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The valuation and calibration of convertible bonds

by

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DECLARATION

I, the undersigned, hereby declare that the dissertation submitted herewith for the degree Magister Scientiae to the University of Pretoria contains my own, independent work and has not been submitted for any degree at any other university.

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ABSTRACT

The valuation and calibration and of convertible bonds

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A convertible bond (CB) is a hybrid security possessing the characteristics of both debt and equity. It gives the holder the right to convert the bond into a pre-specified number of shares (usually by the same issuer of the CB) until maturity of the bond, and may also contain additional features such as callability and putability. CB's along with all hybrid securities are difficult to value due to their uncertain income stream. In this dissertation several convertible bond valuation models are suggested, but with particular attention to the calibration of the underlying inputs into the model and also by taking default risk into account, which is extremely important given the sub-ordination of convertibles. The models range from the basic component models that decompose the CB into a straight bond and an exchange/call option; to more sophisticated ones consisting of stochastic interest rates, default risk, volatility structures, and even some exotics such as exchangeable and inflation-linked convertibles. An important aspect often missed by CB valuation models is the presence of negative convexity for extremely low share prices. As such a credit spread function dependent upon the underlying share price is introduced into the Tsiveriotis and Fernandes, and Hung and Wang models which improve upon the accuracy of the original models. Once a reliable model has been developed it becomes necessary to take advantage of convertible arbitrage trading strategies if they exist. The typical delta hedge, gamma hedge and option strategies that many convertible hedge funds employ are explained including the underlying risks with respect to the "Greeks".

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1. Introduction

1.1 Objectives

This dissertation describes the characteristics of convertible bonds from their valuation, calibration, underlying risks and finally setting up convertible arbitrage trading strategies. The majority of convertible bond models fail to take into account the negative convexity present at extremely low share prices. The aim of this dissertation is to solve this predicament by implementing a credit spread function into the popular Tsiveriotis and Fernandes [69], and Hung and Wang [36] equity models. Calibration is also an important consideration as it exogenously incorporates current market prices into the valuation model.

The valuation models that will be covered generally fall into two broad categories, namely equity valuation models and structural models. Equity models use the underlying share price as the independent variable, whilst structural models use the unobservable firm value. Equity valuation models are discussed first, beginning with simple piecewise models such as the Margrabe European (MEE) and American Exchange (MAE) [49] models to the more complex Component Exchange (CompEx) [29] models with Vasicek interest rate term structures [70] and forced conversion dates. Next, the Quadrinomial tree [52] approach is introduced with stochastic interest rates and share prices including parameter sensitivities. In building this Quadrinomial tree a calibrated binomial interest rate tree is reviewed by using simple BDT [13] calibration and more accurate Arrow Debreu calibration techniques developed by Jamshidian [39]. Credit risk is an important factor to consider when valuing subordinated instruments as their probability of default increases dramatically if the issuing firm is close to insolvency or put under stress. As such credit risk is implemented into several models by assuming a constant credit spread, although a reduced-form[1] approach is examined by using implied default probabilities. The other equity models reviewed are the Tsiveriotis and Fernandes (TF) [69] decoupled PDE model; the Goldman-Sachs (GS) conversion adjusted discount rate model and the Hung and Wang (HW) [36] default risk model. The other popular CB valuation

[1] The reduced-form approach and firm value approach is seen as subsets of structural models.

method is the firm value approach that uses the firm value as the underlying variable and assumes default to occur as soon as the firm value drops below a pre-specified barrier. These models were difficult to use in practice due to the unobservable firm value, although new models such as the Tan and Cai [68] risk equilibrium model and Gheno [31] volatility structure model incorporate some calibration to actual market data, thereby improving the estimation process. The CB market has developed dramatically over the past few years with some exotic convertibles being issued, such as exchangeable convertibles that convert into shares different than that of the issuer and inflation-linked convertible bonds that have their principle and coupon adjusted by the consumer price inflation index, which are explained briefly in Chapter 8. In recent times convertible arbitrage trading has become a popular strategy for many hedge funds. In Chapter 9 a detailed explanation of these strategies is discussed by focussing on the important underling risks such as delta, gamma, rho, theta, and omicron. A few popular hedging strategies are considered such as delta hedging, dynamic hedging, gamma hedging and various option strategies. To conclude I comment on a few improvements to the current models and summarise the important aspects of each of the models.

1.2 General Introduction

The first issue of a convertible bond (CB) was in 1881 with J.J. Hill [17], the railroad magnate, who needed to raise capital for its railroad project but needed some form of cheaper financing, other than the ordinary debt issues. Since then the market has grown to over $460-billion worldwide[2] with even more innovative and exotic types of convertible instruments being issued. The United States and Japan have the largest convertible bond market in terms of market capitalization but the US is seen as relatively more liquid. Japan has a number of small issuers of CB’s but cumulatively they account for a substantial share of the market. The European CB market has become an important source of financing and also seen as relatively higher credit quality to the rest of the markets.

[2] “The $460bn Global Convertible Bond Market”, Neale Safaty, KBC Financial Products - 25 Jul 2001 - Originally published in Fixed Income Market Review (http://www.gtnews.com/article/3383.cfm)

The CB's popularity with issuers came about by the fact that it offered a cheap form of financing. If managers could not persuade investors to reevaluate the risk in their business so that they will invest, then it would cost the firm much more to raise funds (higher coupon payments) if it raised straight debt. By issuing CB's this cost could be greatly reduced. Convertibles give the issuer the ability to sell their equity at a premium to their market value, thereby extending the payback period[3] for the investors.

A convertible bond (CB) is a hybrid instrument due to its fixed-income and equity like nature. In its plain vanilla form a convertible bond is a bond that pays a frequent coupon like any other bond but also entitles the investor the right to purchase a certain number of shares of the issuing firm during the life of the bond. It can be viewed as a plain vanilla bond with a relatively low coupon and a warrant on the equity of the firm. If the investor does exercise the warrant the bond will be replaced by just the equity of the firm. So, depending on the performance of the company, if the stock is more volatile the warrant/equity part becomes more valuable and the CB behaves like equity, on the other hand if the stock is less volatile the bond part is more valuable and as such the CB begins to mimic a fixed-income instrument.

The benefits to investors are that the CB provides higher current yield, greater downside protection, and seniority over common stock with regard to income payments and liquidation but are seen as subordinated debt and thus below senior debt.

The factors that affect the bond component are:

1. Interest rates
2. Credit rating/Spreads
3. Coupon, and
4. Duration

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[3] The payback period is the average amount of time in years it will take the investor to recoup the premium paid for the security, in this case the income differential between the coupon on the CB and the dividend on ordinary equity.

The factors that affect the warrant/option component are:

1. Share price
2. Embedded strike price
3. Common share dividend yield and growth rate
4. Share volatility, and
5. Life of warrant/call protection

Convertible securities are seen as management's eagerness to issue equity in the future. It allows the issuer to pay lower dividends/coupons and hence increase its interest coverage ratios. They can generally be grouped into two broad categories: mandatory conversion and optional conversion. With mandatory conversion the investor has no say in the conversion decision, and thus the security is converted automatically at maturity into common stock, whereas with optional conversion the investor can choose whether or not to convert. With mandatory convertibles the equity benefits increase as time to maturity draws closer as the security is progressively turning into common stock. In-the-money optional convertibles behave just like equity although the investor will choose not to convert, but rather continue to receive a fixed coupon stream and then convert only if forced to do so.

1.3 Literature Review

Generally speaking there are two approaches to valuing convertible bonds, using a firm value or "structural" approach where the asset value of the firm is modelled, and an equity value approach where the CB is modelled via the issuing firms share price. The early developments of the "structural" approach were instigated by Merton [54], Black and Cox [11], and Longstaff and Schwartz [45] where they first valued risky non-convertible debt. The firm's debt and equity are seen as contingent claims on the firm value, and options on the debt and equity are compound options on the firm value. By and large default occurs when the firm value drops to very low levels, such that it is unable to meet its financial obligations. It was seen as an attractive methodology but suffered the same criticisms as that of Jarrow and Turnbull's [40] risky debt approach. That is to say it is often difficult to measure the value of the firm due to the absence of market data, and to make matters worse estimates of the mean and volatility of the firm value are also required. Also, any liabilities senior to the convertible bond issue will also have to be modelled, creating more complication. One of the advantages of the firm value approach however, is that it takes into account the negative convexity[4] of the CB price during the distressed debt phase. When the share price/firm value drops to extremely low levels the probability of default increases dramatically, and due to the sub-ordination of CB's in the firm's capital structure this causes the CB value to follow suit.

To avoid these complications authors introduced new models that use the underlying share price of the issuing firm as the predominant factor. These models are easier to implement due to the easy access of share prices, and without incorporating other senior claims. An early example of this approach was implemented by McConnell and Schwartz [50]. One of the downfalls was the fact that it ignored the possibility of bankruptcy, although McConnell and Schwartz [50] accounted for this by using a risky discount rate instead of a risk-free rate. Others that consider a trivial risky rate are Cheung and Nelken [20], and Ho and Pfeffer [33]. This approach was considered better than structural models, but created further complications as the debt and equity components embedded in the hybrid CB security were discounted at different rates.

An improvement upon this was to value the CB by splitting it into two separate components, that is a vanilla bond component with the same characteristics as the CB and an option to convert the bond into a fixed number of shares. This was initially made popular by Ingersoll [37] and Brennan and Schwartz [14] with the assumption that each component could be traded separately in a perfect and complete market[5]. According to Brennan and Schwartz [14] the value of the CB is dependent upon interest rates and the value, risk and capital structure of the firm. This option/warrant can be seen as a call option on the assets of the firm, and the firm value is assumed to follow a lognormal distribution with Geometric Brownian motion. Closed form solutions were found to exist for the non-callable, European CB, but as soon as callability, putability and American type conversion were introduced into the model a numerical approximation of the PDE had to be used.

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[4] The concept of negative convexity will be explained in detail in Chapter 6, but at this stage can be thought of as the bond price decreasing (below its floor value) for extremely low share prices.

[5] This means that the markets are perfectly liquid without transaction costs and short-sale restrictions.

The Margrabe Exchange model[6] as defined by William Margrabe [49], which values an option to exchange one asset for another asset, was used to price the option component while the regular discounting of cashflows was used to value the vanilla bond component. Thus, the sums of these two components lead to the value of the CB. The option component can be European or American. The American feature can be valued using a slight modification[7] of the Margrabe Model that relies on finding the critical share price for exercising the option to convert using a Barone-Adesi and Whaley [8] approximation.

1.3.1 Credit Risk

What is important to note in using this separation procedure is that the bond component is discounted using the risk-free rate plus a credit spread to allow for default risk of the issuer, whilst the share is discounted using only the risk-free rate. Tsiveriotis and Fernandes [69] suggest that "the equity potential has zero default risk as the issuer can always deliver its own stock" whilst coupons, principle, calls and puts depend on the issuers availability of cash which introduces credit risk. They propose splitting the CB into a "cash only" part, which is subject to credit risk, and an equity part, which is not. They have implemented this in their pricing formula by using two partial differential equations for each of the components, and combined these together using each of their boundary conditions.

A disadvantage of this model is that default is not explicitly modelled, but rather seen as an estimated credit spread of the issuer. It assumes that upon default the share value jumps to zero and there is no recovery. This assumption is a bit absurd as empirical evidence suggests that shares do not instantaneously jump to zero upon default but rather involve a gradual erosion of the stock price prior to the default event and then a sudden jump[8]. An improvement since then has been the development of reduced form models and structural models that model credit risk explicitly. Jarrow and Turnbull [40] introduced a reduced form model using a Poisson jump process to model default.

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[6] It can be seen as a generalization of the familiar Black-Scholes-Merton European call option model.

[7] The modification is in fact an iterative process (e.g. Newton Rhapson).

[8] This jump is significant but much less than 100%.

In this framework once default occurs the share does not drop to zero, but to a very low level, enabling the holder to receive some recovery value. The Merton [54] structural model is similar in that it models default using the assets, equity and liabilities of a firm to find a default barrier, which once reached imposes the condition that the firm is in default. We will use the reduced form approach, as it is difficult to gather information upon the firm value.

Tsiveriotis and Fernandes [69] do not mention what happens to the share price in the case of default and assume that it is unaffected. Ayache, Forsyth and Vetzal [6] improve upon this and define exactly what happens by developing a PDE that explicitly models recovery in the case of default using a hazard rate for the probability of default. Authors such as Davis and Lischka [26], Takahashi et al. [67], Hung and Wang [36] and Anderson and Buffum [3] are others that have applied this approach.

1.3.2 Valuation Models – Lattice

The process of valuing convertible bonds might seem simple if we decompose it into its components and value each item independently, but this approach does not take into account interest rate volatility and also makes the incorrect assumption that each component can be traded separately. As such a Quadrinomial tree as used by Hull [35] and Tsiveriotis and Fernandes [69] that values the share price and interest rate movements together provides a more complete and accurate method of valuing CB's. This method can also be adapted to handle various other embedded features that exist in CB's namely, call protection, putability, stepped coupons or sinking funds.

Rubinstein [60] also uses a lattice model to value an exchange option, but due to a neat adjustment he develops a binomial tree with 2 outcomes from each node as opposed to a Quadrinomial with 4 outcomes. Rubinstein [61] extends this binomial method into a 3-dimensional binomial lattice. This bivariate binomial model is extremely useful for pricing options with two or more factors and converges much quicker than the straight binomial model. American and European type options and several embedded options can be handled by either approach. A downfall however of these approaches is that default risk is not modelled and only achieved through a credit spread in the term structure.

1.3.3 Optimal Call and Conversion Strategies

Most CB's are issued with a call option that the issuer can exercise during a specified interval or throughout the life of the bond. This enables the issuer to control the price of the CB and if necessary refinance the debt with a new cheaper issuance. As both parties now have options[9] this introduces an optimal trading strategy for each of the two parties concerned. The firms' objective is to try and maximize shareholder equity, this equates to minimizing the price of the CB, whilst the bondholders' objective is to maximize the price of the CB or minimize shareholder wealth.

This is a two-person zero sum game and is initially dealt with by Sîrbu [65] although the CB was assumed to have infinite maturity and so time was not taken into account. A companion paper to Sîrbu [65] was Sîrbu and Shreve [63] that did consider time as a factor. However a further complication is that the dividend policy of the firm be dependent on the CB. As such the CB price reduces to a non-linear second order partial differential equation with no closed form solution. Nyborg [56] develops this approach by considering the earlier work of Brennan and Schwartz [14] and factoring senior debt into the firm value with coupon and dividend payments.

Due to the dual nature of the call option[10] optimal trading strategies for both the investor and issuer exists. What makes valuing this option difficult is that the investors' optimal conversion strategy depends on the firms call strategy, and vice-versa. Thus in trying to establish an accurate value of the CB we need to correctly define the trading strategies of the parties involved. Brennan and Schwartz [14] show that the optimal call strategy is to call the bond as soon as the conversion value reaches the call price. One of the assumptions corresponds to the Miller-Modigliani [47] assumption of "symmetric market rationality" where neither party can improve their position by adopting any other strategy; and that firm value is independent of these optimal trading decisions.

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[9] The holder has the option to convert into a pre-specified number of shares

[10] CB's are usually issued with a call option giving the issuer the right to call the bond and the investor the right to convert into equity.

There is also the question of instantaneous conversion or a call notice period. In reality a call notice period is often issued that introduces additional risk to the firm because during the notice period the conversion value might fall below the call price causing the issuer to pay more than necessary for the CB. Ingersoll [37] considers incorporating a safety premium[11] to the conversion price and only calling the CB if it is above this new value.

Ingersoll [37] and Brennan and Schwartz [14] argue that it is not optimal to allow the investor to convert voluntarily before maturity except immediately before a dividend payment. Constantinides [21] on the other hand argues that for a large warrantholder it can be optimal to exercise a few warrants before maturity in the absence of dividends. His rational is that the exercised warrants will inject new capital into the firm. This will increase the volatility of the firm value and in so doing the leftover warrants will also increase in value. This increase could potentially offset the loss of the converted warrants. However, this only applies to coupon paying CB's, large warrantholders and assuming the firm invests in risky assets.

1.3.4 Phases of Convertible Bonds

In developing a model it is important to capture all of its phases, so that a true reflection of all CB risk factors are taken into account. Below is a chart depicting the various phases that the CB price follows:

[11] The safety premium can for example be 20% of the conversion value.

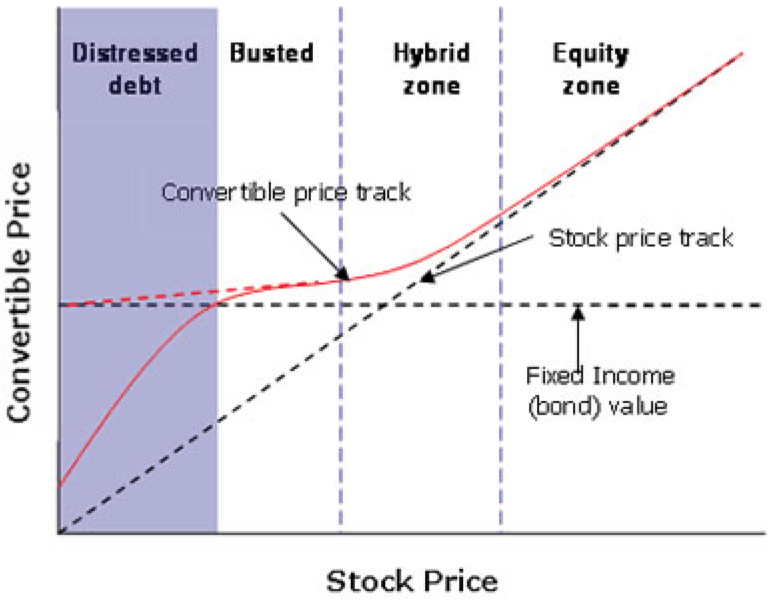

Figure 1– Different phases along the convertible bond track.

1. *Distressed debt* - In this region the convertible is close to a default event. If a default event occurs, a sum proportional to the recovery rate R is paid out to the holder of a convertible. The value of the convertible is highly sensitive to the credit spread in this region. Due to the sub-ordination of the CB, if the share price drops to very low levels, credit risk increases and so the CB value falls below its straight bond value, and actually possesses some negative gamma, vega and increasing delta with respect to the share price.
2. *Busted Convertible* - A term often used to describe a convertible that is out-of-the money but above the distressed zone, and shows characteristics similar to a pure bond.
3. *Hybrid zone* - The convertible displays behaviour akin the stock and a straight bond.
4. *Equity zone* - The convertible price is more equity-like than debt. Credit risk factors become insignificant since the companies credit rating is high due to the high stock value.

2. Simple Piecewise Models

In this Chapter piecewise CB models are discussed, beginning with the simple component model that values the CB using a straight bond plus call option technique. An improvement on this model by way of a floating strike rate for the call option is introduced in Sections 2.2 and 2.3 with the European and American Margrabe Exchange models [49] (MEE and MAE) respectively. The parameter sensitivities of the component, MEE and MAE models are reviewed in Section 2.4. Section 2.5 analyses Finnertys' [29] more advanced component model which incorporates embedded call options and default risk, and concludes with a numerical example.

Piecewise models are models that value the convertible bond as two separate components, namely a plain vanilla bond and a call option on the conversion value αS_t with the strike being the vanilla bond value b_t of the issuing firm. Thus the price of the convertible bond is obtained by adding these two components together. The following notation will be applied throughout this dissertation with new notation being introduced as required:

t	current time
χ_t	fair value of convertible bond
T	maturity
N	face value of convertible bond
s_t	value of underlying share at time t
b_t	bond floor value at time t
$\sigma_{s,t}$	share volatility at time t
$\sigma_{b,t}$	vanilla bond volatility at time t
ρ	correlation between b and s
q	continuously compounded dividend yield
α_t	conversion ratio (assumed constant)
αS_t	conversion value
$r_{t,T}$	continuously compounded yield/riskless rate from time t to T
$\gamma_{t,T}$	continuously compounded credit spread from time t to T
c_t	call price at time t
θ_t	safety premium
η	final redemption ratio (% of face value)
τ_{c1}	start of call period
τ_{c2}	end of call period
τ_{p1}	start of put period
τ_{p2}	end of put period
$\tau_{\alpha 1}$	start of conversion period
$\tau_{\alpha 2}$	end of conversion period

2.1 Component Model

A popular and simplistic model to value convertible bonds is the Component model where the convertible bond is valued as a straight, vanilla bond plus a call option with a fixed strike rate[12]. It is however very restrictive in that it assumes that conversion only takes place at maturity, which is short-sighted since conversion can take place at anytime during the life of CB and so has an American, not a European feature, also

[12] This fixed strike rate is usually the Conversion Price = Face Value/Conversion Ratio

that the strike rate is fixed. The usual Black-Scholes-Merton (BSM) framework with constant interest rates and volatility is applied to value this instrument.

If the share price follows the usual lognormal dynamics

$$dS_t = (r_{t,T} - q)S_t dt + \sigma S_t dW_t, \tag{2.1}$$

with dW being an increment of a standard Brownian motion under the risk-neutral probability measure Q, the arbitrage free value for the call option c_t with payoff at maturity of[13]

$$c_T = Max(\alpha S_T - X)^+, \tag{2.2}$$

is given by

$$c_t = \alpha S_t e^{-q(T-t)}\Phi(d_1) - Xe^{-r_{t,T}(T-t)}\Phi(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{\alpha S_t}{X}\right) + \left(r_{t,T} - q + \frac{\sigma_{S,t}^2}{2}\right)(T-t)}{\sigma_{S,t}\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma_{S,t}\sqrt{T-t},$$

and $\Phi(.)$ is the cumulative standard normal distribution function. Thus the final price of the convertible bond is

$$\chi_t = c_t + b_t. \tag{2.4}$$

Analysing this approach once more it can be seen that the strike price of the call option $X=b_t$ is in fact a stochastic process, and as such the future (maturity) strike price will not be known for certain as it depends on interest rate and credit spread

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[13] Note that $Max(\alpha S_T - B_T)^+ = Max(\alpha S_T - B_T, 0)$

movements. Also the callability and putability features that may exist in some CB issues cannot be handled by this pricing approach.

2.2 Margrabe Model – European Exchange

The Margrabe Model (Exchange Model) as mentioned in [49] is similar to the Component Model in terms of its European characteristic. What is different however is that this model values an option to exchange one asset for another asset at maturity, it can be seen as a generalization of the Black-Scholes-Merton model. Thus we have two stochastic processes that have correlated Brownian motions. If we assume that the vanilla bond b_t follows a Geometric Brownian motion similar to the share we can value the CB as the sum of a straight bond and an option to exchange the straight bond for αS_t shares.

Given that the share follows the following process,

$$dS_t = (r_{t,T} - q)S_t dt + \sigma_S S_t dW_t^S, \tag{2.5}$$

and the bond a similar process with continuous coupon rate c,

$$dB_t = (r_{t,T} - c)B_t dt + \sigma_B B_t dW_t^B, \tag{2.6}$$

with risk neutral probability measures W^S and W^B respectively, and correlation coefficient ρ, then the fair value of the Margrabe option φ_t with payoff

$$\varphi_T = Max(\alpha S_T - B_T)^+,$$

is given by,

$$\varphi_t = \alpha S_t e^{-q(T-t)}\Phi(d_1) - B_t e^{-c(T-t)}\Phi(d_2), \tag{2.7}$$

where

$$d_1 = \frac{\ln\left(\frac{\alpha S_t}{B_t}\right) + \left(c - q + \frac{\sigma_t}{2}\right)(T-t)}{\sigma_t\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma_t\sqrt{T-t}$$

$$\sigma_t = \sqrt{\sigma_{S,t}{}^2 + \sigma_{B,t}^2 - 2\rho\sigma_{S,t}\sigma_{B,t}}\,.$$

The value of the straight bond with continuously compound coupon c, credit spread γ, face value N and final redemption ratio η is found by computing

$$B_t = Ne^{-(r_{t,T}+\gamma_{t,T}-c)(T-t)} + N\eta e^{-(r_{t,T}+\gamma_{t,T})(T-t)}\,. \tag{2.8}$$

Thus the value of the CB is once again

$$\chi_t = \varphi_t + B_t\,. \tag{2.9}$$

Analysing the Margrabe European Exchange (MEE) Model [49], it seems a bit more realistic than the component model as it accounts for the stochastic nature of the strike (bond price) and also includes a correlation coefficient between the share and the bond, which is tremendously important as will be discussed in Section 2.4. However, the stochastic bond process with Brownian motion may not be a realistic assumption, as it does not take mean reverting interest rates into account. Also additional call and put features cannot be incorporated into the model. The European payoff is a limitation and is not usually associated with CB's, but as Subrahmanyam [66] suggests "it is sub-optimal to exercise a Margrabe [49] option prior to maturity if there is a so called "yield advantage", i.e., the cash flows of the exchangeable instrument is greater than the cash flows of the obtained asset at each point." As such it is not advantageous to exercise a Margrabe [49] option prior to maturity as long as $c<q$[14]. This makes sense as the lower discount rate (namely c) will produce a higher

[14] CB's are usually issued with a very low coupon than an otherwise identical vanilla bond

present value, and as such holding the bond until maturity will be the most profitable decision

2.3 Margrabe Model – American Exchange

The American version of the Margrabe option is introduced here that builds on the previous European option mentioned in the previous section. The difference lies in the calculation of the critical share price which controls the conversion of the CB. Contrary to Margrabe [49] that American options are not more valuable than their European counterparts, Bjerksund and Stensland [10] approximate the value of an American Exchange option using Barone-Adesi and Whaley [8] American approximation and show that it actually is greater than its European counterpart. The American approximation can simply be viewed as a European option adjusted for an early exercise premium.

The relevant formula for the American call option C_t with strike X is given by

$$C_t = \begin{cases} c_t + A_2\left(\dfrac{S_t}{S^*}\right)^{m_2} & , \quad S_t < S^* \\ S_t - X & , \quad S_t \geq S^*. \end{cases} \tag{2.10}$$

where c_t is the European call option, S^* is the solution of the equation

$$S^* - X = c_t(S^*, X, T-t) + \left(1 - e^{-q(T-t)}N(d_1)\right)\left(\frac{S^*}{q_2}\right),$$

and the other variables defined as

$$A_2 = \frac{S^*\left(1-e^{q(T-t)}N(d_1)\right)}{m_2}$$

$$m_2 = \frac{1-n+\sqrt{(n-1)^2+4k}}{2}$$

$$n = \frac{2(r-q)}{\sigma^2}$$

$$k = \frac{2r}{\sigma^2\left(1-e^{-r(T-t)}\right)}.$$

Denoting C_t = $American_{Call}$ (S_t,X,r,q,σ,T); then the value of the American Margrabe option is given as $C^t_{exchange}$ = $American_{Call}$ (S_t,b_t,q,c,σ,T).

Hence the value of the CB (with American optionality) is given as

$$\chi_t = C^t_{exchange} + B_t \,,$$

where B_t is defined as before. In this instance the conversion ratio α is set to 1, but by replacing S with S^* and αS with αS^*, the conversion ratio can be included. Correlation is not taken into account and volatility is only assumed for the share, but a slight modification using the European Exchange method

$$\sigma_t = \sqrt{{\sigma_{S,t}}^2 + \sigma_{B,t}^2 - 2\rho\sigma_{S,t}\sigma_{B,t}} \,,$$

for the share volatility input can be made to solve this issue. This is an improvement on the European feature, but once again callability and putability features may not be used in the model and the stochastic bond process is not ideal. There is also the limitation of the constant interest rate and volatility.

2.4 Parameter Sensitivities

In this section we will compare the sensitivities of the parameters to the American and European Margrabe Exchange models, MAE and MEE respectively. We will look at:

- The share price
- Share and bond price volatilities, and
- The correlation, ρ between the share and bond prices.

We will use Parameter Set 1 in Appendix A1 to view the relations of these parameters. Since the MAE model has an American exercise nature, it contains all the features of the MEE model including optionality throughout the term. Thus we should expect the MAE model to always value the CB at a premium relative to the MEE model, which is precisely what we get from the model outputs. Taking the CB price for differing share values, we see in Figure 2 that the CB increases monotonically, with the MAE model pricing at a higher premium to the MEE model as the share increases. The models resort to their bond floor value when the share price gets close to zero, indicating that no explicit default risk is taken into account except for the additional credit spread γ. It is also interesting to note that for deep in-the-money CB's the MEE model dips a little below its parity value, because of the assumption that the embedded conversion option can only be exercised at maturity. This is as a result of the time premium decay of the option, which reduces the options' value as it becomes closer to maturity.

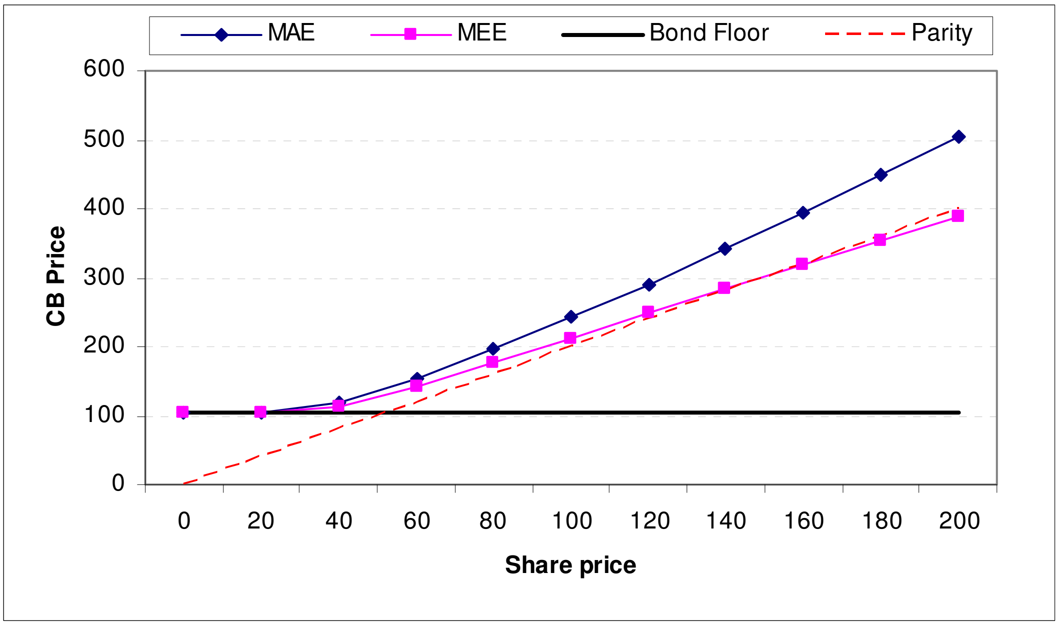

Figure 2 – MAE and MEE models for differing share values, with ρ=-0.2 and γ=0.25%.

As the volatilities of both the share price and bond price increase so to does the CB price. This is simply because the embedded options increase in value as the volatilities increase. Figure 3 and Figure 4 illustrate the share and bond price volatility respectively with the CB price converging to similar values for large volatility increases.

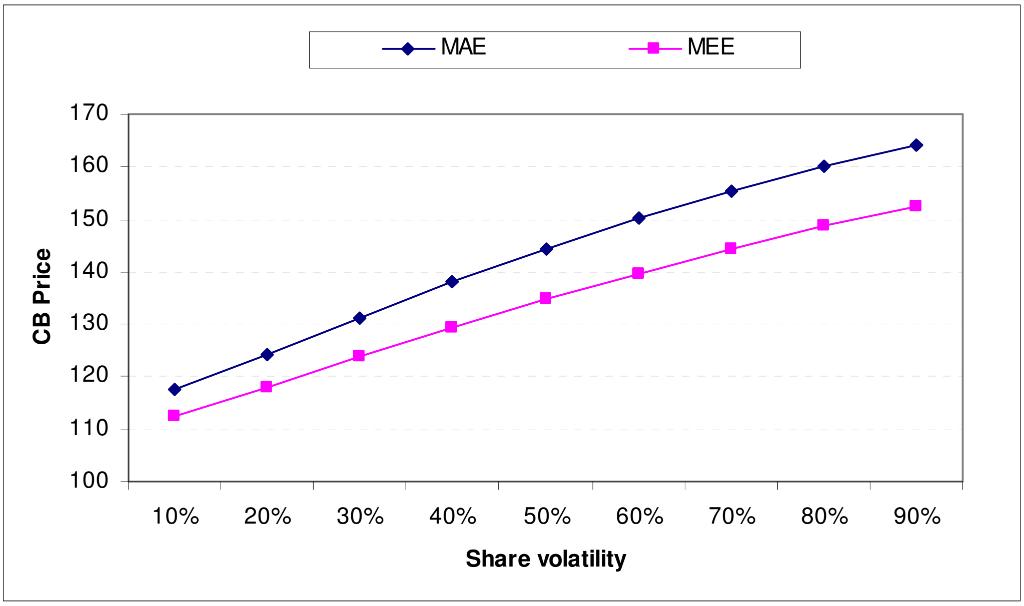

Figure 3 – MAE and MEE model prices for changes in share volatility with S=40, ρ=-0.2 and γ=0.25%.

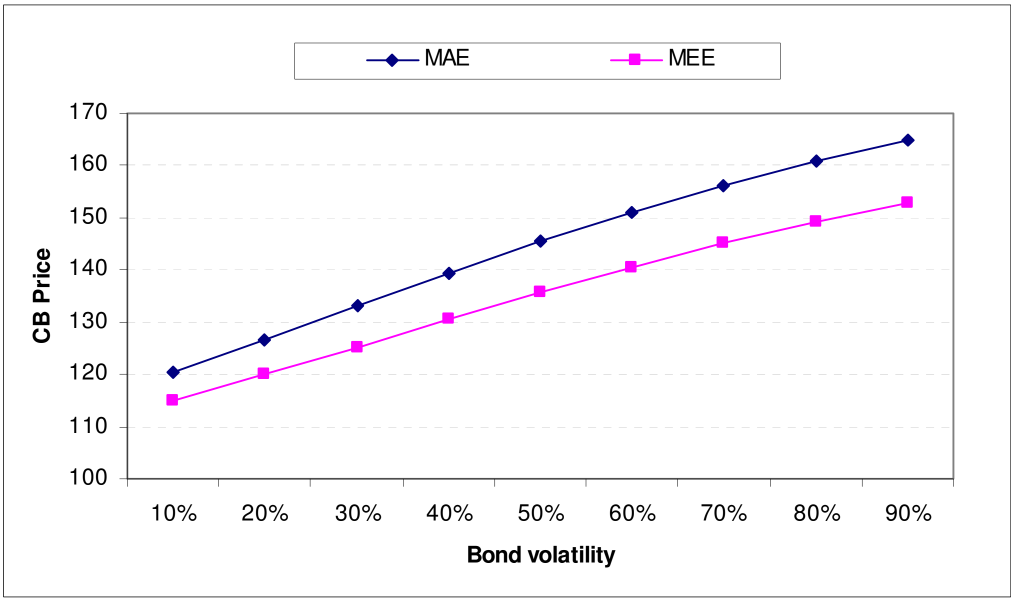

Figure 4 – MAE and MEE model prices for changes in bond price volatility with S=40, ρ=-0.2 and γ=0.25%.

Finally the correlation between share and bond prices is examined to see what effect it has on the CB price. If the correlation between the share and bond price increases, then the return potential will not be that great, relative to lower correlation, as both the assets will move in the same direction offsetting each others gains and losses. In so doing, the exchange option in the MAE and MEE models decreases in value resulting in a lower CB price. If on the other hand the correlation decreases, then the return potential from the embedded exchange option increases, as the gains and losses are magnified due to the inverse relationship of the share and bond. As a result the exchange option increases in value causing the CB value to become greater. Figure 5 demonstrates this concept for both the MAE and MEE models.

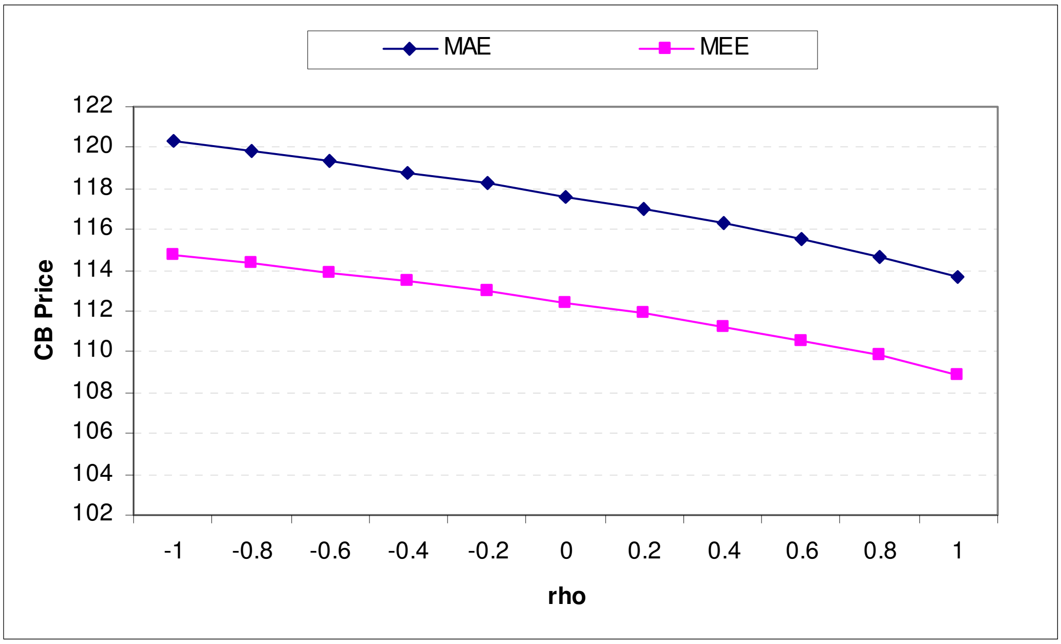

Figure 5 – MAE and MEE CB values for changes in the correlation between the share and bond prices with S=40 and γ=0.25%.

2.5 Component Model

In this section Finnerty's [29] Component model is discussed, which is an extension to the Margrabe Exchange [49] model. It develops a closed-form solution of the embedded exchange option by incorporating stochastic risk-free interest rates, credit spreads and share prices. Section 2.5.1 introduces the basic structure of the zero-coupon Component Exchange model with stochastic interest rates and share prices. Correlation between the two processes is incorporated with duration and convexity playing a role in the volatility of the CB. Section 2.5.2 takes into account default risk by way of a stochastic credit spread and Section 2.5.3 makes mention of discrete coupon payments. The optimal conversion time is a major factor in Finnerty's [29] model and as such is discussed in Section 2.5.4 with 2.5.5 expanding on this concept and quantifying the loss to the investor for early conversion. Section 2.5.6 encompasses it with an example and compares the results to the piecewise models discussed earlier.

2.5.1 Basic Model

This method builds on the earlier, basic component model and the Margrabe exchange model. It differs in that the strike price of the embedded call option is the market value of the CB, not its principal value. If the face value is used, then when the conversion price exceeds the stock price[15], the warrant strike price is overstated and the exchange option and CB are undervalued.

The value of the CB is calculated as the sum of its straight bond value and exchange option subject to the issuer's ability to force early conversion. Early conversion by the firm is modelled as a stopping time problem, where the optimal time to force conversion is by minimizing the cost of the CB. The short-term riskless interest rate r_t follows the usual mean-reverting Hull and White [35] model,

$$dr_t = \left[\Theta_r(t) - \kappa_r r_t\right]dt + \sigma_r dZ_r, \tag{2.11}$$

with κ_r and σ_r constants, dZ_r is the increment to a Gauss-Wiener process and mean reversion level $\Theta_r(t)/\kappa_r$. The share price follows a similar Geometric Brownian motion process

$$\frac{dS}{S} = (\mu_t - q)\,dt + \sigma_e dZ_e, \tag{2.12}$$

[15] This is the usual occurrence at issuance.

with μ_t the mean, q the continuous dividend yield and σ_e the volatility of the share price. There exists some correlation between the two processes[16], namely

$$dZ_e dZ_r = \rho_{12} dt . \tag{2.13}$$

Defining $B(r_t, t, T)$ to be the price at time t of a zero-coupon bond maturing at time T_m, when the short rate is r_t. By applying Itô's lemma twice we get the following dynamics for the logarithm of the bond price

$$d \ln B_t = \left(r_t - \frac{1}{2} D(t, T_m)^2 \sigma_r^2 \right) dt - D(t, T_m) \sigma_r dZ_r , \tag{2.14}$$

where $D(t, T_m) = -(\partial B / \partial r) / B = \left(1 - e^{-\kappa_r (T_m - t)}\right) / \kappa_r$, is the modified duration at time t of a zero-coupon bond maturing at T_m. Finnerty [29] assumes the bond is converted or redeemed at $T_2 \le T_m$, where the investor's optimal conversion date is $T_2^* \le T_2$. From now onwards the CB price will be a function of T_2, which does not depend on the straight bond price. Letting the conversion ratio be α, the logarithm dynamics of α shares are given as

$$d \ln \alpha S_t = \left(r_t - q - \frac{\sigma_e^2}{2} \right) dt + \sigma_e dZ_e . \tag{2.15}$$

Using the equivalent martingale measure in the risk-neutral framework and B as the numeraire, leads to

$$\frac{\alpha S_t e^{-q(T_2 - t)}}{B_t} = E^B \left[\frac{\alpha S_{T_2}}{B_{T_2}} \right]. \tag{2.16}$$

Letting E_t be the value of the exchange option[17] with boundary condition

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[16] The correlation between interest rates and equity values is generally viewed as negative due to the dividend/cashflow discount models employed by equity analysts. Just as the inverse relation between bond prices and interest rates.

$$E_{T_2} = \max\left(\alpha S_{T_2} - B_{T_2}, 0\right). \tag{2.17}$$

From the earlier argument it follows that

$$E_t = B_t E^B\left[\max\left(\alpha S_{T_2} - B_{T_2}, 0\right) / B_{T_2}\right] = B_t E^B\left[\max\left(\frac{\alpha S_{T_2}}{B_{T_2}} - 1, 0\right)\right]. \tag{2.18}$$

Thus the value of the exchange option is given as

$$E_{T_1} = \alpha S_t e^{-q(T_m - t)} H\left(\frac{\mu_c + \sigma_z^2}{\sigma_z}\right) - B_t H\left(\frac{\mu_c}{\sigma_z}\right), \tag{2.19}$$

where $H(.)$ is the standard normal cdf,

$$\mu_c = \ln\left[\frac{\alpha S_t e^{-q(T_2 - t)}}{B_t}\right] \tag{2.20}$$

$$\sigma_Z^2 = Var\left[\ln\left(\frac{\alpha S_{T_2}}{B_{T_2}}\right)\right]$$

(2.21)

$$= \left(\sigma_e^2 + \lambda_{er}\right)\left(T_2 - t\right) - \lambda_{er}\left[D\left(t, T_m\right) - D\left(T_2, T_m\right)\right] - \frac{\sigma_r^2}{2\kappa_r}\left[D\left(t, T_m\right)^2 - D\left(T_2, T_m\right)^2\right]$$

and $$\lambda_{er} = \frac{\sigma_r^2}{\kappa_r^2} + 2\frac{\rho_{12}\sigma_e\sigma_r}{\kappa_r}. \tag{2.22}$$

$D\left(t, T_m\right)$ is the modified duration at time t, and $D\left(t, T_m\right)^2 = \left(\partial^2 B / \partial r^2\right) / B$ is the convexity at time t of a zero-coupon bond maturing at time T_m. σ_z^2 is the variance of the price ratio $\alpha S_t e^{-d(T_2 - t)} / B_t$.

[17] That is to exchange the bond for α underlying shares

Ignoring call options and the optional redemption feature to redeem the bonds for cash, which issuers typically use to force early conversion, we have that the value of the CB at time *t* is,

$$\begin{aligned} C(r_t, B_t, S_t, T_2, T_m) &= B_t + E_t \\ &= \alpha S_t e^{-q(T_2-t)} H\left(\frac{\mu_c + \sigma_z^2}{\sigma_z}\right) + B_t H\left(-\frac{\mu_c}{\sigma_z}\right), \end{aligned} \tag{2.23}$$

with final condition, $C(r_t, B_t, S_t, T_2, T_m) = \max\left[F, \alpha S_{T_m}\right]$ and $E_{T_m} = \max\left[NS_{T_m} - F, 0\right]$ where F [18] is the face value of the bond. Traditional CB's can be seen from a bond investors or equity investors' perspective. From a bond investors viewpoint it is seen as a straight bond plus a warrant giving the holder the right to exchange the bond for a pre-specified number of shares. Using put-call parity the equity investors view the CB as a long share, European put to sell the shares back to the issuer at maturity in exchange for a straight bond and a swap paying the difference between the dividends on the underlying share and the coupons from the bond. The holder of the CB has a call (put) option on the underlying shares, which he can exercise by exchanging the bond (underlying stock) for the stock (bond). This can be seen by substituting into (2.23) and rearranging terms to get

$$\begin{aligned} C(r_t, B_t, S_t, T_m) &= B_t + E_t \\ &= \alpha S_t e^{-q(T_2-t)} + B_t H\left(-\frac{\mu_c}{\sigma_z}\right) - \alpha S_t e^{-q(T_2-t)} H\left(-\frac{\mu_c + \sigma_z^2}{\sigma_z}\right). \end{aligned} \tag{2.24}$$

The underlying stock is worth $\alpha S_t e^{-q(T_2-t)}$, due to dividends not been collected during $[t, T_m]$. The CB equals the underlying stock plus the option to exchange the stock for a bond currently worth B_t. When valuing the CB using the bond plus exchange option approach, many investors use the face value as the strike price of the embedded call option. Due to the CB always been issued with a conversion price

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[18] $F = N\eta$ to simply the notation a little in this section.

premium, this increases the strike price of the option, decreases the call option value and thus underestimates the CB price. As can be seen in the above equation the strike price is the present value of the forward bond price[19]. So the call option payoff is the share price reduced by the present value of the dividends before the anticipated conversion date and the present value of the forward bond price. The option value can be either in/out-the-money depending on whether the dividend-adjusted share value is greater or less than the present value of the forward bond price.

2.5.2 Default Risk

Defining s_t to be the short-term credit spread for bonds with the same credit rating as the issuers CB, it follows that the credit spread process is represented as

$$ds_t = \left[\Theta_s(t) - \kappa_s s_t\right]dt + \sigma_s dZ_s \tag{2.25}$$

Where $dZ_e dZ_s = \rho_{13}dt$ and $dZ_r dZ_s = \rho_{23}dt$. Similarly as with interest rates the credit spread process reverts to its long-term mean $\Theta_s(t)/\kappa_s$. Credit spreads are generally negatively correlated with equity and treasury yields, thus $\rho_{13} < 0$ and $\rho_{23} < 0$. Assuming now that the bond price B is a function of time, interest rates and credit spreads, $B(t,r,s)$ requires us to use Itô's lemma in order to obtain the logarithm of the bond price. Applying Itô's lemma twice to the bond price gives us the following

$$d\ln B_t = \mu_B(t,T_m)dt - \left(D_r(t,T_m)\sigma_r dZ_r + D_s(t,T_m)\sigma_s dZ_s\right), \tag{2.26}$$

where $\mu_B(t,T_m) = r_t - \frac{1}{2}D_r(t,T_m)^2\sigma_r^2 - \frac{1}{2}D_s(t,T_m)^2\sigma_s^2 - \rho_{23}\sigma_r\sigma_s D_r(t,T_m)D_s(t,T_m)$, $D_r(t,T_m) = -(\partial B/\partial r)/B = \left(1 - e^{-\kappa_r(T_m - t)}\right)/\kappa_r$ is the modified risk-free interest rate

[19] The forward bond price is the present value of the payment schedule after the anticipated conversion date.

duration and $D_s(t,T_m) = -(\partial B/\partial r)/B = \left(1-e^{-\kappa_s(T_m-t)}\right)/\kappa_s$ is the modified credit spread duration at time *t* for a zero-coupon bond with maturity T_m.

This new representation of the bond price *B* assumes that it is now a defaultable zero-coupon bond due to the credit spread. As such the exchange option equation (2.19) has a slight modification to its variance term σ_z^2. The new variance term is σ_D^2 which is given in Finnerty [29], and the CB price is once again given by (2.23) with σ_z^2 replaced by σ_D^2.

2.5.3 Coupon Bearing Bonds

Introducing coupon-bearing bonds into the calculation requires us to break down the scheduled payment stream into a series of defaultable zero-coupon bonds. Assume the bond pays *c* per semi-annual period, and accrued interest is not received when the bond is converted. Thus each bond pays *cF* until the bond is redeemed or converted, and *(1+c)F* on maturity. Defining $\hat{B}(r_t, s_t, t, T_m)$ to be the value at time *t* of the *m* serial zero coupon bonds outstanding after the exchange date T_2. I_t is the present value of the interest payments to be received between *t* and T_2. So $\hat{B}_t$, where $t \le T_2$, is the sum of the *m* serial zero-coupon bonds, $\hat{B} = \sum_{j=1}^{m} B_j$.

The duration of the coupon-bearing bond is calculated as the market-value-weighted average duration of the *m* serial zero-coupon bonds[20]. As such the dynamics of $\hat{B}$ are

$$\frac{d\hat{B}}{\hat{B}} = r_t dt - D_r\left(\hat{B}\right)\sigma_r dZ_r - D_s\left(\hat{B}\right)\sigma_s dZ_s, \tag{2.27}$$

with the obvious boundary conditions $\hat{B}\left(r_{T_m}, s_{T_m}, T_m, T_m\right) = (1+c)F$ and

[20] The convexity of the coupon bearing bond is also calculated as the market-value weighted average convexities.

$\hat{B}\left(r_{T_m}, s_{T_m}, T_2, T_m\right) = \hat{F}\left(T_2\right)$ which is the bond price at time T_2. $D_r\left(\hat{B}\right)$ and $D_s\left(\hat{B}\right)$ are the modified riskless interest rate duration and credit spread duration of the straight coupon-bearing bond due after T_2. Once again using Itô's Lemma gives us the logarithm coupon-bearing bond price in the risk-neutral world

$$d\ln\hat{B}_t = \left[r_t - \frac{1}{2}D_r\left(\hat{B}\right)^2\sigma_r^2 - \frac{1}{2}D_s\left(\hat{B}\right)^2\sigma_s^2\right]dt - \frac{1}{2}D_r\left(\hat{B}\right)\sigma_r dZ_r - \frac{1}{2}D_s\left(\hat{B}\right)\sigma_s dZ_s. \quad (2.28)$$

$D_r\left(\hat{B}\right)^2$ and $D_s\left(\hat{B}\right)^2$ are the riskless interest rate and credit spread convexities respectively at time t of a straight coupon-bearing bond paying coupons from T_2 to T_m. With a change of numeraire in equation (2.9) from B to $\hat{B}$, the new value of the exchange option E_t with mean $\mu_c = \ln\left[\alpha S_t e^{-q(T_2-t)} / \hat{B}_t\right]$ and volatility σ_c^2 given in Finnerty [29]. So the value of the coupon-bearing CB is

$$\begin{aligned} C\left(r_t, s_t, \hat{B}_t, S_t, T_m\right) &= I_t + \hat{B}_t + E_t \\ &= I_t + \alpha S_t e^{-q(T_2-t)} H\left(-\frac{\mu_c + \sigma_c^2}{\sigma_c}\right) + \hat{B}_t\, H\left(-\frac{\mu_c}{\sigma_z}\right), \end{aligned} \quad (2.29)$$

where I_t is the present value of the interest payments due between t and $\hat{T}_2$, and $I_t + \hat{B}_t$ is current price of the bond. So equation (2.29) is the bond value plus the exchange option for an otherwise identical but noncallable bond minus the value of the firms call option.

2.5.4 Optimal Conversion Time T_2

CB investors have an American exchange option to convert the CB into the underlying equity. As such they will choose T_2^* to maximize the value of the CB, and so assuming that there is no forced conversion risk, we have that the value of the CB at time t is the maximum of $C\left(r_t, s_t, \hat{B}_t, S_t, T_2, T_m\right)$ over all possible future conversion dates. Thus T_2^* is the projected voluntary conversion date that maximizes C.

Taking into account the call option held by the issuer, which is also American, allows the issuer to force early conversion upon the investor. Defining the effective redemption price as the call price plus accrued interest, we have that rational CB holders will choose to rather convert if the conversion value is above the redemption price, even if they do not receive the full time value of their exchange options.

As such the firm will seek to minimize the cost of the CB by removing the maximum option time premium from it. As Finnerty [29] states, "If the firm could redeem the bonds instantaneously, then in a perfect market, the issuer would call the CB for redemption as soon as the conversion value reaches the effective call price." The option is considered at-the-money and so has a maximum time premium, although this strategy does not work in reality due to market imperfections and agency costs. Most bond indentures require the issuer to give at least 30 days notice prior to redemption date. In practice the issuer usually waits for the bond price to rise above the redemption price by about 20 percent as mentioned in Asquith [5] and Asquith and Mullins [4]. This redemption cushion is seen as a safety premium to prevent the issuer from overpaying for the CB and resulting in a busted forced conversion.

CB's are usually issued with a call deferment period in which the issuer cannot call the bond.[21] Observing the income streams it seems obvious that investors of CB's will want to hold the bond up until maturity if the present value of the coupons payments is greater than the present value of the dividend stream. Even if the dividend income is greater than the interest income, investors will not voluntarily convert as they would forgo the time value of the call option and put option value[22]. Finnerty [29] assumes that forced conversion is a stopping time problem, and so assumes the issuer

[21] Depending on the maturity of the CB, shorter maturity issues have weak call protection whilst longer dated issues have stronger call protection.

[22] Assuming the convertible bond is issued with embedded call and put options.

will call the CB at $\hat{T}_2$, when the conversion value NS_{T_2} touches the upper barrier for the first time. The upper barrier is equal to the effective redemption price R_t multiplied by one plus the safety premium θ. Carr, Reiner and Rubinstein [18] calculate the first passage time density function for $\ln NS_t$ to hit the barrier $\ln \theta R_t$ as,

$$g(t) = \frac{Z_t}{\sqrt{2\pi\sigma_e^2 t^3}} \exp\left(-\frac{1}{2\sigma_e^2 t} \left[Z_t - (\mu - d) t \right]^2 \right), \tag{2.30}$$

with $Z_t = \ln(\theta R_t / \alpha S_t)$. The expected forced conversion date is

$$\hat{T}_2 = T_1 + Z_t^{\frac{1}{\mu - q}}, \qquad \forall \qquad T_R \le \hat{T}_2 \le T_m, \tag{2.31}$$

where T_R is the earliest call date permitted by the bond indenture. As the dividend rate increases $\hat{T}_2$ also increases, which makes sense according to the payment stream argument of Asquith and Mullins [4] by delaying the call date.

2.5.5 Loss of Value Due to Forced Early Conversion

The value lost from the CB because of forced early conversion is the remaining time value of the conversion option plus the difference between interest and dividend payments between $\hat{T}_2$ and T_2^*. The firm forces conversion when the CB's price reaches the barrier θR_t. In a risk-neutral world, the expected loss of value to CB investors resulting from forced early conversion is

$$\begin{aligned} L &= \int_{T_R}^{T_2^*} \left[C\left(r_t, s_t, \hat{B}_t, S_t, T_2^*, T_m \right) - \theta R_t B(r_t, t, u) \right] g(u)\, du \\ &\cong C\left(r_t, s_t, \hat{B}_t, S_t, T_2^*, T_m \right) - C\left(r_t, s_t, \hat{B}_t, S_t, \hat{T}_2, T_m \right) \\ &= E_t\left(T_2^*\right) - E_t\left(\hat{T}_2\right) + I_t\left(\hat{T}_2, T_2^*\right) \end{aligned} \tag{2.32}$$

This loss value assumes that the firm forces immediate conversion when the effective call price first hits the barrier and that the option to force conversion does not affect the probability that the bonds are held to maturity and redeemed for cash. The value is equal to the difference between the CB value occurring at times $\hat{T}_2$ and T_2^*. It can also be viewed as the difference between the values of the exchange options[23] and the present value of interest between $\hat{T}_2$ and T_2^*.

The price of the CB equals:

  The value of the straight bond

+ The value of the exchange option (assuming voluntary conversion)

- The value of the firms' option to force early conversion.

Equation (2.32) combines the last two components into one by using an expected date of forced conversion, as opposed to taking the expected bond price across all permissible conversion dates. The value of this combined option in (2.29) is a convex function of time until forced conversion. As such it tends to overstate the value of the forced conversion option and so understate the CB price.

2.5.6 Numerical Example

To illustrate the differences in each of the piecewise models it makes sense to try out an example on the Margrabe European Exchange model (MEE), the Margrabe American Exchange model (MAE) and the Component model with Coupon and Credit Risk (CompEx). Using the Parameter Set 1 in Appendix A1, Figure 6 shows the sensitivity of each of the models to changes in the underlying share price. As expected the MAE model is above the MEE model due to the American option instead of the European option. For deep in-the-money CB's the MEE dips below parity, this is plausible as the CB can only be converted at maturity of the bond, and so loses out on some of its value due to the discounting of its future conversion value. The MAE model estimates the convertible price to be quite a bit higher than parity for

[23] The exchange option negates any dividends on common stock between $\hat{T}_2$ and T_2^*.

in-the-money CB's to avoid arbitrage opportunities, with both MEE and MAE models reverting to their straight bond value for out-the-money CB's. The CompEx model takes into account stochastic interest rates. As such its bond floor value will vary, depending on the term structure of interest rates. For deep in-the-money CB's its value will trade close to parity with a little premium, and for out-the-money CB's it will trade at its straight bond value which is again dependent on the term structure which uses a one-factor Vasicek interest rate model for zero-coupon bond prices.

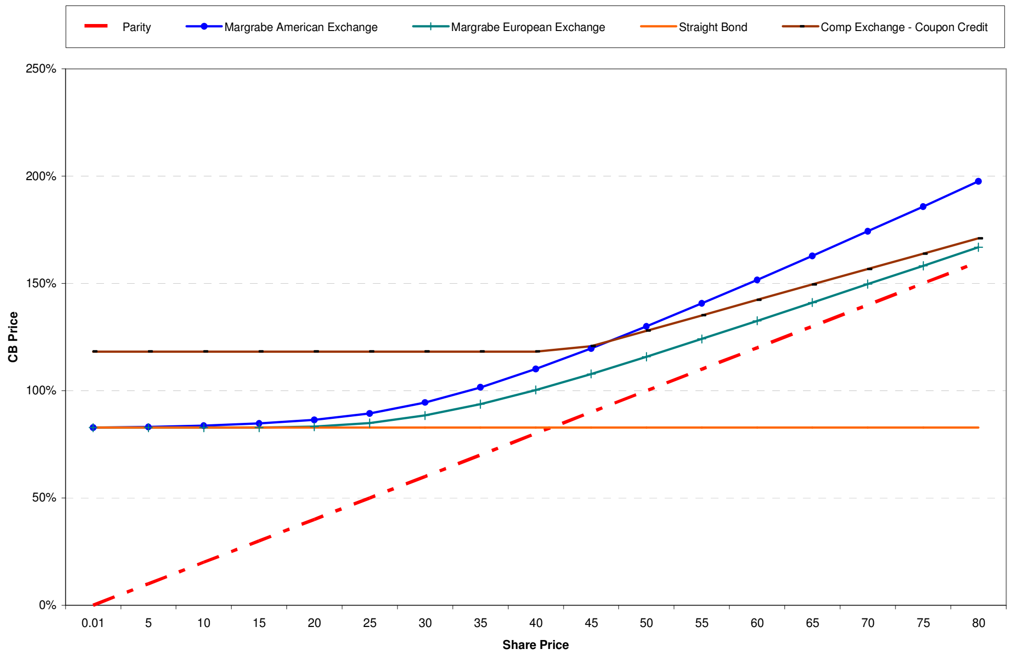

Figure 6– Comparison of the piecewise models MEE, MAE and CompEx models for different share values.

In Chapter 2 the conventional piecewise models were discussed beginning with the simple component model, where the embedded option was valued according to a BSM European call option with a fixed strike rate. The assumption of a fixed strike rate is not plausible, so the Margrabe European and American Exchange (MEE and MAE) models were introduced. These models assume that the embedded option is an exchange option and can be seen as a generalization of the BSM option pricing model. The advantage of this approach over the simple component is that it incorporates a floating strike rate although the dubious assumption is that the bond price follows a lognormal distribution similar to the share price. The American exchange option uses a critical share price adopted from Bjerksund and Stensland [10] to determine optimal conversion. The MAE model would seem to suffice but the constant interest rate assumption and default risk by way of a constant credit spread maybe improved upon. As such Finnerty's [29] Component Exchange (CompEx) model is reviewed which incorporates stochastic interest rates, credit spreads and share prices. In addition the issuing firms call option to purchase the bond is accounted for by using a stopping time problem to find the forced conversion date. The numerous inputs required by the CompEx model reduce its tractability.

The downfall of using piecewise models is that they assume that each component trades separately (i.e. bond + option) which continuously over-estimates the CB's price. Also the embedded call and put options are not easily incorporated, however the CompEx model does provide some flexibility in this regard, but estimating the large amount of inputs is tricky. The major advantage of piecewise models is that they are easy to implement and provide an estimate of the CB's value. Another important property is the inclusion of the correlation coefficient between the share and bond price, which was shown in Section 2.4 to be crucial in estimating the convertible.
This Chapter concludes the piecewise valuation approach but will still be referred to throughout this dissertation. The next approach will be equity valuation models but before we move onto them in Chapter 5, it is important to introduce the fundamental properties of interest rate and share price lattices in Chapter 3 and default risk in Chapter 4.

3. Binomial Trees

To solve the more advanced equity valuation models that will be discussed in Chapter 5 it is important to discuss the construction of the binomial share price and interest rate trees. As such Chapter 3 begins with Section 3.1 which sets up the basic binomial share tree with the risk-neutral probability framework. From here it is extended to the no-arbitrage binomial interest rate tree in Section 3.2. Calibration is critical to the construction of the binomial interest rate tree and as such is discussed in Section 3.3 with a simple calibration scheme, and ends with a more precise Arrow-Debreu calibration scheme.

Binomial trees provide a generalized (explicit) numerical method to solve PDEs and were first proposed by Cox, Ross and Rubinstein [23] in 1985. It is essentially a discrete version of the BSM European option formula under the risk-neutral valuation framework, and gives good approximation values for assets without dividends/coupons to the BSM model albeit with many time steps. Binomial trees can become unstable that lead to sporadic values if the time step is not small enough, but their tractability for valuing simple financial instruments with two variables is what makes them appealing. Extensions of the lattice can be made to incorporate other variables, although this leads to extremely large trees making them unattractive relative to other numerical schemes.

3.1 Share Price

In the Black-Scholes-Merton framework the share price process follows the typical SDE under the real world probability measure (P)

$$dS_t = \mu S_t dt + \sigma S_t dW_t .$$

When working in an arbitrage-free world the share price process follows a similar SDE under the risk-neutral measure (Q)

$$dS_t = rS_t dt + \sigma S_t dW_t .$$

This risk-neutral measure (Q) is also referred to as an equivalent martingale measure because it allows the share to grow at a constant risk-free rate and as such

$$E^Q\left[S_T\right] = S_t e^{r(T-t)} .$$

When dealing with a binomial tree it might seem appropriate to adjust the probabilities for certain situations which will change the up/down moves of the share price. When working under the risk-neutral measure the following set of equations can be found[24]

$$p^* Su + (1-p^*)Sd = Se^{r\Delta t} \tag{3.1}$$

$$p^* u^2 + (1-p^*)d^2 - (p^* u + (1-p^*)d)^2 = \sigma^2 \Delta t , \tag{3.2}$$

where p^* is the probability that the share price will move up under the risk-neutral measure Q, and $\sigma^2\Delta t$ is the variance of the share price returns. Thus, by adjusting the probabilities $p^* \ \varepsilon \ \left[0,1\right]$, the corresponding up and down movements of the share price can be found by solving equations (3.1) and (3.2). Solving (3.1) for p^* we find that

$$p^* = \frac{e^{r\Delta t} - d}{u-d}$$

Therefore substituting p^* into (3.2) and ignoring terms in Δt^2 and higher powers of Δt, the following equations for u and d are produced[25]

$$u = e^{\sigma\sqrt{\Delta t}} \tag{3.3}$$

$$d = e^{-\sigma\sqrt{\Delta t}} . \tag{3.4}$$

[24] This is the Cox, Ross and Rubenstein [23] derivation that uses the variance of share returns as $V(X)=E(X^2)-E(X)^2$.

[25]We assume the Taylor series expansion of $e^x = 1 + x + \left(x^2/2!\right) + \left(x^3/3!\right) + \left(x^4/4!\right) + ...$

As can be seen the up and down movements of the share price are dependent on the volatility of the share price returns.

3.2 Interest Rates

Binomial interest rate trees are a bit more difficult to model than share prices, as they cannot take on arbitrary values, since this would be inconsistent with the arbitrage valuation framework. Due to the presence of bonds in the market a term structure of interest rates can be formed and thus calibrated to the binomial interest rate tree.

The study of interest rate models was pioneered by Merton [54] in 1974. The most popular models were the ones created by Vasicek [70] and Cox, Ingersoll and Ross [22] in1985.The most general model of interest rates takes the following form

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dW_t,$$

or in discrete time

$$\Delta \ln r_t = \ln r_{t+\Delta t} - \ln r_t = \mu\Delta t + \sigma\sqrt{\Delta t}Z_t, \tag{3.5}$$

where Z_t is a random variable that takes on the value +1 and -1 with risk-neutral probability q^* and $\left(1-q^*\right)$ respectively[26]. The variable Z_t can be seen as a random walk. Thus there are two movements from each node

$$\ln r_{t+\Delta t} = \begin{cases} \ln r_t + \mu\Delta t + \sigma\sqrt{\Delta t} & , \quad \text{with } q^* = 0.5 \\ \ln r_t + \mu\Delta t - \sigma\sqrt{\Delta t} & , \quad \text{with } \left(1\text{-}q^*\right) = 0.5. \end{cases} \tag{3.6}$$

As a result there is a multiplicative ratio at each time tick in the binomial tree as depicted in Figure 7. Generally speaking at time period i there are j possible rates,

$$r_i, r_i v_i, r_i v_i^2, \ldots, r_i v_i^{j-1},$$

---

[26] q^* is the risk-neutral probability associated with interest rates, while p^* is the risk-neutral probability associated with the share price. Both belong to the arbitrage-free measure Q.

where r_i is the baseline rate and v_i is defined as

$$v_i = \frac{r_u}{r_d} = e^{2\sigma_i\sqrt{\Delta t}}. \tag{3.7}$$

The volatility can be constant over time or implied from interest caps and floors trading in the market.

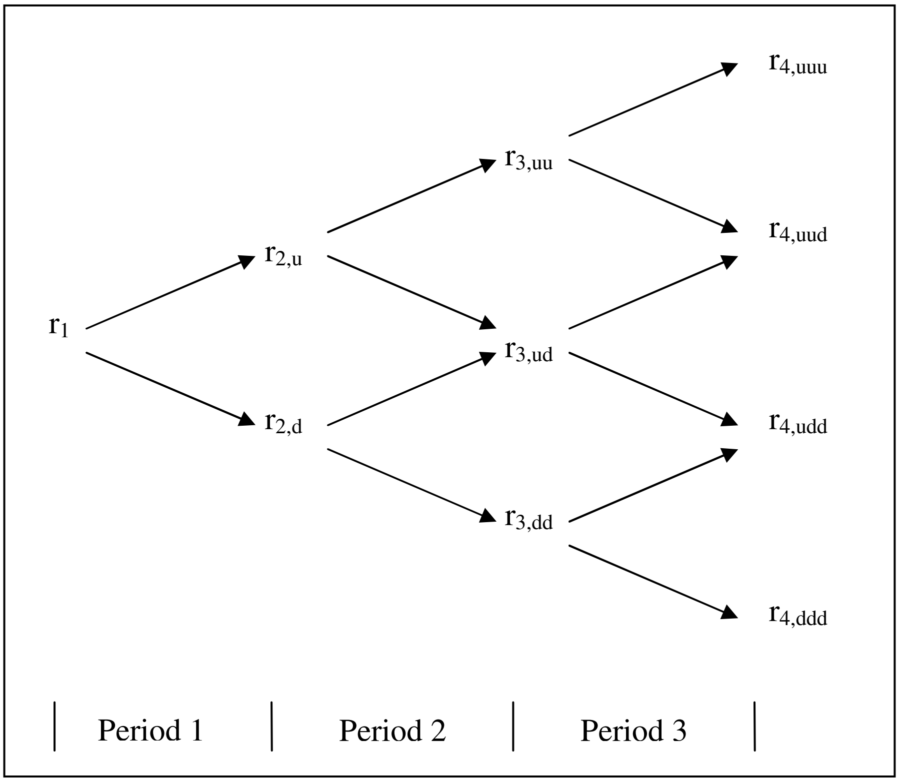

Figure 7– Binomial interest rate tree.

3.3 Calibration of Interest Rate Trees

Calibration is an important factor to consider when building any model; it allows the model to give results consistent with what is observed from market prices. When dealing with any interest-rate sensitive securities it is vital that the term structure of interest rates are some how incorporated into the model to value the security correctly. Thus two methods of calibrating binomial interest rate trees is described, the simple one-factor calibration and the Arrow Debreu calibration schemes.

3.3.1 Simple Term Structure Calibration

This form of calibration is easy to implement and widely used to value interest-rate sensitive securities. The basic variable that is calibrated is the annualised one-period short rate. The inputs required for the calibration is a term structure of yields from zero-coupon bonds together with their prices, and their volatility structure. The volatility can be assumed to be constant for convenience, but using implied volatility estimates from interest rate caps and floors provide more accurate results. This is due to the fact that mean reversion levels change over time, and so should be represented by the volatility structure.

Defining $m(0,t)$ to be the price of a risk-free zero-coupon bond with maturity t, and r_t to be the t-period spot rate. So, if we have the two-year zero-coupon bond price and volatility, we can seek out the forward one-year spot rate, $r_{2,d}$, by using the binomial interest rate tree as described in the previous section. Thus the equation is

$$\begin{aligned} m(0,2) &= \frac{1}{2}e^{-r_1}100\left(e^{-r_{2,u}} + e^{-r_{2,d}}\right) \\ &= \frac{1}{2}e^{-r_1}100\left(e^{-2r_{2,d}\sigma_2} + e^{-r_{2,d}}\right). \end{aligned} \tag{3.8}$$

Similarly, the one-year forward rate in two years time is given as,

$$\begin{aligned} m(0,3) &= \frac{1}{2}e^{-r_1}100\left[\frac{1}{2}e^{-r_{2,u}}100\left(e^{-r_{2,uu}} + e^{-r_{2,ud}}\right) + \frac{1}{2}e^{-r_{2,d}}100\left(e^{-r_{2,ud}} + e^{-r_{2,dd}}\right)\right] \\ &= \frac{1}{2}e^{-r_1}100\left[\frac{1}{2}e^{-r_{2,u}}100\left(e^{-4r_{2,dd}\sigma_3} + e^{-2r_{2,dd}\sigma_3}\right) + \frac{1}{2}e^{-r_{2,d}}100\left(e^{-2r_{2,dd}\sigma_3} + e^{-r_{2,dd}}\right)\right]. \end{aligned} \tag{3.9}$$

As such a recursive formula can be built to find the complete calibrated interest rate tree.

3.3.2 Arrow Debreu Calibration

When calibrating the binomial tree to a term structure, the tree prices have to match the observed market prices. There is generally two ways that a binomial tree can be calibrated, using forward induction or backward induction. Backward induction is a popular and easy method as described by Black, Derman and Toy [13] but at times leads to inaccurate matching of the observed market prices. The forward induction method introduced by Jamshidian [39] in 1991 calibrates the tree by considering how much a security with a payoff of R1 at that particular node and zero elsewhere will cost today. A security that pays R1 in a single state and zero elsewhere is referred to as an Arrow Debreu security. The backward induction method is essentially equating the implied one-period forward rates to the expected value of the short rates. Defining f_j to be the implied forward rate in period j, and r_i as the base line[27] rate in period i (Figure 8). Since each short rate r_i can occur with probability

$$2^{-(j-1)}\binom{j-1}{i-1}, \forall \ 1 \le i \le j, j > 1,$$

which means that

$$\sum_{i=1}^{j} 2^{-(j-1)}\binom{j-1}{i-1} r_j v_j^{i-1} = f_j,$$

and solving for r_j gives,

$$r_j = \left(\frac{2}{1+v_j}\right)^{j-1} f_j, \tag{3.10}$$

where $\binom{j-1}{i-1}$ is the shorthand notation for $(j-1)!/((i-j)!(i-1)!)$

[27] The base line rate is thehighest short rate found at each time tick of the binomial interest rate tree.

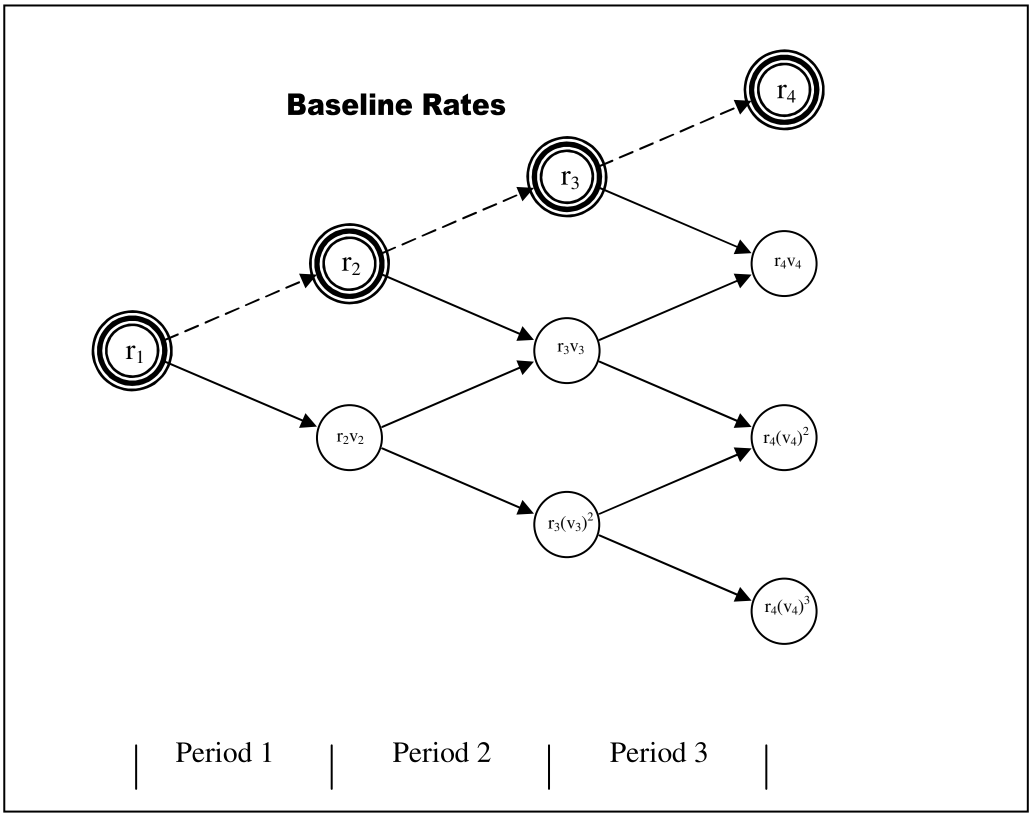

Figure 8 - Baseline rates using Arrow Debreu forward calibration.

The forward induction method is a little more tedious but produces much better calibration. Defining *AD(n,s)* as the price of an Arrow Debreu security in state *s* at time *n*, and *P(n)* as the price of an *n*-period zero coupon bond that pays R1 in all states at time *n* the following relation occurs

$$P(n) = \sum_{s=0}^{n} AD(n,s)\,.$$

Thus, the zero-coupon bond can be seen as a portfolio of Arrow Debreu securities. As forward induction is the expected value (under the risk-neutral measure *Q*) of the Arrow Debreu security it can be computed as

$$AD(n+1,s) \;=\; E_{n+1}^{Q}\left[AD(n,s)/\Im_n\right]$$

$$=\; AD(n,s)\,m(n,s)\left(1-\psi(n,s)\right)+AD(n,s-1)\,m(n,s-1)\,\psi(n,s-1),$$

where $m(n,s)$ is the price of the zero coupon bond maturing at time n in state s and is given by $1/(1+r(n,s))$, and $\psi(n,s)$ is the risk-neutral probability of an up move from state s at time n. If a state is unattainable or non-existent $AD(n,s)\equiv 0$. $\Im_n$ contains all the information of the previous Arrow Debreu securities from time 0 to n[28].

As an example if we are at time i with $i+1$ nodes, given a baseline rate for period i of r_i, and state prices for a prior period of $P_1,P_2,\dots,P_i$ with corresponding rates of $r_i,r_iv_i,\dots,r_iv_i^{i-1}$ we have to solve for $r_i=r$,

$$\frac{1}{(1+S(i))^i}=\sum_{k=1}^{i}\frac{P_k}{(1+rv^{k-1})} \tag{3.11}$$

by some iterative process (e.g. Newton Rhapson). $S(i)$ is the i-period spot (short) rate from observed market prices, whilst r_i is the baseline rate used in the calibrated binomial interest rate tree in Figure 8.

Having discussed the binomial share price and calibrated interest rate trees, the basic building blocks into solving the equity valuation PDEs in Chapter 5 have been laid. These interest rate trees were assumed to be risk-free rates[29], but in reality this may not be the case as the majority of CB issuers are not risk-free. As such an extension of the binomial trees with credit risk is discussed next in Chapter 4.

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[28] Formally it is known as a filtration $\left\{\Im_n^s\right\}_{n\geq 0}$ and denotes all the information generated by s on the interval [0,n], so that we know the exact path of the Arrow Debreu security.

[29] The interest rates that apply to government issued securities are considered to be risk-free.

4. Credit Risk

By definition credit risk "is the possibility that a bond issuer will default, by failing to repay principal and interest in a timely manner. Bonds issued by the federal government, for the most part, are immune from default (if the government needs more money it can just print more). Bonds issued by corporations are more likely to be defaulted on, since companies often go bankrupt. Municipalities occasionally default as well, although it is much less common."[30]

In Chapter 4 an extension to the binomial interest rate tree is formulated to account for risky interest rates. This is a short Chapter commencing with a brief literature review of credit risk models in Section 4.1 and a quick review of calibrating a risk-free term structure to a binomial tree in Section 4.2. Section 4.3 concludes with the construction of the risky interest rate tree from an observed, issuer specific term structure and the calculation of the implied (conditional) default probability

4.1 Introducing Default Probabilities

A recent Moody's sample between 1970 and 2000[31] indicated that default rates for rated CB issuers are higher than those without CB's in their capital structure, which is why credit risk has to be taken into consideration when valuing CB's. Traditionally default risk has been modelled using two distinct classes, structural-form models with the underlying being the asset value of the firm, and reduced-form models where default occurs via a specified process with an assumed recovery rate.

The Merton [53] model, which makes use of the Black, Scholes and Merton (BSM) European option model, proposes that a company will default when its asset value reaches a specific barrier. It is considered the first structural model and assumes that default can only happen at maturity of the option. McConnell and Schwartz's [50] model is considered the first reduced form model and usually assumes default to occur according to a Poisson process. Its downfall is that it doesn't take total market value

[30] http://www.investorwords.com/1210/credit_risk.html

[31] Hamilton, T., Stumpp, P. and Cantor, R., Moody's Investor Service (Global Credit Research), "Default and Recovery Rates of Convertible Bond Issuers: 1970 – 2000", July 2001

of the firm into account, which obviously precludes bankruptcy as the share price always stays above zero.

Generally speaking structural models are harder to implement than reduced form models as the underlying variable, firm asset value is unobservable in the market, whilst with reduced form models the issuers' stock price and volatility can be observed. When dealing with credit risk, structural models provide a link between the credit quality of a firm and the firms' economic and financial conditions, which means that defaults are determined endogenously. However, reduced form models derive their value in an explicit manner and hence produce a probability of default exogenously.

4.2 Calibrating Risk-Free Interest Rates (Quick Review)

The binomial risk free interest rate tree is constructed in the usual manner by letting the rate go up or down by a constant factor along each time tick with a guaranteed amount of 1 being received upon maturity. As such it can be calibrated to any market term structure by adjusting the interest rates at the bottom of each time tick. This is because the rates above differ by a factor of $e^{4\sigma}$, from each node below it as was seen in Figure 7. Figure 9 illustrates a 2 year zero coupon bond paying 1 at maturity with an associated probability π[32] representing the probability that interest rates will move up in the next time period. As discussed previously the risk-free interest rate tree can be calibrated to any term structure by using the risk-free zero-coupon bonds.

[32] In section 3 this probability was referred to as q^*, a risk-neutral probability and set to 0.5. Here it is the probability for both risk-free and risky interest rates to move up.

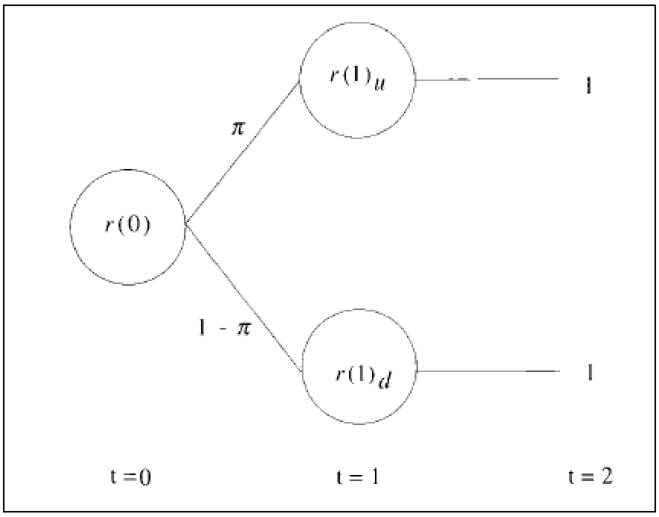

Figure 9 – Risk-free (non-defaultable) 2-year zero-coupon bond

The risky interest rate tree is determined in a slightly different manner, due to the defaultable states included in the tree with a constant recovery if default should occur as shown in Figure 10. Similarly these risky zero-coupon bonds can be obtained from market prices, making it possible to once again calibrate the risky tree to market data.

4.3 Calibrating Risky Interest Rates (Derivation of $\lambda_{\mu t}$)

To incorporate default risk we will adopt Jarrow and Turnbull's [40] model for the risk-free and risky discount rate process. They combine both processes into one tree with $\lambda_{\mu t}$ representing the probability that a firm will default during the interval *(t-1,t]* and δ a constant, fractional recovery rate of the nominal amount. A simple two period tree is shown in Figure 10 which is a Quadrinomial tree used to value defaultable bonds. π is a pseudo probability for interest rates to move up, as mentioned in the Section 4.2. Due to its discretisation it can handle issuer calls, investor put options and the underlying bond coupons. One of the disadvantages is that for longer maturities calculating the implied default probabilities becomes cumbersome due to the complex structure of the tree, although writing a recursive programme will do the trick.

Letting $m^*(0,t)$ be the value of a defaultable zero coupon bond and $m(0,t)$ a non-defaultable zero coupon bond both with a *t*-period maturity. Using the values obtained in the market we can derive $\lambda_{\mu 0}$ by simply solving the following equation

$$\begin{aligned} m^*(0,1) &= e^{-r(0)}\left[\lambda_{\mu 0}\times\delta+\left(1-\lambda_{\mu 0}\right)1\right] \\ &= m(0,1)\left[\lambda_{\mu 0}\times\delta+\left(1-\lambda_{\mu 0}\right)1\right]. \end{aligned} \tag{4.1}$$

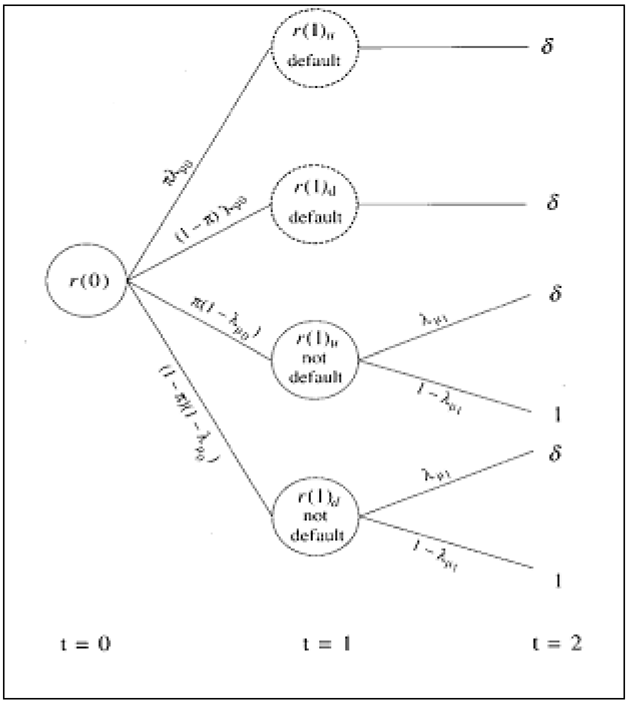

Figure 10 – Risky (defaultable) 2-year zero-coupon bond.

To derive the value of $\lambda_{\mu 1}$ requires us to solve a slightly more difficult equation shown below

$$m^*(0,2)=m(0,2)\left\{\lambda_{\mu 0}\delta+\left(1-\lambda_{\mu 0}\right)\left[\lambda_{\mu 1}\delta+\left(1-\lambda_{\mu 1}\delta\right)1\right]\right\}, \tag{4.2}$$

where

$$m(0,2)=e^{-r(0)}\left[\pi e^{-r(1)_u}+\left(1-\pi\right)e^{-r(1)_d}\right]. \tag{4.3}$$

By continuing this recursive methodology the implied default probabilities for any issuer can be derived so long as there are market prices of both defaultable[33] and non-defaultable zero – coupon bonds for the chosen maturities as outlined in [40]

[33] That is an issuer specific term structure to calculate the risky zero-coupon bonds, and government term structure for the risk-free zero-coupon bonds.

The important aspect to take away from this Chapter is the construction of the risky interest rate tree that can be calibrated to any term structure, if sufficient maturities exist. This is of course a generalized credit risk model that can be integrated into any discrete valuation model. As such it is included in the Hung and Wang CB model (HW) which will be examined in detail in Section 5.4. Now that the pre-requisites of equity valuations models have been addressed, the next Chapter will delve into the details of this valuation approach.

5. Equity Valuation Models

In this Chapter the popular equity valuation models will be discussed. Section 5.1 briefly describes the major advantage of these models, beginning with the Quadrinomial model in Section 5.2, which incorporates stochastic share prices and interest rates, including a correlation coefficient between the two, which is shown in Section 2.4 to play a vital role in building an accurate model. The calculation of the probabilities is reviewed including the boundary conditions, rollback value and parameter sensitivities. Section 5.2 analyses the popular Tsiveriotis and Fernandes [69] splitting model (TF) with different discount rates. The model is derived using the arbitrage free, probability of default and present value approaches with special mention of the boundary conditions including a trivial example. The TF model is seen as an improvement to the traditional model[34] due to the different discount rates used but does not give details about the share price upon default. As such Section 5.3 assesses the Ayache, Forsyth and Vetsal [6] (AFV) model, which develops upon the TF model by specifying what happens to the share price upon default. The basic AFV model with and without credit risk is discussed including a derivation of the risky bond price and inconsistencies with the TF model. The Goldman-Sachs [32] conversion adjusted discount rate model is proposed in Section 5.5 with a quick description and example. Section 5.6 applies the credit risk approach of Chapter 4 into the TF model by way of the Hung and Wang [36] (HW) default risk model. A discretisation of the continuous HW model is setup from the terminal nodes and backward induction process to an adjusted risk-neutral measure and practical example. Finally a comparison of the various equity models is evaluated in Section 5.7 with an illustrative example.

5.1 Background to Equity Valuation Models

Essentially there are two valuation methods, a firm value approach and an equity value approach. As with default risk the firm value approach uses the underlying value of the firm as the stochastic variable, while the equity value approach uses the firms' share price as the underlying variable. Once again firm value models are

[34] The traditional model follows the same procedure as the TF model with the splitting into two components, however, only one discount rate is applied, as opposed to the two in the TF model.

difficult to implement due to the unobservable value of the firm, which has to be estimated. Equity value models enjoy the benefit of having an observable variable, namely the share price trading in the market.

5.2 Quadrinomial Model

The process of valuing convertible bonds might seem simple if we decompose it into its components and value each item independently, but this approach does not take into account interest rate volatility and also makes the incorrect assumption that each component can be traded separately. As such a Quadrinomial tree that values the share price and interest rate movements together provides a more complete and accurate method of valuing CB's. It is a slight modification to the method used by Cox, Ross and Rubenstein [22] in 1979 to value options using a simple binomial tree. This method can also be adapted to handle various other embedded features that exist in CB's namely, call protection, putability, stepped coupons or sinking funds.

5.2.1 Constructing the Quadrinomial Tree

Just like binomial trees that have two moves from each node a Quadrinomial tree has four moves from each node, except the terminal nodes (Figure 11). This is because in a binomial tree there is just one underlying factor, the share price, whilst in a Quadrinomial tree there are two factors, the share price and interest rate. As a result at each time tick there are $(n+1)^2$ nodes, where $n \in \mathbb{Z}$ is the time tick. Due to the added complexity of having four moves from each node, an alternative is to re-number the tree so as to decompose it into two binomial trees. This is achieved by numbering the nodes according to the number of down moves of each factor, as can be seen in Figure 12. This method can be a disadvantage as the Quadrinomial tree will at times not discretise state space, but this simplifies the procedure and enables us to work in a two-dimensional plane.

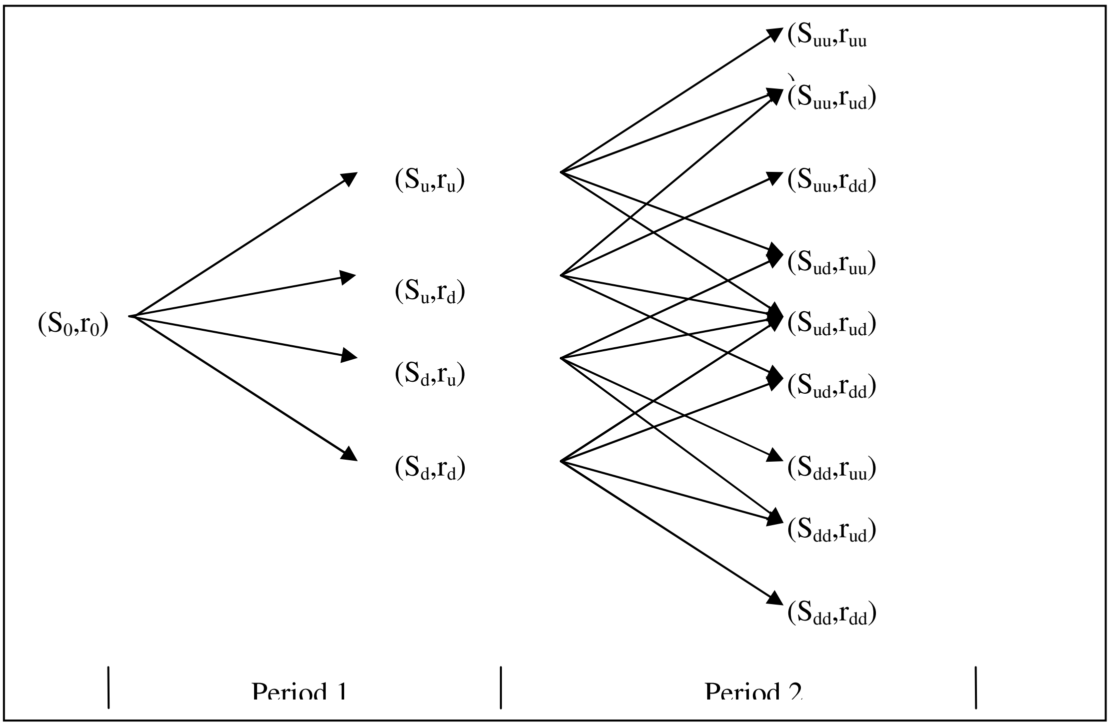

Figure 11 – Convertible bond tree with stochastic interest rates and share prices.

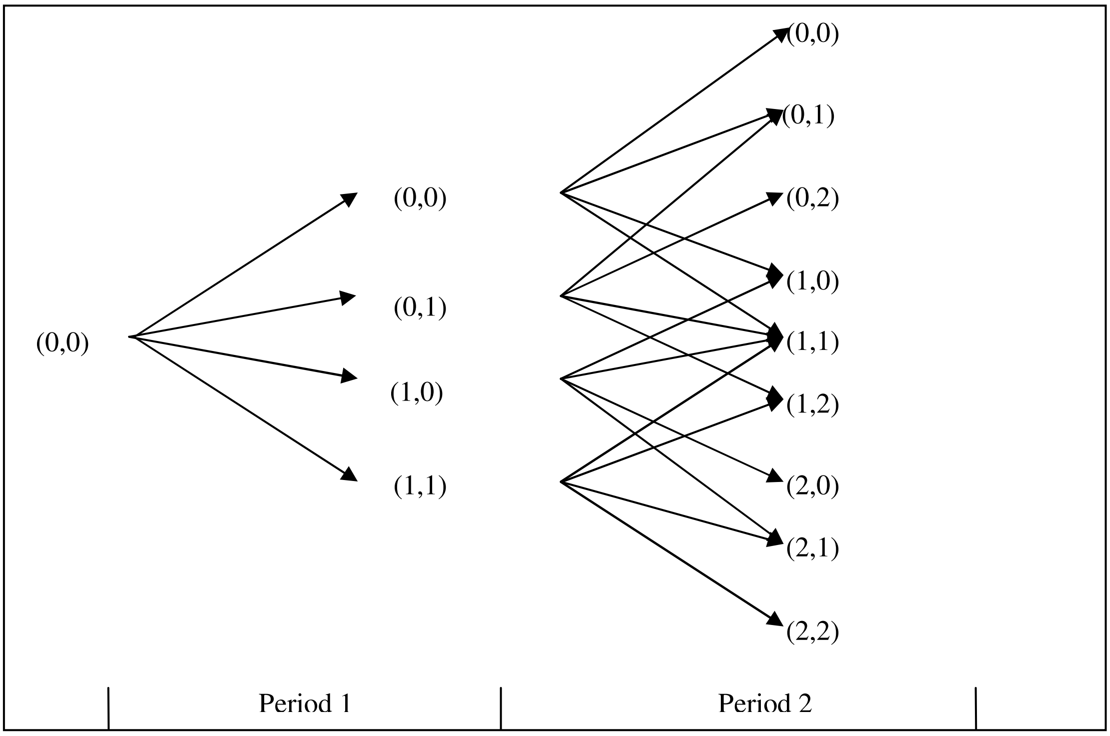

Figure 12 - Re-numbering Quadrinomial tree with two binomial trees.

The big advantage of this method is that if we wanted to find the children[35] of any particular node we would just need to add the fundamental moves to that node. The fundamental moves are $(0,0)$; $(0,1)$; $(1,0)$; and $(1,1)$. To understand the notation a little better the node (1,0) would be the node where the share price had one down move and the interest rate had zero down moves. As an example if we are at node $(1,0)$ and wanted to find out what and where in the tree its children are, we would add each of the fundamental moves to the node $(0,1)$ and arrive at the child node (e.g. $(0,1)+(0,1)=(1,1)$ for one of the children).

5.2.2 Correlated Rollback Value

When dealing with a Quadrinomial tree, the two factors that drive the tree might have some degree of correlation between them. Since we decomposed the tree into two independent binomial trees this correlation was lost. In order to create a correlated Quadrinomial tree from these two independent binomial trees, we need to determine the expected value of the joint process of S and r (i.e. $E^Q[Sr]$).

Defining the $rollback_{parent}$ in mathematical terms as

$$rollback_{parent} = e^{-r_{parent}\Delta t} E^Q\left[\chi_{child}\right],$$

and the children nodes, χ_{Child} as:

$$\chi^{uu}_{Child} \text{ - } (S_u, r_u) \text{ node}$$

$$\chi^{ud}_{Child} \text{ - } (S_u, r_d) \text{ node}$$

$$\chi^{du}_{Child} \text{ - } (S_d, r_u) \text{ node}$$

$$\chi^{dd}_{Child} \text{ - } (S_d, r_d) \text{ node}.$$

[35] The children in this sense refer to the nodes one time step ahead and have a positive probability of occurring from the current node.

Since we always assumed probabilities $p^* = q^* = 0.5$ and $(1-p^*) = (1-q^*) = 0.5$, for both up and down movements respectively of the share price and interest rate, each of the children occurs with probability[36] 0.25 due to their independence property.

So by definition

$$\begin{aligned} &Cov[S,r] = E[Sr] - E[S]E[r] \\ &\Rightarrow E[Sr] = Cov[S,r] + E[S]E[r]. \end{aligned} \tag{5.1}$$

The expected values and variances of S and r are easily found to be

$$E[S] = \frac{1}{2}[S_u + S_d] \tag{5.2}$$

$$E[r] = \frac{1}{2}[r_u + r_d] \tag{5.3}$$

$$\sigma_r^2 = \frac{1}{4}[r_u - r_d]^2 \tag{5.4}$$

$$\sigma_S^2 = \frac{1}{4}[S_u - S_d]^2. \tag{5.5}$$

With the covariance being found using the formula

$$Cov[S,r] = \rho_{Sr}\sigma_S\sigma_r .$$

Substitution of (5.4) and (5.5) into the above equation gives

$$Cov[S,r] = \frac{1}{4}\rho_{Sr}[S_u - S_d][r_u - r_d]. \tag{5.6}$$

Finally after substituting (5.2), (5.3) and (5.6) into (5.1), we arrive at

---

[36] P[XY]=P[X]P[Y] if the random variables X and Y are independent.

$$E[Sr] = \frac{1}{4}(1+\rho_{Sr})\left(\chi_{child}^{uu} + \chi_{child}^{dd}\right) + \frac{1}{4}(1-\rho_{Sr})\left(\chi_{child}^{ud} + \chi_{child}^{du}\right). \tag{5.7}$$

What is so important about this equation is that the probabilities for each child node can explicitly be seen as

	S_u	S_d
r_u	$\frac{1}{4}(1+\rho_{Sr})$	$\frac{1}{4}(1-\rho_{Sr})$
r_d	$\frac{1}{4}(1-\rho_{Sr})$	$\frac{1}{4}(1+\rho_{Sr})$

Table 1 - Probability moves for each state in the Quadrinomial tree.

Thus, the $rollback_{parent}$, now with correlation, can be given as

$$rollback_{parent} = e^{-r_{parent}\Delta t}\left[\frac{1}{4}(1+\rho_{Sr})\left(\chi_{child}^{uu} + \chi_{child}^{dd}\right) + \frac{1}{4}(1-\rho_{Sr})\left(\chi_{child}^{ud} + \chi_{child}^{du}\right)\right]. \tag{5.8}$$

5.2.3 Backward Valuation Process

At maturity of the CB the investor must choose whether to convert the bond into equity or receive the redemption value of the bond. If the conversion ratio is α then the conversion value is αS_T and the bonds redemption value is $N\eta$, where N is the par value and η is the redemption amount in percentage. Therefore a rational investor will choose

$$\chi_T = \max\left[\alpha_T S_T, N\eta\right]. \tag{5.9}$$

By working backwards through the tree we can arrive at time zero and thus find a value of the CB. To obtain the arbitrage-free prices at each parent node we need to discount the expected value from each of the children nodes. This value as defined in (5.8) is the $rollback_{parent}$, so that the value of the CB at time t is

$$\chi_t = \max\left[\min\left(rollback_{parent}, c_t + \theta_t\right), \alpha_t S_t, b_t\right], \tag{5.10}$$

where

$$c_t = \begin{cases} c_t & , \quad t \in [\tau_{c1}, \tau_{c2}] \\ \infty & , \quad t \notin [\tau_{c1}, \tau_{c2}], \end{cases}$$

and similarly

$$\alpha_t = \begin{cases} \alpha_t & , \quad t \in [\tau_{\alpha 1}, \tau_{\alpha 2}] \\ 0 & , \quad t \notin [\tau_{\alpha 1}, \tau_{\alpha 2}]. \end{cases}$$

b_t is the price of a vanilla bond with the same characteristics of the company issuing the CB. The price of the CB has to at least be equal to or greater than b_t since this would create an arbitrage opportunity. The safety premium θ_t is included due to the empirical fact by Ingersoll [37] that firm's only call when the price of the security is above the call price by a certain premium. This is because the issuing firm will want to make sure that at the end of the conversion period the price of the CB is still above the call price. In this dissertation the safety premium is ignored and set to zero.

5.2.4 Parameter Sensitivities

In this section we will look at what effect changes to some parameters have on the value of the convertible bond using the Quadrinomial tree model. We assume the following values for the parameters when determining the sensitivities[37]:

[37] The yield curve used for the values obtained in this section can be found in the Appendix A4.

T = *maturity*	5
Share Price	50.00
N = *Par Value*	100.00
Annual Coupon Rate	6%
Δt = *Time Tick*	0.125
q = *Dividend Rate*	1%
α_t = *Conversion Ratio*	2
$\gamma_{t,T}$ = *Credit Spread*	0.25%
c_t = *Call Price*	120.00
θ_t = *Safety Premium*	0
η = *Redemption Value*	1
$\tau_{\alpha 1}$ = *Call Start Date*	0
$\tau_{\alpha 2}$ = *Call End Date*	5

The three important factors that drive the CB price are:

- The share price
- Volatility of the share price, and
- The correlation between the share price and interest rates.

We compare these three factors against the price of the CB using the Quadrinomial (Quad) model with the accompanying graphs. It is strange to see that interest rates are not a significant factor, although they become important when the CB is trading deep out-the-money in the distressed zone. Credit spread is more of an issue here with the result being that a higher credit spread reduces the CB price; likewise a lower credit spread increases the price. Brennan and Schwartz [14] looked at the price difference between using constant interest rates and stochastic ones for the same CB. They concluded that "for a reasonable range of interest rate levels the errors from the certain interest rate model are likely to be slight, and therefore, for practical purposes it may be preferable to use this simpler model [constant interest rates] for valuing convertible bonds." As such interest rates by themselves are not a major factor; however the correlation that exists between the share price and interest rates are crucial as explained next.

It may seem ignorant to ignore the interest rate volatility, but looking at Figure 13 we see that as the interest rate volatility increases the CB price decreases, although not by much. As such most researchers suggest that constant interest rates are plausible as the net difference is not large enough to warrant the additional complexity. In fact Brennan and Schwartz's [14] claim maybe right for the primary bond market, but in the secondary market things are little more complicated. As an example if we have a busted, callable CB, which is trading deep out-the-money, with a worthless conversion option, then we would expect it to behave like an ordinary callable bond. As such it would be ludicrous to value this bond in the normal callable bond market without taking stochastic interest rates into account. Thus interest-rate sensitivity for CB's become increasingly important for out-the-money issues.

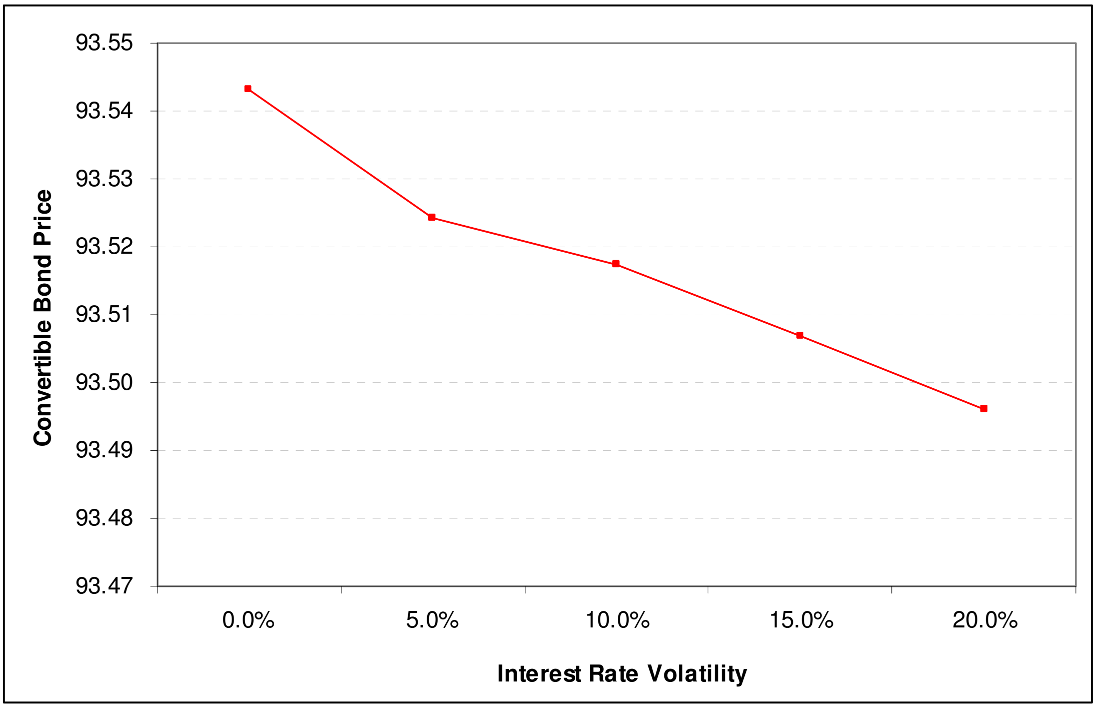

Figure 13 - CB price sensitivity to interest rate volatility using Quadrinomial model with call interval [0,5], S=50 and ρ=-0.5.

Another important point overlooked by practitioners and researchers is the correlation that exists between interest rates and share prices. In the BSM framework it is assumed that the share price follows a lognormal distribution with Brownian motion, and grows at the risk-free rate (drift). What this implies is that if interest rates go up so do share prices, but everyone knows that interest rates are negatively correlated with the equity market (generally speaking of course), as such the share price will initially drop and then grow at a higher drift rate. In so doing negative correlation will lead to lower CB prices whereas positive correlation will lead to higher prices. A CB's fixed-income (debt) value can be seen as an average value across all interest rate scenarios, with high values when interest rates are low, and low values when interest rates are high. If there is positive correlation, the case where interest rates are high does not play that significant a role as the higher share price causes a higher conversion value, and thus CB price. Figure 14 clearly illustrates the direct relationship between correlation and the CB price, with the CB price falling from 115.75 when $\rho=1$ to 54.69 when $\rho=-1$, which is a 53% decrease in value. The bond floor is calculated to be 90.85, so if we limit the lower value of the CB price to this value, the decrease in the CB price is not as dramatic but still substantial. Thus correlation should not be taken lightly.

As mentioned in Section 2.4, the value of the convertible bond when using the Margrabe exchange model is extremely elastic to the correlation between the bond and share price. Since the Margrabe model deals with share and bond price correlation, and the Quadrinomial model deals with share price and interest rate correlation, it might seem that the two models contradict each other when reviewing their sensitivity on the CB price. This is because an increase in correlation will lead to an increase in the CB price when using the Quadrinomial model, but a decrease in CB price when using the Margrabe model and vice versa. This is a result of the inverse relationship between bond prices and interest rates, and in effect leads to the same conclusion, albeit with different underlying variables.

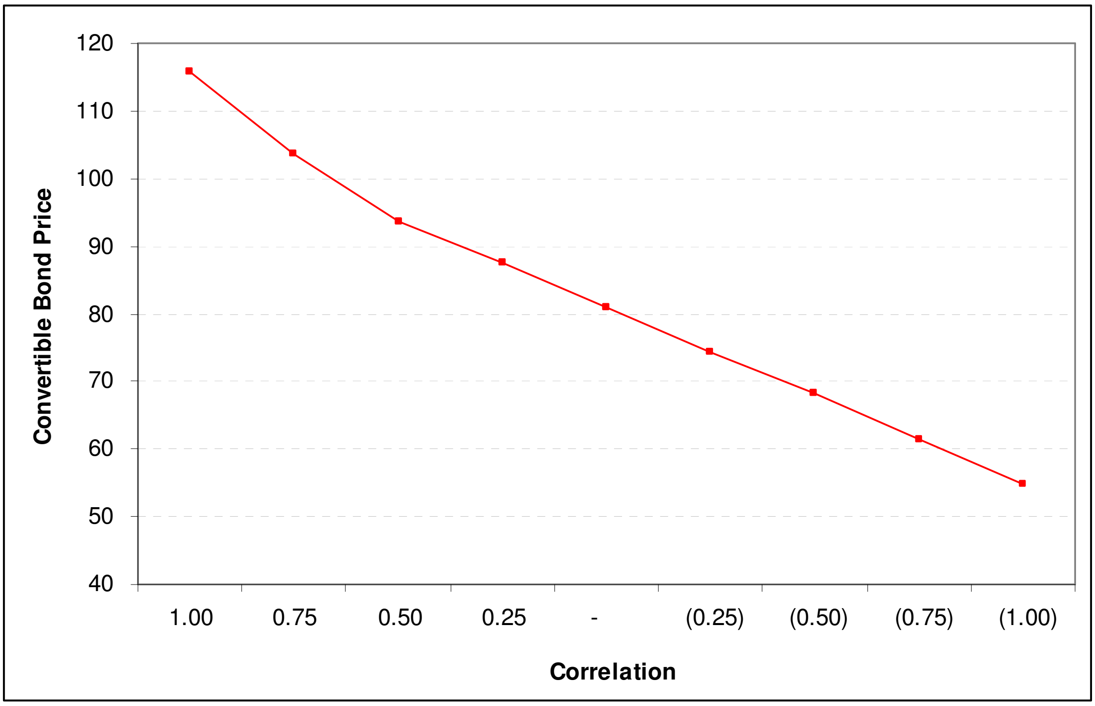

Figure 14 – CB price sensitivity to correlation between interest rates and share price using Quadrinomial model with call interval [0,5], interest rate volatility of 6% and S=50.

A likeable characteristic of the Quadrinomial model is the negative convexity present for low share values. Most CB models do not take this into account and assume that the CB takes on its investment/straight bond value, but due its sub-ordination in the capital structure of the firm, it should in theory take on a value lower then the straight bond floor when in the distressed zone. In Figure 15 the negative convexity becomes apparent for share values less than the conversion price of 50[38]. Negative convexity in CB's will be discussed in more depth in Chapter 9. As mentioned earlier, correlation is an important factor to consider as the CB price track for differing values of rho is also shown in Figure 15. It can clearly be seen that the higher the correlation, the higher the CB price and the lower correlation, the lower the price. It is interesting to note that when $\rho<1$ it always trades below is conversion value. As discussed earlier, when the correlation is not exactly 1, there is an initial drop in the share value before it starts growing with a higher drift rate[39]. This initial drop suppresses the CB price,

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[38] The conversion price is calculated as the face value of the convertible bond divided by its conversion ratio.

[39] It will take on a higher drift rate if the correlation is positive, else a lower drift rate if negative correlation is present.

although in reality the rational investor will convert the CB, which will increase its value back to the conversion value and may also include some premium.

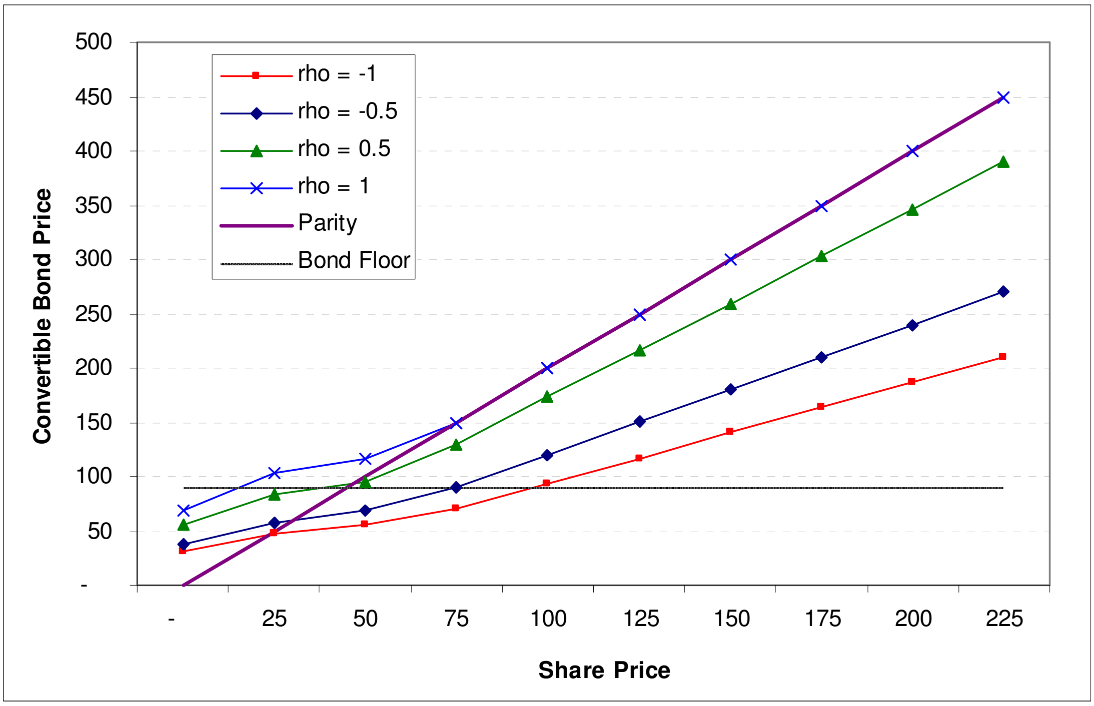

Figure 15 – CB price sensitivity to share price using Quadrinomial model with call interval [0,5], interest rate volatility of 6% and share volatility of 20%.

Finally, looking at Figure 16 we see that low share price volatility leads to low CB prices and higher volatility leads to higher CB prices. This is quite intuitive as the higher volatility increases the value of the embedded conversion option and call option. It is difficult to see which option will increase in value more relative to the other, as the call option is to the benefit of the issuer and so decreases the CB price, whilst the conversion option is to the benefit of the investor and increases the CB price. As such the conversion price is set to 50 to indicate the at-the-money approximation of the CB. When $S=50$ the CB price increases and decreases as the share volatility moves higher, but generally tends to increase. When $S=40$, the CB is considered out-the-money, and so increases much more radically[40] for higher values of volatility. The in-the-money CB does not increase much for higher volatility when $S=60$ and needs a substantially large increase in volatility to achieve greater CB

[40] The rate of increase, or first derivative is much higher for the out-the-money CB price when volatility increases relative to at/in-the-money first derivatives.

prices. This concludes the Quadrinomial model with the discussion moving onto the TF model in the next section.

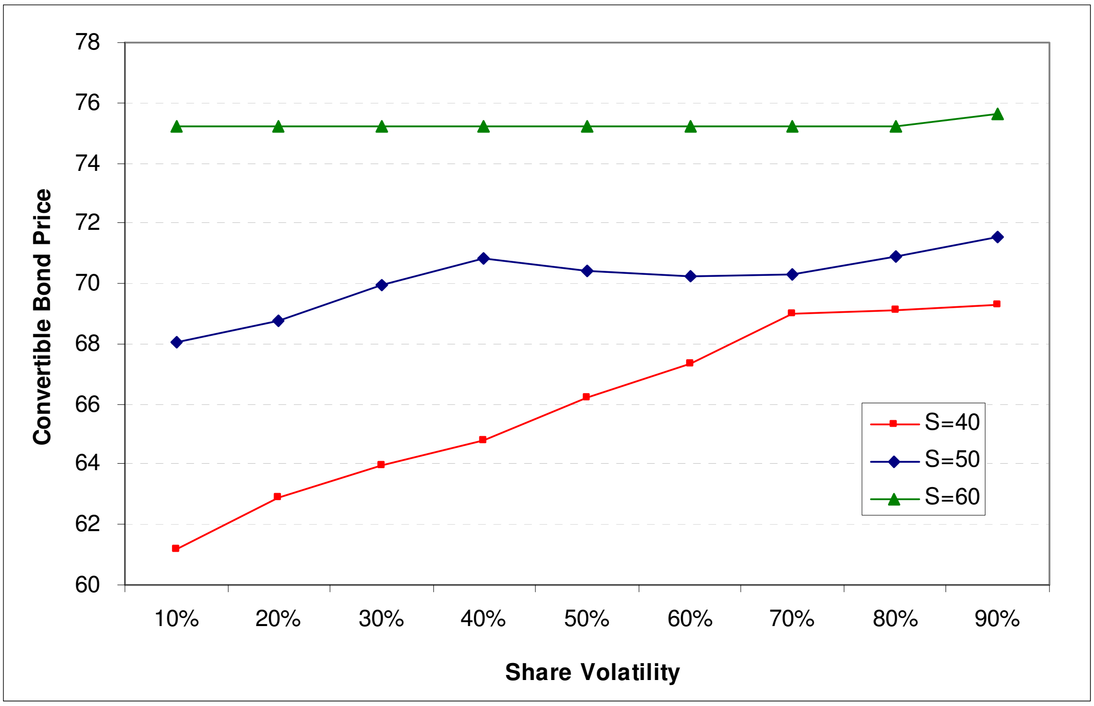

Figure 16 – CB price sensitivity to share price volatility using Quadrinomial model with call interval [0,5], interest rate volatility of 6% and ρ=-0.5.

5.3 Tsiveriotis and Fernandes

According to Tsiveriotis and Fernandes [69] CB's are viewed as derivatives of two underlying variables, namely equity and interest rates. As such a Black-Scholes-Merton model would be an appropriate choice. However due to the small enhancement in the model by allowing stochastic interest rates, it is not warranted and thus constant rates are used. The BSM model can incorporate credit risk by introducing a credit spread into the model although this would be unreasonable since the equity portion should not be exposed to credit, only the cash part. As stated by Tsiveriotis and Fernandes "the equity upside has no-default risk since the issuer can always deliver its own stock [whereas] coupons and principal payments … depend on the issuer's timely access to required cash amounts, which introduces credit risk". As such the CB is split into two components: a cash-only (COCB) part, which is subject to default risk and an equity part, which is not. The COCB entitles the holder "to all cash flows[41], and no equity flows, that an optimally behaving holder of the corresponding CB would receive." Since the CB is a function of the share, the COCB is also a function of the share and thus should also be represented by a BSM equation.

5.3.1 The Model

Letting the price of the COCB be v and the price of the CB be g[42], we have that $w = (g - v)$ is the value of the CB related to payments from equity[43] and as such discounted at the risk-free rate r. This leads to a pair of coupled differential equations that can be solved to find the value of a CB.

Letting the value of the CB be represented as $g(S,t)$ and $f(t)$ be the present value of all coupon payments, leads us to the following BSM equation by applying Itô's Lemma,

$$\frac{\partial g}{\partial t} + (r-q)S\frac{\partial g}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 g}{\partial S^2} = rg\,. \tag{5.11}$$

Similarly equations for w and v can be expressed as

$$\text{Equity Only}: \quad \frac{\partial w}{\partial t} + (r-q)S\frac{\partial w}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 w}{\partial S^2} = rw\,. \tag{5.12}$$

$$COCB: \quad \frac{\partial v}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 v}{\partial t^2} + \mu S\frac{\partial v}{\partial S} - (r + r_c)v + f(t) = 0\,. \tag{5.13}$$

respectively where r_c is the credit spread. Therefore the value of the CB is given by $g = v + w$, which when valued in the usual arbitrage-free setting is set equal to a

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[41] These cashflows consist of coupon payments, principal value, and cash payments received by exercising the embedded put options.

[42] The notation of the CB price was given as χ in Chapter 2. Just to clarify $g = \chi$ and is done to improve the readability of the forthcoming equations.

[43] The equity payments relate to the conversion value if the CB is converted and the call amount received if the embedded call option is exercised by the issuer.

combination of the risk-free and risky interest rates less any future coupon payment stream.

$$
\begin{aligned}
CB: &\quad \frac{\partial g}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 g}{\partial t^2}+\mu S\frac{\partial g}{\partial S}=rw+v\left(r+r_c\right)-f\left(t\right)\\
\Rightarrow &\quad \frac{\partial g}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 g}{\partial t^2}+\mu S\frac{\partial g}{\partial S}-rw-v\left(r+r_c\right)+f\left(t\right)=0\\
\Rightarrow &\quad \frac{\partial g}{\partial t}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 g}{\partial t^2}+\mu S\frac{\partial g}{\partial S}-g\left(r+\hat{s}\right)+f\left(t\right)=0,
\end{aligned}
\tag{5.14}
$$

$\hat{s}=r_c\left(v/g\right)$. As can be seen the interest rate used to discount the CB in equation (5.14) is $\left(r+\hat{s}\right)$. Intuitively this means that as the cash-only part (*v*) of the CB gains value, $\hat{s}$ increases, and so does the interest rate. As the equity component (*w*) increases, $\hat{s}$ decreases and as a result so does the discount rate. This is to be expected, because if the CB is in the money it behaves like equity and so its cash flows should be discounted at the risk-free rate. On the other hand if the bond is out the money, it behaves like debt and as such subject to default risk, which causes the interest rate to increase. At first glance it may seem that equation (5.13) is independent of (5.14) but they are coupled due to their boundary conditions and American optionality features. Reviewing some basic concepts once again we assume that the share price follows the usual Geometric Brownian motion

$$dS=\mu Sdt+\sigma SdW, \tag{5.15}$$

with μ,σ and dW being the drift, volatility and Wiener process respectively. Defining some function $F\left(S,t\right)$ and using Itô's Lemma gives the familiar[44]

$$dF=\left(\mu F\frac{\partial F}{\partial S}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 F}{\partial S^2}+\frac{\partial F}{\partial t}\right)dt+\sigma S\frac{\partial F}{\partial S}dW+o\left(dt\right). \tag{5.16}$$

[44] *o(dt)* is the shorthand notation representing all higher order terms involving *dt* that are a result of the Itô's lemma.

The COCB price, v and the CB price, g are financial derivatives dependent only on time and the share price with a constant risk-free interest rate, thus ignoring credit risk, gives us

$$dg = \left(\mu g \frac{\partial g}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + \frac{\partial g}{\partial t} \right) dt + \sigma S \frac{\partial g}{\partial S} dW \,, \tag{5.17}$$

$$dv = \left(\mu v \frac{\partial v}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 v}{\partial S^2} + \frac{\partial v}{\partial t} \right) dt + \sigma S \frac{\partial v}{\partial S} dW \,. \tag{5.18}$$

As mentioned previously v includes g, and v-g is the value arising from convertibility.

5.3.2 Arbitrage Free Approach

For a small change in time it will make sense to assume that the change in the price of the CB be given as in (5.18). If we include credit risk it becomes apparent that the CB will lose some of its value due to default and so a portion $r_c v dt$, where r_c is the credit spread, will be lost over the small time interval dt. So, adjusting (5.18) to take into account the credit spread, we have

$$dg = \left(\mu S \frac{\partial g}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + \frac{\partial g}{\partial t} \right) dt + \sigma S \frac{\partial g}{\partial S} dW - r_c v dt \,. \tag{5.19}$$

We need to set up a portfolio and replicate the outcomes in each state to find a fair value for the CB. Consider a portfolio π consisting of a long position in one CB, and a short position in Δ shares of stock, i.e. $\pi = g - \Delta S$. Thus the change in the portfolio is given as

$$d\pi = dg - \Delta dS \,. \tag{5.20}$$

Substituting (5.15) and (5.19) into (5.20), and setting $\Delta = \partial g / \partial S$, to eliminate the Wiener Process dW, gives us

$$d\pi = \left(\frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + \frac{\partial g}{\partial t} - r_c v \right) dt\,. \tag{5.21}$$

In an arbitrage-free world we would expect this portfolio to earn the risk-free rate, so

$$d\pi = r\pi dt\,, \tag{5.22}$$

Thus,

$$\begin{aligned} r\pi dt &= \left(\frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + \frac{\partial g}{\partial t} - r_c v \right) dt \\ \Rightarrow r\pi &= \frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + \frac{\partial g}{\partial t} - r_c v. \end{aligned} \tag{5.23}$$

Since $\pi = g - (\partial g / \partial S) S$, we have

$$r\left(g - S\frac{\partial g}{\partial S} \right) = \frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + \frac{\partial g}{\partial t} - r_c v\,, \tag{5.24}$$

which can be re-written as

$$\frac{\partial g}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + rS\frac{\partial g}{\partial S} - rg - r_c v = 0\,. \tag{5.25}$$

This is the PDE from (5.14); in a similar replicating portfolio with g replaced with v, equation (5.13) can also be derived.

5.3.3 Probability of Default Approach

As with all probability approaches we need to find the convertible PDE using an expectations technique; however we need to assume that default risk is diversifiable. To define risk and return Markowitz's [50] first introduced the efficient frontier and mean-variance portfolio selection in 1952. Sharpe [63] later expanded on Markowitz's earlier work and developed the famous capital asset pricing model (CAPM) that described total risk to be attributable to two elements, systematic and unsystematic risk. Systematic risk (market risk) is the non-diversifiable[45] risk that exists in a portfolio which an investor is rewarded for. Unsystematic risk (firm-specific risk) is the diversifiable risk inherent in a portfolio. It has to do with the idiosyncratic risk of the individual asset, which according to Sharpe [63] is not rewarded because it can be eliminated through a well-diversified portfolio. As such we need to assume that our constructed portfolio π is well diversified so that default risk (unsystematic risk) is eliminated. Thus referring to the previous portfolio π in equation (5.20) we have,

$$d\pi = dg - \Delta dS\,. \tag{5.26}$$

Substituting equations (5.15) and (5.17) into (5.26), and setting $\Delta = \partial g / \partial S$ gives

$$d\pi = \left(\frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + \frac{\partial g}{\partial t} \right) dt\,, \tag{5.27}$$

which assumes no default risk. To introduce credit risk we use a slight modification from the arbitrage free approach. We assume that,

- The probability of default in the time interval $[t, t+dt]$ is γdt.
- Upon default the holder will lose the entire COCB value, v[46].
- The share price is unchanged upon default, which is a little unrealistic but assumed so for simplicity.

So, assuming only two outcomes in the next time interval, equation (5.13) becomes

$$d\pi = (1 - \gamma dt)\, d\pi \left(\frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + \frac{\partial g}{\partial t} - r_c v \right) dt - \gamma dt v\,. \tag{5.28}$$

This simplifies to[47]

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[45] Market convention is to describe this as the beta of the asset.

[46] This cash-only portion can be seen as the loss due to default.

[47] Ignoring terms involving dt^2 when dt is assumed to be small.

$$d\pi = \left(\frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + \frac{\partial g}{\partial t} \right) dt - \gamma dtv \, . \tag{5.29}$$

We assume that default risk is diversifiable, meaning that we do not get rewarded for additional risk, so that the portfolio earns the risk-free rate as follows[48],

$$E(d\pi) = r\pi dt = r(g - \Delta S) dt \, . \tag{5.30}$$

Equating (5.29) and (5.30) leaves us with the following PDE

$$\frac{\partial g}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 g}{\partial S^2} + rS \frac{\partial g}{\partial S} - rg - \gamma v = 0 \, . \tag{5.31}$$

Assuming that γ is the credit spread gives us the PDE derived in equation (5.14). (5.13) can be derived in a similar fashion.

5.3.4 Present Value Approach

In this approach the fundamental principle in finance is employed, namely using the expected present value method. Taking a future point in time $t + dt$, the value of the COCB is $v(S + dS, t + dt)$. Assuming that the discount rate for the COCB is ε, then the value today is

$$v(S,t) = \frac{1}{1 + \varepsilon dt} E\big(v(S + dS, t + dt)\big) , \tag{5.32}$$

rearranging gives us

$$\varepsilon v(S,t) dt = E\big(v(S + dS, t + dt)\big) - v(S,t) \, . \tag{5.33}$$

Since

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[48] $E(.)$ denotes the expected value of the portfolio.

$$E\left(v\left(S+dS,t+dt\right)\right)-v\left(S,t\right)=E\left(v\left(S,t\right)+dv\right)-v\left(S,t\right)=E\left(dv\right), \tag{5.34}$$

and

$$\varepsilon v\left(S,t\right)dt=E\left(dv\right), \tag{5.35}$$

then according to Itô's Lemma we have,

$$dv=\left(\mu S\frac{\partial v}{\partial S}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 v}{\partial S^2}+\frac{\partial v}{\partial t}\right)dt+\sigma S\frac{\partial v}{\partial S}dW, \tag{5.36}$$

and noting that $E\left(dW\right)=0$, implies that

$$E\left(dv\right)=\left(\mu S\frac{\partial v}{\partial S}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 v}{\partial S^2}+\frac{\partial v}{\partial t}\right)dt. \tag{5.37}$$

Combining (5.35) into (5.37) gives the familiar looking PDE

$$\frac{\partial v}{\partial t}+\mu S\frac{\partial v}{\partial S}+\frac{\sigma^2 S^2}{2}\frac{\partial^2 v}{\partial S^2}-\varepsilon v=0. \tag{5.38}$$

This is equation (5.13) with ε replaced with $\left(r+r_c\right)$ and no discounted coupon stream $f\left(t\right)$. Since this is the risky bond component of the CB it makes sense to include the risky rate in the calculation. (5.14) can be derived in a similar manner although it must be noted that $\left(g-v\right)$ is the equity component and so should be discounted at the risk-free rate, r only.

5.3.5 Boundary Conditions

Using the familiar notation in Chapter 2 with τ_{c1} and τ_{p1} being defined as the start date for call period and start date for put period respectively, the terminal conditions for w and v at expiration are given by[49]:

$$w(S,T)=\begin{cases}\alpha S_T & \text{for } S\geq \dfrac{N}{\alpha}\\ 0 & \text{elsewhere}\end{cases} \tag{5.39}$$

$$v(S,T)=\begin{cases}0 & \text{for } S\geq \dfrac{N}{\alpha}\\ N & \text{elsewhere.}\end{cases} \tag{5.40}$$

Interim boundary conditions due to:

1. Conversion

$$w(S,t)\geq \alpha S_t \quad \mathit{for}\ t\in[0,T] \tag{5.41}$$

$$v(S,t)=0 \quad \mathit{if}\ w\leq \alpha S_t \quad \mathit{for}\ t\in[0,T] \quad , \tag{5.42}$$

2. Callability of the CB issuer, where the holder still has the option to convert if the bond is called after the call start time τ_{c1}

$$w(S,t)\leq \max(c_t,\alpha S_t) \quad \mathit{for}\ t\in[\tau_{c1},T] \tag{5.43}$$

$$v(S,t)=0 \quad \mathit{if}\ w\geq c_t \quad \mathit{for}\ t\in[\tau_{c1},T], \tag{5.44}$$

3. Putability by the CB holder becomes active after the put start time τ_{p1}

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[49] Assuming no coupons for simplicity. If coupons were present then the critical decision would be whether the conversion value were greater than the face value plus coupon at maturity. As such the critical decision applies only to the face value.

$$w(S,t) \geq p_t \quad for\ t \in [\tau_{p1}, T] \tag{5.45}$$

$$v(S,t) = p_t \quad if\ w \leq p_t \quad for\ t \in [\tau_{p1}, T]. \tag{5.46}$$

A binomial lattice is most commonly used to solve this system of PDEs, although other finite difference schemes could be used to improve the accuracy.

5.3.6 Numerical Example

To solve the continuous TF model a discretisation method is required. The most common approach is to discretise both time and state space into a uniform rectangular mesh by using finite difference methods (FDM). There are various FDMs that one can use, but the simplest is the explicit scheme using a binomial tree, which relies on values calculated in a prior time period.

Assume the following values are given for the parameters of the share and bond, with no embedded call or put options,

Stock	
Current Share Price	50.00
Share price volatility	30%
Risk-free rate	7.00%
Share uptick move	1.35
Dividend rate	0%
Bond	
Face value	100
Bond yield	10.00%
Coupon rate	8.00%
Coupon frequency	1
Credit Spread	0.50%
Maturity	3 yrs
Conversion Ratio	2
No. of time steps	3

Figure 17 gives a discrete 3-period version of the continuous TF model with the binomial share price tree overlayed. A,B,C and D for each of the nodes represent the share price, equity component, debt (cash-only) component and CB price respectively. The CB price is the sum of both the equity and debt components (D=B+C). The equity component (B) consists of the conversion value or embedded put option strike, whereas the debt component (C) consists of the redemption value or embedded call option strike. At time period 3 the CB investor is assumed to be rational and choose the maximum of the redemption or conversion value. As such in block B the conversion value is 319.60 and the redemption value is 108, so the investor converts and the equity component equals 319.60 whilst the debt component equals 0, leaving the CB price to be 319.60. The opposite is true for block E where the share price is relatively low resulting in a conversion value of 67.88, thus it would be wiser to redeem the CB for 108. Therefore the debt component takes on the redemption value of 108 and the equity component equals zero giving the CB a value of 108. Moving backward through the tree, we need to calculate the expected present value of each component. The risk-neutral probability measure is

$$p^* = \frac{e^{0.07} - (1.35)^{-1}}{1.35 - (1.35)^{-1}} = 0.5446$$

As such blocks F and G are calculated as follows

F: $e^{-0.07}\left[134.99(0.5446) + 0(1-0.5446)\right] = 68.54$

G: $e^{-(0.07+0.005)}\left[0(0.5446) + 108(1-0.5446)\right] + 8.00 = 53.63$

Take note that block G contains the discounted expected present value plus the coupon of 8. If the time periods were smaller in size, and a coupon payment occurred between the intervals then the discounted coupon would need to be included in the debt component for each node in that time interval. It is important to note that the debt component (G) is discounted at a higher interest rate[50] compared to the equity component (F), which is discounted at the risk-free interest rate. This is one of the critical points of the TF model that assumes the issuer is always able to deliver its

[50] The higher interest rate includes the risk-free rate and credit spread.

underlying equity, but when any debt servicing or similarly related cash payment is required, an element of risk exists. As such the debt component is discounted by the risk-free rate and the credit spread to account for the additional risk.

Figure 18 shows the binomial tree if an embedded call option and put option with strikes of 120, are present. As can be seen the price of the CB falls from 128.21 to 119.24 due to the issuers call option being exercised. Block H is the debt component, which has dropped from 8 to 0. This is due to the presence of the call option, which the issuer has exercised and forced conversion upon the holder, thereby missing out on the coupon. As can be seen blocks I and J are quite different from G and H. The value of the CB in the middle node in Figure 17 is given as 122.17, which is higher than the call strike of 120. Thus in Figure 18 the issuer exercises the call option resulting in the equity component realizing the strike rate and the debt component equal to 0. The reason the call value is seen as risk-free is because the issuer is purchasing the CB at a price, lower than its market value. The firm can call the CB and refinance it with a cheaper issue.

In Figure 17, the lower node at time period 1 gives a CB value of 115.61. The corresponding node (block K) in Figure 18 gives a CB value of 120. This is due to the holder exercising the put option, as the strike value of 120 is greater than the current CB value of 115.61. As such this put value is incorporated into the debt component as it indicates some stress on the part of the issuing firm to purchase the CB at price higher than market value. The equity component takes on a value of 0.

Using this decoupled PDE approach and valuing each component according to its level of risk makes intuitive sense but the subject on default risk does not seem to be adequately answered, i.e. what happens to the share price upon default. According to Tsiveriotis and Fernandes [69] the share price is unchanged which is not a realistic assumption. Thus we address this matter in the next approach with Ayache, Forsyth and Vetsal [6].

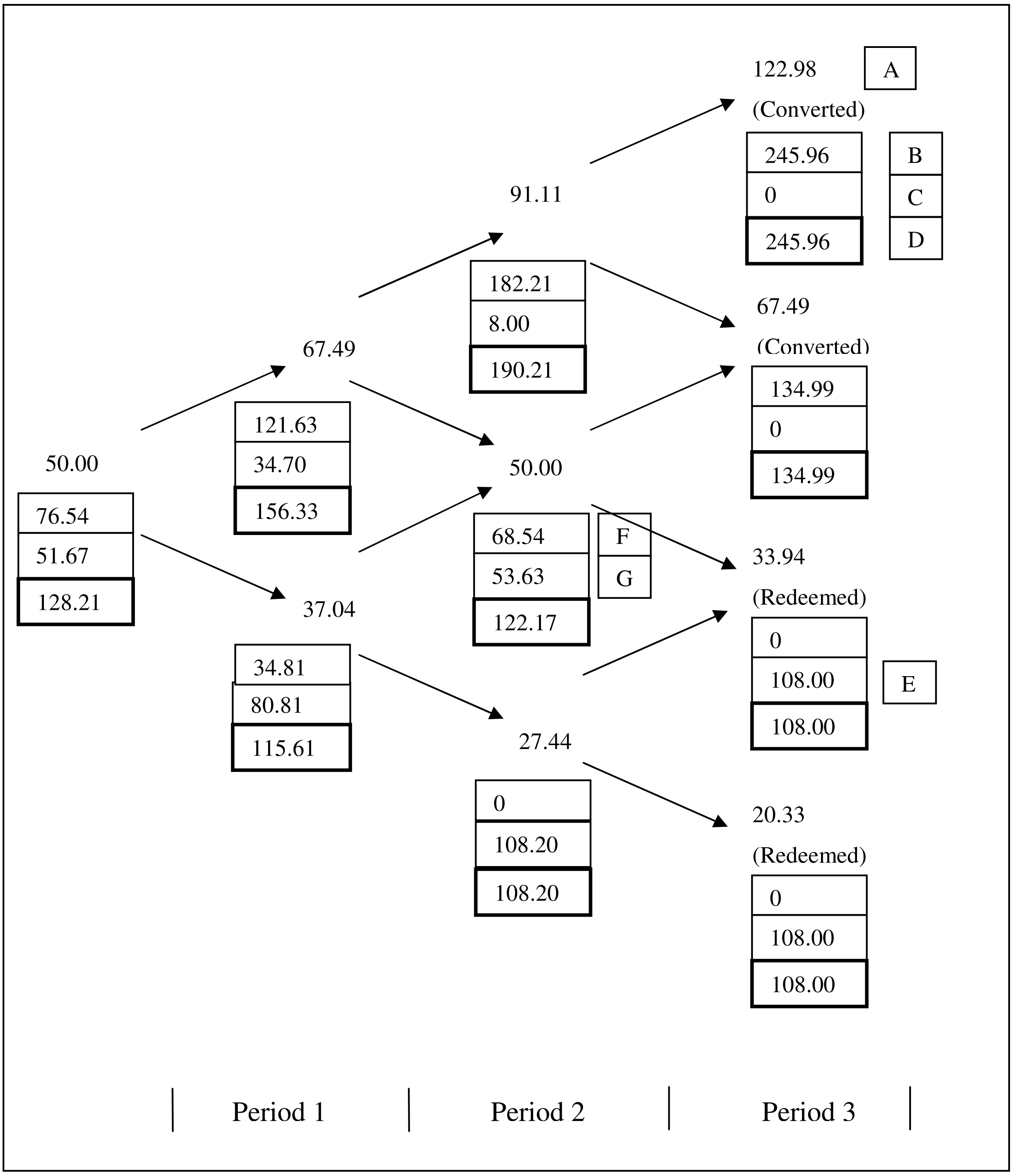

Figure 17 – 3-period binomial tree of the TF model with the conversion option exercised at maturity. The value without a rectangular border is the share price for each node. The first value in the border is the equity component, the second is the debt (cash-only) component and the final border contains the CB value which is the sum of the two components.

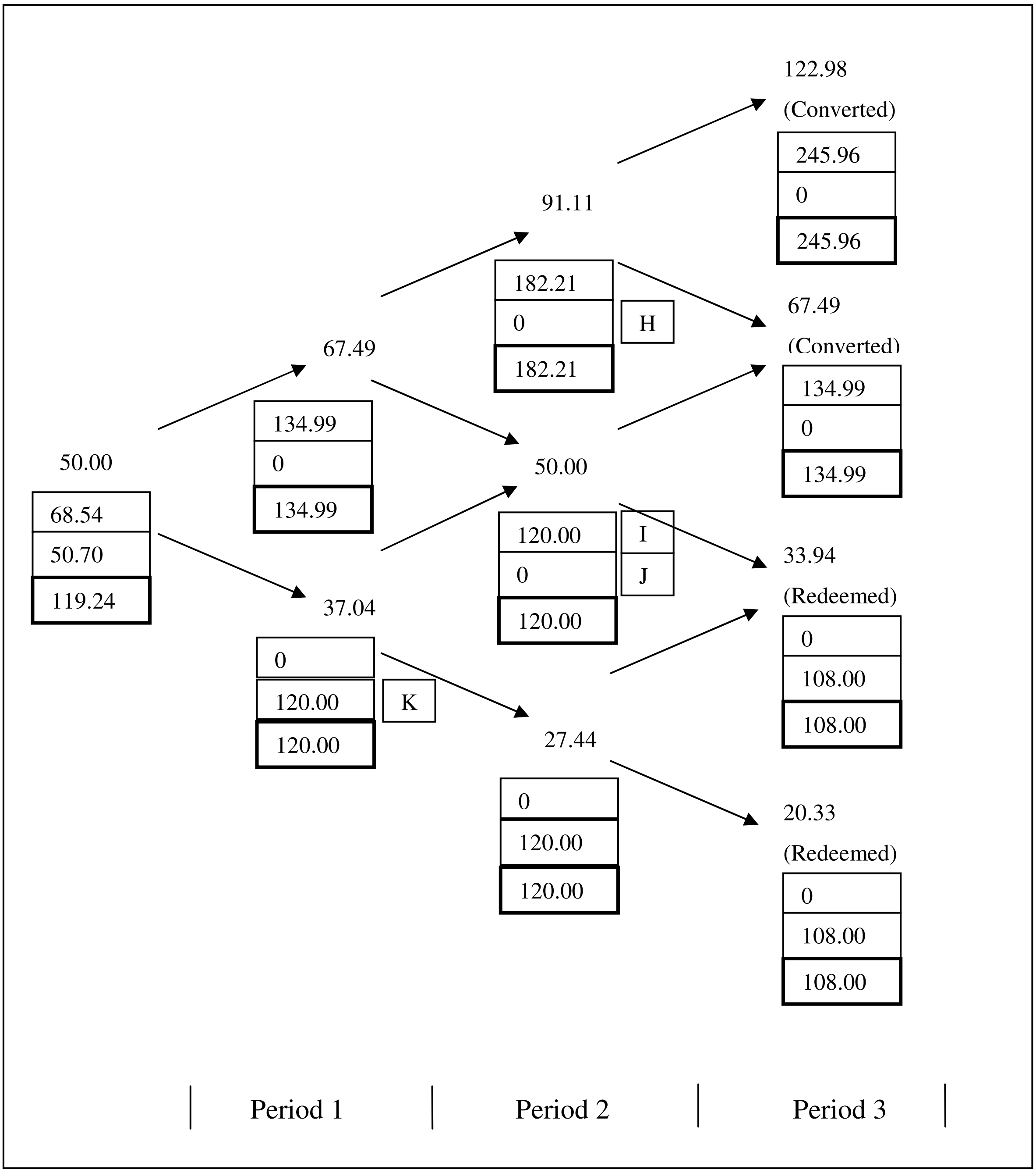

Figure 18– 3-period binomial tree of the TF model with conversion option exercised at maturity and embedded call and put options with strike values of 120.

5.4 Ayache, Forsyth and Vetzal

As mentioned earlier the assumption that the share value drops instantaneously to zero upon default is not a very realistic one. Thus Ayache, Forsyth and Vetzal [7] (AFV) adjust the familiar TF model by allowing some recovery value on the share price upon default. Ayache et al. [7] show that there are internal inconsistencies in the TF model such as cases where an issuer call before expiry causes the CB to be independent of the issuers' credit risk, and also where implied hedging strategies may not be self-financing.

5.4.1 Basic Model without Credit Risk

To introduce the structure of the PDE problem we start with a basic model of the convertible bond without any credit risk and follow up in the next section by introducing credit risk.

Once again assuming the dynamics of the share price follow the SDE in (5.15), we have that the value of any claim $g(S,t)$ is given by

$$g_t + \left(\frac{\sigma^2 S^2}{2} S^2 g_{SS} + \left(r(t) - q \right) S g_S - r(t) g \right), \tag{5.47}$$

where $r(t)$ is the deterministic interest rate and q is the dividend yield. Assuming that the CB has the usual put provision with price p_t, call provision with price c_t and conversion provision with conversion ratio α with these options being exercisable at discrete time points. By considering the cases where $c_t \le \alpha S_t$ and $c_t > \alpha S_t$ and letting

$$Lg \equiv -g_t - \left(\frac{\sigma^2 S^2}{2} S^2 g_{SS} + \left(r(t) - q \right) S g_S - r(t) g \right), \tag{5.48}$$

leads to a CB linear complementary problem given by

$$\text{If } c_t > \alpha S_t, \quad \begin{pmatrix} Lg = 0 \\ \left(g - \max\left(p_t, \alpha S\right)\right) \geq 0 \\ \left(g - c_t\right) \leq 0 \end{pmatrix} \text{ Continuation Region} \vee \begin{pmatrix} Lg \geq 0 \\ \left(g - \max\left(p_t, \alpha S\right)\right) = 0 \\ \left(g - c_t\right) \leq 0 \end{pmatrix} \text{ Call Region} \vee \begin{pmatrix} Lg \leq 0 \\ \left(g - \max\left(p_t, \alpha S\right)\right) \geq 0 \\ \left(g - c_t\right) = 0 \end{pmatrix} \text{ Put Region }, \tag{5.49}$$

And $c_t \leq \alpha S_t$

$$g = \alpha S . \tag{5.50}$$

The continuation region is when $Lg = 0$ subject to the constraints

$$g \geq \max\left(p_t, \alpha S\right)$$
$$g \leq \max\left(c_t, \alpha S\right).$$

So that neither the call nor put provision are active, or $Lg \geq 0$ and the put is active or $Lg \leq 0$ and the call is active. When $S = 0$ the PDE is

$$Lg = -\left(g_t - r\left(t\right) g\right) \quad ; \quad S \to 0 , \tag{5.51}$$

While $S \to \infty$ results in an unconstrained linear solution in S

$$Lg \equiv V_{SS} \quad ; \quad S \to \infty . \tag{5.52}$$

The terminal condition is given as

$$g\left(S, t = T\right) = \max\left(F, \alpha S\right), \tag{5.53}$$

with $F = \eta N$ being the face value of the bond.

5.4.2 Derivation of Risky Bond

Duffie and Singleton [27] value a simple non-defaultable coupon-bearing bond, which Ayache et al. [7] extend to incorporate default risk using simple hedging arguments. For ease of explanation we assume that default risk is diversifiable so that real and risk-neutral default probabilities will be equal[51]. Defining the probability of default in time period $t \to t+dt$, conditional on no default in $[0,t]$, to be $\gamma(S,t)\,dt$ where $\gamma(S,t)$ is the deterministic hazard rate.

To begin we start by constructing the hedging portfolio π consisting of a risky corporate bond $m^*(S,t)$

$$\pi = m^* - \Delta S\,. \tag{5.54}$$

In the absence of default, and choosing $\Delta = \partial g / \partial S$, gives

$$d\pi = \left[m^*_{\ t} + \frac{\sigma^2 S^2}{2} m^*_{\ SS} \right] dt + o(dt)\,. \tag{5.55}$$

Making the assumptions that

- The probability of default in $t \to t+dt$ is γdt
- The value of the bond immediately after default is RX, where $0 \le R \le 1$ is the recovery factor[52].
- The share price is unchanged upon default.

So that equation (5.55) becomes

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[51] This is a rather bold assumption, but it illustrates the point about the introduction of default risk into bond valuation.

[52] The value of X can be the face value for coupon-bearing bonds or the accreted value for zero-coupon bonds.

$$\begin{aligned} d\pi &= (1-\gamma dt)\left[m^*{}_t + \frac{\sigma^2 S^2}{2} m^*{}_{SS} \right] dt - \gamma dt\left(m^* - RX\right) + o(dt) \\ &= \left[m^*{}_t + \frac{\sigma^2 S^2}{2} m^*{}_{SS} \right] dt - \gamma dt\left(m^* - RX\right) + o(dt). \end{aligned} \tag{5.56}$$

Assuming default risk is diversifiable forces the portfolio to equal the risk-free rate under expectation theory as

$$E(d\pi) = r\pi dt\,, \tag{5.57}$$

so that equation (5.56) becomes

$$m^*{}_t + r(t)Sm^*{}_S + \frac{\sigma^2 S^2}{2} m^*{}_{SS} - \left(r(t)+\gamma\right)m^* + \gamma RX = 0\,. \tag{5.58}$$

If we let the hazard rate $\gamma(S,t) = \gamma$ and $X = m^*$, then the solution to (5.58) for a risky zero-coupon bond with face value F at t=T is

$$m^* = F\exp\left(-\int_t^T \left(r(u) + \gamma(u)(1-R) \right) du \right), \tag{5.59}$$

which implies a spread of $s = \gamma(1-R)$. If we consider that the share price jumps to zero upon default then (5.56) becomes

$$\begin{aligned} d\pi &= (1-\gamma dt)\left[m^*{}_t + \frac{\sigma^2 S^2}{2} m^*{}_{SS} \right] dt - \gamma dt\left(m^* - RX - \Delta S\right) + o(dt) \\ &= \left[m^*{}_t + \frac{\sigma^2 S^2}{2} m^*{}_{SS} \right] dt - \gamma dt\left(m^* - RX - \Delta S\right) + o(dt). \end{aligned} \tag{5.60}$$

Equating (5.60) to the risk-free rate and substituting $\Delta = \partial g / \partial S$ gives the following PDE for a defaultable zero-coupon bond

$$m^*_t + \left(r(t)+\gamma\right) S m^*_S + \frac{\sigma^2 S^2}{2} m^*_{SS} - \left(r(t)+\gamma\right) m^* + \gamma RX = 0 . \tag{5.61}$$

What is interesting to note is that the drift and discounting terms include the hazard rate γ, which proves that assumptions about the share price on default changes the valuation of the risky corporate bond, albeit with the use of a simple example.

5.4.3 Basic Model with Credit Risk

Using the same notation as defined in Section 5.3, we let S^+ be the stock price immediately after the default event, S^- be the stock price before the default event, and κ be the loss on the stock price due to default, such that

$$S^+ = S^-\left(1-\kappa\right), \tag{5.62}$$

where $0 \le \kappa \le 1$. The case where $\kappa = 1$ is known as the "total default" case (zero recovery of the share price), and when $\kappa = 0$ it is known as the "partial default" case[53]. As usual, we construct the hedging portfolio π

$$\pi = g - \Delta S , \tag{5.63}$$

with $\Delta = \frac{\partial g}{\partial S}$ to eliminate the uncertainty of the Wiener Process. If we assume no credit risk $(\gamma = 0)$ then the change in the portfolio reduces to

$$d\pi = \left[g_t + \frac{\sigma^2 S^2}{2} g_{SS} \right] dt + o\left(dt\right). \tag{5.64}$$

To incorporate credit risk we have that $\gamma > 0$, and make the following assumptions upon default

[53] The firm defaults, although the share price is unaffected, which is the assumption made by Tsiveriotis and Fernandes, as the name suggests it is a partial default.

- The share price jumps according to equation (5.62).
- The convertible holders have the option of receiving either
    - An amount RX, where $0 \le R \le 1$ is the recovery factor[54], or
    - Shares worth $\alpha S(1-\kappa)$.

So with these assumptions in mind, the change in portfolio value from $t \to t+dt$ is

$$\begin{aligned} d\pi &= (1-\gamma dt)\left[g_t + \frac{\sigma^2 S^2}{2} g_{SS}\right]dt - \gamma dt\left(g - \Delta S\kappa\right) + \gamma dt \max\left(\alpha S(1-\kappa), RX\right) + o(dt) \\ &= \left[g_t + \frac{\sigma^2 S^2}{2} g_{SS}\right]dt - \gamma dt\left(g - g_S S\kappa\right) + \gamma dt \max\left(\alpha S(1-\kappa), RX\right) + o(dt). \end{aligned} \tag{5.65}$$

Under the risk-neutral framework, the expected return on the portfolio has to equal the risk-free rate, as such we equate (5.65) to the risk free rate over the period *dt* to give

$$\begin{aligned} & r\left[g - Sg_S\right]dt = \left[g_t + \frac{\sigma^2 S^2}{2} g_{SS}\right]dt - \gamma dt\left(g - g_S S\kappa\right) + , \\ & \gamma dt \max\left(\alpha S(1-\kappa), RX\right) + o(dt) \end{aligned} \tag{5.66}$$

which simplifies to

$$g_t + \left(r(t) + \gamma\kappa\right)Sg_S + \frac{\sigma^2 S^2}{2} g_{SS} - \left(r(t) + \gamma\right)g + \gamma \max\left(\alpha S(1-\kappa), RX\right) = 0. \tag{5.67}$$

A closer inspection of (5.67) reveals that the drift term contains $\left(r(t)+\gamma\kappa\right)$ and the discounting term contains $\left(r(t)+\gamma\right)$. Letting *R=0*, $\kappa = 1$ will result in the total default model with no recovery and reduce to a simple PDE to solve.

Defining

[54] X can be chosen to be the pre-default value of the Convertible bond or the face value.

$$Mg \equiv -g_t - \left(\frac{\sigma^2 S^2}{2} g_{SS} + \left(r(t) + \gamma\kappa - q \right) S g_S - \left(r(t) + \gamma \right) g \right), \tag{5.68}$$

we can write (5.67) when the share pays a continuous dividend yield q as

$$Mg - \gamma \max\left(\alpha S (1-\kappa), RX \right) = 0 . \tag{5.69}$$

After arriving at the PDE with credit risk and partial recovery, we are now in a position to complete the problem for CB's with risky debt.

If $c_t > \alpha S_t$

$$\begin{aligned} &\begin{pmatrix} Mg - \gamma \max\left(\alpha S(1-\kappa), RX \right) = 0 \\ \left(g - \max\left(p_t, \alpha S \right) \right) \geq 0 \\ \left(g - c_t \right) \leq 0 \end{pmatrix} \quad \text{Continuation Region} \\ \vee &\begin{pmatrix} Mg - \gamma \max\left(\alpha S(1-\kappa), RX \right) \geq 0 \\ \left(g - \max\left(p_t, \alpha S \right) \right) = 0 \\ \left(g - c_t \right) \leq 0 \end{pmatrix} \quad \text{Call Region} \\ \vee &\begin{pmatrix} Mg - \gamma \max\left(\alpha S(1-\kappa), RX \right) \leq 0 \\ \left(g - \max\left(p_t, \alpha S \right) \right) \geq 0 \\ \left(g - c_t \right) = 0 \end{pmatrix} \quad \text{Put Region ,} \end{aligned} \tag{5.70}$$

and $c_t \leq \alpha S_t$

$$g = \alpha S . \tag{5.71}$$

Although the linear complementarities in equations (5.70) and (5.71) look complicated, breaking them down as in (5.49) will explain the basic concept. The value of the CB is given as in equation (5.69) subject to the constraints,

$$\begin{aligned} g &\geq \max\left(p_t, \alpha S \right) \\ g &\leq \max\left(c_t, \alpha S \right) . \end{aligned} \tag{5.72}$$

So that we are either in the continuation region, call region or put region as expressed in (5.70), or will choose to convert. Due to the hedging of the Brownian motion risk, we will refer to equations (5.70) and (5.71) as the basic Hedge model.

5.4.4 Inconsistencies with TF Model

The TF model uses a similar splitting technique to the AFV model as mentioned in Ayache, et al. [6], although there is some discrepancy in their approach due to the unclear notion of what happens to the share price upon default, and the decomposition of the CB. In this section two undesirable features of the TF model relative to the AFV model will be addressed.

For simplicity we will assume that there are no put provisions and coupons, $\alpha = 1$, the only conversion time is at maturity or at the call date where the bond can only be called the instant before maturity i.e. $t = T^-$. If the call price is $c_t = F - \varepsilon$, $\varepsilon > 0$, $\varepsilon << 1$, and the bond is called at $t = T^-$, then according to the TF model we have

$$\begin{aligned} &Lg + \gamma m^* = 0 \\ &g\left(S, T^-\right) = \max\left(S, F - \varepsilon\right) \\ &Lm^* + \gamma m^* = 0 \\ &m^*\left(S, T^-\right) = 0. \end{aligned} \tag{5.73}$$

The solution of m^* using the last two equations of (5.73) is $m^* \equiv 0$, $\forall\ t < T^-$, and with $m^* = 0$ at $t = 0$, we have that the solution to the CB is

$$\begin{aligned} Lg &= 0 \\ g\left(S, T^-\right) &= \max\left(S, F - \varepsilon\right). \end{aligned} \tag{5.74}$$

This is unusual since the hazard rate does not have any affect on the CB value, as one of the constraints in the TF model requires that $m^* = 0$ if $g = c_t$ even if the difference between the call and bond face value is infinitesimally small. What this is saying is that the instant before maturity when the CB is called, the bond becomes independent of the issuers credit risk, which is absurd.

In the second case we will show how hedging with the TF model in the real world leads to a non self-financing portfolio. In the real world we have two measures of uncertainty, namely uncertainty in the share price and default risk. To hedge out these risks we have to introduce another security into our hedging portfolio, which can be another bond by the same issuer. Let this new contingent claim be denoted by $I(S,t)$. Working in the real world means that default risk is not diversifiable, and so λdt is the actual probability of default in the interval $[t,t+dt]$, whereas γdt is the risk-adjusted value. Using the same arguments as previously done, we construct the hedging portfolio

$$\pi = g - \Delta S - \Delta^{'} I + A, \tag{5.75}$$

where A is the cash portion and is given by $A = -\left(g - \Delta S - \Delta^{'} I\right)$ and $\Delta^{'}$ is the number of I security's held. Assuming a real world process of

$$dS = (\mu + \lambda\kappa) Sdt + \sigma Sdz - \kappa Sdq, \tag{5.76}$$

where μ is the drift rate and the Poisson default process is given by

$$dq = \begin{cases} 1 \ , \text{ with probability } \lambda dt \\ 0 \ , \text{ with probabilty } (1-\lambda dt). \end{cases} \tag{5.77}$$

If we choose

$$g_S - \Delta^{'} I - \Delta = 0, \tag{5.78}$$

then by using Itô's Lemma and equations (5.76) and (5.78) we have

$$
\begin{aligned}
d\pi = &\left[\frac{\sigma^2 S^2}{2} g_{SS} + g_t - \Delta'\left(\frac{\sigma^2 S^2}{2} I_{SS} + I_t\right)\right]dt \\
&+\left(\Delta S + \Delta' I - g\right) r dt \\
&+\left[\text{change in } \pi \text{ on default}\right] dq.
\end{aligned} \tag{5.79}
$$

For simplicity we will assume that the recovery rate of the bond component is zero ($R=0$) and that the second contingent claim, I defaults at the same time as the CB. If none of the call, put, or conversion options are exercised and default does not occur in $[t, t+dt]$ then

- The hedge model from (5.67) gives

$$
g_t + \frac{\sigma^2 S^2}{2} g_{SS} = -\left[\left(r+\gamma\kappa\right) S g_S - \left(r+\gamma\right) g + \gamma\alpha S\left(1-\kappa\right)\right], \tag{5.80}
$$

$$
I_t + \frac{\sigma^2 S^2}{2} I_{SS} = -\left[\left(r+\gamma\kappa\right) S I_S - \left(r+\gamma\right) I + \gamma\alpha' S\left(1-\kappa\right)\right], \tag{5.81}
$$

- The TF model from (5.14) without any coupons gives

$$
g_t + \frac{\sigma^2 S^2}{2} g_{SS} = -\left[r S g_S - r g - \gamma m^*\right], \tag{5.82}
$$

$$
I_t + \frac{\sigma^2 S^2}{2} I_{SS} = -\left[r S I_S - r I - \gamma m'\right]. \tag{5.83}
$$

Where α' is the number of shares that the claim I will receive upon default and m' is the bond component of I. Assuming that $\Delta = g_S - \Delta' I_S$ in all cases, we have that

$$
\begin{aligned}
\left[\text{change in } \pi \text{ on default}\right] &= \alpha S(1-\kappa) - \Delta S(1-\kappa) - \Delta'\alpha' S(1-\kappa) - \left(g - \Delta' I - \Delta S\right) \\
&= \alpha S(1-\kappa) + \left(g - \Delta' I\right) S\kappa - \Delta'\alpha' S(1-\kappa) - g + \Delta' I.
\end{aligned} \tag{5.84}
$$

Substituting (5.84) into (5.79) gives

$$
\begin{aligned}
d\pi = & \left[\frac{\sigma^2 S^2}{2} g_{SS} + g_t - \Delta' \left(\frac{\sigma^2 S^2}{2} I_{SS} + I_t \right) \right] dt \\
& + \left(\left(g_S - \Delta' I_S \right) S + \Delta' I - g \right) r dt \\
& + \left[\alpha S (1-\kappa) + \left(g - \Delta' I \right) S\kappa - \Delta' \alpha' S (1-\kappa) - g + \Delta' I \right] dq.
\end{aligned} \tag{5.85}
$$

For the hedge model, using equations (5.80) and (5.81) in (5.85) gives

$$
\begin{aligned}
d\pi = & -\gamma dt \left[S g_S \kappa - g + \alpha S (1-\kappa) - \Delta' \left(\kappa S I_S - I + \alpha' S (1-\kappa) \right) \right] \\
& + dq \left[\alpha S (1-\kappa) + \left(g - \Delta' I \right) S\kappa - \Delta' \alpha' S (1-\kappa) - g + \Delta' I \right] \\
= & -\gamma dt \left[S g_S \kappa - g + \alpha S (1-\kappa) - \Delta' \left(\kappa S I_S - I + \alpha' S (1-\kappa) \right) \right] \\
& + dq \left[\alpha S (1-\kappa) + g_S S\kappa - g - \Delta' \left(\kappa S I_S - I + \alpha' S (1-\kappa) \right) \right].
\end{aligned} \tag{5.86}
$$

Setting

$$
\Delta' = \frac{S g_S - g + \alpha S (1-\alpha)}{\left(\kappa S I_S - I + \alpha' S (1-\kappa) \right)}, \tag{5.87}
$$

will make (5.86) equal to zero, so that the hedging portfolio is risk-free and self-financing under the real world measure.

Looking at the TF model, and substituting (5.82) and (5.83) into (5.85) we have that

$$
d\pi = \gamma dt \left(m^* - \Delta' m' \right) + dq \left[\alpha S (1-\kappa) + g_S S\kappa \right] - g + \Delta' \left(I - \kappa S I_S - \alpha' S (1-\kappa) \right). \tag{5.88}
$$

Setting Δ' as in equation (5.87) and substituting into (5.88) will give

$$
d\pi = \left[m^* - \Delta' m' \right] p dt, \tag{5.89}
$$

which shows that the hedging portfolio is no longer self-financing. Imposing another restriction on the portfolio, namely $E\left[d\pi\right] = 0$, and using equations (5.78) and (5.88) gives us

$$\beta' = \frac{-\lambda\left(\alpha S(1-\eta)+g_S S\eta - g\right)-\gamma m^*}{\lambda\left(I - I_S S\eta - \alpha'(1-\eta)S\right)-\gamma m'}, \tag{5.90}$$

which is dependent on λ. Choosing β' as in (5.90) we see that the variance in the hedging portfolio is given as

$$Var[d\pi] = E\left[(d\pi)^2\right], \tag{5.91}$$

which is non-zero, and thus not risk-free. The hedge model produces a self-financing hedging, zero risk portfolio under the real world measure, unlike the TF model which produces a hedging portfolio that is neither risk-free nor self-financing. As mentioned at the beginning of this section, the hedge model specifies what occurs on default, which keeps it consistent with the default model.

In the next section the Goldman-Sachs conversion adjusted probability model is discussed, which introduces a slight variation to the TF and AFV models by using a conversion probability lattice.

5.5 Goldman-Sachs

The Goldman-Sachs [32] model is similar to the Tsiveriotis and Fernandes [69] model as it uses a blended discount rate. The difference is that the interest rate is adjusted according to the probability of conversion as opposed to the cashflow as in the TF model. As such the Goldman-Sachs model uses a weighted risk-free and risky interest rate, weighted by the probability of conversion in the discounting of its cashflows. Also the Goldman-Sachs model does not use a decoupled PDE approach as everything is computed using a single PDE, making computation a little easier.

5.5.1 The Model

Assuming once again that p^* is the probability of an up move of the share price under the familiar risk-neutral measure Q as discussed in Chapter 3, we can construct a simple conversion probability binomial tree. Letting *e(S,T)* denote the values at the terminal nodes of the conversion probability tree at maturity *T*, so that

$$e(S,T)=\begin{cases}1 & \text{for } S \geq \dfrac{B}{\alpha}\\ 0 & \text{elsewhere.}\end{cases} \tag{5.92}$$

Thus, the conversion probability at time *T-1* in state *S* is given as

$$e(S,T-1)=E^{Q}\left[e(S,T)\right]=p^{*}e(S_u,T)+(1-p^{*})e(S_d,T). \tag{5.93}$$

Working backward through the tree gives us a conversion probability for each node to be used in calculating the weighted average interest rate for discounting. The weighted average interest rate is calculated in a similar manner by constructing a binomial tree and calculating the discount rate for each node. As such at time $(T-1)$ where $0 \leq (T-1) < T$ and state $S \in \mathbb{R}$ the corresponding node is given as

$$r(S,T-1)=re(S,T)+(r+r_c)(1-e(S,T)). \tag{5.94}$$

Once the appropriate discount rate is calculated the usual convertible bond tree can be constructed by using the following interim boundary conditions,

$$w(S,t) \geq \alpha S_t \quad for\ t\ \varepsilon\ [0,T], \tag{5.95}$$

$$w(S,t) \leq \max(c_t,\alpha S_t) \quad for\ t\ \varepsilon\ [\tau_{c1},T], \tag{5.96}$$

$$v(S,t)=p_t \quad if\ w \leq p_t \quad for\ t\ \varepsilon\ \left[\tau_{p1},T\right], \tag{5.97}$$

with the last two equations being the callability and putability conditions where τ_{c1} is the call start date and τ_{p1} is the put start date. The CB value at each point in the binomial lattice must be discounted using the equivalent interest rate computed previously, until time *0* to calculate a fair value for CB.

5.5.2 Numerical Example

A simple example will demonstrate the conversion probability tree and adjusted discount rate tree. Letting the parameters be given as

Stock	
Current Share Price	50.00
Share price volatility	30%
Risk-free rate	7.00%
Dividend rate	0%
Bond	
Face value	100
Bond yield	10.00%
Coupon rate	8.00%
Coupon frequency	2
Maturity	3 yrs
Conversion Ratio	2
No. of time steps	3

Then the 3-period conversion probability tree and discount rate tree are shown in Figure 19 and Figure 20 respectively.

Point A is calculated as, $1(0.545)+0(1-0.545)=0.545$, and

Point B is calculated as, $0.07(0.545)+0.1(1-0.545)=0.08365$

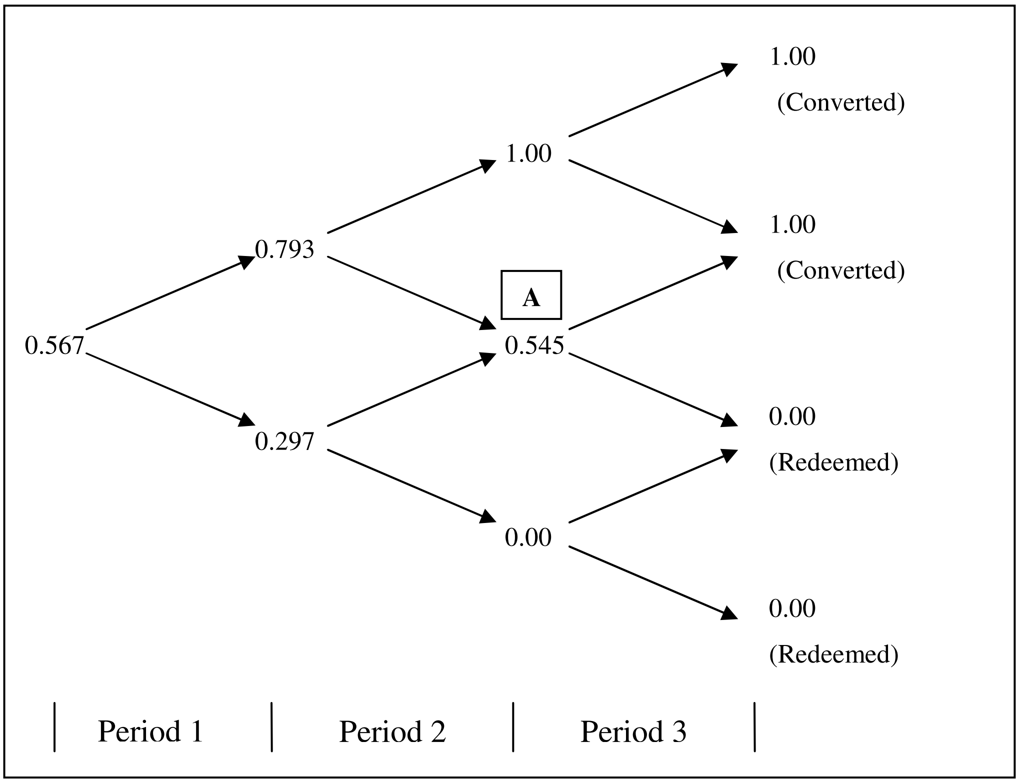

Figure 19 – 3-period conversion probability tree with p^*=0.545.

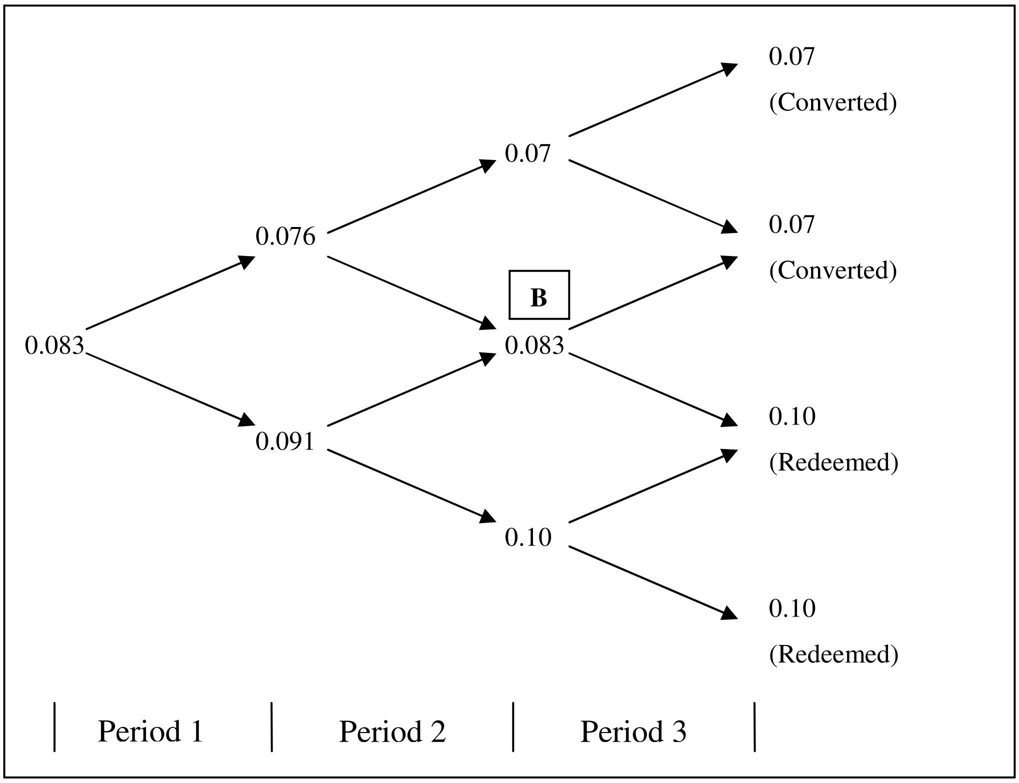

Figure 20 – 3-period adjusted discount rate tree with q^*=0.545, r=7% and r_c=3%.

To calculate the final convertible value we construct a binomial tree applying all the relevant conditions in (5.95), (5.96) and (5.97). Figure 21 illustrates the share price and rollback convertible bond price using the probability adjusted interest rates.

Point C is calculated as $e^{-0.083}\left(134.99(0.545)+104(1\text{-}0.545)\right)=119.20$.

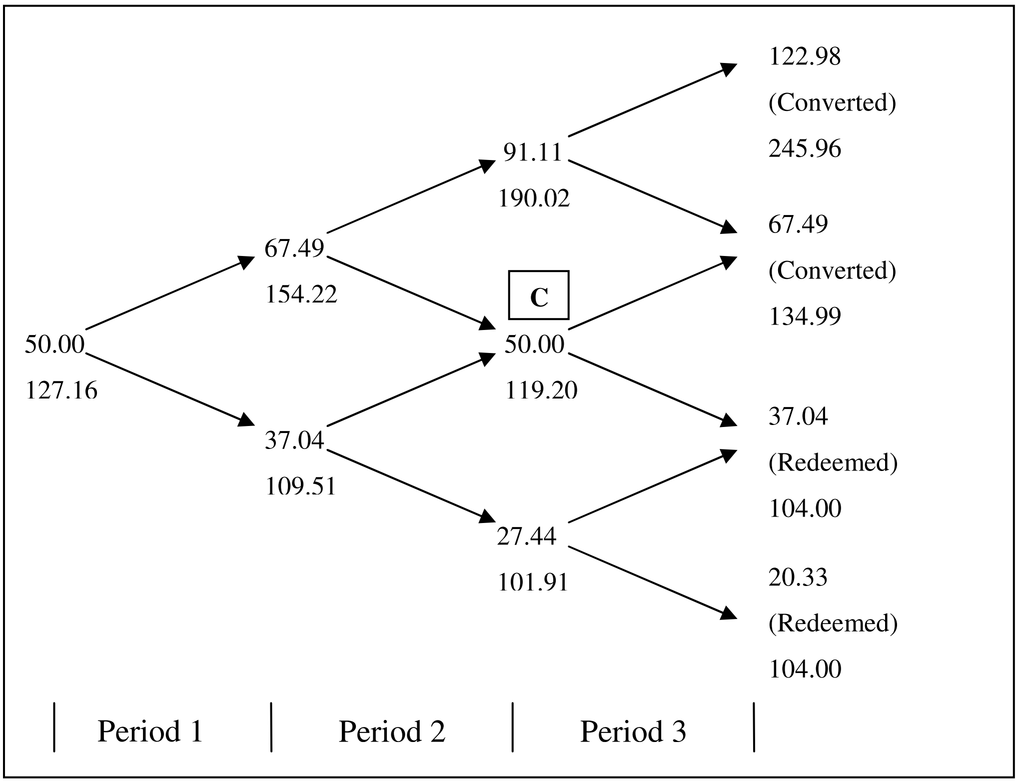

Figure 21 – 3-period binomial tree of the share value and the rollback convertible bond price using the probability-adjusted discount rates in Figure 20. The first number in each node is the share price whilst the second is the rollback CB value.

Embedded calls and puts can easily be handled by the model and due to the blended discount rate used, we should expect the value to be less than or equal to the Tsiveriotis and Fernandes model for out-the-money CB's and below the Tsiveriotis and Fernandes model for in-the-money CB's. A comparison and explanation of this concept will be discussed in Section 5.7.

5.6 Hung and Wang

The Traditional model to incorporate default risk into CB pricing is to use binomial trees with constant risk-free and risky interest rates and value it with an equity and bond component as in Tsiveriotis and Fernandes [69]. The risk-free rates are used to discount the stock whilst the risky rates are used to discount the bond component. As many CB's do not have long maturities they are less prone to interest rate sensitivity and as such without much loss of generality keeping interest rates constant is an acceptable assumption. An illustration of the Traditional model can be seen in Figure 22. This helps reduce the complexities of the model as we now have fewer paths to travel on the lattice, although a theoretical demonstration with stochastic interest rates will be shown.

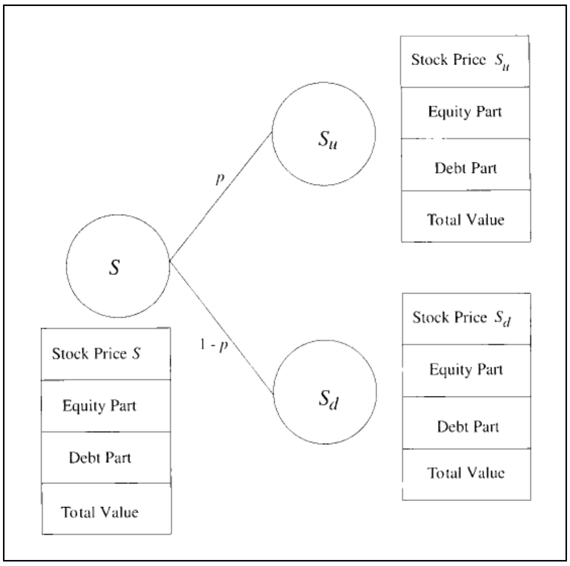

Figure 22 – A simple illustration of the Traditional CB model.

5.6.1 Terminal nodes of Traditional CB Tree

When dealing with binomial trees we start at the terminal nodes to determine the payoffs and then rollback through the tree to the present to establish a fair value for the CB as before. Since the issuer usually issues the CB with a call option the optimal value at each node at maturity has to satisfy the following condition for the Traditional model:

Rollback value = max [min (Market Value, Call Price), Conversion Value],

If a put option is also embedded into the CB the value has to satisfy:

Rollback value = max [min (Market Value, Call Price), Conversion Value, Put Price], Turning our attention to the valuation model with stochastic interest rates, things are a little more complicated. Jarrow and Turnbull [40] use a single binomial tree to account for risky and risk-free interest rates. The risk-free interest rates are stochastic and grow according to a binomial tree. The risky rates use the same binomial tree but are adjusted by a stochastic default probability $\lambda_{\mu t}$ (as explained earlier in section 4.3) in each period and can be implied from any issuer specific term structure. If the issuer were to default, a constant recovery amount in all time periods was assumed.

5.6.2 Default Risk CB Model

When dealing with stochastic interest rates the tree deals with 3 variables each taking on 2 values resulting in 6 branches arising from each node. Thus, the tree becomes very large in only a few time steps, which is illustrated partially in Figure 23 with 3 time steps. The probabilities associated with each branch are illustrated in Table 2.

Branch	Path	Probability
1	Default occurs, r goes up, S=0	$P_1 = \lambda\pi$
2	Default occurs, r goes down, S=0	$P_2 = \lambda(1-\pi)$
3	No-default, r goes up, S goes up	$P_3 = \lambda(1-\pi)\,p^*$
4	No-default, r goes up, S goes down	$P_4 = \lambda(1-\pi)(1-p^*)$
5	No-default, r goes down, S goes up	$P_5 = (1-\lambda)(1-\pi)\,p^*$
6	No-default, r goes down, S goes down	$P_6 = (1-\lambda)(1-\pi)(1-p^*)$

Table 2 – Probabilities for states from each node (assuming stochastic interest rates, default probabilities and share prices).

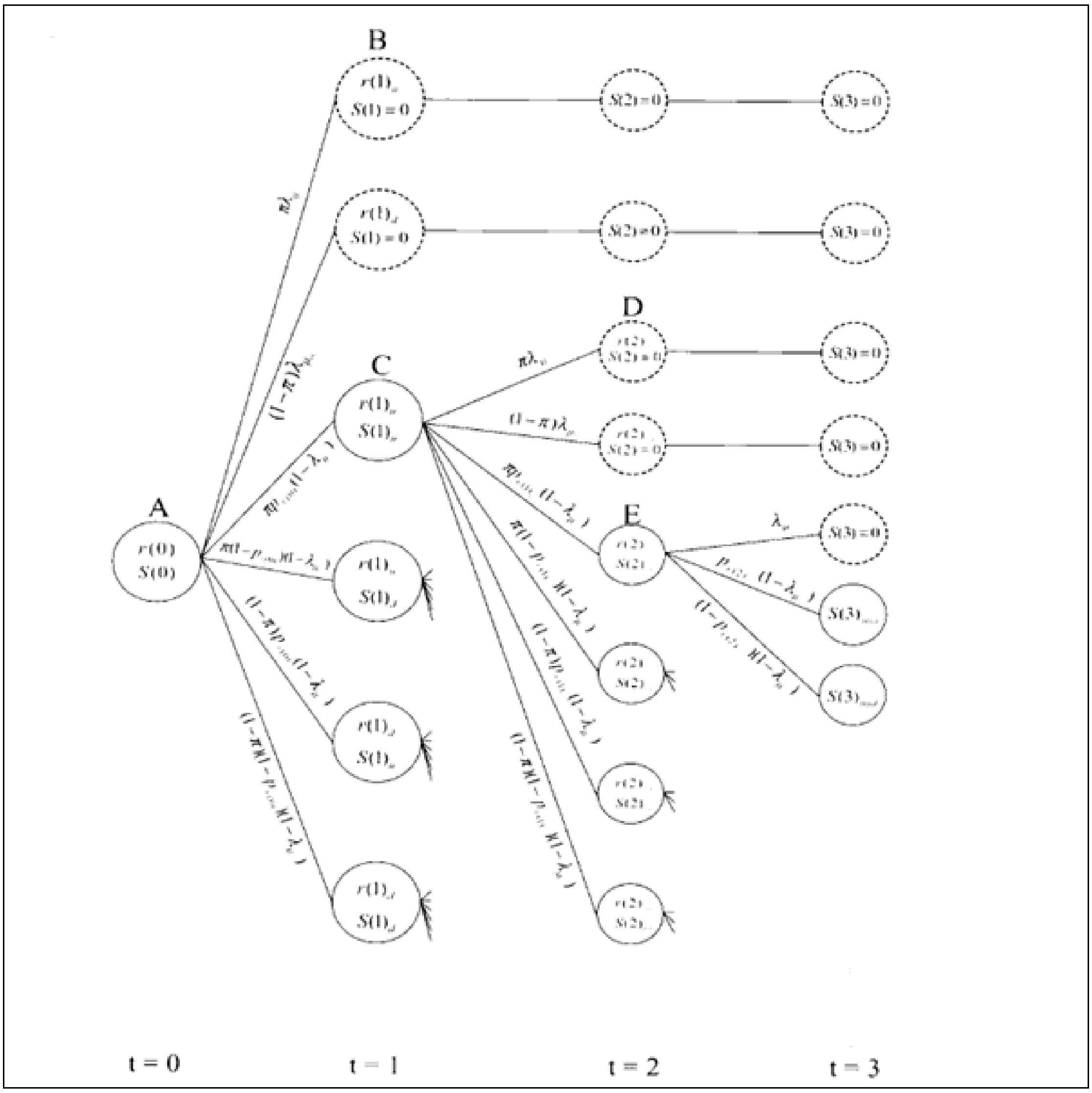

Figure 23 – Reproduction of the Hung and Wang CB model with stochastic interest rates, default probabilities and share prices.

5.6.3 Derivation of Risk-Neutral Measure

Hung and Wang [36] incorporate default risk into interest rates by way of an implied default probability. To incorporate default into the associated risk-neutral probabilities p^* for the share price, we need to adjust their probabilities. Chambers and Lu [19] improve upon the HW model by introducing a correlation parameter between the share price and interest rates, and include the adjusted probability $\tilde{p}$ for the share price. The outputs from their model were not significantly different from the HW model but incorporation of the correlation parameter and default probability is a definite advantage of their model. Using the fundamental concept of finance that the value of a security (z) today is the present value of its expected cash flows, i.e. $z_0 = E^Q\left[z_T / \Im_0\right]$[55] for some time $T > 0$ under the risk-neutral probability measure Q, it is possible to find the probability measure to value this CB. The derivation[56] proposes that the probability $\tilde{p}$ for the share price to move up is given as

$$\tilde{p} = \frac{e^{r_f \Delta t} / (1-\lambda) - d}{u - d}. \tag{5.98}$$

If we assume that interest rates are constant, which was previously justified, and that the only stochastic variables are the share price and default probabilities, the calculations become more tractable, with the associated probabilities being given in Table 3. Figure 24 illustrates the new simplified two-period tree.

Branch	Path	Probability
1	Default occurs, S=0	$P_1 = \lambda \tilde{p}$
2	Default occurs, S=0	$P_2 = \lambda(1 - \tilde{p})$
3	No-default, S goes up	$P_3 = (1-\lambda)\tilde{p}$
4	No-default, S goes down	$P_4 = (1-\lambda)\left(1 - \tilde{p}\right)$

Table 3 – Probabilities for states from each node (assuming constant interest rates, stochastic default probabilities and share prices).

[55] $\Im_0$ is the filtration on the process z_T and contains all the information about z up until time 0.

[56] The derivation can be seen in the Appendix A2.

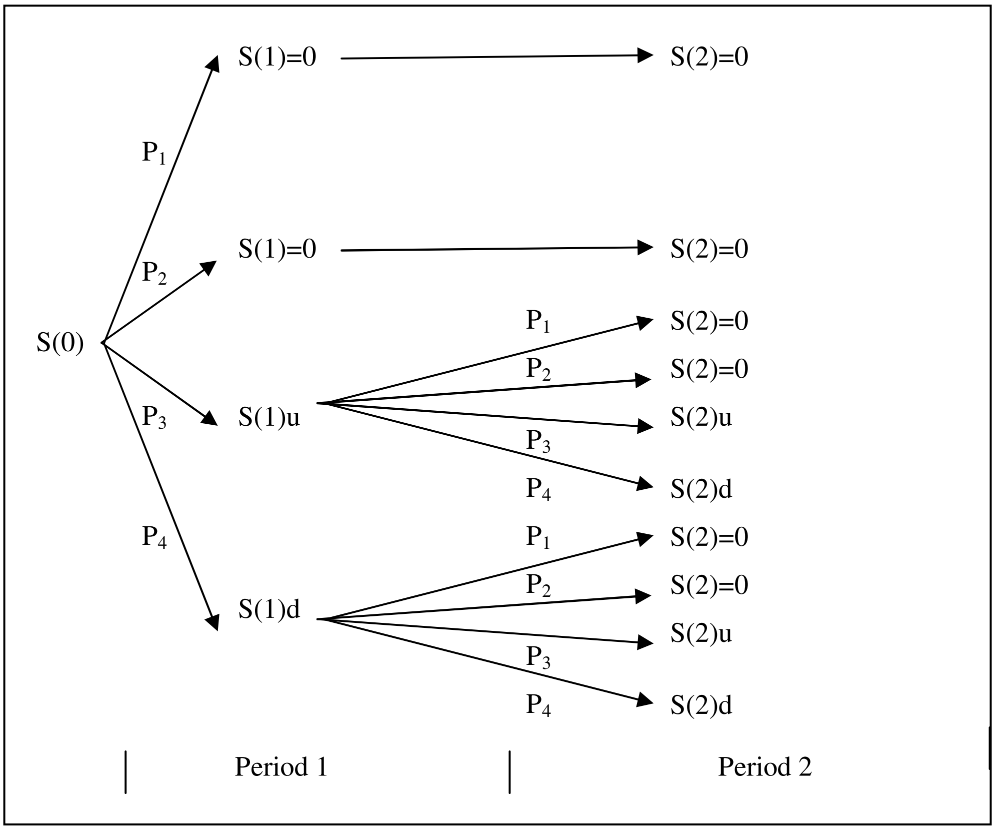

Figure 24 – Hung and Wang CB model with constant interest rates, and stochastic default probabilities and share prices.

5.7 Numerical Example

The Tsiveriotis and Fernandes (TF), Goldman-Sachs (GS) and Hung and Wang (HW) models, might seem similar due to their adjusted discount rates but a numerical example will illustrate the important credit risk aspect brought about by the default probabilities in the HW model. Using risk-free zero-coupon bond and risky zero-coupon bond market data, the $\lambda_{\mu t}$'s can be extracted as described in Chapter 4. The implied default rates[57] as shown in Figure 25 have been used in the calculations to follow and the parameters for the models are given in Appendix A1.

As can be seen the curve is not monotone increasing or decreasing, which can be expected when using actual market data. Each of the $\lambda_{\mu t}$'s is the conditional

[57] The implied default rates have been calculated by using a risk-free government curve and a risky interpolated issuer specific curve. The absolute values are not important, but the credit spread between the two curves is the significant determinant of the default probabilities.

probability meaning that given the bond survived *t-1* years; the probability that it will default in year *t* is $\lambda_{\mu t}$.

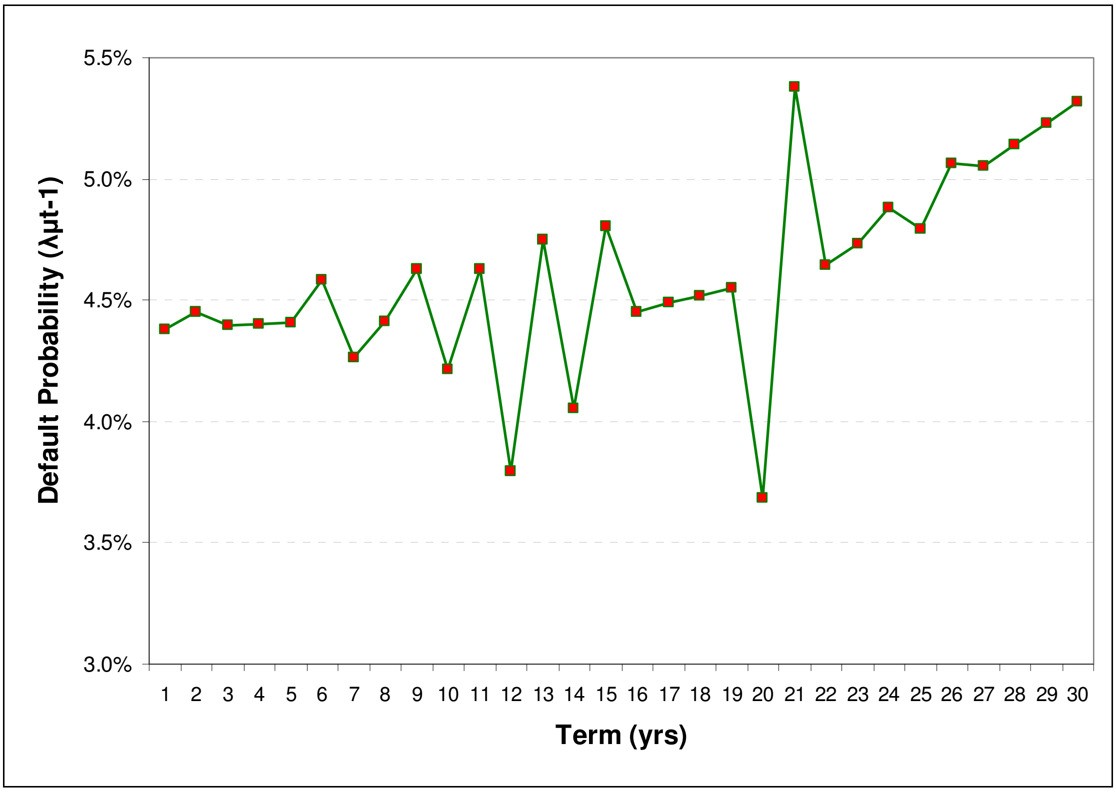

Figure 25 – Implied conditional default rates extracted from market data.

Figure 26 is a vital graph for the HW model as it clearly illustrates the difference between the implied default probabilities in the HW model relative to the TF and GS adjusted discount rate models. What is important to note is that as the share price reaches very low levels the probability of default increases sharply. Since CB's are lower in seniority than straight bonds, if a firm's share price were to drop to very low levels, then theoretically its CB value should drop below its straight bond (bond floor) value as indicated in the Figure 26 for the HW model. For the TF and GS models it takes on the same value as its straight bond value for low share values which is not entirely correct due to the subordination mentioned earlier. With the HW model, there is a bond floor present but due to the additional risk of the subordination, this floor is set much lower than that of an equivalent senior vanilla bond. Another interesting observation is that the HW model tends to value the CB a little more than the TF and GS models for large share prices, which is plausible since the probability of default should decrease as the share increases thereby increasing the CB value. The GS model seems to give lower price estimates than the TF model for increasing share values. This is justified since the discount rate used in the GS model is a smoothed rate while the discount rate used in the TF model is step like. So, if the CB is in-the-money, more often than not the TF model uses the risk-free rate for discounting cash flows, whilst the GS model uses a blended rate, which is higher than the risk-free rate, thus producing a lower CB value relative to the TF model. When the CB moves out-the-money, both the TF and GS models use the same risky rate and so equate to the same CB prices.

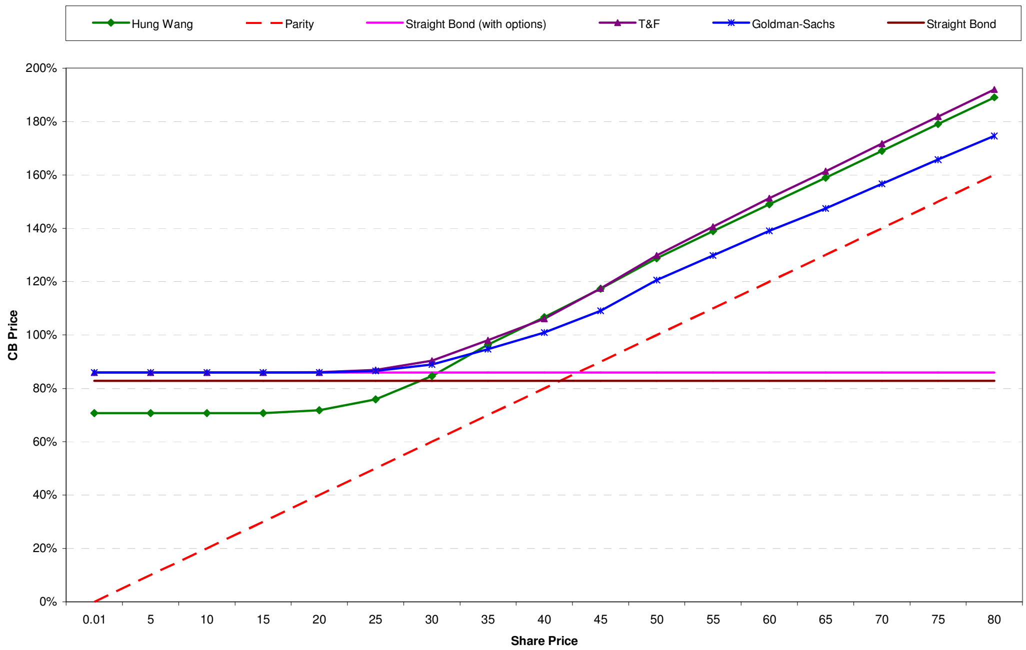

Figure 26 – CB values for different share prices for the TF, GS and HW models including the bond floor value with and without options.

Chapter 5 analysed several equity valuation models including the TF and HW models. These models take into account the hybrid nature of CB's by splitting them into a bond and equity component with the HW model catering for credit risk aswell. The new generation of CB models should however accommodate the negative convexity that becomes apparent for extremely low share prices in the distressed debt phase as depicted in Figure 1. As such Chapter 6 introduces a credit spread function dependent on the share price and incorporates it into the TF model.

6. *k*-factor Credit Spread Model

Referring to Figure 1 the CB price track possesses negative convexity whilst in the distressed debt phase. The TF and HW models failed to model this aspect. Thus, Chapter 6 proposes an improvement to these models by developing a *k*-factor credit spread model which adjusts the credit spread depending on the underlying share price. Section 6.1 gives a concise introduction and origins of the *k*-factor model. Implementation of the *k*-factor into the TF model and a sensitivity analysis of the parameters are discussed in Sections 6.2 and 6.3 respectively. The Chapter concludes with a suggested calibration of the *k*-factor variables in Section 6.4.

6.1 Introduction to *k*-factor model

As briefly discussed earlier, when the share value of the issuing firm drops to extremely low levels[58] there is increasing pressure on the firm to repay its shareholders. Due to the subordination of CB's there is a high probability of default risk and fractional recovery value of the notional amount. As such the CB will experience negative convexity, increasing delta and negative vega for share values close to zero. All models to date tend to over-estimate gamma and vega, and under-estimate delta when the CB is in the distressed zone. This is because the credit spread is independent of the share value, which therefore results in overestimation of the CB price at these low share prices. A good way of incorporating this trait in any model is to adjust the credit spread or risky interest rate used for discounting the debt component with the share price. With share values close to zero, a much higher risky rate would be used than for CB's in-the-money.

In this Chapter the *k*-factor[59] credit spread model for convertible bonds will be discussed. The model was first proposed by Muromachi [55] in which he studied the relation between credit spreads and share values for convertible bonds in the Japanese market. He found a functional form similar to equation (6.1), provides a reasonable fit

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[58] There is usually some pre-specified trigger price, but generally speaking a share value that is very close to zero.

[59] The model was described in a research report by Barclays [58] and given the name *k*-factor credit spread model which will be used from now on.

for bonds rated BB+ and below with the *k*-factor historically ranging between -1.2 and -2.0. Particular attention to the negative convexity for extremely low share prices will be addressed, and an explanation of how to incorporate this into the TF model will be discussed. In addition calibration of the model to actual data will be shown, and the sensitivities of each of the parameters especially the *k*-factor will be illustrated.

6.2 Implementation into TF Model

It is convenient to define the following parameters:

h_∞	=	Minimum Credit Spread
h_0	=	Current Credit Spread
S_0	=	Trigger Share Price
k	=	Decay Factor

In creating a credit spread function $h(t, S(t))$ dependent on the share value, we require that the function $h(t, S(t))$ have some basic properties.

- If the share value drops to zero, then the credit spread should go to infinity, i.e. $S \to 0$ then $h \to \infty$, although the formula can be adjusted to deal with a recovery value.
- The spread should be a monotonic decreasing function of the share price because as the equity capitalization of the firm grows, so should its credit strength due to improved asset coverage and lower financial gearing. Thus,

  $$\frac{dh}{dS} \leq 0 \ , \forall \quad S > 0$$

- There has to exist some positive, minimum credit spread limit h_∞ when the share price becomes very large, which represents the minimum credit risk premium for this level of debt

  $h \to h_\infty$ as $S \to \infty$

- The current spread h_0 and stock price S_0 can be reliably calibrated.

  $h(S_0) = h_0$ where $h_0 \geq h_\infty$.

A simple function that satisfies all these properties with a single factor *k* is,

$$h(S) = h_\infty + (h_0 - h_\infty)(S_0 / S)^{-k}, \text{ where } k \geq 0 \tag{6.1}$$

The decay factor *k*, measures the sensitivity of the credit spread to the share price, a higher *k*-factor increases the dependency and the credit spread, whilst a lower *k*-factor decreases the dependency and hence the credit spread. To implement this *k*-factor credit spread model into the TF model is quite easy. The model is the same with the only difference being that the debt component be discounted using the *k*-factor model, instead of the constant credit spread previously employed.

6.3 Parameter Sensitivities

To illustrate the impact of the *k*-factor model the spread changes for various values of *k* will be represented graphically, and the effect it has on the TF model.

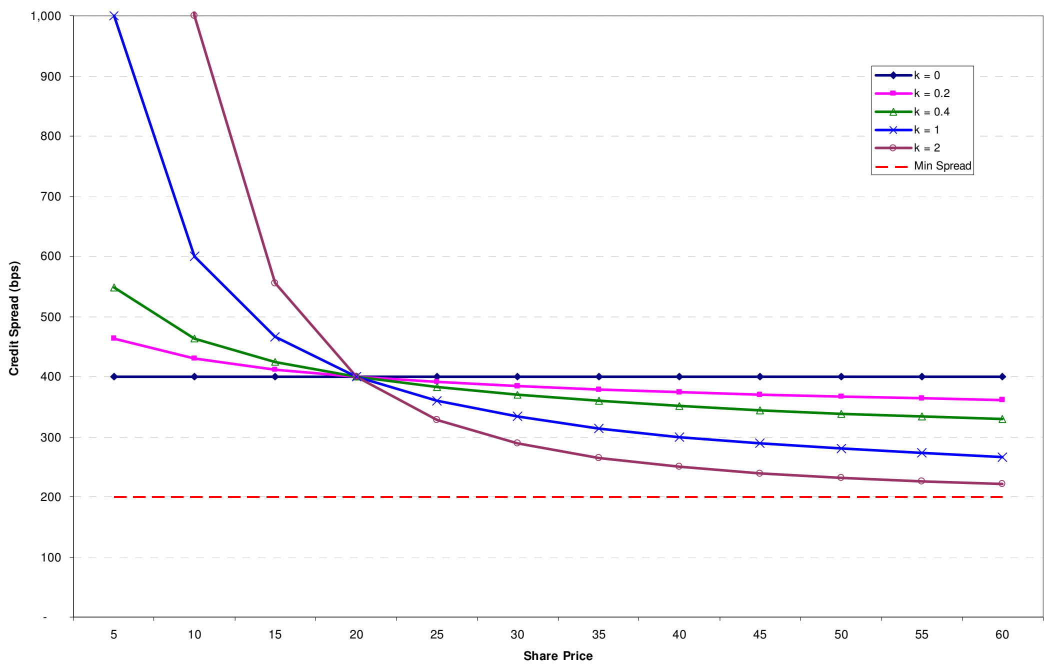

Figure 27 – Credit spread values for changes in the share price and various *k*-factors. Trigger price =20, Min credit spread = 200 bps and Current spread = 400 bps.

In Figure 27 it can be seen that an increase in the k-factor increases the sensitivity of the credit spread to the share price. When the share value increases beyond the trigger price $S_0 = 20$, the credit spread decreases toward the minimum spread $h_\infty = 200$ bps. For share values less than the trigger price the credit spread increases beyond the current spread of 400 bps and for k = 2 this spread increases dramatically, reaching extremely high values.

Using Parameter Set 2 in Appendix A3 and incorporating the k-factor credit spread into the TF model, we see that the bond floor no longer plays any role in the CB price. As soon as the share value drops below the trigger price of 5, the CB experiences negative convexity and vega, and positive delta. For share values above the trigger price the TF k-factor model seems to give slightly higher values relative to the original TF model. Intuitively this makes sense as credit spreads decrease for in-the-money CB's, lending to the lower risky discount rate and hence a higher CB price. A graphical representation of this can be seen in Figure 28.

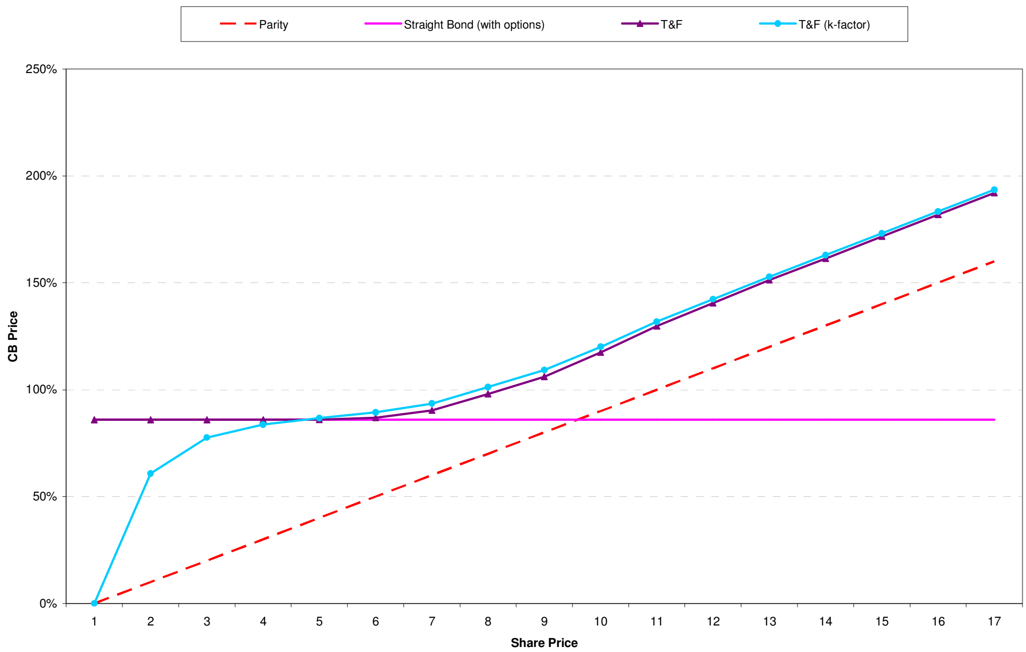

Figure 28 – Difference between the basic TF model and the TF with *k*-factor model for various share values. Trigger price =5, Min credit spread = 200 bps and Current spread = 400 bps.

6.4 Calibration of *k*-factor model

To calibrate the k parameter to actual data requires us to find historic data on the relationship between the share price and credit spread of the issuing firm. Assuming that the other parameters are given, we can regress the function $h(t, S(t))$ on the data, and find an implied *k*-factor. As an example, the convertible bond issued by Barclays for the period 16-Mar-2005 to 16-Jan-2008 will be used, giving 1037 daily observations with the data obtained from Bloomberg. The details of the bond are as follows:

Issuer	Barclays Bank PLC
Underlying Equity	TFI FP
Market of Issue	Great Britain, Euro
Rating	AA+
Issue Date	24 March 2005
Maturity Date	24 March 2010
Annual Coupon	0.5%
Call/Put	None
Conversion Ratio	3.4321

In Figure 29 there seems to be a clear inverse relation between the credit spread and underlying share price. The change in spread seems to be quite dramatic when the share price drops below €18.90, increasing from around 70 bps to 130 bps. Fitting the *k*-factor model to the following observations by minimizing the sum of squares, we get the following parameter values:

h_0 = Current Spread	62.6 bps
h_∞ = Min Spread	39.0 bps
S_0 = Trigger Share Price	€18.90
k - Factor	4.8

Table 4 – Calibrated parameters for the Barclays convertible bond issue.

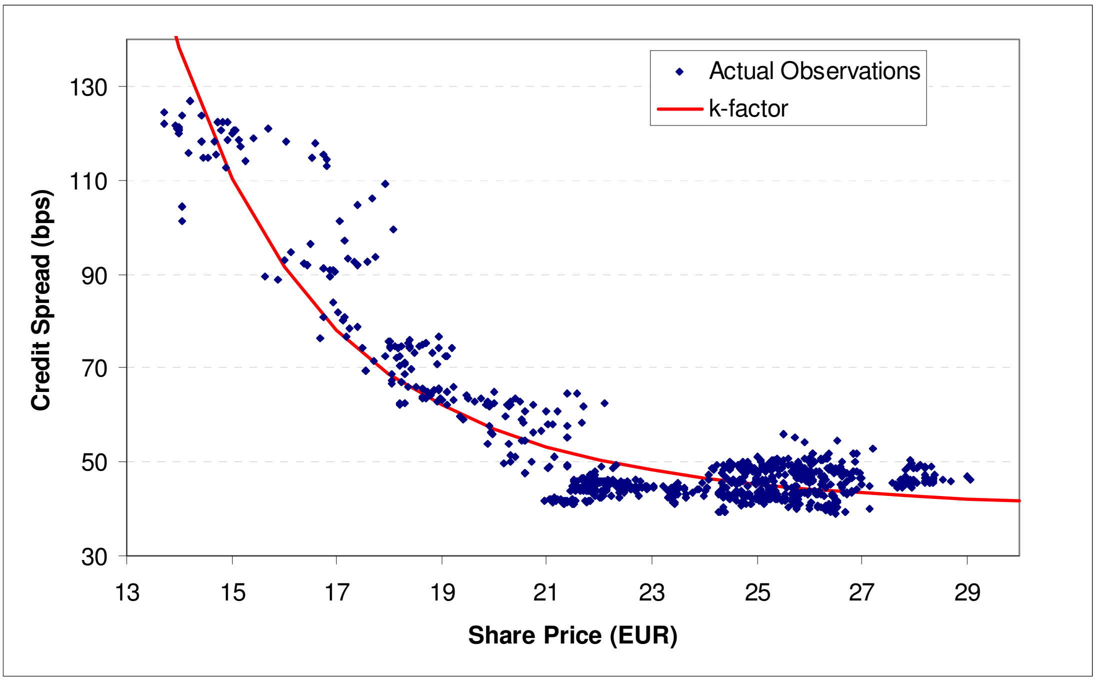

Figure 29 – Scatter plot of the credit spread and share price of the Barclays convertible bond, issued in Euros for the period 16 March 2005 to 16-Jan-2008.

The implied *k*-factor is 4.8 which is quite high given that historically it ranges between 0.5 - 2.5 [58]. This is attributable to the fact that the credit spread increases sporadically once below its implied trigger price of €18.90, but settles to its min spread of 39.0 bps above the implied trigger price.

Volatility plays an interesting role when dealing with the *k*-factor model. If we use the same parameters from Table 4 and assume different values for *k*, we see that the CB price actually decreases for large values of *k* and share volatility as shown in Figure 30.

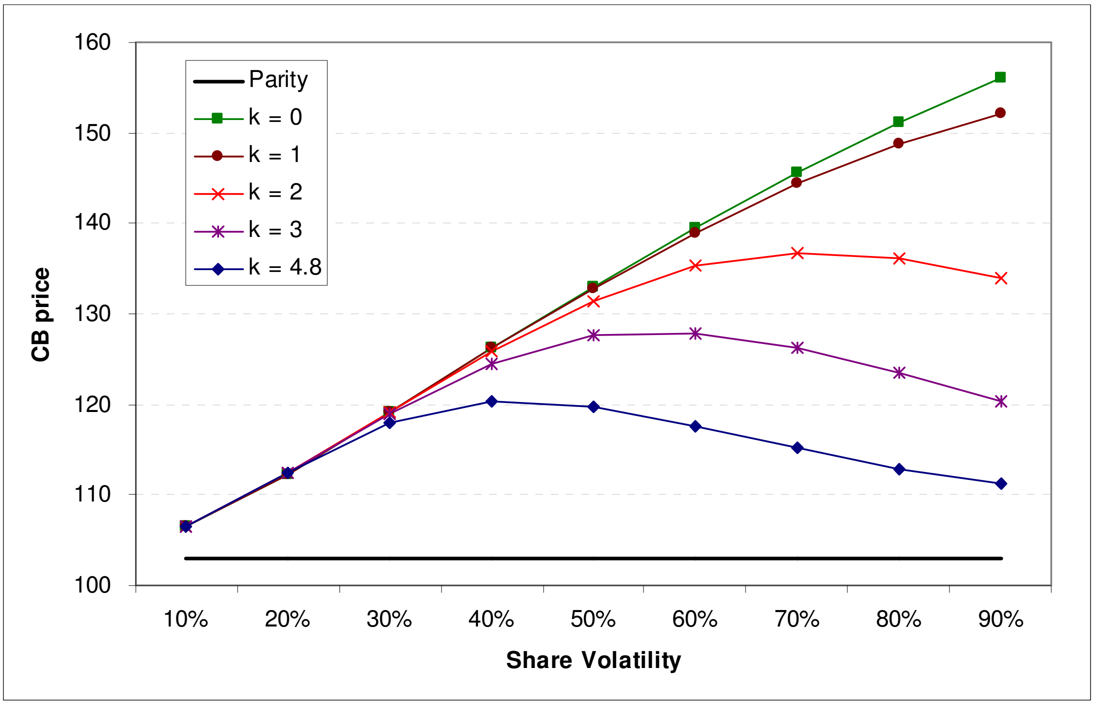

Figure 30 – CB values using the TF (k-factor) model with S=30, maturity = 5yrs and using the calibrated market values in Table 4.

Traditionally, decreasing volatility is considered to be unfavorable to hedgers who seek to profit from volatility through delta hedging, as the lower volatility depresses the CB price thereby reducing profitability. For extremely volatile stocks, bonds with a higher *k*-factor would outperform those with a lower *k* in a falling volatility set-up, and may even gain value in absolute terms. This may also explain why CB's have lower implied volatility relative to historic share volatility and implied levels from other listed options. For example, assuming that the market price of the Barclays convertible bond is €120 and referring to Figure 30, we see that for $k=0$, the implied volatility would be around 30%, whereas for $k=4.8$ it would be closer to 40%. This goes to show that when valuing the CB without a stochastic credit spread model, the implied volatility could be underestimated. Convertible investors have subjectively stated that, for some bonds, they would never pay above a certain level of implied volatility, which would agree with the *k*-factor model, since for large values of *k*, as in the case of $k=4.8$ the maximum value of the CB is around €120. Further volatility increases stifle the CB price and it actually stabilizes to €110. As such convertible investors will not benefit much from purchasing CB's with high-implied volatility due to the low levels of vega.

Convertible investors and hedgers have several methods in which to manage this relationship between credit spreads and underlying equity values. They include

- Shorting more underlying shares than implied by traditional CB models to take advantage of the higher delta.
- Longing put options on the underlying shares to hedge the higher delta and negative gamma, and
- Employing credit protection instruments such as asset or default swaps, to reduce exposure to the widening credit spread for low share prices.

The *k*-factor credit spread can be seen as a general model as with the credit risk model in Chapter 4, and in so doing can be adapted to a wide variety of models that possess default risk and are dependent upon the underlying share price. Calibration can also be incorporated if historic data between the underlying share price and financial instrument[60] exists.

Having discussed equity valuation models in Chapter 5, the next Chapter introduces firm value models with the underlying variable being the value of the firm. It is important to review these models as they form the initial building blocks of the modern day CB models. The reason for the shift away from firm value to equity models is that the latter requires the share price which is more readily available whereas the former requires the firm value which has to be estimated thereby creating potential errors.

[60] The financial instrument is the security that is a function of the underlying share which is trying to be modeled.

7. Firm Value Models

The initial attempts to model CB's lead to the creation of firm value models. What this entails is that the actual value of the firm (assets and liabilities) is used as the underlying variable. To begin with, Section 7.1 gives a quick literature review of firm value models and moves onto 7.2 that introduces the Tan and Cai Risk Equilibrium model. Two prominent features of this model is the concept of Corporation Market Value Allocation (CMVA) and the Risk Burden Ratio, which will be addressed later in the Chapter. The parameter sensitivities and Greeks are also reviewed. Section 7.3 presents the Andrea Gheno volatility structure model that calibrates the term structure of interest rates and implied volatilities[61] from market data. The model is established by setting the boundary conditions, spot rates and asset volatility trees, leading to the construction of the asset value and CB price trees. To understand the application of the model a numerical example with parameter sensitivities is given at the end.

7.1 Background to Firm Value Models

The firm value model was first proposed by Black and Scholes [12] in 1973 and extended by Merton [53] in a more detailed paper in 1974. The model is essentially a simple model for valuing credit risk. It proposes that a company defaults when its share price (or firm value) falls below a prescribed barrier. Default can be defined as either the first time that the share price hits the barrier, in which case the actual dynamics of the share price path become important[62], or at the final share price. If the share price/firm value is below the barrier then the company is considered to be in a state of default. The advantage of the firm value approach is the ability to hedge defaults using the counterparty share price.

7.2 Tan and Cai

Moving on to the assumptions of the model, they are the same as those suggested by Merton but for two main differences. The first is that the model assumes that total

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[61] The implied volatility structure can be obtained from options on the underlying stock that trade in the market.

[62] This is called the structural model, whilst the model that considers only the end share value is termed the firm value model.

market value of the firm follows Geometric Brownian motion whilst Merton does not state whether book value or market value is used. The second is that payouts by the firm are not of concern and can be relaxed.

7.2.1 Risk-Equilibrium Model

Letting the total market value of the firm (A) follow a Geometric Brownian motion dw, then

$$dA = \mu A dt + \sigma A dw, \tag{7.1}$$

thus representing the CB as χ, to be a function of A and t, and using Itô's Lemma gives us the usual PDE

$$dC = \left(\chi_t + \chi_A \mu A + \frac{1}{2} \chi_{AA} A^2 \sigma^2 \right) dt + \chi_A A \sigma dw, \tag{7.2}$$

simplifying the equation into

$$\begin{gathered} \frac{d\chi}{\chi} = \mu^\chi dt + \sigma^\chi dw \\ where \quad \begin{cases} \mu^\chi = \dfrac{\chi_t + \chi_A \mu A + \frac{1}{2} \chi_{AA} A^2 \sigma^2}{\chi} \\ \sigma^\chi = \dfrac{\chi_A A}{\chi} \sigma \end{cases}. \end{gathered} \tag{7.3}$$

Constructing a three security portfolio, x with W_1 dollars of total firm market value, W_2 dollars of CB's, and $W_3 = -(W_1 + W_2)$ dollars of riskless debt with risk-free rate r, we have that the instantaneous dollar return of the portfolio is

$$\begin{aligned} dx &= W_1 \frac{dA}{A} + W_2 \frac{d\chi}{\chi} + W_3 r dt \\ &= \left[W_1 (\mu - r) + W_2 (\mu^\chi - r) \right] dt + \left(W_1 \sigma + W_2 \sigma^\chi \right) dw. \end{aligned} \tag{7.4}$$

In order for the portfolio to be riskless the *dw* term needs to disappear. Also since the portfolio cost nothing to construct and according to arbitrage theory, the expected return of the portfolio has to be zero. Thus we have two equations

$$\begin{cases} W_1\sigma + W_2\sigma^{\chi} = 0 & , \ \textit{Riskless} \\ W_1(\mu - r) + W_2(\mu^{\chi} - r) = 0 & , \ \textit{No arbitrage.} \end{cases}$$

A non-trivial solution exists if and only if

$$\frac{\mu - r}{\sigma} = \frac{\mu^{\chi} - r}{\sigma^{\chi}} = \lambda,$$

where λ is the risk premium per unit of risk. So we can calculate the return on the CB if we have σ^{χ} as follows,

$$\mu^{\chi} = r + \lambda\sigma^{\chi}. \tag{7.5}$$

This means that we have to find σ^{χ}.

By using Itô's Lemma again, the dynamics of $\ln A$ and $\ln \chi$ are given by

$$\begin{cases} d(\ln A) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dw \\ d(\ln \chi) = \left(\dfrac{\chi_t + \chi_A \mu A + \frac{1}{2}\chi_{AA}A^2\sigma^2}{\chi} - \dfrac{\chi_A^2 A^2 \sigma^2}{2\chi^2}\right)dt + \dfrac{\chi_A A}{\chi}\sigma dw, \end{cases} \tag{7.6}$$

Tan and Cai [68] define the Risk Burden Ratio for convertibles as

$$a(A,t) = \frac{\chi_A A}{\chi}. \tag{7.7}$$

It gives an indication of how much risk; convertible investors are willing to take on with respect to the total amount of risk of the firm. Referring to equation (7.3) we see that the Risk Burden Ratio is

$$\sigma^{\chi} = a\sigma \,. \tag{7.8}$$

To calculate a we need some more expressions, so

$$\begin{aligned} \operatorname{cov}\left(d\left(\ln A\right), d\left(\ln \chi\right)\right) &= \operatorname{cov}\left(\sigma dw, a\sigma dw\right) \\ &= a\sigma^2 \operatorname{cov}\left(dw, dw\right) \\ &= a\sigma^2 dt. \end{aligned} \tag{7.9}$$

So we can calculate the return of the CB by using the following set of equations:

$$\begin{cases} a = \dfrac{\operatorname{cov}\left(d\left(\ln A\right), d\left(\ln \chi\right)\right)}{\sigma^2 dt} \\ \sigma^{\chi} = a\sigma \\ \mu^{\chi} = r + \lambda\sigma^{\chi}. \end{cases}$$

The Risk Burden Ratio is a percentage with respect to the standard deviation of return of the total firm market value. When the firm faces bankruptcy (convertible investors have first rights onto the assets of the firm so $\chi = A$), or when stock price is far above conversion price (CB, equity and total market value all have the same return) so *a=1* and the CB has the same risk as the total firm value. If the stock is far below conversion price and the probability of bankruptcy of the firm is very small, *a=0* implies the CB takes on no risk and can be treated as risk-free debt. *a* often lies somewhere between 0 and 1.

7.2.2 Theory of Corporation Market Value Allocation

In the first structural model Merton [53] assumed that total market value of the firm followed a Geometric Brownian motion and the capital structure of the firm was irrelevant (Modigliani and Miller (MM) Theory [48]) and so exogenous to any allocation between convertible and equity investors. The difference is that with Corporation Market Value Allocation (CMVA) there is a clear distinction between convertible investors and stock investors, because if convertible investors invest more, stock investors have relatively invested less and vice versa. This results in competition between the two investors and is caused by rational investors seeking return with the lowest possible risk. MM Theory [48] might seem similar to CMVA but whereas MM Theory assumes capital structure does not affect firm value, CMVA treats firm value as exogenous and divided by different investor types.

In developing the model to value the convertible debt we require a firm value as an input. Since this is unobservable we have to use our model and observable data implicitly to help in obtaining a CB price. The share price of the issuer is observable so we can find total equity market value and propose a CB price. The model assumes that CB value is a function of total firm value so

$$\chi = f\left(X_A\right) \quad , \quad X_A \in \left[0, \infty\right).$$

where X_A is a function dependent on the firm value A.

According to CMVA the claim on the firms' assets for equity investors is,

$$S = X_A - \chi = X_A - f\left(X_A\right). \tag{7.10}$$

Since total stock value S^* can be observed and firm value is unobservable, substituting S^* into equation (7.10) and solving implicitly for X_A to get A^*, we have that total convertible value is given as $\chi^* = A^* - S^*$, so that the convertible price is given by

$$\chi^* = \frac{A^* - S^*}{N_\chi}, \tag{7.11}$$

where N_χ is the total number of convertibles issued by the firm.

7.2.3 Numerical Example

Looking at an example might help in explaining what this Risk Burden Ratio is all about. If we let the underlying inputs be given as in Table 5

Firm Value	
Firm volatility	20%
Mean firm return	6.00%
Equity	
No. of shares issued	20 million
Risk-free rate	5.00%
Convertible Bond	
No. of CB's issued	1 million
Risky rate	7.00%
Maturity	3 years
Face Value	100.00
Risk Premium	2.00%
Coupon	8.00%
Coupon Frequency	2
No. of time steps	35
Conversion Ratio	1.00

Table 5 – Inputs to be used for the Tan and Cai Risk Equilibrium model

Observing the CB price track for various share values in Figure 31, we see that the CB price actually falls below its straight bond value with the enlargement shown in Figure 32, indicating the presence of negative gamma for extremely low share values (close to 0). As the share value increases so does the convertible bond and trades at a premium to its straight bond and conversion value. The Risk Burden Ratio for the CB is also included and increases dramatically as the share value approaches 0. During the busted phase (when the convertible behaves like a straight bond) the Risk Burden Ratio takes on 0, as the firm will not have to payout any major proceeds to the investors and so does not have any immediate obligations. Once in the hybrid and equity phases the Risk Burden Ratio increases towards 1, indicating that there is pressure on the firm to make payments to its CB holders as the investors have a high probability of converting.

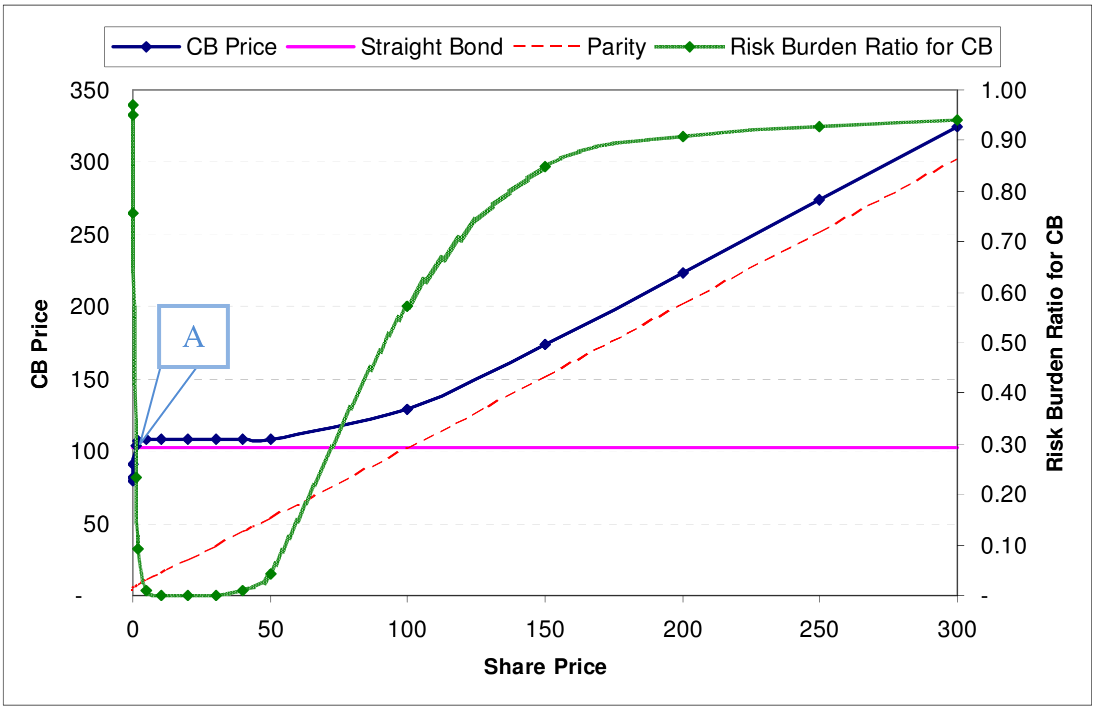

Figure 31 – CB price track using the inputs in Table 5 and the Risk Equilibrium model together with the Risk Burden Ratio for CB.

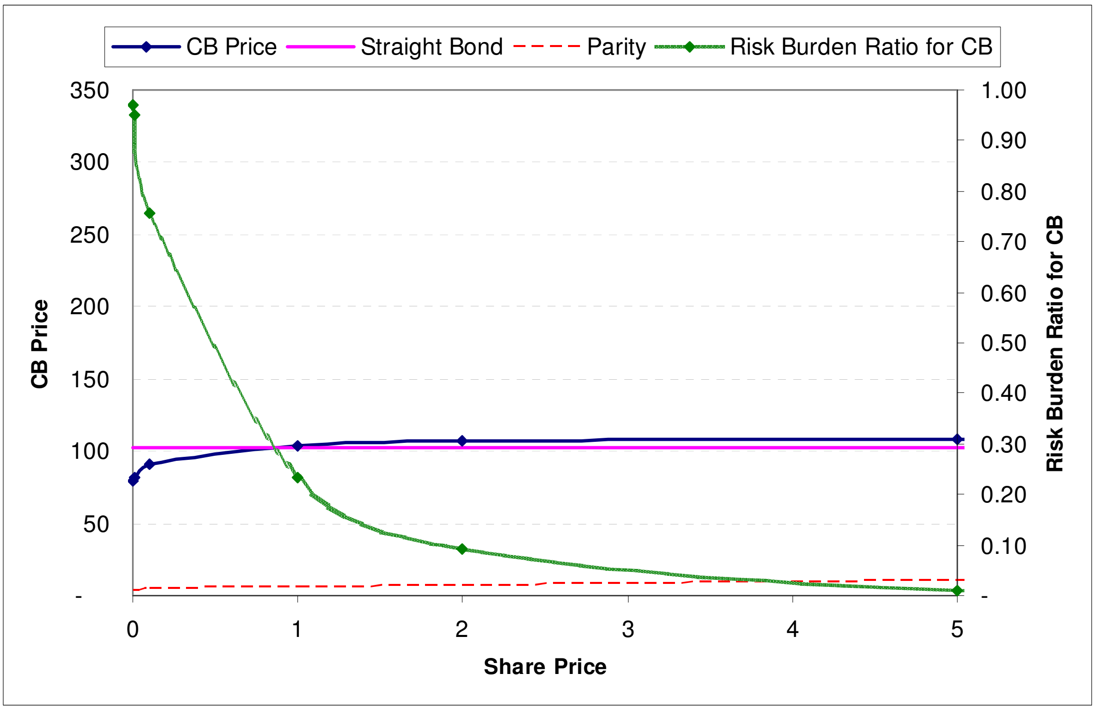

Figure 32 – Enlargement of Section A in Figure 31 showing the CB price falling below its straight bond value for the Risk Equilibrium model and the rapid increase in the Risk Burden Ratio to 1.

The Risk Premium (λ) in equation (7.5) is the discount rate applied to the convertible bond; as such we would expect the convertible to decrease in value if the Risk Premium were to increase. For deep in-the-money CB's the increase in the Risk Premium decreases the CB price but for deep out-the-money issues, the Risk Premium does not play a significant roll. This is because the Risk Burden Ratio is close to zero implying that σ^{χ} also takes on a small value, so that the discount rate doesn't change by much. A graphical depiction is given in Figure 33.

Figure 34 shows the delta and gamma of the Risk Equilibrium model, and it is interesting to note that the delta takes on a shape similar to the Risk Burden Ratio. For low share values the delta reverses and heads towards infinity as the share value approaches 0, signalling the increased default risk that is present in the firm. The gamma reaches its maximum at the conversion price of 100, and falls away to 0 for larger share values. For extremely low share values the gamma goes negative indicating the decrease in the CB below its straight bond value in the distressed zone.

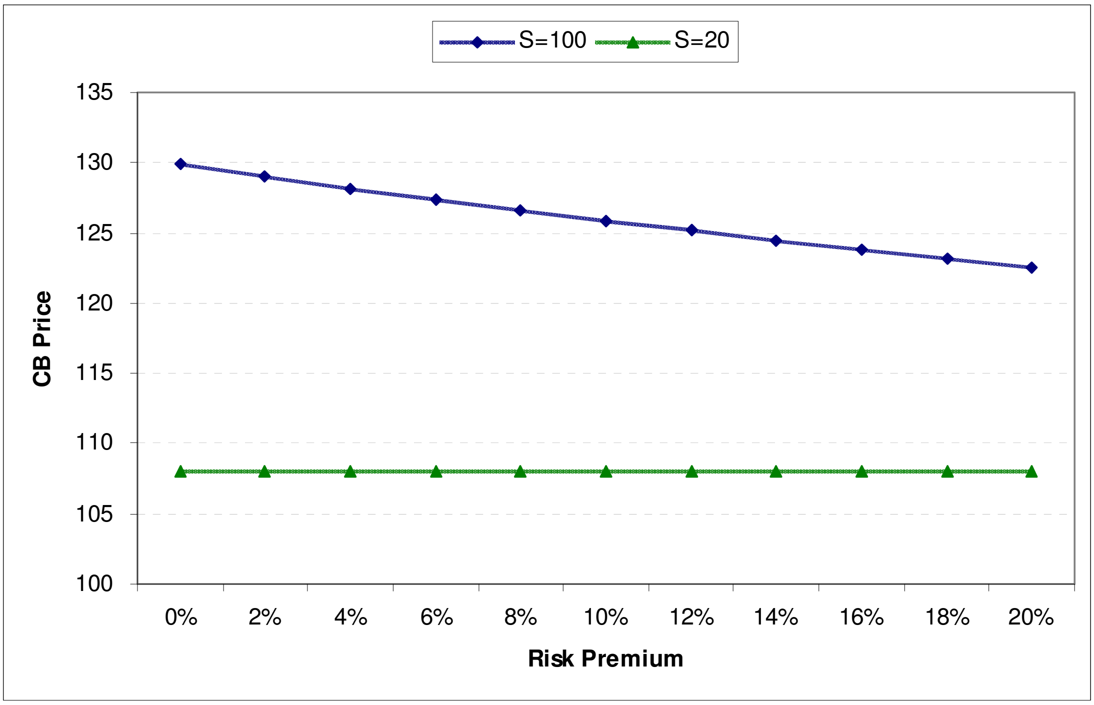

Figure 33 – Convertible bond price for changes in Risk Premium using the Risk Equilibrium model. Graphs are given for an in-the-money CB (where share value is 100) and an out-the-money CB (where share value is 20).

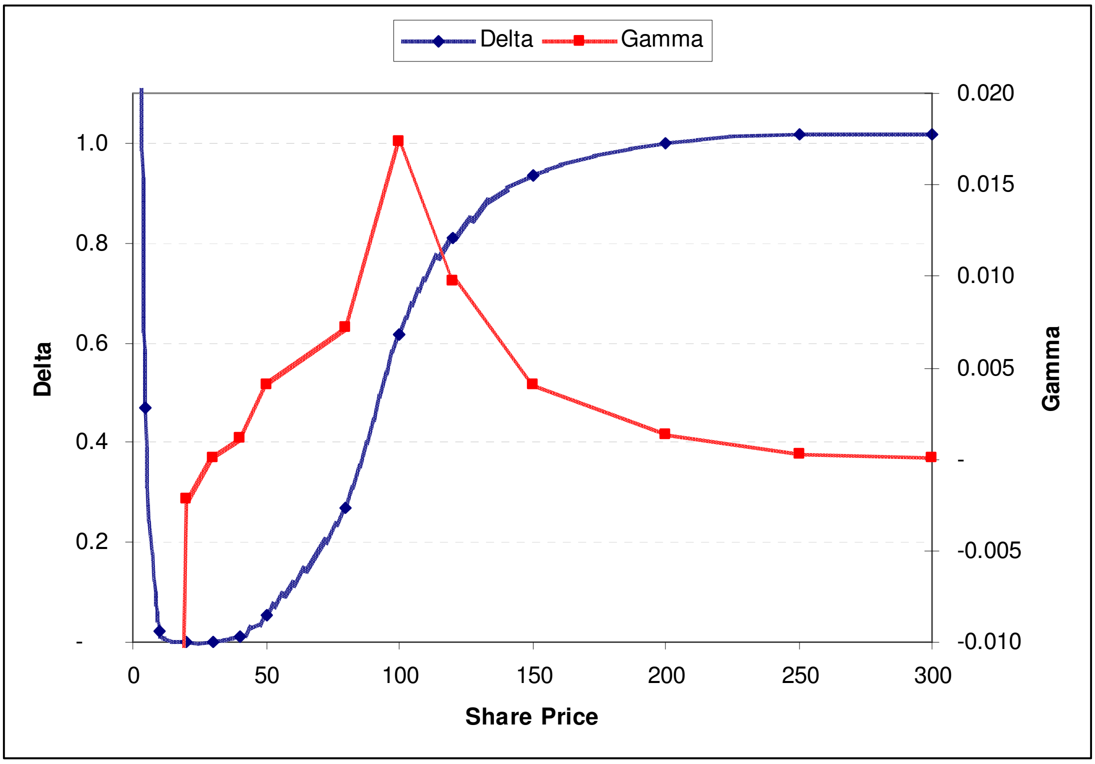

Figure 34 – Delta and gamma of the Risk Equilibrium model showing the reversal of the delta and negative gamma for extremely low share values.

7.3 Andrea Gheno

Calibration in CB valuation models is important but as yet not much literature exists on calibration. By using stochastic variables such as the risk-free spot rate and firm asset value; a two-factor CB valuation model can be constructed. In an earlier paper Gheno [31] shows that the CB price is highly correlated to the volatility structure of the firms' asset value returns and weakly correlated to the volatility structure of future spot rates. As such we need to develop the volatility structure of the firm asset value, and can ignore the volatility structure of future spot rates since this can be implied from bond and cap market prices. The following notation will be used throughout Section 7.3:

$V(t)$	asset value at time t
$\chi(t)$	CB value at time t
$S(t)$	equity value at time t
N	face value of CB
I	annual coupon rate
p	number of CB's issued at t=0
s	number of shares at t=0
α	conversion ratio
c	call price
$r(t,t_k)$	interest rate between t and t_k

7.3.1 Boundary Conditions

Defining the conversion value at time t_j as

$$\alpha \frac{V(t_j)}{p\alpha + s} \quad , \qquad j = 0,1,\ldots,m, \tag{7.12}$$

a rational investor will choose to convert if the present value of his coupons and principal is less than the conversion value, thus we have

$$\alpha \frac{V(t_j)}{p\alpha + s} \quad > F\left[\sum_{k=j+1}^{m} Ie^{-r(t_{k-1},t_k)(t_k - t_{k-1})} + e^{-r(t_{m-1},t_m)(t_m - t_{m-1})}\right], \tag{7.13}$$

otherwise the investor will continue to receive the payment stream. If there are any call features present then the issuer will call the bond if

$$\alpha \frac{V(t_j)}{p\alpha + s} > c, \tag{7.14}$$

and force early conversion upon the investor.

7.3.2 Spot Rates and Asset Value Trees

The spot rate tree is developed and calibrated by using the Black-Derman-Toy [13] (BDT) binomial model. The calibration scheme is the same as the method discussed in Chapter 3. The volatility structure can be obtained from market data and thus a calibrated binomial spot rate tree is formed. This tree will be used to find the risk-neutral probabilities and discount factors when rolling back through the final CB tree.

It is difficult in practice to calculate the asset value of a firm and the asset return volatility. Assuming Modigliani-Miller [48] propositions where the capital structure is comprised only of debt and equity, we must have that the asset value at time *t* is given as

$$V(t) = D(t) + S(t). \tag{7.15}$$

where $D(t)$ is the market value of debt. A popular method as mentioned in [31] in trying to estimate the asset return volatility is to use

$$\sigma_V = \sqrt{a^2\sigma_D^2 + (1-a)^2\sigma_S^2 + 2a(1-a)\rho\sigma_S\sigma_D}, \tag{7.16}$$

where $a = D(t)/V(t)$ is the leverage, σ_D and σ_S are the debt and stock volatilities respectively and ρ is the correlation between the debt and stock returns. This is seen as a little nuisance as we have to estimate three more parameters ρ, σ_D and σ_S, although these values can easily be obtained from market data. Also when a firm pays coupons its asset value decreases which results in increasing the asset return volatility. Thus the amortisation schedule of the firm is an important factor in determining the asset return volatility.

The firm value *V(t)* is modelled using a binomial tree due to the discrete coupon payments at each time tick. It is a non-standard CRR model since the term structure is calibrated using a BDT tree with stochastic interest rates $r(t_k, t_{k+1})$. Since these interest rates and the volatility structure σ_V are stochastic, the risk-neutral probabilities are also stochastic.

7.3.3 Asset Return Volatility Tree

The return volatility tree can be constructed in a similar manner to that used in Black and Scholes [13] with discrete dividend payments. To build the tree we require an initial estimate of the asset value, which we will assume as given. So the volatility returns tree is given by

$$\sigma_V\left(t_k, t_{k+1}; i\right) = \sigma_V\left(t_{k-1}, t_k; i-1\right) \frac{\tilde{V}\left(t_k; i\right)}{\tilde{V}\left(t_k; i\right) - \tilde{V}_D\left(t_k; i\right)},$$

$$where \quad k = 1, \ldots, m \text{ and } i = 1, \ldots, n \quad , \tag{7.17}$$

with $\tilde{V}_D\left(t_k; i\right)$ being the expected present value of the debt payments at time t_k in state i, $\tilde{V}\left(t_k; i\right)$ being the asset value at time t_k in state i inferred from the asset value tree one period prior and $\tilde{\sigma}_V\left(t_0, t_1; 1\right) = \sigma_V\left(t_0, t_1\right) = \sigma_V$. Thus the asset return volatility and asset value trees can be built recursively from each other taking into account debt related payments via the volatility structure.

7.3.4 Asset Value Tree

The asset value tree is constructed using a CRR process with stochastic volatility, although the numbers of nodes are dramatically increased. In the first step there are two nodes, in the second there are eight and in the k^{th} there are $2 \times 4^{k-1}$. Defining $V\left(t_k; i\right)$ as the asset value at time t_k in state i, means that $V\left(t_{k+1}; *\right)$ can take on four possible values, namely

$$V\left(t_{k+1}; \left[\frac{i}{4}\right] 4 + h\right) \tag{7.18}$$
$$where \quad h = 0, \ldots, 3.$$

$[i/4]$ is defined as the integer of $i/4$, so for *h=0,1* the asset value moves to the lower state and for *h=2,3* it moves to the upper state. As mentioned earlier the risk-neutral probability used for the rollback value is non-constant due to the stochastic interest rates and volatility. Thus the risk-neutral probability at time t_k in state i, $p^*(t_k;i)$ is

$$p^*(t_k;i)=\frac{e^{r(t_k,t_{k+1};i)}-e^{-\sigma_V(t_k,t_{k+1};i)}}{e^{\sigma_V(t_k,t_{k+1};i)}-e^{-\sigma_V(t_k,t_{k+1};i)}}. \tag{7.19}$$

7.3.5 Convertible Bond Tree

Stipulating the boundary conditions at maturity we have that the CB value must satisfy

$$\chi(t_m;i)=\max\left[F;\alpha\frac{V(t_m;i)}{\alpha p+s}\right]+IF \qquad , \qquad i=1,\ldots,n. \tag{7.20}$$

Rolling backward through the tree at any time $t_k<t_m$ gives us the recognizable expected present value algorithm without call provisions,

$$\chi(t_k;i)=\max\left[\alpha\frac{V(t_k;i)}{\alpha p+s};RB(t_k;i)\right]+IF \quad , \tag{7.21}$$

where

$$RB(t_k;i)=\frac{1}{2}\left(down\_state(t_k;i)+up\_state(t_k;i)\right), \tag{7.22}$$

and

$$down\_state(t_k;i)=\left(\frac{\chi\left(t_{k+1};\left[\frac{i}{4}\right]4+0\right)p^*\left(t_{k+1};\left[\frac{i}{4}\right]4+0\right)+\chi\left(t_{k+1};\left[\frac{i}{4}\right]4+1\right)p^*\left(t_{k+1};\left[\frac{i}{4}\right]4+1\right)}{e^{r(t_k,t_{k+1};2i)}}\right), \tag{7.23}$$

$$up\_state(t_k;i)=\left(\frac{\chi\left(t_{k+1};\left[\frac{i}{4}\right]4+2\right)p^*\left(t_{k+1};\left[\frac{i}{4}\right]4+2\right)+\chi\left(t_{k+1};\left[\frac{i}{4}\right]4+3\right)p^*\left(t_{k+1};\left[\frac{i}{4}\right]4+3\right)}{e^{r(t_k,t_{k+1};2i+1)}}\right). \tag{7.24}$$

$r(t_k, t_{k+1}; 2i)$ and $r(t_k, t_{k+1}; 2i+1)$ are the interest rates at time t_k in the lower and upper states respectively. Introducing the call provision into the tree produces a slight adjustment to the above formula,

$$\chi(t_k;i) = \min\left[c, \max\left\{\alpha\frac{V(t_k;i)}{\alpha p + s}; RB(t_k;i)\right\}\right] + IF\ , \tag{7.25}$$

with $RB(t_k;i)$, $up\_state(t_k;i)$ and $down\_state(t_k;i)$ as defined previously. Using the familiar backward induction approach on a binomial tree gives us our final answer $C(t_0)$.

7.3.6 Numerical Example

To illustrate the Gheno model a simple 3 step example showing the various tree diagrams will be constructed. Defining the inputs for the underlying variables as

Convertible Bond		
F	= 100.00	Face Value of CB
I	= 5.00%	Annual Coupon Rate
p	= 1.00	No. of CB's Issued
t_n	= 3.00	Maturity
Firm Value		
s	= 10.00	No. of Shares at *t=0*
S	= 100.00	Share Price at *t=0*
σ_V	= 10.00%	Asset Value Volatility
Zero-Coupon Bond		
	Price	*Volatility*
P(0,1)	97.00	11.00%
P(0,2)	94.00	13.00%
P(0,3)	91.00	15.00%
Embedded Options		
α	= 1.00	Conversion Ratio
Put	= 75.00	Put Value
c	= 120.00	Call Value

Table 6 – Variable inputs to be used in the Gheno volatility structure model.

To begin with we need to calibrate the binomial interest rate tree, with the given ZCB prices and volatility, which can be obtained from market data. The calibration is carried out in the same fashion as in Chapter 3. So using the above data we have a calibrated interest rate tree as follows

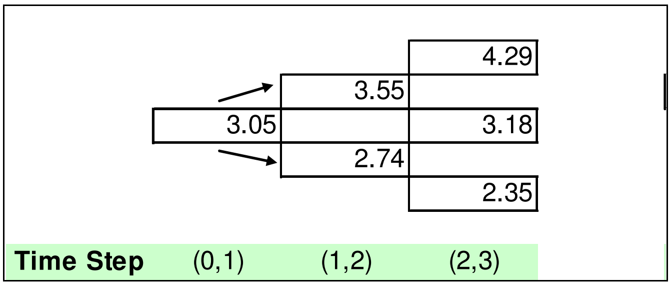

Figure 35 – Calibrated interest rate tree using the simple calibration technique and the given market data from Table 6.

The asset return volatility and asset value trees are constructed simultaneously with the outputs of the asset value tree used as inputs into the asset return volatility tree in a continuous cycle. With the starting asset return volatility given as 10%, the asset return volatility tree is shown in Figure 36, and the asset value tree is given in Figure 37

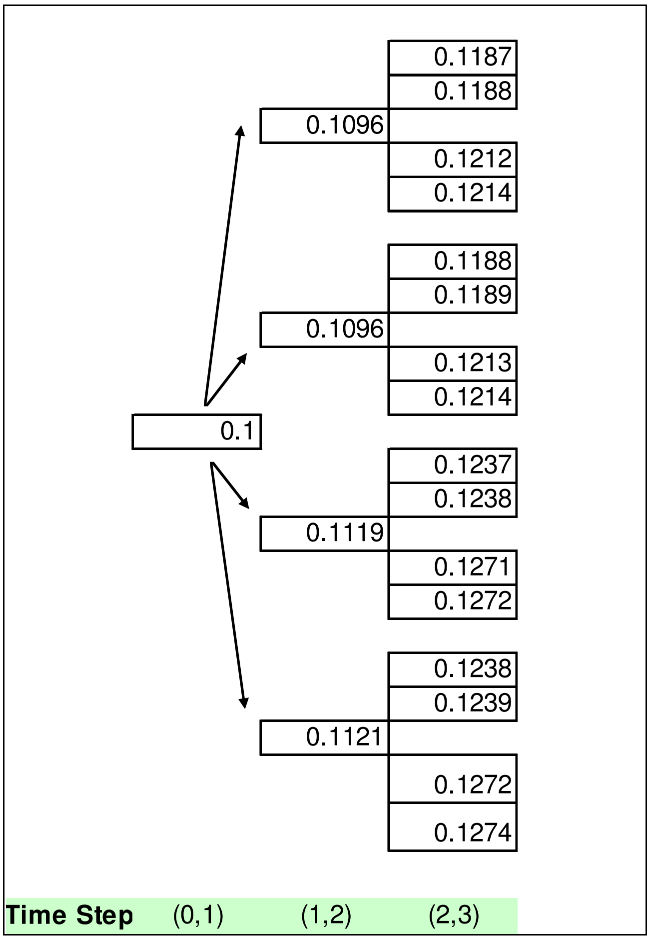

Figure 36 – Asset return volatility tree with a starting volatility of 10%.

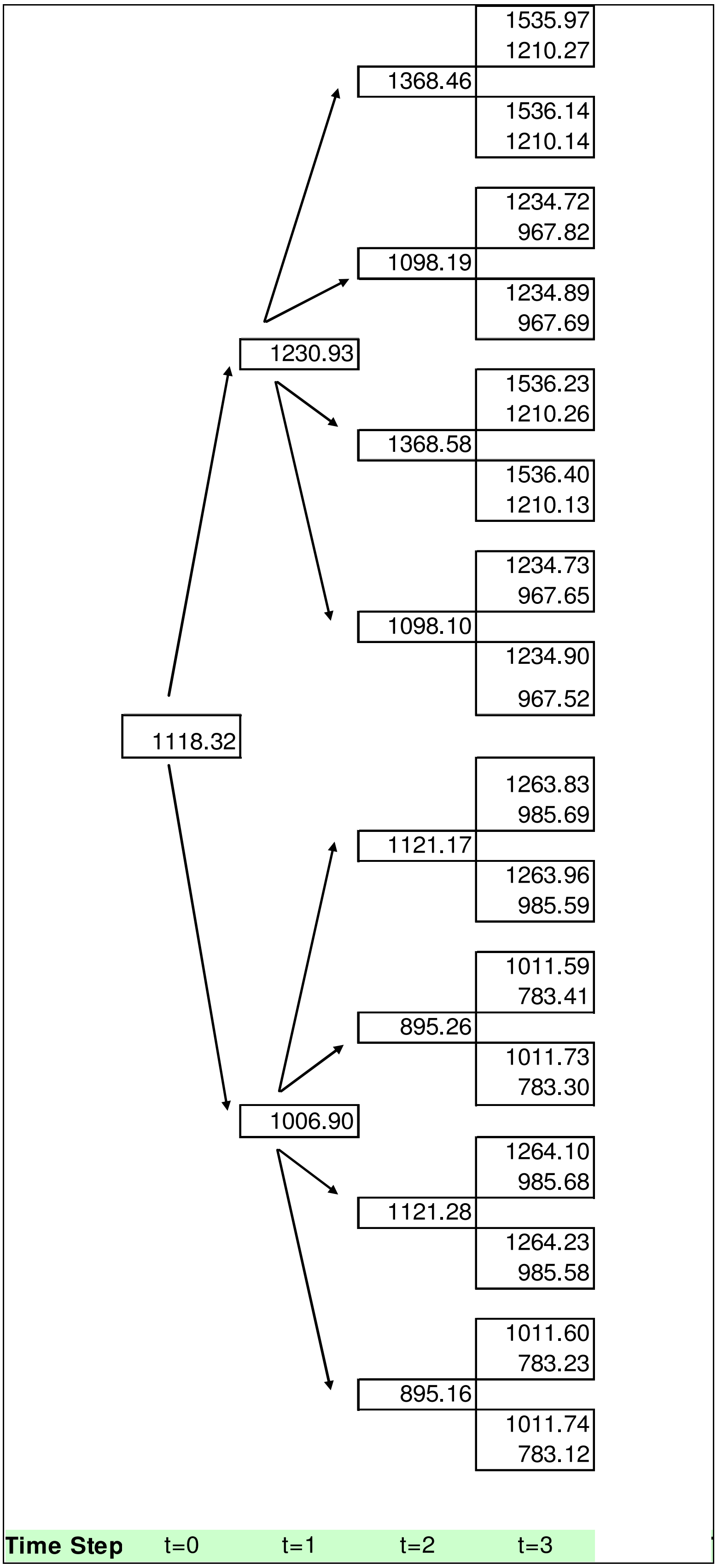

Figure 37 – Asset value tree constructed simultaneously with the asset return volatility tree in Figure 36 over 3 time ticks.

Once the asset return volatility and asset value trees have been calculated the risk-neutral probabilities need to be calculated since the volatility and interest rates are both stochastic leading to a risk-neutral probability tree illustrated below in Figure 38.

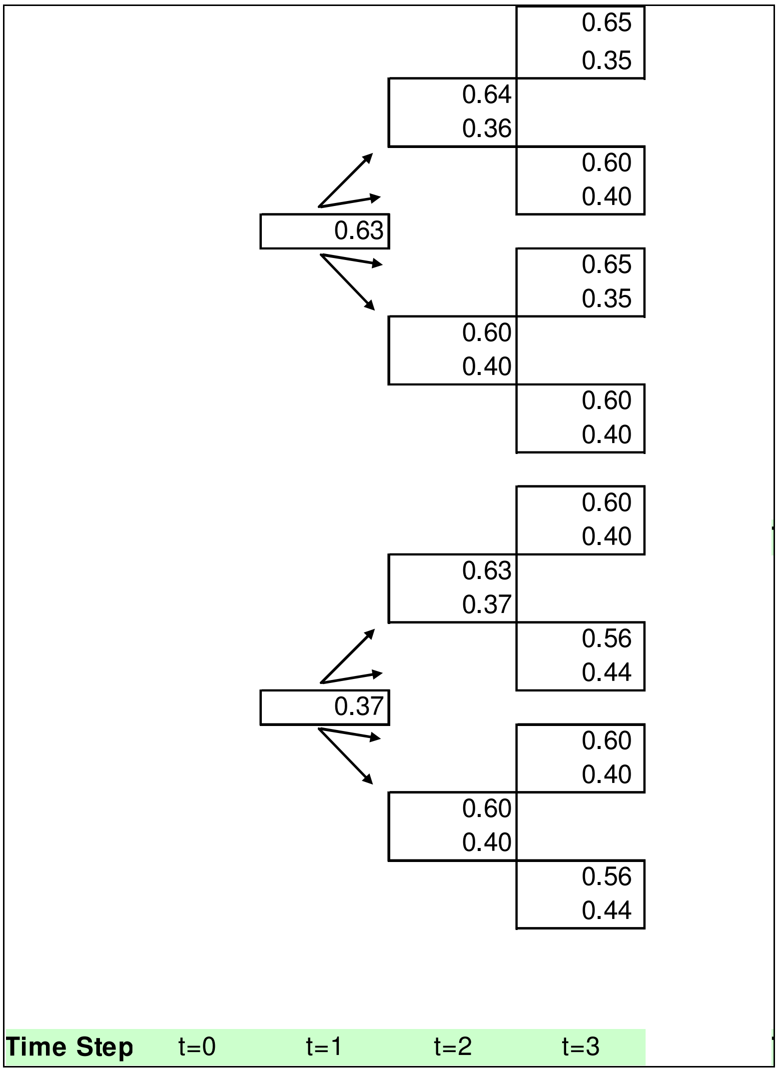

Figure 38 – Risk-neutral probability tree constructed with the use of the asset return volatility (Figure 36) tree and calibrated interest rate tree (Figure 35). The two cell values in time steps 2 and 3 correspond to the up and down probabilities.

Finally we can construct the convertible bond tree by using the asset value, risk-neutral probability and calibrated interest rate trees calculated earlier. The convertible bond tree is given in Figure 39 as

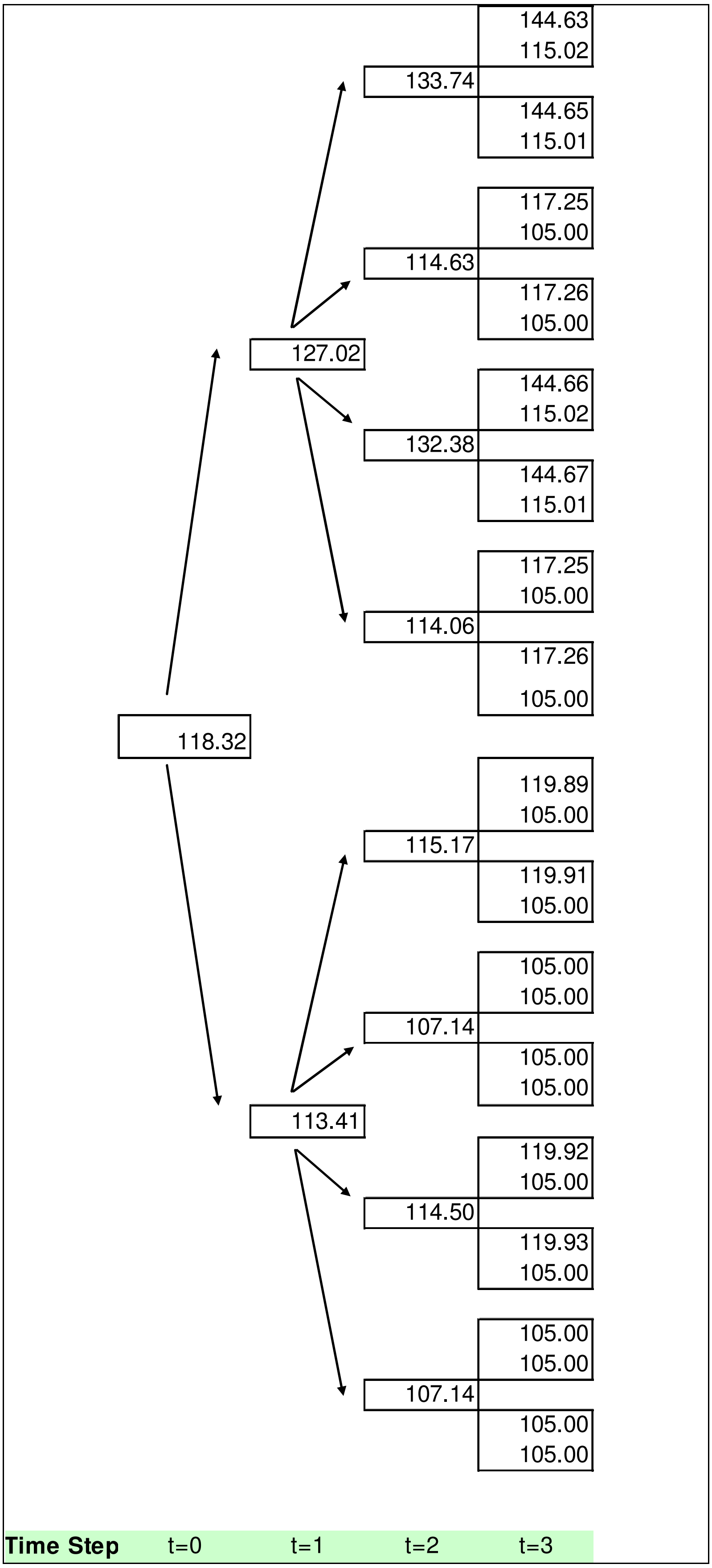

Figure 39 – Convertible bond tree using the Gheno volatility structure model.

7.3.7 Parameter Sensitivities

In this short section the effects of the asset return volatility, and call and put features on the CB value will be undertaken including the delta and gamma of the Gheno [31] model. Letting the variable inputs be the same as those given in Table 6, the CB price track for different share prices is shown in Figure 40

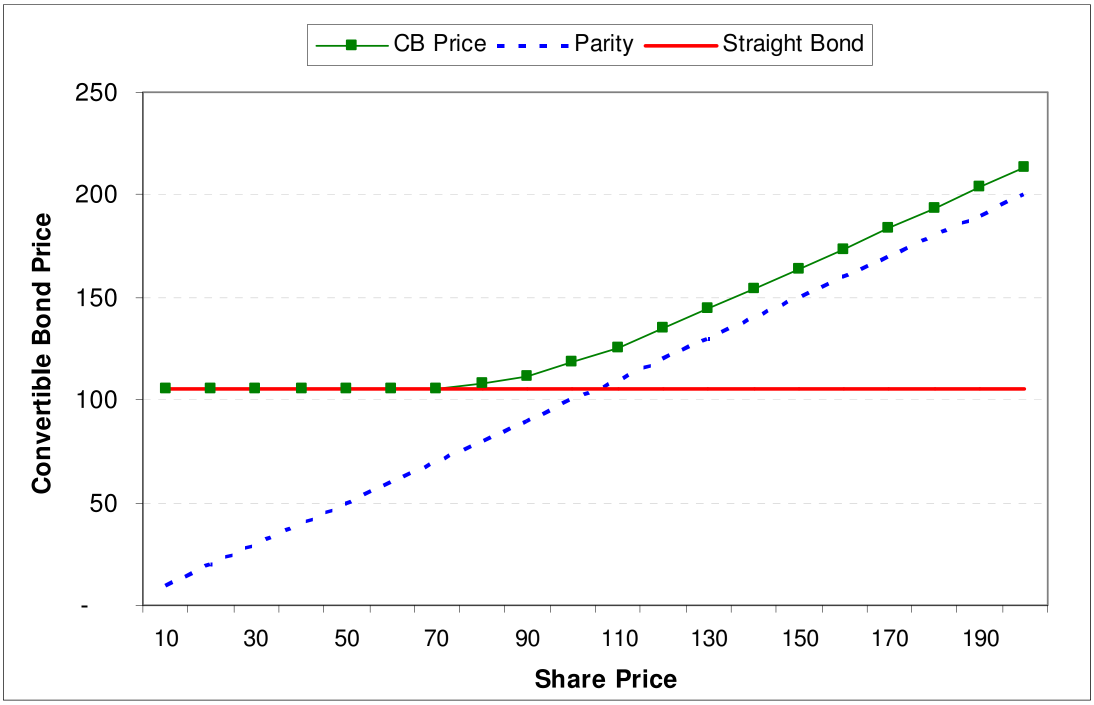

Figure 40 – Convertible bond price using the Gheno model for different share values.

Embedded call options are usually of benefit to the issuer and so decrease the security's value, whereas put options are of value to the investor and thus increase the security's price. As such we would expect the same to happen when we incorporate these embedded options into the Gheno model. If we let the call value be 150, then we see in Figure 41 that the upside gain on the CB is limited to the conversion value since for deep in-the-money CB's the issuer will most certainly force conversion, thereby limiting the CB price. For deep out-the-money issues the CB with the call option converges to the CB without the call option since the call option becomes worthless. Letting the put value equal 120, we can see in Figure 42 the appreciation in the CB price[63] for low share values since the investor can exercise the option if it is deep out-the-money. For deep in-the-money, share values the put option becomes worthless and the CB with the embedded put option tracks the CB price without the put option.

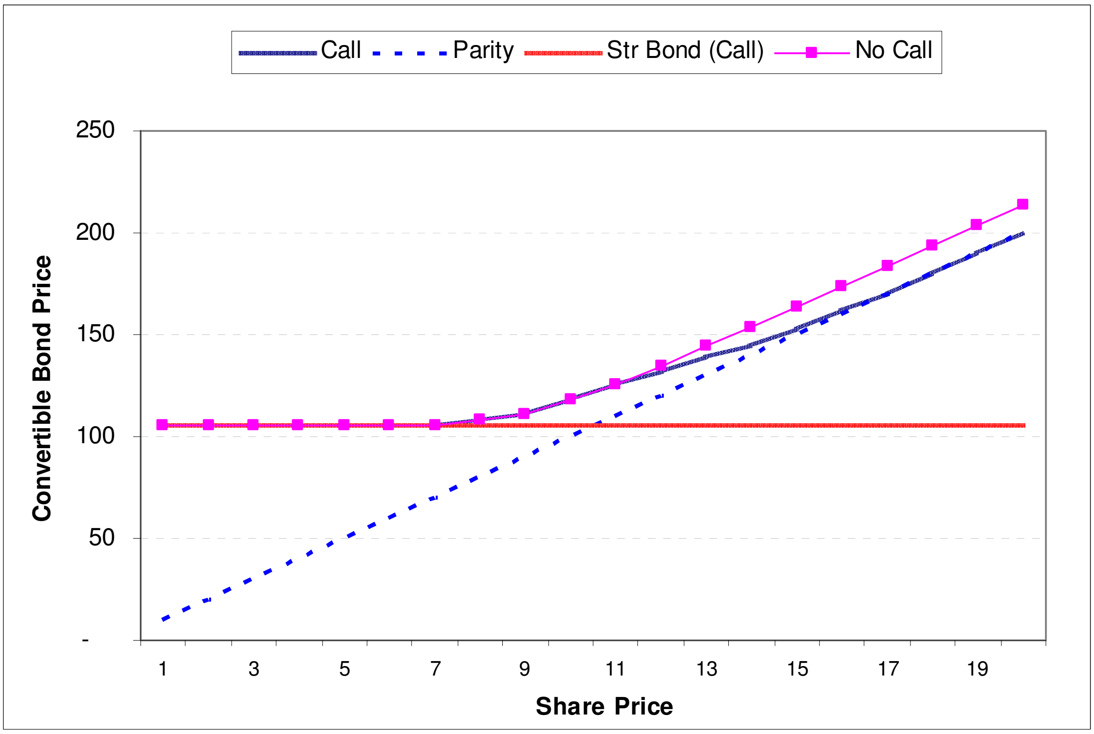

Figure 41 – Convertible bond price with embedded call option versus no call option with call value of 150. The Str Bond (Call) is the value of a straight bond with an embedded call option.

[63] The appreciation is relative to the CB price without any embedded options.

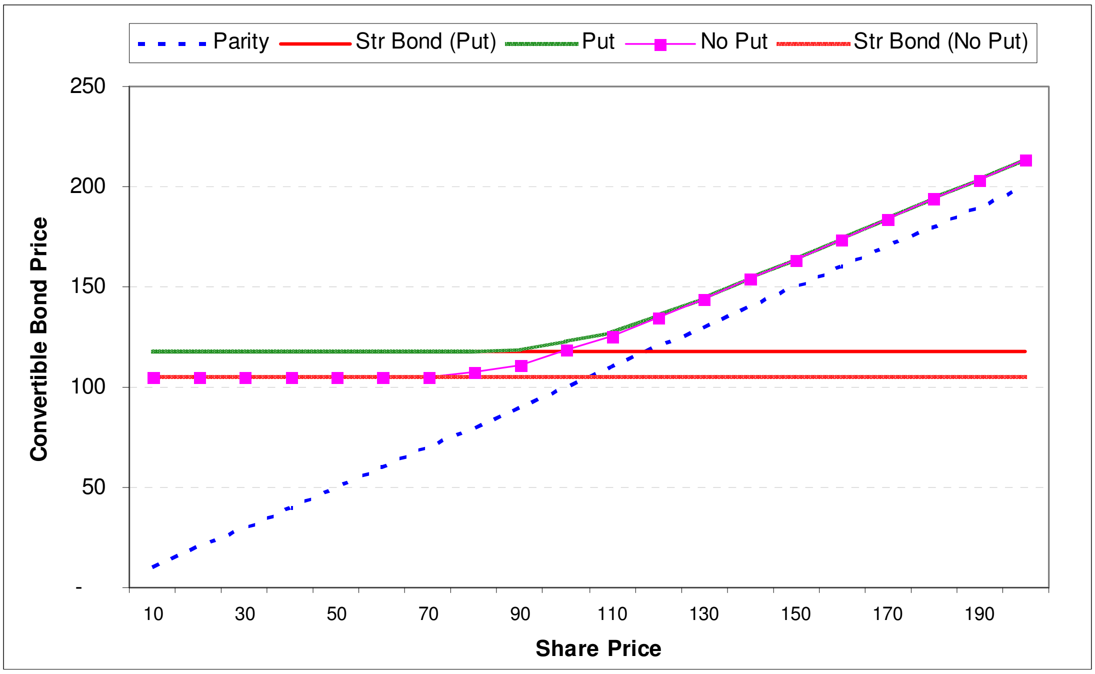

Figure 42 - Convertible bond price with embedded put option versus no put option with put value of 150. The Str Bond (Put) value is the value of a straight bond with an embedded put option and the Str Bond (No Put) is the value of a straight bond without an embedded put option.

The delta and gamma of the Gheno model take the familiar shape with the delta increasing to 1 (using a conversion ratio of 1) as the share price increases and the gamma reaching a maximum when at-the-money, and diminishing away to zero when in/out-the-money. Theoretically the delta and gamma are considered smooth functions but due to the explicit approximation used, there are irregularities in Figure 43, although the general shape can be ascertained.

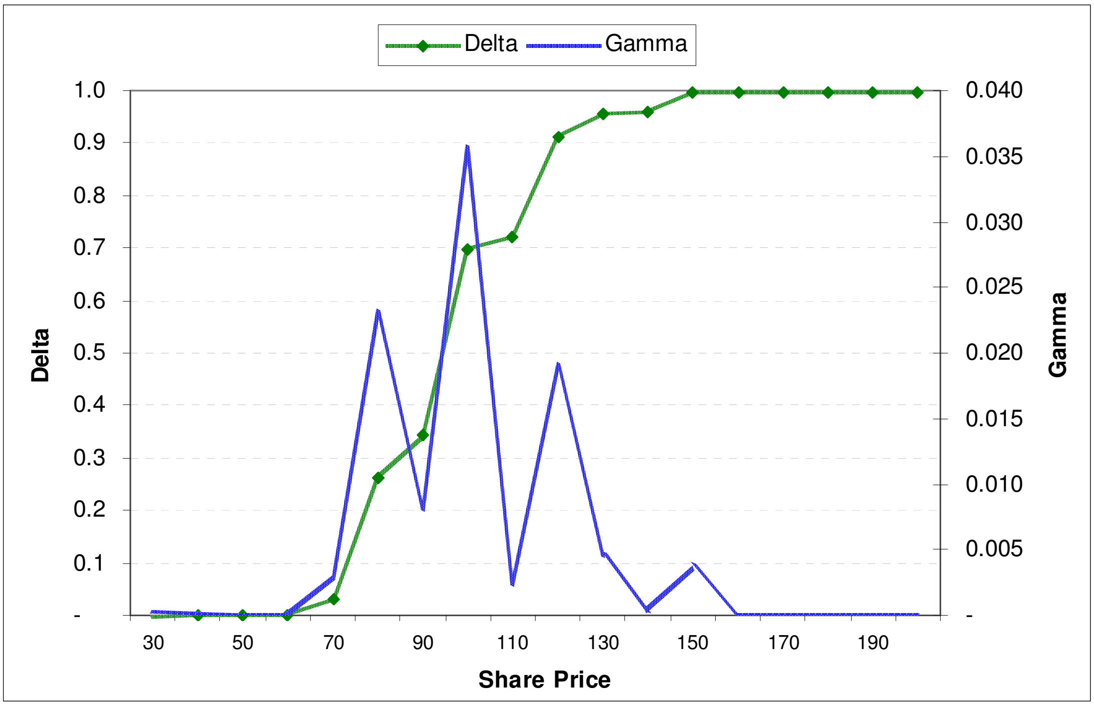

Figure 43 – Delta and gamma of the Gheno model without any embedded options.

Including call and put options into the Gheno model we see that the delta changes somewhat. Inclusion of the call option in Figure 44 shows that the upside delta is lower relative to the no call option delta. As mentioned previously this is due to the issuer exercising the option when the CB moves in-the-money and decelerates the delta on its way towards 1.

The put option delta is interesting since it is always less than the no put option delta in Figure 45. The reason behind this is that the CB price with the embedded put option reaches its bond floor value much sooner than the CB without the put option. This is due to the investor exercising the put option if the CB falls out-the-money. As such the rate of change in the CB price is not as large and so the put option delta always hangs below the no put option delta[64].

[64] This only applies if the share price falls below the strike of 150.

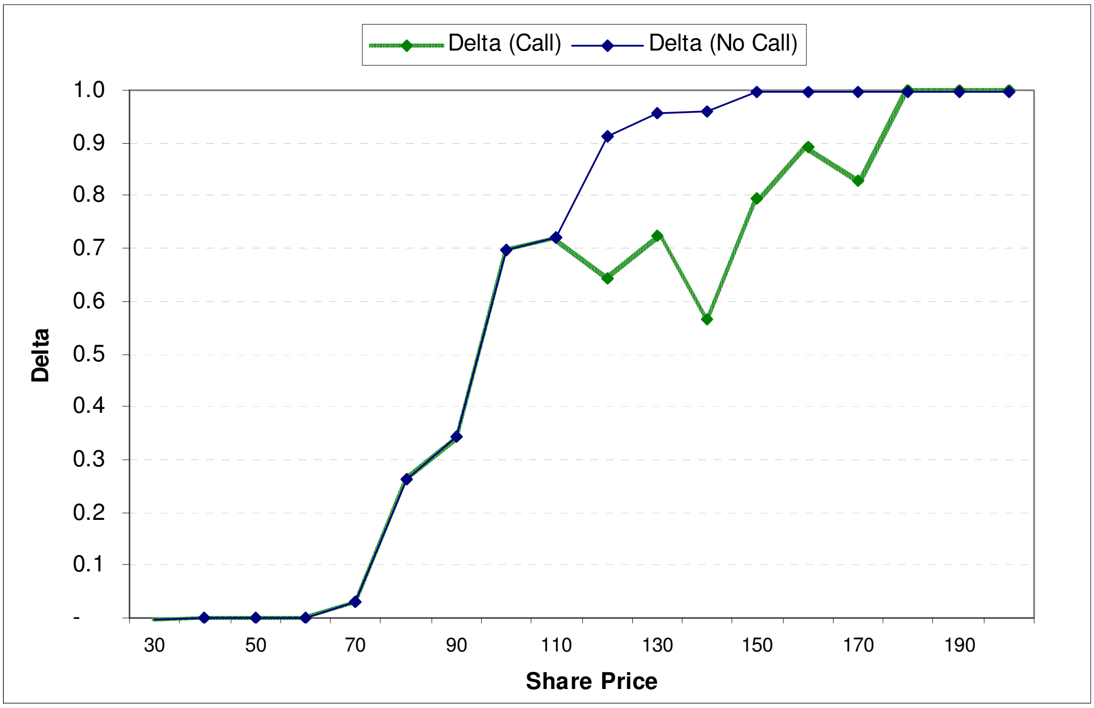

Figure 44 – Delta of the Gheno model with and without and embedded call option with call value of 150.

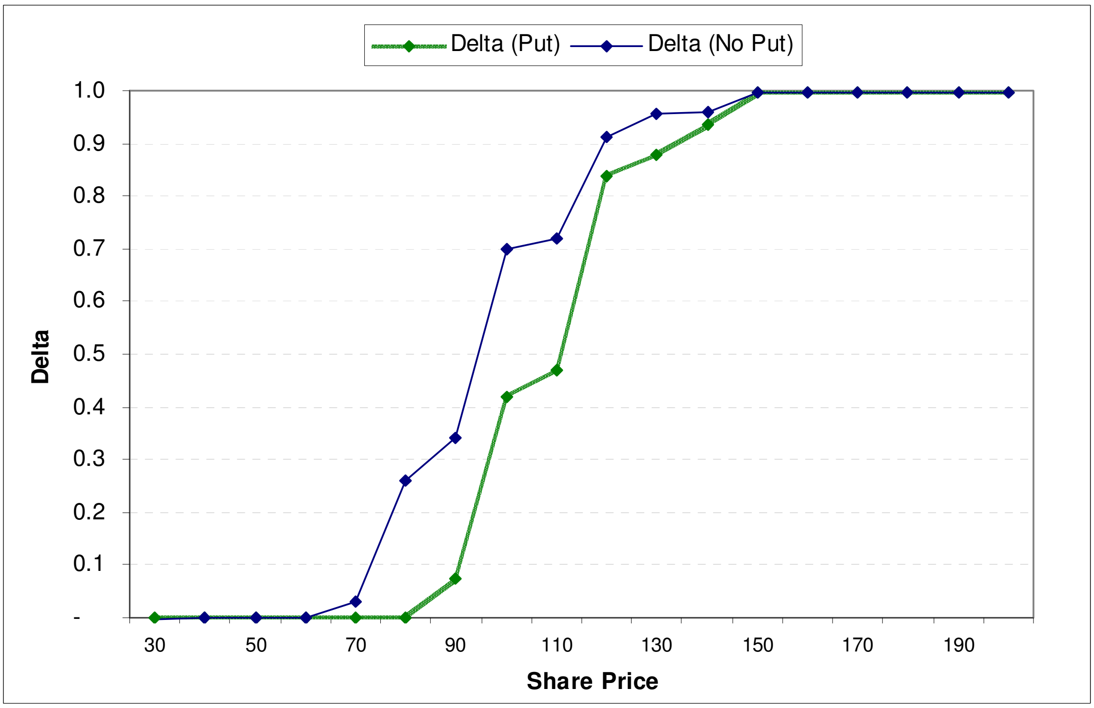

Figure 45 - Delta of the Gheno model with and without and embedded put option with put value of 150.

When dealing with volatility in any option, it is trivial that an increase in volatility will increase the value of the option. This is also true when dealing with firm (structural) models that incorporate asset return volatility, so that an increase in asset return volatility will lead to an increase in the CB price which is shown in Figure 46. The CB price increases dramatically for volatility values greater than 90% and trades well above its parity value and bond floor value. If an embedded call option were present the issuer would force conversion and the CB would inflect and move down towards it parity value for large volatility increases.

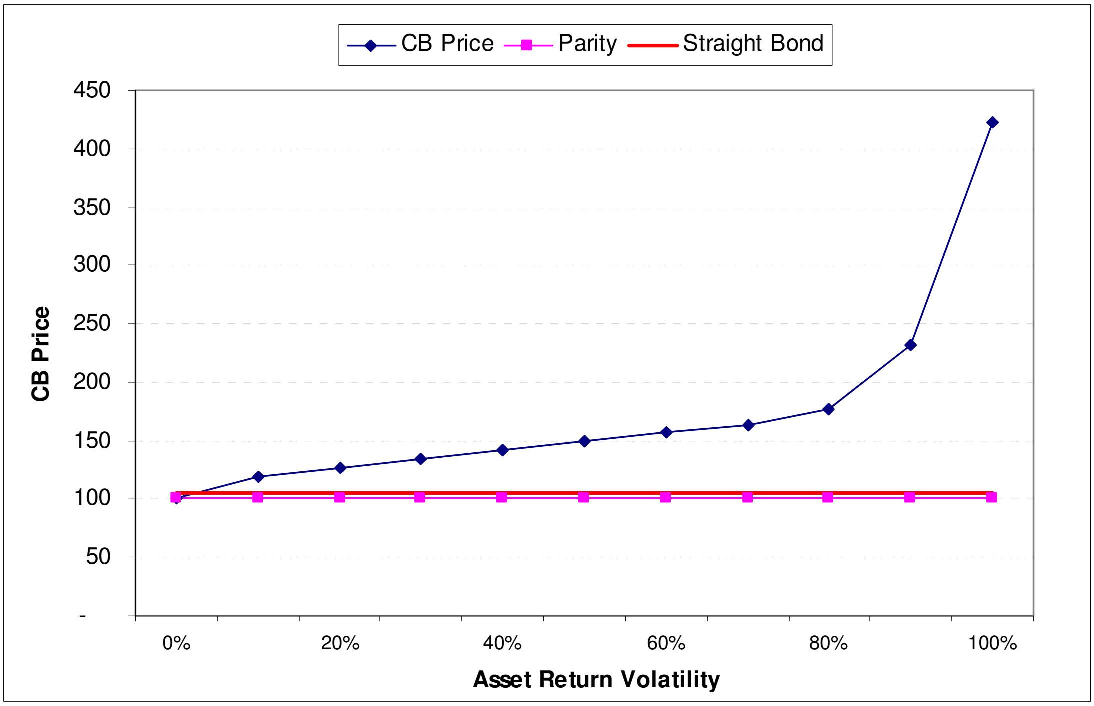

Figure 46 – CB price for changes in the asset return volatility using the Gheno model.

As already mentioned, the application of firm value models has the major downfall of having to estimate the assets and liabilities of the firm, and also modelling them to improve the accuracy of the final CB price. In saying this, these models can incorporate the negative convexity in the distressed debt phase by using a barrier whereas most equity valuation[65] models do not. Due to the sophistication and

[65] The Quadrinomial, TF and HW models do not possess the negative convexity property but as shown in Chapter 6 with the TF model, this is easily solved by the inclusion of the *k*-factor credit spread.

development of the CB market with issuers continually searching for cheaper means of financing, many exotic CB's have become available. Chapter 8 suggests two such variations of the vanilla CB, namely the exchangeable CB and the inflation linked CB that have been used by issuers of late.

model. The AFV model with credit risk does actually possess negative convexity but also by using the *k*-factor.

8. Exotic Convertible Bond Models

The past few years have shown that convertibles continue to evolve as issuers use them for a variety of purposes. This reflects the term "mainstreaming" of the CB asset class, as yesterday's exotic option becomes today's vanilla product, so to are the complexities of CB structures. Section 8.1 describes exchangeable CB's where the underlying equity is not the issuers but that of another firm. Inflation linked CB's are discussed in Section 8.2 where the principal amount is adjusted according to an inflation index, which preserves the value of the income stream. The valuation and inflation models are discussed together with the boundary conditions. A numerical approximation scheme is also suggested to solve the continuous PDEs.

8.1 Exchangeable Convertibles

Exchangeable CB's differ from ordinary CB's in that they allow the holder to convert into shares other than the issuers, unlike ordinary CB's. In this way the holders are allowed to exchange the bond for a different entity's' equity. Thus holders are exposed to the credit risk of the issuer and equity risk of the entity, whereas ordinary CB holders are exposed to both equity and credit risk of the issuer. The model that will be discussed is by Realdon [59].

Some comparisons of exchangeable CB's relative to ordinary CB's are that even when the issuer is experiencing financial distress the "exchange option[66]" is still valuable and worth exercising, whereas with ordinary CB's the conversion option is usually "out-the-money" and worthless. The exchange option can usually be exercised at any time and in combination with the pledged shares, which causes the CB to be insensitive to the issuer's credit risk. Higher volatility of the issuer's assets decreases the value of exchangeable CB's, although it increases the conversion option and in doing so increases the ordinary CB's value. Interestingly the exchangeable CB usually decreases in value as the correlation between the pledged shares and issuers assets rises. This is to be expected as the credit risk is increased. The major influence for issuing exchangeable CB's is to dispose of the underlying shares, which we assume is

[66] The exchange option in this case is the embedded conversion option, but is called an exchange option as the bond is exchanged for the underlying shares (from another entity) not the CB issuers' equity.

owned by the issuer. Another motivating factor is that when the call price of the exchangeable CB is high and the issuer is under distress, investors know that the CB's are unlikely to be called so there is a higher probability that the underlying shares will be disposed. In turn what this means is that the issuer will now have significantly lower bankruptcy costs and may even avoid bankruptcy altogether.

8.1.1 The Valuation Model

The default process of the firm is determined in an explicit way by using a structural model for credit risk. Assuming the value of the exchangeable CB is represented as $E(V,S,t)$, where V is the total value of the issuers' assets, S the share price of the underlying shares which the bond can be exchanged for and t is time. It is also assumed that S and V follow the usual lognormal dynamics,

$$dS = rSdt + S\sigma_S dW_S \tag{8.1}$$

$$dV = V(r-q)dt + V\sigma_V dW_V , \tag{8.2}$$

where r is the risk-free interest rate, σ_S and σ_V are the volatility of the shares and firm asset value respectively, q is the issuers' dividend payout rate, and dW_S and dW_V are the Wiener processes driving the share price and asset value respectively with $dW_S dW_V = \rho dt$. It is assumed that the underlying shares do not pay any dividends although the bond coupon payments are paid continuously. This means that default is triggered as soon as V reaches the following barrier,

$$V_d = \frac{IF + I_0 F_0}{b}(1-\tau) \tag{8.3}$$

where τ is the corporate tax rate, I_0 and F_0 are the coupon rate and face value of the exchangeable bond and, I and F are the coupon rate and face value of the other outstanding debt of the issuing firm. Using Itô's lemma we have that the PDE of the exchangeable CB is represented as,

$$\frac{\partial E(V,S,t)}{\partial t}+\frac{\partial^2 E(V,S,t)}{\partial V^2}\sigma_V^2 V^2+\frac{\partial^2 E(V,S,t)}{\partial V\partial S}\rho\sigma_V V\ \sigma_S S\ +\frac{\partial^2 E(V,S,t)}{\partial S^2}\sigma_S^2 S^2$$
$$+\frac{\partial E(V,S,t)}{\partial V}(r-q)V+\frac{\partial E(V,S,t)}{\partial S}rS-rE(V,S,t)+I=0, \tag{8.4}$$

subject to the following boundary conditions

$$E(V\to\infty,S,t)=\frac{I}{r}\left(1-e^{-r(T-t)}\right)+Fe^{-r(T-t)}+O(S,t) \tag{8.5}$$

$$E(V_d,S,t_d)=\max\left(R\left(V_d,S_{t_d}\right),S_{t_d}\right) \tag{8.6}$$

$$E\left(V,S=Y^*,t\right)=Y^* \tag{8.7}$$

$$E(V,S\to 0,t)\to D(V,t) \tag{8.8}$$

$$E(V,S,t=T)=\max(S,F). \tag{8.9}$$

(8.5) states that if the firm value were extremely high then the exchangeable CB would trade as a straight bond. If the firm were to default as in (8.6) then the greater of the recovery $R\left(V_d,S_{t_d}\right)$ amount or conversion value would be paid. The issuer would force conversion if the share price reached some trigger level Y^* as in (8.7). If the share price were to reach zero then the exchangeable CB would be valued as a seriously discounted straight debt instrument, independent of the share price as in (8.8). (8.9) is the usual maturity decision, where the greater of the redemption amount or conversion value is paid to the investors. Using the implicit finite difference scheme we can rewrite the above PDE $\forall i,j,k\in\mathbb{N}$ and $k\in[0,n-1]$ as[67],

[67] The complete derivation can be found in the paper by Landskroner and Raviv [42], the final result has only been shown here.

$$
\begin{aligned}
\frac{E_{i,j}^{k+1}-E_{i,j}^{k}}{dt} &= \frac{1}{2}\left(jdS\sigma_S\right)^2 \frac{E_{i,j+1}^{k+1}-2E_{i,j}^{k+1}+E_{i,j-1}^{k+1}}{\left(dS\right)^2} \\
&+\frac{1}{2}\left(jdV\sigma_V\right)^2 \frac{E_{i+1,j}^{k+1}-2E_{i,j}^{k+1}+E_{i-1,j}^{k+1}}{\left(dV\right)^2} \\
&+\left(jdS\sigma_S\right)\left(idV\sigma_V\right)\rho \frac{E_{i+1,j+1}^{k+1}-E_{i-1,j+1}^{k+1}-E_{i+1,j-1}^{k+1}-E_{i-1,j-1}^{k+1}}{4dSdV} \\
&+\left(r-q\right)idV \frac{E_{i+1,j}^{k+1}-E_{i-1,j}^{k+1}}{2dV}+rjdS\frac{E_{i,j+1}^{k+1}-E_{i,j-1}^{k+1}}{2dS}-rE_{i,j}^{k+1}.
\end{aligned}
\tag{8.10}
$$

8.2 Inflation-Indexed Convertible Bonds with Credit Risk

Inflation-Indexed convertible bonds (IICB) differ slightly in the make-up to ordinary convertible bonds. The difference lies in the fact that the principal amount is adjusted in accordance to an index, in this particular case the consumer price index or inflation rate as it is widely known. As always it is important to incorporate some sort of default risk into the model to account for the further financial distress brought about by the IICB's to the issuing firm. Landskroner and Raviv [42] implement two approaches to incorporate default risk. The first is the traditional decoupling approach of the PDEs into a cash-only (debt) part and an equity part as illustrated earlier by Tsiveriotis and Fernandes [69]. The second is by using McConnell and Schwartz [50] structural methodology of firm value.

8.2.1 Default Trigger

As mentioned previously the TF model splits the CB price into an equity component and a cash-only component. The equity component is discounted using the risk-free rate, so there is no default risk assumed since the issuer can always issue more of its equity. This will however cause adverse effects on the price of its equity due to dilution and negative market sentiment. The cash-only component is discounted using the risk-free rate plus a constant credit spread, as these are obligations that the firm has to make. The firm has to pay these cash obligations to its investors, which will stress the firm, and as such investors would require a credit spread due to the exposure of additional risk.

The McConnell and Schwartz [50] (MS) default approach focuses on the capital structure of the firm where default is triggered if the firm's assets fall below its debt face value. The firm value is assumed to follow lognormal dynamics, although the total firm value is not observable in the market and thus causes a minor drawback to the approach. Default risk can also be achieved by adding a constant credit spread to the risk-free rate to discount the bond. The downfall is that this method does not account for the fact that default risk of the CB varies according to its moneyness[68].

8.2.2 Valuation Model

The stock price and inflation process are assumed to follow the usual lognormal dynamics and possess some correlation. The governing PDE and boundary conditions have to be stipulated and default risk is incorporated by both the TF and MS approaches. Finally the resultant PDE is solved using Rubinstein's [61] three-dimensional binomial tree, which is easier to implement than finite difference methods. Landskroner and Raviv [42] show that the underlying model can be extended to value foreign exchange CB's by simply replacing the inflation variable by a foreign exchange variable. The familiar assumptions of the model are:

- Investors trade continuously in a complete, frictionless and arbitrage-free financial market.
- No transaction costs, restrictions on short selling or taxes.
- Uncertainty in the economy is given by the probability space (Ω, F, P), where Ω is a state space, F is the set of possible outcomes and P is the objective martingale measure on (Ω, F).

The share price S and inflation process I[69] are assumed to follow the usual SDEs respectively,

[68] Moneyness refers to the phase of the CB, whether it is out/at/in-the-money.

[69] The coupons were defined as *I* in 8.1.1 but change notation to *C* in this sub-section to avoid confusion with the inflation process *I*.

$$\frac{dS}{S} = (\mu_S - q)\,dt + \sigma_S dW_S \tag{8.11}$$

$$\frac{dI}{I} = \mu_I dt + \sigma_I dW_I \,, \tag{8.12}$$

where μ_S is the expected return on the issuers' stock, q is the continuously compounded dividend payout rate, μ_I is the expected inflation rate, σ_S and σ_I are the instantaneous volatility measures of the underlying share and inflation rate respectively. dW_S and dW_I are the standard Wiener processes with the following correlation $dW_S dW_I = \rho_{SI} dt$.

Looking at a foreign currency analogy, real prices correspond to foreign prices, nominal prices relate to domestic prices in local currency, and the CPI relates to the spot exchange rate. Garman and Kohlhagen [30] assume that the foreign currency follows the same process as the stock with dividend q replaced with the foreign risk free rate r_f. By the same reasoning the CPI drift must have a similar drift to the foreign exchange drift with $\mu_I = (r - r_R)$, where r_R is the domestic real interest rate.

Let $U(S, I, t)$ be the value of the IICB[70] at some time $t \in [0, T]$, with fixed coupons C and principal F that is inflation-adjusted, and convertible throughout its life into α shares. In the absence of default risk the value of the IICB is represented as

$$\begin{aligned} rU + f(t) &= U_t + U_S (r - q) S + U_I (r - r_R) I \\ &+ \frac{1}{2}\left(U_{SS}\sigma_S^2 S^2 + U_{II}\sigma_I^2 I^2 + 2U_{IS}\sigma_I\sigma_S\rho_{IS} IS\right), \end{aligned} \tag{8.13}$$

where $f(t)$ is the coupon payment function, and U_j and U_{jj} are the first and second order partial derivatives $(j=I,S)$. Incorporating default risk by applying the MS approach gives the slightly adjusted PDE

[70] The IICB price U can be seen as the vanilla CB price χ.

$$\begin{aligned}(r+cs)U+f(t)&=U_t+U_S(r-q)S+U_I(r-r_R)I\\&+\frac{1}{2}\left(U_{SS}\sigma_S^2S^2+U_{II}\sigma_I^2I^2+2U_{IS}\sigma_I\sigma_S\rho_{IS}IS\right).\end{aligned} \tag{8.14}$$

where *cs* is the credit spread of the issuers pari passu-convertible debt, relative to a similar maturing treasury bond.

By incorporating default risk using the TF approach requires us to decompose the PDE in (8.14) into a debt-only part and an equity part. Letting *V* represent the debt-only part, it can be considered a contingent claim with the stock being its only underlying variable. As such *V* should follow the BSM equation with the credit spread being as defined above. The equity part being represented as *(U-V)* is considered as the CB payments related to equity and as such should be discounted using the risk-free rate. The respective equations of each part are given as:

CB price:

$$r(U-V)+(r+cs)V-f(t)=U_t+\frac{1}{2}U_{SS}\sigma_S^2S^2+U_S(r-q)S \tag{8.15}$$

Debt – Only:

$$(r+cs)V-f(t)=V_t+\frac{1}{2}V_{SS}\sigma_S^2S^2+V_SrS\ . \tag{8.16}$$

To find the IICB PDE we have to take into account the CPI from (8.13) and add all the CPI terms to each of the above equations, which leaves us with

$$\begin{aligned}r(U-V)+(r+cs)V-f(t)&=U_t+\frac{1}{2}U_{SS}\sigma_S^2S^2+U_S(r-q)S\\&+\frac{1}{2}\left(U_{SS}\sigma_S^2S^2+U_{II}\sigma_I^2I^2+2U_{IS}\rho_{IS}\sigma_S\sigma_ISI\right)\end{aligned} \tag{8.17}$$

$$\begin{aligned}(r+cs)V-f(t)&=V_t+\frac{1}{2}V_{SS}\sigma_S^2S^2+V_SrS\\&+\frac{1}{2}\left(U_{SS}\sigma_S^2S^2+U_{II}\sigma_I^2I^2+2U_{IS}\rho_{IS}\sigma_S\sigma_ISI\right)\end{aligned} \tag{8.18}$$

8.2.3 Boundary Conditions

As always we have to stipulate boundary conditions for the PDEs, although in this simple model we ignore callability and putability. At maturity the boundary conditions for the debt-only part, *V* and CB price, *U* are,

$$U(S,I,T)=\begin{cases}\alpha S & , \quad \alpha S \geq (F+C)\dfrac{I_T}{I_0} \\ (F+C)\dfrac{I_T}{I_0} & , \quad elsewhere\end{cases} \tag{8.19}$$

$$V(S,I,T)=\begin{cases}0 & , \quad \alpha S \geq (F+C)\dfrac{I_T}{I_0} \\ (F+C)\dfrac{I_T}{I_0} & , \quad elsewhere\end{cases} \tag{8.20}$$

where I_0 is the CPI value on the issue date of the CB. As the CB is convertible throughout its life it contains free moving boundary conditions. To the upside it has to satisfy

$$U \geq \alpha S \quad , \quad \forall t \in [0,T] \tag{8.21}$$

$$V = 0 \quad , \quad \forall t \in [0,T] \tag{8.22}$$

When using the MS model the boundary conditions are condensed into equations (8.19) and (8.21).

The equations in (8.17) and (8.18) are very general, and so in that sense may seem a little complex. This can be simplified by letting some of the variables tend to zero.

- If $q=0$ and $cs=0$, then the underlying asset pays no dividends and credit risk is eliminated from the model. Since the asset pays no dividends and the investor holds an American call option to exchange the CB for equity, it would never be optimal for the investor to exercise his call option prior to maturity. As such this model will revert to the Margrabe [49] exchange option model.

- $cs=0$ reduces the model to Rubenstein [60] for valuing American options dependent on two correlated assets.
- $\sigma_I \to 0$ and $r=r_R$ switches off the inflation process and the models revert to the usual TF and MS models.
- $\sigma_I \to 0$, $r=r_R$, $cs=0$ reduces the two factor model into the one factor Cox-Ross-Rubenstein (CRR) model.

8.2.4 Numerical Approximation

As the previous equations have no closed form solutions a numerical approximation needs to be implemented. TF use explicit finite differences, which is a little easier to implement than Crank-Nicholson or implicit methods, although the explicit method can become conditionally unstable meaning that the time steps have to be really small for accurate results. In so doing we construct a Rubinstein [61] three-dimensional binomial tree with the stochastic variables being the CPI process and the share price. As with all binomial tree methods a recursive backward in time algorithm is applied to calculate the present value of the IICB. Splitting the time interval $[0,T]$ into n sub-intervals Δt of equal length which will be denoted by i $(i=0,1,\ldots,n)$.

The four possible moves from each node are $A=uu$, $B=ud$, $C=du$ and $D=dd$ for each of the processes S and I that each have an equal probability of up and down moves. The combined probabilities are,

$$\begin{aligned}
A &= \exp\left[\varpi_I \Delta t + \sigma_I \sqrt{\Delta t}\left(\rho_{SI} + \sqrt{1-\rho_{SI}^2}\right)\right] \\
B &= \exp\left[\varpi_I \Delta t + \sigma_I \sqrt{\Delta t}\left(\rho_{SI} - \sqrt{1-\rho_{SI}^2}\right)\right] \\
C &= \exp\left[\varpi_I \Delta t - \sigma_I \sqrt{\Delta t}\left(\rho_{SI} - \sqrt{1-\rho_{SI}^2}\right)\right] \\
D &= \exp\left[\varpi_I \Delta t - \sigma_I \sqrt{\Delta t}\left(\rho_{SI} + \sqrt{1-\rho_{SI}^2}\right)\right],
\end{aligned} \tag{8.23}$$

where $\varpi_I = (r - r_R) - \sigma_I^2 / 2$. For the lattice to recombine we have to impose the condition $AD=BC$. If $A \neq C$ and $B \neq D$ it becomes possible to construct non-zero correlation between the two processes.

From any node *(i,S,I)*, the lattice progresses to four nodes, namely, $(i+1, Su, IA), (i+1, Su, IB), (i+1, Sd, IC)$ and $(i+1, Sd, ID)$, where *IA*, *IB*, *IC* and *ID* are the different values that the CPI takes at each of the new nodes. The CPI at each node, any time period *i*, with *j* up moves of the share price, is equal to

$$I(i,j,k) = I_0 e^{\varpi_I i\Delta t + \sigma_I \sqrt{\Delta t}\left[\rho_{SI}(2j-i)+\left(\sqrt{1-\rho_{SI}^2}\right)(2k-i)\right]}, \tag{8.24}$$

where $k=0,1,\ldots,i$.

The four nodes are assumed to occur with probability 0.25, and at each time tick there are 2^{i+1} distinct nodes with a total of $(i+1)^2$ nodes throughout the entire tree.

At maturity the investor is faced with the decision to either redeem/hold the bond or convert by applying condition (8.19) and (8.20). If the investor chooses to hold the bond it is denoted as $UH_{i,j,k}$ and if the investor chooses to convert, the value of the CB is denoted as $UC_{i,j,k}$. Thus the rollback CB value is $U_{i,j,k} = \max\left[UH_{i,j,k}, UC_{i,j,k}\right]$.

To incorporate credit risk into the model using TF we need to discount the debt-only part using a risky interest rate, $r^{*(i,i+1)}$ that of the issuer, and a risk-free rate $r^{(i,i+1)}$ to discount the equity part $E_{i,j,k}$. Therefore at each node the CB price is equal to $U_{i,j,k} = V_{i,j,k} + E_{i,j,k}$ which is the sum of the two components. Using MS to include credit risk requires a minor adjustment whereby the risk-free rate and credit spread are used to discount the debt-only part and the equity part. At maturity the holding value and conversion value are given by equations (8.25) and (8.26) below,

$$UC_{N,j,k} = \alpha S u^j d^{N-j} \tag{8.25}$$

$$UH_{N,j,k} = (F+C) e^{\varpi_I i\Delta t + \sigma_I \sqrt{\Delta t}\left[\rho_{SI}(2j-i)+\left(\sqrt{1-\rho_{SI}^2}\right)(2k-i)\right]}. \tag{8.26}$$

In doing so the equity part at maturity is given as

$$E_{N,j,k} = \begin{cases} UC_{N,j,k} & , \quad UC_{N,j,k} \geq UH_{N,j,k} \\ 0 & , \quad elsewhere \end{cases} , \tag{8.27}$$

and the debt part is

$$V_{N,j,k} = \begin{cases} UH_{N,j,k} & , \quad UH_{N,j,k} \geq UC_{N,j,k} \\ 0 & , \quad elsewhere \end{cases} . \tag{8.28}$$

Due to the equal probability of each of the four possibilities occurring, the rollback holding value is calculated as per usual using discounted expected value. The credit risk (according to TF) is factored into the model by discounting the components using different interest rates. This leads us to the following equation,

$$\begin{aligned} UH_{i,j,k} &= \frac{1}{4} e^{-r^*(i,i+1)\Delta t} \left(V_{i,j,k} + V_{i,j,k+1} + V_{i,j+1,k} + V_{i,j+1,k+1} \right) \\ &\quad + \frac{1}{4} e^{-r(i,i+1)\Delta t} \left(E_{i,j,k} + E_{i,j,k+1} + E_{i,j+1,k} + E_{i,j+1,k+1} \right). \end{aligned} \tag{8.29}$$

As previously mentioned the conversion value at any time prior to maturity is

$$UC_{i,j,k} = \alpha S u^j d^{i-j} . \tag{8.30}$$

Finally, the rollback debt-only and equity parts are also obtained using discounted expected values. Thus at time $k \in [0,n]$ and $\forall i, j \in [0,n-1]$, we have that the equity part is calculated as

$$E_{i,j,k} = \begin{cases} UC_{i,j,k} & , \quad UC_{i,j,k} \geq UH_{i,j,k} \\ \frac{1}{4} e^{-r(i,i+1)\Delta t} \left(E_{i,j,k} + E_{i,j,k+1} + E_{i,j+1,k} + E_{i,j+1,k+1} \right) & , \quad elsewhere. \end{cases} \tag{8.31}$$

If conversion takes place the debt-only part takes on zero, else it is computed similarly to the equity component although with a risky interest rate,

$$V_{i,j,k} = \begin{cases} 0 & , \quad UC_{i,j,k} \geq UH_{i,j,k} \\ \frac{1}{4} e^{-r^*(i,i+1)\Delta t} \left(V_{i,j,k} + V_{i,j,k+1} + V_{i,j+1,k} + V_{i,j+1,k+1} \right) & , \quad elsewhere. \end{cases} \tag{8.32}$$

To take into account coupons, the coupon is inflation adjusted by multiplying the nominal coupon by equation (8.24) at the appropriate time tick and adding it to the debt-only part. If a coupon flow occurs between two time ticks the present value is obtained using the risky interest rate.

The valuation of vanilla CB's is by no means a simple task as has been outlined in this dissertation. Trying to model exotic CB's will pose an even greater challenge, but as the market becomes more sophisticated and valuation methodologies advance, this will hopefully improve the accuracy of the models.

In step with the growth in the convertibles market there has been a rapid increase in the participation of hedge funds for CB issuance, continually looking to exploit arbitrage opportunities. Therefore the next Chapter gives a detailed description of the underlying risks of CB's and many popular trading strategies employed by hedge funds.

9. Convertible Bond Arbitrage Strategies

As the appetite for alternative investments grow (especially non-cyclical assets) so too will the complexity of the trading strategies. Hedge funds have become major players in the CB space and in doing so have constructed several arbitrage strategies which will be analysed in this Chapter. Section 9.1 establishes the underlying risks of CB's with respect to their "Greeks". Once the basic risks have been introduced the various hedging strategies are proposed, beginning with the delta hedge in Section 9.2, dynamic hedge in 9.3, gamma hedge in 9.4 and option hedge in 9.5. Included in each of the hedging strategies are simple examples showing the potential profit or loss.

9.1 Convertible Bond "Greeks"

When trading, hedging or investing in convertible bonds it is important for any valuation model to take into account the underlying risks, or parameter sensitivities. Generally speaking the measures of risk of CB's (or any option) are referred to as "The Greeks". In its simplistic form it gives the investor or arbitrageur an estimate of the change in value of the CB for a given change in an underlying variable. An arbitrageur by definition is a person who attempts "to profit by exploiting price differences of identical or similar financial instruments, on different markets or in different forms. The ideal version is riskless arbitrage."[71]

9.1.1 Delta *(Δ)*

The most important measure has to be the CB price sensitivity to changes in the underlying share price, i.e. $\partial\chi/\partial S$, where χ is the CB price. The delta is used by arbitrageurs to determine the number of shares to short against the long position in the CB to create a delta-neutral hedge. The delta is however not constant and varies according to whether the CB is in/out-the-money. For high grade CB issuers the delta usually moves between 0 and the conversion ratio, and is seen as positively correlated with the share price. When the CB is deep in-the-money the delta approaches the conversion ratio rapidly, whilst if it is out-the-money, the delta moves closer to 0. When the share price drops to extremely low levels close to 0 for low grade issuers or CB's in the distressed debt phase; the delta can actually move back up towards the

[71] http://www.investorwords.com/245/arbitrage.html

conversion ratio due to the negative convexity (explained in Chapter 6 and clearly illustrated in the distressed debt zone in Figure 1) due to the increased bankruptcy risk of the issuer. The risky delta is the apparent delta that takes the negative convexity into account, whereas the normal delta is the delta from the model assuming the convertible reverts to its straight bond value for low share values. Thus, assuming the CB reverts to its straight bond value for low share values can be costly to the arbitrage hedge if the risky delta proves to be correct.

Figure 48 illustrates the modified delta for the TF, TFk, HW and GS models. The actual graphs are not as smooth as Figure 47 due to the approximation of the delta, but the shapes are similar. It is interesting to note that the only model to emphasize the change in direction of the delta is the TFk model, which follows the path of the risky delta. All the others possess regular deltas, implying that credit concern is not a problem for low share values.

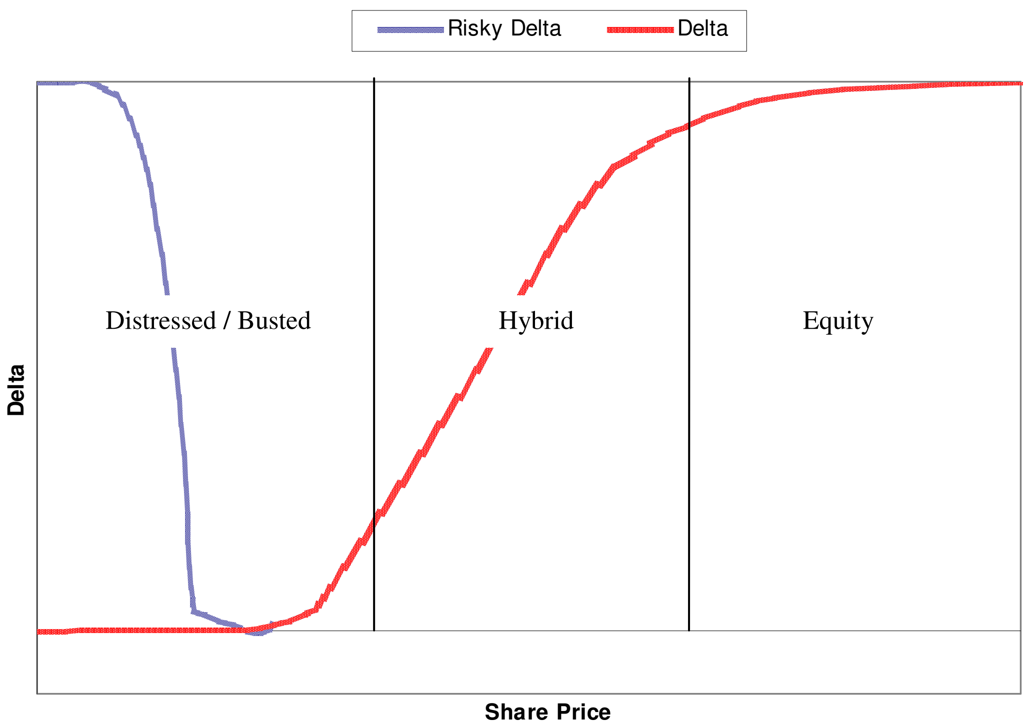

Figure 47 – Typical Delta for a convertible bond and Risky Delta for a convertible bond displaying negative convexity (gamma) across the various share price ranges.

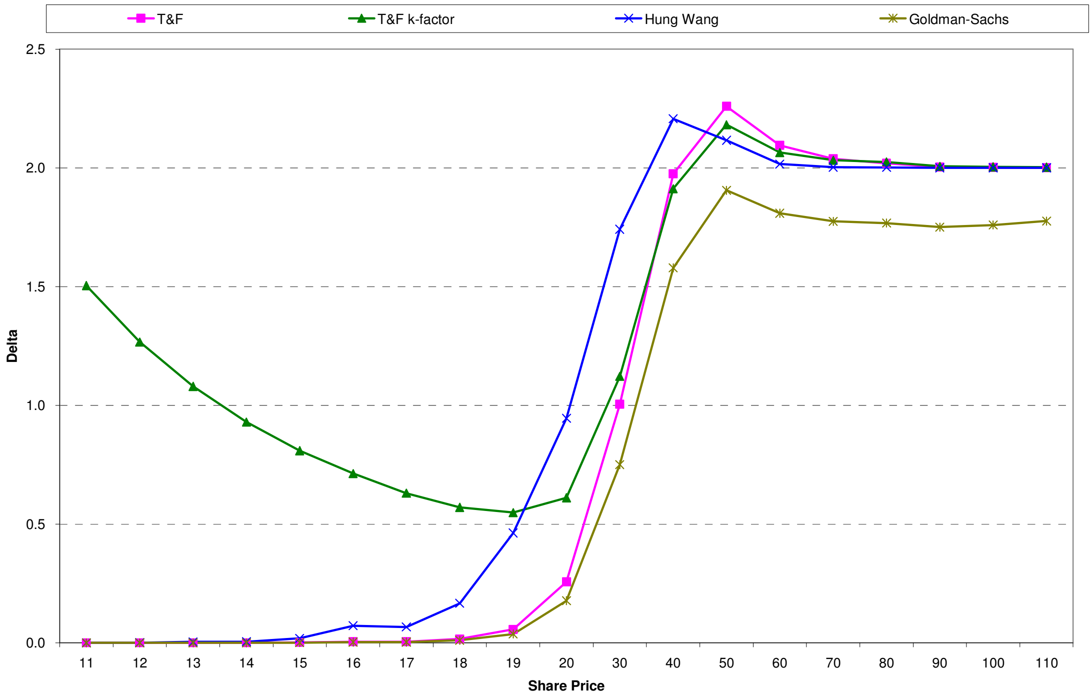

Figure 48 – Delta for TF, TFk, HW and GS models.

9.1.2 Gamma *(Γ)*

As can be seen the delta and risky delta are not constant and change according to the moneyness of the CB, as such it becomes important to look at the rate of change of the delta (i.e. gamma)

The rate of change of delta with respect to the share value is known as gamma and is computed as $\partial\Delta/\partial S$, also as the second-order partial derivative $\partial^2\chi/\partial S^2$. It is basically the convexity of the CB price track. For arbitrageurs a spot gamma will help in determining an appropriate stock range to hedge their position. The higher the gamma, the more delta changes and the more the hedge needs to be rebalanced. The gamma is at a maximum when the CB is at its conversion price, and progressively decreases to 0 when it moves in/out-the-money. Thus, there is upside gamma and downside gamma. Upside gamma is seen as the change in delta for an upward move in the share price, and similarly downside gamma is the change in gamma from a downward move in the share price. The downward gamma allows the arbitrageur the opportunity to capture additional profits if the share drops without much risk (assuming there is no negative gamma) and hedging with fewer shares if the downside gamma is less than the upside. As mentioned earlier for high grade issuers the delta is viewed as positive and so is the gamma, however for low grade issuers or any CB in the distressed phase, the CB breaks down to its liquidation value, the delta reverts to its conversion ratio and the gamma becomes negative. This is important when considering neutral hedges, but also requires a solid understanding of the credit quality of the issuer.

Figure 49 shows the usual gamma for a convertible bond and the risky gamma for models incorporating the negative gamma. The negative gamma comes into effect when the CB is in the distressed zone, and can cause serious damage to an arbitrageur's hedge profile if not taken into consideration, as he will continuously underestimate the delta and so not achieve his optimal hedge strategy. Figure 50 shows the approximate values of the gamma achieved by the TF, TFk, HW and GS models. Once again the curves are not smooth as in Figure 49 due to the estimation, but they give the same results with the TF, HW and GS models all displaying regular gamma's whilst the TFk model expresses the negative gamma and so represents the risky gamma. The gamma is highest at the conversion price with the HW model giving the greatest value for gamma due to the reduction in default risk once the CB moves in-the-money.

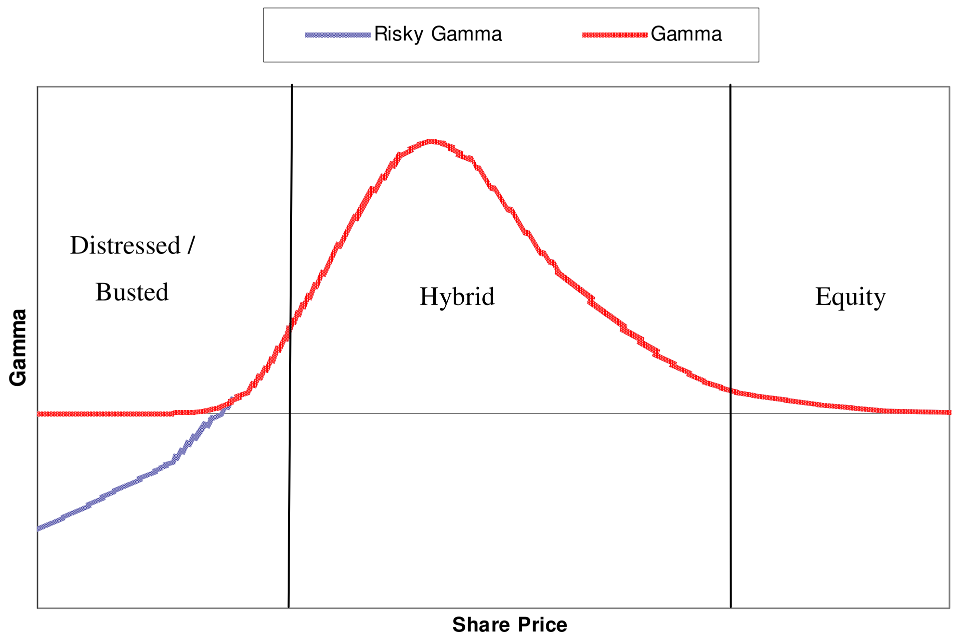

Figure 49 - Typical Gamma for a convertible bond and Risky Gamma for a convertible bond displaying negative convexity (gamma) across the various share price ranges.

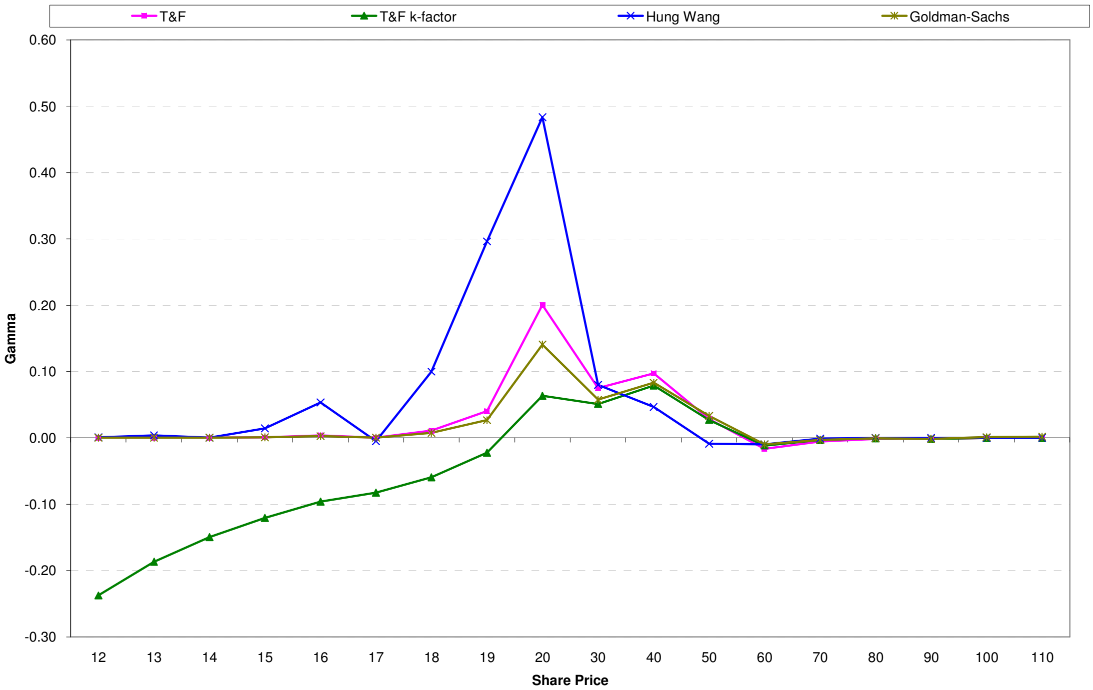

Figure 50 – Gamma for TF, TFk, HW and GS models. Maximum gamma is achieved at the conversion price, except for the TFk model which achieves maximum gamma at a higher share value and also displays negative gamma for low share values.

9.1.3 Vega (v)

The sensitivity of the CB price to changes in implied volatility is referred to as vega risk $(\partial\chi/\partial\sigma)$. Generally speaking an undervalued CB trades with a lower implied volatility than what the arbitrageur is anticipating. The vega gives an estimate of the change in price for a unit change in implied volatility. Volatility is mean reverting but the mean is non-constant and the reversion time may vary, giving way to trading opportunities. To help estimate the volatility, the arbitrageur can look at historical volatility changes and the volatility of volatility. This is a cause for concern as implied volatility is forward looking not historically traded, as such if there are options on the underlying stock available, those would be more accurate estimates to use in the valuation model. As with gamma, vega is at its highest near the “hybrid” zone or conversion price, as the CB is continually moving in/out-the-money. Vega risk is important when hedging, as most strategies are long volatility, although this can be reduced to a certain extent by investing in put options on the CB so that if the volatility were to fall, the options would be in-the-money and counter the lost value in the CB price.

Figure 51 shows the regular vega for CB's without credit spread sensitivity, and risky vega for CB's that take credit spread sensitivity into account. The vega is similar to the gamma graph since it reaches its maximum at the conversion price and falls away towards zero for deep in and out-the-money convertibles. It is worthwhile to take a look at Figure 52, which shows that all the models possess some negative vega in the distressed zone, indicating a risky vega. Although the TFk model also shows negative vega for increases in volatility. This was discussed in Section 6.4, in that the *k*-factor is relatively large and so suppresses the CB price for large increases in volatility.

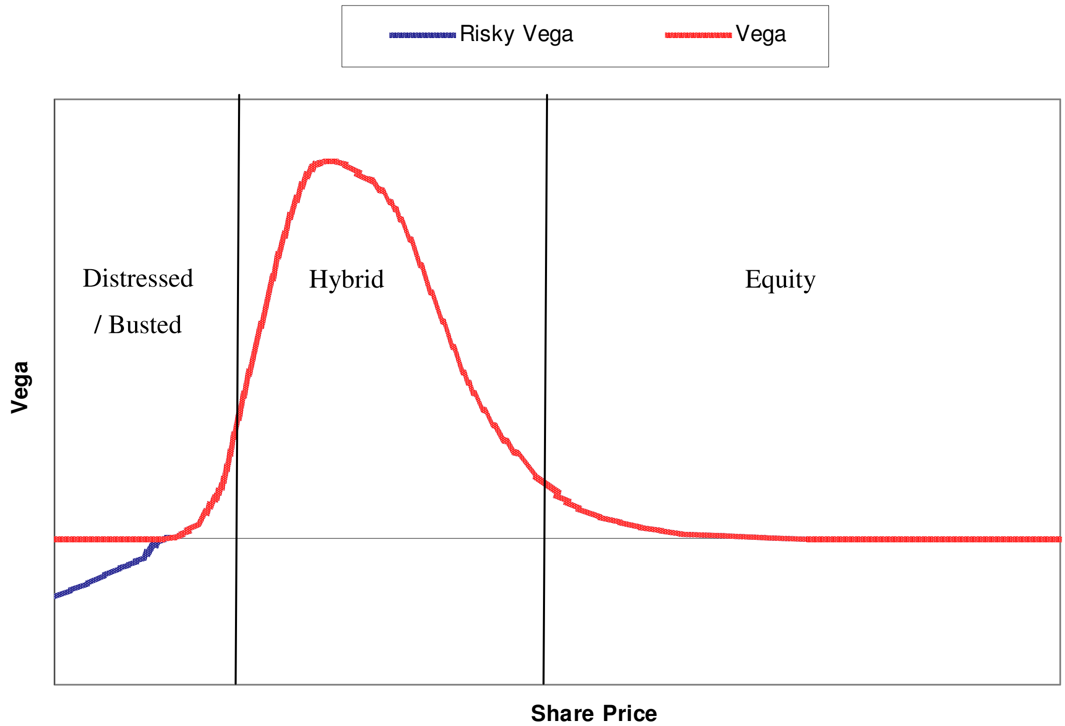

Figure 51 - Typical Vega for a convertible bond and Risky Vega for a convertible bond displaying negative convexity (gamma) across the various share price ranges.

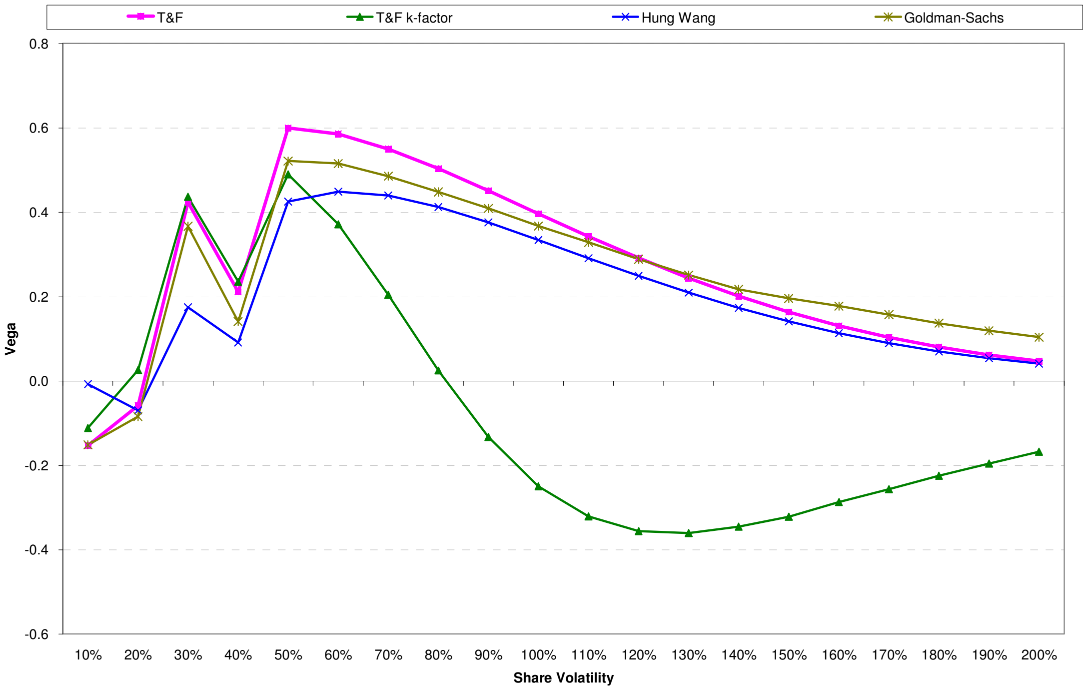

Figure 52 – This graph is slightly different to Figure 51 in that it plots vega against share volatility as opposed to share price. The TF, TFk, HW and GS models display negative vega for extremely low share volatility, with the TFk model giving negative vega for high volatility aswell. The maximum vega is attained at 50% share volatility in this case.

9.1.4 Theta (τ)

Changes in the value of the CB price to changes with respect to maturity (time to maturity) is referred to as theta $\left(\partial\chi/\partial t\right)$. Theta is generally[72], positively related to the option maturity, so that as the maturity of the option increases, the option value also increases, and vice versa. There exists time-premium decay which eats away at the embedded option premium as the CB draws closer to maturity, although this is a little more complicated with CB's since a portion of the CB is affected by the maturity date and the exercisable call option of the issuer. Unlike regular options, CB's that are out-the-money with no call protection and a low coupon payment will most likely remain outstanding by the issuer, thus being exposed to very little theta risk. However if interest rates were to decrease, the issuer will be tempted to refinance the issue with a new, cheaper one having a lower coupon payment, thereby increasing the theta risk of

[72] The theta of an option also depends on whether the option is out-the-money or in-the-money, with the latter having a greater theta relative to the out-the-money option. To keep matters simple, in this dissertation the use of bond carry for theta is ignored.

the existing issue. Theta risk should also be recalculated with respect to implied volatility, as the volatility influences the embedded option value and thus the amount of theta risk. Theta risk varies according to the degree of moneyness of the CB as indicated in Figure 53, which shows the theta time decay. For in-the-money issues there is little theta risk as there is not much conversion premium to lose due to the equity behaviour of the CB, but any remaining conversion premium will evaporate in the last few months before maturity. At-the-money issues have the greatest theta risk as the conversion premium is the greatest and the CB price is trading above par value and call price, incentivizing the issuer to call the bond, and thus eliminating the CB's entire conversion premium. As with in-the-money issues the theta risk accelerates during the last few months before maturity. Out-the-money issues have very low theta risk except for the pull to par effect at maturity, driving the bond to its par value. Call protection also plays a role in pricing the CB as the longer the call protection is, the higher its value and the higher the theta risk when the CB's call protection expires.

Figure 54 emphasizes the high theta risk for at-the-money convertibles and the low theta risk for in/out-the-money issues. It is important for the arbitrageur to monitor the theta risk of his position since this can erode away any potential profits if not taken into consideration, albeit not as dramatic as delta, gamma or vega, but still a factor.

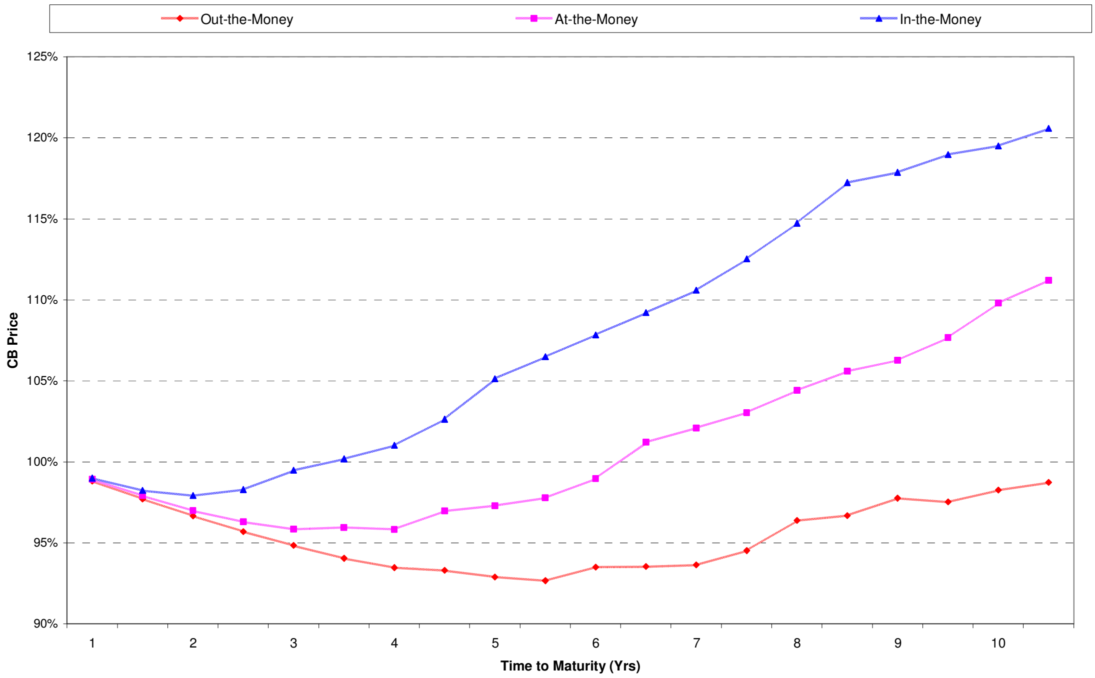

Figure 53 – Theta time decay using the TFk model for in-the-money, at-the-money and out-the-money convertible bonds.

Figure 54 – Theta risk for convertible bonds illustrating the significant risk at the conversion price for at-the-money Convertibles.

9.1.5 Rho *(ρ)*

Also referred to as duration risk when dealing with straight bonds, the change in CB value to changes in interest rates is known as rho$\left(\partial\chi/\partial r\right)$. As is commonly known, an increase in interest rates decreases the value of the CB due to its decrease in straight bond/investment value, but unlike regular bonds this can be offset by the gain in the option value. Generally speaking rho is negative for most share price ranges along the CB price track. Deep in-the-money issues have very little rho risk, whilst out-the-money issues experience the highest level of rho risk due to the lower investment value. Rho is at its maximum for an out-the-money issue when the embedded option is worth very little. When the CB price moves into the distressed area however, there is very little rho risk present as the dominant factors affecting the CB are the issuers' credit worthiness and probability of bankruptcy. In the distressed region macro factors play very small roles in computing the CB value; business related factors take full control of valuation. The typical rho risk assumes parallel yield curve shifts, which at times can be misleading but educates the arbitrageur on the sensitivity of the hedge to interest rates. Rho2 risk for in-the-money CB's relate to the short end of the yield curve for risk-free government bonds, whilst Rho1 risk relates to the corporate yield curve, or credit spread widening. This is important for arbitrageurs at a portfolio level who need to be aware of both measures in calculating their cost of capital from the short end and hedge on margins.

The convertible bond price for changes in interest rates can be seen in Figure 55 with an out-the-money issue having the greatest sensitivity to interest rates due to its debt component taking on a straight bond value. Figure 56 (plotted on a negative scale) is confirmation of the highest rho risk for an out-the-money convertible. The decrease in the rho for higher interest rates is attributable to the convexity[73] of the bond component in the CB, becoming inelastic to increasing rates.

This rho risk may seem contradictory because earlier it is mentioned that stochastic interest rates are not a dominant factor in CB valuation models. What is meant by this

[73] We assume that positive convexity is only possible in this simplistic example. If negative convexity is assumed for convertibles trading in the distressed zone, the rho risk would increase dramatically (that is to say becoming more negative) as the credit spread would widen substantially.

is that interest rate volatility does not play a significant role in CB prices as demonstrated with the Quadrinomial model. It does however improve the accuracy of the valuation model but given its additional complexities, assuming constant interest rates with zero volatility will produce a more tractable model.

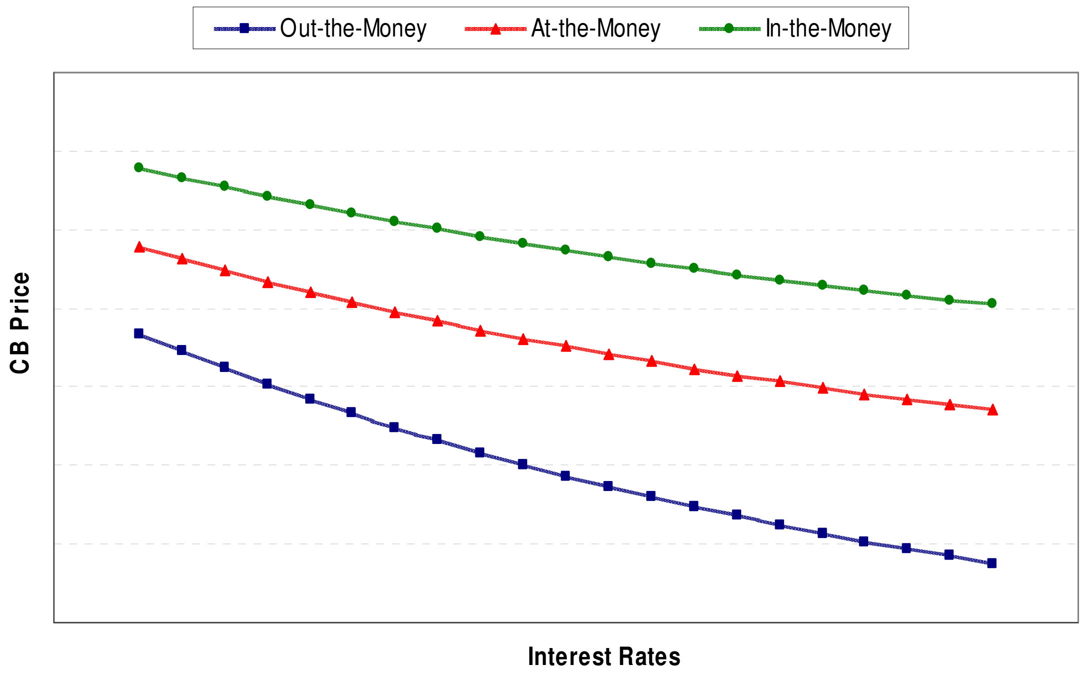

Figure 55 – Convertible price for changes in interest rates. The convexity of the debt component in the CB is evident with an out-the-money issue illustrating the steepest slope and hence the greatest interest rate sensitivity.

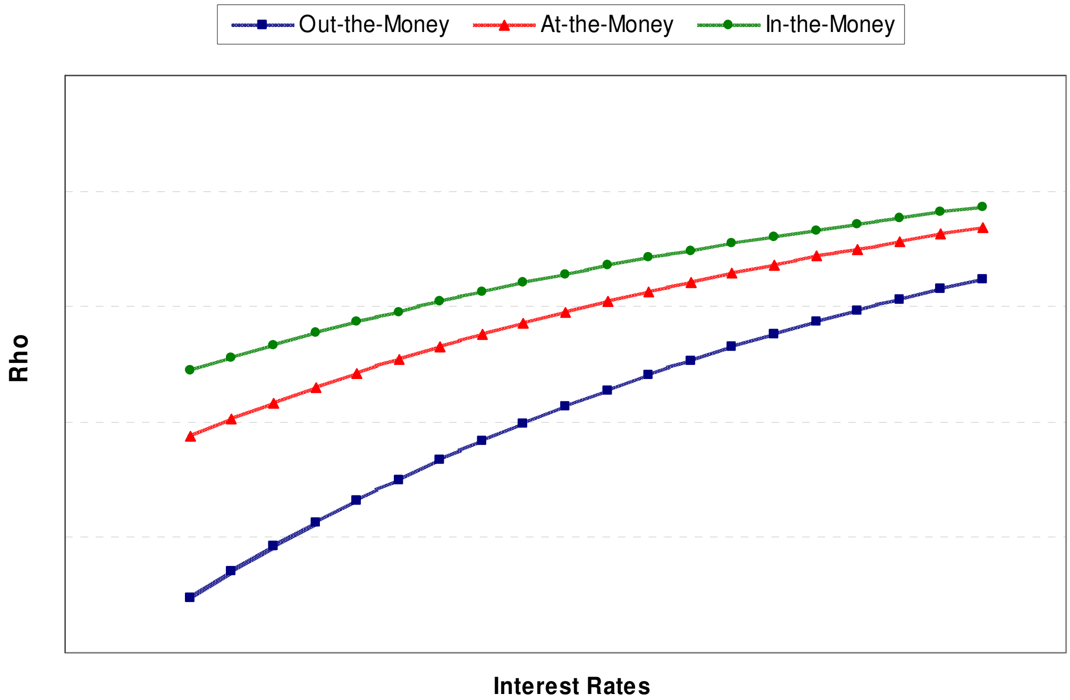

Figure 56 – Rho risk of a convertible bond on a negative scale with the out-the-money issue possessing the greatest risk.

9.1.6 Omicron *(o)*

Omicron $(\partial\chi/\partial o)$ measures the change in CB price to changes in the credit spread of the issuer. As was mentioned in Chapter 6 with the *k*-factor model, the omicron risk can increase substantially when the CB trades in the distressed zone. Generally speaking if the credit spread widens, the CB loses value and if the credit spread narrows, the CB rises in value. For both high and low grade issuers, deep in-the-money CB's have low omicron risk, as the driving factor becomes the share price. For at-the-money and out-the-money convertibles the omicron risk increases with most sensitivity occurring out-the-money.

9.2 Delta Hedging

One of the most common hedging strategies is the delta neutral hedge as it is considered lower risk relative to the other hedging strategies, because of its ability to reduce the equity sensitivity while taking advantage of the equity volatility. As the share price changes so to do the risks and hedge opportunities. Figure 57 together with Table 7 shows the various Greek sensitivities along the CB price track. Figure 58 and

Table 8 depict the risk measures for the CB price track with negative convexity present.

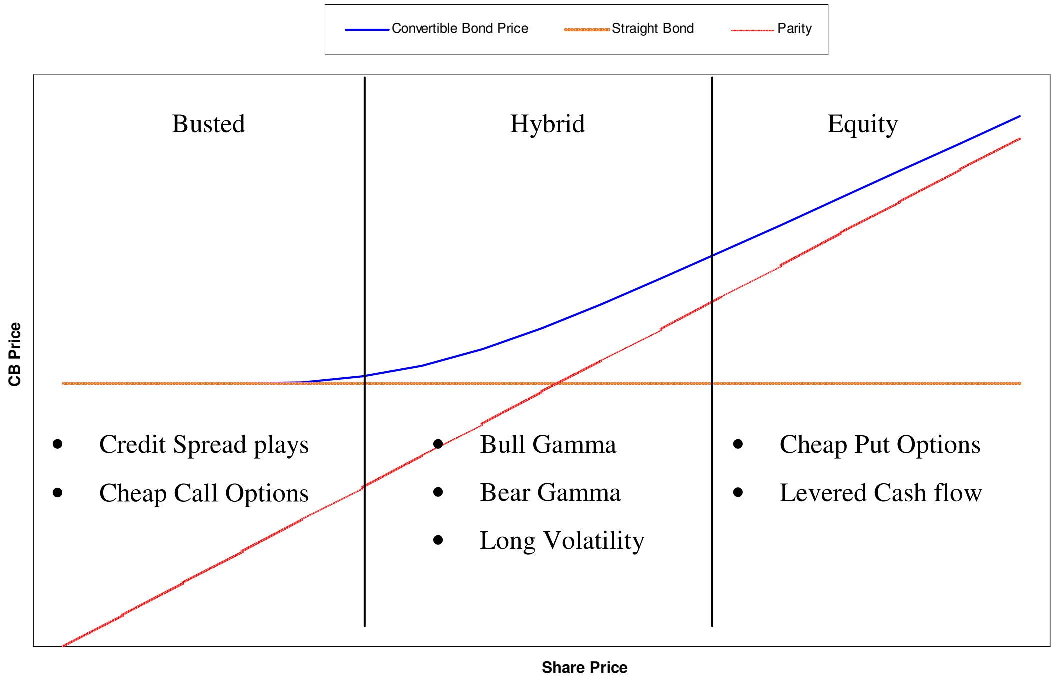

Figure 57 – Trade strategies at various point along the convertible price track (No credit spread sensitivity to share price).

Greek Risks:	Busted	Hybrid	Equity
Delta	Low	Medium	High
Gamma	Low	High	Low
Vega	Low	High	Low
Theta	Low	High	Low – Medium
Rho	High	Medium	Low
Credit	High	Medium	Low

Table 7 – Greek exposure through various stages of the convertible bond price track (No credit spread sensitivity to share price).

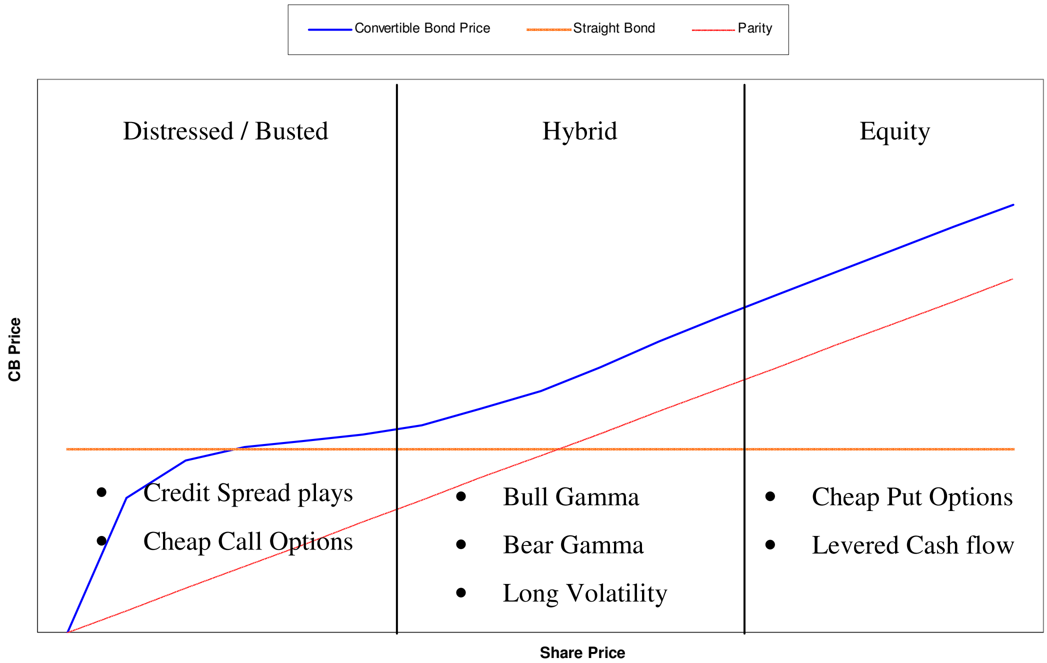

Figure 58 - Trade strategies at various points along the convertible bond price track (credit spread sensitivity to share price).

Greek Risks:	Busted	Hybrid	Equity
Delta	High	Medium	High
Gamma	Medium (Negative)	High	Low
Vega	Low (Negative)	High	Low
Theta	Low	High	Low – Medium
Rho	High	Medium	Low
Credit	High	Medium	Low

Table 8 - Greek exposure through various stages of the convertible bond price track (credit spread sensitivity to share price).

Deep in-the-money CB's experience low amounts of rho, omicron and gamma risk, allowing the arbitrageur to purchase cheap put options on the CB and take on more leveraged positions. The distressed CB is similar but has high omicron risk that offers cheap call options on the underlying and a credit spread play related to the undervaluation of the CB. Depending on the credit quality of the issuer, low grade (and at times high grade) issuers experience negative gamma in the distressed zone, which could significantly eat away into the hedge strategy profits. The CB is now highly correlated with the share price and the negative gamma causes the delta to reverse and move closer to 1 (assuming a conversion ratio of 1). A strong credit assessment needs to be undertaken when it comes to low grade issuers, especially when they are trading in the distressed zone.

The long volatility convertible hedge is a popular delta neutral hedge involving longing the CB and shorting the underlying stock at the current delta. The hedge is constructed so that there are no gains or losses from small stock price movements; the majority of the income comes from the CB's yield and short interest rebate[74]. The long volatility hedge neutralizes the equity risk (delta) although leaves the arbitrageur exposed to volatility (vega) and interest rate risk (rho). The arbitrageur is taking a view that implied volatility is undervalued relative to his expectations and so implements this long volatility hedge in the hope that volatility does in fact increase. This is where the majority of the hedge's income will stem from, if on the otherhand the volatility decreases the only income would be the coupon from the CB and the short interest rebate. As such the more volatile the implied volatility, the more trading opportunities exist. An important note on volatility is the effect it has on the time value of an option, increasing volatility has the effect of reversing the time value of an option whilst decreasing volatility has the opposite effect. As such CB's with little or no call protection can be exposed to these irregular volatility effects. The following criteria help in identifying the long volatility hedge:

- Implied volatility or credit mis-pricing
- Yield advantage greater than risk-free rate
- Expectation that implied volatility will increase for low cash positions, and increase a fraction or stay the same for high cash position.
- Stable or improving credit opinion
- Minimal liquidity risk
- Ample stock borrowing available
- Vega higher than omicron

[74] The short interest rebate is the interest earned on the short position and margin gains and losses.

The risks to consider are:

- Implied volatility decreases
- Credit spread widens
- Surprise call from issuer
- Yield curve shifts up at short end
- Theta eats away vega gains

Delta neutral hedges are constructed to capture the cash flows available on the convertible position as well as profit from longing the cheap volatility in the embedded option. The correct hedge ratio depends on the expected changes in implied volatility, gamma and delta. The modified delta[75] is a more accurate delta to use when using large share price moves. The arbitrageur has to determine a stock price range within a specified time interval for the hedge to be held or rebalanced. Figure 59 illustrates the delta-neutral hedge with the hedge offering downside protection of 4.4% compared to the unhedged CB return of 2.9% and upside return of 8.0% compared to the unhedged CB's 9.8%.

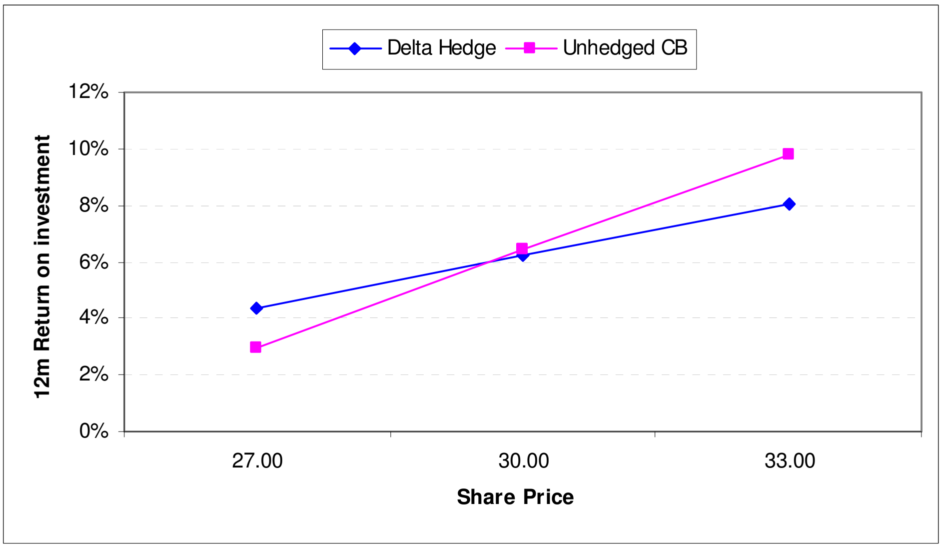

Figure 59 – Delta neutral hedge vs. unhedged convertible bond return profile for a selection of share prices.

[75] The modified delta is calculated as $\left(\chi_{S^+} - \chi_{S^-}\right) / \left(S^+ - S^-\right)$ with χ_{S^+} being the value of the convertible when the share moves up and χ_{S^-} being the convertible value when the share moves down.

9.3 Dynamic Hedging

In theory dynamic hedging requires the arbitrageur to create a maximum or minimum Greek exposure and have to continuously rebalance to maintain a delta neutral hedge. The less frequent the hedging the higher the equity correlation and the lower the potential return. This does not mean that the hedge should be rebalanced frequently because the trading costs involved will become significant and affect the profitability of the position. Static hedging or infrequent balancing would be acceptable if trading costs were high in an un-levered portfolio and the total income flow was low, but to achieve a higher return some hedging is required to lock in the gains. In essence a hedge position needs to be rebalanced more frequently when it has a high gamma due to the accelerated change in the delta for stock price moves and also when the underlying stock is considered volatile. Similarly a low gamma will result in less frequent rebalancing due to a small change in the delta when the share price moves.

The only situation where the active hedge underperforms relative to a passive hedge is when the share price only moves in one direction, which occurs very rarely. The constant rebalancing will inflate the trading costs and destroy the benefits of the hedge, leaving the income flow and convergence to theoretical value as the only source of income.

9.4 Gamma Hedging

Due to the convexity of the CB price track an opportunity for convertible arbitrageurs exists. They can capture this gamma with less frequent rebalancing and minimal risk to increase the alpha in their portfolios. If the convertible arbitrageur has a strong market viewpoint he can achieve even greater returns with a slight directional bias if it pays off of course. In this section the most common gamma hedging tilts used in the industry will be discussed, namely bullish and bearish gamma hedging. Bullish gamma hedging will be defined as using a delta less than the true delta neutral, and bearish gamma hedging will be defined as a delta greater than the true delta neutral hedge.

To start off with a delta neutral gamma hedge can benefit despite the movement in the underlying stock price, as the hedge will be exposed before rebalancing is done.

However, if a passive hedge is held over time the investor will be exposed to substantial equity market volatility. In practice frequent rebalancing is done to capture smaller amounts of gamma with lower volatility and as a drawback, the returns will not be as significant. Less frequent rebalancing increases the returns and gamma capture but also the equity volatility. The neutral gamma (delta neutral) hedge is most advantageous when the CB possesses a high gamma and the underlying share is volatile. This equates to large share price moves over short time intervals ensuring that the hedge is exposed to this volatility and also before the hedge is rebalanced, thereby creating higher returns.

If the CB has a high gamma it might be worthwhile just to have a delta neutral hedge, but for CB's that do not possess enough gamma, a gamma hedge would suffice but directional bets on the market are required for any meaningful returns.

The bullish un-leveraged gamma hedge captures upside gamma and relies on an appreciation of the share price. The hedge ratio is set slightly below the delta neutral ratio depending on the investors' confidence of the market (lower if he is more confident). It works best when the upside gamma is smaller than the downside gamma, providing the investor with downside protection if the market moves in the opposing direction. The 12-month return profile with accompanying breakdown of returns is given in Figure 60 and Table A- 1 respectively.

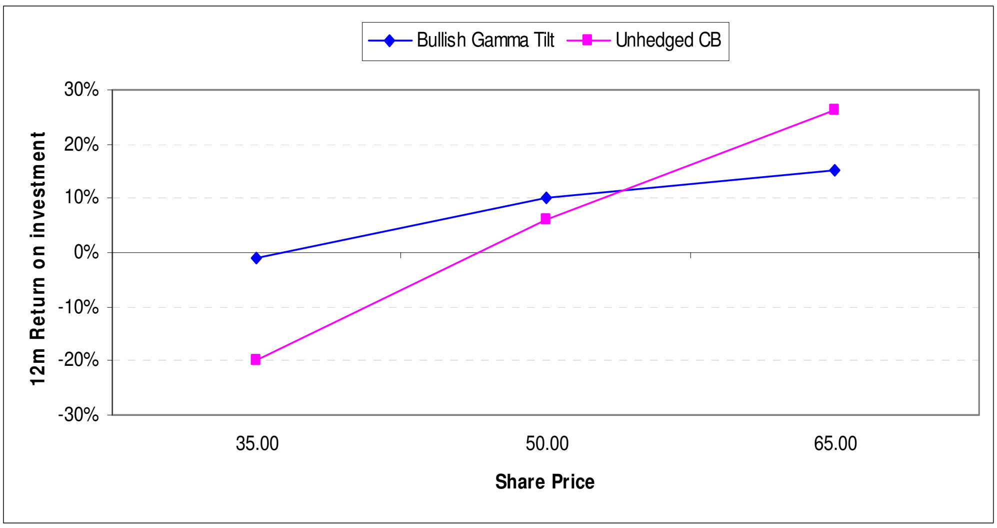

Figure 60 – Bullish gamma hedge (without leverage) vs. unhedged convertible bond return profile for a selection of share values.

The bullish leveraged gamma hedge also captures upside gamma and relies on the share price increasing but takes on additional risk due to the leveraged position. The hedge ratio is set slightly higher than the bullish un-leveraged gamma hedge but less than the neutral hedge to reduce some of the volatility. More aggressive investors can increase it as desired to improve the upside potential. As the stock increases the gamma starts to decline and the delta increases, requiring the convertible arbitrageur to add to the hedge so that the hedge ratio closes on the delta neutral hedge until the stock price objective is reached. The disadvantage is of course the short interest[76] missed due to the lighter hedge ratio. The bullish gamma hedge with leverage is shown in Figure 61 with the detail in Table A-2. As can be seen the additional leverage increases the upside returns but at the expense of greater downside loss, with the unhedged CB still producing greater downside returns.

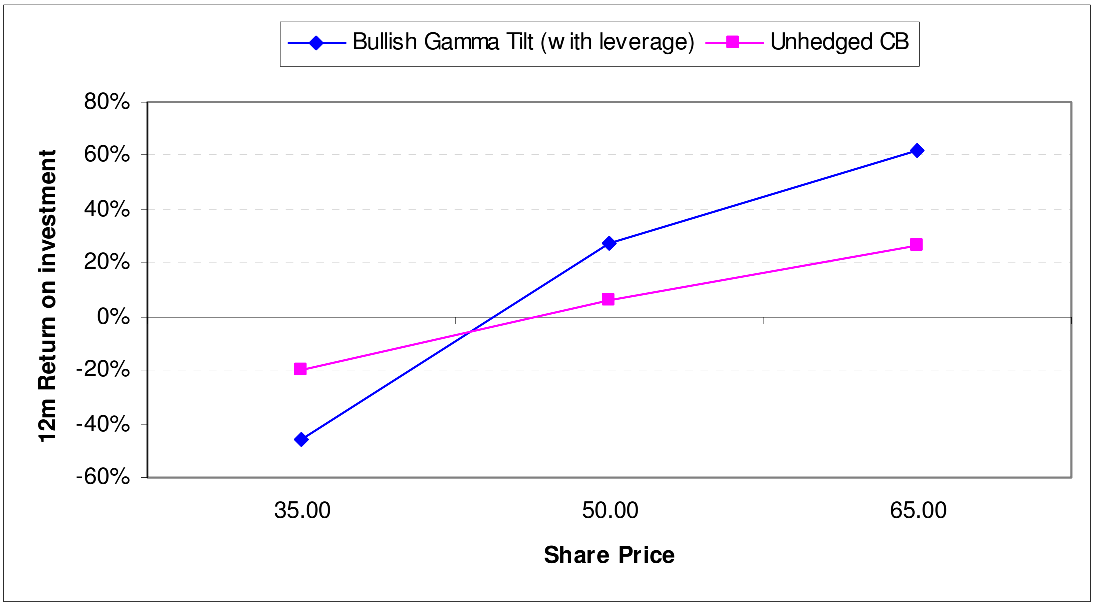

Figure 61 - Bullish gamma hedge (with leverage) vs. unhedged convertible bond return profile for a selection of share values.

The bearish un-leveraged gamma hedge captures downside gamma and relies on the share depreciating in value. The hedge ratio is set above the delta neutral hedge ratio, once again depending on the investors risk tolerance. This strategy works best if

[76] The short interest is the interest earned on the short stock proceeds that have been invested in the bank at the risk-free rate.

upside gamma is less than downside gamma, and can achieve high returns if the share moves in line with the investors' expectations. The 12 month returns are given in Figure 62 with details in Table A- 3, with the downside producing higher returns for the bearish un-leveraged gamma hedge at the cost of upside returns.

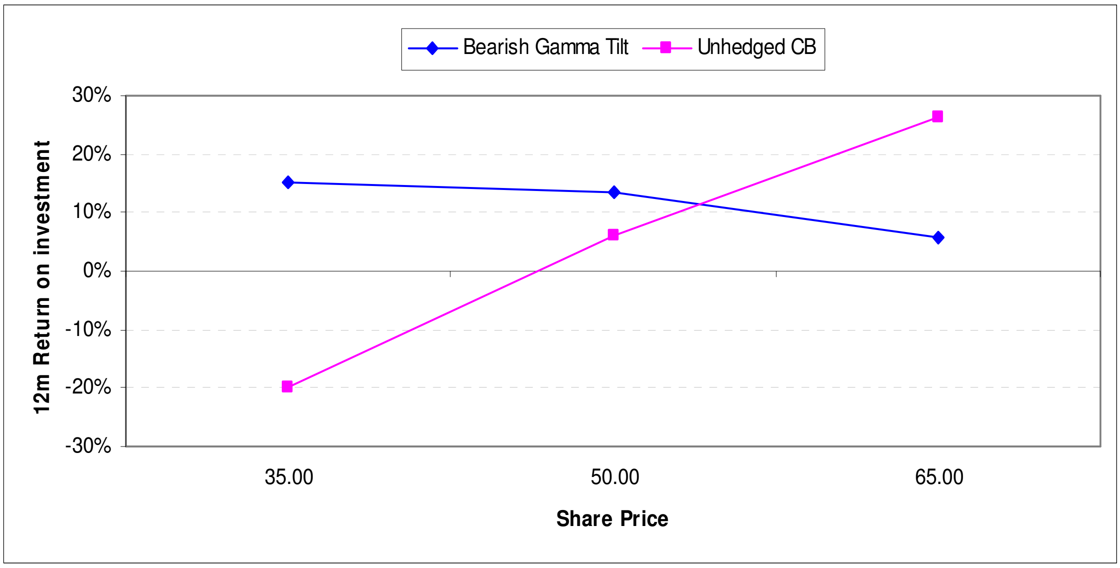

Figure 62 - Bearish gamma hedge (without leverage) vs. unhedged convertible bond return profile for a selection of share values.

The bearish leveraged gamma hedge is similar to the un-leveraged one except as the name implies, it relies on more leverage to increase alpha. The hedge ratio is set above the un-leveraged hedge and even higher depending on the investors' confidence. As the hedge ratio increases, the upside potential decreases and the downside increases, due to the downside gamma being higher than the upside gamma. This dramatically increases the downside returns but at the expense of decreasing the upside returns as shown in Figure 63 and Table A-4.

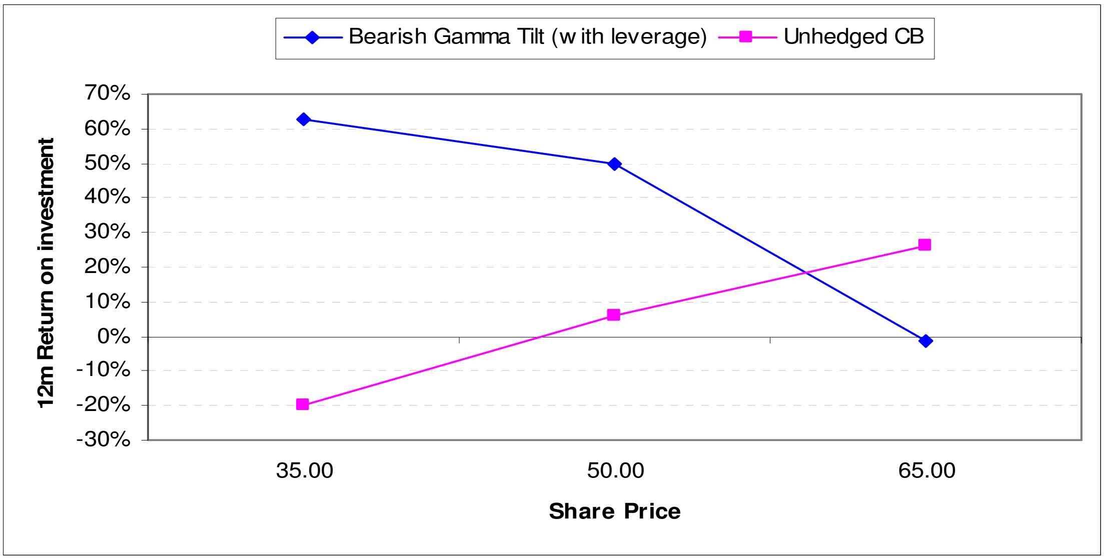

Figure 63 - Bearish gamma hedge (with leverage) vs. unhedged convertible bond return profile for a selection of share values.

As can be seen the bearish gamma hedge looks to take advantage of a decline in the share price whilst the bullish gamma hedge benefits from an increase in the share price. Looking at Figure 64 and Table 9 we can see that the bearish hedge returns the highest with 15.32% and the bullish hedge returns the lowest with -0.89% which is appropriate due to the lighter hedge ratio on the bullish hedge, reducing the potential gain from the short interest and stock positions. When the share price increases, the bullish hedge outperforms the bearish hedge with a return of 15.19% and 5.79% respectively because of the lighter short stock position. The gains from the CB are not being eroded by the loss on the short stock. The delta neutral hedge falls in-between these two strategies, so depending on the directional hedge that the arbitrageur wants to take, he can enhance his return relative to the delta neutral hedge and even more so with leverage, but obviously at a greater cost if the stock moves in the opposing direction.

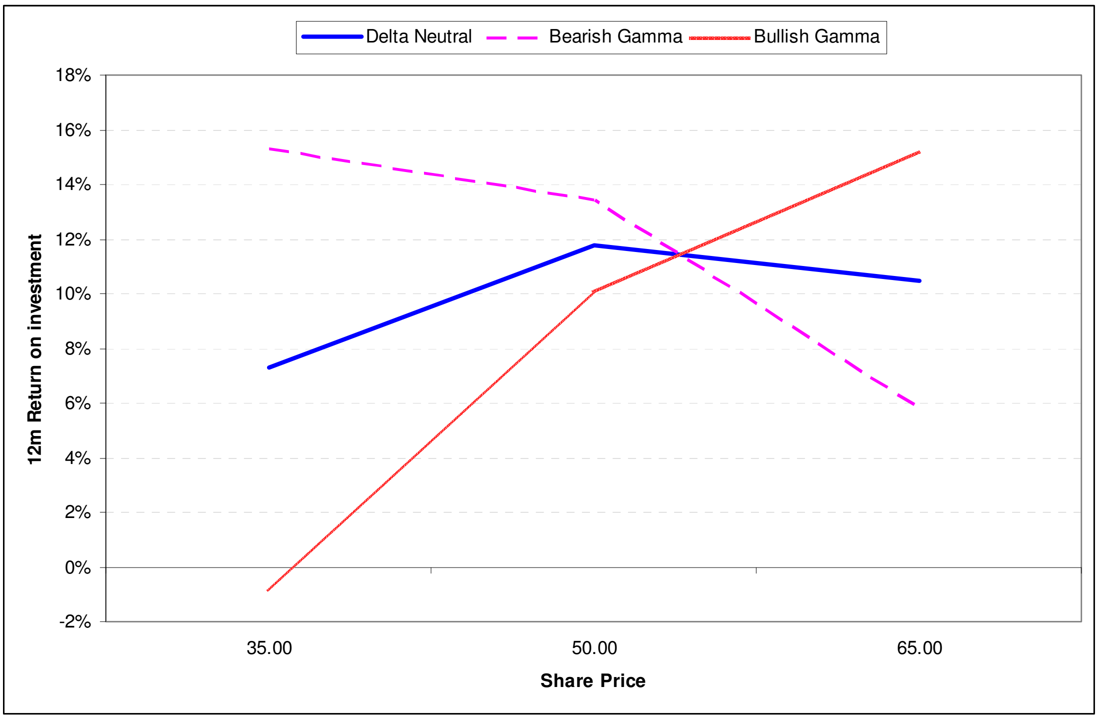

Figure 64 – Comparison of the bearish and bullish gamma hedges relative to the delta-neutral hedge.

Share Price	35.00	50.00	65.00
Bullish Gamma	-0.89%	10.04%	15.19%
Bearish Gamma	15.32%	13.44%	5.79%
Delta Neutral	7.28%	11.75%	10.45%

Table 9 – 12 month returns for the various hedging strategies, with a graphical representation given in Figure 64.

When adopting these gamma strategies into a portfolio the convertible arbitrageur should have a very confident outlook of the market supported by sufficient equity research. If so these hedging strategies will reduce trading costs as they allow more passive hedging to be achieved as opposed to delta hedging. The convertible arbitrageur should also attempt to make a market neutral portfolio by setting complementary positions and creating a delta-adjusted beta exposure of zero. The delta-adjusted beta is important as it quantifies the exposure of the hedge relative to the amount of market risk (beta) present in the underlying stock. The best way of describing this is through an example.

Bull Gamma

Position	Position Size	Beta	Theoretical Delta	Hedge Delta	Delta Exposure	Delta Adj Beta
Issuer A	100,000	1.1	0.70	0.60	10,000	11,000
Issuer B	100,000	1.4	0.58	0.50	8,000	11,200
Issuer C	100,000	0.9	0.77	0.65	12,000	10,800
					Net Beta Exposure =	33,000

Bear Gamma

Position	Position Size	Beta	Theoretical Delta	Hedge Delta	Delta Exposure	Delta Adj Beta
Issuer D	100,000	1.5	0.70	0.78	-8,000	-12,000
Issuer E	100,000	1.18	0.58	0.68	-10,000	-11,800
Issuer F	100,000	0.84	0.77	0.88	-11,000	-9,240
					Net Beta Exposure =	**-33,040**

Portfolio's net bull and bear gamma beta exposure =	**-40**

Table 10 – Delta-adjusted beta portfolio with a zero delta-adjusted beta to take advantage of the non-linear relationship between the underlying stock and convertible cash flows.

Referring to Table 10 we can see a simple portfolio of convertible bonds that is setup for a bullish and bearish gamma hedge but has a delta-adjusted beta of almost zero. Walking through Table 10, we first have the bullish gamma hedge that sets up a lighter delta relative to its theoretical delta to provide more downside protection. The delta exposure is the difference between the two deltas multiplied by the position size. Thus the delta-adjusted beta is equal to the delta exposure multiplied by the stock beta. Likewise the bearish gamma hedge is positioned with a relatively larger delta to participate in share price depreciation. In doing so the net delta-adjusted beta exposure of the portfolio is -40 and so is effectively zero. This gamma hedge allows the convertible arbitrageur to add additional alpha while controlling the delta exposure. Arbitrageurs that have a strong equity support team can also take more aggressive directional bets in a diversified convertible hedge portfolio.

9.5 Option Hedging

Since the CB contains an embedded call option and in some cases issued with a put option giving the holder of the CB the benefit of exercising, many arbitrage opportunities exist. These opportunities arise due to the mis-pricing between the issuers listed option and the embedded option in the convertible. To actually quantify whether the implied volatility in the CB is under/overvalued requires the investor/arbitrageur to possess a precise valuation model. The basic idea is to sell the more expensive option and purchase the cheaper option. Most arbitrageurs use the option hedge technique to take advantage of volatility time skews. Usually the listed option has a shorter term to maturity and so has less implied volatility whereas the embedded option has a longer term to maturity and a relatively higher implied volatility.

The typical option hedge strategy involves shorting a call option on the issuers stock and investing in the CB. This is referred to as a covered or partially covered convertible call option hedge, depending on the risk-return profile of the arbitrageur, and whether to write out-the-money or in-the-money call options. Writing out-the-money calls improves the upside potential but limits downside profits due to the low premium, on the otherhand writing in-the-money options improves downside profits but exposes the hedge to greater upside losses. The call option hedge is usually constructed for only a few months, until the written call options expire. This strategy takes a view that the share value is going to move sideways or even down, because if the share value were to increase beyond the strike value plus the premium, the gain in the CB would be eroded by the written calls being exercised. If the share value were to move sideways or even down, the profits would arise from the bond coupon and the premium on the written calls. Shorting a greater number of calls, which is dependent on the arbitrageurs risk appetite, could enhance the returns even further. The ideal situation to implement this option strategy is when:

- The CB is undervalued[77] and has higher upside gamma than downside gamma which reduces the cost of rebalancing the hedge,
- The call options implied volatility is greater than the expected volatility and the CB's implied volatility, and
- The share is overvalued, showing weak technical support with no upside movement.

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[77] The undervalued CB occurs when the implied volatility in the embedded option is less than the expected volatility.

Of course if the share were to increase above the strike and premium this would cause unlimited upside loss with only limited downside profit. Also if the volatility time skew were to widen with shorter maturity volatility increasing and longer term volatility decreasing, this would cause the hedge to breakdown. A 4-month return profile is shown in Figure 65 with the details given in Table A-5.

There is some downside protection evident with the covered convertible call hedge offering a return of -10.6% relative to the unhedged convertible return of –15.8%. As the CB moves into the money the profits from the hedge are eroded with the unhedged convertible yielding 17.5% and the covered convertible call hedge yielding 12.5% at 72.60.

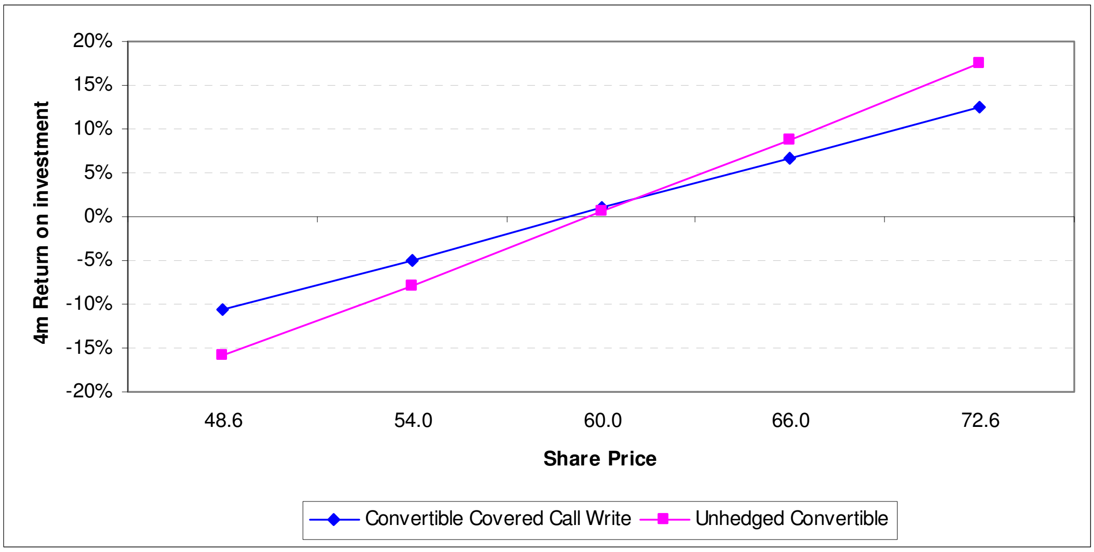

Figure 65 – Convertible covered call write hedge vs. unhedged convertible bond for a selection of share prices showing downside protection but sacrificing upside potential relative to the convertible bond.

As mentioned earlier the traditional convertible hedge involves longing the CB and shorting the stock, but if the arbitrageur has a negative view of the share price he can enhance the return by shorting call options on the underlying as well. This strategy is called the convertible stock hedge with call write overlay, and takes advantage of mean-reverting implied volatility in the written call option. This position once again takes a directional view for a short period of time assuming that the implied call volatility is trading above its expected value and the CB's implied volatility. The assumption is that the implied call volatility will soon normalise by decreasing and the implied CB volatility increase, with the share trading in a narrow price range. It may also be difficult to short additional stock when establishing or rebalancing the delta-neutral hedge, justifying the call overlay hedge. As mentioned previously if the volatility skew were to widen or the share increase in value, this would erode the profit potential of the hedge. An example of the return profile under the call overlay hedge is given in Figure 66 with the corresponding details given in Table A-6. As can be seen there is more downside protection with a return of 0.1% when the share moves to 48.60 but this comes at a price with the upside yielding only 3.2% at 72.60 compared to the unhedged convertible yielding 17.5%.

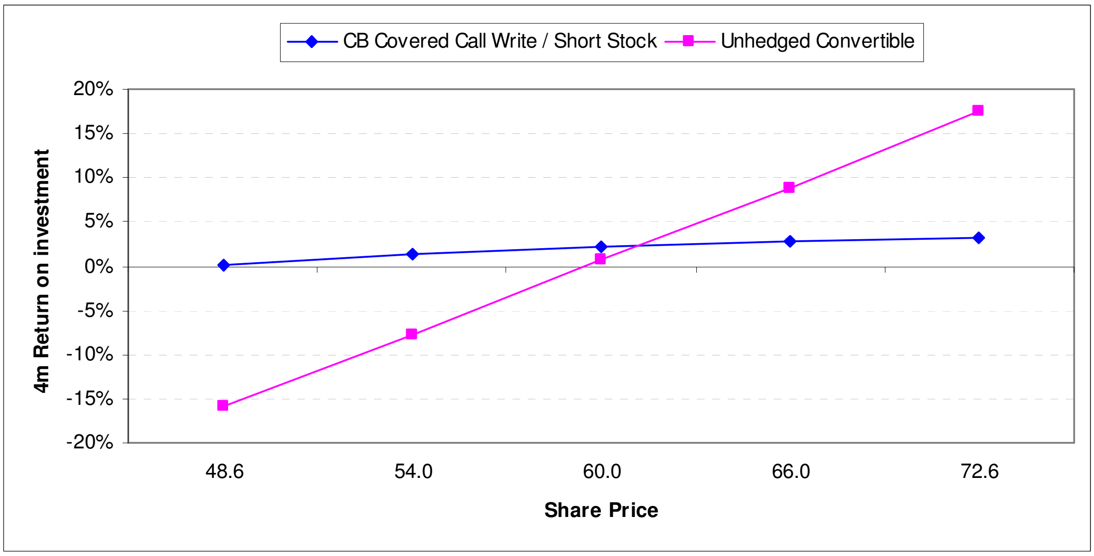

Figure 66 - Convertible covered call write with additional short stock hedge vs. unhedged convertible bond for a selection of share prices showing downside protection but sacrificing upside potential relative to the convertible bond.

As previously discussed, when the share value drops to very low levels the CB experiences negative gamma and falls below its investment value. An out-the-money put option on the CB can be purchased quite cheaply to limit the downside risk should the share drop to these levels. As the strike price on the embedded call option in the CB is a moving target it would be difficult to decide at what share value the CB moves into the distressed zone so that a strike price for the put can be determined. Due to the negative gamma the delta reverses and starts moving up towards its maximum, which can be devastating to the return profile due to the equity correlation, and rebalancing required. As such using a put strategy is very effective since it is perfectly correlated to the stock but lacks all the credit risk ensuring that rebalancing produces short stock gains and downside protection. The upside is of course the same but decreased by the put option premium. Criteria to lookout for, for this hedge is whether

- The CB has low downside gamma risk,
- High volatility,
- Good yield carry,
- High liquidity with low implied put option volatility and
- The CB is of low-grade[78] but not trading in the distressed zone just yet, with a pessimistic view on the share price.

The risks are that if the put options strike is calculated incorrectly or below the level needed to prevent a loss, it could seriously reduce the overall returns of the strategy. The return profile is illustrated in Figure 67 with the detail shown in Table A-7. The return profile is similar to the covered convertible call write with short stock but a little more risky due to the lower downside return of –3.1% but higher upside return of 6.4%.

[78] Investment grade issues can also be considered but generally speaking there is little chance of these issues defaulting, although if the arbitrageur has a strong equity support team with convincing evidence of a possible default, then an opportunity may arise.

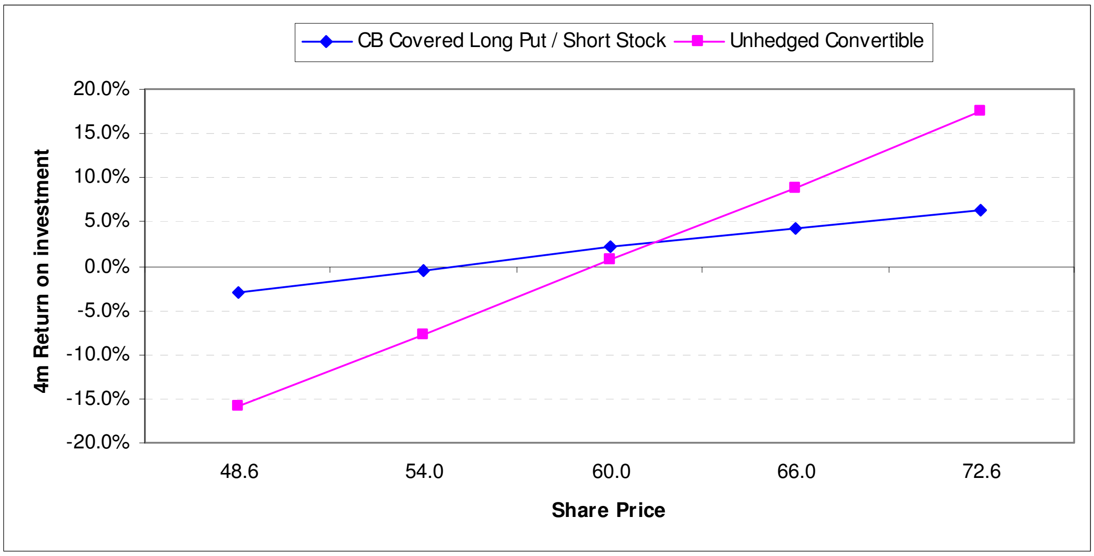

Figure 67 - Convertible covered long put with additional short stock hedge vs. unhedged convertible bond for a selection of share prices showing downside protection but sacrificing upside potential relative to the convertible bond.

The option hedging techniques offer great opportunities depending on the risk appetite and level of conviction of the arbitrageur. As can be seen from all these hedging strategies, the greater the upside potential, the greater the downside risk and vice versa. Taking on additional risk has its consequences if the market moves against the position, as such it is critical to do substantial research and develop a reasonable basis before constructing a hedging strategy. The reader should bear in mind that these are not the only hedging strategies that exist, but only a few of the most popular ones. In reality there are numerous hedging strategies that exist.

After explaining the various CB models and hedging strategies, one asks the question as to which of the models is the most reliable. As such a comparison of several models is presented relative to actual market prices of a traded CB. The firm value models were not included due to the predicament of finding an accurate firm asset value.

10. Numerical Results

In the final Chapter a comparison of the accuracy of several models is done. The Barclays CB is used as the benchmark. Section 10.1 describes how the precision of each model is determined with 10.2 giving the results.

10.1 Description

A good starting block to assess the accuracy of the above models is by testing their estimates against actual data. The convertible bond that will be used is the Barclays issue introduced in Chapter 6. The models used to obtain estimate values are, the Margrabe American Exchange (MAE), Tsiveriotis and Fernandes (TF), Tsiveriotis and Fernandes with *k*-factor (TFk), Hung and Wang (HW), Goldman-Sachs (GS) and the Component Exchange with Coupon and Credit (CompEx). The inputs for the models can be found in the Appendix A3. To evaluate the accuracy of each model the average absolute pricing error (AAPE) defined as $\sum_{i=1}^{n}|X_i - O_i|/n$ will be used, where X_i is the model output, O_i is the observed value for the CB in the modulus and n is the number of observations. It can be viewed as the standard deviation of the differences between the predicted and actual outcomes. The model with the lowest AAPE should be the most accurate according to this crude evaluation technique. The results are given below in Table 11.

Model	AAPE (%)
MAE	41.0
GS	16.5
CompEx	8.3
HWk	5.5
TF	4.6
TFk	4.4
HW	3.8

Table 11 - AAPE of the various CB models relative to the Barclays issue.

10.2 Results

The GS and CompEx models are shown in Figure 68 with the GS model underestimating with an AAPE of 16.5% which is substantial, but it could be because the credit spread is a little high increasing the risky discount rate. The bond is trading in the debt zone so is most likely using this higher rate to discount the cash flows. The CompEx model provides a better estimate with an AAPE of 8.3%, mostly with overestimation. The improvement is due to the calibration of the term structure of interest rates and credit spreads to a one-factor Vasicek interest rate model, with the overestimation due to the separation of the straight bond and exchange option, which causes additional premium of the CB. This also applies to the MAE model in Figure 69, which shows the large overestimation with an AAPE of 41.0%. The TF model (Figure 70) like the MAE and CompEx model also overestimates although improves the pricing estimate with an average absolute pricing error of 4.6%. This is a little peculiar seen as the GS model uses a smoothing discount rate whereas the TF model uses either the risk-free or risky rate for discounting, and as such should cause the CB price to be lower than the GS model. A possible explanation is that due to the decoupling PDEs that separate the CB into a cash-only part (equity) and debt part, the equity part is being weighted a little more and so increasing the CB value because of the lower risk-free interest rate. The TFk improves upon this slightly with an AAPE of 4.4% as can be seen in

Figure 70. This seems to suggest that the inclusion of the *k*-factor credit spread has improved the estimation on average by 0.2%. The HW model in Figure 71 gives the best results with an AAPE of 3.8%, although the HFk actually increases the AAPE to 5.5% so that the incorporation of the *k*-factor into the HW model does not improve the accuracy. The pricing improvements in the HW model can be anticipated because of the implied default probabilities calibrated from the market term structure. It is interesting to note how the models move in tandem with the share value, which emphasizes the importance of the share price as a dominant factor in the CB. Another observation with respect to the TF and HW models is that the inclusion of the *k*-factor decreases the value of the model output with respect to the TFk and HWk models. This is even more prevalent when the share value drops to the trigger share value of 18.90, between Aug 05 to Dec 05 and after Nov 07. During this period the TFk and

HWk models underestimate quite substantially from the TF and HW models respectively, because of the higher risky discount rate.

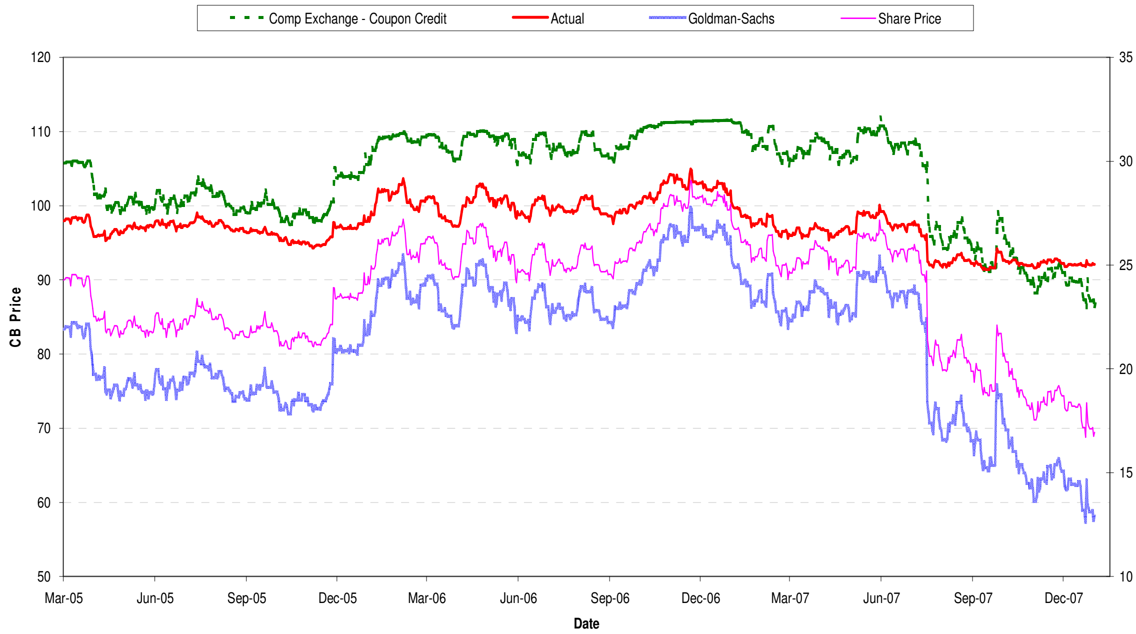

Figure 68 - Time series of the Barclays CB vs. the CompEx and GS models, including the underlying share price in euros.

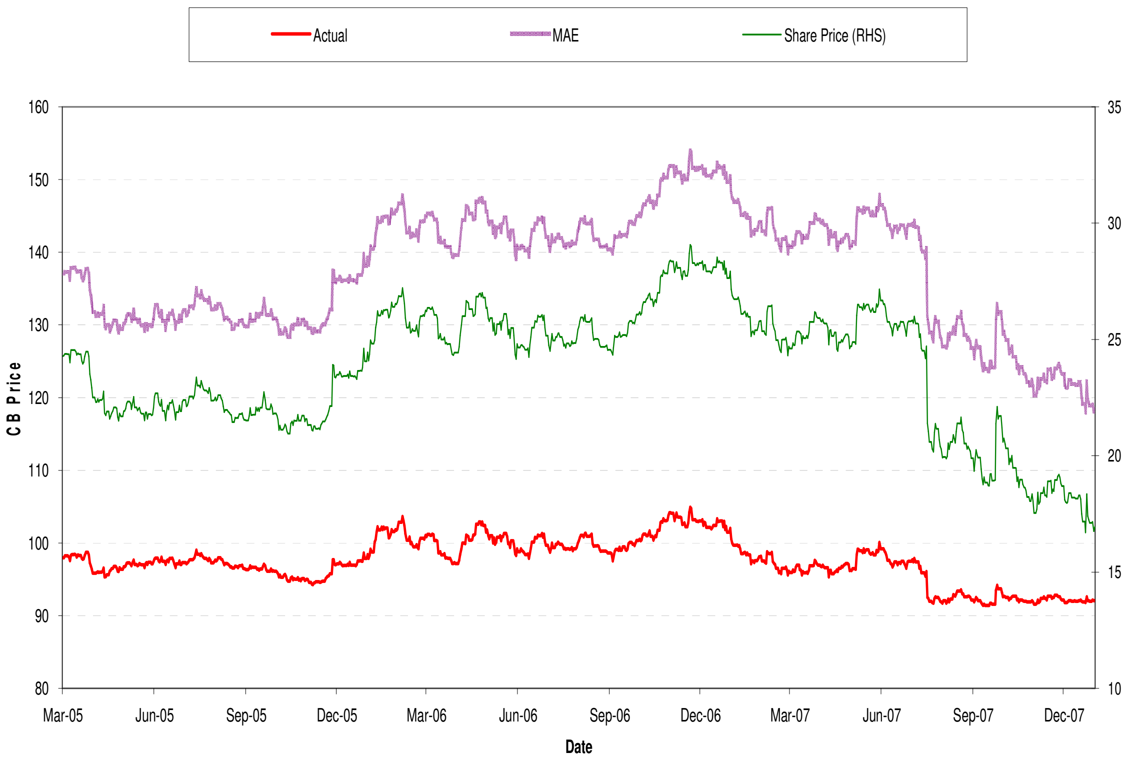

Figure 69– Time series of the Barclays CB vs. the MAE model, including the underlying share price in euros.

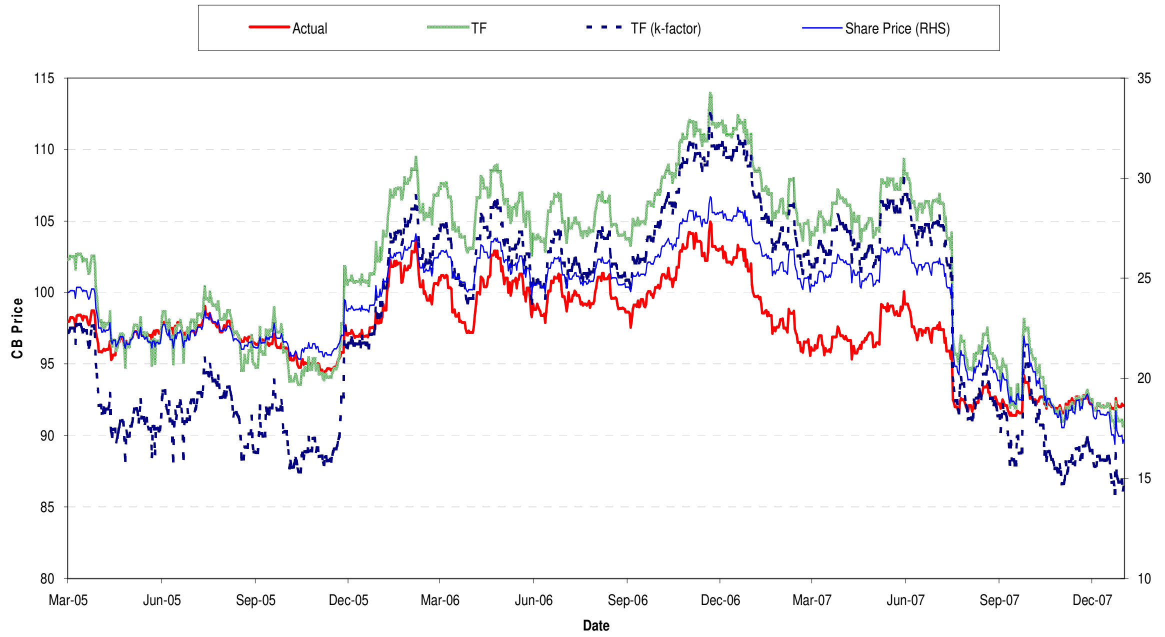

Figure 70 - Time series of the Barclays CB vs. the TF and TFk models, including the underlying share price in euros.

Figure 71 - Time series of the Barclays CB vs. the HW and HWk models, including the underlying share price in euros.

Conclusion

Convertible bonds can be an extremely rewarding asset class if the inherent risks and trading strategies are managed correctly. Since a forecast of the convertible bonds return is required to see if the hedge/trading strategy provides satisfactory performance, the accuracy of the model used in these forecasts is vital. In this dissertation a variety of CB models have been reviewed from the most basic piecewise models, to more complex models that deal with credit and default risk.

The simple piecewise models that value the CB as a straight bond plus call option are easy to compute and implement and provide an estimate value for the CB, but their downfall is the fact that the strike value of the embedded call option is fixed and that the bond and option components can each be traded separately, thus constantly over-estimating the value of the CB[79]. The MEE and MAE models use an exchange option instead of a call option, which corrects the fixed strike value, by using a floating strike instead. However these models also assume that the components trade separately once again over-estimating the CB price. They also assume that the straight bond follows a lognormal distribution which is a little subjective and in addition do not account for embedded call and put options. The Component model improves on this by allowing embedded call and put options, with some default risk as well, but is much more difficult to implement due to the large amount of inputs required including a Voluntary Conversion Date which is tricky to approximate. A desirable property of these piecewise models however, is the inclusion of correlation between interest rates and share prices, which have shown to cause a significant impact on the CB price.

Next a binomial interest rate tree is calibrated to a given term structure using the BDT and more precise Arrow-Debreu techniques. This is required for the equity valuation models beginning with the Quadrinomial tree model which improves the CB price by taking into account share price and interest rate correlation, the stochastic nature of interest rates, and treating the CB as a single financial security; although as shown in Chapter 5.2, the volatility of interest rates do not play a dominant role in the value of

[79] This is because the straight bond and embedded option in a single convertible bond will more often than not cost less than the sum of the individual components.

the CB. The downfall of this model is that credit risk is introduced in a very simplistic manner by using a constant credit spread to rollback through time. The TF model provides a more robust model by using a decoupled PDE approach breaking the CB into a debt and equity part and discounting each part with a risky and risk-free interest rate respectively. This improves on the credit risk aspect but does not specify what happens to the share value upon default and assumes the share price is unaffected. The GS model uses a blended discount rate dependent on the probability of conversion, which gives results very similar to the TF model but once again the share price is unaffected upon default. The AFV and HW models take default risk into account by specifying exactly what happens to the share price upon default and thereby improve the estimation of the CB, especially for low share values where default risk increases. These models also incorporate the negative convexity that becomes evident for low share values. An improvement to the TF model was also discussed that included a stochastic credit spread dependent upon the level of the share value (*k*-factor credit spread model), which also facilitated in the negative convexity. This increased the results of the TFk model to some degree, but not incorporating any default risk is seen as a downfall. The major advantage of the equity and structural models is that they value the CB as a single security thereby increasing the accuracy of the estimates.

The structural models described consisted of the Tan and Cai model and the Gheno model. These models use the firm value as the explanatory variable and incorporate default risk by using a barrier approach ascribing to the fact that the firm will default if the debt value of the firm rises above its asset value. Although this approach does have its advantages in that it takes into account the negative convexity of the CB, finding data on the firms' asset returns and asset volatility is a difficult prospect and so calibration becomes a problem. The calibration of the asset value trees using the volatility structure of options trading in the market is a real plus for the Gheno model but getting reliable values for the inputs in the model is questionable. Also the other securities issued by the firm have to be modelled as well, thereby reducing the tractability of this approach. As such only the basics of these models were discussed.

Convertible arbitrage has become a popular hedging strategy and as such a few important strategies have been highlighted from the basic delta hedge to more creative gamma and option hedging techniques. Also included is some incite into the "Greeks" of convertible bonds such as the delta, gamma, theta, rho and omicron.

The above CB models capture the important factors, namely credit risk, default risk and calibration but improvements can be made. One such is the negative convexity or gamma that occurs when the CB is in the distressed debt phase. Although incorporation of credit risk does make it clear that the CB value drops below an equivalent straight bond value during this phase[80], its drop in value is not as dramatic as observed in real world situations. A superior model should take this subordination factor into account as was illustrated by the TFk, HW, Tan and Cai and AFV models. The correlation between the interest rates and share price is also an important parameter to consider in developing these models. The piecewise and Quadrinomial model take this into account but the other equity and structural models should also account for this. Another assumption that can be reviewed is the conversion option. In all these models when the conversion value is above the CB price it is assumed that all investors[81] will convert at that time (block conversion). In reality it is not true since there are sequential conversion dates as described by Bühler and Koiziol [16], where only a percentage of investors will choose to convert. The other investors choose to receive the coupon payments associated with the bond to enhance their returns and convert later.

With the introduction of hybrid securities the gap between debt and equity is becoming narrower. These new instruments are difficult to value and require a mixture of both debt and equity methodologies. As financial markets become more sophisticated with this relatively new asset class, additional concepts and approaches will arise assisting in the development of more accurate valuation models. Monte Carlo simulation is proving to be a popular method for valuing securities with uncertain income streams such as American-style options. In essence it requires knowledge of the distribution of the random variables being modelled, so that thousands of samples from the distribution can be drawn. Once these samples have

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[80] This can be seen with the Hung and Wang Default Risk model

[81] The assumption made is that all investors behave rationally and choose to maximize the value of their convertible bond holdings.

been drawn numerous paths of the underlying variables can be simulated. The present value of the security is calculated along each of these numerous paths[82] with the final value being given as the average of the paths. The simulation approach is well suited for modelling discrete coupon and dividend payments, for including realistic dynamics of the underlying state variables, and for taking into account path-dependent call features[83]. As mentioned by Ammann, Kind and Wilde [2] another advantage is that "the relationship between the number of state variables and computing time is almost linear in our Monte Carlo framework and this can become advantageous when multiple state variables need to be modelled." Of late there has been an extensive literature review on using Monte Carlo simulation to value convertible bonds, In the meantime separating the asset into debt and equity parts amid appropriate discount rates, with credit spread sensitivity and default risk, seems to be the latest instalment.

[82] The usual interim and final boundary conditions will apply throughout the life of the security to calculate its value along each simulated path.

[83] An example of path dependency may arise due to early redemption only being allowed when the share price is above a pre-specified barrier for 20 out of the last 30 days.

Appendix

A1 – Parameter Set 1

Stock	
Share price volatility	11%
Risk-free rate	7.00%
Dividend rate	2.50%
Bond	
Face value	100.00
Bond yield	10.00%
Conversion Ratio	2
Coupon rate	8.00%
Coupon frequency	2
Recovery rate	32%
Call/Put	None
No. of time steps	35
Margrabe Parameters	
Continuous coupon	7.84%
Final redemption ratio	1
Bond price volatility	5%
Correlations	
Share / Interest rates	-0.2
Interest rates / Credit spreads	-0.3
Share / Credit spreads	0.15
Vasicek – Interest Rates	
Initial r	6.50%
Theta -r	7.85%
Kappa -r	0.14
Sigma – r	0.24%
Vasicek – Spreads	
Initial s	0.65%
Theta -s	0.79%
Kappa -s	0.14
Sigma – s	0.24%
Safety Premium (K)	1.05
Voluntary Conversion Date (T^*_2)	6 years

A2 – Derivation of Risk Neutral Measure with Default Risk

We are going to first assume that the risk-neutral probability measure is given by $\left\{\lambda, \tilde{p}(1-\lambda), \left(1-\tilde{p}\right)(1-\lambda)\right\}$, which is when the firm is in default, no default share moves up and no default share moves down respectively. If further we know that the CB pays g_u when share moves up, g_d when share moves down and $g_{default}$ when firm defaults, the value of the CB today is:

$$
\begin{aligned}
g &= e^{-r_f \Delta t}\left[g_{default}\lambda + g_u \tilde{p}(1-\lambda) + g_d\left(1-\tilde{p}\right)(1-\lambda)\right] \\
&= e^{-r_f \Delta t}\left[g_{default}\lambda + g_u \frac{e^{r_f \Delta t}/(1-\lambda) - d}{u-d}(1-\lambda) + g_d \frac{u - e^{r_f \Delta t}/(1-\lambda)}{u-d}(1-\lambda)\right] \\
&= e^{-r_f \Delta t}\left[g_{default}\lambda + g_u \frac{e^{r_f \Delta t} - d(1-\lambda)}{u-d} + g_d \frac{u(1-\lambda) - e^{r_f \Delta t}}{u-d}(1-\lambda)\right] \\
&= e^{-r_f \Delta t}\left[g_{default}\lambda + \frac{g_u e^{r_f \Delta t} - g_u d(1-\lambda) + g_d u(1-\lambda) - g_d e^{r_f \Delta t}}{u-d}\right] \\
&= e^{-r_f \Delta t}\left[g_{default}\lambda + \frac{e^{r_f \Delta t}(g_u - g_d) + (1-\lambda)(g_d u - g_u d)}{u-d}\right] \\
&= g_{default}\lambda e^{-r_f \Delta t} + \frac{(g_u - g_d)}{u-d} + + \frac{e^{-r_f \Delta t}(1-\lambda)(g_d u - g_u d)}{u-d}.
\end{aligned}
\qquad (A.1)
$$

By constructing the following portfolio we can prove that the above risk-neutral measure holds true:

- Long a risky bond with a value of x in one year
- Short a riskless bond with a value of y in one year
- Long Δ shares of stock

Since we want to replicate the payoffs of the CB we must have that in each state the portfolio is equal to the CB, thus:

$$\Delta S_u - y + x = g_u \tag{A.2}$$

$$\Delta S_d - y + x = g_d \tag{A.3}$$

$$x\delta - y = g_{default}\,. \tag{A.4}$$

Solving for Δ using equations (A.2) and (A.3), we have that

$$\Delta = \frac{g_u - g_d}{S_u - S_d}\,, \tag{A.5}$$

Substituting (A.4) into (A.5) and solving for x gives us

$$x = \frac{g_u - g_{default} - \Delta S_u}{1-\delta}\,. \tag{A.6}$$

The value of the portfolio at $t=0$ is given by:

$$V = \Delta S - ye^{-r_f \Delta t} + xe^{-r_b \Delta t}\,. \tag{A.7}$$

Substituting equation (A.4) into (A.7) implies

$$\Delta S - x\delta e^{-r_f \Delta t} + xe^{-r_b \Delta t} g_{default} e^{-r_f \Delta t}\,. \tag{A.8}$$

Replacing x with equation (A.6) into (A.8) leads to

$$\Delta S + \frac{g_u - g_{default} - \Delta S_u}{1-\delta}\left(e^{-r_b \Delta t} - \delta e^{-r_f \Delta t}\right) + g_{default} e^{-r_f \Delta t}\,. \tag{A.9}$$

Assuming that the expected cash flows of a risky bond is equal to its face value discounted at the risky rate; we must have the following equation,

$$1e^{-r_f \Delta t}\left(1-\lambda\right) + \delta\lambda e^{-r_f \Delta t} = 1e^{-r_b \Delta t}\,. \tag{A.10}$$

Subtracting $e^{-r_f \Delta t}$ from both sides of (A.10) and simplifying the LHS

$$e^{-r_f \Delta t}(1-\delta)(1-\lambda) = \delta e^{-r_f \Delta t} + e^{-r_b \Delta t}. \qquad \text{(A.11)}$$

Inserting equation (A.11) into (A.9) and simplifying, we have

$$\Delta S + (g_u - \Delta S_u)(1-\lambda) e^{-r_f \Delta t} + g_{default} e^{-r_f \Delta t} \lambda. \qquad \text{(A.12)}$$

Finally inserting equation (A.5) for Δ and rearranging terms proves our risk-neutral measure, which is equal to (A.1),

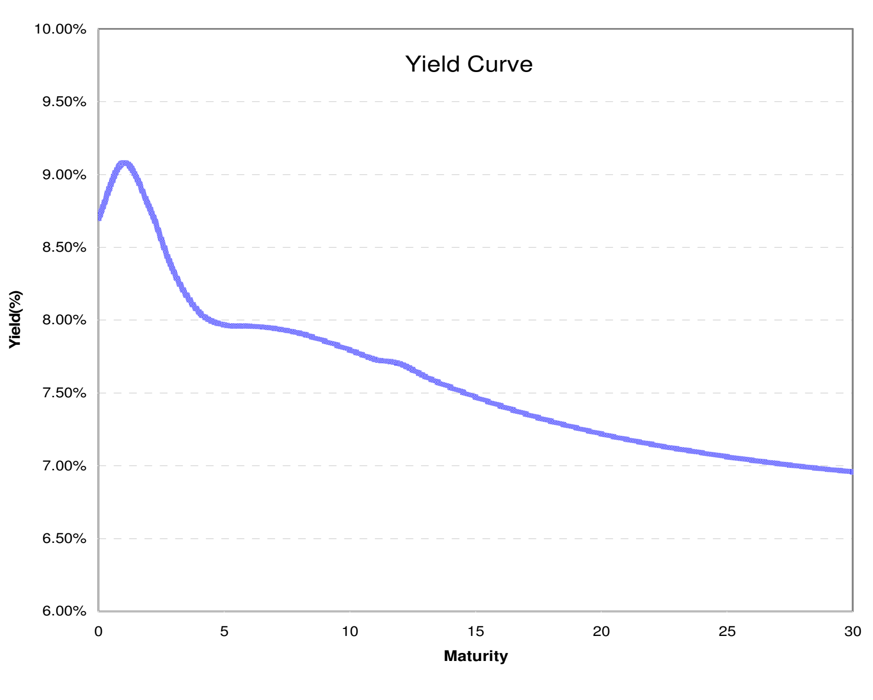
$$\frac{g_u - g_d}{u-d} + e^{-r_f \Delta t}\left(\frac{ug_d - dg_u}{u-d}\right)(1-\lambda) + e^{-r_f \Delta t} g_{default} \lambda . \qquad \text{(A.13)}$$

A3 - Parameter Set 2

Stock	
Share price volatility	30%
Risk-free rate	4.35%
Dividend rate	0.23%
Bond	
Face value	100
Bond yield	4.50%
Coupon rate	0.50%
Coupon frequency	2
Recovery rate	40%
No. of time steps	30
Margrabe Parameters	
Continuous coupon	0.5%
Final redemption ratio	1
Bond price volatility	5%
Correlations	
Share / Interest rates	-0.2
Interest rates / Credit spreads	-0.3
Share / Credit spreads	0.15
Vasicek – Interest Rates	
Initial r	3.81%
Theta -r	4.85%
Kappa -r	0.14
Sigma – r	0.24%
Vasicek – Spreads	
Initial s	0.38%
Theta -s	0.48%
Kappa -s	0.14
Sigma – s	0.24%
Credit Spreads	
Current Spread (bps)	400
Min Spread (bps)	200
Trigger Share Price	25
k-factor	1.5
Safety Premium (K)	1.05
Voluntary Conversion Date (T^*_2)	24 March 2010

A4 – Yield Curve for Quadrinomial Tree Model (Chapter 5)

Term (Year)	Spot Curve
0	8.70%
1	9.08%
2	8.79%
3	8.33%
4	8.06%
5	7.97%
6	7.96%
7	7.94%
8	7.91%
9	7.86%
10	7.80%
11	7.73%
12	7.70%
13	7.61%
14	7.54%
15	7.47%
16	7.41%
17	7.36%
18	7.31%
19	7.26%
20	7.22%
21	7.18%
22	7.15%
23	7.12%
24	7.09%
25	7.06%
26	7.04%
27	7.02%
28	6.99%
29	6.98%
30	6.96%
31	6.94%



A5 – Convertible Arbitrage Return Breakdown

Bullish Gamma Hedge

Hedge Ratio	**0.54**				
Risk-Free Rate	**7%**				
Risk Tolerance	**0.7**				
Curr Implied Vol	**10%**				
End Implied Vol	**10%**				
	t=5		t=6		t=5
	Downside Trigger Price		**Current Price**		**Upside Trigger Price**
Share Price	35.00	-30%	50.00	30%	65.00
CB Price	97.77		131.84		158.30
Delta	0.28		0.77		0.87
Gamma	0.040		0.013		0.002
Vega	0.246		0.199		0.037
Long Bonds	100.00	bonds @	131.84	Total invest	13,183.93
Borrow	-			Net invest	13,183.93
Short Stock	-141.00	shares @	53.00		-7,473.00
P/L CB	-3,407.35		-		2,646.21
P/L Share	2,115.00		-		-2,115.00
CB Income	800.00		800.00		800.00
Share Div	-		-		-
Short Credit Int	375.06		523.11		671.16
Margin Int	-		-		-
Total P/L	-117.29		1,323.11		2,002.37
Bullish Gamma Tilt	**-0.89%**		**10.04%**		**15.19%**
Unhedged CB	-19.78%		6.07%		26.14%

Table A- 1 – Return breakdown of a bullish gamma hedge, without leverage, when the share price increases and decreases by 30%.

Bullish Gamma Hedge (with leverage)

Hedge Ratio	**0.54**				
Risk-Free Rate	**7%**				
Risk Tolerance	**0.7**				
Curr Implied Vol	**10%**				
End Implied Vol	**10%**				
	t=5		t=6		t=5
	Downside Trigger Price		**Current Price**		**Upside Trigger Price**
Share Price	35.00	-30%	50.00	30%	65.00
CB Price	97.77		131.84		158.30
Delta	0.28		0.77		0.87
Gamma	0.040		0.013		0.002
Vega	0.246		0.199		0.037
Long Bonds	100.00	bonds @	131.84	Total invest	13,183.93
Borrow	85% of LMV @ 7%		11,206.34	Net invest	1,977.59
Short Stock	-141.00	shares @	53.00		-7,473.00
P/L CB	-3,407.35		-		2,646.21
P/L Share	2,115.00		-		-2,115.00
CB Income	800.00		800.00		800.00
Share Div	-		-		-
Short Credit Int	375.06		523.11		671.16
Margin Int	-784.44		-784.44		-784.44
Total P/L	-901.74		538.67		1,217.92
Bullish Gamma Tilt (with leverage)	**-45.60%**		**27.24%**		**61.59%**
Unhedged CB	-19.78%		6.07%		26.14%

Table A- 2 - Return breakdown of a bullish gamma hedge, with leverage, when the share price increases and decreases by 30%.

Bearish Gamma Hedge

Hedge Ratio	**1.00**				
Risk-Free Rate	**7%**				
Risk Tolerance	**1.3**				
Curr Implied Vol	**10%**				
End Implied Vol	**10%**				
	t=5		t=6		t=5
	Downside Trigger Price		**Current Price**		**Upside Trigger Price**
Share Price	35.00	-30%	50.00	30%	65.00
CB Price	97.77		131.84		158.30
Delta	0.28		0.77		0.87
Gamma	0.040		0.013		0.002
Vega	0.246		0.199		0.037
Long Bonds	100.00	bonds @	131.84	Total invest	13,183.93
Borrow	-			Net invest	13,183.93
Short Stock	-262.00	shares @	53.00		-13,886.00
P/L CB	-3,407.35		-		2,646.21
P/L Share	3,930.00		-		-3,930.00
CB Income	800.00		800.00		800.00
Share Div	-		-		-
Short Credit Int	696.92		972.02		1,247.12
Margin Int	-		-		-
Total P/L	2,019.57		1,772.02		763.33
Bearish Gamma Tilt	**15.32%**		**13.44%**		**5.79%**
Unhedged CB	-19.78%		6.07%		26.14%

Table A- 3 - Return breakdown of a bearish gamma hedge, without leverage, when the share price increases and decreases by 30%.

Bearish Gamma Hedge (with leverage)

Hedge Ratio	**1.00**				
Risk-Free Rate	**7%**				
Risk Tolerance	**1.3**				
Curr Implied Vol	**10%**				
End Implied Vol	**10%**				
	t=5		t=6		t=5
	Downside Trigger Price		**Current Price**		**Upside Trigger Price**
Share Price	35.00	-30%	50.00	30%	65.00
CB Price	97.77		131.84		158.30
Delta	0.28		0.77		0.87
Gamma	0.040		0.013		0.002
Vega	0.246		0.199		0.037
Long Bonds	100.00	bonds @	131.84	Total invest	13,183.93
Borrow	85% of LMV @ 7%		11,206.34	Net invest	1,977.59
Short Stock	-262.00	shares @	53.00		-13,886.00
P/L CB	-3,407.35		-		2,646.21
P/L Share	3,930.00		-		-3,930.00
CB Income	800.00		800.00		800.00
Share Div	-		-		-
Short Credit Int	696.92		972.02		1,247.12
Margin Int	-784.44		-784.44		-784.44
Total P/L	1,235.12		987.58		-21.12
Bearish Gamma Tilt (with leverage)	**62.46%**		**49.94%**		**-1.07%**
Unhedged CB	-19.78%		6.07%		26.14%

Table A- 4 – Return breakdown of a bearish gamma hedge, with leverage, when the share price increases and decreases by 30%.

Convertible Covered Call Write

Common Stock	
Price	60
Volatility	10%
Delta	0.84
Risk-Free Rate	7.00%
Dividend Yield	2.50%
Convertible Bond	
Price	152.86
Quantity	100
Adj. Strike	14.093
Short Call	
Price	35.08
Quantity	65
Strike	25
Time to Expiration	0.33
Implied Vol	10%
Risk Tolerance	0.6
Share Move	10%
Hedge Ratio	0.50
Holding Period	0.331
Start (time to mat)	6.00
End (time to mat)	5.67

Share Price	48.6	54.0	60.0	66.0	72.6
CB Price	128.68	140.88	153.92	166.23	179.63
% Chg CB	-15.8%	-7.8%	0.7%	8.7%	17.5%
Call Price	23.60	29.00	35.00	41.00	47.60
% Chg Call	-32.7%	-17.3%	-0.2%	16.9%	35.7%
P&L Convertible	-2418.84	-1198.22	105.99	1336.57	2676.55
Short Interest	53.20	53.20	53.20	53.20	53.20
P&L Call	746.02	395.04	5.07	-384.91	-813.88
Total P&L	-1619.62	-749.98	164.27	1004.87	1915.88
ROI	**-10.6%**	**-4.9%**	**1.1%**	**6.6%**	**12.5%**
Annualized	-20.1%	-9.6%	2.2%	13.6%	26.6%

Table A- 5 - Return breakdown of a convertible covered call write strategy when share price increases and decreases by 10% and 20%.

Convertible Covered Call Write / Short Stock

Short Stock	
Price	60
Quantity	128
Volatility	10%
Delta	0.84
Risk-Free Rate	7.00%
Dividend Yield	2.50%
Convertible Bond	
Price	152.86
Quantity	100
Adj. Strike	14.093
Short Call	
Price	35.08
Quantity	65
Strike	25
Time to Expiration	0.33
Implied Vol	10%
Risk Tolerance	0.6
Share Move	10%
Hedge Ratio	0.50
Holding Period	0.33
Start (time to mat)	6.00
End (time to mat)	5.67

Share Price	48.6	54.0	60.0	66.0	72.6
% Chg Share	-19.0%	-10.0%	0.0%	10.0%	21.0%
CB Price	128.68	140.88	153.92	166.23	179.63
% Chg CB	-15.8%	-7.8%	0.7%	8.7%	17.5%
Call Price	23.60	29.00	35.00	41.00	47.60
% Chg Call	-32.7%	-17.3%	-0.2%	16.9%	35.7%
P&L Convertible	-2418.84	-1198.22	105.99	1336.57	2676.55
Short Interest	232.40	232.40	232.40	232.40	232.40
P&L Call	746.02	395.04	5.07	-384.91	-813.88
P&L Share	1459.20	768.00	0.00	-768.00	-1612.80
Total P&L	18.78	197.22	343.47	416.07	482.28
ROI	**0.1%**	**1.3%**	**2.2%**	**2.7%**	**3.2%**
Annualized	0.2%	2.6%	4.5%	5.5%	6.4%

Table A- 6 - Return breakdown of a convertible covered call write strategy, with a short stock position when share price increases and decreases by 10% and 20%.

Short Stock	
Price	60
Quantity	152
Volatility	10%
Delta	0.99
Risk-Free Rate	7.00%
Dividend Yield	2.50%
Convertible Bond	
Price	152.86
Quantity	100
Adj. Strike	14.093
Long Put	
Price	0.000
Quantity	90
Strike	25
Time to Expiration	0.33
Implied Vol	10%
Risk Tolerance	0.6
Share Move	10%
Hedge Ratio	0.59
Holding Period	0.33
Start (time to mat)	6.00
End (time to mat)	5.67

Convertible Covered Long Put / Short Stock

Share Price	48.6	54.0	60.0	66.0	72.6
% Chg Share	-19.0%	-10.0%	0.0%	10.0%	21.0%
CB Price	128.68	140.88	153.92	166.23	179.63
% Chg CB	-15.8%	-7.8%	0.7%	8.7%	17.5%
Put Price	0.00	0.00	0.00	0.00	0.00
% Chg Put	-100.0%	-100.0%	-100.0%	-100.0%	-100.0%
P&L Convertible	-2418.84	-1198.22	105.99	1336.57	2676.55
Short Interest	212.80	212.80	212.80	212.80	212.80
P&L Put	0.00	0.00	0.00	0.00	0.00
P&L Share	1732.80	912.00	0.00	-912.00	-1915.20
Total P&L	-473.24	-73.42	318.79	637.37	974.15
ROI	**-3.1%**	**-0.5%**	**2.1%**	**4.2%**	**6.4%**
Annualized	-6.1%	-1.0%	4.2%	8.5%	13.2%

Table A- 7 - Return breakdown of a convertible covered long put strategy, with a short stock position when share price increases and decreases by 10% and 20%.

A6 – VBA source code for several convertible bond models

A6.1 Tsiveriotis and Fernandes model

```
Option Explicit

'//Valuation of Convertible Bond
   'Parameters:
   'S_0: Stock Price at time 0
   'Sigma: volatility of stock price
   'IntRate: risk free rate
   'IntRateRisky: required bond yield (to discount debt component of CB)
   ' DividendRateContinuous: dividend rate p.a. (for cont dividend model)
   'Conversion_Ratio: number of stocks per face value CB
   'Maturity: time to CB maturity
   'Face Value CB
   'Coupon_rate (total annual %)
   'Coupon Frequency p.a.
   'Call_Value (optional): value at which bond can be called by another company (at any time)
   'Put_Value (optional): value at which bond can be redeemed by holder (at any time)
   'Call_Start_Time (optional): when the call option becomes active
   'Put_Start_Time (optional): when the put option becomes active

   'No_t_Steps(optional): number of discrete t steps in grid
   'Optional AmeEurFlag: European("e") - default or American "a" with early conversion option possible.
   ''InitialNoConversionPeriod: if American exercise option, time at which early exercise begins

   'Version (optional)
   '-"Full": full tree with
   '1st Row: stock price
   '2nd row: equity component (if CB is converted)
   '3rd row: debt component (if CB is NOT converted)
   '4th row: value of CB
   '-"Node": all components of initial node omitted or any value - CB value only

'*************************************************************************************************************

Function ConvertibleBinomial( _
      S_0 As Double, _
      sigma As Double, _
      IntRate As Double, _
      IntRateRisky As Double, _
      DividendRateContinuous As Double, _
      Conversion_Ratio As Single, _
      Maturity As Double, _
      Face_value As Double, _
      coupon_rate As Double, _
      Optional coupon_frequency As Integer = 1, _
      Optional call_value As Double = 0, _
      Optional put_value As Double = 0, _
      Optional No_t_Steps As Single, _
      Optional AmeEurFlag As String = "e", _
      Optional InitialNoConversionPeriod As Double, _
      Optional Version As String = "", _
      Optional Call_Start_Time As Double = 0.00000001, _
      Optional Put_Start_Time As Double = 0.00000001)

On Error GoTo error_handling
Dim u As Double, D As Double, p As Double
Dim ConversionValue As Double
Dim delta_t As Double, df As Double, DfRisky As Double
Dim coupon_time As Double, i As Integer, k As Integer

ReDim nodevalue(0 To No_t_Steps, 0 To No_t_Steps, 0 To 2)

'defining parameters tobe used
delta_t = Maturity / No_t_Steps
u = Exp(sigma * Sqr(delta_t))
D = 1 / u
p = (Exp((IntRate - DividendRateContinuous) * delta_t) - D) / (u - D)
df = Exp(-(IntRate - DividendRateContinuous) * delta_t): DfRisky = Exp(-IntRateRisky * delta_t)

'Set up vector of PV of coupons to be added to the debt component to particular column of the tree
ReDim coupon_schedule(0 To No_t_Steps)
coupon_schedule(No_t_Steps) = coupon_rate / coupon_frequency
coupon_time = Maturity - 1 / coupon_frequency
For k = No_t_Steps - 1 To 0 Step -1
   Do While coupon_time >= k * delta_t And coupon_time > 0
      coupon_schedule(k) = coupon_schedule(k) + _
      Exp(-IntRateRisky * (coupon_time - k * delta_t)) * _
      coupon_rate / coupon_frequency
   coupon_time = coupon_time - 1 / coupon_frequency
   Loop
Next k
'initialize final (right) nodes of tree
For i = 0 To No_t_Steps
   nodevalue(i, No_t_Steps, 0) = S_0 * u ^ i * D ^ (No_t_Steps - i)
   ConversionValue = Conversion_Ratio * nodevalue(i, No_t_Steps, 0) / Face_value
   If ConversionValue > 1 + coupon_schedule(No_t_Steps) Then
      nodevalue(i, No_t_Steps, 1) = ConversionValue ' set final equity values
   Else
      nodevalue(i, No_t_Steps, 2) = 1 + coupon_schedule(No_t_Steps) ' set final debt values
   End If
Next i

'Loop thru tree back
```

```
For k = No_t_Steps - 1 To 0 Step -1
   For i = 0 To k
      nodevalue(i, k, 0) = S_0 * u ^ i * D ^ (k - i)
      ConversionValue = Conversion_Ratio * nodevalue(i, k, 0) / Face_value

         nodevalue(i, k, 1) = (p * nodevalue(i + 1, k + 1, 1) + _
                     (1 - p) * nodevalue(i, k + 1, 1)) * df 'Equity component
         nodevalue(i, k, 2) = coupon_schedule(k) + _
                     (p * nodevalue(i + 1, k + 1, 2) + _
                     (1 - p) * nodevalue(i, k + 1, 2)) * DfRisky 'Debt component

         'If American and inside the early exercise window, check whther early conversion increases value
         If AmeEurFlag = "a" And InitialNoConversionPeriod <= k * delta_t Then

            If ConversionValue > (nodevalue(i, k, 1) + nodevalue(i, k, 2)) Then
               nodevalue(i, k, 1) = ConversionValue         'Equity component
               nodevalue(i, k, 2) = 0                       'Debt component
            End If
         End If

         'Test whether company should call the bond, i.e. force conversion
         'call if...
         '-conversion value exceeds the call value
         '-the CB value at particular node exceeds the conversion value
         '- cannot call at time 0 or maturity & only after start time
         'NOTE: The TF model treats the call as equity part, not debt part,whereas the put is treated as debt
         If (call_value > 0) And ConversionValue > call_value And k > 0 _
                    And (nodevalue(i, k, 1) + nodevalue(i, k, 2)) > ConversionValue _
                    And InitialNoConversionPeriod <= k * delta_t _
                    And delta_t * k >= Call_Start_Time Then
                 nodevalue(i, k, 1) = ConversionValue  'Equity component
                 nodevalue(i, k, 2) = 0               'Debt component
         ElseIf (call_value > 0) And (nodevalue(i, k, 1) + nodevalue(i, k, 2)) > call_value _
                      And k > 0 And call_value > ConversionValue _
                      And delta_t * k >= Call_Start_Time Then
                 nodevalue(i, k, 1) = call_value    'Equity component
                 nodevalue(i, k, 2) = 0            'Debt component
         End If
         'Test whether holder should put bond
         'cannot put at time 0 or maturity & only after start time
         If (put_value > 0) And (nodevalue(i, k, 1) + nodevalue(i, k, 2)) < put_value And k > 0 _
                   And delta_t * k >= Put_Start_Time And put_value > ConversionValue Then
                 nodevalue(i, k, 1) = 0               'Equity component
                 nodevalue(i, k, 2) = put_value        'Debt component
         End If
      Next i
   Next k

If Version = "Full" And No_t_Steps > 35 Then Version = "Node"

Select Case Version
Case "Full"
   'initialize FullTree() - conditional formatting needs "" values to blank out non relevant cells
   ReDim fulltree(0 To (4 * (No_t_Steps + 1) - 1), 0 To No_t_Steps)
   For k = 0 To UBound(fulltree, 2)
      For i = 0 To UBound(fulltree, 1)
         fulltree(i, k) = ""
      Next i
   Next k

   'write to FullTree()
   For k = No_t_Steps To 0 Step -1
      For i = 0 To k
      fulltree(4 * (k - i), k) = nodevalue(i, k, 0)
      fulltree(4 * (k - i) + 1, k) = nodevalue(i, k, 1)
      fulltree(4 * (k - i) + 2, k) = nodevalue(i, k, 2)
      fulltree(4 * (k - i) + 3, k) = nodevalue(i, k, 1) + nodevalue(i, k, 2)
      Next i
   Next k
   ConvertibleBinomial = fulltree
Case "Node"
   ReDim initialnode(0 To 3) As Double
   initialnode(0) = S_0
   initialnode(1) = nodevalue(0, 0, 1)
   initialnode(2) = nodevalue(0, 0, 2)
   initialnode(3) = initialnode(1) + initialnode(2)
   ConvertibleBinomial = Application.Transpose(initialnode)
Case Else
   ConvertibleBinomial = nodevalue(0, 0, 1) + nodevalue(0, 0, 2)
End Select

Exit Function
error_handling:
   ConvertibleBinomial = "Error #" & str(Err.Number) & "was generated by " _
      & Err.Source & Chr(13) & Err.Description

End Function


'find PV of coupons between the time ticks
Function couponschedule(Maturity As Double, _
                  No_t_Steps As Double, _
                  delta_t As Double, _
                  coupon_rate As Double, _
                  coupon_frequency As Double, _
                  IntRateRisky As Double)

Dim coupon_time As Double, k As Integer, i As Integer
ReDim coupon_schedule(0 To No_t_Steps)

coupon_schedule(No_t_Steps) = coupon_rate / coupon_frequency
```

```
'check if coupon falls between 2 timeticks and if so discount to earlier time
coupon_time = Maturity - 1 / coupon_frequency
For k = No_t_Steps - 1 To 0 Step -1
    Do While coupon_time >= k * delta_t And coupon_time > 0
        coupon_schedule(k) = coupon_schedule(k) + _
        Exp(-IntRateRisky * (coupon_time - k * delta_t)) * _
        coupon_rate / coupon_frequency
    coupon_time = coupon_time - 1 / coupon_frequency
    Loop
Next k

couponschedule = coupon_schedule
End Function
```

A6.2 Goldman-Sachs Conversion Probability Model

```
Option Explicit

'//Valuation of Convertible Bond using Goldman-Sachs Conversion Prob Tree
    'Parameters:
    'S_0: Stock Price at time 0
    'Sigma: volatility of stock price
    'IntRate: risk free rate
    'IntRateRisky: required bond yield (to discount debt component of CB)
    ' DividendRateContinuous: dividend rate p.a. (for cont dividend model)
    'Conversion_Ratio: number of stocks per face value CB
    'Maturity: time to CB maturity
    'Face Value CB
    'Coupon_rate (total annual %)
    'Coupon Frequency p.a.
    'Call_Value (optional): value at which bond can be called by another company (at any time)
    'Put_Value (optional): value at which bond can be redeemed by holder (at any time)
    'Call_Start_Time (optional): when the call option becomes active
    'Put_Start_Time (optional): when the put option becomes active

    'No_t_Steps(optional): number of discrete t steps in grid
    'Optional AmeEurFlag: European("e") - default or American "a" with early conversion option possible.
    ''InitialNoConversionPeriod: if American exercise option, time at which early exercise begins

    'Version (optional)
    '-"Full": full tree with
    '1st Row: stock price
    '2nd row: equity component (if CB is converted)
    '3rd row: debt component (if CB is NOT converted)
    '4th row: value of CB
    '-"Node": all components of initial node omitted or any value - CB value only

'*************************************************************************************************************

Function ConvBinomial_Goldman( _
        S_0 As Double, _
        sigma As Double, _
        IntRate As Double, _
        IntRateRisky As Double, _
        DividendRateContinuous As Double, _
        Conversion_Ratio As Single, _
        Maturity As Double, _
        Face_value As Double, _
        coupon_rate As Double, _
        Optional coupon_frequency As Integer = 1, _
        Optional call_value As Double = 0, _
        Optional put_value As Double = 0, _
        Optional No_t_Steps As Single, _
        Optional AmeEurFlag As String = "e", _
        Optional InitialNoConversionPeriod As Double, _
        Optional Version As String, _
        Optional Call_Start_Time As Double = 0.00000001, _
        Optional Put_Start_Time As Double = 0.00000001)

On Error GoTo error_handling
Dim u As Double, D As Double, p As Double
Dim ConversionValue As Double
Dim delta_t As Double, df As Double, DfRisky_u As Double, DfRisky_d As Double
Dim coupon_time As Double, i As Integer, k As Integer

ReDim nodevalue(0 To No_t_Steps, 0 To No_t_Steps, 0 To 3)

'defining parameters to be used
delta_t = Maturity / No_t_Steps
u = Exp(sigma * Sqr(delta_t))
D = 1 / u
p = (Exp((IntRate - DividendRateContinuous) * delta_t) - D) / (u - D)

'Set up vector of PV of coupons to be added to the debt component to particular column of the tree
ReDim coupon_schedule(0 To No_t_Steps)
coupon_schedule(No_t_Steps) = coupon_rate / coupon_frequency
coupon_time = Maturity - 1 / coupon_frequency
For k = No_t_Steps - 1 To 0 Step -1
    Do While coupon_time >= k * delta_t And coupon_time > 0
        coupon_schedule(k) = coupon_schedule(k) + _
        Exp(-IntRateRisky * (coupon_time - k * delta_t)) * _
        coupon_rate / coupon_frequency
    coupon_time = coupon_time - 1 / coupon_frequency
    Loop
Next k
'initialize final (right) nodes of tree
For i = 0 To No_t_Steps
    nodevalue(i, No_t_Steps, 0) = S_0 * u ^ i * D ^ (No_t_Steps - i)
```

```
    ConversionValue = Conversion_Ratio * nodevalue(i, No_t_Steps, 0) / Face_value
    If ConversionValue > 1 + coupon_schedule(No_t_Steps) Then
        nodevalue(i, No_t_Steps, 1) = ConversionValue ' set final values
        nodevalue(i, No_t_Steps, 2) = 1  ' set final conversion prob to either 1 or 0
        nodevalue(i, No_t_Steps, 3) = IntRate * nodevalue(i, No_t_Steps, 2) + _
                              IntRateRisky * (1 - nodevalue(i, No_t_Steps, 2)) 'Conv adjusted discount rate
    Else
        nodevalue(i, No_t_Steps, 1) = 1 + coupon_schedule(No_t_Steps) ' set final values
        nodevalue(i, No_t_Steps, 2) = 0  ' set final conversion prob to either 1 or 0
        nodevalue(i, No_t_Steps, 3) = IntRate * nodevalue(i, No_t_Steps, 2) + _
                              IntRateRisky * (1 - nodevalue(i, No_t_Steps, 2)) 'Conv adjusted discount rate
    End If
Next i

'Loop thru tree backwards
For k = No_t_Steps - 1 To 0 Step -1
    For i = 0 To k
        nodevalue(i, k, 0) = S_0 * u ^ i * D ^ (k - i)
        ConversionValue = Conversion_Ratio * nodevalue(i, k, 0) / Face_value

            nodevalue(i, k, 2) = p * nodevalue(i + 1, k + 1, 2) + _
                          (1 - p) * nodevalue(i, k + 1, 2) 'Rollback Conversion Prob
            nodevalue(i, k, 3) = IntRate * nodevalue(i, k, 2) + _
                          IntRateRisky * (1 - nodevalue(i, k, 2)) 'Rollback Conversion Adj Discount Rate
            DfRisky_u = Exp(-nodevalue(i + 1, k + 1, 3) * delta_t)       ' Prob adjusted discount rate (up)
            DfRisky_d = Exp(-nodevalue(i, k + 1, 3) * delta_t)           ' Prob adjusted discount rate (down)
            nodevalue(i, k, 1) = coupon_schedule(k) + _
                          (p * DfRisky_u * nodevalue(i + 1, k + 1, 1) + _
                          (1 - p) * DfRisky_d * nodevalue(i, k + 1, 1)) 'Rollback CB value

            'If American and inside the early exercise window, check whther early conversion increases value
            If AmeEurFlag = "a" And InitialNoConversionPeriod <= k * delta_t Then

                If ConversionValue > nodevalue(i, k, 1) Then
                    nodevalue(i, k, 1) = ConversionValue          'Check for conversion
                End If
            End If

            'Test whether company should call the bond, i.e. force conversion
            'call if...
            '-conversion value exceeds the call value
            '-the CB value at particular node exceeds the conversion value
            '- cannot call at time 0 or maturity & only after start time
            If (call_value > 0) And ConversionValue > call_value And k > 0 _
                          And nodevalue(i, k, 1) > ConversionValue _
                          And InitialNoConversionPeriod <= k * delta_t _
                          And delta_t * k >= Call_Start_Time Then
                      nodevalue(i, k, 1) = ConversionValue  'Converted
            ElseIf (call_value > 0) And nodevalue(i, k, 1) > call_value _
                            And k > 0 And call_value > ConversionValue _
                            And delta_t * k >= Call_Start_Time Then
                      nodevalue(i, k, 1) = call_value        'Called
            End If
            'Test whether holder should put bond
            'cannot put at time 0 or maturity & only after start time
            If (put_value > 0) And nodevalue(i, k, 1) < put_value And k > 0 _
                         And delta_t * k >= Put_Start_Time And put_value > ConversionValue Then
                      nodevalue(i, k, 1) = put_value        'CB is Put by holder
            End If
        Next i
    Next k

If Version = "Full" And No_t_Steps > 35 Then Version = "Node"

Select Case Version
Case "Full"
    'initialize FullTree() - conditional formatting needs "" values to blank out non relevant cells
    ReDim fulltree(0 To (4 * (No_t_Steps + 1) - 1), 0 To No_t_Steps)
    For k = 0 To UBound(fulltree, 2)
        For i = 0 To UBound(fulltree, 1)
            fulltree(i, k) = ""
        Next i
    Next k

'check for immediate conversion
nodevalue(0, 0, 1) = Application.Max(nodevalue(0, 0, 1), Conversion_Ratio * S_0 / Face_value)
    'write to FullTree()
    For k = No_t_Steps To 0 Step -1
        For i = 0 To k
        fulltree(4 * (k - i), k) = nodevalue(i, k, 0)
        fulltree(4 * (k - i) + 1, k) = nodevalue(i, k, 3)
        fulltree(4 * (k - i) + 2, k) = nodevalue(i, k, 2)
        fulltree(4 * (k - i) + 3, k) = nodevalue(i, k, 1)
        Next i
    Next k
    ConvBinomial_Goldman = fulltree
Case "Node"
    ReDim initialnode(0 To 3) As Double
    initialnode(0) = S_0
    initialnode(1) = nodevalue(0, 0, 3)
    initialnode(2) = nodevalue(0, 0, 2)
    initialnode(3) = Application.Max(initialnode(1), Conversion_Ratio * S_0 / Face_value)
    ConvBinomial_Goldman = Application.Transpose(initialnode)
Case Else
    ConvBinomial_Goldman = Application.Max(nodevalue(0, 0, 3), Conversion_Ratio * S_0 / Face_value)
End Select

Exit Function
error_handling:
    ConvBinomial_Goldman = "Error #" & str(Err.Number) & "was generated by " _
        & Err.Source & Chr(13) & Err.Description
```

End Function

A6.3 Tsiverotis and Fernandes *k*-factor model

```
Option Explicit

'//Valuation of Convertible Bond with k-factor credit spread methodology for negative convexity
'at extremely low share values
    'Parameters:
    'S_0: Stock Price at time 0
    'Sigma: volatility of stock price
    'IntRate: risk free rate
    'IntRateRisky: required bond yield (to discount debt component of CB)
    ' DividendRateContinuous: dividend rate p.a. (for cont dividend model)
    'Conversion_Ratio: number of stocks per face value CB
    'Maturity: time to CB maturity
    'Face Value CB
    'Coupon_rate (total annual %)
    'Coupon Frequency p.a.
    'Call_Value (optional): value at which bond can be called by another company (at any time)
    'Put_Value (optional): value at which bond can be redeemed by holder (at any time)
    'Call_Start_Time (optional): when the call option becomes active
    'Put_Start_Time (optional): when the put option becomes active

    'No_t_Steps(optional): number of discrete t steps in grid
    'Optional AmeEurFlag: European("e") - default or American "a" with early conversion option possible.
    '"InitialNoConversionPeriod: if American exercise option, time at which early exercise begins

    'Version (optional)
    '-"Full": full tree with
    '1st Row: stock price
    '2nd row: equity component
    '3rd row: debt component
    '4th row: Risky discount rate (used to discount debt component)
    '5th row: value of CB
    '-"Node": all components of initial node omitted or any value - CB value only

'***************************************************************************************************************

Function ConvertibleBinomial_CreditSpread( _
        S_0 As Double, _
        sigma As Double, _
        IntRate As Double, _
        Credit_Spread As Object, _
        DividendRateContinuous As Double, _
        Conversion_Ratio As Single, _
        Maturity As Double, _
        Face_value As Double, _
        coupon_rate As Double, _
        Optional coupon_frequency As Integer = 1, _
        Optional call_value As Double = 0, _
        Optional put_value As Double = 0, _
        Optional No_t_Steps As Single, _
        Optional AmeEurFlag As String = "e", _
        Optional InitialNoConversionPeriod As Double, _
        Optional Version As String = "", _
        Optional Call_Start_Time As Double = 0.00000001, _
        Optional Put_Start_Time As Double = 0.00000001)

On Error GoTo error_handling
Dim u As Double, D As Double, p As Double
Dim ConversionValue As Double
Dim delta_t As Double, df As Double, DfRisky As Double, IntRateRisky
Dim coupon_time As Double, i As Integer, k As Integer
Dim Curr_Spread As Double, min_spread As Double, k_factor As Double, trig_share_price As Double

ReDim nodevalue(0 To No_t_Steps, 0 To No_t_Steps, 0 To 3), IntRateRisky(0 To No_t_Steps, 0 To No_t_Steps)

'defining parameters to be used
Curr_Spread = Credit_Spread(1): min_spread = Credit_Spread(2): trig_share_price = Credit_Spread(3): k_factor = Credit_Spread(4)
delta_t = Maturity / No_t_Steps
u = Exp(sigma * Sqr(delta_t))
D = 1 / u
p = (Exp((IntRate - DividendRateContinuous) * delta_t) - D) / (u - D)
df = Exp(-(IntRate - DividendRateContinuous) * delta_t)


'Set up vector of PV of coupons to be added to the debt component to particular column of the tree
ReDim coupon_schedule(0 To No_t_Steps)
coupon_schedule(No_t_Steps) = coupon_rate / coupon_frequency
coupon_time = Maturity - 1 / coupon_frequency

'initialise binomial share tree and Risky Interest rate tree dependent on the stock level using k-factor spread model
For k = No_t_Steps - 1 To 0 Step -1
    For i = 0 To k
    nodevalue(i, k, 0) = S_0 * u ^ i * D ^ (k - i)
    IntRateRisky(i, k) = IntRate + 1 / 10000 * (min_spread + (Curr_Spread - min_spread) * (nodevalue(i, k, 0) / trig_share_price) ^ (-k_factor))
    nodevalue(i, k, 3) = IntRateRisky(i, k)
    Next i
Next k

'PV coupons for each time tick (assumes simple constant spread for discounting coupons at each time tick)
Dim Const_RiskyRate As Double
Const_RiskyRate = Credit_Spread(1) / 10000 + IntRate
For k = No_t_Steps - 1 To 0 Step -1
    Do While coupon_time >= k * delta_t And coupon_time > 0
        coupon_schedule(k) = coupon_schedule(k) + _
        Exp(-Const_RiskyRate * (coupon_time - k * delta_t)) * _
```

```
        coupon_rate / coupon_frequency
    coupon_time = coupon_time - 1 / coupon_frequency
    Loop
Next k
'initialize final (right) nodes of tree
For i = 0 To No_t_Steps
    nodevalue(i, No_t_Steps, 0) = S_0 * u ^ i * D ^ (No_t_Steps - i)
    ConversionValue = Conversion_Ratio * nodevalue(i, No_t_Steps, 0) / Face_value
    If ConversionValue > 1 + coupon_schedule(No_t_Steps) Then
        nodevalue(i, No_t_Steps, 1) = ConversionValue ' set final equity values
    Else
        nodevalue(i, No_t_Steps, 2) = 1 + coupon_schedule(No_t_Steps) ' set final debt values
    End If
Next i

'Loop thru tree back
For k = No_t_Steps - 1 To 0 Step -1
    For i = 0 To k
        nodevalue(i, k, 0) = S_0 * u ^ i * D ^ (k - i)
        ConversionValue = Conversion_Ratio * nodevalue(i, k, 0) / Face_value
        'IntRateRisky = IntRate + 1 / 10000 * (Min_Spread + (Curr_Spread - Min_Spread) * (nodevalue(i, k, 0) / Trig_share_price) ^ (-k_factor))
        DfRisky = Exp(-IntRateRisky(i, k) * delta_t)
        ConversionValue = Conversion_Ratio * nodevalue(i, k, 0) / Face_value

            nodevalue(i, k, 1) = (p * nodevalue(i + 1, k + 1, 1) + _
                         (1 - p) * nodevalue(i, k + 1, 1)) * df 'Equity component
            nodevalue(i, k, 2) = coupon_schedule(k) + _
                         (p * nodevalue(i + 1, k + 1, 2) + _
                         (1 - p) * nodevalue(i, k + 1, 2)) * DfRisky 'Debt component

            'If American and inside the early exercise window, check whther early conversion increases value
            If AmeEurFlag = "a" And InitialNoConversionPeriod <= k * delta_t Then

                If ConversionValue > (nodevalue(i, k, 1) + nodevalue(i, k, 2)) Then
                    nodevalue(i, k, 1) = ConversionValue          'Equity component
                    nodevalue(i, k, 2) = 0                        'Debt component
                End If
            End If

            'Test whether company should call the bond, i.e. force conversion
            'call if...
            '-conversion value exceeds the call value
            '-the CB value at particular node exceeds the conversion value
            '- cannot call at time 0 or maturity & only after start time
            If (call_value > 0) And ConversionValue > call_value And k > 0 _
                        And (nodevalue(i, k, 1) + nodevalue(i, k, 2)) > ConversionValue _
                        And InitialNoConversionPeriod <= k * delta_t _
                        And delta_t * k >= Call_Start_Time Then
                    nodevalue(i, k, 1) = ConversionValue  'Equity component
                    nodevalue(i, k, 2) = 0                'Debt component
            ElseIf (call_value > 0) And (nodevalue(i, k, 1) + nodevalue(i, k, 2)) > call_value _
                          And k > 0 And call_value > ConversionValue _
                          And delta_t * k >= Call_Start_Time Then
                    nodevalue(i, k, 1) = call_value         'Equity component
                    nodevalue(i, k, 2) = 0                  'Debt component
            End If
            'Test whether holder should put bond
            'cannot put at time 0 or maturity & only after start time
            If (put_value > 0) And (nodevalue(i, k, 1) + nodevalue(i, k, 2)) < put_value And k > 0 _
                       And delta_t * k >= Put_Start_Time And put_value > ConversionValue Then
                    nodevalue(i, k, 1) = 0                'Equity component
                    nodevalue(i, k, 2) = put_value        'Debt component
            End If
        Next i
    Next k

If Version = "Full" And No_t_Steps > 35 Then Version = "Node"

Select Case Version
Case "Full"
    'initialize FullTree() - conditional formatting needs "" values to blank out non relevant cells
    ReDim fulltree(0 To (5 * (No_t_Steps + 1) - 1), 0 To No_t_Steps)
    For k = 0 To UBound(fulltree, 2)
        For i = 0 To UBound(fulltree, 1)
            fulltree(i, k) = ""
        Next i
    Next k

    'write to FullTree()
    For k = No_t_Steps To 0 Step -1
        For i = 0 To k
        fulltree(5 * (k - i), k) = nodevalue(i, k, 0)
        fulltree(5 * (k - i) + 1, k) = nodevalue(i, k, 1)
        fulltree(5 * (k - i) + 2, k) = nodevalue(i, k, 2)
        fulltree(5 * (k - i) + 3, k) = nodevalue(i, k, 3)
        fulltree(5 * (k - i) + 4, k) = nodevalue(i, k, 1) + nodevalue(i, k, 2)
        Next i
    Next k
    ConvertibleBinomial_CreditSpread = fulltree
Case "Node"
    ReDim initialnode(0 To 4) As Double
    initialnode(0) = S_0
    initialnode(1) = nodevalue(0, 0, 1)
    initialnode(2) = nodevalue(0, 0, 2)
    initialnode(3) = nodevalue(0, 0, 3)
    initialnode(4) = initialnode(1) + initialnode(2)
    ConvertibleBinomial_CreditSpread = Application.Transpose(initialnode)
Case Else
    ConvertibleBinomial_CreditSpread = nodevalue(0, 0, 1) + nodevalue(0, 0, 2)
End Select
```

```
Exit Function
error_handling:
    ConvertibleBinomial_CreditSpread = "Error #" & str(Err.Number) & "was generated by " _
        & Err.Source & Chr(13) & Err.Description

End Function

'k-factor credit spread model
Function k_factor_model(current_spread As Double, _
                    current_share_price As Double, _
                    min_spread As Double, _
                    trig_share_price As Double, _
                    k_factor As Double)

k_factor_model = min_spread + (current_spread - min_spread) / (trig_share_price / current_share_price) ^ (-k_factor)

End Function
```

A6.4 Hung and Wang Default Risk model

```
Option Explicit

'//Valuation of Convertible Bond with Default Risk using Hung and Wang model
    'Parameters:
    'S_0: Stock Price at time 0
    'Sigma: volatility of stock price
    'IntRate: risk free rate
    'IntRateRisky: required bond yield (to discount debt component of CB)
    ' DividendRateContinuous: dividend rate p.a. (for cont dividend model)
    'Conversion_Ratio: number of stocks per face value CB
    'Maturity: time to CB maturity
    'Face Value CB
    'Coupon_rate (total annual %)
    'Coupon Frequency p.a.
    'Default_Prob:the prob of default vector to be used in the calc
    'Call_Value (optional): value at which bond can be called by another company (at any time)
    'Put_Value (optional): value at which bond can be redeemed by holder (at any time)
    'Call_Start_Time (optional): when the call option becomes active
    'Put_Start_Time (optional): when the put option becomes active
    'Default_Share_Prob (optional): adjusting the share prob moves with default

    'No_t_Steps(optional): number of discrete t steps in grid
    'Optional AmeEurFlag: European("e") - default or American "a" with early conversion option possible.
    '"InitialNoConversionPeriod: if American exercise option, time at which early exercise begins

    'Version (optional)
    '-"Full": full tree with
    '1st Row: stock price
    '2nd row: option type exercised
    '3rd row: equity value (without option exercised)
    '4th row: bond value (without option exercised)
    '5th row: equity value (option exercised)
    '6th row: bond value (option exercised)
    '7th row: Total CB value
    '-"Node": all components of initial node omitted or any value - CB value only

'*****************************************************************************************************************

Function ConvBond_DefaultRisk( _
        S_0 As Double, _
        sigma As Double, _
        IntRate As Double, _
        IntRateRisky As Double, _
        DividendRateContinuous As Double, _
        Conversion_Ratio As Single, _
        Maturity As Double, _
        Face_value As Double, _
        coupon_rate As Double, _
        Recovery_value As Double, _
        Optional coupon_frequency As Integer = 1, _
        Optional call_value As Double = 0, _
        Optional put_value As Double = 0, _
        Optional No_t_Steps As Single, _
        Optional InitialNoConversionPeriod As Double, _
        Optional Call_Start_Time As Double = 0.00000001, _
        Optional Put_Start_Time As Double = 0.00000001, _
        Optional Version As String = "Full")

On Error GoTo error_handling
Dim u As Double, D As Double, p As Double
Dim ConversionValue As Double, Default_Prob, Default_Prob_vec As Object
Dim delta_t As Double, df As Double, DfRisky As Double
Dim coupon_time As Double, i As Integer, k As Integer, Default_Share_Prob

'defining parameters to be used
Default_Share_Prob = Range("Default_Share_Prob")
delta_t = Maturity / No_t_Steps
u = Exp(sigma * Sqr(delta_t))
D = 1 / u
df = Exp(-(IntRate - DividendRateContinuous) * delta_t): DfRisky = Exp(-IntRateRisky * delta_t)

Set Default_Prob_vec = Sheets("Default Prob Calc").Range("C2:C32")
ReDim Default_Prob(No_t_Steps)
For i = 0 To No_t_Steps
Default_Prob(i) = inter2(Sheets("Default Prob Calc").Range("A2:A32"), Default_Prob_vec, (i + 1) * delta_t)
Next i

ReDim nodevalue(0 To No_t_Steps, 0 To No_t_Steps, 0 To 6)
```

```
'Set up vector of PV of coupons to be added to the debt component to particular column of the tree
ReDim coupon_schedule(0 To No_t_Steps)
coupon_schedule(No_t_Steps) = coupon_rate / coupon_frequency
coupon_time = Maturity - 1 / coupon_frequency
For k = No_t_Steps - 1 To 0 Step -1
    Do While coupon_time >= k * delta_t And coupon_time > 0
        coupon_schedule(k) = coupon_schedule(k) + _
        Exp(-IntRateRisky * (coupon_time - k * delta_t)) * _
        coupon_rate / coupon_frequency
    coupon_time = coupon_time - 1 / coupon_frequency
    Loop
Next k
'set up vector of PV of Default amount to be used in the default states
ReDim Rollback_recovery(No_t_Steps)
For k = No_t_Steps - 1 To 0 Step -1
Rollback_recovery(No_t_Steps) = Recovery_value / Face_value
Rollback_recovery(k) = DfRisky * Rollback_recovery(k + 1)
Next k
'initialize final (right) nodes of tree
For i = 0 To No_t_Steps
    nodevalue(i, No_t_Steps, 0) = S_0 * u ^ i * D ^ (No_t_Steps - i)
    ConversionValue = Conversion_Ratio * nodevalue(i, No_t_Steps, 0) / Face_value
    If ConversionValue > (1 + coupon_schedule(No_t_Steps)) Then
        nodevalue(i, No_t_Steps, 1) = "Voluntary Conversion" ' Conversion option exercised
        nodevalue(i, No_t_Steps, 2) = ConversionValue        ' set final equity values
        nodevalue(i, No_t_Steps, 3) = 0                      ' set final debt values
    Else
        nodevalue(i, No_t_Steps, 3) = 1 + coupon_schedule(No_t_Steps) ' set final debt values
        nodevalue(i, No_t_Steps, 2) = 0                               ' set final equity values
        nodevalue(i, No_t_Steps, 1) = "Redemption"                    ' Let bond be redeemed
    End If
        'at terminal nodes without option exercise and option exercise are same value
        nodevalue(i, No_t_Steps, 4) = nodevalue(i, No_t_Steps, 2)
        nodevalue(i, No_t_Steps, 5) = nodevalue(i, No_t_Steps, 3)
Next i

Dim Default As Double, no_default As Double
'Loop thru tree backwards
For k = No_t_Steps - 1 To 0 Step -1
    For i = 0 To k
        Default = Default_Prob(k + 1) * delta_t
        no_default = 1 - Default
        'adjust prob of up share moves with default risk
        If Default_Share_Prob = 1 Then
        p = (Exp((IntRate - DividendRateContinuous) * delta_t) / no_default - D) / (u - D)
        Else
        p = (Exp((IntRate - DividendRateContinuous) * delta_t) - D) / (u - D)
        End If
        nodevalue(i, k, 0) = S_0 * u ^ i * D ^ (k - i)
        ConversionValue = Conversion_Ratio * nodevalue(i, k, 0) / Face_value

            nodevalue(i, k, 2) = (p * nodevalue(i + 1, k + 1, 2) * no_default + _
                         (1 - p) * nodevalue(i, k + 1, 2) * no_default) * df  'Equity component
            nodevalue(i, k, 3) = coupon_schedule(k) + _
                         (p * nodevalue(i + 1, k + 1, 3) * no_default + _
                         (1 - p) * nodevalue(i, k + 1, 3) * no_default) * DfRisky + _
                         Rollback_recovery(k) * Default                'Debt component
        nodevalue(i, k, 4) = nodevalue(i, k, 2)
        nodevalue(i, k, 5) = nodevalue(i, k, 3)
            'If American and inside the early exercise window, check whther early conversion increases value
            If InitialNoConversionPeriod <= k * delta_t Then

                If ConversionValue > (nodevalue(i, k, 4) + nodevalue(i, k, 5)) Then
                    nodevalue(i, k, 1) = "Voluntary Conversion"   'Conversion option exercised
                    nodevalue(i, k, 4) = ConversionValue          'Equity component
                    nodevalue(i, k, 5) = 0                        'Debt component
                End If
            End If

            'Test whether company should call the bond, i.e. force conversion
            'call if...
            '-conversion value exceeds the call value
            '-the CB value at particular node exceeds the conversion value
            '- cannot call at time 0 or maturity & only after start time
            If (call_value > 0) And ConversionValue > call_value And k > 0 _
                        And (nodevalue(i, k, 4) + nodevalue(i, k, 5)) > ConversionValue _
                        And InitialNoConversionPeriod <= k * delta_t _
                        And delta_t * k >= Call_Start_Time Then
                    nodevalue(i, k, 1) = "Forced Conversion"  'Issuer forces holder to convert
                    nodevalue(i, k, 4) = ConversionValue      'Equity component
                    nodevalue(i, k, 5) = 0                    'Debt component
            ElseIf (call_value > 0) And (nodevalue(i, k, 4) + nodevalue(i, k, 5)) > call_value _
                          And k > 0 And call_value > ConversionValue _
                          And delta_t * k >= Call_Start_Time Then
                    nodevalue(i, k, 1) = "Called"         'Holder lets bond be called
                    nodevalue(i, k, 4) = call_value       'Equity component
                    nodevalue(i, k, 5) = 0                'Debt component
            End If
            'Test whether holder should put bond
            'cannot put at time 0 or maturity & only after start time
            If (put_value > 0) And (nodevalue(i, k, 4) + nodevalue(i, k, 5)) < put_value And k > 0 _
                        And delta_t * k >= Put_Start_Time And put_value > ConversionValue Then
                    nodevalue(i, k, 1) = "Put"           'Holder puts the bond to issuer
                    nodevalue(i, k, 4) = 0               'Equity component
                    nodevalue(i, k, 5) = put_value       'Debt component
            End If
            nodevalue(i, k, 6) = nodevalue(i, k, 4) + nodevalue(i, k, 5) 'CB value at each node after options exercised
        Next i
Next k

Select Case Version
```

```
Case "Full"

'initialize FullTree() - conditional formatting needs "" values to blank out non relevant cells
    ReDim fulltree(0 To (7 * (No_t_Steps + 1) - 1), 0 To No_t_Steps)
    For k = 0 To UBound(fulltree, 2)
        For i = 0 To UBound(fulltree, 1)
            fulltree(i, k) = ""
        Next i
    Next k

    'write to FullTree()
    For k = No_t_Steps To 0 Step -1
        For i = 0 To k
        fulltree(7 * (k - i), k) = nodevalue(i, k, 0)
        fulltree(7 * (k - i) + 1, k) = nodevalue(i, k, 1)
        fulltree(7 * (k - i) + 2, k) = nodevalue(i, k, 2)
        fulltree(7 * (k - i) + 3, k) = nodevalue(i, k, 3)
        fulltree(7 * (k - i) + 4, k) = nodevalue(i, k, 4)
        fulltree(7 * (k - i) + 5, k) = nodevalue(i, k, 5)
        fulltree(7 * (k - i) + 6, k) = nodevalue(i, k, 6)
        Next i
    Next k
    ConvBond_DefaultRisk = fulltree

Case "Node"
    ConvBond_DefaultRisk = nodevalue(0, 0, 4) + nodevalue(0, 0, 5)

End Select

Exit Function
error_handling:
    ConvBond_DefaultRisk = "Error #" & str(Err.Number) & "was generated by " _
        & Err.Source & Chr(13) & Err.Description

End Function

'function used in theHung and Wang default risk model for nodes that have defaulted and need to bediscounted to time 0
Function Rollback_recovery(Rec_Rate As Double, Face_value As Double, _
                    Maturity As Double, No_t_Steps As Double, _
                    IntRateRisky As Double)

Dim i As Integer, delta_t, DfRisky As Double
ReDim rollback_vec(No_t_Steps)
delta_t = Maturity / No_t_Steps
DfRisky = Exp(-IntRateRisky * delta_t)

For i = No_t_Steps - 1 To 0 Step -1
rollback_vec(No_t_Steps) = Rec_Rate * Face_value
rollback_vec(i) = DfRisky * rollback_vec(i + 1)
Next i

Rollback_recovery = rollback_vec

End Function

'find coupon values on annual dates in Default Risk Model
Function total_coup_amt(Annual_coup As Double, coup_frequency As Integer, rate As Double, Face_value As Double)

Dim i As Integer, n As Integer, sum_coups

sum_coups = (Annual_coup / coup_frequency * Face_value)
For i = 1 To coup_frequency - 1
    sum_coups = sum_coups + (Annual_coup / coup_frequency * Face_value) * (1 + rate) ^ (-i / coup_frequency)
Next i

total_coup_amt = sum_coups

End Function
```

A6.5 Margrabe American and European Exchange models

```
Option Explicit
Option Base 1

'usual BS European call option formula
Function EuropeanCall(spot As Double, strike As Double, r As Double, q As Double, sigma As Double, tau As Double)

Dim d1 As Double, d2 As Double, Nd1 As Double, Nd2 As Double, delta As Double, b As Double

b = r - q

d1 = (Log(spot / strike) + (b + 0.5 * sigma ^ 2) * tau) / (sigma * Sqr(tau))
d2 = d1 - sigma * Sqr(tau)
Nd1 = WorksheetFunction.NormSDist(d1)
Nd2 = WorksheetFunction.NormSDist(d2)

EuropeanCall = spot * Exp(-q * tau) * Nd1 - strike * Exp(-r * tau) * Nd2

End Function

'Barone-Adesi & Whaley american call option approximation
Function AmericanCall(spot As Double, strike As Double, r As Double, q As Double, sigma As Double, tau As Double)

Dim X As Double, f As Double, f_prime As Double, A2 As Double, q2 As Double, n As Double, k As Double, Nd1 As Double
Dim d1 As Double, cream As Double, delta As Double, ECall As Double, d2 As Double, Nd2 As Double, b As Double

' b is defined in the actual paper, delta was defined using some other notation
b = r - q
delta = -q
```

```
n = 2 * (r - delta) / sigma ^ 2
k = 2 * r / (sigma ^ 2 * (1 - Exp(-r * tau)))
q2 = (1 - n + Sqr((n - 1) ^ 2 + 4 * k)) / 2

'using Newton Rhapson to arrive at the trigger price to exercise call - certain degreee of error
X = spot
Do
    d1 = (Log(X / strike) + (b + 0.5 * sigma ^ 2) * tau) / (sigma * Sqr(tau))
    Nd1 = WorksheetFunction.NormSDist(d1)
    cream = 1 - (Exp(-delta * tau) * Nd1)

    f = strike - X + EuropeanCall(X, strike, r, q, sigma, tau) + cream * (X / q2)
    f_prime = Exp(delta * tau) * Nd1 - 1 + (1 / q2) * cream - (1 / q2) * ((sigma * Sqr(tau * 2 * WorksheetFunction.Pi)) ^ -1) * Exp(-delta * tau - 0.5 * d1 ^ 2)

X = X - f / f_prime
Loop Until Abs(f) < 0.0000001

A2 = X * (1 - Exp(delta * tau) * Nd1) / q2

' depends on the trigger price and spot price if its worth exercising now or not
If spot < X Then
AmericanCall = EuropeanCall(spot, strike, r, q, sigma, tau) + A2 * (spot / X) ^ q2
ElseIf spot >= X Then
AmericanCall = spot - strike
End If

End Function

Function EuropeanExchange(spot1 As Double, spot2 As Double, q1 As Double, q2 As Double, sigma1 As Double, sigma2 As Double, tau As Double, rho As Double,
conv_ratio As Double)

Dim d1 As Double, d2 As Double, Nd1 As Double, Nd2 As Double, sigma As Double

sigma = Sqr(sigma1 ^ 2 + sigma2 ^ 2 - 2 * rho * sigma1 * sigma2)
d1 = (Log(conv_ratio * spot1 / spot2) + (q2 - q1 + 0.5 * sigma ^ 2) * tau) / (sigma * Sqr(tau))
d2 = d1 - sigma * Sqr(tau)
Nd1 = WorksheetFunction.NormSDist(d1)
Nd2 = WorksheetFunction.NormSDist(d2)

EuropeanExchange = (spot1 * conv_ratio * Exp(-q1 * tau) * Nd1) - (spot2 * Exp(-q2 * tau) * Nd2)

End Function

Function AmericanExchange(spot1 As Double, spot2 As Double, div1 As Double, div2 As Double, sigma1 As Double, sigma2 As Double, tau As Double, rho As Double,
conv_ratio As Double)

Dim X As Double, f As Double, f_prime As Double, A2 As Double, q2 As Double, n As Double, k As Double, Nd1 As Double, num As Integer
Dim d1 As Double, cream As Double, delta As Double, ECall As Double, d2 As Double, Nd2 As Double, b As Double, sigma As Double

' b is defined in the actual paper, delta was defined using some other notation
b = conv_ratio * div1 - div2
delta = -div2

sigma = Sqr(sigma1 ^ 2 + sigma2 ^ 2 - 2 * rho * sigma1 * sigma2)

n = 2 * (div1 - delta) / sigma ^ 2
k = 2 * div1 / (sigma ^ 2 * (1 - Exp(-div1 * tau)))
q2 = (1 - n + Sqr((n - 1) ^ 2 + 4 * k)) / 2

'using Newton Rhapson to arrive at the trigger price to exercise call - certain degree of error
X = spot1
Do
    d1 = (Log(conv_ratio * X / spot2) + (b + 0.5 * sigma ^ 2) * tau) / (sigma * Sqr(tau))
    Nd1 = WorksheetFunction.NormSDist(d1)
    cream = 1 - (Exp(-delta * tau) * Nd1)

    f = spot2 - conv_ratio * X + EuropeanExchange(X, spot2, div1, div2, sigma1, sigma2, tau, rho, conv_ratio) + cream * (X * conv_ratio / q2)
    f_prime = conv_ratio * Exp(delta * tau) * Nd1 - conv_ratio + (1 / q2) * cream - (1 / q2) * ((sigma * Sqr(tau * 2 * WorksheetFunction.Pi)) ^ -1) * Exp(-delta * tau - 0.5 *
d1 ^ 2)

X = X - f / f_prime
    num = num + 1
    If num > 200 Then Exit Do
Loop Until Abs(f) < 0.0000001
A2 = X * (1 - Exp(delta * tau) * Nd1) / q2

Dim tmp1 As Double, tmp2 As Double
 'depends on the trigger price and spot price if its worth exercising now or not
If (spot1) < X Then
AmericanExchange = EuropeanExchange(spot1, spot2, div1, div2, sigma1, sigma2, tau, rho, conv_ratio) + A2 * (spot1 / X) ^ q2
ElseIf (spot1) >= X Then
AmericanExchange = conv_ratio * spot1 - spot2
End If

'calc breaks down at critical stock price, so have to include this safety precaution to find max option value
tmp1 = EuropeanExchange(spot1, spot2, div1, div2, sigma1, sigma2, tau, rho, conv_ratio) + A2 * (spot1 / X) ^ q2
tmp2 = conv_ratio * spot1 - spot2

AmericanExchange = Application.Max(tmp1, tmp2)

End Function
```

A6.5 Component Exchange model

```
Option Explicit
Option Base 1

'used to calc the modified duration in the component exchange  model
```

```
Function dur_comp_model(start_date As Double, end_date As Double, kappa As Double)

dur_comp_model = (1 - Exp(-kappa * (end_date - start_date))) / kappa

End Function
'used to calc the convexity in the component exchange  model
Function convex_comp_model(start_date As Double, end_date As Double, kappa As Double)

convex_comp_model = dur_comp_model(start_date, end_date, kappa) ^ 2

End Function

'used to calc the volatility in the component exchange  model (zero-coupon CB)
Function sigma_z(sigma_e As Double, sigma_r As Double, kappa_r As Double, rho As Double, _
                T As Double, T2 As Double, Tm As Double)

Dim temp1 As Double, temp2 As Double, temp3 As Double

temp1 = (sigma_e ^ 2 + sigma_r ^ 2 / kappa_r ^ 2 + (2 * rho * sigma_r * sigma_e) / kappa_r) * (T2 - T)
temp2 = -(sigma_r ^ 2 / kappa_r ^ 2 + 2 * rho * sigma_r * sigma_e / kappa_r) * _
        (dur_comp_model(T, Tm, kappa_r) - dur_comp_model(T2, Tm, kappa_r))
temp3 = -sigma_r ^ 2 * (convex_comp_model(T, Tm, kappa_r) - convex_comp_model(T2, Tm, kappa_r)) / (2 * kappa_r)

sigma_z = Sqr(temp1 + temp2 + temp3)

End Function

'used to calc the volatility in the component exchange  model (zero-coupon credit spread CB)
Function sigma_d(sigma_e As Double, sigma_s As Double, sigma_r As Double, kappa_r As Double, kappa_s As Double, rho_es As Double, _
                rho_rs As Double, rho_er As Double, T As Double, T2 As Double, Tm As Double)

Dim temp1 As Double, temp2 As Double, temp3 As Double, temp4 As Double

temp1 = (sigma_e ^ 2 + sigma_r ^ 2 / kappa_r ^ 2 + sigma_s ^ 2 / kappa_s ^ 2 + (2 * rho_er * sigma_r * sigma_e) / kappa_r + _
       (2 * rho_rs * sigma_s * sigma_r) / (kappa_r * kappa_s) + (2 * rho_es * sigma_e * sigma_s / kappa_s)) * (T2 - T)
temp2 = -(sigma_r ^ 2 / kappa_r ^ 2 + (2 * rho_rs * sigma_r * sigma_s) / (kappa_r * kappa_s) + 2 * rho_er * sigma_r * sigma_e / kappa_r) * _
        (dur_comp_model(T, Tm, kappa_r) - dur_comp_model(T2, Tm, kappa_r))
temp3 = -(sigma_s ^ 2 / kappa_s ^ 2 + (2 * rho_rs * sigma_r * sigma_s) / (kappa_r * kappa_s) + 2 * rho_es * sigma_s * sigma_e / kappa_s) * _
        (dur_comp_model(T, Tm, kappa_s) - dur_comp_model(T2, Tm, kappa_s))
temp4 = (2 * rho_rs * sigma_r * sigma_s) * (dur_comp_model(T, Tm, kappa_r + kappa_s) - dur_comp_model(T2, Tm, kappa_r + kappa_s)) / (kappa_r * kappa_s) _
       - sigma_r ^ 2 * (convex_comp_model(T, Tm, kappa_r) - convex_comp_model(T2, Tm, kappa_r)) / (2 * kappa_r) _
       - sigma_s ^ 2 * (convex_comp_model(T, Tm, kappa_s) - convex_comp_model(T2, Tm, kappa_s)) / (2 * kappa_s)

sigma_d = Sqr(temp1 + temp2 + temp3 + temp4)

End Function

'used to calc the volatility in the component exchange model (coupon credit spread CB)
Function sigma_c(sigma_e As Double, sigma_s As Double, sigma_r As Double, kappa_r As Double, kappa_s As Double, rho_es As Double, _
                rho_rs As Double, rho_er As Double, cflows, risk_free As Double, spread As Double, _
                Freq, T, T2, Tm, xirr As Double)

Dim temp1 As Double, temp2 As Double, temp3 As Double, temp4 As Double
Dim dur_r_t, dur_r_T2, dur_s_t, dur_s_T2, dur_rs_t, dur_rs_T2, convex_r_t, convex_r_T2, convex_s_t, convex_s_T2

dur_r_t = DurModCon(T, T2, cflows, risk_free, 0, xirr, Freq)(1, 1)
dur_r_T2 = DurModCon(T2, T2, cflows, risk_free, 0, xirr, Freq)(1, 1)
dur_s_t = DurModCon(T, T2, cflows, 0, spread, xirr, Freq)(1, 1)
dur_s_T2 = DurModCon(T2, T2, cflows, 0, spread, xirr, Freq)(1, 1)
dur_rs_t = DurModCon(T, T2, cflows, risk_free, spread, xirr, Freq)(1, 1)
dur_rs_T2 = DurModCon(T2, T2, cflows, risk_free, spread, xirr, Freq)(1, 1)
convex_r_t = DurModCon(T, T2, cflows, risk_free, 0, xirr, Freq)(3, 1)
convex_r_T2 = DurModCon(T2, T2, cflows, risk_free, 0, xirr, Freq)(3, 1)
convex_s_t = DurModCon(T, T2, cflows, 0, spread, xirr, Freq)(3, 1)
convex_s_T2 = DurModCon(T2, T2, cflows, 0, spread, xirr, Freq)(3, 1)

temp1 = (sigma_e ^ 2 + sigma_r ^ 2 / kappa_r ^ 2 + sigma_s ^ 2 / kappa_s ^ 2 + (2 * rho_er * sigma_r * sigma_e) / kappa_r + _
       (2 * rho_rs * sigma_s * sigma_r) / (kappa_r * kappa_s) + (2 * rho_es * sigma_e * sigma_s / kappa_s)) * (T2 - T) / 365
temp2 = -(sigma_r ^ 2 / kappa_r ^ 2 + (2 * rho_rs * sigma_r * sigma_s) / (kappa_r * kappa_s) + 2 * rho_er * sigma_r * sigma_e / kappa_r) * _
        (dur_r_t - dur_r_T2)
temp3 = -(sigma_s ^ 2 / kappa_s ^ 2 + (2 * rho_rs * sigma_r * sigma_s) / (kappa_r * kappa_s) + 2 * rho_es * sigma_s * sigma_e / kappa_s) * _
        (dur_s_t - dur_s_T2)
temp4 = ((2 * rho_rs * sigma_r * sigma_s) * (dur_rs_t - dur_rs_T2)) / (kappa_r * kappa_s) _
       - sigma_r ^ 2 * (convex_r_t - convex_r_T2) / (2 * kappa_r) _
       - sigma_s ^ 2 * (convex_s_t - convex_s_T2) / (2 * kappa_s)

sigma_c = Sqr(temp1 + temp2 + temp3 + temp4)

End Function


'calc no. of coupons between 2 dates
Function Num_Coupon_Pmts(valuation_date As Date, NCD As Date, edate As Date, frequency As Single)

Dim i As Integer, n As Integer, coup_dates(61), PV_fut_coups, j As Integer

coup_dates(1) = NCD
'calc the number of coupon pmts
For i = 1 To 60
coup_dates(i + 1) = DateAdd("m", 12 / frequency, coup_dates(i))
    If coup_dates(i) <= edate And valuation_date <= (coup_dates(i) - 10) Then
    n = i
    Else
    'Nothing
    End If
Next i

Num_Coupon_Pmts = n

End Function
```

```
'function to create duration, mod duration and convexity of variable series of cash flows
Function DurModCon(t0, force_conv_date, cflows As Variant, risk_free As Double, spread As Double, xirr As Double, i)

'i is the NAC of the specified spot rate
Dim k As Integer, n As Integer, LinSpot() As Variant
Dim df_t() As Variant, cf_df() As Variant, cf_df_t() As Variant
Dim df() As Variant, num As Single, denom As Single
Dim cf_df_t_t1() As Variant 'Note t1 is(t+1)
Dim stow(3, 1), i2 As Single 'i2 is NACS of xirr

n = Application.Count(cflows) / 2
ReDim LinSpot(n)
ReDim df_t(n)
ReDim cf_df(n)
ReDim cf_df_t(n)
ReDim df(n)
ReDim cf_df_t_t1(n)

For k = 1 To n
   If cflows(k, 1) < force_conv_date Then
       'Nothing
   Else
       LinSpot(k) = risk_free + spread
       'LinSpot(k) = inter2(term, spot, cflows(k, 1))
       df(k) = (1 + LinSpot(k) / (i * 100)) ^ (-i * (force_conv_date - t0) / 365) * _
           (1 + LinSpot(k) / (i * 100)) ^ (-i * k)
       df_t(k) = df(k) * k
       cf_df(k) = cflows(k, 2) * df(k)
       cf_df_t(k) = cflows(k, 2) * df(k) * k
       cf_df_t_t1(k) = cflows(k, 2) * df(k) * k * (k + 1)
   End If
Next k

num = Application.Sum(cf_df_t)
denom = -cflows(1, 2)

i2 = i * ((1 + xirr) ^ (1 / i) - 1)

stow(1, 1) = num / denom / (i * (1 + i2))
stow(2, 1) = num / denom / i
stow(3, 1) = Application.Sum(cf_df_t_t1) / (denom) / (1 + i2) ^ i

DurModCon = stow
End Function

'returns the Vasicek (imod=1) or CIR (imod=2) zero-coupon bond value
Function VasicekCIRZeroValue(imod As Integer, a As Double, b As Double, r As Double, nowyr, zeroyr, sigma As Double)

Dim syr, sig2, Asyr, Bsyr, rinf, gamma, c1, c2
syr = (zeroyr - nowyr)
sig2 = sigma ^ 2

'Vasicek ZCB value
If imod = 1 Then
   If a = 0 Then
       Bsyr = syr
       Asyr = Exp((sig2 * syr ^ 3) / 6)
   Else
       Bsyr = (1 - Exp(-a * syr)) / a
       rinf = b - 0.5 * sig2 / (a ^ 2)
       Asyr = Exp((Bsyr - syr) * rinf - ((sig2 * Bsyr ^ 2) / (4 * a)))
   End If

'CIR ZCB value
ElseIf imod = 2 Then
   gamma = Sqr(a ^ 2 + 2 * sig2)
   c1 = 0.5 * (a + gamma)
   c2 = c1 * (Exp(gamma * syr) - 1) + gamma
   Bsyr = (Exp(gamma * syr) - 1) / c2
   Asyr = ((gamma * Exp(c1 * syr)) / c2) ^ (2 * a * b / sig2)
End If

VasicekCIRZeroValue = Asyr * Exp(-Bsyr * r)

End Function

'calculate the value for zero-coupon CB
Function ZC_Component_Exchange(s0 As Double, bond_price As Double, div As Double, conv_ratio As Double, _
                        r As Double, theta_r As Double, kappa_r As Double, sigma_r As Double, sigma_e As Double, _
                        rho_er As Double, valuation_date As Double, forced_conv_date As Double, Maturity As Double)

Dim str_bond_value As Double, exchange_option As Double, mean_c As Double, sigma As Double, mu_sigma As Double, breakdown(3, 1)

mean_c = Log(conv_ratio * s0 / bond_price * Exp(-div * (forced_conv_date - valuation_date)))
sigma = sigma_z(sigma_e, sigma_r, kappa_r, rho_er, valuation_date, forced_conv_date, Maturity)
str_bond_value = 100 * VasicekCIRZeroValue(1, kappa_r, theta_r, r, valuation_date, Maturity, sigma_r)
mu_sigma = mean_c / sigma

exchange_option = conv_ratio * s0 * Exp(-div * (forced_conv_date - valuation_date)) * WorksheetFunction.NormSDist(mu_sigma + sigma) _
                        - str_bond_value * WorksheetFunction.NormSDist(mu_sigma)

breakdown(1, 1) = str_bond_value
breakdown(2, 1) = exchange_option
breakdown(3, 1) = str_bond_value + exchange_option

ZC_Component_Exchange = breakdown

End Function

'calculate the value for zero-coupon CB with credit risk
Function ZC_Credit_Component_Exchange(s0 As Double, bond_price As Double, div As Double, conv_ratio As Double, _
```

```
                r As Double, theta_r As Double, kappa_r As Double, sigma_r As Double, sigma_e As Double, _
                S As Double, theta_s As Double, kappa_s As Double, sigma_s As Double, _
                rho_er As Double, rho_rs As Double, rho_es As Double, _
                valuation_date As Double, forced_conv_date As Double, Maturity As Double)

Dim str_bond_value As Double, exchange_option As Double, mean_c As Double, sigma As Double, mu_sigma As Double, breakdown(3, 1)

mean_c = Log(conv_ratio * s0 / bond_price * Exp(-div * (forced_conv_date - valuation_date)))
sigma = sigma_d(sigma_e, sigma_s, sigma_r, kappa_r, kappa_s, rho_es, rho_rs, rho_er, valuation_date, forced_conv_date, Maturity)
str_bond_value = 100 * (VasicekCIRZeroValue(1, kappa_r, theta_r, r, valuation_date, Maturity, sigma_r) _
            - (1 - VasicekCIRZeroValue(1, kappa_s, theta_s, S, valuation_date, Maturity, sigma_s)))
mu_sigma = mean_c / sigma

exchange_option = conv_ratio * s0 * Exp(-div * (forced_conv_date - valuation_date)) * WorksheetFunction.NormSDist(mu_sigma + sigma) _
                - str_bond_value * WorksheetFunction.NormSDist(mu_sigma)
breakdown(1, 1) = str_bond_value
breakdown(2, 1) = exchange_option
breakdown(3, 1) = str_bond_value + exchange_option

ZC_Credit_Component_Exchange = breakdown

End Function

'calculate the value for coupon CB with credit risk
Function Credit_Component_Exchange(s0 As Double, div As Double, conv_ratio As Double, _
                r As Double, theta_r As Double, kappa_r As Double, sigma_r As Double, sigma_e As Double, _
                S As Double, theta_s As Double, kappa_s As Double, sigma_s As Double, _
                rho_er As Double, rho_rs As Double, rho_es As Double, Annual_coup, par_value, _
                valuation_date As Date, forced_conv_date As Date, Maturity As Date, next_coup_date As Date, Freq As Single, _
                term As Object, yield_curve As Object, Optional cap_option = True, Optional Version As String)

Dim str_bond_value As Double, exchange_option As Double, mean_c As Double, sigma As Double, mu_sigma As Double, breakdown(3, 1)
Dim i As Integer, n As Integer, accrued_int As Double

n = Num_Coupon_Pmts(valuation_date, next_coup_date, Maturity, Freq)
ReDim cashflow_schedule(n, 5), pv_vec(n)

For i = 1 To n
    If i = 1 Then
    cashflow_schedule(i, 2) = next_coup_date
    Else
    cashflow_schedule(i, 2) = DateAdd("m", 12 / Freq, cashflow_schedule(i - 1, 2))
    End If
  cashflow_schedule(i, 1) = (cashflow_schedule(i, 2) - valuation_date) / 365
    If i = n Then
    cashflow_schedule(i, 3) = (1 + Annual_coup / Freq) * par_value
    Else
    cashflow_schedule(i, 3) = Annual_coup / Freq * par_value
    End If
  cashflow_schedule(i, 4) = (1 + inter2(term, yield_curve, cashflow_schedule(i, 1)) / (par_value * Freq)) ^ (-Freq * cashflow_schedule(i, 1))
    If i = 1 Then
    cashflow_schedule(1, 5) = (next_coup_date - DateAdd("m", -12 / Freq, next_coup_date)) / 365 * cashflow_schedule(i, 3)
    accrued_int = cashflow_schedule(1, 5)
    Else
    cashflow_schedule(i, 5) = cashflow_schedule(i, 3) * cashflow_schedule(i, 4)
    End If
pv_vec(i) = cashflow_schedule(i, 5)
Next i
str_bond_value = Application.Sum(pv_vec)

're-mapping to cflows to be used in duration of cashflow calc -> also used in mean and sigma calc
ReDim cflows(n + 1, 2)
For i = 1 To n + 1
  If i = 1 Then
  cflows(1, 1) = valuation_date
  cflows(1, 2) = -str_bond_value
  Else
  cflows(i, 1) = cashflow_schedule(i - 1, 2)
  cflows(i, 2) = cashflow_schedule(i - 1, 3)
  End If
Next i

Dim riskless_vec As Double, spread_vec As Double, risky_vec As Double
riskless_vec = DurModCon(valuation_date, forced_conv_date, cflows, 100 * r, 0, r, Freq)(1, 1)
spread_vec = DurModCon(valuation_date, forced_conv_date, cflows, 0, 100 * S, S, Freq)(1, 1)
risky_vec = DurModCon(valuation_date, forced_conv_date, cflows, 100 * r, 100 * S, r + S, Freq)(1, 1)

mean_c = r - sigma_e ^ 2 / 2 - rho_er * sigma_r * sigma_e * riskless_vec - rho_es * sigma_s * sigma_e * spread_vec
sigma = sigma_c(sigma_e, sigma_s, sigma_r, kappa_r, kappa_s, rho_es, rho_rs, rho_er, cflows, r, S, Freq, valuation_date, forced_conv_date, Maturity, r)
mu_sigma = mean_c / sigma

exchange_option = conv_ratio * s0 * Exp(-div * (forced_conv_date - valuation_date) / 365) * WorksheetFunction.NormSDist(mu_sigma + sigma) _
                + (str_bond_value - accrued_int) * (WorksheetFunction.NormSDist(-mu_sigma) - 1)
Select Case Version
Case "Full"
breakdown(1, 1) = str_bond_value
  'Limit min value of option to zero
  If cap_option = True Then
  breakdown(2, 1) = Application.Max(exchange_option, 0)
  Else
  breakdown(2, 1) = exchange_option
  End If
breakdown(3, 1) = breakdown(1, 1) + breakdown(2, 1)

Credit_Component_Exchange = breakdown
Case Else
breakdown(1, 1) = str_bond_value
  'Limit min value of option to zero
  If cap_option = True Then
  breakdown(2, 1) = Application.Max(exchange_option, 0)
  Else
```

```
   breakdown(2, 1) = exchange_option
   End If
breakdown(3, 1) = breakdown(1, 1) + breakdown(2, 1)

Credit_Component_Exchange = breakdown(3, 1)
End Select

End Function

'value a vanilla bond using a term structure
Function Bond_value(valuation_date As Date, next_coup_date As Date, Annual_coup, Freq As Single, par_value, Maturity As Date, term As Object, yield_curve As Object)
Dim cashflow_schedule, i As Integer, n As Integer

n = Num_Coupon_Pmts(valuation_date, next_coup_date, Maturity, Freq)
ReDim cashflow_schedule(n, 5)

For i = 1 To n
      If i = 1 Then
      cashflow_schedule(i, 2) = next_coup_date
      Else
      cashflow_schedule(i, 2) = DateAdd("m", 12 / Freq, cashflow_schedule(i - 1, 2))
      End If
   cashflow_schedule(i, 1) = (cashflow_schedule(i, 2) - valuation_date) / 365
      If i = n Then
      cashflow_schedule(i, 3) = (1 + Annual_coup / Freq) * par_value
      Else
      cashflow_schedule(i, 3) = Annual_coup / Freq * par_value
      End If
   cashflow_schedule(i, 4) = (1 + inter2(term, yield_curve, cashflow_schedule(i, 1)) / (par_value * Freq)) ^ (-Freq * cashflow_schedule(i, 1))
      If i = 1 Then
      cashflow_schedule(1, 5) = (next_coup_date - DateAdd("m", -12 / Freq, next_coup_date)) / 365 * cashflow_schedule(i, 3)
      Else
      cashflow_schedule(i, 5) = cashflow_schedule(i, 3) * cashflow_schedule(i, 4)
      End If
Bond_value = Bond_value + cashflow_schedule(i, 5)
Next i

End Function
```

A6.6 Valuation of Vanilla Bond with embedded Call and Put Options

```
Option Explicit

'//Valuation of Bond using binomial tree - takes into account call and put features
   'Parameters:
   'IntRateTree: calibrated interest rate tree to be used
   'Maturity: time to bond maturity
   'Face Value CB
   'Coupon_rate (total annual %)
   'Coupon Frequency p.a.
   'Call_Value (optional): value at which bond can be called by another company (at any time)
   'Put_Value (optional): value at which bond can be redeemed by holder (at any time)
   'Call_Start_Time (optional): when the call option becomes active
   'Put_Start_Time (optional): when the put option becomes active
   'No_t_Steps(optional): number of discrete t steps in grid

   '1st Row: int rate used to discount backwards in tree
   '2nd row: bond value without option exercised
   '3rd row: option type exercised
   '4th row: bond value (with option exercised)
'*********************************************************************************************************

Function Bond_Tree_Valuation( _
      IntRate As Double, _
      Maturity As Double, _
      Face_value As Double, _
      coupon_rate As Double, _
      sigma As Double, _
      Optional coupon_frequency As Integer = 1, _
      Optional call_value As Double = 0, _
      Optional put_value As Double = 0, _
      Optional No_t_Steps As Single, _
      Optional Call_Start_Time As Double = 0.00000001, _
      Optional Put_Start_Time As Double = 0.00000001, _
      Optional Version As String)

On Error GoTo error_handling
Dim u As Double, D As Double, p As Double
Dim delta_t As Double, DfRisky
Dim coupon_time As Double, i As Integer, k As Integer

ReDim nodevalue(0 To No_t_Steps, 0 To No_t_Steps, 0 To 3)
delta_t = Maturity / No_t_Steps
u = Exp(sigma * Sqr(delta_t))
D = 1 / u
p = (Exp(IntRate * delta_t) - D) / (u - D)
DfRisky = Exp(-IntRate * delta_t)
call_value = call_value / Face_value
put_value = put_value / Face_value

'Set up vector of PV of coupons to be added to the debt component to particular column of the tree
ReDim coupon_schedule(0 To No_t_Steps)
coupon_schedule(No_t_Steps) = coupon_rate / coupon_frequency
coupon_time = Maturity - 1 / coupon_frequency
For k = No_t_Steps - 1 To 0 Step -1
   Do While coupon_time >= k * delta_t And coupon_time > 0
      coupon_schedule(k) = coupon_schedule(k) + _
      Exp(-IntRate * (coupon_time - k * delta_t)) * _
      coupon_rate / coupon_frequency
```

```
    coupon_time = coupon_time - 1 / coupon_frequency
    Loop
Next k

'initialize final (right) nodes of tree
For i = 0 To No_t_Steps
    nodevalue(i, No_t_Steps, 3) = 100 * (1 + coupon_schedule(No_t_Steps)) / Face_value   ' Let bond be redeemed
    nodevalue(i, No_t_Steps, 2) = "Matured"                                   ' Option type exercised
    nodevalue(i, No_t_Steps, 1) = 100 * (1 + coupon_schedule(No_t_Steps)) / Face_value   ' Let bond be redeemed
    nodevalue(i, No_t_Steps, 0) = ""                                          ' Maturity int rates = 0
Next i

Dim Default As Double, no_default As Double
'Loop thru tree backwards
For k = No_t_Steps - 1 To 0 Step -1
    For i = 0 To k
        nodevalue(i, k, 0) = IntRate
        nodevalue(i, k, 1) = coupon_schedule(k) + _
                    (p * nodevalue(i + 1, k + 1, 3) + _
                    (1 - p) * nodevalue(i, k + 1, 3)) * DfRisky          'Discounted rollback value
            If nodevalue(i, k, 1) >= call_value And delta_t * k >= Call_Start_Time Then   'check if rollback value greater than call value
            nodevalue(i, k, 2) = "Called"
            nodevalue(i, k, 3) = call_value
            ElseIf nodevalue(i, k, 1) <= put_value And delta_t * k >= Put_Start_Time Then 'check if rollback value less than put value
            nodevalue(i, k, 2) = "Put"
            nodevalue(i, k, 3) = put_value
            Else
            nodevalue(i, k, 2) = ""
            nodevalue(i, k, 3) = nodevalue(i, k, 1)
            End If
        Next i
Next k

Select Case Version
Case "Full"
'initialize FullTree() - conditional formatting needs "" values to blank out non relevant cells
    ReDim fulltree(0 To (4 * (No_t_Steps + 1) - 1), 0 To No_t_Steps)
    For k = 0 To UBound(fulltree, 2)
        For i = 0 To UBound(fulltree, 1)
            fulltree(i, k) = ""
        Next i
    Next k

    'write to FullTree()
    For k = No_t_Steps To 0 Step -1
        For i = 0 To k
        fulltree(4 * (k - i), k) = nodevalue(i, k, 0)
        fulltree(4 * (k - i) + 1, k) = nodevalue(i, k, 1)
        fulltree(4 * (k - i) + 2, k) = nodevalue(i, k, 2)
        fulltree(4 * (k - i) + 3, k) = nodevalue(i, k, 3)
        Next i
    Next k
    Bond_Tree_Valuation = fulltree
Case "Node"
    ReDim initialnode(0 To 3) As Double
    initialnode(0) = nodevalue(0, 0, 0)
    initialnode(1) = nodevalue(0, 0, 1)
    initialnode(2) = nodevalue(0, 0, 2)
    initialnode(3) = initialnode(1) + initialnode(2)
    Bond_Tree_Valuation = Application.Transpose(initialnode)
Case Else
    Bond_Tree_Valuation = nodevalue(0, 0, 1)
End Select

Exit Function
error_handling:
    Bond_Tree_Valuation = "Error #" & str(Err.Number) & "was generated by " _
        & Err.Source & Chr(13) & Err.Description

End Function
```

A6.7 Tan and Cai Risk Equilibrium model

```
Option Explicit


'//Valuation of Convertible Bond using Tan & Cai Risk Equilibrium Model
    'Parameters:
    'A_0: Total Firm Mkt Value at time 0
    'Sigma: volatility of Firm Value
    'IntRate: risk free rate
    'IntRateRisky: required bond yield (to discount debt component of CB)
    ' DividendRateContinuous: dividend rate p.a. (for cont dividend model)
    'Conversion_Ratio: number of stocks per face value CB
    'Maturity: time to CB maturity
    'Face Value CB
    'Coupon_rate (total annual %)
    'Coupon Frequency p.a.
    'Call_Value (optional): value at which bond can be called by another company (at any time)
    'Put_Value (optional): value at which bond can be redeemed by holder (at any time)
    'Call_Start_Time (optional): when the call option becomes active
    'Put_Start_Time (optional): when the put option becomes active

    'No_t_Steps(optional): number of discrete t steps in grid
    'Optional AmeEurFlag: European("e") - default or American "a" with early conversion option possible.
    '"InitialNoConversionPeriod: if American exercise option, time at which early exercise begins

    'Version (optional)
```

```
    '-"Full": full tree with
    '1st Row: stock price
    '2nd row: equity component (if CB is converted)
    '3rd row: debt component (if CB is NOT converted)
    '4th row: value of CB
    '-"Node": all components of initial node omitted or any value - CB value only

'*******************************************************************************************************

Function ConvBinomial_RiskEquil( _
        A_0 As Double, _
        Sigma As Double, _
        meanreturn As Double, _
        num_convertibles As Double, _
        num_shares As Double, _
        IntRate As Double, _
        coupon_rate As Double, _
        coupon_frequency As Double, _
        Conversion_Ratio As Single, _
        maturity As Double, _
        Face_value As Double, _
        lambda As Double, _
        Optional call_value As Double = 0, _
        Optional put_value As Double = 0, _
        Optional No_t_Steps As Single, _
        Optional InitialNoConversionPeriod As Double, _
        Optional Version As String = "", _
        Optional Call_Start_Time As Double = 0.00000001, _
        Optional Put_Start_Time As Double = 0.00000001)

'On Error GoTo error_handling
Dim u As Double, d As Double, p As Double, expA, expC, cov_AC, CBMkt_Value
Dim ConversionValue As Double, NomValue As Double, sigma_conv, tot_call_value
Dim delta_t As Double, Df As Double, DfRisky As Double, tot_put_value As Double
Dim coupon_time As Double, i As Integer, k As Integer, j As Integer

ReDim nodevalue(0 To No_t_Steps, 0 To No_t_Steps, 0 To 3), a(0 To No_t_Steps, 0 To No_t_Steps)
ReDim sigma_conv(0 To No_t_Steps, 0 To No_t_Steps), meanconvret(0 To No_t_Steps, 0 To No_t_Steps)
delta_t = maturity / No_t_Steps
u = Exp(Sigma * Sqr(delta_t))
d = 1 / u
p = (Exp(meanreturn * delta_t) - d) / (u - d)

'Set up vector of PV of coupons to be added to the debt component to particular column of the tree
ReDim coupon_schedule(0 To No_t_Steps)
coupon_schedule(No_t_Steps) = coupon_rate / coupon_frequency
coupon_time = maturity - 1 / coupon_frequency
For k = No_t_Steps - 1 To 0 Step -1
    Do While coupon_time >= k * delta_t And coupon_time > 0
        coupon_schedule(k) = coupon_schedule(k) + _
        Exp(-IntRate * (coupon_time - k * delta_t)) * _
        coupon_rate / coupon_frequency
    coupon_time = coupon_time - 1 / coupon_frequency
        Loop
Next k
'initialize final (right) nodes of tree
For i = 0 To No_t_Steps
    nodevalue(i, No_t_Steps, 0) = A_0 * u ^ i * d ^ (No_t_Steps - i)
    NomValue = num_convertibles * Face_value * (1 + coupon_rate / coupon_frequency)
    If NomValue > nodevalue(i, No_t_Steps, 0) Then
        nodevalue(i, No_t_Steps, 1) = nodevalue(i, No_t_Steps, 0) ' set final CB values
    Else
        nodevalue(i, No_t_Steps, 1) = NomValue ' set final debt values
    End If
Next i
'Loop thru tree backwards
For k = No_t_Steps - 1 To 0 Step -1
    For i = 0 To k
        nodevalue(i, k, 0) = A_0 * u ^ i * d ^ (k - i)

            'defining the call, put and conversion parameters
            If k * delta_t >= Call_Start_Time Then
            tot_call_value = call_value * Face_value * num_convertibles
            Else
            tot_call_value = 1E+91
            End If
            If k * delta_t >= Put_Start_Time Then
            tot_put_value = put_value * Face_value * num_convertibles
            Else
            tot_put_value = 0
            End If
            If k * delta_t >= InitialNoConversionPeriod Then
            ConversionValue = (Conversion_Ratio * num_convertibles * nodevalue(i, k, 0)) / (num_shares + num_convertibles * Conversion_Ratio)
            Else
            ConversionValue = 0
            End If

            'using expected values to compute covariance and thus find risk burden ratio
            expA = p * WorksheetFunction.Ln(nodevalue(i + 1, k + 1, 0)) + _
                (1 - p) * WorksheetFunction.Ln(nodevalue(i, k + 1, 0))    'Firm Value component
            expC = p * WorksheetFunction.Ln(nodevalue(i + 1, k + 1, 1)) + _
                (1 - p) * WorksheetFunction.Ln(nodevalue(i, k + 1, 1))  'CB component
            cov_AC = p * (WorksheetFunction.Ln(nodevalue(i + 1, k + 1, 0)) - expA) * (WorksheetFunction.Ln(nodevalue(i + 1, k + 1, 1)) - expC) + _
                 (1 - p) * (WorksheetFunction.Ln(nodevalue(i, k + 1, 0)) - expA) * (WorksheetFunction.Ln(nodevalue(i, k + 1, 1)) - expC)
            a(k - i, k) = cov_AC / (Sigma ^ 2 * delta_t)
            sigma_conv(k - i, k) = a(k - i, k) * Sigma
            meanconvret(k - i, k) = IntRate + lambda * sigma_conv(k - i, k)     'appropriate discount rate to be used

            nodevalue(i, k, 2) = meanconvret(k - i, k)
            nodevalue(i, k, 3) = a(k - i, k)
            CBMkt_Value = (Face_value * num_convertibles * coupon_schedule(k)) + _
```

```
                    Exp(-meanconvret(k - i, k) * delta_t) * (p * nodevalue(i + 1, k + 1, 1) + (1 - p) * nodevalue(i, k + 1, 1))

            nodevalue(i, k, 1) = Application.Min(CBMkt_Value, nodevalue(i, k, 0), tot_call_value)
            nodevalue(i, k, 1) = Application.Min(Application.Max(nodevalue(i, k, 1), ConversionValue, tot_put_value), nodevalue(i, k, 0))

            If nodevalue(i, k, 1) = ConversionValue Then
            nodevalue(i, k, 2) = "Conversion"
            ElseIf nodevalue(i, k, 1) = tot_put_value Then
            nodevalue(i, k, 2) = "Put"
            ElseIf nodevalue(i, k, 1) = tot_call_value Then
            nodevalue(i, k, 2) = "Called"
            Else
            nodevalue(i, k, 2) = ""
            End If
    Next i
Next k

If Version = "Full" And No_t_Steps > 35 Then Version = "Node"

Select Case Version
Case "Full"
    'initialize FullTree() - conditional formatting needs "" values to blank out non relevant cells
    ReDim fulltree(0 To (4 * (No_t_Steps + 1) - 1), 0 To No_t_Steps)
    For k = 0 To UBound(fulltree, 2)
        For i = 0 To UBound(fulltree, 1)
            fulltree(i, k) = ""
        Next i
    Next k

    'write to FullTree()
    For k = No_t_Steps To 0 Step -1
        For i = 0 To k
        fulltree(4 * (k - i), k) = nodevalue(i, k, 0)
        fulltree(4 * (k - i) + 1, k) = nodevalue(i, k, 1)
        fulltree(4 * (k - i) + 2, k) = nodevalue(i, k, 2)
        fulltree(4 * (k - i) + 3, k) = nodevalue(i, k, 3)
        Next i
    Next k
    ConvBinomial_RiskEquil = fulltree
Case "Node"
    ReDim initialnode(0 To 3) As Double
    initialnode(0) = A_0
    initialnode(1) = nodevalue(0, 0, 1)
    initialnode(2) = nodevalue(0, 0, 2)
    initialnode(3) = nodevalue(0, 0, 3)
    ConvBinomial_RiskEquil = Application.Transpose(initialnode)
Case Else
    ConvBinomial_RiskEquil = nodevalue(0, 0, 1)
End Select

Exit Function
error_handling:
   ConvBinomial_RiskEquil = "Error #" & Str(Err.Number) & "was generated by " _
       & Err.Source & Chr(13) & Err.Description

End Function
```

A6.8 Other Functions

```
' Calculates the value of an European Option (Black-Scholes)
' Typ -> Call or Put
' Command -> Price, Delta, Gamma, Theta, Vega, Rho

Function phi(X As Double) As Double
    phi = 1 / Sqr(2 * Application.Pi()) * Exp(-X ^ 2 / 2)
End Function


Function fi(X As Double) As Double
    fi = 1 / Sqr(2 * Application.Pi()) * Exp(-X ^ 2 / 2)
End Function


Function Gauss(X As Double) As Double
    Gauss = Application.NormSDist(X)
End Function

Function option_e(Put_Call As String, S As Double, e As Double, Tmt As Double, r As Double, q As Double, sigma As Double, Command As String) As Double
' Calculates the value of an European Option (Black-Scholes)
' Typ -> Call or Put
' Command -> Price, Delta, Gamma, Theta, Vega, Rho

    Tmt = Application.Max(0.00001, Tmt)

    Dim Sign As Integer
    Dim d1 As Double, d2 As Double, ed1 As Double, ed2 As Double
    Select Case UCase$(Left$(Put_Call, 1))
        Case "C": Sign = 1
        Case "P": Sign = -1
        Case Else: MsgBox "Not valid."
                option_e = 0
                Exit Function

    End Select

    If (sigma * S * Tmt > 0) Then
        d1 = (Log(S / e) + (r - q + 0.5 * sigma * sigma) * Tmt) / (sigma * Sqr(Tmt))
        ed1 = Gauss(Sign * d1)
        d2 = d1 - sigma * Sqr(Tmt)
```

```
        ed2 = Gauss(Sign * d2)
    Else
        If (S < e) Then
            ed1 = 0.5 * (1 - Sign)
        Else
            ed1 = 0.5 * (1 + Sign)
        End If
        ed2 = ed1
    End If
    Select Case UCase$(Left$(Command, 1))
        Case "P": option_e = Sign * (S * Exp(-q * Tmt) * ed1 - e * Exp(-r * Tmt) * ed2) ' Price
        Case "D": option_e = Sign * Exp(-q * Tmt) * ed1 ' Delta
        Case "G":
                    If S * sigma * Tmt > 0 Then
                       option_e = Exp(-q * Tmt) * phi(d1) / (sigma * S * Sqr(Tmt))
                    Else: option_e = 0
                    End If ' Gamma
        Case "T":
              option_e = (-0.5 * S * sigma * Exp(-q * Tmt) * phi(d1) / Sqr(Tmt) _
                      + Sign * (q * S * Exp(-q * Tmt) * ed1 - r * e * Exp(-r * Tmt) * ed2)) / 365.25 ' Theta 1/Day
        Case "V": option_e = 0.01 * Exp(-q * Tmt) * phi(d1) * S * Sqr(Tmt) ' Vega [1/%]
        Case "Q": option_e = -0.01 * Sign * S * Tmt * Exp(-q * Tmt) * ed1 ' Divi-Yield-Sensitivität [1/%]
        Case "R": option_e = 0.01 * Sign * e * Tmt * Exp(-r * Tmt) * ed2 ' Rho [1/%]
        Case Else: MsgBox "Illegal output query '" & Command & "'"
                 option_e2 = 0
    End Select
End Function

'Linear Interpolation (e.g. of interest rates or volatilities): x_v = Datevector,y_v = valuevector, X = date

Function inter2(x_v As Object, y_v As Object, X As Variant) As Double
    Dim n As Long, ind As Long, i As Long
    n = x_v.Rows.Count
    If X < x_v(1) Then
        inter2 = y_v(1)
    ElseIf X > x_v(n) Then
        inter2 = y_v(n)
    Else
        For i = 1 To n
            If x_v(i) <= X Then
                ind = ind + 1
            Else
                ind = ind
            End If
        Next
        Dim X1 As Variant, X2 As Variant, Y1 As Double, Y2 As Double
        X1 = x_v(ind)
        X2 = x_v(ind + 1)
        Y1 = y_v(ind)
        Y2 = y_v(ind + 1)
        inter2 = (Y1 * (X2 - X) + Y2 * (X - X1)) / (X2 - X1)
    End If
End Function
```

References

1. Ammann, M., Kind, A.H. and Wilde, C., *"The Pricing of Convertible Bonds – An Analysis of the French Market"*, Discussion Paper, University of St. Gallen, Department of Economics, 2001.
2. Ammann, M., Kind, A.H. and Wilde, C., *"Simulation-Based Pricing of Convertible Bonds"*, Journal of Empirical Finance, Vol. 15, no.2, 2008, pp 310-331.
3. Andersen, L. and Buffum, D, *"Calibration and implementation of convertible bond models"*, Working paper, Banc of America Securities.
4. Asquith, P. and Mullins, D.W Jr., *"Convertible Debt: Corporate Call Policy and Voluntary Conversion"*, Journal of Finance, Vol. 46, No. 3, 1991, pp 1273-1290.
5. Asquith, P., *"Convertible Bonds are not Called late"*, Journal of finance, Vol. 50, No. 3, 1995, pp 1275-1289.
6. Ayache, E., Forsyth, P.A. and Vetzal, K.R., *"Valuation of Convertible Bonds with Credit Risk"*, Journal of Derivatives, Vol. 11, No. 1, 2003, pp 9-29.
7. Ayache, E., Forsyth, P.A. and Vetzal, K.R., *"Next Generation models for convertible bonds with credit risk",* Wilmott Magazine, pp 68-77, (2002).
8. Barone-Adesi, G. & Whaley, R. *"Efficient Analytic Approximation of American Option Values",* Journal of Finance, (June '87).
9. Barone-Adesi, G., Bermudez A. and Hatgioannides J. *"Two-Factor Convertible Bonds Valuation using the Method of Characteristics/Finite Elements"*, Journal of Economic Dynamics and Control, Vol. 27, (2003), pp 1801-1831.
10. Bjerksund, P. & Stensland, G**.**, *"Closed-Form Approximation of American Options"*, Scandinavian Journal of Management, Vol.9, (1993), pp 87-99.
11. Black, F. and Cox, J.C., *"Valuing Corporate Securities: Some Effects of Bond Indenture Provisions"*, Journal of Finance 32, pp 1699-1715.
12. Black, F. and Scholes, S, *"The pricing of options and corporate liabilities"*, Journal of Political Economy, Vol. 81, pp 637-654.
13. Black, F., Derman, E. and Toy W, *"A One-Factor Model of Interest Rates and its Application to Treasury Bond Options"*, Financial Analysts Journal, Vol. 46, 1990, pp 33-39.
14. Brennan, M. and E. Schwartz, *"Analyzing convertible bonds"*, J. Financial Quantitative Analysis 15 (1980), pp 907–929.
15. Brennan, M. and E. Schwartz, *"Convertible bonds: valuation and optimal strategies for call and conversion"*, J. Finance 32, (1977), pp 1699 –1715.

16. Bühler, W. and Koziol, C, *"Valuation of Convertible Bonds with Sequential Conversion"*, Schmalenbach Business Review, Vol. 54, Oct 2002, pp 302-334.
17. Calamos, N.P, *"Convertible Arbitrage"*, John Wiley & Sons, (2003).
18. Carr, P., Reiner, E. and Rubinstein, M., *"Appendix: Useful Facts about Normal Distributions"*, Cornell University, (1995).
19. Chambers, D.R. and Lu, Q., *"A Tree Model for Pricing Convertible Bonds with Equity, Interest Rate, and Default Risk",* Journal of Derivatives, Summer 2007, pp 25-46.
20. Cheung, W. and Nelken, I., *"Costing the converts"*, Riak Magazine, July (1994), pp 47-49.
21. Constantinides, G. M., *"Warrant exercise and bond conversion in competitive markets"*, Journal of Financial Economics, 13, pp 371 – 397, (1984).
22. Cox, J.C., Ingersoll, J.E. and Ross, S.A., *"A Theory of the Term Structure on Interest Rates",* Econometrica, Vol. 53, pp 385 – 407 (1985).
23. Cox, J.C., Ross, S.A. and Rubinstein, M., *"Option Pricing: A Simplied Approach"*, Journal of Financial Economics, 7, pp 229 – 264, (1979).
24. McCracken, D.D., and Dorn, W.M., *"Numerical Methods and Fortran Programming"*, John Wiley & Sons, Inc, (1964).
25. Das, S.R., Sundaram, R.K. and Sundaresan, S.M., *"A Simple Unified Model for Pricing Derivative Securities with Equity, Interest-Rate, and Default Risk"*, AFA 2004 San Diego Meetings, Jan 2003.
26. Davis, M. and Lischka F.R., "*Convertible bonds with market risk and credit risk"*, Working paper, Tokyo Mitsubishi International plc (1999).
27. Duffy, D. and Singleton, K.J, *"Modeling Term Structures of Defaultable Bonds"*, Review of Financial Studies, Vol. 12, pg 687 – 720, (1999).
28. Duffy, D.J., *"Finite Difference Methods in Financial Engineering"*, First Edition, Wiley (2006).
29. Finnerty, J.D., *"Valuing Convertible Bonds and the Option to Exchange Bonds for Stock"*, Finnerty Economic Consulting, May 2007.
30. Garman, M.B. and Kohlhagen, S.W., *"Foreign Currency Option Values"*, Journal of International Money and Finance Vol.2, (1983), pg 231 – 237.
31. Gheno, A., *"Convertible Bonds and Volatility Structure"*, Working Paper no. 57, University of Roma, 2005.
32. Goldman-Sachs, *"Valuing convertible bonds as derivatives"*, Quantitative Strategies Research Notes, Goldman-Sachs, (1994).
33. Ho, T. and Pfeffer, D., *"Convertible bonds: Model, value attribution and analytics"*, Financial Analysts Journal, 52, 35 – 44, (1996).

34. Huang, H. and Liao, S., *"Valuation and Optimal Strategies of Convertible Bonds"*, Journal of Futures Markets, Vol. 26, No. 9, 2006, pp 895 – 922.

35. Hull, J.C., *"Options, futures and other derivative securities"*, Fourth Edition, Prentice-Hall, (2000).

36. Hung, M.W. and J.Y. Wang, *"Pricing convertible bonds subject to default risk"*, Journal of Derivatives 10, (2002), pp 75 – 87.

37. Ingersoll J.E., *"An examination of corporate call policy on convertible securities"*, Journal of Finance, Vol. 32, (1977).

38. Ingersoll, J.E., *"A contingent-claims valuation of convertible securities"*, J. Financial Economics. 4, (1977), pp 289 – 322.

39. Jamshidian, F., *"Forward Induction and Construction of Yield Curve Diffusion Models"*, Journal of Fixed Income 1, (1991), pp 62 – 74.

40. Jarrow, R.A. and S.M. Turnbull, *"Pricing Derivative Securities on Financial Securities subject to Credit Risk"*, Journal of Finance 50, (1995), pp 53 – 85.

41. Kifer, I., *"Game options"*, Finance and Stochastics 4, (2002), pp 443 – 463.

42. Landskroner, Y. and Raviv, A., *"The Valuation of Inflation-Indexed and FX Convertible Bonds"*, Unpublished Paper (2002).

43. Levy, G., *"Computational Finance - Numerical Methods for Pricing Financial Instruments"*, First Edition, Elsevier Butterworth-Henniemann (2004).

44. Lischka, R. and Davis, M., *"Convertible Bonds with Market and Credit Risk"*, AMS IP Studies in Advanced Mathematics, Vol. 26, 2002, pp 45 – 58.

45. Longstaff, F.A. and Schwartz E.S., *"A Simple Approach to Valuing Risky Fixed and Floating Rate Debt"*, Journal of Finance 50, pp 789 – 819.

46. Lvov, D., Yigitbasioglu, A.B. and Bachir, N.E, *"Pricing Convertible Bonds by Simulation"*, ISMA Centre Discussion Papers in Finance, University of Reading, 2004.

47. Miller M.H. and Modigliani F., *"Dividend policy, growth, and the valuation of shares"*, Journal of Business, Volume 34 (1996).

48. Miller M.H. and Modigliani F., *"The Cost of Capital, Corporation Finance, and the Theory of Investment"*, American Economic Review, Volume 48, (1958).

49. Margrabe, W., *"The value of an option to exchange one asset for another"*, Journal of Finance, 33, (1978), pp 177 – 186.

50. Markowitz, H.M., *"Portfolio Selction"*, Journal of Finance, March 1952, pp 77 – 91.

51. McConell, J. and Schwartz, E., *"LYON Taming"*, Journal of Finance, Vol. 50, (1986), pp 53 – 85.

52. McIntyre, R.A., *“Two approaches to the pricing of convertible bonds”*, Unpublished Honours Project, (2002).

53. Merton, R.C., *“A dynamic general equilibrium model of the asset market and its application to the pricing of the capital structure of the firm”*, Unpublished manuscript, (1970).

54. Merton, R.C., *“On the Pricing of Corporate Debt: The Risk Structure of Interest Rates”*, Journal of Finances 29, (1974), pp 449 – 470.

55. Muromachi, Y., *“The growing recognition of credit risk in corporate and financial bond markets”*, NLI Research Institute, Paper No. 126, (1999).

56. Nyborg, K.G., *“The Use and Pricing of Convertible Bonds”*, Journal of Applied mathematical Finance, Vol. 3, 1996, pp 167-190.

57. Olsen, L., *“Convertible Bonds: A Technical Introduction”*, Convertible Bonds Research – Barclays Capital, Jan 2002.

58. Olsen, L., Beattie, H. and Allison, A., *“Convertibles and the k-factor, implications of the credit-equity relationship in the current market”*, Convertible Bonds Research - Barclays Capital, Aug 2007.

59. Realdon, M., *“Valuation of Exchangeable Convertible Bonds”*, International Journal of Theoretical and Applied Finance, Vol.7, No.6, (2004), pp 701 – 721.

60. Rubinstein, M., *"One for Another"*, Risk Magazine 4, (1991), pp 30-32.

61. Rubinstein, M., *"Return to Oz"*, Risk Magazine 7, 11, (1994).

62. Sarkar, S., *“Duration and Convexity of Zero-Coupon Convertible Bonds”*, Journal of Economics and Business, Vol. 51, (1999), pp 175 – 192.

63. Sharpe, W.F., *“Capital asset prices: A theory of market equilibrium under conditions of risk”*, Journal of Finance, Vol. 19, no.3, (1964), pp 425 – 442.

64. Sîrbu, M and Shreve, S.E., *“A Two-Person Game for Pricing Convertible Bonds”*, (2005)

65. Sîrbu, M., Pikovsky I. and Shreve S., *“Perpetual convertible bonds”*, SIAM J. Optimal Control. 43, (2004), pp 58 – 85.

66. Subrahmanyam, M.G., *“The early exercise feature of American options”*, in Subrahmanyam, M.G., Figlewski, S. and Silber, W.L., *“Financial Options – From Theory to Practice”*, Irwin, Burr Ridge, Illinois (1990).

67. Takahashi, A., Kobayashi T., and Nakagawa N., *“Pricing convertible bonds with default risk”*, Journal of Fixed Income 11, (2001), pp 20 – 29.

68. Tan, Z. and Cai, Y., *“Risk Equilibrium Binomial Model for Convertible Bonds Pricing”*, Unpublished Paper, South West University of Finance and Economics, 2007.

69. Tsiveriotis, K. and Fernandes, C., *"Valuing convertible bonds with credit risk"*, Journal of Fixed Income 8, (1998), pp 95 – 102.

70. Vasicek, O., *"An Equilibrium Characterization of the Term Structure",* Journal of Financial Economics, Vol. 5, pg 177 – 188.

71. Yagi, K. and Sawaki, K., *"The Valuation and Optimal Strategies of Callable Convertible Bonds"*, Pacific Journal of Optimization, Vol. 1, No. 2, May 2005, pp 375 – 386.