Chapter 1

Introduction

The analysis of modern designs and systems is becoming ever more complex. Closed form analytical solutions to problems, although extremely important, are often limited to relatively simple geometries or restricted to specific loadings, and are therefore difficult to generalise to industrial applications. For some time now numerical solution techniques, implemented on digital computers, have been an invaluable tool in the analysis of complex physical problems. Indeed, commercial, general purpose, finite element analysis (FEA) and computational fluid dynamics (CFD) software are now commonplace in the engineering community.

Although numerical methods are commonly used commercially in the *analysis* of mechanical designs, they are often not fully exploited in the *design process* itself. Generally this process starts with the designer conceiving an initial concept design which is analysed, possibly using numerical methods, and the results are judged based on predetermined objectives or criteria. If the design is not satisfactory the designer then, based on past experience, predicts an improved design. This process is repeated until a satisfactory design is found, upon which the process is terminated. Some of the disadvantages of such a procedure include:

- There is no guarantee that the design is optimal, or even good, although it satisfies the prescribed requirements. That is to say, unknown to the designer, significant improvements may still be possible.
- The process relies on the experience of the designer (and often trial-and-error) to predict design improvements. This process may not be systematic, and in fact the designer's intuition or experience could at times be misleading.
- The process is not repeatable, and is difficult to describe as a fixed procedure. If a designer gains experience with a given problem, it is would take a significant effort to pass this experience on, or document a procedure, to repeatably come up with a good design for similar problems.

Structural optimization holds the potential to improve upon or optimize designs conceived by designers in a *systematic*, *repeatable* manner. Furthermore, using structural topology optimization, the design process itself can largely be described and automated[1]. Structural

[1] Within reason and for specific components only.

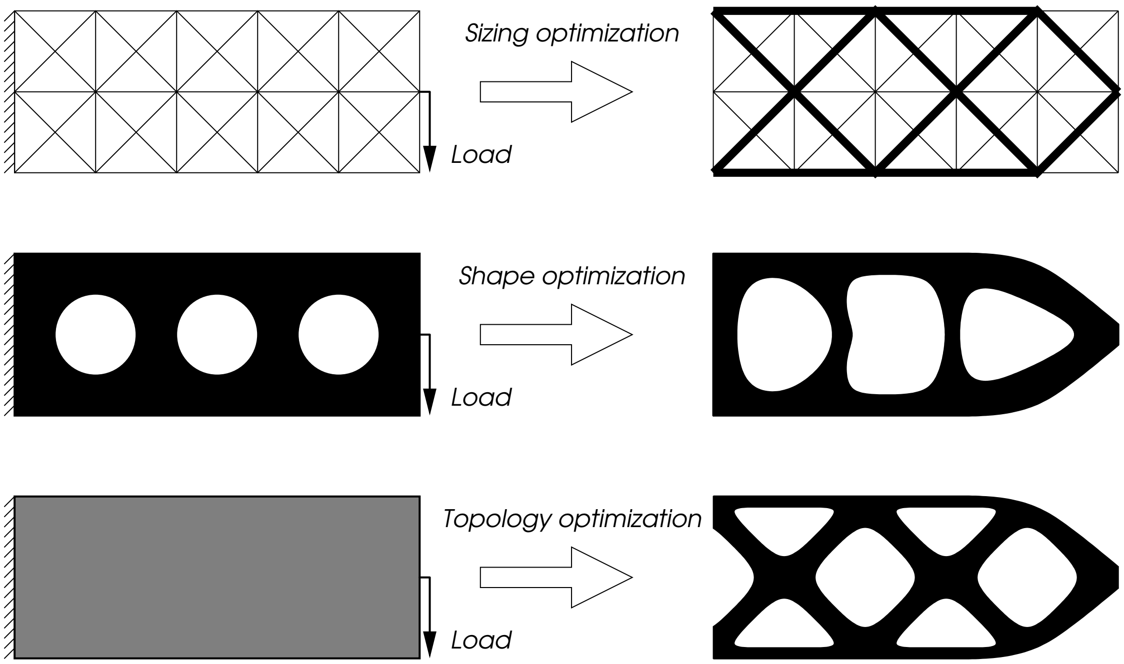

Figure 1.1: Three categories of structural optimization. Initial design and problem description on the left, optimal designs on the right. Note that optimal designs in this figure are for illustrative purposes only.

optimization is therefore an important tool which can assist in finding not only good, but *optimal* solutions to structural design problems.

1.1 Structural topology optimization

Structural optimization seeks to find the *best* design (out of all possible designs) which is capable of satisfying a number of prescribed criteria. The measure of how good the design is and the criteria which the design is required to fulfil are usually conflicting. A simple example of such a problem is to determine the stiffest possible structure, under given loading and support conditions, such that the weight of the structure is limited. The requirements are conflicting because physically the more material the structure contains, the stiffer it will generally be, but the heavier it becomes.

Broadly speaking, structural optimization problems can be classified into one of three categories, namely *sizing*, *shape* or *topology* optimization. These three classes are schematically depicted in Figure 1.1.

Sizing optimization generally uses truss or grillage member cross-sectional areas, or membrane, plate or shell component thicknesses as design variables in the optimization process. More specifically, in sizing optimization problems the shape and topology of the design analysis domain considered are fixed, and do not change during the optimization process, as shown in Figure 1.1. Therefore, if a finite element model is used, the nodes remain in fixed positions throughout the process.

On the other hand, shape optimization generally involves finding the analysis domain shape which optimally performs a given function, subject to certain constraints. The design domain is usually determined parametrically by the design variables. If gradient based optimization methods are employed, considerable difficulties would be encountered if holes were to merge during the process, and it is certainly not a trivial task to determine the effect of adding or removing holes *a priori*, or indeed to determine *where* to introduce additional holes if necessary. Therefore, during this process the topology is generally fixed. For example, in the shape optimization illustration in Figure 1.1, the starting design has three holes and the final design also has three holes.

Finally, topology optimization can be considered the most general type of structural optimization. In topology optimization, the optimal boundary and connectivity, as well as the optimal size, shape, location and number of features (including holes) in an analysis domain are sought. Topology optimization is therefore also sometimes referred to as generalised shape or layout optimization.

These classes of problems can be combined in procedures to exploit the advantages of the different techniques. For example, some authors have combined shape and topology optimization to take advantage of the salient features of both classes simultaneously, especially in the design of generally curved shell structures [1, 2]. In this work, however, attention is exclusively focused on the topology optimization problem.

Furthermore, the optimization of low volume fraction, inherently discrete, structures (such as truss structures) is not considered, and attention is focused on continuum structural problems only. Having said that, much of the theory and many techniques employed in the optimization of continuum structures are derived from the topology optimization of truss and grillage problems, see [3, 4] for examples. Furthermore, many of the difficulties encountered are similar, for example singularity problems with stress constraints [4, 5, 6, 7].

The iterative process of structural topology optimization used in this study is schematically depicted in Figure 1.2. In the remainder of this section, an informal description of the process is presented in more detail for readers unfamiliar with the process.

Initial design and design parameterization

The first step in the process is to select an initial or starting design. This design could be based on the experience of the designer, but could also simply be randomly selected within a given design domain or could be the result of a previous (more general) optimization procedure.

At this stage the parameters that describe the design, and which will be allowed to change during the process (the *design variables*, $\boldsymbol{x}$) are defined. The initial design is accordingly described by an initial design vector $\boldsymbol{x}^0$. The type of parameterization chosen will dictate (or is dictated by) the type or category of structural optimization that is employed. The design variables used in topology optimization typically describe the material distribution within the design domain using a density function.

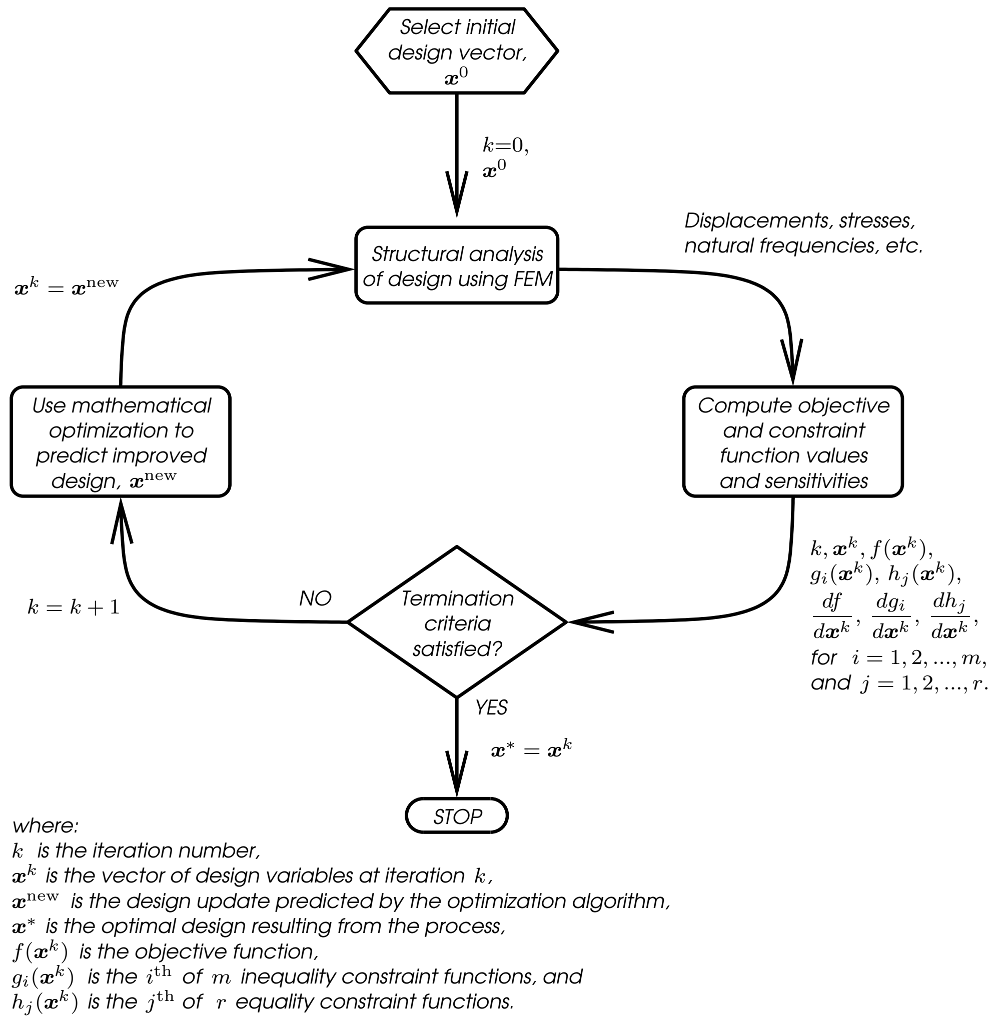

Figure 1.2: Schematic of the process of structural topology optimization.

Structural analysis

After the physical problem has been identified and a representative mathematical model constructed, the initial design can be analysed. If necessary, simplifying assumptions are made in the construction of the mathematical model. Often, the model of the physical problem results in a system of differential equations. These equations can be solved using classical analytical methods, the finite element method (FEM), the boundary element method (BEM) or any of numerous other numerical methods. In this work, the finite element method is exclusively used to perform the structural analyses.

Design evaluation and sensitivity analysis

The next step in the process involves the computation of the measure of how good the design is (the *objective function* denoted $f(\boldsymbol{x})$) and the criteria which the design is required to fulfil (the *constraint functions*, with inequality and equality constraints denoted $g_i(\boldsymbol{x})$ and $h_j(\boldsymbol{x})$

respectively).

Often the structural analysis is rather general, especially if general purpose finite element software is used, and it may therefore be necessary to post-process the results in order to extract the specific information used to construct the objective and constraint functions.

Furthermore, if a gradient based optimization procedure is to be employed, gradient or sensitivity information is also required. The process by which sensitivity information is obtained is usually referred to as design *sensitivity analysis*. For the problems considered in this work, sensitivity information is relatively easily and numerically inexpensively obtained, and therefore solely gradient based methods will be used.

Systematic design improvement

The step in which a prediction is made for an improved design is at the heart of the structural optimization process. Mathematical optimization techniques are employed to systematically generate a series of improving designs. Mathematical optimization can be described as consisting of the formulation and the solution of a constrained optimization problem of the general mathematical form:

$$\begin{aligned} &\min_{\boldsymbol{x}} \ f(\boldsymbol{x}) \\ \text{subject to : } &g_i(\boldsymbol{x}) \leq 0, i = 1, 2, ..., m \\ &h_j(\boldsymbol{x}) = 0, j = 1, 2, ..., r \end{aligned} \tag{1.1}$$

where $\boldsymbol{x} = [x_1, x_2, ..., x_n]^T \in \mathbb{R}^n$ is the vector of real numbered design variables. The real scalar functions f, g and h represent the objective function and inequality and equality constraint functions respectively. For more details see for example Haftka *et al.* [8] or Snyman [9].

Over the years, many techniques have been developed to solve the optimization problem, each well suited to particular problems. Many of the classical methods for solving convex problems are gradient-based, including classical line search methods and sequential approximation methods.

In this study, two different gradient based methods are used, namely the Method of Moving Asymptotes (MMA) of Svanberg [10], and the well known heuristic updating scheme based on Optimality Criteria (OC) [4].

Stopping criterion

The above process is repeated until some termination criterion has been satisfied, upon which the process is stopped. The stopping criterion can be based on the number of iterations performed, the norm of the change in function value, the norm of the change in design variables, the gradient of the objective function or any combination of the above. If it is not possible to further improve upon this design with a given algorithm, it is considered a *local optimal* solution, usually denoted $\boldsymbol{x}^*$.

For some classes of problems, such as problems with strictly convex objective and constraint functions, this local optimum may be shown to also be the *global optimal* solution. However, if this is not possible (as is the case with topology optimization) this local optimum is often accepted as a reasonable approximation to the global optimum.

1.2 Background to the study

Arguably one of the most popular and immediately useful applications of structural topology optimization is in the synthesis of compliant mechanisms. Whereas traditional mechanisms employ rigid links connected by movable joints, compliant mechanisms are mechanical devices used to transfer or transform motion, force or energy via mobility gained through the deflection of flexible members. Manually designing compliant mechanisms to carry out all but the simplest of tasks is a challenging exercise. Therefore, finite element modelling and topology optimization have become increasingly useful design tools of late.

Compliant mechanisms have a number of advantages over conventional jointed mechanisms. For example, they have fewer moving parts, reducing or eliminating backlash, play, noise as well as weight. Furthermore, compliant mechanisms are easily miniaturized making them ideal for use in Micro Electro Mechanical Systems (MEMS) or precision engineering applications where smart materials are regularly used as transducers.

Piezoelectric materials in particular are very attractive as actuator smart materials for compliant mechanisms due to their high energy density, large force capacity and excellent operational bandwidth. However their low induced strain (typically in the order of 0.1−0.2%[2]) means that output displacements are small, limiting their direct application. To convert these small induced strains to usable displacements, piezoelectric ceramics commonly employ some form of mechanical amplification. Often, this mechanical amplification takes the form of a compliant mechanism.

In 2003 the Department of Mechanical Engineering, and in particular the Structural Optimization Research Group (SORG) at the University of Pretoria, and the Centre for Integrated Sensing Systems (CISS)[3] at the Council for Scientific and Industrial Research (CSIR) in South Africa became jointly involved in a project aimed at investigating the development of novel finite elements and structural optimization techniques for application in compliant, piezoelectrically actuated, micropositioning systems.

The goal of this study was to develop accurate finite elements, or procedures used in their calculation, which could be used for (but are not limited to) the modelling of piezoelectrically driven compliant mechanisms. Furthermore, the study aimed to show how the salient features of these specially developed finite elements and procedures could be exploited in a topology optimization environment to overcome common numerical instabilities such as checkerboarding. In the remainder of this section a very brief background to relevant topics in topology optimization and the finite element method, is presented.

[2]However, new relaxor ferroelectric single crystals (PZN-PT and PMN-PT) can reportedly deliver in excess of 1% strain [11, 12].

[3]Now known as Sensor Science and Technology (SST).

1.2.1 Topology optimization

In topology optimization, there are several acknowledged fundamental, theoretical issues which need to be appropriately addressed if sensible results are to be achieved [13]. These issues include *non-existence* of the solution (mesh dependency), *multiple local optima*, and *non-uniqueness* of the solution.

Firstly, it is well known that the 0-1 problem statement in topology optimization lacks existence of solution in a continuum setting [4]. This problem exists due to a lack of closedness of the set of admissible designs.[4] In a discrete (finite element) setting, this problem manifests itself as a mesh dependency problem. One possible way to deal with this problem is to *a priori* allow composite materials constructed from the original isotropic material (and void). This method extends the design space and sufficiently *relaxes* the original problem. Alternatively, the design space can be *restricted* by limiting local or global variations in material distribution, thereby sufficiently closing the the set of admissible designs.

Another common complication is that of multiple (local) optima. If one observes the many different optimal solutions which have been published for benchmark problems, for example the well known MBB beam problem, it is clear that there are many local optima present. This is due to the fact that most topology optimization problems are non-convex. Unfortunately, there is no way to overcome this problem, although a popular method to alleviate non-convexity is the use of *continuation methods*. Continuation methods gradually change problems from artificial (strictly) convex, or nearly convex, problems to the original non-convex problem.

Finally, problems with multiple globally optimal solutions are termed non-unique. An example commonly cited is that of a structure under uniaxial tension, in which only the cross-sectional area is of importance and not the topology. The only sensible way to deal with this problem is to impose manufacturing preference constraints.

Unfortunately these fundamental issues cannot be resolved solely through the use of finite element technology, no matter how sophisticated the finite element. Therefore attention in this study is rather focused on problems which occur *locally* in material distributions, and which can be attributed to the numerical deficiencies of the finite element model, such as *checkerboarding* and *one-node connected hinges*.

The checkerboarding problem is characterised by material in 'optimal topologies' being distributed in alternating solid and void elements, similar to the pattern created by the squares on a checkerboard. Checkerboarding is largely a result of poor numerical modelling of this spurious material distribution, as shown by Díaz and Sigmund [14]. In essence, the numerical behaviour of this material distribution is over-stiff and is therefore especially common in the solutions of minimum compliance problems.

The one-node connected hinge is characterised by four elements surrounding a node, where two diagonally opposite elements are solid and the other two are void, see for example

[4] As more holes are introduced without changing the volume of the structure, the efficiency of the structure is generally improved. Eventually, variations on the microstructural level are introduced (requiring composite material descriptions) which cannot be described by the original problem which permits isotropic material only.

Poulsen [15]. This material distribution is somewhat common in the design of compliant mechanisms such as those for which this work is ultimately aimed. This is due to the fact that in compliant mechanism design, solid state hinges are employed to achieve the required motion and the numerical model of a one-node hinge employing standard elements is ideal (albeit unrealistic) since it offers zero resistance to rotation about the common node.

For a more detailed introduction to the topology optimization problem, the reader is referred to Appendix A.

1.2.2 The finite element method

The finite element method (FEM) is essentially a numerical method for the solution of differential equations [16]. Originally, FEM gained popularity among engineers as a method for structural stress analysis. With the advent of ever faster and less expensive digital computers, FEM found application in other fields including heat transfer, fluid dynamics, as well as electric, piezoelectric and electromagnetic analysis. In fact, many commercial finite element codes now support these types of analyses as standard and some even have structural optimization algorithms included.

FEM topics for consideration in this study were identified based on the topology optimization problems selected for attention, namely checkerboarding and one-node connected hinges. Finite element procedures which could alleviated or eliminate the numerical modelling deficiencies associated with, or leading, to these material distributions were sought.

Since checkerboarding has been attributed to an over-stiff numerical model of a checkerboard layout, a procedure to effectively soften elements in this layout was sought. One common technique used to soften (especially higher order deformation modes) is to employ a reduced numerical integration scheme in the calculation of elemental quantities.

In the finite element method, the equilibrium equations involve integration over the element volume. This is also true for the expressions for consistent nodal loads, mass matrices, penalty matrices, etc. For simple elements the integrand may be formed explicitly, resulting in *exact* integration. However, numerical integration schemes are necessary when element geometries are distorted, of which the Gaussian rules are possibly the best known and most frequently employed. The effects of numerical integration schemes are summarized in a clear manner by Cook *et al.* [17].

A lower-order quadrature rule, called reduced integration, may be desirable for two reasons. Firstly, since the expense of generating the stiffness matrix by numerical integration is proportional to the number of sampling points, fewer points results in lower computational cost. Secondly, a low order rule tends to soften an element, thus countering the overly-stiff behavior associated with assumed displacement fields. (The displacement based finite element method is monotonically convergent from below.) Softening comes about because certain higher-order polynomial terms happen to vanish at Gauss points of a low-order rule. Simply stated, with fewer sampling points, some of the more complicated displacement modes offer less resistance to deformation.

Conversely, the numerical model of a one-node hinge comprised of standard displacement-based elements is known to possess little or no stiffness in rotation. In fact, for planar problems most elements do not possess in-plane rotational degrees of freedom at all. However, elements with in-plane rotational degrees of freedom do exist, and their use in schemes to effectively stiffen one-node connected hinges is evaluated.

In recent times, elements with in-plane rotational (drilling) degrees of freedom have become quite popular. Apart from enrichment of the displacement field, which increases element accuracy, drilling degrees of freedom allow for the modelling of, for instance, folded plates and beam-slab intersections.

The membrane elements used in this study account for in-plane rotations based on a continuum mechanics definition of rotation. The approach relies on a variational formulation employing an independent rotation field, as presented by Hughes and Brezzi [18]. It utilizes the skew-symmetric part of the stress tensor as a Lagrange multiplier to enforce equality of independent rotations and the skew-symmetric part of the displacement gradient in a weak sense. The stress tensor is therefore not *a priori* assumed to be symmetric.

1.3 Objectives of the study

There has been a steady stream of publications reporting on improvements and advances in finite element technology since the 1970's, when FEM was conceived as a general computer implementation [19]. Improvements can be judged on several measures, including accuracy, numerical robustness or computational cost. Unfortunately though, as with most things, there is 'no free lunch', and generally accuracy is traded for robustness or computational effort, or *vice versa*.

Advances in finite element technology are of particular interest to practitioners of structural optimization, since multiple finite element analyses (at least one per iteration) are required to determine the optimal structure. Procedures which increase accuracy or decrease computational effort are of significant benefit in such an iterative environment, and are therefore *per se* of interest.

However, finite element formulation does not only affect the accuracy (and/or numerical effort) required to perform the structural analyses in topology optimization, it can actually influence the resulting topology itself. For example, the use of higher order elements are known to lessen or eliminate checkerboarding, common in minimum compliance solutions employing 4-node planar elements. Although it may be argued that topology results containing checkerboarding are physically unreasonable, the conclusion is that under certain circumstances finite element formulation indeed affects the results of a topology optimization procedure.

The goal of this work is essentially to investigate this observation further. In principal, the intention is to develop new finite elements and/or procedures (or to identify existing elements and procedures) and to exploit their unique characteristics to overcome numerical instabilities or deficiencies in a topology optimization setting.

Firstly, traditional planar elements do not possess nodal rotational degrees of freedom. Therefore if two of these elements are connected at a single node, the assembly offers no resistance to rotation about the common node. Although this model of a compliant hinge is completely unrealistic and therefore undesirable in topological results, one-node hinges are regularly encountered in compliant mechanism design using topology optimization. Consequently, the idea to exploiting the nodal rotational stiffness associated with elements with drilling degrees of freedom to prevent (or at least improve the numerical model of) a one-node connected hinge is investigated.

Ultimately, the practical application of this investigation is in the design of piezoelectrically driven compliant mechanisms. Therefore new piezoelectric elements with drilling degrees of freedom are required. Furthermore, although it was decided to focus the finite element development on 2-D planar elements, planar membrane elements also form part of flat shell elements, and therefore developments in planar element technology are not restricted to planar problems. Accordingly, the effects of membrane (and plate) element formulations on flat shell topology optimization results are also investigated.

Secondly, since the model of a checkerboard patch of elements is commonly known to exhibit over-stiff behaviour, a method to soften the finite element model is sought. A popular method of softening (especially higher-order) deformation modes associated with elements is to employ reduced integration schemes. Subsequently, the notion of employing alternative reduced order integration schemes to soften patches of elements arranged in a checkerboard layout is investigated.

In summary, the primary objectives of this study are to:

1. Develop new finite elements, or elemental procedures, which not only improve model accuracy, robustness and/or efficiency, but can also be used in schemes which alleviate or eliminate the numerical instabilities, or improper modelling, associated with spurious material layouts in topology optimization. Attention is focused on the following topics in particular:

    - Determining the sensitivity of elements with drilling degrees of freedom to the penalty parameter, usually denoted γ.
    - The formulation, implementation and evaluation of planar piezoelectric finite elements with drilling degrees of freedom.
    - Investigating the use of reduced order integration rules in higher order elements to enhance element accuracy.

    Note that application of these finite element developments is not restricted to structural topology problems. Their increased accuracy, modelling capacity or numerical efficiency could also be applied in general purpose finite element codes.

2. To develop procedures which exploit the salient features of the new finite elements, or elemental procedures, in order to overcome or alleviate the numerical instabilities, or modelling deficiencies, leading to spurious material layouts in topology optimization. More specifically, the topics under consideration include:

    - Exploiting elements with drilling degrees of freedom in schemes to prevent undesirable local material distributions in topology optimization.

- Evaluating the effect of element formulation in plate and shell topology optimization problems.
- Quantifying the effect of reduced order integration rules on the stiffness of a checkerboard patch of elements.

1.4 Thesis overview and list of contributions

The thesis is divided into six main research chapters. Each of these chapters represent scientific contributions (in the form of conference or journal papers) made during the course of this work. As such, each chapter is intended to be self-contained and can be read independently of the other chapters. Note also that the notation in each chapter is therefore slightly different.

A schematic depiction of the thesis layout is presented in Figure 1.3. As indicated in the figure, the thesis body is divided into two parts. The first part contains Chapters 2 to 4, and details contributions made in finite element development and technology. The second, comprising Chapters 5 to 7, explores the application of finite element technology in a topology optimization environment. Furthermore, the finite element developments presented in Chapters 2 and 3 are related to topology optimization topics communicated in Chapters 5 and 6. Similarly the finite element developments offered in Chapter 4 find application in Chapter 7.

In **Chapter 2** an introduction to elements with drilling degrees of freedom (DOFs) is presented. Furthermore, the effect of the parameter which relates in-plane displacements and rotations, usually denoted γ, is studied. A unique feature of this study is the novel use of the skew symmetric part of the stress tensor to assess element accuracy. This contribution has been published in *Finite Elements in Analysis and Design*, with the full reference:

- C.S. Long, S. Geyer, and A.A. Groenwold. A numerical study of the effect of penalty parameters for membrane elements with independent rotation fields and penalized equilibrium. *Finite Elements in Analysis and Design*, 42:757–765, 2006.

Chapter 3 details the development of planar 4-node piezoelectric elements with drilling DOFs. Firstly, two families of variational formulations accounting for piezoelectricity and in-plane rotations are derived. The first retains the skew-symmetric part of the stress tensor, while in the second, the skew part of stress is eliminated from the functional. The finite elements derived from the variational formulations are then benchmarked against existing elements.

Much of the work presented in this chapter is summarised in the journal article:

- C.S. Long, P.W. Loveday, and A.A. Groenwold. Planar four node piezoelectric elements with drilling degrees of freedom. *International Journal for Numerical Methods in Engineering*, 65:1802–1830, 2006.

However, due to space limitations in the journal article, not all variational formulations are given in detail and only two of the eight elements presented in this chapter are evaluated.

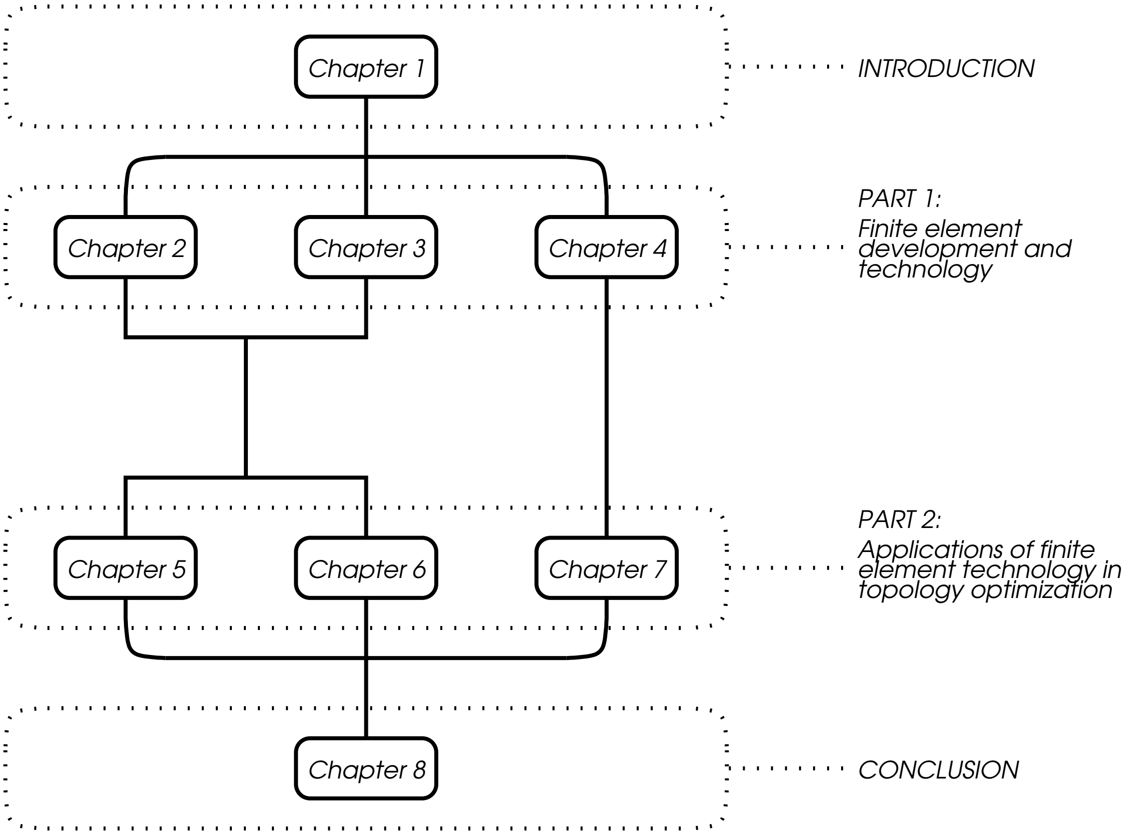

Figure 1.3: Schematic of thesis layout.

Here, the complete compilation of variational formulations, as well as a numerical evaluation of all eight elements (with and without assumed stress and electric flux density) are presented.

Chapter 4 explores the application of modified (5- and 8-point) integration schemes in higher order Q8 and Q9 elements. Reduced integration schemes are often employed to enhance element accuracy. Application of a 4-point reduced integration rule in quadratic elements however, results in spurious zero energy modes. The application of modified integration rules in elemental calculations of Q8 and Q9 elements is shown to suppress these spurious modes while, maintaining element accuracy comparable to that of their under-integrated counterparts.

The journal article resulting from this work is

- C.S. Long and A.A. Groenwold. Reduced modified quadrature for quadratic membrane finite elements. *International Journal for Numerical Methods in Engineering*, 61:837–855, 2004.

Chapter 5 explores the use of elements with drilling DOFs in developing new schemes to prevent checkerboarding, one-node connected hinges and diagonal members. As an application, topology optimization is applied to the design of a piezoelectrically driven mirror scanning device. The new method to deal with one-node connected hinges and diagonal members is employed in order to improve upon the designs achieved using conventional Q4 elements and filter strategies.

The presentation in this chapter is largely adapted from work that was presented at the 5^{th} and 6^{th} World Congress of Structural and Multidisciplinary Optimization (WCSMO5 and WCSMO6). For further details, the reader is referred to the articles:

- C.S. Long, P.W. Loveday, and A.A. Groenwold. On membrane elements with drilling degrees of freedom in topology optimization. In *Proc. Fifth World Congress on Structural and Multidisciplinary Optimization*, Lido di Jesolo, Venice, Italy, May 2003. Paper no. 83.

- C.S. Long, P.W. Loveday, and A.A. Groenwold. Design of a piezoelectric mirror scanning device using topology optimization. In *Proc. Sixth World Congress on Structural and Multidisciplinary Optimization*, Rio de Janeiro, Brazil, May 2005. Paper no. 4031.

Chapter 6 deals with generally curved shell problems in topology optimization. The differences in optimal topologies obtained when employing *ad hoc* versus mathematically sound methods to include in-plane rotations, are investigated. The sensitivity of optimal topologies to the parameter γ, studied in Chapter 2, for membrane and shell problems is quantified. Furthermore, the effect of the plate component of flat shell elements on optimal topologies is determined. Differences between the popular Discrete Kirchhoff Quadrilateral (DKQ) elements, and two frequently employed Mindlin Reissner plate elements, are presented.

Parts of this work were presented at the Second International Conference on Structural Engineering, Mechanics and Computation held in Cape Town

- C.S. Long, A.A. Groenwold, and P.W. Loveday. Implications of finite element formulation in optimal topology design. In A. Zingoni, editor, *Progress in Structural Engineering, Mechanics and Computation*, pages 1015–1019, Cape Town, South Africa, July 2004.

Furthermore, the work in this chapter is to form the basis of an article:

- C.S. Long, A.A. Groenwold, and P.W. Loveday. Effect of element formulation on membrane, plate and shell topology optimization problems. *Finite Elements in Analysis and Design*, 2007. To be submitted.

Chapter 7 reports on the application of reduced order integration schemes in topology optimization. It is widely known that a patch of elements, with material distributed in a checkerboard pattern, exhibits artificially high stiffness. This is generally accepted as the cause of checkerboarding, a phenomena characterised by significant areas of 'optimal' topologies in which material is distributed in a checkerboard-like pattern. Therefore, particular attention is paid to the effect of reduced order integration on the stiffness of higher order (Q8 and Q9) checkerboard patches.

A journal article summarising this work is in preparation:

- C.S. Long, A.A. Groenwold, and P.W. Loveday. Effect of reduced order integration schemes on checkerboard patterns in topology optimization. *Structural and Multidisciplinary Optimization*, 2007. To be submitted.

Finally, in **Chapter 8** a retrospective summary of the work is offered. Some concluding remarks regarding the study are offered, as well as recommendations for future work.

In **Appendix A** a brief introduction to topology optimization is offered, while **Appendix B** presents some additional topology optimization results not included in Chapter 6.