p^2\, \epsilon}{\exp\left(\frac{\epsilon}{T}\right) - 1}. \tag{C.11}$$

For relativistic bosons with m=0, the energy density is given by

$$u = \frac{d}{2\,\pi^2} \int_0^\infty \frac{dk\, k^3}{\exp\left(\frac{k}{T}\right) - 1}. \tag{C.12}$$

Letting $x = \frac{k}{T}$, one obtains

$$u = \frac{d\,T^4}{2\,\pi^2} \int_0^\infty \frac{dx\, x^3}{\exp(x) - 1}. \tag{C.13}$$

Evaluating the integral yields

$$u = \frac{1}{30}\, d\, \pi^2\, T^4\,. \tag{C.14}$$

The pressure, which is $P = \frac{1}{3}\,u$, is given by

$$P = \frac{1}{90}\, d\, \pi^2\, T^4\,. \tag{C.15}$$

The energy density residing in the quarks alone, or the antiquarks alone, can not be calculated analytically in the general case, $\mu, T \neq 0$. However, the sum of both yields a simple analytical formula as can be derived in the following way:

The energy density carried by quarks is given by

$$u_Q = \frac{d_Q}{2\,\pi^2} \int_0^\infty \frac{dk\, k^3}{\exp\beta(k - \mu) + 1}. \tag{C.16}$$

Letting $x = \beta(k-\mu)$, one obtains

$$u_Q = \frac{d_Q}{2\,\pi^2\,\beta^4}\int_{-\beta\,\mu_Q}^{\infty}\frac{dx\,(x+\beta\,\mu_Q)^3}{\exp(x)+1}. \tag{C.17}$$

The only change occurring in the same expression for antiquarks is the replacement of μ by (-μ). Up on letting $x = \beta(k+\mu)$, one obtains

$$u_{\bar{Q}} = \frac{d_Q}{2\,\pi^2\,\beta^4}\int_{\beta\,\mu_Q}^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1}. \tag{C.18}$$

The integral in (C.18) can be split up in the following way:

$$\int_{\beta\,\mu_Q}^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} = \int_{0}^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} - \int_{0}^{\beta\,\mu_Q}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1}. \tag{C.19}$$

Substituting x by (-x) in the second integral and using the relation

$$\frac{1}{\exp(-x)+1} = 1 - \frac{1}{\exp(x)+1} \tag{C.20}$$

yields

$$\int_{\beta\,\mu_Q}^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} = \int_{0}^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} + \int_{0}^{-\beta\,\mu_Q}\frac{dx\,(x+\beta\,\mu_Q)^3}{\exp(x)+1} - \int_{0}^{-\beta\,\mu_Q} dx\,(x+\beta\,\mu_Q)^3. \tag{C.21}$$

Therefore, the energy density of the QGP, which is the sum of the energy density of quarks, antiquarks and gluons is given by

$$u_{QGP} = \frac{d_Q}{2\,\pi^2\,\beta^4}\left\{\int_{-\beta\,\mu_Q}^{\infty}\frac{dx\,(x+\beta\,\mu_Q)^3}{\exp(x)+1} + \int_{0}^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} + \int_{0}^{-\beta\,\mu_Q}\frac{dx\,(x+\beta\,\mu_Q)^3}{\exp(x)+1} + \int_{-\beta\,\mu_Q}^{0} dx\,(x+\beta\,\mu_Q)^3\right\} + \frac{d_G}{2\,\pi^2\,\beta^4}\int_{0}^{\infty}\frac{dx\,x^3}{\exp(x)-1} + B. \tag{C.22}$$

Combining the first and the third integrals yields

$$u_{QGP} = \frac{d_Q}{2\,\pi^2\,\beta^4}\left\{\int_{0}^{\infty}\frac{dx\,(x+\beta\,\mu_Q)^3}{\exp(x)+1} + \int_{0}^{\infty}\frac{dx\,(x-\beta\,\mu_Q)^3}{\exp(x)+1} + \right.$$

$$\int_{-\beta\,\mu_Q}^{0} dx\,(x+\beta\,\mu_Q)^3\Bigg\} + \frac{d_G}{2\,\pi^2\,\beta^4}\int_0^\infty \frac{dx\,x^3}{\exp(x)-1} + B\,. \tag{C.23}$$

Expanding and rearranging these integrals, one obtains

$$u_{QGP} = \frac{d_Q}{2\,\pi^2\,\beta^4}\Bigg\{2\int_0^\infty \frac{dx\,x^3}{\exp(x)+1} + 6\,\beta^2\,\mu_Q^2\int_0^\infty \frac{dx\,x}{\exp(x)+1} +$$
$$\int_{-\beta\,\mu_Q}^{0} dx\,(x+\beta\,\mu_Q)^3\Bigg\} + \frac{d_G}{2\,\pi^2\,\beta^4}\int_0^\infty \frac{dx\,x^3}{\exp(x)-1} + B\,. \tag{C.24}$$

Evaluation of these integrals yields

$$u_{QGP} = \frac{\pi^2}{30}\left(d_G + \frac{7}{4}\,d_Q\right)T^4 + \frac{d_Q\,\mu^2\,T^2}{36} + \frac{d_Q\,\mu^4}{648\,\pi^2} + B\,. \tag{C.25}$$

The pressure, which is $\frac{1}{3}\,(u_{QGP} - 4B)$, is given by

$$P_{QGP} = \frac{\pi^2}{90}\left(d_G + \frac{7}{4}\,d_Q\right)T^4 + \frac{d_Q\,\mu^2\,T^2}{108} + \frac{d_Q\,\mu^4}{1944\,\pi^2} - B\,. \tag{C.26}$$

The baryon number density is given by

$$n_{QGP} = \frac{1}{3}\,(n_Q - n_{\bar{Q}})\,, \tag{C.27}$$

where

$$n_Q = \frac{d_Q}{2\,\pi^2}\int_0^\infty \frac{dk\,k^2}{\exp\beta(k-\mu)+1}$$

and

$$n_{\bar{Q}} = \frac{d_Q}{2\,\pi^2}\int_0^\infty \frac{dk\,k^2}{\exp\beta(k+\mu)+1}\,.$$

Using the same procedure as the energy density, it can be shown that

$$n_{QGP} = d_Q\left(\frac{\mu\,T^2}{54} + \frac{\mu^3}{486\,\pi^2}\right). \tag{C.28}$$