Chapter 2

Generalized Statistics

New developments in statistical mechanics have shown that in the presence of *long-range* forces and/or in irreversible processes related to microscopic long-time memory effects, the extensive thermodynamics, based on the conventional BG thermostatistics, may not be correct and, consequently, the equilibrium particle distribution functions can show different shapes from the conventional well-known distributions [27, 28].

An interesting generalization of the conventional BG statistics has been proposed by Tsallis [13] and proves to be able to overcome the shortcomings of the conventional statistical mechanics in many physical problems, where the presence of *long-range* interactions, *long-range* microscopic memory, or *fractal space-time* constraints hinders the usual statistical assumptions.

In the past few years the non-extensive form of statistical mechanics proposed by Tsallis has found applications in astrophysical self-gravitating systems [29], solar neutrinos [30, 31], high energy nuclear collisions [27, 28], cosmic microwave back ground radiation [32], high temperature superconductivity [33, 34] and many others. In these cases a small deviation of the Tsallis parameter q ($\approx 10\%$) from one (BG statistics) reduces the discrepancies between experimental data and theoretical models.

The generalized entropy proposed by Tsallis [13] takes the form:

$$S_q = \mathcal{K}\left(\frac{1-\sum_{i=1}^{W} p_i^q}{q-1}\right) \qquad (q \in \Re), \tag{2.1}$$

where $\mathcal{K}$ is a positive constant (from now on set equal to 1), W is the total number of microstates in the system, p_i are the associated probabilities with $\sum_{i=1}^{W} p_i = 1$, and the Tsallis parameter (q) is a real number.

The new entropy has the usual properties of positivity, equiprobability, concavity and irreversibility, preserves the whole mathematical structure of thermodynamics (Legendre transformations) and reduces to the conventional BG logarithmic entropy, $S = -\sum_{i=1}^{W} p_i \ln p_i$, in the limit $q \to 1$. Only in this limit is the ensuing statistical mechanics extensive [13, 26, 35]. For general values of q, the measure S_q is non-extensive. That is, the entropy of a composite system $A \oplus B$ consisting of two subsystems A and B, which are statistically independent in the sense that $p_{i,j}^{(A\oplus B)} = p_i^{(A)} p_j^{(B)}$, is not equal to the sum of the individual entropies associated with each subsystem. Instead, the entropy of the composite system is given by Tsallis' q-additive relation [13],

$$S_q(A \oplus B) = S_q(A) + S_q(B) + (1-q) S_q(A) S_q(B)\,. \tag{2.2}$$

The quantity $|1-q|$ can be regarded as a measure of the degree of non-extensivity exhibited by S_q.

Suppose that the set of W microstates is arbitrarily separated into two subsets having W_L and W_M microstates ($W_L + W_M = W$) and define their corresponding probabilities as $p_L \equiv \sum_{i=1}^{W_L} p_i$ and $p_M \equiv \sum_{i=W_L+1}^{W} p_i$ with $p_L + p_M = 1$. It can be shown that [36]

$$S_q(\{p_i\}) = S_q(p_L, p_M) + p_L^q\, S_q(\{p_i/p_L\}) + p_M^q\, S_q(\{p_i/p_M\}), \tag{2.3}$$

where the sets $\{p_i/p_L\}$ and $\{p_i/p_M\}$ are the conditional probabilities. This is a generalization of the famous Shanon's property except for the appearance of p_L^q and p_M^q instead of p_L and p_M in the second and third terms of the right hand side of (2.3). Since the probabilities $\{p_i\}$ are normalized, $p_i^q > p_i$ for $q < 1$ and $p_i^q < p_i$ for $q > 1$. As a consequence the values $q < 1$ ($q > 1$) will favor rare (frequent) events, respectively [27, 28].

Starting from the one parameter deformation of the exponential function $\exp_{\{\kappa\}}(x) = (\sqrt{1+\kappa^2x^2} + \kappa x)^{\frac{1}{\kappa}}$, a generalized statistical mechanics has been recently constructed by Kaniadakis [37, 38], which reduces to the ordinary BG statistical mechanics as the deformation parameter κ approaches to zero. The difference between Tsallis and Kaniadakis statistics is that: Tsallis statistics is non-extensive and reduces to BG statistics (extensive) as the Tsallis parameter q tends to one. On the other hand, Kaniadakis statistics is extensive and tends to BG statistics as the deformation parameter κ tends to zero. The distribution functions for fermions and bosons can be derived from maximum entropy principle [37, 39]. The κ-entropy is linked to Tsallis entropy $S_q^{(T)}$ through the following relationship [37]:

$$S_\kappa = \frac{1}{2}\frac{\alpha^\kappa}{1+\kappa}S_{1+\kappa}^{(T)} + \frac{1}{2}\frac{\alpha^{-\kappa}}{1-\kappa}S_{1-\kappa}^{(T)} + const. \tag{2.4}$$

where α is a real positive constant. Here we consider the generalized statistics proposed by the Tsallis to represent the dominant part of the *long-range* interactions among the constituents in the QGP[1].

The standard quantum mechanical distributions can be obtained from a maximum entropy principle based on the entropic measure [40, 41],

$$S = -\sum_i[\bar{n}_i \ln \bar{n}_i \mp (1 \pm \bar{n}_i)\ln(1 \pm \bar{n}_i)], \tag{2.5}$$

[1]The phase diagram for Kaniadakis statistics is shown in fig. 3.8.

where the upper and lower signs correspond to bosons and fermions, respectively, and $\bar{n}_i$ denotes the number of particles in the i^{th} energy level with energy ϵ_i. The extremization of the above measure under the constraints imposed by the total number of particles,

$$\sum_i \bar{n}_i = N \tag{2.6}$$

and the total energy of the system,

$$\sum_i \bar{n}_i \epsilon_i = E, \tag{2.7}$$

leads to the standard quantum distributions (see Appendix B),

$$\bar{n}_i = \frac{1}{\exp \beta(\epsilon_i - \mu) \mp 1}, \tag{2.8}$$

where $\beta = \frac{1}{T}$, μ is the chemical potential which is associated with the number of particles and the upper and lower signs correspond to the Bose-Einstein and Fermi-Dirac distributions, respectively.

To deal with non-extensive scenarios (characterized by $q \neq 1$), the extended measure of entropy for fermions proposed in [33, 39] is:

$$S_q^{(F)} = \sum_i \left[\frac{\bar{n}_i - \bar{n}_i^q}{q-1} + \frac{(1-\bar{n}_i) - (1-\bar{n}_i)^q}{q-1} \right], \tag{2.9}$$

which for $q \to 1$ reduces to the entropic functional (2.5) (with lower signs).

The constraints

$$\sum_i \bar{n}_i^q = N \tag{2.10}$$

and

$$\sum_i \bar{n}_i^q \epsilon_i = E \tag{2.11}$$

lead to (see Appendix B)

$$\bar{n}_i = \frac{1}{[1 + (q-1)\beta(\epsilon_i - \mu)]^{\frac{1}{q-1}} + 1}. \tag{2.12}$$

In the limit $q \to 1$ one recovers the usual Fermi-Dirac distribution (2.8) (with lower sign).

Similarly,

$$\bar{n}_i = \frac{1}{[1 + (q-1)\beta(\epsilon_i - \mu)]^{\frac{1}{q-1}} - 1} \tag{2.13}$$

for bosons.