Ideal perturbation of elements in C*-algebras

by

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Contents

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PREFACE

The examples in this thesis serve not only to illustrate the abstract ideas of the thesis but to introduce notation that shall be used consistently throughout the thesis. The index of symbols follows after this preface. Hence many concrete examples of C*-algebras will be given.

The first chapter of the thesis sets the ground work for the thesis and has three dominant themes.

The first theme (Chapter 1, section 1) is the anti-unitization of a C*-algebra, where we embed a C*-algebra as a closed 2-sided ideal of another C*-algebra which lacks an identity. This embedding gives us licence to treat any C*-algebra as a C*-algebra without an identity for our purposes. This of fundamental importance in the thesis : it is an appropriate first theme.

The second theme (Chapter 1, section 2) is the representation of an abstract C*-algebra in terms of more concrete C*-algebras : the C*-algebra of all continuous complex valued functions which vanish at infinity on a locally compact Hausdorff space, the C*-algebra of all bounded operators on a Hilbert space and the Double Centralizer Algebra. These representations formed the frameworks in which the problems of the thesis was solved. The bulk of Chapter 1, section 2 focusses on the latter two representation theories. The second representation theory which is well established is attacked from the viewpoint of theory of a *-representation, a *-algebraic concept void of the norm. In particular for C*-algebras, we look at irreducible and non-degenerate *-representations, the former being the stronger condition. The well known second representation theory is a non-degenerate *-representation which uses irreducible cyclic *-representations in its construction : the crux is the bijective correspondence between irreducible cyclic *-representations and pure states. We furnish amongst the concrete examples, an example of a *-representation which is non-degenerate but far from the being irreducible.

The Double Centralizer Algebra Representation theory is the least known. We introduce it as an improved left regular representation by showing the inadequacy of the left regular representation. Indeed, the solution to this inadequacy is a triumph of the school of solving the problem by looking at it in a more abstract setting. Hence, we build up the theory from the very general theory of the category of semigroups which will provide reasons for the definitions of a double centralizer, which otherwise would appear as if it were plucked out of the air. We then move the theory up into the category of rings where we vindicate our efforts by demonstrating the preservation of the ideal structure of the original C*-algebra in the Double Centralizer Algebra. Finally, we move the theory up into the category of C*-algebras where we show that it solves the problem associated with the left regular representation. Moreover, the structure inherits all the benefits that are associated with the former two categories. To demystify the Double Centralizer Algebra Representation theory, we furnish concrete examples of the Double Centralizer Algebra Representation of two well known C*-algebras.

We end off the second theme appropriately by contrasting the Universal Representation and the Double Centralizer Algebra Representation. We first represent the Double Centralizer algebra of a C*-algebra as a subspace of the C*-algebra of all bounded operators on the same Hilbert space used in the representation of the original C*-algebra. However, not to give the false impression that the Double Centralizer Algebra Representation is a special case of the Universal Representation, we furnish a concrete example which establishes the Double Centralizer Algebra Representation as having its own merits over the Universal Representation and will therefore be regarded as a representation theory in its own right.

Just as much as the existence or absence of an identity element played a central role, there are other special types of elements of the C*-algebra, namely, the normal elements. These have a representation theory that we call the Functional Calculus since the normal elements can be represented as functions of a function algebra. We develop three corollaries which will be used extensively in the thesis. One of these corollaries involves a factorization of a normal element. To redress the bias towards normal elements, we resort to the Universal Representation which enables *any* element in a C*-algebra to be viewed as a bounded operator on a Hilbert space, enabling us to apply a factorization or decomposition known as the Polar Decomposition which we take as the second local representation of the arbitrary element, normal or non normal. Just as in the case of normal elements, a list of corollaries used extensively in the thesis is developed. We end off by relating the two local representations in an important result and applying the Functional and Polar Decomposition Theorem in the context of the quotient C*-algebra to yield small but important results as demanded by the mathematics which follows in the thesis.

We start Chapter 2 off by proving the lifting of the problem of zero divisors affirmatively. The proof rests on a bootstrapping argument: we first quickly prove the result for the case of positive zero divisors using the Orthogonal Decomposition Corollary and then prove for the general case by virtue of the Polar Decomposition with the aid of the Functional Calculus. We further prove the result affirmatively for the lifting of self adjoint zero divisors. The remainder of the chapter is occupied with proving the lifting problem of the more general case of n-zero divisors. We start by first proving it for the case of a commutative C*-algebra by an elegant interplay between the two global representation theories of the Universal Representation and the C*-algebra represented as the C*-algebra of all continuous complex valued functions which vanish at infinity on a locally compact Hausdorff space. The result is then next proved affirmatively in the case of a Von Neumann C*-algebra where the proof is short and simple by virtue of the abundance of projections in a Von Neumann C*-algebra. In fact, we isolate the property, the Von Neumann Lifting Lemma, which makes the lifting easy in the Von Neumann C*-algebra case. We then step into the fundamentally important paradigm of Non Commutative Topology which gives birth to a special C*-algebra which we call a SAW*-algebra that has a special property which mimicks the Von Neumann Lifting Lemma responsible for lifting n-zero divisors in Von Neumann C*-algebras. The special property in question is the property of possessing orthogonal local units. This is the primary motivation of the SAW*-algebra which eventually is the dominant theme in proving not only the problem of lifting n-zero divisors in the general C*-algebra but also the problem of lifting the property of the nilpotent element. We show the importance of the paradigm of Non Commutative Topology by demonstrating how the important C*-algebraic properties of a σ-unital C*-algebra, possessing orthogonal local units, being a Von Neumann C*-algebra and most of all a Corona C*-algebra, a special kind of SAW*-algebra, originate from this paradigm. We use a specific case to demonstrate the commutative origin of the Corona C*-algebra, the key to the affirmative lifting of the property of n-zero-divisors in any C*-algebra.

Before showing the significance of the construction of the corona of a C*-algebra to solving the lifting property of n-zero-divisors in any C*-algebra, we demonstrate properties of the corona C*-algebra that point to this direction : local properties of the corona translate into global properties of the finer double centralizer algebra representation. The significance of the construction of the corona of a C*-algebra with regards to the lifting of the property of n-zero divisors is then pinpointed : the corona of every non-unital σ-unital C*-algebra is a SAW*-property and the local unit associated with the SAW*-algebra like its counterpart in the case of an identity element of a C*-algebra with an identity has a norm of exactly one.

The lifting of the property of n-zero divisors in the corona of any non-unital σ-unital C*-algebra is initiated by the SAW*-algebra property of possessing orthogonal local units, very similar in approach to the case of a Von Neumann C*-algebra. The orthogonality of the pair is exploited by a use of the Polar Decomposition Theorem to produce the desired perturbations.

When we prove the lifting of the property of n-zero divisors in the general C*-algebra, we essentially reduce the problem to the case of the corona by setting up the same scenario as in the case of the corona. Namely, we construct closed essential ideals from the given closed ideal and show that it is without loss of generality that the general C*-algebra can be taken as a non-unital σ-unital C*-algebra. In constructing closed essential ideals from the given closed ideal, we make use of what we call the pseudo-pythagorean inequality. In constructing σ-unitalness, we work purely in the C*-algebra generated by the finitely many elements defining the lifting problem. In constructing the non-unitalness of the C*-algebra, we resort to the stable algebra contruction which embeds the original C*-algebra in the non-unital stable algebra as a 2-sided closed ideal. Intuitively, this stable algebra is the C*-algebra of the infinite matrices whose entries are elements of the C*-algebra. The reduction occurs when we construct the corona of the essential ideal which is a non-unital and σ-unital C*-algebra and hence lift the property of n-zero divisor. The construction by annihilators used in the closed essential ideal then dumps all the elements not of the original C*-algebra safely away, retaining the bona fide elements of the ideal of the original C*-algebra to do their job of lifting the property of n-zero divisors affirmatively.

Buoyed by our success in lifting the property of n-zero divisors, we attack a closely related property : the property of a nilpotent element. This is the theme of Chapter 3, and to start off we prove the result affirmatively in simple cases where the degree of nilpotency is 2. Much of the machinery developed in chapter 2 is used again to shoot down these simple cases. For the general case of lifting nilpotent elements of any degree n, we prove this affirmatively by first lifting the property in the corona of a non-unital, σ-unital C*-algebra and then reducing the problem of lifting it in the general C*-algebra exactly as in the case of lifting n-zero divisors. To prove the result in the corona a non-unital, σ-unital C*-algebra, once again, the Von Neumann algebra was the benchmark. The key was in establishing a triangular form for a nilpotent element relative to a finite commutative set of elements in the corona. This triangular form for a nilpotent element occurs naturally in a Von Neumann algebra. More importantly, we can lift this triangular form although not totally with a clever use of the properties of a hereditary C*-subalgebra. However for the purposes of proving the lifting of nilpotent elements, this partial lifting suffices. The approach directly constructs from the coset of the nilpotent element of the corona, an element of the double centralizer algebra which is nilpotent. The trick is to construct the element as a sum of elements which annihilate each other. These summands are constructed via the functional calculus all within the framework of the finite commutative set involved in the triangular form of the nilpotent element.

For the general case of lifting nilpotent elements in the general C*-algebra, the reduction of the problem to the case of the corona of a non-unital, σ-unital C*-algebra proved successful using an identical argument as in the case of lifting n-zero divisors.

In chapter 4, we explore the new frontier of lifting the more general property of a polynomially ideal element in a general C*-algebra. We quickly show that this is not possible : we demonstrate the topological obstruction to this lifting in the C*-algebra of all continuous functions on the unit interval. The topological obstruction is the property of connectedness in the complex plane. We however salvage the situation by establishing a criterion under which polynomially ideal elements can be lifted : when the property of a finite orthogonal family of projections can be lifted. This criterion rests on our ability to lift nilpotent elements along with the lifting of the commutative property associated with an invertible element which we prove by a lovely interplay between Tietze's extension theorem and the Stone Weierstrass theorem as well as the lifting of positive invertible elements of which we give two independent proofs.

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Chapter 1

Introduction

1.1 C*-algebra : Preliminaries

For the purposes of making this dissertation as self contained as possible and fixing the basic definitions as used in this dissertation, we state the following definitions and some of the basic theorems which are of fundamental importance in the thesis.

An *algebra* over a field $\mathbf{F}$ is a ring A that is simultaneously a vector space over $\mathbf{F}$ with the same addition; the ring multiplication and the scalar multiplication are related as follows:

$$(\alpha x)y = \alpha(xy) = x(\alpha y) \tag{1.1}$$

for all $x, y \in$ A and $\alpha \in \mathbf{F}$. *All vector spaces and algebras will be taken over the complex number field* $\mathbf{C}$ *unless otherwise stated.*

An algebra A is *commutative* if $xy = yx$ for all $x, y \in A$.

Example 1 *Let $M_n(\mathbf{C})$ denote the set of all the $n \times n$ matrices with entries taken from the complex number field $\mathbf{C}$. Then $M_n(\mathbf{C})$ is an algebra over the field $\mathbf{C}$: under matrix addition and multiplication, it forms a non-commutative ring over $\mathbf{C}$; it is an n^2 - dimensional vector space over $\mathbf{C}$ when equipped with the usual scalar multiplication; the ring multiplication and the scalar multiplication are related as in (1.1).*

A **-algebra* is an algebra over $\mathbf{C}$ with a map $x \mapsto x^*$ on A into itself such that for all $x, y \in$ A and complex $\lambda \in \mathbf{C}$:

$$
\begin{aligned}
&(a) \quad (x+y)^* = x^* + y^* \\
&(b) \quad (\lambda x)^* = \bar{\lambda} x^*
\end{aligned}
$$

$$(c) \quad (xy)^* = y^* x^*$$
$$(d) \quad x^{**} = x$$

where the map $x \mapsto x^*$ is called an involution. A *-homomorphism of a *-algebra A into a *-algebra B is a linear mapping which preserves the ring multiplication and the involution.

Example 2 *Consider the map* $*$*:*$M_n(\mathbf{C}) \to M_n(\mathbf{C})|x \mapsto x^*$ *where* x^* *is the conjugate transpose of the* $n \times n$ *matrix* x*. Then,* $M_n(\mathbf{C})$ *is a *-algebra over* $\mathbf{C}$ *under the operation *.*

A *normed algebra* is a normed vector space A which is an algebra where the multiplication is related to the norm as follows:

$$\| xy \| \leq \| x \| \| y \| \tag{1.2}$$

A *normed *-algebra* is a normed algebra which is a *-algebra. If the algebra is complete with respect to the norm, it is called a *Banach *-algebra*.

The involution in a normed *-algebra is *continuous* if there exists a constant $M > 0$ such that $\| x^* \| \leq M \cdot \| x \|$ for all x; the involution is *isometric* if $\| x^* \| = \| x \|$ for all x. Two normed *-algebras are *isometrically *-isomorphic* if there exists a *-isomorphism (bijective *-homomorphism) $\phi : A \to B$ such that $\| \phi(x) \| = \| x \|$ for all x in A.

A norm on a *-algebra satisfies the *C*-condition* if

$$\| x^* x \| = \| x \| \| x^* \| \quad (x \in A) \tag{1.3}$$

Consequently, if the involution is isometric, the norm satisfies the *strong C*-norm* condition:

$$\| x^* x \| = \| x \|^2 \quad (x \in A) \tag{1.4}$$

We shall call a Banach *-algebra a *C*-algebra* if the strong C*-norm condition (1.4) is satisfied.

Example 3 *Consider each* $x \in M_n(\mathbf{C})$ *as an operator on the* n *- dimensional vector space* $\mathbf{C}^n$*. Consider the operator norm on* $M_n(\mathbf{C})$ *defined by:*

$$\| x \| = max_{|\gamma| \leq 1} |x(\gamma)|$$

where $\gamma \in \mathbf{C}^n$ *and* $|\cdot|$ *is the usual Euclidean norm on the* n *- dimensional vector space* $\mathbf{C}^n$*. By virtue of the Cauchy-Schwartz inequality on* $\mathbf{C}^n$*,* $\| x \| \leq \sqrt{n} K$ *where* K *is the maximum of the Euclidean norms of the rows of the matrix* x *taken as elements of* $\mathbf{C}^n$*.*

Then $M_n(\mathbf{C})$ *is a C*-algebra.*

The strong C*-norm condition (1.4) implies that the involution is isometric: $\| x \|^2 = \| x^*x \| \leq \| x^* \| \| x \|$ whence $\| x \| \leq \| x^* \|$. Analogously $\| x^* \|^2 = \| (x^*)^*x^* \| = \| xx^* \| \leq \| x \| \| x^* \|$ whence $\| x^* \| \leq \| x \|$. Hence the C*-algebra condition (1.4) is equivalent to both the conditions that the involution is isometric and the C*-condition (1.3). In fact, it was a highly non-trivial problem to show that a Banach *-algebra which *only* satisfies the C*-condition (1.3) has an isometric involution [Chapter 3, Theorem 16.1 [8]]. Consequently the C*-norm condition and the strong C*-norm condition is equivalent in the setting of a Banach *-algebra and a C*-algebra can be defined equivalently as a Banach *-algebra that satisfies the C*-norm condition (1.3). We shall however continue to define a C*-algebra as a Banach *-algebra that satisfies the the strong C*-norm condition (1.4) since this strong C*-norm condition is useful in the solution of many problems of the thesis.

Example 4 *We can view $M_n(\mathbf{C})$ as the space of all the bounded operators on the finite dimensional Hilbert space $\mathbf{C}^n$. More generally, the space of all bounded operators on a Hilbert space $\mathbf{H}$, $B(\mathbf{H})$, is a C*-algebra: the operator norm on $B(\mathbf{H})$ satisfies the strong C*-norm condition [see Theorem 2.4.2 [10]].*

If there exists an element e in the C*-algebra A such that $xe = ex = x$ for all x in A, then A is called a *C*-algebra with identity*. We call e the identity element. Not all C*-algebras have an identity. The latter case is more common among C*-algebras which occur in applications.

Example 5 (Non Commutative C*-algebra with an identity) *$M_n(\mathbf{C})$ is a C*-algebra with the $n \times n$ unit matrix defined by:*

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

as the identity element, e.

Example 6 (Non Commutative C*-algebra without an identity) *A compact operator on a Hilbert space $\mathbf{H}$, is an operator with the property that image of the unit ball of the Hilbert space, $\mathbf{H}$, is relatively compact in $\mathbf{H}$. Since all compact sets are bounded, a compact operator is a bounded operator. In fact, the set of all compact operators,$K(\mathbf{H})$, is a norm-closed *-subalgebra of $B(\mathbf{H})$.*

If $\mathbf{H}$ is an infinite dimensional Hilbert space, then the identity operator $e(x) = x$ for all $x \in \mathbf{H}$ is not a compact operator since the unit ball of $\mathbf{H}$, $B_{\mathbf{H}}$, is compact if and only if $\mathbf{H}$ is a finite dimensional normed space [Chapter II, Theorem 1.2.6 [25]]. Hence, the space of all compact operators on an infinite dimensional Hilbert space $\mathbf{H}$ is a non-commutative C-algebra without an identity.*

Example 7 (Commutative C*-algebra with an identity) *The Banach space $C(K)$ of continuous complex-valued functions on a compact Hausdorff space K with the sup norm is a C*-algebra under point-wise multiplication and point-wise complex conjugation :* $(fg)(t) = f(t)g(t)$ *and* $f^*(t) = \overline{f(t)}$*, is a commutative C*-algebra with identity e, the constant 1 - map on X, defined by:*

$$e(t) = 1 \text{ for all } t \text{ in } X.$$

Example 8 (Commutative C*-algebra without an identity) *Let $C_0(X)$ be the Banach *-algebra of all continuous complex-valued functions on a locally compact Hausdorff space which vanishes at infinity, with respect to the operations of point-wise multiplication, point-wise addition, point-wise complex conjugation and the supremum norm. Then $C_0(X)$ is a C*-algebra which will not possess an identity unless X is compact.*

In 1967, B.J. Vowden showed how to embed a C*-algebra A without an identity into a C*-algebra which has an identity, A_e. A_e is the direct sum $A \oplus \mathbf{C}$ as vector spaces, with the following operations:

$$\begin{array}{ll} (a) & (x,\lambda)(y,\mu) = (xy + \mu x + \lambda y, \lambda\mu) \\ (b) & (x,\lambda)^* = (x^*, \bar{\lambda}) \\ (c) & \| (x,\lambda) \| = \| x \| + |\lambda| \end{array}$$

The element (0, 1) which we denote as e, is the identity of A_e and the map $i : A \to A_e | x \mapsto (x, 0)$ is an isometric *-isomorphism (embedding). Formally:

Theorem 1 (Unitization) *(Chapter 1.1, Proposition 1.5, [19]) If A is a C*-algebra without an identity, there exists a norm on A_e which satisfies the strong C*-norm condition (1.4) and extends the original norm on A (A is isometrically embedded).*

Since we have assumed the strong C*-norm condition (1.4) on A, A will trivially have an isometric involution. A is a maximal closed 2-sided ideal of A_e.

Up into the 1960's, much of the work on C*-algebras centered around the *representation theory* of these algebras. Let A be a C*-algebra. We often analyze the C*-algebra A by means of an isometric *-isomorphism Φ on A onto a more concrete C*- algebra. We call the isometric *-isomorphism Φ, a *representation* of the C*-algebra A. The representation of a C*-algebra is a major topic of this chapter and will be studied in depth in the next section, Chapter 1.2. Some of the key results of the thesis has been the result of the interplay of different ways of representing the entire C*-algebra A.

The Commutative Case:

Theorem 2 (Gelfand Naimark Theorem I) *(Chapter 1.1, [8]) Let A be a commutative C*-algebra. Then there exists a locally compact Hausdorff space X such that A is isometrically *-isomorphic to the C*-algebra $C_0(X)$. In the case that A has an identity, X is compact and hence A is isometrically *-isomorphic to $C(X)$, the C*-algebra of all continuous complex-valued functions on a compact Hausdorff space X.*

We construct the locally compact Hausdorff space X as follows: Let $X = A^\dagger$, the *Gelfand space* of the C*-algebra A, denote the set of all the non-zero complex-valued algebra-homomorphisms on A. The Gelfand space, $A^\dagger$, is a subset of the unit ball of the continuous dual of A, taken as a normed space [Chapter V.7, Proposition 7.3 [9]]. The topology on $A^\dagger$, is the topology of point-wise convergence or equivalently, the weak*-topology. Now recall the fact that the unit ball of the continuous dual of A is weak*-compact. Further, depending on whether or not the zero - homomorphism belongs to the weak*-closure of $A^\dagger$, the Gelfand space $A^\dagger$ becomes locally compact or compact, respectively. The latter case arises when A has an identity [Chapter V, Proposition 5 [9]].

The representation is the *Gelfand transform* which is an isometric *-isomorphism of A onto $C_0(X)$. The Gelfand transform, $\hat{}$, is the map $\hat{} : A \to C_0(X) | a \mapsto \hat{a}$ where $\hat{a}$ is the point-evaluational functional restricted to the Gelfand space, $A^\dagger \subseteq A^*$, evaluated at a : $\hat{a} : A^\dagger \to \mathbf{C} | \phi \mapsto \phi(a)$.

The Non-commutative case :

The hint for a representation of a non-commutative C*-algebra is offered by the finite dimensional case. The C*-algebra, $M_n(\mathbf{C})$ [Chapter 1.1, Example 3], is a prototype of a finite dimensional C*-algebra, of dimension n^2: there exists a finite dimensional Hilbert space $\mathbf{C}^n$ such that $M_n(\mathbf{C})$ can be identified as the C*-algebra of all bounded operators on the Hilbert space $\mathbf{C}^n$. In fact, if A is a finite dimensional C*-algebra, then A is isometrically *-isomorphic to *C*-direct sum* $\sum_{k=1}^m \bigoplus A_k$, where each A_k is isomorphic to the matrix algebra of $n_k \times n_k$ complex matrices, $M_{n_k}(\mathbf{C})$ [Chapter VI.3, Proposition 3.14 [9]]. Consequently, A has an identity.

The C*-direct sum $\sum_{k=1}^m \bigoplus A_k$ is vector space direct sum of all the A_k's, where we define addition, multiplication, addition and scalar multiplication on $\sum_{k=1}^m \bigoplus A_k$ point-wise and the C*-norm as follows:

$$\| x \| = \sup_{1 \leq k \leq m} \| x_k \|$$

where $x = (x_1, \ldots, x_k, \ldots, x_m)$ with $x_k \in A_k$.

Therefore each *finite dimensional* C*-algebra is a isometrically *-isomorphic to a *-subalgebra of the space of all bounded operators $B(\mathbf{H})$ on some *finite dimensional* Hilbert space $\mathbf{H} = \sum_{k=1}^{m} \bigoplus \mathbf{C}^{n_k}$. Consider each A_k as the space of all operators on the Hilbert space $\mathbf{C}^{n_k}$. Set $\mathbf{H}$ as the finite Hilbert direct sum $\sum_{k=1}^{m} \bigoplus \mathbf{C}^{n_k}$. Take the *-subalgebra as the operators on $\sum_{k=1}^{m} \bigoplus \mathbf{C}^{n_k}$ of the form of the direct sum $\sum_{k=1}^{m} \bigoplus T_k$ where $T_k \in A_k = B(\mathbf{C}^{n_k})$.

In fact each *separable* C*-algebra A is isometrically *-isomorphic to a *-subalgebra of the space of all bounded operators $B(\mathbf{H})$ on some *separable* Hilbert space $\mathbf{H}$ [Chapter VI.22 Proposition 22.13, [9]]. For the general C*-algebra A, we fortunately have the following analogous representation:

Theorem 3 (Gelfand Naimark Theorem II) *(Chapter 1.1, [8]) Let A be any C*-algebra. Then A is isometrically *-isomorphic to a norm-closed *-subalgebra of bounded linear operators on some Hilbert space.*

For a detailed introduction of theorem 2, the reader is referred to [9]. A detailed explanation of theorem 3 is given in section 1.2.

The rich structure on a C*-algebra, A, allows us to define the following concepts of a *self adjoint*, *normal*, *projection*, *positive* and *invertible* element as well as the concept of the *spectrum* of an element in a C*-algebra by analogy to the theory of operators on Hilbert spaces:

An element x in A is called *self adjoint* if $x^* = x$; x is *normal* if $x^*x = xx^*$; x is *idempotent* if $x^2 = x$; x is a *projection* if $x^* = x = x^2$; x is *invertible* if $xy = yx = 1$ for some y in A and provided A is unital; if A is unital, then $1^* = 1$ and consequently, if x is invertible the $(x^*)^{-1} = (x^{-1})^*$.

Clearly self-adjoint elements are normal and hence the C*-algebra generated by these elements are commutative. In fact, x is normal if and only if x belongs to some commutative *-subalgebra of A.

For an element x in A, the *spectrum of x in A* , $\sigma_A(x)$, is defined to be the set of all those complex numbers λ such that $x - \lambda 1$ has no inverse **in** A. In the case that, A has no identity, the spectrum $\sigma_A(x)$ is defined to be the same as $\sigma_{A_e}(x)$ where A_e is the C*-algebra which is the unitization of A [see Theorem 1]. We can do away with a case by case definition of the spectrum using the concept of an *adverse*. Define a new operation $\circ$ on A as follows: $x \circ y = x + y - xy$.

This operation is a measure of the difference between the sum and the product of two elements. Now, the operation is associative and has 0 as the unit. We say x has an adverse y if and only if $x \circ y = y \circ x = 0$. For the case of A having an identity, x has an adverse if and only if $1 - x$ is invertible. In the case where A has no identity, y is an adverse of x in A, forces y to be in A. Hence we define the spectrum of x in A, $\sigma_A(x)$, as follows:

Definition 1 *(Chapter 5, section 5, Proposition 5.8 [9]) Let x be an element in the C*-algebra A. Then a non-zero complex number λ belongs to $\sigma_A(x)$ if and only if $\lambda^{-1}x$ has no adverse in A. Further, $0 \notin \sigma_A(x)$ if and only if A has an identity and x^{-1} exists in A.*

The element x in the C*-algebra A is *positive*, $x \geq 0$, provided that x is hermitian and its spectrum in A, $\sigma_A(x)$, consists entirely of nonnegative real numbers. Kaplansky showed that this is equivalent to the existence of a y in A such that $x = y^*y$. If A^+ denotes all the positive elements, then A^+ forms a *closed (convex) cone with vertex 0* [Chapter 3, section 12, Theorem 12.3 [8]]. Recall that a *cone with vertex 0*, is a non-empty subset K of the vector space A, such that:

$$(a) \quad If\ x \in K, then\ \lambda x \in K\ for\ all\quad \lambda \geq 0\quad (the\ ray\ 0x\ belongs\ to\ K)$$
$$(b) \quad Whenever\ x, y \in K, x + y \in K.$$

Hence K is closed with respect to convex combinations (i.e convex). Further, the cone is *proper*: it cannot contain both a and $-a$ unless $a = 0$ ($\lambda \in \mathbf{C}$ is in the spectrum of a if and only if $-\lambda$ is in the spectrum of $-a$.)

Treating the C*-algebra A as a real vector space, the fixed cone $K = A^+$ defines a partial order $\leq$ (the antisymmetry of the partial order $\leq$ is by virtue of the cone being proper) as follows:

$$\text{If } a, b \in A, \text{ then } a \leq b \text{ if and only if } b - a \in A^+.$$

The resulting (partial) order allows us to treat A as an ordered vector space [Chapter III.1 Proposition 1.1.1 [25]] and satisfies the following conditions:

Definition of Order by the Cone

$$(a) \quad a \geq 0 \Leftrightarrow a \in A^+ \tag{1.5}$$

Partial Order Axioms

$$(b) \quad a \leq a \tag{1.6}$$
$$(c) \quad a \leq b \leq c \Rightarrow a \leq c \tag{1.7}$$
$$(d) \quad a \leq b \leq a \Rightarrow a = a \tag{1.8}$$

Compatibility with vector space operation Axioms

$$(e) \quad a \leq b,\ 0 \leq \lambda \in \mathbf{R} \Rightarrow \lambda a \leq \lambda b \tag{1.9}$$
$$(f) \quad a \leq b,\ c \in A \Rightarrow a + c \leq b + c \tag{1.10}$$

By the Gelfand Naimark Theorem I (theorem 2) we can add a further two inequalities which we shall use later on in the thesis [see proof of Chapter 2, Theorem 2.2.5, [13]].

$$(g) \quad If\ 0 \leq a \leq b,\ then\ \parallel a \parallel \leq \parallel b \parallel \tag{1.11}$$

$$(h) \quad If\ A\ has\ an\ identity\ and\ a, b \in A^+, then\ a \leq b \Rightarrow 0 \leq b^{-1} \leq a^{-1} \tag{1.12}$$

We have seen examples of C*-algebras without an identity. However, all C*-algebras A have an *approximate identity.* An approximate identity is a fixed net of elements e_α in A such that $\lim_\alpha e_\alpha x = \lim_\alpha x e_\alpha = x$ for any x in A. If the net can be indexed by the set of positive counting numbers then the C*-algebra is $\sigma - unital$. In fact, the approximate identity can be chosen such that it is *bounded by 1* ($\parallel e_\alpha \parallel \leq 1$ for all α), and is *increasing* (if $\alpha \leq \beta$, then $e_\alpha \leq e_\beta$). [Chapter 1, Proposition 13.1, [8]].

Prior to the advent of K-theory, it was desirable that every C*-algebra had an identity; if it did not, an identity was immediately adjoined to it by embedding it with a *-isometric isomorphism into a C*-algebra with an identity [Theorem 1]. In a total reverse of this trend, with the advent of K-theory and crossed products, if A is a C*-algebra, it is desirable that it does not have an identity. In the case that it does have an identity, we remove the identity by embedding it with a *-isometric isomorphism into the *stable algebra* $A \bigodot K(\mathbf{H})$, where $K(\mathbf{H})$ denotes the C*-algebra of all compact operators on a separable infinite dimensional Hilbert Space. The stable algebra $A \bigodot K(\mathbf{H})$ is a C*-algebra without an identity, and can be regarded as the spatial tensor product of the C*-algebras A and $K(\mathbf{H})$. In the thesis, this recent trend is of fundamental importance and we formally note:

Theorem 4 (Anti-Unitization) *If A is a C*-algebra with an identity, there exists a C*-algebra, the stable algebra $A \bigodot K(\mathbf{H})$, such that A embeds by a *-isometric isomorphism into $A \bigodot K(\mathbf{H})$ as a closed 2-sided ideal.*

The proof and description of this anti-unitization process is found in appendix B of the thesis.

1.2 C*-algebra : Global Representations

Some of the key results of the thesis has been the result of the interplay of different ways of representing the entire C*-algebra. This is what we mean by the title 'global representation', as opposed to the representation of the individual elements of the C*-algebra, the topic of the next section, Chapter 1.3. In section 1.2.1, we define the concept of a *-representation of a C*-algebra along with the different types of *-representations. In section 1.2.2. we then go onto describing the Universal representation and in section 1.2.3, we describe the Double Centralizer Algebra representation. These are both particular types of global representations critical to the thesis. Finally, in section 1.2.4, we relate these two global representations.

1.2.1 The Concept of a *-Representation

Let A be an algebra. We often analyze the algebra A by means of a homomorphism Φ on A into a more concrete algebra: the algebra of all linear endomorphisms on some vector space **H**. We call the homomorphism Φ, an *operator set on* **H** *indexed by the set* A : A behaves as an indexing set. Alternatively, we call the homomorphism Φ, *a representation of the algebra* A. If the dimension of **H** is $n \in \mathbf{N}$, we call Φ an n-dimensional representation of A. Formally, *a representation of the algebra* A is an operator set Φ indexed by A such that:

$$\begin{array}{ll} (a) & \Phi_{\lambda a} = \lambda \Phi_a \\ (b) & \Phi_{a+b} = \Phi_a + \Phi_b \\ (c) & \Phi_{ab} = \Phi_a \Phi_b \end{array}$$

If Φ is one-to-one, that is, $\mathrm{Ker}(\Phi) = 0$, Φ is called *faithful*.

Example 1 (A representation of a real algebra) *Let A be the complex number field* **C** *viewed as a 2 dimensional vector space over* **R** *with basis vectors* $\{1, i\}$. *We shall now represent each complex number z of the algebra A more concretely as a linear endomorphism on the vector space* $\mathbf{R}^2$.

Treating A as a two dimensional **R** *- vector space, for each fixed element z of the algebra A, let L_z denote the left multiplication map on the finite dimensional vector space A:*

$$L_z : A \to A | x \mapsto zx$$

Consequently, if $z = a + bi$ then the matrix associated with the linear transformation L_z with respect to the ordered basis $(1, i)$ will be the matrix $\mathcal{M}_z$:

$$\mathcal{M}_z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Then the homomorphism $\Phi : A \to M_2(\mathbf{R}) | z \mapsto \mathcal{M}_z$ is a faithful representation of A : all the conditions (a) - (c) above are met, where $\lambda \in \mathbf{R}$.

We now construct more examples of representations of algebras. For this, we resort to the concept of a group algebra of a group where an algebra is constructed from a given group. We now give an example of a group algebra:

Example 2 (Group Algebra) *Consider the finite additive group* $\mathbf{Z}_2 = \{0, 1\}$. *We shall use the multiplicative notation to denote the group operation of addition modulo 2. Let* A *be the group algebra* $\mathbf{CZ}_2$ *of the finite group* $\mathbf{Z}_2$. *As a vector space,* $\mathbf{CZ}_2$ *is the set of all the complex valued functions which has the set* $\mathbf{Z}_2$ *as its domain, denoted by* $\mathbf{C}^{\mathbf{Z}_2}$, *equipped with the usual vector space addition of point-wise addition and vector space scalar multiplication of point-wise scalar magnification.*

Consider $0^*, 1^* \in \mathbf{C}^{\mathbf{Z}_2}$ *where* $0^*, 1^*$ *are the characteristic functions* $\chi_{\{0\}}$ *and* $\chi_{\{1\}}$ *respectively. Then* $\{0^*, 1^*\}$ *forms a basis for the* $\mathbf{C}$ *- vector space* $\mathbf{C}^{\mathbf{Z}_2}$. *Hence the typical elements of* $\mathbf{C}^{\mathbf{Z}_2}$ *are the formal sums* $\alpha 0^* + \beta 1^*$ *where* $\alpha, \beta \in \mathbf{C}$.

In order to construct an algebra, we define vector multiplication in terms of the basis elements 0^* *and* 1^**; if* $a, b \in \mathbf{Z}_2$ *and their sum in* $\mathbf{Z}_2$ *is* ab, *then:*

$$a^* b^* = (ab)^*$$

Example 3 (A representation of a complex algebra) *Let* A *be the complex algebra* $\mathbf{CZ}_2$ *as defined in Example 2. We treat* A *as a 2 dimensional vector space over* $\mathbf{C}$ *with basis vectors* $\{0^*, 1^*\}$. *We shall now represent each element* z *of the algebra* A *more concretely as a linear endomorphism on the vector space* $\mathbf{C}^2$.

Treating A *as a two dimensional* $\mathbf{C}$ *- vector space, for each fixed element* z *of the algebra* A, *let* L_z *denote the left multiplication map on the finite dimensional vector space* A*:*

$$L_z : A \to A | x \mapsto zx$$

Consequently, if $z = \alpha 0^* + \beta 1^*$ *then the matrix associated with the linear transformation* L_z *with respect to the ordered basis* $(0^*, 1^*)$ *will be the matrix* $\mathcal{M}_z$*:*

$$\mathcal{M}_z = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

Then the homomorphism $\Phi : A \to M_2(\mathbf{R}) | z \mapsto \mathcal{M}_z$ *is a faithful representation of* A *: all the conditions (a) - (c) above are met, where* $\lambda \in \mathbf{C}$.

Analogously, if A is a *-algebra, we represent a *-algebra A by means of a *-homomorphism Φ on A into the concrete *-algebra of all bounded linear operators on some Hilbert space $\mathbf{H}$, $B(\mathbf{H})$: a *-representation is a representation with the additional conditions that it preserves the involution and the vector space $\mathbf{H}$ is a Hilbert space.

Formally, let A be a *-algebra. Then the *-homomorphism Φ on A into the *-algebra of all bounded linear operators on some Hilbert space $\mathbf{H}$, $B(\mathbf{H})$, is called *a *-representation of the *-algebra* A. Formally, *a *-representation of the *-algebra* A is an operator set Φ indexed by A where each Φ_a is a bounded linear operator, such that:

$$(a) \quad \Phi_{\lambda a} = \lambda \Phi_a \tag{1.12}$$
$$(b) \quad \Phi_{a+b} = \Phi_a + \Phi_b \tag{1.13}$$
$$(c) \quad \Phi_{ab} = \Phi_a \Phi_b \tag{1.14}$$
$$(d) \quad \Phi_{a*} = (\Phi_a)^* \tag{1.15}$$

Equivalently, a representation Φ of the underlying algebra A on $\mathbf{H}$, a Hilbert space with inner product $(\cdot|\cdot)$, is a *-representation provided that:

$$(\Phi_a(\xi)|\eta) = (\xi|\Phi_{a^*}(\eta)) \quad for\ all\ \xi, \eta \in \mathbf{H}$$

where each Φ_a is a bounded linear operator on $\mathbf{H}$.

Example 4 *Let A be the complex number field* $\mathbf{C}$ *as defined in Example 1. If we equip A with the operation of complex conjugation as the involution , then A is a *-algebra over the real field* $\mathbf{R}$. *We take* $\mathbf{B}(\mathbf{H})$ *as* $M_2(\mathbf{R})$. *Taking* Φ *as in Example 1, to show that* Φ *is a *-representation, it suffices to show that* $\Phi_{z*} = (\Phi_z)^*$. *This is immediate on noticing that:*

$$\mathcal{M}_{z^*} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

and that the adjoint of the matrix $\mathcal{M}_z$ *[see Example 1] is nothing but the transpose (we are working in a real vector space) and hence yields the matrix* $\mathcal{M}_{z^*}$.

Analogously,we define a representation of an abstract C*-algebra as a *bounded* *-homomorphism into the concrete C*-algebra of all bounded linear operators on some Hilbert space $\mathbf{H}$, in order to remain in the category of C*-algebras. Fortunately it is equivalent to work in the purely algebraic category of *-algebras: any *-homomorphism from a C*-algebra into a C*-algebra is bounded [Chapter VI.3 Theorem 3.7 [9]].

Example 5 (Representation of a C*-algebra) *Consider the Gelfand space* $A^\dagger$ *of a* **commutative** *C*-algebra A with an identity e. Let* Φ *be any one of the* **non-zero** *algebra homomorphism on A. We shall also call* Φ *a non-zero multiplicative linear functional. Treating* $\mathbf{C}$ *as* $M_1(\mathbf{C})$, Φ *is a one - dimensional representation of the C*-algebra A, viewed as an algebra:* $\Phi : A \to M_1(\mathbf{C})|z \mapsto \Phi(z)$ *where* $\Phi(z)$ *is a* 1×1 *matrix.*

Now, Φ *turns out to be a *-representation of A :* $\Phi(a^*) = \overline{\Phi(a)}$ *for all* $a \in A$. *This follows as a consequence [Chapter VI.2, Proposition 2.5 [9]] of the fact that* *every C*-algebra is symmetric [Chapter VI.7 Theorem 7.11 [9]] :* $(1 + a^*a)^{-1}$ *exists for all* $a \in A$ *[see Chapter VI.2, Proposition 2.5 [9] for other characterizations of symmetry].*

Examples 1 and 3, suggest a natural way of constructing *-representations of a C*-algebra A via the left multiplication map. Such representations are not too far from being exhaustive [1] in the following sense: suppose for a given C*-algebra A, there are two *-representations Φ and Φ' on the Hilbert spaces $\mathbf{H}$ and $\mathbf{H}'$ respectively; we do not distinguish between Φ and Φ', calling them *equivalent*, if there exists a vector space isomorphism $f : \mathbf{H} \longrightarrow \mathbf{H}'$ such that

$$f \circ \Phi_a = \Phi'_a \circ f \quad \textit{for all } a \in \ A.$$

In the more general case of f being a vector space homomorphism, we call f an *Φ, Φ' intertwining Hilbert space homomorphism.*

Example 6 (Equivalent Representation) *Let A be the complex number field* $\mathbf{C}$ *as defined in Example 4. Then we take* $\mathbf{B}(\mathbf{H})$ *as* $M_2(\mathbf{R})$ *and* Φ *as in Example 4 as our *-representation. Consider another *-representation* $\Phi : A \rightarrow M_2(\mathbf{R}) | z \mapsto \mathcal{M}'_z$ *where* $\mathcal{M}'_z$ *is the same matrix of transformation as defined in Example 1 except that it is written with respect to a different ordered basis* $(1, -i)$.

Then the change of basis matrix from the basis $(1, i)$ *to the basis* $(1, -i)$ *establishes the equivalence of the *-representations* Φ *and* Φ'.

We have now defined and illustrated the concept of a *-representation of a C*-algebra. In practise, we require the *-representation to fulfill more stringent requirements for it to be useful as a tool to represent the original C*-algebra. The two conditions are the condition of being an *irreducible* *-representation and the condition of being *non-degenerate* *-representation.

[1] Chapter IV.3, Proposition 3.9 [9]. This fact rests upon the existence of a fixed vector η_0 in the Hilbert space $\mathbf{H}$ of the representation Φ which single handedly generates the entire Hilbert space $\mathbf{H}$ as follows:

$$\mathbf{H} = \bigcup_{a \in A} \{\Phi_a(\eta_0)\}$$

1.2.1.1 Irreducible *-representation

Given a *-representation Φ, a vector subspace U, of the Hilbert space **H**, is an $\Phi -$ *invariant subspace of the operator set* Φ when $\Phi_a[U] \subset U$ for all a in A. The *-representation Φ , is called *irreducible* when no nontrivial subspace of the Hilbert space is a Φ - invariant subspace.

Example 7 *Any *-representation on a one dimensional Hilbert space is irreducible. Hence for any commutative C*-algebra A with an identity, every $\Phi \in A^\dagger$ is an irreducible *-representation [Example 6].*

If the above example is of any value, we need to assume that the Gelfand space $A^\dagger$ is non-empty. We now give examples of Gelfand spaces of commutative C*-algebras with identity that are non-empty and a Gelfand space of commutative Banach *-algebra that is empty:

Example 8 (Non-empty singleton Gelfand space) *Let A be the complex number field* **C** *viewed as a C*-algebra. Then the identity map $e(x) = x$ for all $x \in \mathbf{C}$ is a non-zero multiplicative linear functional.*

This is the only one. Note that $A^\dagger$ is one dimensional: the Gelfand space $A^\dagger$ is a subset of the unit ball of the continuous dual of the one dimensional **C***-vector space,* **C***. The algebraic dual is one-dimensional. Therefore, the continuous dual is also one-dimensional and is spanned by any non-zero vector, in particular, $e(x) = x$. Now $\Phi(xy) = \Phi(x)\Phi(y)$ iff $\lambda xy = \lambda x \lambda y$ iff $(\lambda^2 - \lambda)xy = 0$ for all $x, y \in \mathbf{C}$ iff $\lambda = 0$ or $\lambda = 1$.*

Note that the C*-algebra **C** is the only C*-algebra which is a field. This follows from the 1-1 correspondence between the maximal ideals of a C*-algebra which has an identity and the non-zero multiplicative linear functionals [Chapter V.7 Proposition 7.4 [9]]: any field only has the trivial ideal $\{0\}$ as the sole maximal ideal; consequently, by the Gelfand Naimark Theorem 1 (Theorem 2), the C*-algebra is a $C(K)$ where K is a singleton set.

We give another more sophisticated example of a non-empty Gelfand space where the cardinality of the Gelfand space is exactly the cardinality of the continuum.

Example 9 (Non-Empty Gelfand space) *Let the C*-algebra $A = C(K)$. Then the non-zero multiplicative linear functionals are precisely the point-evaluational functional Θ_x where $x \in K$:*

$$\Theta_x : C(K) \to \mathbf{C} | f \mapsto f(x)$$

If we define K as the compact unit interval $[0, 1]$ of the real line **R***, then each point-evaluational functional is distinct: the point e where $e(x) \mapsto x$ is the identity map on $[0, 1]$, separates the functionals: $\Theta_x(e) = x$.*

As a non-example, we give an example of an empty Gelfand space of a commutative Banach *-algebra. Our example must come from a (commutative) Banach *-algebra that does not have an identity since every such Banach *-algebra A has at least one multiplicative linear functional: we can assume A is not a field [the Banach *-algebra $\mathbf{C}$ is the only Banach *-algebra which is a field] and hence has a non-invertible element which generates a principal ideal which by Zorn's Lemma can be embedded into a maximal ideal [Chapter 12, Lemma 12.3, [28]].

Example 10 (Empty Gelfand space) *(Chapter II, example 9.3 [16]) Let A be $L^1([0,1])$, the Banach space of all complex valued functions which are absolutely summable (Riemann integrable) on the compact unit interval $[0,1]$. The norm is the usual Riemann integral:*

$$\| f \| = \int_0^1 |f(x)|dx \text{ for all } f \in \mathbf{C}^{[0,1]}$$

Note that f Riemann integrable implies f bounded. We take as our multiplication operation, the convolution product, $f * g$, of the functions f and g of $L^1([0,1])$ which is the function defined as the following indefinite integral:

$$(f * g)(x) = \int_0^x f(t)g(x-t)dt \text{ for all } x \in [0,1]$$

*Then $\| (f * g) \|$ is an iterated integral which is dominated by $\| f \| \| g \|$. Defining the involution as point-wise complex conjugation, $L^1([0,1])$ is a Banach *-algebra with an isometric involution.*

First note that the functions in $L^1([0,1])$ which are identically zero on some neighbourhood of 0, $[0,\epsilon]$, form a dense subset of $L^1([0,1])$. By the nature of the involution, such a function, f, is nilpotent: there exists a natural number n such that $f^n = 0$ since $f^k(x) = 0$ for all $x \in [0, k\epsilon]$. Hence if Φ is a multiplicative linear functional, then $\Phi(f) = 0$ for all such f. Therefore Φ is zero for all $g \in L^1([0,1])$ since it is continuous.

Note that $L^1([0,1])$ does not satisfy the strong C^*-norm condition (1.4): set $f(x) = \chi_{[0,\frac{1}{2}]}$. Then $f^* = f$ and $(f * f^*) = 0$. Hence $\| f \| = \frac{1}{2}$ but $\| f * f^* \| = 0$.

1.2.1.2 Non-degenerate *-representation

The *-representation Φ , is called *non-degenerate* when the ranges of Φ_a, $a \in A$, span $\mathbf{H}$ and $\{\Phi_a | a \in A\}$ separates points in the Hilbert space $\mathbf{H}$, that is, $\bigcap_{a \in A} Ker(\Phi_a) = 0$. In fact, we shall show later that the two conditions are logically equivalent to each other.

Proposition 1 (Irreducible Implies Non-degenerate) *All irreducible -representations are non-degenerate.*

Proof. Define the *null space of a *-representation* Φ, $N(\Phi)$, as the set $\{\eta \in \mathbf{H} | \Phi_a(\eta) = 0 \; for \; all \; a \in \; A\}$. Note that $N(\Phi) = \bigcap_{a \in A} Ker(\Phi_a)$. Hence $N(\Phi)$ is closed and trivially Φ - invariant subspace of $\mathbf{H}$. If Φ is irreducible, then its null space $N(\Phi)$ is either the trivial subspace $\{0\}$ or the entire Hilbert space $\mathbf{H}$. The latter case is not possible since Φ is non-zero. The former case shows that $\{\Phi_a | a \in A\}$ separates points in the Hilbert space $\mathbf{H}$.

Q.E.D

Example 11 *All irreducible representations are non-degenerate representations. Hence, any element of the Gelfand space of a C*-algebra is a non-degenerate representation [see Example 7].*

The following example is a non-degenerate *-representation on a C*-algebra A which fails to be anywhere near to being *-irreducible. We define the C*-algebra A as follows:

Definition 1 *Let $(\Omega, \mathcal{B}(\Omega), \mu)$ is a σ-finite measure space: $\mathcal{B}(\Omega)$ is the smallest σ - algebra of subsets of Ω which supersets the set of all open sets of Ω. Then let A be the C*-algebra $L^\infty(\Omega, \mathcal{B}(\Omega), \mu)$ of all essentially bounded Borel measurable complex valued functions defined on a set Ω equipped with a locally compact Hausdorff topology where μ is a complex valued regular Borel measure defined on $\mathcal{B}(\Omega)$.*

A function f is essentially bounded if and only if $|f| \leq M$ μ-almost everywhere. The L^∞ - norm : $\| f \|_\infty = inf\{M| \;\; |f| \leq M \;\;\; \mu - \;\; almost \; everywhere\}$, satisfies the strong C-norm condition [Chapter 1.1, equation 1.4]. The involution is pointwise-conjugation.*

The assumption that Ω is a locally compact Hausdorff space is needed to establish the existence of a regular Borel measure on Ω. This results from the fact that $C_0(\Omega)^* = M(\Omega)$ where $M(\Omega)$ denotes all the regular Borel measures on Ω which follows from the local compactness of Ω [Theorem 9.16 [21]] and the fact that there are non-zero bounded linear functionals on $C_0(\Omega)$: $C_0(\Omega)^* \neq \emptyset$ [Hahn Banach Theorem for normed spaces, Proposition 6.1.7 [20]].

The assumption of a σ-finite measure space, $(\Omega, \mathcal{B}(\Omega), \mu)$, is needed in anticipation of an application of the Radon Nikodyn theorem for Complex measures [Chapter 6, Theorem 6.12 [23]]which is based on the Radon Nikodyn theorem [Chapter 6, Theorem 6.10 [21]] where the σ-finiteness of μ cannot be dropped: the measure space $(\mathbf{N}, \mathcal{A}, \mu)$ where $\mathbf{N}$ is the set of all natural numbers, $\mathcal{A} = \{\mathbf{N}, \emptyset, \{1\}, \mathbf{N}\backslash\{1\}\}$ and μ is the counting measure; any measure on $\mathbf{N}$ is absolutely continuous with respect to μ.

Example 12 (Non-degenerate but far from being irreducible) *Consider the Hilbert space $\mathbf{H} = L^2(\Omega, \mathcal{B}(\Omega), \mu)$ of all square integrable functions on the measure space $(\Omega, \mathcal{B}(\Omega), \mu)$ [Definition 1]. Now, for each $g \in L^\infty(\Omega, \mathcal{B}(\Omega), \mu)$ we define the associated multiplication operator $M_g \in B(\mathbf{H}$:*

$$M_g : u \mapsto gu \text{ where } gu(x) = g(x)u(x) \quad \forall x \in \Omega.$$

since $gu \in L^2(\Omega, \mathcal{B}(\Omega), \mu)$ since $|g(x)| \leq M$ for all $x \in \Omega$ except on a measure zero set for some $M > 0$. The map

$$\Phi : L^\infty(\Omega, \mathcal{B}(\Omega), \mu) \to B(\mathbf{H}) | g \mapsto M_g$$

*is a non-degenerate *-representation of the C^*-algebra $L^\infty(\Omega, \mathcal{B}(\Omega), \mu)$ which is far from irreducible.*

Firstly, we show each $M_g \in B(\mathbf{H})$. Let us denote the norm on the Hilbert space $\mathbf{H} = L^2(\Omega, \mathcal{B}(\Omega), \mu)$ and the operator norm of the bounded operators on $\mathbf{H} = L^2(\Omega, \mathcal{B}(\Omega), \mu)$ as $\| \cdot \|_H$ and $\| \cdot \|$ respectively. Then $\| M_g \| = \sup_{\|u\|_H \leq 1} \| gh \|_H \leq \| g \|_\infty$:

$$\| gu \|_H^2 = \int_\Omega |gu|^2 d\mu \leq \| g \|_\infty^2 \int_\Omega |u|^2 d\mu = \| g \|_\infty^2 \| u \|_H.$$

*Since $M_g^* = M_{\bar{g}}$, $M_{af+bg} = aM_f + bM_g$, $M_{fg} = M_f M_g$ we conclude that Φ is a *-representation.*

The map Φ is non-degenerate : the constant 1 -map, $e(x) = 1$ for all $x \in \Omega$, is the identity for $L^\infty(\Omega, \mathcal{B}(\Omega), \mu) \subset L^2(\Omega, \mathcal{B}(\Omega), \mu)$; M_e is hence the identity operator on $\mathbf{H} = L^2(\Omega, \mathcal{B}(\Omega), \mu)$; therefore Φ separates points in the Hilbert space.

*The map Φ is far from *-irreducible: let S be any measurable subset of Ω of non-zero measure; set $V = \{u \in L^2(\Omega, \mathcal{B}(\Omega), \mu) | \ \ u(s) = 0 \ \ \text{for almost all } s \in S\}$; then V is a subspace of $\mathbf{H}$ and is Φ-invariant since $M_g[V] \subset V$ for all $g \in L^\infty(\Omega, \mathcal{B}(\Omega), \mu)$.*

The σ - finiteness of the measure space is needed in showing that $\mathbf{H}$ is non trivial : it contains all the constant functions on Ω since μ is a complex measure. This follows from the string of equalities

$$\int k^2 d\mu = \int k^2 h d|\mu| \leq k^2 \int d|\mu| = k^2 |\mu|(\Omega) < \infty \quad k \in \mathbf{R}$$

since there exists a complex valued Borel measureable function h such that $|h(x)| = 1$ for all $x \in \Omega$ which is the Radon Nikodyn derivative of μ with respect to the *bounded* positive measure $|\mu|$ [Theorem 6.12 [23]].

As promised earlier, we show that the two conditions defining a non-degenerate *-representation are equivalent. Let the orthogonal complement of $N(\Phi)$, $N(\Phi)^{\perp}$, be called the *essential space of the *-representation* Φ. Now let $R(\Phi)$ be the closed linear span of $\{Range(\Phi_a) | a \in A\}$. Now $N(\Phi)^{\perp} = R(\Phi)$ [Chapter VI, Prop 9.6 [9]] and we have a decomposition of the Hilbert space $\mathbf{H}$: $\mathbf{H} = N(\Phi) \oplus R(\Phi)$ into two closed $\Phi -$ invariant subspaces. Therefore, the two conditions of Φ being non-degenerate is satisfied by $N(\Phi) = 0$. From now on, we require

*All *-representations to be at least non-degenerate.*

1.2.1.3 Irreducible Cyclic *-representations

We now relate the concepts of an irreducible *-representation and a multiplicative linear functional as hinted at by Example 7 for a commutative C*-algebra with an identity. In the context of an arbitrary C*-algebra, we need to impose more restrictions on the irreducible *-representations and the multiplicative linear functionals:

In any C-algebra, there is a bijective correspondence between the irreducible cyclic *-representations and pure states.*

Let A be an arbitrary C*-algebra. A *pure state* of A is a special type of *positive functional* on A. A positive functional on A is a linear functional p such that $p(x^*x) \geq 0$ for all positive elements x^*x in A. A *cyclic representation* Φ of a C*-algebra A is a *-representation of the C*-algebra A on a Hilbert space $\mathbf{H}$ which has the additional property of the existence of a vector $x_0 \in \mathbf{H}$ such that $\{\Phi_a(x_0) | a \in A\}$ is dense in $\mathbf{H}$. We call this vector x_0 a *cyclic vector.* This then forces Φ to be non-degenerate. The above correspondence is the crux of the famous Gelfand Naimark Theorem II [Chapter 1.1, Theorem 3].

We first describe the concept of a positive functional on a C*-algebra as integrals before we describe the concept of a pure state.

Example 13 (Positive Functionals: Integrals) *Let A be the C*-algebra $C([0,1])$, of the complex valued functions on the unit interval $[0,1]$. Then $p : C([0,1]) \to \mathbf{C}$ where:*

$$p : f \mapsto \int_0^1 f(x)dx \text{ where } \int_0^1 f(x)dx \text{ is the usual Riemann integral}$$

is a positive linear functional on $C([0,1])$: the positive elements of $C([0,1])$ are precisely the real valued functions whose range is a subset of $\mathbf{R}^{+0}$.

Since the Riemann Integral was developed from first defining the integral of a step function on the interval $[0,1]$, the Riemann Integral depends on the concept of length of special sets: intervals, which reside on the ordered real line. By virtue of the Lebesgue measure on the real line, the concept of length was extended to sets which are not intervals: the Lebesgue measure of the Cantor set is zero. Further, the Lebesgue integral extended the Riemann integral to allow the integration of Lebesgue measurable functions: non Riemann integrable functions like $\chi_{\mathbf{Q}\cap[0,1]}$ has a Lebesgue integral of 0. As a further generalization of the Riemann integral, the abstract Lebesgue integral allows us to integrate functions defined on an arbitrary measure space: continuity hence has been relaxed and the sets can assume forms which are not necessarily intervals; the sets nevertheless still conform to the geometric requirement of a measurable space:

Example 14 (Positive Functionals: Integrals without intervals) *Let A be the C*-algebra $L^{\infty}(\mu)$ of all essentially bounded $\mathcal{A}$ - measurable complex valued functions defined on a set Ω where $(\Omega, \mathcal{A}, \mu)$ is a measure space: $\mathcal{A}$ is a σ - algebra of subsets of Ω and μ is a positive measure defined on $\mathcal{A}$.*

Then $p : L^{\infty}(\mu) \to \mathbf{C}$ where:

$$p : f \mapsto \int_{\Omega} f(x)d\mu \text{ where } \int_{\Omega} f(x)d\mu \text{ is the abstract Lebesgue integral}$$

is a positive linear functional on $L^{\infty}(\mu)$: the positive elements of $L^{\infty}(\mu)$ are precisely the real valued functions whose range is a subset of $\mathbf{R}^{+0}$.

In order to completely abandon the sets which have a geometric character and hence integrate functions over arbitrary sets, we resort to the concept of a positive linear functional.

Example 15 (Positive Functionals: Integrals over arbitrary sets) *Let A be the C*-algebra $B(\mathbf{H})$ of all the bounded operators on some Hilbert space $\mathbf{H}$. Then $p : B(\mathbf{H}) \to \mathbf{C}$ where x_0 is a fixed vector in the Hilbert space $\mathbf{H}$:*

$$p_{x_0} : T \mapsto (Tx_0|x_0) \text{ where } (Tx|x) \text{ is the quadratic form associated with the sesquilinear form corresponding to the operator } T \in A$$

is a positive linear functional on $B(\mathbf{H})$: T is a positive operator in $B(\mathbf{H})$ if and only if $(Tx|x) \geq 0$ for each $x \in \mathbf{H}$ [Chapter 2.4 Definition 2.4.5 [10]].

If T is a positive operator in $B(\mathbf{H})$, then we can regard $(Tx_0|x_0)$ as the norm of the fixed vector x_0 with respect to the sesquilinear form induced by the positive operator T : the sesquilinear form corresponding to a positive operator T induces a semi-norm $\| \cdot \|$ on $\mathbf{H}$, $\| x_0 \| = (Tx_0|x_0)$ [Chapter II.3 Proposition 3.1.7, Chapter VI.2 Proposition 2.4.11 [25]].

The above example shows that for a given C*-algebra, positive functionals or integrals abound: for each fixed vector x_0 of the Hilbert space $\mathbf{H}$ we can associate a positive functional. Further, by the above example, we can associate a positive linear functional or integral with a *-representation of a C*-algebra :

Example 16 (Positive Functionals: Integrals associated with *-representation) *Let Φ be a *-representation of a C*-algebra A on a Hilbert space $\mathbf{H}$. Let x_0 be a fixed vector in the Hilbert space $\mathbf{H}$. Then we define a positive functional p_{x_0} on the C*-algebra A as follows:*

$$p_{x_0} : A \to \mathbf{C} | a \mapsto (\Phi_a(x_0)|x_0)$$

where $(\Phi_a(x)|x)$ is the quadratic form associated with the sesquilinear form corresponding to the operator Φ_a: Φ_a is a positive operator when a is a positive element.

The converse of Example 16 is true for a general C*-algebra, A:

*Given a positive functional p on A, there exists a *-representation Φ of A on some Hilbert space $\mathbf{H}$ with a vector $x_0 \in \mathbf{H}$ such that $p_{x_0} = p$ where p_{x_0} is constructed from Φ as in Example 15. [Chapter VI.19, Proposition 19.6 [9]]*

In fact, the construction used in example 16 establishes a 1- 1 correspondence between the set of all positive linear functionals on a C*-algebra A and the set of all equivalence classes of *cyclic *-representations* Φ of the C*-algebra A [Chapter VI. 19, Theorem 19.9 [9]] : set x_0 in the construction to be the cyclic vector of the cyclic *-representation.

If we impose a further condition on the set of positive functionals, we can in fact establish a 1-1 correspondence with the set of all equivalence classes of irreducible cyclic *-representations [Chapter VI.20, Theorem 20.4 [9] / Proposition 4.5.3 [10]]. The condition required of the positive functional is it to be *indecomposable*. We call such a positive linear functional a *pure state*. A positive functional p on a C*-algebra A is indecomposable if any positive linear functional q on A it *dominates* is of the form λp for some $0 \leq \lambda \leq 1$. p dominates q if the restriction of p to the positive cone A^+ is greater then the restriction of q to the positive cone A^+ as positive real valued functions.

The construction of the unique (up to equivalent classes) irreducible cyclic *-representation associated with a pure state p on the C*-algebra A rests on the following construction:

Theorem 1 *(Chapter 4, Proposition 4.5.1 [10]) Let p be a state of a C*-algebra A. Then the set $L_p = \{a \in A | p(a^*a) = 0\}$ is a closed left ideal of A and in particular, $p(b^*a) = 0$ whenever $a \in L_p$ and $b \in A$. We call the set L_p the left kernal of the pure state p. The equation:*

$$\Big(a + L_p | b + L_p \Big) = p(b^*a) \text{ where } a, b \in A$$

defines a positive definite inner product $\Big(\cdot | \cdot \Big)$ on the the quotient linear space A/L_p.

We take the unique completion of the pre-Hilbert space A/L_p as the Hilbert space $\mathbf{H}$ associated with the cyclic *-representation Φ constructed from the pure state p .

1.2.2 The Universal Representation : A Non-Degenerate *-Representation

We now describe the universal representation of a C*-algebra A. The essence of this representation is to represent the C*-algebra A as a norm-closed *-subalgebra of the space $B(\mathbf{H})$ of all bounded operators on some Hilbert space $\mathbf{H}$. The key to establishing an isometric *-isomorphism with this *-subalgebra lies in the abundance of *-representations of A which *distinguishes* points of the C*-algebra A:

Definition 2 (Reduced C*-algebra) *Let A be a C*-algebra. Then let $S = \{\Phi^\alpha | \alpha \in \Lambda\}$ be a family of *-representations of A, where Λ is the indexing set.*

We say S separates or distinguishes points if for any non-zero element a in the C-algebra, there exists a *-representation $\Phi^{\alpha_0} \in S$ such that $\Phi_a^{\alpha_0} \in B(\mathbf{H}^{\alpha_0})$ is not the zero-operator on $\mathbf{H}^{\alpha_0}$. Equivalently, $a \neq a'$ implies that there exists an $\alpha_0 \in \Lambda$ such that $\Phi_a^{\alpha_0} \neq \Phi_{a'}^{\alpha_0}$ as operators on $\mathbf{H}^{\alpha_0}$.*

*If A has such a set S of *-representations, we say that A is reduced.*

Theorem 1 *(Chapter VI.10 Proposition 10.6 [9]) If A is a reduced C*-algebra then it is isometrically *-isomorphic with a norm-closed *-subalgebra of $B(\mathbf{H})$ for some Hilbert space $\mathbf{H}$. The isometric *-isomorphism is an isometric one-to-one *-representation of the C*-algebra A.*

Proof. Let $S = \{\Phi^\alpha | \alpha \in \Lambda\}$ be a family of *-representations of the C*-algebra A , where Λ is the indexing set. Then

$$\mathbf{H} = \Sigma \oplus_{\alpha \in \Lambda} \mathbf{H}^\alpha$$

the Hilbert space direct sum of the Hilbert spaces $\mathbf{H}^\alpha$ associated with each of the *-representations Φ^α. Our candidate for the *-representation $\boldsymbol{\Theta}$ will be

$$\boldsymbol{\Theta} : A \to B(\mathbf{H}) |\ a \mapsto \sum \bigoplus \Phi_a^\alpha$$

where $\boldsymbol{\Theta}$ maps each element a in A to the Hilbert direct sum $\sum \bigoplus \Phi_a^\alpha$ of the bounded operators $\Phi_a^\alpha \in B(\mathbf{H}^\alpha)$ and $\boldsymbol{\Theta}_a \in$ B($\mathbf{H}$) is of the form $\Sigma \oplus_{p \in \Lambda} \Phi_a^p$, the direct sum of the family $\{\Phi_a^p\ \in B(\mathbf{H}^p) |\ a \in A\}$:

$$\sum \bigoplus \Phi_a^\alpha : \mathbf{H} \to \mathbf{H} |\ (x_\alpha)_{\alpha \in \Lambda} \to (\Phi_a^\alpha(x_\alpha))_{\alpha \in \Lambda}$$

We can take the direct sum since each $\Phi_a^p\ \in B(\mathbf{H}^p)$ is bounded by $\| a \|$: each *-representation Φ^p is a *-homomorphism on a C*-algebra A into a C*-algebra $B(\mathbf{H}^p)$; hence each Φ^p is continuous and $\| \Phi^p \| \leq 1$ [Chapter VI Theorem 3.7 [9]]; it follows that $\| \Phi^p(a) \| \leq \| a \|$.

Then $\mathbf{\Theta}$ is a *-homomorphism between the C*-algebra A and the C*-algebra $B(\mathbf{H})$ by equations (1.12) - (1.15), Chapter 1.2.1. Once we show that $\mathbf{\Theta}$ is one-to-one then we are done since any one-to-one *-homomorphism from a C*-algebra into a C*-algebra is an isometric *-isomorphism [Chapter VI.8, Proposition 8.8 [9]]. Then the range is complete and hence closed. $\mathbf{\Theta}$ is 1-1 by virtue of the fact that S separates the points of A: for any non-zero a in the C*-algebra, there exists a *-representation $\Phi^{\alpha_0} \in S$ such that $\Phi_a^{\alpha_0} \in B(\mathbf{H}^{\alpha_0})$ is not the zero-operator on $\mathbf{H}^{\alpha_0}$.

Q.E.D

Now every C*-algebra A is reduced. In fact, A is reduced by the family of all irreducible cyclic *-representations or equivalently, in the language of functional analysts, has enough pure states on A to separate the points in A.

Let a denote any non-zero element of the C*-algebra A. Let us assume that A has enough pure states to separate its points. Then there exists a pure state p such that $p(a) \neq 0$. The *-representation, Φ^p, of A associated with the pure state p_{x_0} on the Hilbert space $\mathbf{H}^p$ is an irreducible cyclic *-representation with a non-zero cyclic vector x_0 in $\mathbf{H}^p$. Now, $\Phi_a^p \neq 0$ - operator in B($\mathbf{H}^p$) : $(\Phi_a^p(x_0), (x_0)) = p(a) \neq 0$ [see Example 15]. The converse is established similarly.

We now show that every C*-algebra A has enough pure states on A to separate the points of A. We assume without loss of generality that the C*-algebra has an identity since every positive functional p on a C*-algebra A is *extendable*[Chapter VI.19 Proposition 19.9 [9]] : p is extendable if p can be extended to a positive functional on A_e, the C*-unitization of A. We take A_e as A in the case where A does not have an identity. We need the following lemma:

Lemma 1 (Pure states separates positive elements) *Let a_0 be an arbitrary non-zero fixed positive element of the C*-algebra A_e. There exists a pure state p such that $p(a_0) \neq 0$. Equivalently, every C*-algebra A_e has enough irreducible *-representations to separate its positive elements.*

Proof. There exists a positive functional p on A_e such that $p(e) = 1$ and $p(a_0^* a_0) = \| a_0 \|^2 \neq 0$ [Chapter 3, Theorem 18.1, [8]]. Since A_e has an identity and an isometric involution, p is continuous and $\| p \| = p(e) = 1$ [Chapter 4, Theorem 22.11, [8]]. Equivalently, there exists a positive functional p with norm 1 such that $p(a_0) \neq 0$. [Chapter 5, Theorem 29.9, [8]]

Q.E.D

Using the canonical construction of Example 16, we now show that the existence of a pure state that separates the non-zero positive elements of a C*-algebra implies the existence of other pure states that separate arbitrary non-zero elements of the C*-algebra.

Lemma 2 *Let a be an arbitrary non-zero fixed element of the C*-algebra A_e. If there exists a pure state p such that $p(a^*a) \neq 0$ then there exists another pure state p' such that $p'(a) \neq 0$.*

Proof. Let Φ be the irreducible cyclic *-representation with a cyclic vector x_0 on the Hilbert space $\mathbf{H}^p$ associated with the pure state p via the construction used in Example 16. Then, $p(a^*a) \neq 0$ implies that the operator Φ_a is not the 0-operator on $(\mathbf{H}^p, \| \cdot \|)$:

$$\Big(\Phi_{a^*a}(x_0)|x_0 \Big) = \Big(\Phi_{a^*}\Phi_a(x_0)|x_0 \Big) = \Big(\Phi_a(x_0)|\Phi_a(x_0) \Big) = \| \Phi_a(x_0) \|^2 \neq 0.$$

Therefore Φ_a is not the 0-operator on $\mathbf{H}^p$.

Consequently by the Polarization Identity, there exists a vector $y_0 \in \mathbf{H}^p$ such that the quadratic form $(\Phi_a(y_0), y_0) \neq 0$. Then using the construction of Example 16, we construct another positive functional, p_{y_0}, on A for the same *-representation Φ. This completes the proof since $p_{y_0}(a) = (\Phi_a(y_0), y_0) \neq 0$. p_{y_0} plays the role of the required p'.

Q.E.D

We can therefore conclude that :

Theorem 2 (Pure states separates points of C*-algebra) *Let a_0 be an arbitrary non-zero fixed element of the C*-algebra A_e. There exists a pure state p such that $p(a_0) \neq 0$. Equivalently, every C*-algebra A_e has enough irreducible *-representations to separate its positive elements.*

and the famous Gelfand Naimark Theorem II [Theorem 3, Chapter 1.1] follows immediately from Theorem 1:

Gelfand Naimark Theorem II *Any C*-algebra A is isometrically *-isomorphic with a norm-closed *-subalgebra of $B(\mathbf{H})$ for some Hilbert space $\mathbf{H}$. Consequently, the isometric *-isomorphism Π is an isometric one-to-one *-representation of the C*-algebra A.*

Let $S(A)$ be the set of all the pure states on the C*-algebra, A. Let $S = \{\Phi^p | p \in S(A)\}$ be the family of irreducible cyclic *-representations of the C*-algebra A which is in bijection with the set of all pure states $S(A)$ via the construction used in Example 16. The Hilbert space $\mathbf{H}$ is the Hilbert space direct sum $\Sigma \oplus_{p \in S(A)} \mathbf{H}^p$ of the Hilbert spaces $\mathbf{H}^p$ associated with each of the *-representations Φ^p. We call this the *universal Hilbert Space.* The *-representation $\mathbf{\Theta} : A \to B(\mathbf{H}) = B(\Sigma \oplus_{p \in S(A)} \mathbf{H}^p)$ where $\mathbf{\Theta}$ maps each element a in A to the Hilbert direct sum $\sum \bigoplus \Phi^p_a$ of the bounded operators $\Phi^p_a \in B(\mathbf{H}^p)$, is called the *universal representation* of the C*-algebra A.

Finally, we show that the universal representation meets the minimum requirement of being a non-degenerate *-representation:

Theorem 3 *The universal representation is non-degenerate*

Proof. Each irreducible cyclic *-representation is trivially non-degenerate. Therefore each cyclic *-representation Φ^p where $p \in S(A)$ is non-degenerate and hence the universal representation which is the Hilbert direct sum $\Sigma \oplus_{p \in S(A)} \Phi^p$ is also non-degenerate [Chapter VI.9 Theorem 9.15 [9]].

Q.E.D

1.2.3 The Double Centralizer Algebra Representation (DCAR)

Let A be any C*-algebra. Then A can be identified with some norm closed *-subalgebra of the concrete C*-algebra of all bounded operators on the universal Hilbert space $\mathbf{H}$ [Gelfand Naimark Theorem II]. The left multiplication maps provided a natural way, aided by the fact that a one-to-one *-homomorphism from a C*-algebra into a C*-algebra is an isometric *-isomorphism, of constructing isometric *-isomorphic embeddings into some $B(\mathbf{H})$.

Example 17 *Recall the C*-algebras,* $\mathbf{C}$ *and* $L^\infty(\Omega, \mathcal{B}(\Omega), \mu)$ *[Chapter 1.2.1, Examples 1, 12]. In the case of* $L^\infty(\Omega, \mathcal{B}(\Omega), \mu)$*, since* $\mathbf{H} = L^2(\Omega, \mathcal{B}(\Omega), \mu)$ *contains the constant 1 -map,* $e(x) = 1$ *for all* $x \in \Omega$*,* Φ *is a one-to-one *-homomorphism into* $B(\mathbf{H})$*:* $f \neq g \Rightarrow L_f(e) \neq L_g(e)$*.*

Note that unlike in the case of $L^\infty(\Omega, \mathcal{B}(\Omega), \mu)$*, there was no need to construct a Hilbert space associated with the *-representation : the C*-algebra* $\mathbf{C}$ *turned out to be a Hilbert space itself and we took* $\mathbf{C}$ *as the Hilbert space* $\mathbf{H}$ *associated with the representation. We can always do this in the category of Banach spaces.*

1.2.3.1 Left Regular Representation : Category of Banach Spaces

All C*-algebras are Banach spaces and shall be take as such here.

Definition 3 (Left Regular Representation (LRR)) *The left regular representation* $\mathbf{\Theta}$ *of a C*-algebra* A *is the map* $\mathbf{\Theta} : A \to B(A) | a \mapsto L_a$ *where* $L_a : A \to A | \ x \mapsto ax$*.*

where $B(A)$ is the more concrete Banach space of all bounded operators on the C*-algebra A taken as a Banach space [compare Chapter 1.2.1, Example 1]. We need the following lemma to establish that $\mathbf{\Theta}$ is an isometric isomorphism in the category of Banach spaces:

Lemma 3 *(Chapter VIII Proposition 1.8 [24]) Let* A *be a C*-algebra and* a *be any element in* A*. Then:*

$$\| a \| = \sup\{\| ax \| \mid x \in A \ \| x \| \leq 1\} \tag{1.16}$$
$$= \sup\{\| xa \| \mid x \in A \ \| x \| \leq 1\} \tag{1.17}$$

It is now immediate that $\mathbf{\Theta}$ is an isometric isomorphism:

Theorem 4 *Let* A *be any C*-algebra. The left regular representation* $\mathbf{\Theta} : A \to B(A) | a \mapsto L_a$ *where* $L_a : A \to A | \ x \mapsto ax$ *is an isometric isomorphism from the C*-algebra* A *taken as a Banach space into the Banach space* $B(A)$

1.2.3.2 Inadequacy of LRR in Category of C*-algebras

Under the left regular representation, elements of the C*-algebra are represented as bounded operators on the C*-algebra A taken as a Banach space, $B(A)$, rather than as bounded operators on some Hilbert space $B(\mathbf{H})$. Stepping up into the category of C*-algebras where the involution is defined, we have the problem that if T is a bounded operator on A, then the canonical involution T^* is an operator on the dual space, A^*, of all bounded linear functionals on A. The dual A^* need not remain in the category of C*-algebras:

Example 18 *Let A, taken as a Banach space, be* $L^\infty([0,1], \mathcal{M}_{[0,1]}, \lambda_{[0,1]})$*, the Banach space of all essentially bounded functions on the unit interval* $[0,1]$ *endowed with the restricted Lebesgue measure,* $\lambda_{[0,1]}$ *: the Lebesgue measurable set* $[0,1] \subset \mathbf{R}$ *induces the sigma algebra* $\mathcal{M}_{[0,1]} = \{[0,1] \bigcap A | A$ *is Lebesgue measureable subset of* $\mathbf{R}\}$*. Then the dual* $(L^\infty([0,1], \mathcal{M}_{[0,1]}, \lambda_{[0,1]}))^*$ *contains* $L^1[0,1]$ *which when given the convolution as a product no longer satisfies the C*-algebra norm condition [Chapter 1.2.1, Example 10].*

We want to remain in the category of C*-algebras. Therefore the left regular representation of a C*-algebra A falls short in this regard.

1.2.3.3 DCAR : Category of Semigroups

The above example illustrated how the multiplication operation can be responsible for the problem of leaving the category of C*-algebras. To address this problem, we now shift focus on the multiplication operation of the C*-algebra. We shall study the multiplication operation in the general context of the category of a semigroup:

Definition 4 (Semigroup) *A semigroup is a set equipped with a single associative operation which assigns to every pair of element in the set a new element which we call their product.*

We see an immediate benefit of working in the more general category of semigroups: the left regular representation of a *semigroup* remains in the category of semigroups.

Example 19 *Let A be any C*-algebra. Then A with the ring multiplication is a semigroup. The set* $\{L_a | a \in A\}$ *of all left multiplication maps* $L_a : A \to A |\ x \mapsto ax$ *is a semigroup under the operation of function composition.*

In order to solve the problem of leaving the category of C*-algebras, we study the left multiplication maps in the context of a more general class of maps called the *left centralizer maps* .

Definition 5 (Left Centralizer) *A left centralizer map on the semigroup A is a map* $L : A \to A$ *such that*

$$L(x)y = L(xy) \quad for\ all\ x, y \in A \tag{1.18}$$

In order to emphasize the associative law, we define the action of L on x, $L(x)$, as the product of the map L with the element x, Lx, and hence equation (1.18) can be written in the associative notation $(Lx)y = L(xy)$.

We denote the set of all left centralizers by $\Gamma_l(A)$ which is a semigroup under function composition. Hence the set $\{L_a | a \in A\}$ of all left multiplication maps is a *sub-semigroup* of $\Gamma_l(A)$. A sub-semigroup of $\Gamma_l(A)$ is a subset of $\Gamma_l(A)$ which is a semigroup in its own right. The more abstract concept of the left centralizer map will not on its own solve the problem of leaving the category of C*-algebras when we return to viewing A as a C*-algebra. We need a new construct : the *double centralizer* which we shall define in the context of a semigroup. In short a double centralizer is a pair made out of a left and a *right centralizer map*, which we define next. We shall see later that this turns out to be equivalent to defining a double centralizer on a C*-algebra where we enforce the condition that the left and right centralizer maps are bounded operators on the C*-algebra.

Definition 6 (Right Centralizer) *First we define a right centralizer on A on the semigroup A as a map $R : A \to A$ such that*

$$xR(y) = R(xy) \quad for\ all\ x, y \in A \tag{1.19}$$

Example 20 *Let A be any C*-algebra. Then A with the ring multiplication is a semigroup. The set $\{R_a | a \in A\}$ of all right multiplication maps $R_a : A \to A|\ x \mapsto xa$ is a right centralizer on A. Note that the set of all right multiplication maps form a semigroup under the operation of function composition where $R_a R_{a'} = R_{a'a}$. Let us denote the set of all right centralizers by $\Gamma_r(A)$. Then $\Gamma_r(A)$ is a semigroup under function composition and the set $\{R_a | a \in A\}$ of all right multiplication maps is a sub-semigroup of $\Gamma_r(A)$.*

Again, in order to emphasize the associative notation, we define the action of R on x, $R(x)$, as the product of the map R with the element x, xR, so that equation (1.19) can be written in the more suggestive associative notation $x(yR) = (xy)R$. We now define the double centralizer on a semigroup A which solves the problem of leaving the category of C*-algebras:

Definition 7 (Double Centralizer) *The double centralizer on a semigroup A is an ordered pair $(L,\ R)$ of $A \to A$ maps with the condition*

$$xL(y) = R(x)y \quad for\ all\ x, y \in A. \tag{1.20}$$

Equation (1.20) becomes $x(Ly) = (xR)y$ if L is a left centralizer and R is a right centralizer. The set, $\Gamma(A)$, of all double centralizers on the semigroup A forms a semigroup by defining multiplication as follows:

$$(L, R) \cdot (L', R') = (L \circ L', R' \circ L') \tag{1.21}$$

where $\circ$ is function composition.

Example 21 *Let A be any C*-algebra . Then A with the ring multiplication is a semigroup. The ordered pair of maps (L_a, R_a) for some fixed $a \in A$ where L_a and R_a are the left and right multiplication maps defined as in Examples 19 and 20, respectively, is a double centralizer on the semigroup A: $xL_a y = xR_a y$ for all $x, y \in A$. Further, $(L_a, R_a) \cdot (L_{a'}.R_{a'}) = (L_{aa'}, R_{aa'})$.*

Example 22 (Multiplicative Identity) *Let A be any semigroup. Let 1_A denote the identity map on A, $1_A : A \to A | x \mapsto x$. The pair $(1_A, 1_A)$ is a double centralizer on A. The pair $(1_A, 1_A)$ is the multiplicative identity with respect to the semigroup $\Gamma(A)$ of all the double centralizers on A.*

1.2.3.4 Embedding Theorem I : Category of Semigroups

We are now ready to embed the C*-algebra, A , taken as a semigroup into the semigroup, $\Gamma(A)$, of all the double centralizers on A. Let Ψ be the map on A into $\Gamma(A)$ defined as $\Psi : A \to \Gamma(A) |\ a \mapsto (L_a, R_a)$. We call Ψ the *double representation* of the C*-algebra A. It is immediate that Ψ is a one-to-one semigroup-homomorphism from the following proposition which is a corollary of Lemma 3:

Proposition 1 *(Corollary 2.4 [2]) The C*-algebra A is a faithful semigroup. A faithful semigroup is a semigroup which does not have a pair of distinct fixed elements a, b with the property that $ax = bx$ for all $x \in A$ or $xa = xb$ for all $x \in A$. Equivalently, $a \neq b$ implies $L_a \neq L_b$ and $R_a \neq R_b$.*

The faithfulness of the C*-algebra contracts the double centralizer semigroup $\Gamma(A)$ as a sub-semigroup of all left centralizers $\Gamma_l(A)$ [see Theorem 2, [1]] as follows:

Theorem 5 *Let A be a C*-algebra taken as a semi-group with respect to the ring multiplication. If (L, R) is a double centralizer on A, then for a given right (left) centralizer R (L) there is a unique left (right) centralizer L (R) which has the property that (L, R) is a double centralizer, that is, satisfies equation (1.20).*

The map $\pi : \Gamma(A) \to \Gamma_l(A) |\ (L, R) \mapsto L$ is a one-to-one semigroup homomorphism. $\Gamma(A)$ can therefore be regarded as left centralizers that have the additional property of having a right centralizer map that satisfies equation (1.20). In the case that A is commutative, since the concepts of a left and a right centralizer coincides, π is onto and we write $\Gamma(A) = \Gamma_l(A)$.

In fact, $\Gamma(A)$ is the largest sub-semigroup of $\Gamma_l(A)$ which contains the set of all left multiplication maps $\{L_a |\ a \in A\}$ as a two-sided semigroup ideal of $\Gamma_l(A)$. A two-sided semigroup ideal of $\Gamma_l(A)$ is a sub-semigroup of $\Gamma_l(A)$ which absorbs products in $\Gamma_l(A)$: for each element T in the ideal, TL, LT are members of the ideal for each $L \in \Gamma_l(A)$.

Since the map $\mathbf{\Theta} : A \to \Gamma_l(A)|\ a \mapsto L_a$ is a one-to-one semigroup homomorphism [A is faithful], we can identify A with the set $\{L_a|\ a \in A\}$ in the category of semigroups. Therefore:

Theorem 6 (Representation Theory I) *Let A be a C*-algebra taken as a semigroup with respect to the ring multiplication. Then A is a two-sided semigroup ideal of the semi-group $\Gamma_l(A)$ where the sub-semigroup $\Gamma(A)$ of all the double centralizers of A is the largest sub-semigroup of $\Gamma_l(A)$ which contains A.*

Proof. The proof that the sub-semigroup $\Gamma(A)$ of all the double centralizers of A is the largest sub-semigroup of $\Gamma_l(A)$ which contains A, will give the origin of the definition of the double centralizer, equation (1.20).

Let S be any sub-semigroup of $\Gamma_l(A)$ which contains the 2-sided semigroup ideal $I = \{L_a|\ a \in A\}$. Once we show that for each left centralizer L in S, there exists a $R : A \to A$ such that (L, R) is a double centralizer, we are done.

Since I is an ideal, $L_x L = L_{w_{(L,x)}}$ where $w_{(L,x)}$ is an element in A. Consequently,

$$L_x L(y) = w_{(L,x)} \cdot y \text{ for all } y \in A$$

which is precisely equation (1.20).

Q.E.D

In the case where $\Gamma(A) = \Gamma_l(A)$, the above representation theory becomes redundant. This occurs, for instance, when A is commutative. We can still salvage the above representation theory by showing that the double centralizer semigroup $\Gamma(A)$ is still the largest semigroup containing A as a two-sided semigroup ideal in the following sense:

1.2.3.5 Embedding Theorem II : Category of Semigroups

The double representation $\Psi : A \to \Gamma(A)|\ a \mapsto (L_a, R_a)$ is a one-to-one semigroup homomorphism. Hence, we identify the C*-algebra A taken as a semigroup with the semigroup $\Psi(A) \subset \Gamma(A)$ which is a two-sided ideal of $\Gamma(A)$.

Let the C*-algebra A taken as a semigroup be an *essential two-sided ideal* of the semigroup S. We shall call the semigroup S the over-semigroup. An essential two-sided semigroup ideal, A, is a two-sided semigroup ideal with the additional condition of being *essentially faithful with respect to the entire semigroup S* : there are no two distinct pair of elements r, s in the over-semigroup S such that $ra = sa$ or $ar = as$ for all $a \in A$.

Example 23 *(Remark 3.1.3, [13]) The C*-algebra A taken as a semigroup is an essential two-sided ideal of the double centralizer semigroup $\Gamma(A)$.*

Let $R_s : S \to S$ and $L_s : S \to S$ denote the left and right multiplication maps on the over-semigroup S. Since A is a two-sided ideal of S, the restriction of R_s and L_s to the ideal $A \subset S$, which we denote as $L_s|_A$ and $R_s|_A$ respectively, produces a pair of additive centralizers on the semigroup A.

Now, the double representation $\Psi|^S : S \to \Gamma(A)|s \mapsto (L_s|_A, R_s|_A)$ is a semigroup homomorphism from the over-semigroup S into the semigroup of double centralizers $\Gamma(A)$. Since A is an essential two-sided semigroup ideal, $\Psi|^S$ is a one-to-one homomorphism: we identify the over-semigroup S with the sub-semigroup $\Psi|^S(S)$ of the double centralizer semigroup $\Gamma(A)$.

Theorem 7 (Representation Theory II) *Let A be the C^*-algebra taken as a semigroup with respect to the ring multiplication. The double centralizer semigroup $\Gamma(A)$ is the largest semigroup which contains A as an essential two-sided semigroup ideal.*

Having accounted for the origin of equation (1.20), we give an intuitive perspective of the requirement imposed by equation (1.21). The last equation of Example 21 illustrates a reason for the requirement imposed by equation (1.21). There is a deeper reason: equation (1.21) enables many theorems that apply to left centralizers $\Gamma_l(A)$ on commutative semigroups A, to generalize to non-commutative semigroups A, if the double centralizers $\Gamma(A)$ are considered.

Example 24 *If A is a commutative C^*-algebra, then A is a commutative semigroup with respect to the ring multiplication. Then if A has a cancellation law:*

$$xa = xb \Rightarrow a = b \text{ and } ax = bx \Rightarrow a = b \text{ for all } a,\ b,\ x \in H$$

then so does the semigroup, $\Gamma_l(A)$, of left centralizers on A [Theorem 1, [1]].

If the general C^*-algebra A has a cancellation law then so does $\Gamma(A)$ [Theorem 3, [1]].

Example 25 *The commutative C^*-algebra* **C** *has a cancellation law yet the commutative C^*-algebra $C([0,1])$ does not have a cancellation law; define $x(t)$, $a(t)$ and $b(t)$ as follows:*

$$x(t) = \begin{cases} 0 & : \quad 0 \le t \le \frac{1}{2} \\ 2t - 1 & : \quad \frac{1}{2} \le t \le 1 \end{cases}$$

$$a(t) = \begin{cases} 2t & : \quad 0 \le t \le \frac{1}{2} \\ -2t + 2 & : \quad \frac{1}{2} \le t \le 1 \end{cases}$$

$$b(t) = \begin{cases} 1 & : \quad 0 \le t \le \frac{1}{2} \\ -2t + 2 & : \quad \frac{1}{2} \le t \le 1 \end{cases}$$

and note that $xa = xb$ yet $a \neq b$.

We now make note of a simple yet fundamentally important observation:

Theorem 8 *If A is a C*-algebra with an identity e, then $\Gamma_l(A) = \Gamma(A) = \{L_a |\ a \in A\} = A$. Therefore the above representation theories are redundant.*

Proof. Any centralizer $(L,\ R)$ is of the form $(L_{L(e)},\ R_{R(e)})$ since $L(x) = L(ex) = (Le)x = L_{L(e)}(x)$ for all x in A. Similarly, since e is a right identity, $R(x) = R(xe) = xeR = x(eR) = R_{R(e)}(x)$ for all x.

Q.E.D

We now consider the double representation $\Psi : A \to \Gamma(A) \mid a \mapsto (L_a, R_a)$ in the category of rings to highlight the merits of the double representation, Ψ, and move a step closer to showing that the Representation Theory II [Theorem 7] does in fact hold in the category of C*-algebras.

1.2.3.6 DCAR : Category of Rings

Consider the C*-algebra A as a ring. In order to impose a ring structure on the double centralizer semi-group $\Gamma(A)$, we need an additive structure. For this purpose, we restrict our attention to the subset of *additive centralizers* on A : $\Phi(A)$. An additive centralizer on A is a double centralizer ,(L, R), on A with the additional condition that both L and R are additive maps on A. This restriction guarantees the distributivity of the multiplication operation on $\Gamma(A)$ over the addition operation on $\Gamma(A)$.

We then define the addition on $\Phi(A)$ as follows:

$$(L_1, R_2) + (L_2, R_2) = (L_1 + L_2, R_1 + R_2) \tag{1.22}$$

and the subset of additive centralizers $\Phi(A)$ becomes a ring with an identity. The double centralizer $(0_A, 0_A)$ where 0_A is the zero map $A \to A|a \mapsto 0$ is the 0-element of the ring. Further, the set of all the additive left (right) centralizers $\Phi_l(A)$ $(\Phi_r(A))$ is closed with respect to addition and hence is a ring.

It turns out that the restriction to additive centralizers exists only in name [see Theorem 7, [1]]:

Theorem 9 *Let A be a C*-algebra. Then $\Gamma(A) = \Phi(A)$: $\Gamma_l(A) = \Phi(A)$ and $\Gamma_r(A) = \Phi(A)$.*

Just as in the category of semigroups, the faithfulness of the C*-algebra contracts the double centralizer ring $\Gamma(A)$ as a subring of all the left centralizers $\Gamma_l(A)$: replace the term "semigroup" with the term "ring" in theorem 5. Therefore, the representation theory I (theorem 6) carries over word for word, replacing the term "semigroup" with "ring":

Theorem 10 (Representation Theory I) *Let A be a C*-algebra taken as a ring. Then A is a two-sided ring ideal of the ring, $\Gamma_l(A)$, of all the left centralizers, where the sub-ring $\Gamma(A)$ of all the double centralizers of A is the largest sub-ring of $\Gamma_l(A)$ which contains A.*

The same is true for the representation theory II (theorem 7). The double representation $\Psi : A \to \Gamma(A) \mid a \mapsto (L_a, R_a)$ embeds the C*-algebra A taken as a ring into the ring , $\Gamma(A)$, of double centralizers on A : Ψ is a one-to-one ring homomorphism. Therefore we can identify A with the sub-ring $\Psi(A)$:

Theorem 11 (Representation Theory II) *Let A be the C*-algebra taken as a ring. The double centralizer ring $\Gamma(A)$ is the largest ring which contains A as an essential two-sided ring ideal.*

Viewing the double representation Ψ in the category of ring makes good sense in light of the following theorem. We first introduce some terminology:

Definition 8 *Taking the C*-algebra A as a ring, I is an ideal of A if it is a sub-ring (a subset of A which is a ring in its own right) of A and a semigroup ideal of A viewed as a semigroup with respect to ring multiplication. A right (left) ideal of A is an additive subgroup of A closed with respect to right (left) multiplication. A is an additive group with respect to ring addition and an additive subgroup of A is a subset of A that is a group in its own right with respect to the ring addition. An ideal I of A is proper as long as it is not the trivial ideal A.*

A left (right) ideal I of the C-algebra A taken as a ring, is modular if there exists an $e \in A$ such that $A - Ae \subset I$, $(A - eA \subset I)$ or equivalently, $x - xe \in I$ $(x - ex \in I)$ for all $x \in A$: we say e is a right (left) identity for A modulo I. A (two-sided) ideal is modular if it is modular both as a left and as a right ideal. Then it is not hard to show that either the left or right identity modulo I serves as both the left and right identity modulo I*

An ideal is a maximal modular ideal if it is a proper modular ideal with the property that any other modular ideal that contains it, is the entire ring A.

Example 26 (Maximal Modular Ideal) *For the case of a commutative C*-algebra, the map which associates each non-zero multiplicative linear functional in the Gelfand space , $A^\dagger$, of the C*-algebra A, with its kernal is a bijective correspondence between the set of all modular maximal ideals of A and the Gelfand space $A^\dagger$ of the C*-algebra [Chapter V.7 Proposition 7.4 [9]].*

The following theorem, which follows from Theorem 9 [1], has deep implications since a lot of information about a C*-algebra rests upon the structure of its maximal modular ideals [Chapter V, Proposition 5.12 [9]]:

Theorem 12 (Preservation of Modular Ideal Structure) *Let A be a C*-algebra taken as a ring. Then any maximal modular right, left or two-sided ideal J in A can be extended to a maximal ideal J' of the same kind in $\Gamma(A)$ such that $(L_a, R_a) \in J'$ if and only if $a \in J$. This is a one-to-one correspondence between the maximal modular ideals in A and the maximal modular ideals of the same type in $\Gamma(A)$ which do not contain the set $\{(L_a, R_a)|\ a \in A\}$.*

We first furnish examples of ideals in C*-algebras to give substance to Definition 8.

Example 27 (Ideals in Commutative C*-algebra) *Let A be the commutative C*-algebra $C(K)$. By the commutativity of A, the concept of an ideal, right ideal and left ideal coincide. Ideals are plentiful: the set $I = \{f \in C(K) \mid f$ is zero on $S \subset K\}$ is an ideal of A; for any fixed function f in A, $< f >= fA$ is an ideal of A which we call a principal ideal generated by f.*

Example 28 (Ideals in Non-Commutative C*-algebra) *Let A be the non-commutative C*-algebra $M_2(\mathbf{C})$. Then A has no proper ideals except the trivial ideal $\{0\}$.*

Suppose $I \neq \{0\}$ is a proper ideal of A. Choose a non-zero non-invertible matrix x in I:

$$x = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

On pre- and post-multiplying the matrix x by the elementary matrix e_{11} which has 1 as the first row - first column entry and 0 elsewhere, the following matrix exists in the ideal I :

$$\begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

Since the post(pre) multiplication by matrices effect the elementary column (row) operations, we can assume without loss of generality that $a_{11} \neq 0$: if $x \neq 0$ then we can perform the row and column operation to arrange this. Dividing out row 1 by a_{11}, we end up with the matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Effecting the elementary row and column operations we also end up with the matrix:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The sum of the above two matrices is the identity $\mathbf{1}-$ matrix. This contradicts the assumption that I, which is closed under addition, is proper.

However define y as the following matrix:

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Then yA is a proper right ideal of A. Ay^* is a proper left ideal of A.

Note that the above example also shows that $M_n(\mathbf{C})$, the space of all operators on a finite dimensional Hilbert space, has no proper ideals except the trivial ideal $\{0\}$. We therefore search for non-trivial proper ideals for the space of operators on infinite dimensional Hilbert spaces.

Let $\mathbf{H}$ be the infinite dimensional Hilbert space l^2 of all the square summable complex valued sequences. Then it is well known that the set of all the compact operators on l^2 is an ideal of the C-algebra $B(\mathbf{H})$ [see Chapter VI.2.6 Corollary 2.6.3 [25]].*

A compact operator on l^2 maps the open unit ball of l^2 to a relatively compact subset of l^2. Compact operators are the next step up in complexity from the operators of finite rank : operators whose range is a finite dimensional subspace. Hence all finite rank operators are compact and l^2 has a rich supply of them: for every finite dimensional subspace, V, of l^2, the associated projection $P_V : l^2 \to l^2 \mid x \mapsto y$ where $y \in V$ and $z \in V^\perp$ and $x = y + z$ is the unique decomposition of x relative to these closed subspaces, is a finite rank operator. If V has orthonormal basis $(e_i)_{1 \le i \le n}$, then $P_V(x) = \sum_{i=1}^{n} (x, e_i) e_i$.

Since the unit ball B_{l^2} of l^2 is not compact, the ideal of compact operators is void of the identity operator $e : l^2 \to l^2 \mid x \mapsto x$ which is equivalent to it being proper.

We now give the first application of the concept of a modular ideal. Let us consider A as a left (right) A - module over itself ${}^\bullet A$ ($A^\bullet$). Then the left (right) ideal I can be regarded as a left submodule of A or rather ${}^\bullet A$ ($A^\bullet$) and we can construct the factor left (right) A - module, ${}^\bullet A/I$ ($A^\bullet/I$). Then $e+I$ will be the right (left) identity of the factor module ${}^\bullet A/I$ ($A^\bullet/I$). In the case of I being a two-sided ideal, the factor ring A/I will have an identity: $e + I$, where e is the left and right identity modulo I.

Example 29 *Let A be a C*-algebra with an identity e. Then all ideals are modular ideals : the identity e is the identity modulo the ideal. Hence the concept of a modular ideal exists in name only.*

Example 30 *Let A be the C*-algebra $C_0(\Omega)$. Then the ideal $I = \{\mathbf{0}\}$ is not a modular ideal : the only possible candidate for the identity modulo I is the constant one function, $e : \Omega \to \mathbf{C} \mid \omega \mapsto 1$ since we require $x - xe = 0$ for all $x \in C_0(\Omega)$.*

Let $\Omega = \mathbf{R}$ with the usual norm topology. Then Ω is a non-compact locally compact set. Now, the larger the ideal I the greater the chance of the existence of an identity e modulo I. Consider the ideal I of all complex valued functions in $C_0(\Omega)$ which vanish on the set $\{0\} \subset \mathbf{R}$: $I = \{f \in C_0(\mathbf{R}) \mid f(0) = 0\}$. Then any function $e \in C_0(\Omega)$ which evaluates to 1 on the subset $\{0\} \subset \mathbf{R}$ is an identity modulo I : $x - xe$ is zero on $\{0\}$ for all $x \in C_0(\Omega)$.

The concept of a modular ideal has further implications in the spectrum of x in A, $\sigma_A(x)$. In the case that A has an identity, the concept of a modular ideal and an ideal coincide and we have:

Proposition 2 *(Chapter V.5 Proposition 5.11, [9]) An element a of an algebra A with an identity e has a left (right) inverse if and only if a does not belong to any maximal left (right) ideal of A.*

For the case of the C*-algebra A not having an identity e, recall that $0 \neq \lambda$ is in the spectrum of a in A, $\sigma_A(x)$ if and only if $\lambda^{-1}a$ has no adverse in A (Definition 1, Chapter 1.1). We then have:

Proposition 3 *(Chapter V.5 Proposition 5.12, [9]) Let A be a C*-algebra. Then a has a left (right) adverse if and only if a is not a right (left) identity modulo any modular maximal left (right) ideal.*

The proof of proposition 3, provides a method of constructing modular ideals from the element $\lambda^{-1}a$ which does not have an adverse if $0 \neq \lambda \in \sigma_A(a)$.

1.2.3.7 DCAR : Category of C*-algebras

Having vindicated the double representation of the C*-algebra A in the category of rings, we now show that the double representation of the C*-algebra A in the category of C*-algebras exists and has a form which is not too different from the form in the category of rings [see Theorem 11, Representation Theory II]. To this end, we define a Banach *-algebra structure on the ring of double centralizers, $\Phi(A)$, on A which satisfies the C*-norm condition.

We define the involution on the ring of double centralizers $\Phi(A)$ using the involution on the C*-algebra A as follows [see Theorem 6, [1]]:

Proposition 2 *Let A be a C*-algebra. Consider the double centralizer $(L, R) \in \Phi(A)$ on A. We define the involution $(L)^a$ of the left centralizer $L : A \to A$ as the map $(L)^a : A \to A \mid x \mapsto (L(x^*))^*$. Similarly, the involution , $(R)^a$, of the map $R : A \to A$ as the map $(R)^a : A \to A \mid x \mapsto (R(x^*))^*$.*

Since $((L)^a)^a = L$ *and* $((R)^a)^a = R$*, the involution:*

$$^* : \Phi(A) \to \Phi(A) | \{L,\ R\} \mapsto \{(R)^a,\ (L)^a\} \tag{1.23}$$

defines an involution on $\Phi(A)$.

The following example will give a clue to the above definition:

Example 31 *Let* A *be a C*-algebra. Let* $\Phi(A)$ *be the ring of all (additive) centralizers on* A*. Then the double representation* $\Psi : A \to \Phi(A) \mid a \mapsto (L_a, R_a)$ *is a ring homomorphism that preserves the involution:*

$$\Psi : A \to \Phi(A) \mid a^* \mapsto (L_a, R_a)^*$$

since $(L_a, R_a)^* = (L_{a^*}, R_{a^*})$

A vector space structure is imposed by the following scalar multiplication:

$$\lambda(L, R) = (\lambda L, \lambda R) \tag{1.24}$$

where $\lambda \in \mathbf{C}$. The ring of double centralizers $\Phi(A)$ then becomes a *-algebra, and the double representation Ψ a *-algebra homomorphism. We are now left with equipping the double centralizer algebra with a norm that satisfies the strong C*-norm condition [Chapter 1.1, equation 1.4)]. We now give an outline of the strategy.

By theorem 4, we have already established that the left regular representation $\mathbf{\Theta} : A \to B(A)|a \mapsto L_a$ where $L_a : A \to A|\ x \mapsto ax$ is an isometric isomorphism from the C*-algebra A taken as a Banach algebra into the Banach algebra $B(A)$ of all bounded operators on the Banach space A which is the C*-algebra taken as a Banach space. Symmetrically by Lemma 3, the above is true for the right multiplication map R_a where $R_a : A \to A|\ x \mapsto xa$. Consequently, in analogy with the C*-direct sum norm, defining the norm on the pair (L_a, R_a) as follows:

$$\| (L_a, R_a) \| = \sup\{\| L_a \|, \| R_a \|\}$$

forces the double representation $\Psi : A \to \Phi(A)$ to be an isometric *-isomorphism from the normed *-algebra A into the normed *-algebra $\Phi(A)$ *provided that the other centralizers of* $\Phi(A)$ *are bounded operators on the C*-algebra* A *taken as a Banach space. We will then also define the norm of such centralizers by taking the larger of the operator norms of the left and right centralizer of the pair.* By the faithfulness of the C*-algebra A, this is in fact the case [see Lemma 1, Corollary [1]]:

Theorem 13 *Let* A *be a C*-algebra. Then taking* A *as an algebra over the field* $\mathbf{C}$*, if* $T : A \to A$ *is a left (right) centralizer , then* $T : A \to A$ *is homogenous. Consequently, if* T *is an additive left (right) centralizer,* T *is a linear map on the C*-algebra taken as a Banach space.*

Further for any centralizer $(L, R) \in \Phi(A)$, both L and R are bounded operators on the C-algebra taken as a Banach space. This follows from the completeness of the C*-algebra A as a metric space [see Theorem 14, [1]].*

The following defines a norm on the *-algebra of the double centralizers (L, R) on the C*-algebra $\Phi(A)$:

$$\| (L, R) \| = \sup\{\| L \|, \| R \|\} \tag{1.25}$$

The above norm satisfies the C*-norm condition [Chapter 1.1, equation (1.4)] and the *-algebra, $\Phi(A)$, of the double centralizers on the C*-algebra A is a C*-algebra with the identity $e = (1_A,\ 1_A)$ which we shall denote henceforth as $M(A)$. Briefly,

Theorem 14 *Let A be a C*-algebra. Then the double centralizer algebra $M(A)$ is a C*-algebra with an identity $(1_A,\ 1_A)$.*

The double representation $\Psi : A \to M(A) | a \mapsto (L_a, R_a)$ of A is a one-to-one *-homomorphism from the C*-algebra A into the C*-algebra $M(A)$. Hence any C*-algebra, A, is isometrically *-isomorphic to a closed self-adjoint ideal in $M(A)$.

In light of Representation Theory I and II for the category of rings [see Theorem 10, 11], we have the following representation theory in the category of C*-algebras:

Theorem 15 (The Double Centralizer Algebra Representation Theory) *(Theorem 3.1.8 [13]) Let A be a C*-algebra. Let $M(A)$ denote the C*-algebra of all the double centralizers on A. Then the double centralizer algebra, $M(A)$, is the largest C*-algebra with an identity which contains A as an (closed) essential ideal : the C*-algebra A is a closed 2-sided ideal of the C*-algebra $M(A)$ which is essentially faithful with respect to $M(A)$, taken as an over-ring of A.*

We end of this section with two examples of the double centralizer algebra of two well known C*-algebras without identity. The proofs of these two examples can be found in Appendix A of the thesis.

Example 32 (Double Centralizers on Commutative C*-algebra) *Let A be the C*-algebra $C_0(\Omega)$. Then the double centralizer algebra, $M(A)$, is the set of all bounded continuous functions, $C_b(\Omega)$, on the locally compact space Ω.*

Example 33 (Double Centralizers on Non-Commutative C*-algebra) *Let A be the C*-algebra $K(\mathbf{H})$. Then, the double centralizer algebra, $M(A)$, is the set of all bounded operators, $B(\mathbf{H})$, on the Hilbert space $\mathbf{H}$.*

1.2.4 The DCAR and the Universal Representation

We now conclude the section on the global representation of a C*-algebra by relating the two global representations, namely, the Double Centralizer Algebra representation and the Universal representation.

The Universal Representation of the C*-algebra A, $\Phi : A \to B(\mathbf{H})$ where $\mathbf{H}$ is the universal Hilbert space, is a non-degenerate *-representation. By the double centralizer algebra representation, the C*-algebra A is an ideal of the double centralizer C*-algebra $M(A)$. Then

Theorem 16 *(Chapter I, Lemma I.9.14 [12]) There exists a unique *-representation* $\overline{\Phi}$ *of the double centralizer algebra* $M(A)$ *on the same universal Hilbert space* $\mathbf{H}$ *extending* Φ *: the restriction of* $\overline{\Phi}$ *to* A *is* Φ*.*

We construct the *-representation $\overline{\Phi}$ of the double centralizer algebra $M(A)$ as follows:

Construction 1 (Extension of the Universal Representation) *Since the the ranges of* Φ_a*,* $a \in A$*, span the universal Hilbert space* $\mathbf{H}$*, it suffices to define each operator* $\overline{\Phi}_m \in B(\mathbf{H})$ *where* $m \in M(A)$ *on the subset* $V = \bigcup_{a \in A}\{\Phi_a(\mathbf{H})\}$*.*

Now, each vector $v \in V$ *is of the form* $\Phi_{a_1}(u_1)$ *for some* a_1 *in the ideal* A *and* u_1 *in the universal Hilbert space* $\mathbf{H}$*. Then the following definition of* $\overline{\Phi}$ *is well defined, by the existence of the approximate identity of the C*-algebra* A*:*

$$\overline{\Phi}_m(v) = \overline{\Phi}_m(\Phi_{a_1}(u_1)) = \Phi_{ma_1}(u_1)$$

for each $m \in M(A)$ *and* $v \in V$*.*

Although we can associate a *-representation with the Double Centralizer Representation of the C*-algebra A, the Double Centralizer Representation has its own merits. In particular, it preserves the ideal structure of the C*-algebra [see Theorem 12, Chapter 1.2.3]. This is important since the key theme of this thesis is *lifting* [2] properties from the *quotient* C*-algebra to the original C*-algebra. The quotient C*-algebra is related to the *closed* two-sided ideals of the C*-algebra in the following fundamental way:

Proposition 4 *Let* A *denote a C*-algebra and* I *a closed 2-sided ideal of* A*. Then* I *is self-adjoint and* A/I *, endowed with the natural involution algebra structure and the quotient norm, is a C*- algebra. Further* $\pi : A \to A/I$ *the quotient map, is a *-homomorphism where* $\| \pi \| \leq 1$*.*

As opposed to the purely algebraic concept of a 2-sided ideal [Chapter 1.2.3, Examples 27 - 30], we shift our focus onto closed 2-sided ideals.

[2] We shall define this term later on. For now its meaning is not needed.

1.2.4.1 Closed 2-sided Ideals in C*-algebras

With regard to Proposition 4, the closed two-sided ideals of an abstract C*-algebra are of critical importance. Commutative C*-algebras furnish a rich supply of closed two-ideals of a topological nature [cf Chapter 1.2.3, example 27]:

Example 34 (Closed Two-sided Ideals : Commutative C*-algebra with an identity) *(Chapter 3, Theorem 3.4.1 [10]) Let A be the C*-algebra $C(K)$. Then for every closed ideal I of $C(K)$ there exists a closed subset S, of K such that I is the set of all functions vanishing on the S : $I = \{f \in C(K) | f|_K = 0\}$. Conversely, if $S \subset K$ is a closed subset of K, then the set of all functions vanishing on K is a closed ideal of $C(K)$.*

Further, the maximal ideals of $C(K)$ are those closed ideals for which the corresponding closed subset of K (on which all the functions of the ideal vanish) consists of a single point.

The above one-to-one correspondence between the closed subsets of the compact Hausdorff space K and the closed two-sided ideal of $C(K)$ also holds for the C*-algebra $C_0(\Omega)$ [Chapter 7.4, Theorem 7.4.2 [17]] :

Example 35 (Closed Two-sided Ideals : Commutative C*-algebra without an identity) *Let A be the C*-algebra, $C_0(\Omega)$. Then for every closed ideal I, of $C_0(\Omega)$ there exists a closed subset K, of Ω such that $I = \{f \in C_0(\Omega) | f|_K = 0\}$.*

Proof. *This follows from the proof of Theorem 3.4.1 [10], once we establish the existence of a compact set $F_\epsilon \subset \Omega$ such that $|f(p)| \geq \epsilon \ \forall p \in F_\epsilon$ for every $\epsilon \geq 0$ where $f \in C_0(\Omega)$.*

Since f vanishes at infinity, there exists a compact subset $K \subset \Omega$ such that $|f(p)| \leq \epsilon$ for all p outside of K; consequently, the set $\{p \in \Omega \mid |f(p)| \geq \epsilon\} = |f|^{-1}[\mathbf{C} \backslash B(0;\epsilon)]$ is a closed subset of the compact set K by the continuity of $|f|$.

The correspondence between the closed sets and the closed two-sided ideals of $C_0(\Omega)$ is one-to-one.

All compact Hausdorff spaces are normal. A topological space is normal if the topology is fine enough to distinguish disjoint closed sets. We need the normality to apply Tietze's extension theorem to establish the one-to-one correspondence in the case of $C(K)$. For the case of a locally compact space Ω, normality need not follow although the one-to-one correspondence is still valid:

Example 36 [Non Compact, Locally Compact Hausdorff but Not Normal] *(Deleted Tychonoff Plank) The crux of this construction is the non-preservation of normality on taking topological products.*

Consider the ordinal spaces $W^*(\omega_1) = [0, \omega_1]$ *and* $W^*(\omega) = [0, \omega]$ *which consist of all the ordinals from 0 up to and including the first uncountable ordinal* ω_1 *and the first non finite ordinal* ω*, respectively. The first uncountable ordinal* ω_1 *exists in ZFC since all sets have a bijection to a unique cardinal and the set of subsets of the set of all natural numbers* ω *is a set. A cardinal is an ordinal which has no bijection with itself and any section of it. Both these spaces are well ordered by the set membership condition* $\in$*, and are endowed with the interval (order) topology. The ordinal spaces,* $W^*(\omega_1)$ *and* $W^*(\omega)$*, are then compact Hausdorff spaces and hence the product space* $W^*(\omega_1) \times W^*(\omega)$ *is a compact Hausdorff space. We call this space the Tychonoff Plank* T*.*

The Deleted Tychonoff Plank T_∞ *is the subspace topology that results from deleting the point* (ω_1, ω) *from* $T : T_\infty = T \backslash \{(\omega_1, \omega)\}$*. It is therefore an open subspace (points are closed in* T*) of the compact Hausdorff space* T *and we therefore conclude that it is a locally compact Hausdorff space.*

The Deleted Tychonoff Plank T_∞ *is not normal: there can be no disjoint open sets separating the closed sets* $A = \{\omega_1\} \times (W^*(\omega) - \{\omega\})$ *and* $B = (W^*(\omega_1) - \{\omega_1\}) \times \{\omega\}$*.*

However, there is a large class of locally compact Hausdorff spaces which are normal:

Example 37 (Normal Spaces: Metrizable, Non-Metrizable) *All compact Hausdorff spaces and metric spaces are normal.*

The set of all reals, $\mathbf{R}$*, with the usual norm topology along with the set of all rationals* $\mathbf{Q}$ *taken as a subspace of the reals with the norm topology along the Cantor dust, are non-compact metric spaces.*

Let the doubleton set $\{0, 1\}$ *be endowed with the discrete topology. Then, by Tychonoff's theorem, the product space* $\{0, 1\}^{\mathbf{R}}$ *of the compact space ,* $\{0, 1\}$*, is compact. This space is not sequentially compact since* $\mathbf{R}$ *is not countable [Chapter 16, Example 16.38, [29]]. Therefore, it is not metrizable since it is both compact but not sequentially compact.*

Example 38 (Non-Metrizable, Locally compact and Normal) *Consider the ordinal space,* $W(\omega_1)$*, which is the set of all countable ordinals,* $\{\sigma \mid \sigma \in \omega_1\}$*, where* ω_1 *is the first uncountable ordinal.*

The ordinal space, $W(\omega_1)$*, is well ordered by the set membership condition* $\in$*, and it is endowed with the interval (order) topology.*

The ordinal space, $W(\omega_1)$*, is then a non-compact locally compact Hausdorff space. Since it is not first countable :* ω_1 *does not have a countable neighbour-*

hood basis, $W(\omega_1)$ is not metrizable. Therefore, it stands a chance of being not normal. Fortunately, $W(\omega_1)$ is normal [Chapter 5, 5.11 [31]].

Example 39 (Normal but not Locally Compact Hausdorff Spaces) *The set of all rationals* $\mathbf{Q}$ *taken as a subspace of the reals with the norm topology is a normal metric space by example 2. However, since the only compact subsets are finite, compact sets have an empty interior and* $\mathbf{Q}$ *is not locally compact.*

Example 40 (Not Normal and Not Locally Compact) *The set of all reals,* $\mathbf{R}$*, endowed with the cocountable topology is not locally compact: only finite sets can be compact. Further it is trivially not normal since every pair of open sets has a non-empty intersection.*

In contrast to the commutative C*-algebras, non-trivial ideals, let alone closed ideals, in the non-commutative C*-algebra, $B(\mathbf{H})$, of all bounded operators on a Hilbert space $\mathbf{H}$ are rare [Chapter 1.2.3, Example 28].

Example 41 (Closed Ideals in Non-Commutative C*-algebra) *Let A be the C*-algebra, $B(\mathbf{H})$, of all bounded operators on an infinite dimensional Hilbert space* $\mathbf{H}$*. It is well known that the set of all compact operators, $K(\mathbf{H})$, on the Hilbert space* $\mathbf{H}$ *is a closed two-sided ideal of A [Chapter VI.2.6, Corollary 2.6.3 [25]]. If I is a closed ideal of $B(\mathbf{H})$, then $I \supseteq K(\mathbf{H})$ or $I = \{0\}$ [Chapter VIII.4 Proposition 4.10, [24]]. Further, if* $\mathbf{H}$ *is separable, then the set of all compact operators, $K(\mathbf{H})$, is the only non-trivial closed two-sided ideal.*

As opposed to the Double Centralizer Algebra Representation, the Universal Representation falls short with regards to shedding light on the structure of the closed 2-sided ideals of the C*-algebra, A, regarded as a norm closed *-subalgebra of $B(\mathbf{H})$. First and foremost, the universal Hilbert space is extremely large and ideals in bounded operators on this Hilbert space are difficult to compute. We end off with the following examples:

Example 42 *Let A be the C*-algebra, $C[-\pi, \pi]$, of all the complex valued continuous functions on the compact interval $[-\pi, \pi]$. By the Stone-Weierstrass Theorem, this C*-algebra is separable : it is a separable metric space, that is, has a countable dense subset in the form of the polynomials in z and $\overline{z}$ with coefficients taken from* $\mathbf{Q} \times \mathbf{Q} \subset \mathbf{C}$*.*

From the universal representation, we can construct a subspace $\mathbf{U}$ *of the universal Hilbert space* $\mathbf{H}$ *which is separable and A embeds isometrically *-isomorphically into $B(\mathbf{U})$ [Chapter VI.22, Proposition 22.13 [9]].*

There is only one possible non-trivial closed ideal in $B(\mathbf{U})$ while $C[-\pi, \pi]$ has as many closed two sided ideals as there are closed subsets of $[-\pi, \pi]$.

We now construct a *-representation of the separable C*-algebra, $C[-\pi, \pi]$, on a separable Hilbert space:

Example 43 *Let A be the C^*-algebra $C[-\pi,\pi]$. Consider the measure space $\Sigma = ([-\pi,\pi], \mathcal{M}_{[-\pi,\pi]}, \lambda_{[-\pi,\pi]})$ where $\lambda_{[-\pi,\pi]}$ be the Lebesgue measure restricted to the Lebesgue measurable set $[-\pi,\pi] \subset \mathbf{R}$: the sigma algebra $\mathcal{M}_{[-\pi,\pi]} = \{[-\pi,\pi] \bigcap A | A \text{ is Lebesgue measureable subset of } \mathbf{R}\}$.*

Consider the Hilbert space $\mathbf{H} = L^2(\Sigma)$ of all square integrable functions on the measure space Σ . Then $\mathbf{H}$ is non trivial : it contains all the constant functions on $[-\pi,\pi]$ since λ is a finite measure.

Since λ is a finite measure, $C[-\pi,\pi] \subset L^\infty(\Sigma) \subset L^2(\Sigma)$. For each $g \in C([-\pi,\pi]) \subset L^2(\Sigma)$ we define the associated left multiplication operator M_g on $\mathbf{H} = L^2(\Sigma)$ as follows [Chapter 1.2.1, Example 12]:

$$M_g : h \mapsto gh \text{ where } gh(x) = g(x)h(x) \quad \forall x \in [-\pi,\pi].$$

where $gh \in L^2(\Sigma)$ since $L^2(\Sigma)$ is an algebra and M_g is a bounded operator with norm less than or equal to $\| g \|_\infty$:

$$\| gh \|_H^2 = \int_{[-\pi,\pi]} |gh|^2 d\lambda \leq \| g \|_\infty^2 \int_{[-\pi,\pi]} |h|^2 d\lambda \leq 2\pi \| g \|_\infty^2 .$$

where $\| \cdot \|_H$ denotes the norm on the Hilbert space $\mathbf{H} = L^2(\Sigma)$.

*Since $M_g^* = M_{\bar{g}}$, $M_{af+bg} = aM_f + bM_g$, $M_{fg} = M_f M_g$, the map $\Phi : C[-\pi,\pi] \to B(L^2([-\pi,\pi], \mathcal{M}_{[-\pi,\pi]}, \lambda_{[-\pi,\pi]})) \mid g \mapsto M_g$ is a *-representation which is one-to-one since $\mathbf{H}$ has the identity map $e : [-\pi,\pi] \to [-\pi,\pi] \mid x \mapsto x$. We are now done since a *-isomorphism between two C^*-algebras is an isometric *-isomorphism.*

$L^2(\Sigma)$ is a separable Hilbert space since the countably many functions $e_n(t) = e^{int}$ where $n \in \mathbf{N}$ form an orthonormal basis. Hence the only closed two-sided ideal is the set of all compact operators. On the other hand, $C[-\pi,\pi]$ has as many distinct closed two-sided ideals as there are closed subsets of $[-\pi,\pi]$.

1.3 C*-algebra : Local Representations

As mentioned in the beginning of the previous section, Chapter 1.2, we now describe representations of the individual elements of the C*-algebra. Just as the global representation of the C*-algebra played a critical role in the key results of the thesis, the local representations also played a critical role.

The first local representation theory is by virtue of the Gelfand Naimark theorem I, Chapter 1.1, for commutative C*-algebras and hence only applies to a special type of element: a normal element. In fact, an element x in the C*-algebra A is normal if and only if x belongs to some commutative *-subalgebra of A. Then with the aid the Gelfand Naimark theorem I, we can develop a *functional calculus* for normal elements of the C*-algebra. A functional calculus for a fixed normal element x represents the element x as a continuous function on a compact space and allows the definition and computation of the element $f(x)$ which we call the result of applying f to x, where f is a continuous complex valued function on some compact space.

1.3.1 Local Representation Theory I: The Functional Calculus for Normal Elements

The benefit of the functional calculus for normal elements, is the reduction of problems in C*-algebras encountered in this thesis, to problems in the more familiar *function algebras*. A function algebra is a pair (A, K) where K is a fixed compact Hausdorff space and A is a subalgebra of $C(K)$. Further, A is itself a commutative C*-algebra under a norm which may or may not coincide with the supremum norm of $C(K)$ where :

(i) A vanishes nowhere on K : for each x in K, $f(x) \neq 0$ for some $f \in A$.

(ii) A separates points of K : for $x, y \in K$ where $x \neq y$, there exists a $f \in A$ such that $f(x) \neq f(y)$.

Let $C^*(x)$ denote the closed commutative *-subalgebra of the C*-algebra A generated by the normal element x. Then with the aid of the Gelfand Naimark Theorem I, we have the following representation theory:

Theorem 1 [Local Representation Theory I : The Functional Calculus for Normal Elements] *(Chapter 2, Prop 8.4, [8]) Let A be a C*-algebra with an identity, e, and x a normal element of A. Then there exists an onto isometric *-isomorphism $\Phi : C(K) \to C^*(x) \subset A \mid f \mapsto f(x)$ where K is, $\sigma_A(x)$, the spectrum of x in A, which is a non-empty compact set.*

In the case where A does not have an identity, we enforce the additional condition that f vanishes at 0: $f(0) = 0$ if $0 \in \sigma_A(x)$. Then Φ has the following properties:

(a) $\Phi(e) = 1$ *where* e *is the constant one map on* $\sigma_A(x)$

(b) $\Phi(\mathbf{1}_{\sigma_A(x)}) = x$ *where* $\mathbf{1}_{\sigma_A(x)}$ *is the identity map on* $\sigma_A(x)$

(c) *If* $f_i \to f$ *in* $C(\sigma_A(x))$ *uniformly, then* $\Phi(f_i) \to \Phi(f)$ *in* $C^*(x) \subset A$

where property (a) only applies for a C*-algebra with an identity. The identity map, $\mathbf{1}_{\sigma_A(x)} : \omega \mapsto \omega$, on $\sigma_A(x)$, is called a *functional representation* of the normal element x. More generally, the element $f(x)$ in the C*-algebra generated by x, has a functional representation as the continuous complex valued function $f \in C(\sigma_A(x))$. In particular, if x is a positive element of the C*-algebra A, setting f to be the square root function $\surd$ on the spectrum, $\sigma_A(x)$, of x in A, defines the C*-algebra element $\sqrt{x}$ in A: $\sigma_A(x) \subseteq \mathbf{R}^{+0}$ where $\mathbf{R}^{+0}$ is the positive part of the real line and the square root function vanishes at 0.

Example 1 *Let A be the C*-algebra $M_n(\mathbf{C})$. Then the normal elements are precisely the $n \times n$ matrices that are diagonalizable: have an orthornormal basis of eigenvectors. The spectrum consists entirely of the eigenvalues of the matrix.*

Consider the normal matrix x given by :

$$x = \begin{pmatrix} 9 & 5 & -4 \\ -8 & -4 & 4 \\ 2 & 2 & 0 \end{pmatrix}$$

which has 0, 1 and 4 as its eigenvalues. Therefore $K = \sigma_A(x) = \{0, 1, 4\}$ and the square root function on K is precisely the interpolating second order polynomial function $p(x) = a_2x^2 + a_1x^1 + a_0$ with the conditions $p(0) = \sqrt{0}$, $p(1) = \sqrt{1}$ and $p(4) = \sqrt{4}$. Computation yields $a_2 = -\frac{1}{6}, a_1 = \frac{7}{6}$, $a_0 = 0$. Therefore, the square root of x is the matrix $p(x)$ where $p(x) = -\frac{1}{6}x^2 + \frac{7}{6}x$.

We end off this section with three corollaries of the Local Representation Theory I which we shall need frequently later on. We shall phrase these corollaries in the language of *orthogonal* elements. We state a definition of orthogonality in a C*-algebra, suggested by Dr. Duvenhage, based on the following observation. In the proof where a one-to-one correspondence between the pure states and the irreducible cyclic *-representations is established [cf. Chapter 1.2.1, Theorem 1], each pure state induced an inner product $(\cdot|\cdot)$ on the C*-algebra A as follows:

$$(x|y) = p(y^*x) \text{ for each } x, y \in A$$

from which the Hilbert space which is associated with the cyclic *-representation was constructed. Since $(y|x) = p(x^*y)$, setting p to be the identity function gives us the following definition of orthogonality:

Definition 1 (Orthogonal) *Two elements x, y in the C*-algebra A are orthogonal if and only if $x^*y = 0 = 0^* = y^*x$. In the case when x and y are both positive or even self adjoint, the condition is equivalent to $xy = yx = 0$.*

Example 2 *Let A be the C*-algebra* $C[0,1]$ *(example 25, chapter 1.2.3.). Then the positive elements* $x(t)$ *and* $y(t)$ *where* $y(t)$ *is defined as:*

$$y(t) = \begin{cases} -2t+2 & : \quad 0 \leq t \leq \frac{1}{2} \\ 0 & : \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

are orthogonal.

Corollary 1 [Orthogonal Decomposition For Self Adjoint Elements] *(Chapter VI, Section 7, Proposition 7.16 [9]) For every self adjoint element* x *of the C*-algebra A, there exists a unique pair of orthogonal positive elements* x_+ *and* x_- *:* $x_+x_- = 0 = x_-x_+$ *such that:*

$$x = x_+ - x_-$$

We call x_+ *and* x_- *the positive orthogonal parts of* x*. Further* $\| x_+ \| \leq \| x \|$ *and* $\| x_- \| \leq \| x \|$

Corollary 2 (Functional Factorization For Normal Elements) *Let* x *be any normal element in* A *and* f *be any continuous function on* $\sigma_A(x)$ *which vanishes at 0 :* $f(x) \in A$*. If we can write* f *as the point-wise product of two other functions of the same type:* $f = gh$*, then the element* $f(x) = g(x)h(x)$ *with* $g(x), h(x) \in A$*. We call* g *and* h *factors of* f*. The condition of vanishing at 0 is irrelevant if* A *has an identity and* $0 \notin \sigma_A(x)$*.*

Proof Firstly, consider the C*-algebra $C(\sigma_A(x))$ of all complex valued continuous functions on the spectrum, $\sigma_A(x)$, of x in A, which is a compact subset of **C**. Then by the Stone Weierstrass theorem on $C(\sigma_A(x))$, there exists *sequences* of polynomials which do not have constant terms which we shall denote as p_n^g and p_n^h, which converge uniformly to the functions g, h, respectively, on the compact set $\sigma_A(x) \subset \mathbf{C}$:

$$p_n^g \to g \text{ and } p_n^h \to g.$$

Then by the joint continuity of the product on the C*-algebra $C(\sigma_A(x))$:

$$p_n^g p_n^h \to gh = f$$

The map $\Phi : C(\sigma_A(x)) \to A | f \mapsto f(x)$ of the Local representation theory I is a *-homomorphism. Therefore

$$\Phi(p_n^g p_n^h) = \Phi(p_n^g)\Phi(p_n^h) \to \Phi(f) = f(x)$$

Now, invoking the joint continuity of the product in $C^*(x)$, the commutative C*-algebra generated by the normal element x,

$$\Phi(p_n^g)\Phi(p_n^h) \to g(x)h(x)$$

By the uniqueness of the limit we are done.

Q.E.D

Corollary 3 *Let a and b be two positive commuting elements of the C*-algebra A. Then their product ab is positive and $\sqrt{ab} = \sqrt{a}\sqrt{b}$.*

Proof.

(i) **The element ab is positive**.

If a and b are self-adjoint and commute, then $(ab)^* = b^*a^* = ba = ab$. Therefore, ab is self adjoint.

If A does not have an identity, take A as the unitization C*-algebra A_e. Consider $B = C^*(a, b, e)$, the commutative $\|\cdot\|$ - closure of the *-algebra generated by a, b and e. Then the spectrum $\sigma_A(ab)$ of ab in A which is defined as the spectrum, $\sigma_{A_e}(ab)$, of ab in A_e, is exactly the spectrum, $\sigma_B(ab)$, of ab in the C*-subalgebra B [Chapter VI, section 5, proposition 5.2 [9]]. Since B is commutative, $\sigma_B(ab) \subset \sigma_B(a)\sigma_B(b) \subset \mathbf{R}^+$.

(ii) $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Consider ab as a normal (positive) element of the commutative C*-algebra $B = C^*(a, b, e)$. The square root function $\sqrt{\cdot}$, is a continuous function on the spectrum, $\sigma_B(ab) \subset \mathbf{R}^+$, of ab. Therefore $\sqrt{ab}$ is an element of the commutative C*-algebra B : by the Local Representation Theory I, we have an onto isometric *-isomorphism $\Phi : C(\sigma_B(ab)) \to C * (ab) \subset B|\sqrt{\cdot} \mapsto \sqrt{ab}$. Similarly, $\sqrt{a}$ and $\sqrt{b}$ are elements of the commutative C*-algebra B. Therefore $(\sqrt{a}\sqrt{b})^2 = (\sqrt{a})^2(\sqrt{b})^2 = ab$.

Q.E.D

In the next section, we explore another local representation theory. The second local representation theory is by virtue of the Universal representation where the C*-algebra is embedded as a closed *-subalgebra of the space of all bounded operators on the universal Hilbert space $\mathbf{H}$. We can therefore view each element as an operator on a Hilbert space and hence decompose it using the *polar decomposition*. Unlike the Local Representation Theory I, this representation works for all elements, normal or non-normal.

1.3.2 Local Representation II: The Polar Decomposition

In the analogy between $M_n(\mathbf{C})$ and the complex number field $\mathbf{C}$, a complex number z corresponded to an operator $x \in M_n(\mathbf{C})$, the real numbers $z = \overline{z}$ corresponded to self adjoint operators $x^* = x^*$, the positive numbers z which were characterized as having a positive square root, corresponded to the positive operator x which can be characterized as having a positive square root, a complex number, z, which belongs to the unit circle $[z\overline{z} = 1]$ corresponded to a *unitary operator* x defined as $xx^* = x^*x = 1$ (the columns of unitary operators consist of orthonormal vectors and unitary operators are exactly the *isometries* : $\| x(\eta) \| = \| \eta \|$ for all $\eta \in \mathbf{C}^n$) and the decomposition of the complex number z into its real and imaginary parts: $z = Re(z) + Im(z)i$ corresponded to the decomposition of the operator x into $\frac{1}{2}(x + x^*) + i(\frac{1}{2}i(x - x^*))$ where $\frac{1}{2}(x + x^*), \frac{1}{2}(x - x^*)$ are self adjoint operators. The analogy went even further: the polar decomposition of the complex number $z = e^{i\theta}r$ where $e^{i\theta}$ is a complex number on the unit circle and r is the positive number $\sqrt{z^*z}$ corresponded to the decomposition of the operator x into the product up, where p is the positive operator $\sqrt{x^*x}$ and u is an unitary operator, or equivalently, an isometry.

In practice, one computes the positive operator p and the isometry u by first decomposing the $n \times n$ matrix x into the singular value decomposition: $x = wdv$ where v, w are isometries and d is a diagonal matrix whose non-zero entries are positive real numbers. Then, setting $p = v^*dv$ and $u = v^*w^*$ we have $x^* = pu$ so that $x = u^*p$ completes the polar decomposition. Formally:

Polar Decomposition in $M_n(\mathbf{C})$: *Let $\mathbf{H}$ be a n-dimensional Hilbert space. Then any operator $x \in B(\mathbf{H})$ has a polar decomposition up where u is an isometry and p the positive operator $\sqrt{x^*x}$. Hence p is uniquely determined by x but the isometry u need not be unique. If x is invertible, then the isometry u is also uniquely determined by x [Chapter 3, section 83, Theorem 1 [32]].*

Unfortunately, the above breaks down in an infinite dimensional Hilbert space. Consider the separable Hilbert space, l^2, of all square summable sequences with complex-valued entries:

Example 3 *Let $\mathbf{H}$ be the separable Hilbert space, l^2, of all square summable sequences with complex-valued entries. Consider the left shift map $L : l^2 \to l^2 \mid (x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, \ldots)$.*

The positive operator p uniquely defined by the operator L will be the projection $p : l^2 \to l^2 \mid (x_1, x_2, x_3, \ldots) \mapsto (0, x_2, x_3, \ldots)$ since $L^* = R$ where R is the right shift map $R : l^2 \to l^2 \mid (x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, x_3, \ldots)$. Therefore if $L = up$ then u must be the left shift map L which is far from being an isometry since it is not even one-to-one. However, the operator u is an isometry when restricted to the closed subspace $V = \{(x_1, x_2, x_3, \ldots) | x_1 = 0\}$ which is the orthogonal complement of the kernal of L, $(kerL)^\perp$.

Note that unlike in the case of a finite dimensional Hilbert space where the set of unitary operators coincide with the set of all isometries, the adjoint of L, the right shift map R, is an isometry which is not a unitary operator : L is only a left inverse. However, all unitary operators are isometries.

As hinted by the decomposition of the left shift map, L, as $L = up$ where u is an isometry when restricted to $(kerL)^{\perp}$ which we shall call the *initial space* of L, fortunately does hold for all operators on *any* Hilbert space [Chapter VII, section 3, Theorem 3.11 [24]]:

Theorem 2 (Local Representation Theory II : Polar Decomposition) *Let $\mathbf{H}$ be any Hilbert space. Then any operator $x \in B(\mathbf{H})$ has a polar decomposition up where u is an isometry when restricted to the initial space of x, $(ker(x))^{\perp}$ and annihilates the orthogonal complement $ker(x)$, which we call the final space, while p is the positive operator $\sqrt{x^*x}$. Hence p is uniquely determined by x and we call u a partial isometry. If $x = UP$ where P is a positive operator and U shares the same initial and final space as u, then $U = u$ and $P = p$.*

Consequently, any element x in a C*-algebra A, by virtue of the Universal Representation, can be taken as an operator on a Hilbert space and hence has a factorization $x = up$ where u is a partial isometry and p the positive element uniquely determined by x. We shall from now on denote it as $|x|$ and call it the *modulus* of x. Without a Hilbert space in the background, the concept of an isometry makes no sense in the category of an abstract C*-algebra : a Banach algebra which satisfies the C*-norm condition. However the concept of the modulus does, as well as the concept of a unitary element which coincides with the concept of an isometry only in the setting of a finite dimensional Hilbert space. Now, for an *invertible* element x in a C*-algebra with an identity, there is a decomposition up where p is the modulus, $\sqrt{x^*x}$, of x and u is an *unitary* element : $u^*u = uu^* = 1$. Formally:

Theorem 3 (Polar Decomposition for Invertible Element) *(Chapter VI, section 7, proposition 7.24 [9])Let A be a C*-algebra with an identity. Let x be an invertible element. Then there is a unique unitary element u of A and a unique positive element p of A such that $x = up$. In fact, $p = \sqrt{x^*x}$.*

The proof of the above theorem does not require the Universal Representation. The proof rests purely on the Local Representation Theory I of the normal element x^*x.

Another type of element that receives a favourable treatment from the polar decomposition are the self adjoint elements. This is due to the fact that the polar decomposition of the operator x as $x = up$ works hand in hand with the adjoint operator x^* : the adjoint x^* has the polar decomposition $x^* = u^*p$ where $p = |x^*|$.

Theorem 4 *Let $x \in B(\mathbf{H})$ and $x = u|x|$ be the unique polar decomposition of x into the partial isometry u and modulus $|x| = \sqrt{x^*x}$. Then the adjoint x^* has the following unique polar decomposition $x^* = u^*|x^*|$ where $|x^*| = \sqrt{xx^*}$.*

Proof. We prove the decomposition $x^* = u^*|x^*|$ by showing that

$$(u(\eta)|\ |x^*|(\eta)) \;=\; (\eta|\ x^*(\eta)) \quad for\ all \quad \eta \in \mathbf{H}.$$

where $(\cdot|\cdot)$ is the inner product on the Hilbert space $\mathbf{H}$ [Chapter 2, Proposition 2.4.3 [10]]. Consider the decomposition of $\mathbf{H} = Ker(x)^{\perp} \bigoplus Ker(x)$ into the closed subspaces $Ker(x)$ and its orthogonal complement $Ker(x)^{\perp}$ which is identical to the closure, $\overline{Ran(|x|)}$, of $Ran(|x|)$. By the sesquilinearity of the inner product, it suffices to establish the above equation for $\eta \in Ker(x)^{\perp}$ and $\eta \in Ker(x)$ separately. Therefore we shall equivalently show that

$$(u(\eta)|\ |x^*|(\eta)) \;=\; (\eta|\ x^*(\eta)) \quad for\ all \quad \eta \in Ker(x)^{\perp} \tag{1.26}$$

and

$$(u(\eta)|\ |x^*|(\eta)) \;=\; (\eta|\ x^*(\eta)) \quad for\ all \quad \eta \in Ker(x) \tag{1.27}$$

We shall prove, as our first step, equation (1.26) by showing that the decomposition $x^* = u^*|x^*|$ holds in the closed subspace $Ker(x)^{\perp}$. Then we prove equation (1.27) directly.

Step 1. $x^* = u^*|x^*|$ for all $\eta \in Ker(x)^{\perp} = \overline{Ran(|x|)}$.

In the construction of the partial isometry u of the polar decomposition of x [see the proof of Chapter VIII Proposition 3.11 [24]], where $x = u|x|$, the restriction of u to the closed subspace $Ker(x)^{\perp}$ of the Hilbert space $\mathbf{H}$, which we denote as $\overline{u}$, is an onto isometry from $Ker(x)^{\perp}$ onto $\overline{Ran(x)}$. Therefore, the adjoint of $\overline{u}$, $(\overline{u})^* : \overline{Ran(x)} \to Ker(x)^{\perp}$, is the inverse, $(\overline{u})^{-1}$, of $\overline{u}$ and is also an isometry on the closed subspace $\overline{Ran(x)}$ [Chapter 2, Proposition 2.4.5. [10]]. Consequently,

$$u^*u = 1\ for\ all\ \eta \in Ker(x)^{\perp} \tag{1.28}$$

since $(\overline{u})^*$ is the restriction of u^* to $\overline{Ran(x)}$.

But $xx^* = u|x||x|u^* = ux^*xu^*$ for all $\eta \in \mathbf{H}$. Therefore

$$|x^*| = \sqrt{xx^*} = u|x|u^*\ for\ all\ \eta \in \mathbf{H} \tag{1.29}$$

since $u|x|u^*u|x|u^* = u|x||x|u = xx^*$ for all $\eta \in \mathbf{H}$ by (1.28). Furthermore,

$$x = u|x| = u|x|u^*u\ for\ all\ \eta \in Ker(x)^{\perp} \tag{1.30}$$

by (1.28), so that from (1.29),

$$x = |x^*|u \; for \; all \; \eta \in Ker(x)^\perp \tag{1.31}$$

Step 2. $(u(\eta), |x^*|(\eta)) \;=\; (\eta, x^*(\eta)) \quad for \; all \;\; \eta \in Ker(x)$

Since $Ker(x)$ is the final space of the partial isometry u of the polar decomposition of x as $x = up$, $u(\eta) = 0$. Therefore,

$$(u(\eta), |x^*|(\eta)) \;=\; 0 \text{ for all } \eta \in Ker(x).$$

Noting that $Ker(x) = Ran(x^*)^\perp$ [Chapter VII Proposition 2.2.1 (c) [25]], it follows that:

$$(\eta, x^*(\eta)) \;=\; 0 \text{ for all } \eta \in Ker(x).$$

Step 3. u^* is a partial isometry with initial space $Ker(x^*)^\perp$: from Step 1, $(\overline{u})^*$ is an isometry on $\overline{Ran(x)}$, $(\overline{u})^*$ being the restriction of u^* to $\overline{Ran(x)}$ which is identical to $Ker(x^*)^\perp$. We leave it to the reader to show u^* annihilates $Ker(x^*)$.

By the uniqueness of the polar decomposition, we are done.

Q.E.D

To emphasize the favourable status received by self-adjoint elements with regards to the polar decomposition, we note the following corollary:

Corollary 4 *Let x be a self-adjoint element of the C^*-algebra A taken as a C^*-subalgebra of $B(\mathbf{H})$, the space of all bounded operators on its universal Hilbert space $\mathbf{H}$. If $x = u|x|$ be the unique polar decomposition of x into the partial isometry u and modulus $|x| = \sqrt{x^*x}$, then u and $|x|$ commute and the partial isometry u is also self adjoint.*

Proof. By theorem 4 above, if $x = u|x|$ then $x^* = u^*|x^*|$. Since $x = x^*$, $x = u|x| = u^*|x|$ where u and u^* are partial isometries with the same initial and final space. Therefore by the uniqueness of the polar decomposition, $u = u^*$. Finally, $x = u|x|$ implies $x^* = |x|u^* = |x|u$.

Q.E.D

1.3.3 The Functional Calculus [Local Representation Theorem I] and The Polar Decomposition Theorem [Local Representation Theorem II]

The polar decomposition theorem holds for all elements of the C*-algebra unlike the functional calculus which only applies for normal elements. One of the major setbacks with the polar decomposition theorem is the introduction of elements which are not necessarily in the original C*-algebra : let A be the C*-algebra identified as a closed *-subalgebra of the concrete C*-algebra of all bounded operators on the universal Hilbert pace $\mathbf{H}$; then the element x of the C*-algebra can be identified with an operator in $B(\mathbf{H})$ and hence has the polar decomposition up where u is a partial isometry and p is the modulus $\sqrt{x^*x}$; by the functional calculus, p belongs to the original C*-algebra but there is no guarantee that u belongs to the C*-algebra A. This problem of u being an element of the original C*-algebra does not exist for a special type of C*-algebra, Von Neumann C*-algebras, which we shall meet later and for the case of invertible elements [Chapter 1.3.2, Theorem 3]. However, we can alleviate this problem with the following theorem which relates the functional calculus and the polar decomposition into a result we shall need later on:

Theorem 5 (Functional Polar Decomposition theorem) *Consider the C*-algebra A as a closed *-subalgebra of the concrete C*-algebra of all bounded operators on the universal Hilbert pace $\mathbf{H}$ [Gelfand Naimark Theorem II, chapter 1.2.2]. Consider the element x in the C*-algebra $A \subset B(H)$ where $x = u|x|$ is the unique polar decomposition of x into the partial isometry u and modulus $|x| = \sqrt{x^*x}$. Then for any continuous complex-valued function f on $\sigma_A(|x|)$ which vanishes at 0, $uf(|x|) \in A$.*

Proof. The crux of the proof is the fact that $up(|x|)$ is a member of the C*-algebra A for any polynomial p which does not have a constant term : Let $p(z) = \sum_{i=1}^n a_i z^i$ denote any polynomial in z without a constant term. Then $up(|x|) = u\Big(\sum_{i=1}^n a_i|x|^i\Big) = \sum_{i=1}^n a_i u|x|^i = \sum_{i=1}^n a_i x|x|^{i-1}$. Since A is an algebra with the elements $x, |x| \in A$, $up(|x|) = \sum_{i=1}^n a_i x|x|^{i-1}$ is a member of A.

By the Stone-Weierstrass theorem, any continuous function f on the compact subset $\sigma_A(|x|) \subset \mathbf{R}^+$ which vanishes at 0 is the uniform limit of a net of polynomials, p_i, on $\sigma_A(|x|)$ which also vanishes at 0: that is, do not have constant terms. The polynomials will be only in one variable z. Then, applying the Functional Calculus to the self-adjoint element $|x|$ of the C*-algebra A, $\Phi(p_i) \to \Phi(f)$ in A or equivalently $(p_i(|x|)) \to (f(|x|))$ in A. Therefore, by the continuity of the product, $u(p_i(|x|)) \to u(f(|x|))$. Since $u(f(|x|))$ is the limit of the convergent net $u(p_i(|x|))$ in a complete space A, $u(f(|x|)) \in A$.

Q.E.D

1.3.4 Local Representations in the Quotient C*-algebra

In this section we list applications of the Functional Calculus and the Polar Decomposition in the context of the quotient C*-algebra which we shall need later on.

Proposition 1 (Weak Polar Decomposition: Quotient C*-algebra) *If x is a positive [self-adjoint, normal] element of the C*-algebra A, then $\pi(x)$ is a positive [self-adjoint, normal] element of the quotient C*-algebra A/I . In fact, $\pi(x)$ is positive if and only if x is positive: the map π maps the positive cone of A onto the positive cone of A/I. Further, if the pair of elements x, y in the C*-algebra A are orthogonal, then the pair $\pi(x), \pi(y)$ are orthogonal elements of the quotient C*-algebra A/I .*

*Identifying A, A/I with a norm-closed *-subalgebra of the bounded linear operators on their universal Hilbert space, if $x = u|x|$ is the unique polar decomposition of x into the partial isometry u and modulus $|x| = \sqrt{x^*x}$, then $\pi(x) = \pi(u)|\pi(x)|$ provided the partial isometry u belongs to A.*[3]

Proof. The first part of the proof follows immediately from π being a *-homomorphism and $x = a^*a$ for some $a \in A$. We just prove the converse of the first equivalence : note that π is a surjective map on A/I and if $\pi(x)$ is positive then there exists an $a \in A$ such that $\pi(x) = \pi(a)\pi(a)^* = \pi(a)\pi(a^*) = \pi(aa^*)$.

For the second part of the proof, $\pi(|x|) = \pi(\sqrt{x^*x}) = \sqrt{\pi(x^*x)} = \sqrt{[\pi(x)]^*\pi(x)} = |\pi(x)|$ since $\sqrt{x^*x}\sqrt{x^*x} = x^*x \Rightarrow \pi(\sqrt{x^*x}\sqrt{x^*x}) = \pi(\sqrt{x^*x})\pi(\sqrt{x^*x}) = \pi(x^*x)$. Hence, $\pi(x) = \pi(u)|\pi(x)|$.

Q.E.D

Proposition 2 (Quotient Mapping Of The Functional Calculus For Self Adjoint Elements) *For any self-adjoint element x and any continuous function f on $\sigma_A(x)$ which vanishes at 0, we have $\pi(f(x)) = f(\pi(x))$.*

Proof By the Stone-Weierstrass theorem any continuous function f on the compact subset $\sigma_A(x) \subset \mathbf{R}$ which vanishes at 0 is the uniform limit of a net of polynomials in one variable, p_i, on $\sigma_A(x)$ which also vanishes at 0: that is, do not have constant terms. Then, applying the Functional Calculus to the self-adjoint element x of the C*-algebra A, $\Phi(p_i) \to \Phi(f)$ where $\Phi(f) \in C^*(x) \subset A$. Equivalently $(p_i(x)) \to (f(x))$ in A : each p_i has no constant term so $p_i(x)$ is also an element in $A \subset A_e$.

By the continuity of $\pi : A \to A/I$, $\pi(p_i(x)) \to \pi(f(x))$, the convergence taking place in A/I. But π is a *-homomorphism so $\pi(p_i(x)) = p_i(\pi(x))$ which is a net in A/I which converges to the element $f(\pi(x)) \in A/I$.

[3] This occurs for special types of C*-algebras: C*-algebras which are Von Neumann algebras on some Hilbert space [see Theorem 4.1.10 [13]]. We shall explore Von Neumann C*-algebras in Chapter 2.

We now apply the Functional Calculus to the self adjoint element $\pi(x)$: the spectrum of $\pi(x) = x + I$ is a compact subspace of $\sigma_A(x)$; let p_i^r, f^r denote the restrictions of $p_i, f \in C(\sigma_A(x))$ to the compact subspace $\sigma_{A/I}(x+I) \subset \sigma_A(x)$, respectively; then the uniform convergence of p_i in $C(\sigma_A(x))$ to $f \in C(\sigma_A(x))$ implies the uniform convergence of p_i^r in $C(\sigma_{A/I}(x+I))$ to $f^r \in C(\sigma_{A/I}(x))$; hence $p_i^r(\pi(x))$ converges to $f^r(\pi(x))$.

Since $p_i^r(\pi(x)) = \pi(p_i(x) + I)$, $\pi(f(x))$ converges to the limit $f^r(\pi(x))$. We are now done by the uniqueness of the limit.

Q.E.D

Remark. Since the above proposition only used the continuity and the *-homomorphism of π , the above proposition should hold for all *-homomorphisms since all *-homomorphisms of a C*-algebra into a C*-algebra are continuous [Chapter VI. Theorem 3.7 [9]].

Our final corollary follows from the above Quotient Mapping Proposition of the Functional Calculus for Self Adjoint Elements [Proposition 2].

Corollary 5 *For any self-adjoint element* x , $\pi(x_+) = \pi(x)_+$.

Proof By the Functional Calculus, the self-adjoint element x in A is represented as the identity map, $\mathbf{1}_{\sigma_A(x)} : \omega \mapsto \omega$, on $\sigma_A(x)$. Define the following continuous function on $\sigma_A(x)$ which vanishes at 0:

$$f(\lambda) = \begin{cases} \lambda & : \quad \lambda \geq 0 \\ 0 & : \quad \lambda \leq 0 \end{cases}$$

Then $x_+ = \Phi(f) = f(x)$ and applying the Quotient Mapping Of The Functional Calculus For Self Adjoint Elements to the self adjoint element $\pi(x)$ of the C*-algebra A/I we are done.

Q.E.D

Chapter 2

Lifting: Zero Divisors

2.1 Lifting Problem

We now solve with earnest, special types of *lifting problems*. We define the lifting problem as follows:

Definition 1 (Lifting Problem) *Let there be a* n *-ary property,* $P(y_1+I,\ldots,y_n+I)$*, (algebraic, analytical, topological etc) that holds for the elements* $y_1+I,\ldots,y_n+I$ *in the coarse quotient C*-algebra* A/I*. We say that the property* P *lifts to the finer C*-algebra* A *provided that we can find elements* $x_1,\ldots,x_n$ *in the finer C*-algebra,* A*, that are identified with the elements* $y_1+I,\ldots,y_n+I$*, respectively, of the quotient C*-algebra* A/I *under the quotient map, such that* $P(x_1,\ldots,x_n)$ *is true whenever* $P(y_1+I,\ldots,y_n+I)$ *is true.*

The relation $\pi(x_i) = y_i + I$ *for* $i = 1,\ldots,n$ *holds if and only if there exists an element* a_i *in the ideal* I *such that* $x_i + a_i = y_i$*. Consequently, we can rephrase the lifting problem of the* $n-$ *ary property* P *to finding elements* a_i *for* $i = 1,\ldots,n$ *in the ideal* I *which you perturb the* y_i*'s for* $i = 1,\ldots,n$*, respectively, such that the* $n-$ *ary property* P *will hold also for the perturbed elements* x_i*,* $i = 1,\ldots,n$ *whenever it holds for the* y_i*'s.*

We now define a ring theoretic (algebraic) property on the C*-algebra A which we take as a ring:

Definition 2 (Property of Zero Divisor) *Let* x *be an element in the C*-algebra* A *taken as a ring. Let the property* $P(x)$ *be the ring-theoretic property that the element* x *is a zero-divisor. A non-zero element* x *in the ring* A *is a zero divisor when there exists a non-zero element* y *in the ring such that* $xy = 0$*.*

More precisely, x *is a left zero-divisor and* y *is a right zero-divisor. Since a left zero-divisor implies the existence of a right zero-divisor and vice versa, we make no distinction between the two concepts. We say* x *is a zero divisor with respect to* y*. In the context of a commutative ring there is no distinction at all.*

Note that in the factor ring $\mathbf{Z}/n\mathbf{Z}$ since any non-zero element $\overline{x} \neq 0$ of the factor ring is a zero divisor exactly when the greatest common divisor of x and n is greater than one, and invertible exactly when the the greatest common divisor of x and n is one, every non-zero element of the factor ring $\mathbf{Z}/n\mathbf{Z}$ is either invertible or a zero-divisor. In fact, in a commutative *Noetherian ring* R, the set of all zero divisors is precisely the union of finitely many *prime ideals*. A Noetherian ring, A, is a ring where any ascending chain of ideals $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$ is eventually constant: from some point onwards, the I_n's are all equal. A prime ideal I is an ideal such that $ab \in I$ implies $a \in I$ or $b \in I$ which has the equivalent property that A/I is an integral domain.

Example 1 (Lifting Zero Divisors) *Taking the C*-algebra A/I as a ring and a non-zero element $x + I$ in the quotient C*-algebra A/I, the property P of the element $x + I$ being a zero-divisor, $P(x + I)$ is true, lifts when there exists an element a in the ideal I such that the perturbed element $x - a$ in the finer C*-algebra A is also a zero-divisor: $P(x - a)$ is true.*

We prove the lifting problem of zero divisors affirmatively in the next section. We now give examples of zero-divisors in C*-algebras:

Example 2 (Zero-Divisors in Non-Commutative C*-algebra) *Let A be the C*-algebra $M_2(\mathbf{C})$. Then the matrices x and y defined as:*

$$x = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

qualify x and y as zero-divisors: $xy = 0$.

Let A be the C-algebra of all the bounded operators on a separable Hilbert space, $\mathbf{H}$, such as l^2. Equivalently, $\mathbf{H}$ has a countable orthonormal basis and each operator $x \in B(\mathbf{H})$ has a matrix representation $[a_{ij}]$ with countably infinite rows and columns where each row and column is square summable.*

We can embed the above matrices $x, y \in M_2(\mathbf{C})$ as the matrices $diag[x, 0]$ and $diag[y, 0]$ in $B(\mathbf{H})$, respectively. The matrix $diag[x, 0]$ is the matrix where x is embedded in the top left corner with all other entries zero. Then $diag[x, 0]$ is a zero-divisor.

Example 3 (Zero-Divisors in Commutative C*-algebra) *Let A be the commutative C*-algebra $C[0, 1]$ of all continuous complex valued functions defined on the unit interval $[0, 1]$. Define $x(t)$ and $y(t)$ as in Example 2, Chapter 1.3.1. Then $x(t)$ is a zero-divisor.*

Let K be any closed subset of $[0, 1]$ which is disjoint from the union of the zeros of $x(t)$ and $y(t)$ which is the finite subset $\{0, 1\}$. Then let I denote the ideal uniquely associated with the closed set K of all the functions which vanish on K. Then $x(t) + I$ and $y(t) + I$ are zero-divisors of the factor ring A/I.

2.2 Lifting Zero Divisors

Here we lift the property of a zero divisor from the quotient C*-algebra A/I onto the C*-algebra A. Formally:

Theorem 1 (Lifting Zero Divisors) *(Proposition 2.3 [3]) For any C*-algebra A , we can lift the property of a zero divisor from the quotient C*-algebra A/I onto the finer C*-algebra A.*

Suppose $x+I$ is a zero divisor. Then there exists a non-zero element of A/I , $y+I$, such that $xy+I=I$ where I is the zero element of the quotient C-algebra A/I : there exists elements $x, y \in A-I$ such that their product $xy \in I$. There exist elements $a, b \in I$ of the ideal I such that $(x-a)(y-b)=0$ where the perturbed elements $(x-a)$ and $(y-b)$ are non-zero.*

Proof Our plan of attack is as follows: we first show that the above theorem holds for positive elements: $x, y \geq 0$ [Step 1]; then we prove it for arbitrary elements x, y by a 'bootstrapping argument' facilitated by the polar decomposition of x, y into their partial isometries and their positive moduli [Step 2].

Step 1: Lifting positive zero divisors Let $x, y \geq 0$ where $xy \in I$. Let us decompose the difference $x-y$, which is self-adjoint, into its unique *orthogonal* positive parts [Chapter 1.3.1, Corollary 1]:

$$x-y=(x-y)_{+}-(x-y)_{-}$$

where $(x-y)_{+},(x-y)_{-} \geq 0$ and $(x-y)_{+} \cdot (x-y)_{-}=0$. Then the elements a and b defined as $a=x-(x-y)_{+}$ and $b=y-(x-y)_{-}$, are the required ideal perturbations:

(1) a, b $\in$ **I**: It suffices to show $\pi(a)=\pi(x)-\pi((x-y)_{+})=0$. Equivalently, $\pi(x)=\pi((x-y)_{+})$. Note that $a=b$, so that $\pi(a)=0$ if and only if $\pi(b)=0$.

Observe that

$$\pi(x-y)=\pi((x-y)_{+})-\pi((x-y)_{-})$$

is the unique orthogonal decomposition of self-adjoint element $\pi(x-y)$ [Chapter 1.3.4, Corollary 5]. But

$$\pi(x-y)=\pi(x)-\pi(y)$$

is another orthogonal decomposition of $\pi(x-y)$ since the positive elements $\pi(x),\ \pi(y)$ [Chapter 1.3.4, Proposition 1] are orthogonal : $\pi(x) \cdot \pi(y)=0$.

Therefore, by the uniqueness of orthogonal decompositions, $\pi(x)=\pi((x-y)_{+})$ and $\pi(y)=\pi((x-y)_{-})$.

$(\mathbf{2})(\mathbf{x}-\mathbf{a})(\mathbf{y}-\mathbf{b})=\mathbf{0}$: This follows immediately from noting that $(x-a)$, $(y-b)$ are the orthogonal elements $(x-y)_+, (x-y)_-$, respectively.

Step 2: Lifting arbitrary zero divisors Consider the elements $x, y \in A - I$ where their product $xy \in I$.

Identify A with a norm-closed *-subalgebra of the concrete C*-algebra of bounded linear operators on its universal Hilbert space [Gelfand Naimark Theorem II, Chapter 1.2.2]. Therefore we represent the elements x and y in their polar decomposition form *up* [Chapter 1.3.2, Theorem 2] where p is a positive element for which we have already affirmed the lifting of the property of zero divisors. Now, instead of y we consider its dual y^*. This should not pose any difficulty since we can recover y from y^* by taking its adjoint. So we write x and y^* in terms of their polar decomposition form in B($\mathbf{H}$) : $x = u|x|$ and $y^* = v|y^*|$.

Suppose that we can show the existence of the elements $a_1,\ b_1 \in I$ such that:

$$(|x|^{\frac{1}{2}} - a_1)(|y^*|^{\frac{1}{2}} - b_1) = 0. \tag{2.1}$$

Then the elements a and b defined as $a = u|x|^{\frac{1}{2}}a_1$ and $b = b_1|y^*|^{\frac{1}{2}}v^*$ are the required ideal perturbations for x and y :

$$\begin{aligned}
(x-a)(y-b) &= (u|x|-a)(|y^*|v^*-b) \\
&= (u|x|-u|x|^{\frac{1}{2}}a_1)(|y^*|v^*-b_1|y^*|^{\frac{1}{2}}v^*) \\
&= u|x|^{\frac{1}{2}}(|x|^{\frac{1}{2}}-a_1)(|y^*|^{\frac{1}{2}}-b_1)|y^*|^{\frac{1}{2}}v^* \\
&= 0 \qquad\qquad [by (2.1)]
\end{aligned}$$

where the elements a and b are elements of the ideal I since $a_1,\ b_1 \in I$ (I is 2-sided ideal in A) and $u|x|^{\frac{1}{2}}, v|y^*|^{\frac{1}{2}} \in A$ [Chapter 1.3.3, Theorem 5].

We now bootstrap : since the theorem is true for all positive elements, equation (2.1) is valid as long as we show $|x|^{\frac{1}{2}}|y^*|^{\frac{1}{2}} \in I$. This we prove by writing $|x|^{\frac{1}{2}}|y^*|^{\frac{1}{2}}$ as the limit of a convergent sequence in I (I is closed). This will vindicate the use of y^* over y. If the C*-algebra does not have an identity, we replace it with the unitization C*-algebra A_e of A [Chapter 1.1, Theorem 1]. We define the sequence as follows:

$$a_n = (\tfrac{1}{n} + |x|)^{-\frac{3}{2}}(x^*x)(yy^*)(\tfrac{1}{n} + |y^*|)^{-\frac{3}{2}}.$$

Reexpressing each a_n as $\Big((\frac{1}{n} + |x|)^{-\frac{3}{2}}x^*\Big)(xy)\Big(y^*(\frac{1}{n} + |y^*|)^{-\frac{3}{2}}\Big)$, we note that each $a_n \in I \subset A$ thereby allowing us to assume without loss of generality that the C*-algebra has an identity. We show $a_n \in I$.

Identify the commutative C*-algebra, $C^*(|x|, e)$, the closure of the *-algebra generated $|x|$ and e in the unitization C*-algebra A_e of A, with the commutative C*-algebra, $C(K)$ [Chapter 1.1, Theorem 2]. The normal element $(\frac{1}{n} + |x|)$ of $C^*(|x|, e)$ is a strictly positive function in $C(K)$: its spectrum consists purely of positive real numbers. Therefore the element $(\frac{1}{n} + |x|)^{-\frac{3}{2}}$ is a well defined element of $C^*(|x|, e) \subset A_e$ [Chapter 1.3.1, Theorem 1]. Similarly, $(\frac{1}{n} + |y^*|)^{-\frac{3}{2}}$ belongs to $C^*(|y^*|, e) \subset A_e$.

Since A is a 2-sided ideal of A_e, $\left((\frac{1}{n} + |x|)^{-\frac{3}{2}} x^*\right)$ and $\left(y^*(\frac{1}{n} + |y^*|)^{-\frac{3}{2}}\right)$ is an element of the original C*-algebra A. But I is a 2-sided ideal of A (in fact, even of A_e, as direct computation shows). Therefore, we conclude that each a_n is an element of the ideal I.

We are now left with showing that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (\frac{1}{n} + |x|)^{-\frac{3}{2}} (x^*x)(y^*y)(\frac{1}{n} + |y^*|)^{-\frac{3}{2}} = |x|^{\frac{1}{2}} |y^*|^{\frac{1}{2}} \tag{2.2}$$

Case I : Spectrum of $|x|$ and $|y^*|$ do not contain the number 0 We first prove equation (2.2) with the restriction that the spectrum of the positive elements $|x|$ and $|y^*|$ do not contain 0 to motivate the definition of the sequence a_n. This is needed to define the elements $|x|^{-\frac{3}{2}}$ and $|y^*|^{-\frac{3}{2}}$.

Applying the Functional Factorization For Normal Elements corollary [Chapter 1.3.1, Corollary 2] to the normal element $|x|$ where we set $g(x) = x^{-\frac{3}{2}}$ and $h(x) = x^2$ and $f(x) = x^{\frac{1}{2}}$, we note that:

$$|x|^{\frac{1}{2}} = |x|^{-\frac{3}{2}} |x|^2 = |x|^{-\frac{3}{2}} x^* x.$$

Similarly, (this is the reason for working with y^*)

$$|y^*|^{\frac{1}{2}} = |y^*|^2 |y^*|^{-\frac{3}{2}} = yy^* |y^*|^{-\frac{3}{2}}.$$

Therefore,

$$|x|^{\frac{1}{2}} |y^*|^{\frac{1}{2}} = |x|^{-\frac{3}{2}} x^*(xyy^*) |y^*|^{-\frac{3}{2}}.$$

Consider the continuous function $k_n : \mathbf{R}^{+\mathbf{0}} \to \mathbf{R} | x \mapsto (\frac{1}{n} + x)^{-\frac{3}{2}}$. Recall that $g : \mathbf{R}\backslash\{0\} \to \mathbf{R}$ denotes the power function: $g(x) = x^{-\frac{3}{2}}$. Then on any compact subset K of $\mathbf{R}$ which does not contain the element 0, $k_n \to g$, the convergence being uniform in $C(K)$. Consequently, applying the Gelfand Naimark theorem to the compact subsets of $\mathbf{R}$ which are void of the zero element, $C(\sigma_{A_e}(|x|))$ and $C(\sigma_{A_e}(|y^*|))$, we infer that $\Phi(k_n) \to \Phi(g) = g(|x|) = |x|^{-\frac{3}{2}}$ and $\Phi(k_n) \to \Phi(g) = g(|y^*|) = |y^*|^{-\frac{3}{2}}$, respectively. That is,

$$
\begin{aligned}
|x|^{\frac{1}{2}}|y^*|^{\frac{1}{2}} &= |x|^{-\frac{3}{2}}(x^*xyy^*)|y^*|^{-\frac{3}{2}} \\
&= \big(\lim_{n\to\infty}(\frac{1}{n}+|x|)^{-\frac{3}{2}}\big)(x^*xyy^*)\big(\lim_{n\to\infty}(\frac{1}{n}+|y^*|)^{-\frac{3}{2}}\big) \\
&= \lim_{n\to\infty}(\frac{1}{n}+|x|)^{-\frac{3}{2}}(x^*x)(y^*y)(\frac{1}{n}+|y^*|)^{-\frac{3}{2}}.
\end{aligned}
$$

Case II : Spectrum of $|x|$ and $|y|$ contains the number 0 By the joint continuity of the product, it suffices to show that:

$$\lim_{n\to\infty}(\tfrac{1}{n}+|x|)^{-\frac{3}{2}}(x^*x) = |x|^{\frac{1}{2}}$$

since the proof of $\lim_{n\to\infty}(y^*y)(\frac{1}{n}+|y^*|)^{-\frac{3}{2}} = |y^*|^{\frac{1}{2}}$ is similarly done. Although the term $|x|^{-\frac{3}{2}}$ is undefined, we go about this problem as follows:

As before, identify the commutative C*-algebra $C^*(|x|,e)$ with an identity, with $C(K)$ where K is the compact space of all non-zero multiplicative linear functionals of $C^*(|x|,e)$ [Chapter 1.1, Theorem 2]. Let f, f_n denote the functional representation of the C*-algebra element $|x|$, $\frac{1}{n}+|x|$ under the Gelfand transform respectively : $f_n = \widehat{\frac{1}{n}+|x|} \in C(K)$ and $f = \widehat{|x|} \in C(K)$. Since the spectrum of the positive element $|x|$ is exactly the range of f, f is a positive valued function. The function $f_n = \widehat{\frac{1}{n}+|x|} = \hat{\frac{1}{n}} + \widehat{|x|} = K_{\frac{1}{n}} + f$ is the positive valued function f translated by the constant function $K_{\frac{1}{n}} : K \to \mathbf{C}|k \mapsto \frac{1}{n}$. Therefore $f_n \in C(K)$ is a strictly positive function. Consequently, the composite function $g \circ f_n \in C(K)$ where $g(x) = x^{-\frac{3}{2}}$ is a well defined function on K which corresponds to the C*-algebra element $(\frac{1}{n}+|x|)^{-\frac{3}{2}}$ since the Gelfand transform of a polynomial expression is the polynomial expression of the Gelfand transform.

In $C(K)$, the sequence of functions $g \circ f_n$ is an increasing sequence of positive valued functions on the compact space K. Then the increasing sequence of continuous functions $(g\circ f_n)\cdot f^2$ on the compact space K, converges pointwise to $h \circ f$ where $h(x) = x^{\frac{1}{2}}$, which we shall denote as $f^{\frac{1}{2}}$: for all $k \in K$ such that $f(k) \neq 0$, $[(g\circ f_n)\cdot f^2](k)$ converges to $f^{\frac{1}{2}}(k)$; for all $k \in K$ such that $f(k) = 0$, $[(g\circ f_n)\cdot f^2](k) = 0$ which is equal to $f^{\frac{1}{2}}(k)$. Consequently, by Dini's theorem for compact Hausdorff spaces, $(g\circ f_n)\cdot f^2$ converges uniformly to $f^{\frac{1}{2}}$. Equivalently, $\lim_{n\to\infty}(\frac{1}{n}+|x|)^{-\frac{3}{2}}(x^*x) = |x|^{\frac{1}{2}}$. We state the version of Dini's Theorem we used:

Dini's Theorem for compact spaces. *(Chapter 8, Theorem 8.7 [21]) Let K be a compact Hausdorff space. Suppose that $\mathcal{F} \subset C(K,\mathbf{R})$ where $C(K,\mathbf{R})$ is the space of all real valued functions on the compact space K, has the following properties:*

a) $f, g \in \mathcal{F}$ implies that there is an $h \in \mathcal{F}$ such that $h \geq f \bigvee g$.

b) The function f_0 defined by $f_0(k) = sup\{f(k)|f \in \mathcal{F}\}$ is real valued and continuous.

Then given $\epsilon > 0$, there exists an $f \in \mathcal{F}$ such that $\| f - f_0 \| < \epsilon$ where the norm $\| \cdot \|$ is the supremum norm on $C(K)$.

where we set $\mathcal{F} = \{(g \circ f_n) \cdot f^2 | n \in \mathbf{N}\}$ and $f_0 = f^{\frac{1}{2}}$. Consequently, $(g \circ f_n) \cdot f^2$ converges uniformly to $f^{\frac{1}{2}}$. Equivalently, $\lim_{n\to\infty}(\frac{1}{n} + |x|)^{-\frac{3}{2}}(x^*x) = |x|^{\frac{1}{2}}$.

Q.E.D

Buoyed by the success in lifting the property of a zero divisor, using essentially the same techniques, we lift the property of positive zero divisors and self-adjoint zero divisors from the quotient C*-algebra A/I onto the C*-algebra A:

Corollary 1 *(Corollary 2.4, [3]) If $x, y \in A^+$ with $xy \in I$, then there exist $a, b \in I$ with $(x - a)(y - b) = 0$ and $(x - a)$, $(y - b) \in A^+$. Further, if $x, y \in A_{sa}$ with $xy \in I$, then there exist $a, b \in I$ with $(x - a)(y - b) = 0$ and $(x - a)$, $(y - b) \in A_{sa}$ where A_{sa} denotes the self adjoint elements of the C*-algebra A.*

Proof. The first assertion follows from Step 1 of the proof of theorem 1. For the second assertion, we shall construct the required ideal perturbations from the following elements:

Consider A as a uniformly closed self-adjoint subalgebra of operators on its universal Hilbert space $\mathbf{H}$ [Gelfand Naimark Theorem II, Chapter 1.2.2]. Let $x = u|x|$ and $y = v|x|$ denote the polar decompositions of the self adjoint elements x and y, respectively, in $B(\mathbf{H})$. The partial isometries u and v are self-adjoint and they commute with their respective moduli $|x|$ and $|y|$ [Chapter 1.3.2, Corollary 4]. They will therefore commute with the elements $|x|^{\frac{1}{3}}$ and $|y|^{\frac{1}{3}}$, respectively : the element $|x|^{\frac{1}{3}}$ resides in the commutative C*-algebra generated by u and $|x|$; likewise for the element $|y|^{\frac{1}{3}}$.

Now, as in Step 1 of the proof of Theorem 1, consider the decomposition of the difference $|x| - |y|$, which is a self-adjoint element in A, into its unique *orthogonal* positive parts [Chapter 1.3.1, Corollary 1]:

$$|x| - |y| = (|x| - |y|)_+ - (|x| - |y|)_-$$

where $(|x| - |y|)_+, (|x| - |y|)_- \geq 0$ and $(|x| - |y|)_+ \cdot (|x| - |y|)_- = (|x| - |y|)_- \cdot (|x| - |y|)_+ = 0$: the positive parts $(|x| - |y|)_+$ and $(|x| - |y|)_-$ commute and we shall denote them as d_1 and d_2, respectively.

We define the required ideal perturbations a and b as the self-adjoint elements $x - x_1$ and $y - y_1$ respectively, where x_1 and y_1 are defined as:

$$x_1 = (d_1)^{\frac{1}{3}} u|x|^{\frac{1}{3}} (d_1)^{\frac{1}{3}}$$
$$y_1 = (d_2)^{\frac{1}{3}} v|y|^{\frac{1}{3}} (d_2)^{\frac{1}{3}}$$

The elements a, b are self-adjoint elements of A : It suffices to show that x_1 and y_1 are self adjoint. Since the elements $u|x|^{\frac{1}{3}}$ and $v|y|^{\frac{1}{3}}$ are elements of A [Chapter 1.3.3, Theorem 5], $x_1, y_1 \in A$. Now $u|x|^{\frac{1}{3}}$ and $v|y|^{\frac{1}{3}}$ are self-adjoint since the terms of the product commute and u and v are self adjoint. Therefore x_1, y_1 are self adjoint : $(d_1)^{\frac{1}{3}}$ and $(d_2)^{\frac{1}{3}}$ are positive and hence self adjoint.

Since d_1 and d_2 commute:

$$(x - a)(y - b) = x_1 y_1 = 0$$

and $(x - a)$ and $(y - b)$ are self adjoint elements. We are therefore left with showing that:

a, **b** $\in$ **I**: We show that $\pi(a) = \pi(x - x_1) = \pi(x) - \pi(x_1) = 0$ by showing that $\pi(x) = \pi(x_1)$. $\pi(b) = 0$ is done similarly.

By the Quotient Mapping Of the Functional Calculus For Self Adjoint Elements [Chapter 1.3.4, Proposition 2] and its corollary [Chapter 1.3.4, Corollary 5]:

$$\pi(x_1) = [\Big(\pi(|x|) - \pi(|y|)\Big)_+]^{\frac{1}{3}} \pi(u|x|^{\frac{1}{3}}) [\Big(\pi(|x|) - \pi(|y|)\Big)_+]^{\frac{1}{3}} \qquad (2.3)$$

Recall that $|x|^{\frac{1}{2}}|y^*|^{\frac{1}{2}} \in I$ [Step 2 of the proof of Theorem 1]. Since y is self adjoint, $|x|^{\frac{1}{2}}|y|^{\frac{1}{2}} \in I$. Therefore, $|x||y| = |x|^{\frac{1}{2}}(|x|^{\frac{1}{2}}|y|^{\frac{1}{2}})|y|^{\frac{1}{2}}$ is also an element of the ideal I. Consequently the self adjoint elements $\pi(|x|)$ and $\pi(|y|)$ are orthogonal. Hence, $\pi(|x|) - \pi(|y|)$ is the unique orthogonal decomposition of $\pi(|x|) - \pi(|y|)$: $\Big(\pi(|x|) - \pi(|y|)\Big)_+ = \pi(|x|)$. Equation (2.3) then becomes:

$$\begin{aligned} \pi(x_1) &= [\pi(|x|)]^{\frac{1}{3}} \pi(u|x|^{\frac{1}{3}}) [\pi(|x|)]^{\frac{1}{3}} && (2.4)\\ &= [\pi(|x|^{\frac{1}{3}})] \pi(u|x|^{\frac{1}{3}}) [\pi(|x|^{\frac{1}{3}})] && (2.5)\\ &= \pi\Big(|x|^{\frac{1}{3}}(u|x|^{\frac{1}{3}})|x|^{\frac{1}{3}}\Big) && (2.6)\\ &= \pi(x) && (2.7) \end{aligned}$$

the last equation being a consequence of the fact that u commutes with $|x|^{\frac{1}{3}}$.

Q.E.D

2.3 Lifting n-zero divisors : Abelian C*-algebra

2.3.1 Lifting n-zero divisors : Statement of Problem

We generalize the definition of a zero divisor [Chapter 2.1, Definition 2] of a ring to a *n-zero divisor* of a ring as follows:

Definition 1 (Property of n-Zero Divisor) *Consider an n -tuple of elements, $(x_1, \ldots, x_n)$, of a C*-algebra A taken as a ring. Let the n-ary property $P(x_1, \ldots, x_n)$ be the ring-theoretic property that the non-zero n- tuple $(x_1, \ldots, x_n)$ (none of the x_i's are the zero element) is a n-zero divisor. A non-zero n- tuple $(x_1, \ldots, x_n)$ is a n-zero divisor when their product $\prod_{1 \leq i \leq n} x_i = 0$.*

In this section, for special types of C*-algebras, we extend the lifting problem of zero divisors to that of lifting n-zero divisors, which we define formally as:

Definition 2 (Lifting n-Zero Divisors) *Taking the C*-algebra A/I as a ring and a non-zero n - tuple $(x_1 + I, \ldots, x_n + I)$ in the quotient C*-algebra A/I, the n-ary property $P(x_1 + I, \ldots, x_n + I)$ of the n - tuple $(x_1 + I, \ldots, x_n + I)$ being a n-zero divisor, $P(x_1 + I, \ldots, x_n + I)$ is true, lifts when there exists a n - tuple $(a_1, \ldots, a_n)$ in the ideal I such that the perturbed n - tuple $(x_1 - a_1, \ldots, x_n - a_n)$ in the finer C*-algebra A is also a zero-divisor: $P(x_1 - a_1, \ldots, x_n - a_n)$ is true.*

The lifting problem of n-zero divisors from the quotient C*-algebra A/I onto the C*-algebra A for special types of C*-algebras, will serve to motivate the solution of the lifting problem of n-zero divisors for a general C*-algebra.

2.3.2 Lifting n-zero divisors : Commutative C*-algebras

Theorem 1 *(Proposition 2.5 [3])* **Provided A is a commutative C*-algebra**, *if $x_1, \ldots, x_n$ are elements of A with $\prod_{1 \leq i \leq n} x_i \in I$, then there exist $a_1, \ldots, a_n$ in I with $\prod_{1 \leq i \leq n}(x_i - a_i) = 0$.*

Proof. Identify A with the C*-algebra $C_0(\Omega)$. Then the closed ideal I can be identified with closed set $K \subset \Omega$ on which the functions of I vanish : $I = \{f \in C_0(\Omega) \mid f|_K = 0\}$ [Chapter 1.2.4, Example 35]. Therefore, the hypothesis $\prod_{1 \leq i \leq n} x_i \in I$ forces at least one of the x_i to be 0 at each point $p \in K$.

Now identify $C_0(\Omega) \hookrightarrow L^\infty(\Omega, \mathcal{B}(\Omega), \mu)$ [Chapter 1.2.1, Example 12] with the commutative subalgebra S of multiplication operators of $B(\mathbf{H})$ where $\mathbf{H} = L^2(\Omega, \mathcal{B}(\Omega), \mu)$. By the polar decomposition $x_i = u_i|x_i|$ where $|x_i| = x^*x = \overline{x(t)}x(t)$.

Returning to $C_0(\Omega)$, for each x_i, consider its modulus $|x_i|^{\frac{1}{2}} \in C_0(\Omega)$. Define the finite join $w = \bigwedge_{1 \leq i \leq n} |x_i|^{\frac{1}{2}}$. Then $w \in I$ since x is 0 at each point in K. Further for every point in Ω at least one of $(|x_i|^{\frac{1}{2}} - w)$ is zero. That is,

$\prod_{1\le i\le n}(|x_i|^{\frac{1}{2}} - w) = 0$. Setting the element a_i in the ideal I as $u_i|x_i|^{\frac{1}{2}}w$ (note that $w \in I$) completes the proof: each a_i will perturb each of the x_i, $x_i - a_i$, to force $\prod_{1\le i\le n}(x_i - a_i) = 0$:

$$\prod_{1\le i\le n}(x_i - a_i) = \prod_{1\le i\le n}(u_i|x_i| - a_i) = \prod_{1\le i\le n}(u_i|x_i| - u_i|x_i|^{\frac{1}{2}}w) \tag{2.8}$$

$$= \prod_{1\le i\le n} u_i|x_i|^{\frac{1}{2}}\Big((|x_i|^{\frac{1}{2}} - w)\Big) \tag{2.9}$$

$$= \Big(\prod_{1\le i\le n} u_i|x_i|^{\frac{1}{2}}\Big)\Big(\prod_{1\le i\le n}(|x_i|^{\frac{1}{2}} - w)\Big). \tag{2.10}$$

$$= 0. \tag{2.11}$$

The above equations (2.8) - (2.11) all occur in the commutative subalgebra $S \subset B(\mathbf{H})$, where in particular, equation (2.10) follows from the commutativity of this subalgebra.

Q.E.D

The above proof offers no clue to a generalization to the non-abelian case. The case of a Von Neumann C*-algebra sheds more light in this regard. The case of a SAW*-algebra, another special type of C*-algebra, is the key to the required generalization. Before we digress into a study of Von Neumann C*-algebras and SAW*-algebras, we state without proof the following lifting problem:

Theorem 2 (Lifting Finite Positive Orthogonal Sets) *(Proposition 2.6, [3]) Let $O = \{x_1 + I, \dots, x_n + I\} \subset (A/I)^+$ be a finite set of n positive elements of the quotient C*-algebra A/I which are pairwise orthogonal : $(x_i + I)(x_j + I) = 0$ for all $i \neq j$. We call the set O a finite positive orthogonal set of the C*-algebra A/I. The above condition is equivalent to $O' = \{x_1, \dots, x_n\} \subset A^+$ being a finite set of n positive elements of the C*-algebra A such that $x_i x_j \in I$ for all $i \neq j$.*

Then there exists a finite set, $\{a_1, \dots, a_n\} \subset I$, of elements of the ideal I with which we perturb the elements of O' such that the set $\{x_1 - a_1, \dots, x_n - a_n\}$ is a finite positive orthogonal set of the C-algebra A : $\{x_1 - a_1, \dots, x_n - a_n\} \subset A^+$ and $(x_i - a_i)(x_j - a_j) = 0$.*

2.4 Lifting n-zero divisors : Von Neumann C*-algebra

2.4.1 Definition of Von Neumann C*-algebra

Every C*-algebra is an operator-norm closed *-subalgebra of some $B(\mathbf{H})$ [Chapter 1.2.2, Gelfand Naimark Theorem II]. A Von Neumann algebra is a special kind of *-subalgebra of $B(\mathbf{H})$:

Definition 1 (Von Neumann Algebra) *A Von Neumann Algebra A is a *-subalgebra of a $B(\mathbf{H})$ which contains the identity operator $1_{\mathbf{H}}$ and is closed with respect to a locally convex topology which is weaker than the usual operator norm topology. The topology in question is the strong-operator-topology, abbreviated as SOT-topology, and defined in the next definition.*

Definition 2 (Strong-Operator-Topology, Weak-Operator-Topology) *We can define two weaker-than-norm topologies on $B(\mathbf{H})$. These topologies are locally convex topologies on $B(\mathbf{H})$ which have $\{p_x : B(\mathbf{H}) \to \mathbf{R}^+ \mid x \in \mathbf{H}$ and $p_x(A) = \parallel Ax \parallel\}$ and $\{p_{x,y} : B(\mathbf{H}) \to \mathbf{R}^+ \mid x,\ y \in \mathbf{H}$ and $p_{x,y}(A) = |(Ax|y)|\}$ where $(\cdot|\cdot)$ is the inner product on $\mathbf{H}$ as their defining families of semi-norms, respectively. We call these locally convex topologies the strong and the weak operator topology on B(H) respectively, abbreviated as SOT-topology and WOT -topology, respectively.*

The WOT-topology is coarser than the SOT-topology. However, with regards to closed convex sets of $B(\mathbf{H})$ they are indistinguishable:

Theorem 1 *(Theorem 5.1.2 [10]) The SOT and WOT closures of a convex subset K of $B(\mathbf{H})$ coincide*

Therefore we have an alternative definition of a Von Neumann algebra since all subspaces are trivially convex:

Definition 3 (Von Neumann Algebra) *A Von Neumann algebra A is a *-subalgebra of $B(\mathbf{H})$ which contains the identity operator $1_{\mathbf{H}}$ and is WOT - closed.*

We are now in the position to define a Von Neumann C*-algebra as a C*-algebra which is also a Von Neumann algebra:

Definition 4 (Von Neumann C*-algebra) *Let A be a C*-algebra and Φ be a one-to-one non-degenerate *-representation where $\mathbf{H}$ is the Hilbert space associated with the *-representation Φ. Since Φ is a one-to-one *-homomorphism between two C*-algebras, Φ is an isometric *-isomorphism and we can identify A with an operator-norm closed *-subalgebra of all bounded operators, $B(\mathbf{H})$, on the Hilbert space $\mathbf{H}$.*

The C-algebra A is a Von Neumann C*-algebra if A contains the identity* *operator $1_{\mathbf{H}}$ and is closed in the* SOT *- topology or equivalently, in the* WOT *- topology. Therefore, the usual operator norm closure, the* SOT *- topology closure and the* WOT *- topology closure coincide.*

Example 1 (Commutative Von Neumann C*-algebra) *(Chapter 5.1, Example 5.1.6 [10]) Let A be the C*-algebra $L^\infty(\Omega, \mathcal{B}(\Omega), \mu)$ [Example 12, Chapter 1.2.1]. We take as our non-degenerate *-representation, the map $\Phi : L^\infty(\Omega, \mathcal{B}(\Omega), \mu) \to B(\mathbf{H}) | g \mapsto M_g$ where $\mathbf{H} = L^2(\Omega, \mathcal{B}(\Omega), \mu)$.*

A contains the identity operator $1_{\mathbf{H}}$: the constant 1 -map, $e(x) = 1$ for all $x \in \Omega$, is the identity for $L^\infty(\Omega, \mathcal{B}(\Omega), \mu) \subset L^2(\Omega, \mathcal{B}(\Omega), \mu)$; M_e is hence the identity operator on $\mathbf{H} = L^2(\Omega, \mathcal{B}(\Omega), \mu)$ and one can show that A is WOT *- closed.*

In fact, we can *characterize* a Von Neumann C*-algebra as a C*-algebra which has a pre-dual Banach space:

Theorem 2 *([14])A C*-algebra A is a Von Neumann C*-algebra if and only if it is the dual of some Banach space which we denote as A_* and call the pre-dual of A : $A = ((A_*)^*)$.*

Example 1 (Pre-Dual Banach Space) *Consider the Banach sequence spaces c_0, c, l^1 which denote the sequence spaces of null sequences, convergent sequences and absolutely summable sequences, respectively.*

Then $(c_0)^* = l^1$: all linear functionals on the sequence space c_0 are of the form $\varphi_{(a_n)} : c_0 \to \mathbf{C} | (x_n) \mapsto \sum_{n=1}^\infty x(n)a(n)$ where $(a_n) \in l^1$. The map $\Phi : l^1 \to (c_0)^* : (a_n) \mapsto \varphi_{(a_n)}$ is an onto isometric isomorphism.

Further, $(c)^* = l^1$: this follows from the fact that all linear functionals on the sequence space of convergent sequences c are of the form $\rho : c \to \mathbf{C} | (x_n) \mapsto k \lim_n (x_n) + \psi_{a_n}$ where $\psi_{a_n} : c \to \mathbf{C} | (x_n) \mapsto \sum_{n=1}^\infty a(n)x(n)$ and (a_n) is a sequence in l^1.

As the above example illustrates, in general Banach space theory, a dual Banach space need not be the dual of a unique Banach space. This is not the case for C*-algebras : it turns out that Von Neumann C*-algebras have a unique pre-dual.

2.4.2 Property of Von Neumann C*-algebra : Closure Under Range Projection Of Operator

One of the pleasant features of a Von Neumann C*-algebra A is the abundance of (orthogonal) projections in A. Recall that $p \in B(\mathbf{H})$ is an orthogonal projection if and only if $p^2 = p = p^*$. Each closed subspace of the Hilbert space $\mathbf{H}$, corresponds uniquely to an (orthogonal) projection. The closed subspace associated with the projection p is the range of p. To be more precise:

Theorem 3 (Closure Under Range Projection Of Operator) *Let A be a Von Neumann algebra in $B(\mathbf{H})$. For every operator $a \in A$, the unique operator associated with the closed subspace $\overline{Ran(a)}$ where $Ran(a)$ is the range of the operator a, is a member of the A. We call this unique projection the range projection $R(a)$.*

In fact any Von Neumann C*-algebra A is the closed linear span of its projections [Chapter 4, Theorem 4.1.11 [13]]. In contrast, there are C*-algebras which have the trivial projections, the identity and zero operator, as their only projections:

Example 2 (Projectionless C*-algebra) *Consider the C*-algebra $C([0,1])$ as an operator-norm closed *-subalgebra of the space, $B(\mathbf{H})$, of all bounded operators on the Hilbert space $\mathbf{H} = L^2(\lambda_{[0,1]})$ [Chapter 1.2.4, Example 43]. Each $f \in C([0,1])$ is identified with the multiplication operator $M_f : g \in (L^2, \mathcal{M}, \mu) \mapsto fg$.*

M_f is a projection if and only if $f^2 = f$ and f is real-valued. Now $f^2 = f$ implies $f^2(t) = f(t) \Rightarrow f(t) = 0$ or $1 \quad \forall t \in [0,1]$. Since [0, 1] is connected, the range of f is connected, or equivalently is an interval, and therefore cannot assume finitely many values. Hence f is either the constant 1 or 0 function which corresponds to the trivial $1_{\mathbf{H}}$ and 0 - operator on $B(L^2, \mathcal{M}, \mu)$.

In order to prove the Closure Under Range Projection Of Operator theorem [Theorem 3], we need an analogue of Dini's Theorem for compact metric spaces fortunately found in $B(\mathbf{H})$. We quote Dini's Theorem for compact metric spaces, which in effect establishes the equivalence of pointwise and uniform convergence under certain conditions, as follows:

Dini's Theorem for Compact Metric Spaces. *Let X be a compact metric space, and (f_n) an increasing sequence in $C(X)$, the space of all continuous real valued functions on X, that converges simply (pointwise) to a continuous function $f \in C(X)$. Then (f_n) converges uniformly to f.*

Now, instead of an increasing sequence of real-valued functions f_n in $C(X)$ bounded by $f \in C(X)$, we have a *monotone increasing* net of self-adjoint operators H_a in $B(\mathbf{H})$ bounded by $k1_{\mathbf{H}} \in B(\mathbf{H})$. Recall that the set of all self-adjoint operators has the usual operator ordering $\leq$:

Definition 5 (Usual Operator Ordering on $B(\mathbf{H})$) *Let $B(\mathbf{H})$ denote the space of all bounded operators on some Hilbert space $\mathbf{H}$. Recall that an operator $x \in B(\mathbf{H})$ is positive if and only if $(x(\eta), \eta) \geq 0$ for each $\eta \in \mathbf{H}$. The* *set $B(\mathbf{H})^+$ of all the positive operators on the Hilbert space* **H** *forms a proper closed convex cone with vertex 0. Since the set of all self-adjoint operators on the Hilbert space* **H** *forms a* **real** *vector space, the fixed cone $K = B(\mathbf{H})^+$ defines a partial order $\leq$ (the antisymmetry of the partial order $\leq$ is by virtue of the cone being proper) on the set of all self-adjoint operators as follows:*

$$\text{If } a, b \in A, \text{ then } a \leq b \text{ if and only if } b - a \in A^+.$$

We call this partial order the usual operator ordering on $B(\mathbf{H})$.

We define a monotone increasing net of self-adjoint operators H_a in $B(\mathbf{H})$ as follows:

Definition 6 (Monotone Increasing Net Of Self-adjoint Operators) *Let $(H_a | a \in D)$ be a net of self-adjoint operators in $B(\mathbf{H})$ with respect to the usual operator ordering $\leq$ associated with the positive cone $B(\mathbf{H})^+$. The index set D which we now denote as $(D, \leq_D)$, is a directed set : $\leq_D$ is a partial order on set D which need not satisfy the antisymmetry axiom and D is closed with respect to joins: if $m, n \in D$ then there exists a $p \in D$ such that $p \geq m$ and $p \geq n$.*

The net $(H_a | a \in D)$ is monotone increasing if and only if the operator ordering associated with the positive cone $B(\mathbf{H})^+$ coincides with the directed set $\leq_D$- ordering of the net: $H_a \leq H_b \Leftrightarrow a \leq_D b$.

To complete the analogy, instead of point-wise convergence in $C(X)$, we have convergence in the SOT-topology in $B(\mathbf{H})$. This analogy is reasonable since $H_a \overset{SOT}{\to} H$ if and only if $p_x(H_a) \to p_x(H)$ if and only if $H_a(x) \to H(x) \;\; \forall x \in \mathbf{H}$ in the norm topology of **H**. The limit H in analogy with the real valued pointwise limit $f \in C(X)$ is also self-adjoint. In short we have:

Lemma 1 (Dini Theorem Analogue in B(H)) *(Chapter 5, Lemma 5.1.4 [10]) If $(H_a)_{a \in D}$ is a monotone increasing net of self adjoint operators on the Hilbert space* **H** *and $H_a \leq k\, 1_{\mathbf{H}}$ for all a, then the net H_a is strong operator convergent to a self-adjoint operator H , and H is the least upper bound of the set $\{H_a | a \in D\}$.*

The crux of the proof of the Closure Under Range Projection Of Operator theorem [Theorem 3] lies in observing that the positive operator $a \in A$ where A is a Von Neumann C*-Algebra and the positive n-th root $a^{\frac{1}{n}}$ can be treated as the identity and the n-root functions on some $C(K)$, respectively, since A has an identity [Chapter 1.1, Theorem 2].

Lemma 2 *(Chapter 5, Lemma 5.1.5 [10]) If $a \in B(\mathbf{H})$ is a positive operator on the Hilbert space* **H** *and $0 \leq a \leq 1_{\mathbf{H}}$, then $a^{\frac{1}{n}}$ is a monotone increasing sequence of positive operators whose strong-operator limit is the projection on the closure of the range of a.*

Proof. Consider the self-adjoint element $a \in B(\mathbf{H})$, a C*-algebra with an identity, as an element of the commutative C*-subalgebra $C^*(a)$. The sequence $a^{\frac{1}{n}}$ is a monotone increasing sequence of positive operators in $B(\mathbf{H})$ since it corresponds to the monotone increasing sequence of functions $(a^{\frac{1}{n}})$ in C$(\sigma_{B(\mathbf{H})}(a))$ under the order preserving isometric *-isomorphism $\Phi : C(\sigma_{B(\mathbf{H})}(a)) \to C^*(a)$ [Chapter 1.3.1, Theorem 1]. Therefore, the sequence $(a^{\frac{1}{n}})$ has a SOT -topology limit E , which is a bounded self-adjoint operator [Lemma 1]. Once we establish $E^2 = E$ then we have shown that E is an (orthogonal) projection. We must however check a few things.

The sequence $(a^{\frac{1}{n}})$ *corresponds to the monotone increasing sequence of functions* $(a^{\frac{1}{n}})$ *in* $C(\sigma_{B(\mathbf{H})}(a))$. Firstly, each of the elements $a^{\frac{1}{n}}$ is well defined since the n-th root functions are well defined: $a \geq 0 \Rightarrow \sigma_{B(\mathbf{H})}(a) \subset \mathbf{R}^+$. Secondly, the restriction that $a \leq 1_{\mathbf{H}}$ forces the domain, $\sigma_{B(\mathbf{H})}(a)$, of the n-th root functions $x^{\frac{1}{n}}$ to be a subset of $[0,1]$: the spectral radius $r(a) = \sup\{|\lambda| \mid \lambda \in \sigma_{B(\mathbf{H})}(a)\}$ is bounded by $\| a \|$ which in turn is bounded by $\| 1_{\mathbf{H}} \| = 1$ [Chapter 1.1, Equation (1.11)]. Therefore, the sequence $(a^{\frac{1}{n}})$ is a monotone increasing sequence of functions in C$(\sigma_{B(\mathbf{H})}(a)\})$.

E is an orthogonal projection : $E^2 = E$. Observe that the product sequence $(a^{\frac{1}{n}})(a^{\frac{1}{n}}) = (a^{\frac{2}{n}})$ has E^2 as its SOT-topology limit: this follows from the joint continuity of the point-wise product of the C*-algebra $C^*(a)$ which is transferred faithfully by the *-isomorphism Φ. Since $(a^{\frac{1}{n}})$, which is the sequence $(a^{\frac{2}{2n}})$, is a subsequence of the sequence $(a^{\frac{2}{n}})$, $E = E^2$: all subsequences of a convergent sequence share the same limit.

We are now left with showing that $E = R(a)$: the unique closed subspace associated with the projection E is the closed subspace of $Ran(a)$. We prove this by showing that the projection E has the same kernal as the positive operator a, since $ker(a)^{\perp} = \overline{Ran(a)}$ by virtue of the fact that the operator a is self-adjoint [Chapter 7.2, Proposition 2.2.1, [25]].

(1) $ker(a) \subset ker(E)$.

Since the n-th root functions $x^{\frac{1}{n}}$ vanish at 0, each $x^{\frac{1}{n}} \in C(\sigma_A(x))$ is a uniform limit of polynomials $p_i \in C(\sigma_A(x))$, which do not have constant terms [Stone-Weierstrass theorem]. Therefore, $\Phi(p_i) \to \Phi(f)$, the convergence occurring in $C^*(a)$ [Theorem 1, Chapter 1.3]. Since Φ is a *-homomorphism, $\Phi(p_i) = p_i(a)$. Consequently, each $a^{\frac{1}{n}}$ is a limit of a sequence consisting of polynomial expressions purely involving a, the convergence taking place in the operator norm topology of $B(\mathbf{H})$. The convergence will also take place in the coarser SOT - topology:

$$p_i(a)(\eta) \to a^{\frac{1}{n}}(\eta) \text{ for each } \eta \text{ in } \mathbf{H}$$

If $a(\eta) = 0$, then $p_i(a)(\eta) = 0$ for all i. Therefore, $a^{\frac{1}{n}}(\eta) = 0$ for each n. Similarly, $E(\eta) = 0$ since it is the SOT-topology limit of $a^{\frac{1}{n}}$.

(2) $ker(E) \subset \ker(a)$.

Firstly, $E \geq a^{\frac{1}{n}}$. Equivalently, $E - a^{\frac{1}{n}} \in B(\mathbf{H})^+$. Therefore,

$$(E - a^{\frac{1}{n}}(\eta)|\eta) \geq 0 \text{ if and only if } (E(\eta)|\eta) \geq \ (a^{\frac{1}{n}}(\eta)|\eta)$$

for all $\eta \in \mathbf{H}$, where $(\cdot|\cdot)$ is the inner product in $\mathbf{H}$. Since $a^{\frac{1}{n}}$ is a positive operator, $(a^{\frac{1}{n}}(\eta)|\eta) \geq 0$.

The restriction $E|_{ker(E)}$ of E to the Hilbert space $ker(E)$ is the zero operator. Since $E(\eta) = 0$ forces $(a^{\frac{1}{n}}(\eta)|\eta) = 0$ for all η in the Hilbert space $ker(E)$, the restriction of $a^{\frac{1}{n}}$ to the Hilbert space $ker(E)$ is the 0-operator [Polarization identity, Chapter 2, Proposition 2.4.3 [10]].

Q.E.D

We are now ready to prove the Closure Under Range Projection Of Operator theorem [Theorem 3] as an immediate corollary of lemma 2:

Firstly, we can assume that a is positive without loss of generality. This is because the positive operator aa^* is such that $\overline{Ran(aa^*)} = \overline{Ran(a)}$. We can further assume without loss of generality that $a \leq 1_{\mathbf{H}}$: since a is a positive operator, $0 \leq a \leq\parallel a \parallel 1_{\mathbf{H}}$ [Theorem 1, Chapter 1.3.1]; since for all positive real constants λ, $Ran(\lambda a) = Ran(a)$, we can choose $\lambda = \frac{1}{\|a\|}$ and replace a with λa.

Finally, for each positive operator a in the Von Neumann Algebra A, each of the n-th root of a, $a^{\frac{1}{n}}$, belongs to the Von Neumann algebra A. Each $a^{\frac{1}{n}}$ resides in C*(a), the operator norm closure of the set of all polynomial expressions in a. The set of all polynomial expressions in a also resides in the Von Neumann algebra A. Since the SOT - topology is coarser than operator norm topology, it follows that $a^{\frac{1}{n}} \in \mathrm{C}^*(a) \subset A$. Hence the SOT-topology limit E is in the SOT-topology closed Von Neumann algebra A.

Q.E.D

2.4.3 Lifting n-zero divisors : Von Neumann C*-algebra

Let A be a Von Neumann C*-algebra [Chapter 2.4.1, Definition 4]. The abundance of projections [Chapter 2.4.2, Theorem 3] in A enables us to construct a pair of projections in A which have a property that is critically used in the lifting of n-zero products in the Von Neumann algebra A. The property in question is a generalization of Theorem 2.3 of C. Olsen's paper, *A Structure Theorem For Polynomially Compact Operators, Amer. J. Math. 93 (1971), p 686 - 698*, which focusses on the lifting of properties in the C*-algebra $B(\mathbf{H})$, of all bounded operators on a separable Hilbert space $\mathbf{H}$, where the closed 2-sided ideal in question is the space of all the compact operators $K(\mathbf{H})$. We now state and prove this property in the context of a Von Neumann C*-algebra A and an arbitrary closed 2-sided ideal I:

Lemma 3 (Von Neumann Lifting Lemma) *(Proposition 4.1 [3]) Let A be a Von Neumann C*-algebra. Let $x, y \in A$ such that their product $xy \in I$. Then there exists a projection $e \in A$ onto some closed subspace $V \subset \mathbf{H}$ along $V^\perp$ together with its dual $1-e$, denoted e′, a projection on $V^\perp \subset \mathbf{H}$ along V, such that the modified multiplicands xe' and ey are* **also** *in the closed ideal I.*

Proof. By the Lifting Zero Divisors theorem [Chapter 2.2, Theorem 1], there exists elements $a,\ b \in I$ such that $(x-a)(y-b) = 0$. The range projection $R(y-b)$: the projection uniquely associated with the closed subspace $\overline{Ran(y-b)}$ will do the job. We denote it as e'. Let its dual, the orthogonal projection on the orthogonal subspace $\overline{Ran(y-b)}^\perp$, be denoted as e : $1_{\mathbf{H}} - e = e'$.

Clearly $e(y-b)$ is the zero operator. Hence $ey = eb \in I$ since $b \in I$. Similarly $(x-a)e'$ is the zero operator since e' is a projection onto $\overline{Ran(y-b)}$ which is annihilated by the operator $(x-a)$: $(x-a)(y-b) = 0$. Therefore, $xe' = ae' \in I$ since $a \in I$.

Q.E.D

Now armed with the above lemma, we proceed to lift the property of n-zero divisors in a Von Neumann C*-algebra:

Theorem 4 *(Theorem 4.2 [3]) If $x_1, \ \dots\ , x_n$ are elements of the Von Neumann C*-algebra A with $\prod_{1\le i\le n} x_i \in I$, then there exist $a_1, \ \dots\ , a_n$ in I with $\prod_{1\le i\le n}(x_i - a_i) = 0$.*

Proof. [Proof by Induction] The Lifting Zero Divisors Theorem [Theorem 1, Chapter 4] completes the induction step for n = 2. Suppose that the theorem holds for the case $n = n$. We now show that it also holds for the case of $n+1$.

Since $\prod_{1\le i\le n+1} x_i = \left(\prod_{1\le i\le n} x_i\right) \cdot x_{n+1} \in I$, we 'split' the original product which resides in I into the terms $\left(\prod_{1\le i\le n} x_i\right)e'$ and ex_{n+1} which both reside in I, by means of projections $e,\ e'$ in A [Lemma 1]. Now invoke the induction hypothesis on $\left(\prod_{1\le i\le n} x_i\right)e' = \left(\prod_{1\le i\le n-1} x_i\right)(x_n e')$ to assume the existence of the elements $\{a_1, \ldots, a_{n-1}, b\}$ in the ideal I which produce the perturbations such that:

$$\Big(\prod_{1\le i\le n-1}(x_i - a_i)\Big)(x_n e' - b) = 0. \tag{2.12}$$

Define $a_n = be'$ and $a_{n+1} = ex_{n+1}$. Both a_n and a_{n+1} reside in I since $b \in I$ and $ex_{n+1} \in I$ [lemma 1]. Then

$$\textstyle\prod_{1\le i\le n+1}(x_i - a_i) = 0$$

since

$$\begin{aligned}
\prod_{1\le i\le n+1}(x_i - a_i) &= \Big(\prod_{1\le i\le n-1}(x_i - a_i)\Big)(x_n - a_n)(x_{n+1} - a_{n+1}) \\
&= \Big(\prod_{1\le i\le n-1}(x_i - a_i)\Big)(x_n - be')(x_{n+1} - ex_{n+1}) \\
&= \Big(\prod_{1\le i\le n-1}(x_i - a_i)\Big)(x_n - be')e'(x_{n+1}) \\
&= \Big(\prod_{1\le i\le n-1}(x_i - a_i)\Big)(x_n e' - be')(x_{n+1}) \qquad (2.13) \\
&= \Big(\prod_{1\le i\le n-1}(x_i - a_i)\Big)(x_n (e')^2 - be')(x_{n+1}) \qquad (2.14) \\
&= \Big(\prod_{1\le i\le n-1}(x_i - a_i)\Big)(x_n e' - b)e'(x_{n+1}) \qquad (2.15) \\
&= 0.
\end{aligned}$$

where equations (2.9) and (2.10) follow from the idempotency of the projection $e' : (e')^2 = e'$ and the last equality from the induction hypothesis (2.8).

Q.E.D

Many C*-algebras do not have a supply of projections which have the property described in Lemma 1. We next study a new class of C*-algebras, called SAW*-algebras, which have a C*-algebraic property that closely resembles the Von Neumann Lifting Lemma (Chapter 2.4.3, Lemma 3). The C*-algebraic property in question is the possession of orthogonal local units which we shall motivate and define in the next section.

2.5 SAW*-algebra : Corona C*-algebra

2.5.1 The Paradigm of Non Commutative Topology

In the study of rings of continuous functions, one of the goals is the study of the interplay between algebraic (ring-theoretic) properties of the ring, $C(X)$, of all continuous functions on a topological space, X, and the actual topology of the topological space X.

Evidently the topology on X determines the ring $C(X)$ by determining which function on X into $\mathbf{R}$ is continuous. In fact it is without loss of generality that we can assume that X is a *completely regular* topological space. A completely regular topological space is a Hausdorff space such that whenever $F \subset X$ is a closed subset and x is a point outside of it, there exists a function in $C(X)$ such that $f(x) = 1$ and $f[F] = 0$:

Theorem 1 *(Chapter 3.9, Theorem 3.9 [31]) For every topological space X, there exists a completely regular space Y and a continuous mapping τ of X onto Y, such that the mapping $g \mapsto g \circ \tau$ is a ring isomorphism of $C(Y)$ onto $C(X)$.*

Now the converse problem of investigating topological spaces X and Y for which their associated ring of continuous functions $C(X)$ and $C(Y)$, respectively, if ring isomorphic implies the topological isomorphism of X and Y is a central goal of the study of rings of continuous functions.

Theorem 2 *(Chapter 4.9, Theorem 4.9 [31]) Two compact spaces are homeomorphic if and only if their rings C(X) and C(Y) are isomorphic.*

Now every commutative C*-algebra is a $C_0(X)$. We wish to study certain C*-algebraic properties of $C_0(X)$ that are logically equivalent to topological properties on the locally compact Hausdorff space X. Our first example logically equates the C*-algebraic property of $C_0(X)$ being σ-unital and the topological property of X being *σ-compact*. The topological space X is σ-compact if it is the countable union of compact subsets.

Example 1 *([4]) The C*-algebra $C_0(X)$ is σ - unital if and only if X is σ-compact.*

Proof. Here we shall only prove the assertion that if X is σ-compact then $C_0(X)$ is σ-unital.

Write X as a countable union of compact subsets $K_1, K_2, \ldots, K_n, \ldots$: $X = \bigcup_{i=1}^{\infty} K_i$. Since the finite union of compact subsets is compact, we construct an ascending sequence of compact subsets:

$$K_1 \subset K_1 \cup K_2 \subset \ldots \subset \bigcup_{i=1}^{n} K_i \ldots$$

where we shall let S_n denote the compact subset $\bigcup_{i=1}^{n} K_i$. Then arguing as in Appendix A.3, Theorem 3, Step 1, there exists a continuous function $\Delta_{S_n} \in C_0(\Omega)$ acting as a continuous approximant of the characteristic function on the compact set S_n [Urysohn's Lemma For Locally Compact Spaces : Chapter 7, Theorem 7.14, [21]]:

$$\Delta_{S_n}(x) = \begin{cases} 1 & : \quad x \in S_n \\ 0 & : \quad x \notin O \supset S_n \end{cases}$$

where O is a proper open set containing the compact set S_n. Now consider the join function $f_n = \bigvee_{i=1}^{n} \Delta_{S_n}$ which belongs to $C_0(X)$. Then the sequence of functions f_n form an ascending sequence of functions in $C_0(X)$ whose norm are less than or equal to 1. We claim that this sequence (f_n) is an approximate identity for $C_0(X)$:

$$\| f f_n - f \| \to 0 \text{ as } n \to \infty \text{ for each } f \in C_0(X).$$

Since $f \in C_0(X)$, for each $\epsilon > 0$, there exists a compact subset $K \subset X$ such that $|f(x)| \leq \epsilon$ for all x outside of K. Since $\| f_n \| \leq 1$, $|f_n f| \leq \epsilon$ for all x outside of K. Now $X = \bigcup_{i=1}^{\infty} K_i$ so there exists an n high enough such that $K \subset S_n$. Since $|f_n f(x) - f(x)| = 0$ for all $x \in K$ and $|f_n f(x) - f(x)| \leq 2\epsilon$ for all x outside of K, we are done.

Q.E.D

Hence we focus on C*-algebraic properties can be identified with topological properties of X of $C_0(X)$. In the above example, the C*-algebraic property in question is the property of being σ-unital.

In the paradigm of *non-commutative topology*, each non-commutative C*-algebra as taken as a non-commutative $C_0(X)$: the functions of a $C_0(X)$ which do not commute. Non-Commutative Topology is the study of certain C*-algebraic properties on a general C*-algebra. The C*-algebraic properties are those C*-algebra properties on $C_0(X)$ defined by topological properties on the topological space X which will make sense for a general C*-algebra. The C*-algebraic property of being σ-unital is one such example. Hence the identification of the topology X with respect to certain C*-algebraic properties in the C*-algebra $C_0(X)$ gives rise to the term "Non-Commutative Topology". As a further example, consider the C*-algebraic property of being a Von Neumann C*-algebra. Recall that Von Neumann C*-algebras have an identity, so we consider $C_0(X)$'s which have an identity: that is $C(X)$ where X is a compact Hausdorff space:

Example 2 (Stonean - Von Neumann C*-algebra) *The C*-algebra $C(X)$ of all continuous functions on a compact Hausdorf space X is a Von Neumann C*-algebra if and only if X is extremally disconnected. We shall call all compact Hausdorff spaces which are extremally disconnected, Stonean spaces.*

We now introduce the topological concept of an extremally disconnected topological space X. Recall that a topological space X is *connected* if and only if X cannot be written as a disjoint union of open subsets U and V. Although not all topological spaces are connected, such as the set of all rationals $\mathbf{Q}$ with the subspace topology induced by the usual topology on the reals $\mathbf{R}$, all topological spaces are a disjoint union of closed connected subsets (a subset is connected if it is connected with respect to the subspace topology) : for each $x \in X$, consider the collection $\mathcal{C}_x$ of all connected subsets containing x; this collection is non-void since $\{x\} \in \mathcal{C}_x$; $\bigcup \mathcal{C}_x$ is the largest connected set containing x and is therefore closed; it is called a *component* and the set $\{\bigcup \mathcal{C}_x | x \in X\}$ is a partition set of the topological space into disjoint closed connected sets.

Now there are various degrees of disconnectedness. If the only non-empty connected subsets are the trivial singletons, then the space is called *totally disconnected.* Consequently singletons are the only components. There is an even higher degree of disconnectedness : *extremally disconnected.*

Definition 1 (Extremally Disconnected Topological Space) *A topological space X is extremally disconnected if it is Hausdorff and the closure of each open set is open.*

Example 3 (Totally Disconnected but Not Extremally Disconnected) *Consider the set of all rationals $\mathbf{Q}$ as a subspace of the set of all reals $\mathbf{R}$ endowed with the usual topology. Then $\mathbf{Q}$ is totally disconnected since for any pair of distinct points x, y, there exists a pair of clopen subsets separating the two points. Let r denote an irrational number between x and y : $x < r < y$. Then the subset $\mathbf{Q} \cap < r, \infty >$ is a clopen subset of $\mathbf{Q}$ containing y but not x.*

The open subset $\mathbf{Q} \cap < 0, 1 >$ has closure $\mathbf{Q} \cap [0, 1]$ which is not open in the subspace topology.

The property of being extremally disconnected is stronger than the property of being totally disconnected for a locally compact space X. Firstly, in a locally compact space X, for each point $x \in O$ where O is an open set, there exists an open set V such that $x \in V \subset \overline{V} \subset O$ [Chapter 7, Proposition 7.22 [21]]. Since the closure of every open set is open [X is extremally disconnected], each point x has a basic neighbourhood system of clopen sets : X is called *0-dimensional.* Secondly, since X is Hausdorff, for any pair of distinct points x, y, there exists an open set W which contains x but not y. Hence, there exists a clopen set $Y \subset X$ which contains x but not y. Therefore, any subset which has more than two points are disconnected.

Example 4 (Extremally disconnected Compact Hausdorff Space) *Consider the set $\mathbf{N}$ of all the natural numbers with the discrete topology. $\mathbf{N}$ is therefore a locally compact Hausdorff space which is extremally disconnected. Since $\mathbf{N}$ is completely regular, its Stone-Čěch compactification $\beta\mathbf{N}$ is also extremally disconnected.[Chapter 6, Theorem 12 [30]]*

Returning back to Example 2, we can view Example 2 as a passage from a C*-algebra into a Von Neumann algebra. The proof of Example 2, rests on the following property induced by the extremally disconnected compact Hausdorff space X on its associated ring of continuous function $C(X)$:

Theorem 3 (Stonean - Dedekind Complete) *Let X be Stonean. Let its associated ring of continuous functions, $C(X)$, be taken as a lattice. A lattice is a partially ordered set such that every pair of elements has a greatest lower bound (glb) element in the set and dually, a least upper bound (lub) element in the set. The partial order in $C(X)$ is the usual partial order associated with the positive cone of all the positive valued functions in $C(X)$. The lub and glb of any two functions in $C(X)$ is the standard join and meet functions which belong to $C(X)$.*

Now consider the following lattice theoretic property: a lattice is Dedekind complete if and only if every non-empty subset of the lattice which has an upper bound in the lattice has a least upper bound also in the lattice.

It turns out that $C(X)$ is Dedekind complete if and only if X is Stonean.

2.5.2 Non Commutative Topology : Sub-Stonean Spaces and Corona Sets

Here we investigate the C*-algebraic properties of the C*-algebra $C_0(X)$ which are logically equivalent to the topological property of X being sub-Stonean. By default, X is a locally compact Hausdorff space.

Definition 2 (Sub-Stonean Topological Space) *A locally compact σ-compact Hausdorff space X is sub-Stonean if and only if any two disjoint, open, σ-compact subsets of X have disjoint compact closures.*

Note that deleting the term σ-compact takes us back to the definition of an extremally disconnected space.

The following proposition provides a motivation for studying locally compact σ-compact Hausdorff spaces:

Theorem 4 *If X is a locally compact σ-compact Hausdorff space, then every open σ-compact subset is exactly the complement of a zero set, Z_f, for some function $f \in C_0(X)$. A zero set $Z_f = \{x \in X | f(x) = 0\}$.*

Proof.

(i) Every open σ-compact subset Y is the complement of a zero set, Z_f, for some function $f \in C_0(X)$.

Let Y be an open σ-compact subset of the topological space X. Write Y as a countable union of compact subsets $K_1, K_2, \ldots, K_n, \ldots$: $Y = \bigcup_{i=1}^{\infty} K_i$.

Note that for each compact K_n, $K_n \subset Y$ where Y is open. Therefore for each K_n, there exists a continuous function $\Delta_{K_n} \in C_0(X)$ acting as a continuous approximant of the characteristic function on the compact set K_n [Urysohn's Lemma For Locally Compact Spaces : Chapter 7, Theorem 7.14, [21]]:

$$\Delta_{K_n}(x) = \begin{cases} 1 & : \quad x \in K_n \\ 0 & : \quad x \notin Y \end{cases}$$

We piece together these positive valued approximants of characteristic functions by considering the function $\Delta = \sum 2^{-n} \Delta_{K_n}$. Δ is the required function.

(i) $\Delta \in C_0(X)$: Since $\| \Delta_{K_n} \| \leq 1$, $\| 2^{-n} \Delta_{K_n} \| \leq 2^{-n}$. Therefore, the series $\sum 2^{-n} \Delta_{K_n}$ is absolutely convergent and since $C_0(X)$ is a Banach space, the limit Δ exists in $C_0(X)$.

(ii) Z_Δ is $X \backslash Y$: Since $Y = \bigcup_{i=1}^{\infty} K_i$, the zero set of Δ, Z_Δ, is $X \backslash Y$.

(ii) Conversely, every complement of a zero set, Z_f, for some function $f \in C_0(X)$ is an open σ-compact subset.

We first define some topological terms. A subset of a topological space is a G_δ set if and only if it is the countable intersection of open subsets. The dual concept is an F_σ set which is the countable union of closed sets.

(a) *Each zero set, Z_f, is a closed G_δ set.*

Let $B(0, \frac{1}{n})$ denote the open ball of radius $\frac{1}{n}$ centered at the origin of the complex plane. Then, we write Z_f as the countable intersection:

$$Z_f = \bigcap_{n=1}^{\infty} f^{-1}[B(0, \tfrac{1}{n})].$$

of open subsets $f^{-1}[B(0,1)], f^{-1}[B(0, \frac{1}{2})], \ldots, f^{-1}[B(0, \frac{1}{n})], \ldots$. Since f is continuous, each zero set, Z_f, is a closed G_δ set.

Consequently, its complement is an open F_σ set by de Morgan's law.

(b) *An F_σ set is σ-compact.*

Let U be a F_σ set. We write U as a countable union of closed subsets $C_1, C_2, \ldots, C_n, \ldots$:

$$U = \bigcup_{n=1}^{\infty} C_n.$$

Since the topological space X is σ-compact, we write X as a countable union of compact subsets $K_1, K_2, \ldots, K_n, \ldots$:

$$X = \bigcup_{m=1}^{\infty} K_m.$$

Therefore $U = U \cap X$ is a countable union of compact subsets since compactness is closed hereditary:

$$\begin{aligned} U &= \bigcup_{n=1}^{\infty} C_n \bigcap \bigcup_{m=1}^{\infty} K_m \\ &= \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} C_n \cap K_m \end{aligned}$$

Q.E.D

We now have a C*-algebraic property for the C*-algebra $C_0(X)$ which is logically equivalent to the topological property of the locally compact σ-compact Hausdorff space X being sub-Stonean. This C*-algebraic property is analogous to the Von Neumann Lifting Lemma property (Chapter 2.4.3, Lemma 3):

Theorem 5 (Sub-Stonean - Orthogonal Local Units) *(Proposition 1.1 [5])* *The C*-algebra $C_0(X)$ possesses orthogonal local units if and only if X is sub-Stonean. The C*-algebra $C_0(X)$ possesses orthogonal local units if and only if whenever f and g in $C_0(X)$ are orthogonal (that is, $fg = 0$) there are orthogonal functions f_1, g_1 in $C_0(X)$ such that $f_1 f = f$ and $g_1 g = g$.*

Proof. We first prove the assertion that if the locally compact σ-compact Hausdorf space X is sub-Stonean then $C_0(X)$ possesses orthogonal local units .

(i) The complement of the zero sets of the orthogonal functions f and g are disjoint.

The functions f and g are orthogonal if and only if the union of their zero sets is the entire space X : $Z_f \cup Z_g = X$. Equivalently, $X\backslash Z_f \cap X\backslash Z_g = \emptyset$ by De Morgan's Law.

(ii) The closures of the complements of the zero sets of the orthogonal functions f and g are disjoint and compact.

By theorem 4, the complements of the zero sets of the orthogonal functions f and g, $X\backslash Z_f$ and $X\backslash Z_g$, respectively, are disjoint open σ-compact subsets. Their closures $\overline{X\backslash Z_f}$ and $\overline{X\backslash Z_g}$ are disjoint and compact since the topological space X is sub-Stonean.

(iii) There exists local units for the orthogonal functions.

By (ii), we construct the local units as follows:-

Consider the following pairs of compact-open subsets: $\overline{X\backslash Z_f} \subset O_1$ and $\overline{X\backslash Z_g} \subset O_2$ where $O_1 = X\backslash\overline{X\backslash Z_g}$ and $O_2 = X\backslash\overline{X\backslash Z_f}$. Therefore for the compact sets $\overline{X\backslash Z_f}$ and $\overline{X\backslash Z_g}$, there exists continuous functions $\Delta_{\overline{X\backslash Z_f}}, \Delta_{\overline{X\backslash Z_g}} \in C_0(X)$ acting as a continuous approximant of the characteristic function on the compact sets $\overline{X\backslash Z_f}$ and $\overline{X\backslash Z_g}$, respectively [Urysohn's Lemma For Locally Compact Spaces : Chapter 7, Theorem 7.14, [21]]:

$$\Delta_{\overline{X\backslash Z_f}}(x) = \begin{cases} 1 & : \quad x \in \overline{X\backslash Z_f} \\ 0 & : \quad x \in \overline{X\backslash Z_g} \end{cases}$$

$$\Delta_{\overline{X\backslash Z_g}}(x) = \begin{cases} 1 & : \quad x \in \overline{X\backslash Z_g} \\ 0 & : \quad x \in \overline{X\backslash Z_f} \end{cases}$$

Their restrictions, $\Delta_{\overline{X\backslash Z_f}}|_S, \Delta_{\overline{X\backslash Z_g}}|_S$, to the closed subset $S = \overline{X\backslash Z_f} \cup \overline{X\backslash Z_g}$, are orthogonal elements in $C_0(S)$, the space of all continuous complex valued functions on S which vanish at infinity. Note that $\Delta_{\overline{X\backslash Z_f}}|_S \cdot f = f$ and

$\Delta_{\overline{X\backslash Z_g}}|_S \cdot g = g$. They will therefore have orthogonal extensions f_1 and g_1 in $C_0(X)$ [Lemma 1.3 [5]].

In the case of X being compact, the normality of X ensures the existence of disjoint opens U and V such that $\overline{X\backslash Z_f} \subset U$ and $\overline{X\backslash Z_g} \subset V$ enabling us to construct orthogonal local units without resorting to the Extension Lemma [Lemma 1.3 [5]].

We now prove the converse : if the C*-algebra $C_0(X)$ possesses orthogonal local units then the locally compact σ-compact Hausdorff space X is sub-Stonean.

Two disjoint, open, σ-compact subsets of X corresponds to the complements, $X\backslash Z_f, X\backslash Z_g$, of the zero sets of two orthogonal functions $f, g \in C_0(X)$, respectively [Chapter 2.5.2, theorem 4]. We therefore have to show that their closures, $\overline{X\backslash Z_f}, \overline{X\backslash Z_g}$ are disjoint and compact.

(i) *There exists disjoint open σ-compact subsets of X which separate $X\backslash Z_f, X\backslash Z_g$*

By the hypothesis, there exists two orthogonal functions f_1, g_1 in $C_0(X)$ such that $f_1 f = f$ and $g_1 g = g$. But, if $f_1 f = f$, then the complement, $X\backslash Z_{f_1}$, of the zero set of f_1 supersets the complement, $X\backslash Z_f$, of the zero set of f. An identical conclusion holds for g. Formally:

$$X\backslash Z_f \subset X\backslash Z_{f_1}$$
$$X\backslash Z_g \subset X\backslash Z_{g_1}$$

Since f_1 and g_1 are orthogonal functions in $C_0(X)$, the subsets $X\backslash Z_{f_1}$ and $X\backslash Z_{f_1}$ are disjoint open σ-compact subsets of X.

(ii) *There exists closed proper subsets C_f, C_g of $X\backslash Z_{f_1}$ and $X\backslash Z_{g_1}$, respectively, such that $X\backslash Z_f \subset C_f \subset X\backslash Z_{f_1}$ and $X\backslash Z_g \subset C_g \subset X\backslash Z_{g_1}$.*

Firstly, $f_1 = 1$ on $X\backslash Z_f$. Therefore, the continuous function $K_1 - f_1$ where K_1 is the constant 1-function on X, is zero on $X\backslash Z_f$. Let C_f denote the zero set, $Z_{K_1 - f_1}$, of $K_1 - f_1$: $C_f = \{x \in X | f_1(x) = 1\}$. Then $C_f \supset X\backslash Z_f$. Further, C_f is closed since $K_1 - f_1$ is continuous. Finally, $C_f \subset X\backslash Z_{f_1}$ since $K_1 - f_1 = 0$ exactly when $f_1 = 1$.

Secondly, we show that C_f is not a dense subset of $X\backslash Z_{f_1}$. Suppose on the contrary that $C_f = \{x \in X | f_1(x) = 1\}$ is dense in $X\backslash Z_{f_1}$. Then by the continuity of f_1, $f_1(x) = 1$ for all $x \in X\backslash Z_{f_1}$. Therefore, f_1 is the non-continuous characteristic function $\chi_{X\backslash Z_{f_1}}$ on $X\backslash Z_{f_1}$:

$$\chi_{X\backslash Z_{f_1}}(x) = \begin{cases} 1 & : \quad x \in X\backslash Z_{f_1} \\ 0 & : \quad x \in Z_{f_1} \end{cases}$$

which contradicts the fact that $f_1 \in C_0(X)$.

Identically, defining C_g to be the zero set, $Z_{K_1 - g_1}$ completes the proof of (ii).

(iii) *The closures,* $\overline{X \backslash Z_f}, \overline{X \backslash Z_g}$ *are disjoint and compact.*

We only prove the compactness for $\overline{X \backslash Z_f}$ since the proof for the compactness of $\overline{X \backslash Z_g}$ is identical. The disjointedness follows immediately from (ii) and (i)

Firstly, since $X \backslash Z_{f_1}$ is σ-compact, write $X \backslash Z_{f_1}$ as a countable union of compact subsets $K_1, K_2, \ldots, K_n, \ldots$: $X \backslash Z_{f_1} = \bigcup_{i=1}^{\infty} K_i$.

Secondly, since C_f is a non-dense subset of $X \backslash Z_{f_1}$, there exists a $n \in \mathbf{N}$ high enough such that $C_f \subset \bigcup_{i=1}^{n} K_i$:

Suppose not. Then for all $n \in \mathbf{N}$, $C_f \supset \bigcup_{i=1}^{n} K_i$. Consider an arbitrary $x \in X \backslash Z_{f_1} = \bigcup_{i=1}^{\infty} K_i$. Then, there exists a $j \in \mathbf{N}$ such that $x \in K_j \subset \bigcup_{i=1}^{j} K_i$. Consequently, $x \in C_f$ which contradicts the fact that C_f is a proper subset of $X \backslash Z_{f_1}$.

Finally, from (ii), $\overline{X \backslash Z_f} \subset \overline{\bigcup_{i=1}^{n} K_i} = \bigcup_{i=1}^{n} \overline{K_i} = \bigcup_{i=1}^{n} K_i$. Since compactness is preserved under finite unions, $\bigcup_{i=1}^{n} K_i$ is compact and $\overline{X \backslash Z_f}$ is compact since compactness is closed-hereditary.

Q.E.D

Having established the logical equivalence between the C*-algebraic property of the C*-algebra $C_0(X)$ possessing orthogonal local units and the topological property of the locally compact σ-compact Hausdorff space X being sub-Stonean, we now construct examples of sub-Stonean spaces. It turns out that sub-Stonean spaces are large in general:

Theorem 6 *(Theorem 1.10 [5]) The closure of every open, σ-compact subset Y of a sub-Stonean space X is homeomorphic to the Stone-Cěch compactification βY of Y.*

The following construction provides a rich source of examples of sub-Stonean spaces. We first introduce the concept of the corona:

Definition 3 (Corona) *Let X be a locally compact Hausdorff space with Stone-Cěch compactification βX. The remainder $\beta X \backslash X$ is called the corona of the locally compact Hausdorff space X.*

Example 5 (Corona of N) *Let $\mathbf{N}$ be the set of natural numbers with the discrete topology. $\mathbf{N}$ is a locally compact Hausdorff space and is therefore an open subspace of any compactification, in particular, the Stone-Cěch compactification $\beta\mathbf{N}$. The remainder or the corona $\beta\mathbf{N} \backslash \mathbf{N}$ is a closed and hence compact subspace of $\beta\mathbf{N}$.*

The Stone-Cĕch compactification $\beta\mathbf{N}$ is extremally disconnected [Example 4, chapter 2.6.1] but the corona $\beta\mathbf{N}\backslash\mathbf{N}$ is not [Chapter 6.2, Example 4 [30]]. Fortunately, by virtue of $\mathbf{N}$ being σ-compact, it turns out, from the next theorem, that it is sub-Stonean.

Theorem 7 (Sub-Stonean Corona) *(Theorem 3.2 [5]) If X is a locally compact space, σ-compact Hausdorff space then its corona, $\beta X\backslash X$, is a sub-Stonean space. Equivalently, $C_0(\beta X\backslash X)$ possesses orthogonal local units.*

We now state without proof a theorem on the C*-algebra $C_0(X)$ which will give a commutative origin of the concept of the corona C*-algebra, which we shall discuss in the next section.

Theorem 8 *(Chapter 7.4, Theorem 7.4.4 [17]) Consider the C*-algebra $C_0(X)$. If Y is a closed subset of the locally compact Hausdorff space X, then the unique 2-sided closed ideal, I, of $C_0(X)$ associated with the closed subset Y : $I = \{f \in C_0(X) | f|_Y = 0\}$ is denoted by $C_0(X\backslash Y)$ [Chapter 1.2.4, Example 35]. Then we have the following isometric onto isomorphism :*

$$C_0(X)/C_0(X\backslash Y) \cong C_0(Y)$$

We state and prove the following result which we shall need in providing a commutative origin of the concept of the corona C*-algebra:

Proposition 1 *In the notation of Theorem 8, let the locally compact σ-compact Hausdorff space X be the compact space $\beta\mathbf{N}$, the Stone-Cĕch compactification of the set of naturals $\mathbf{N}$ with the discrete topology. Let the closed set Y be the corona $\beta\mathbf{N}\backslash\mathbf{N}$ [Example 5].*

The 2-sided closed ideal I of all the functions in $C_0(\beta\mathbf{N}) = C(\beta\mathbf{N}) = \{f : \beta\mathbf{N} \to \mathbf{C} | f$ is continuous$\}$ which vanish on the corona $Y = \beta\mathbf{N}\backslash\mathbf{N}$, denoted $C_0(\beta\mathbf{N}\backslash Y)$, is exactly $C_0(\mathbf{N})$, the space of all null sequences [Appendix A.3, Example 4].

Proof. The intuition behind this, is the fact that the onto ring isomorphism $\Phi : C_b(\mathbf{N}) \to C(\beta\mathbf{N}) : f \mapsto f^\beta$ enables us to identify each f in $C_b(\mathbf{N})$, the space of all bounded sequences, with the restriction of an $f^\beta \in C(\beta\mathbf{N})$, to the subspace $\mathbf{N}$. The set $C(\beta\mathbf{N})$ is the space of all continuous complex-valued functions on the Stone-Cĕch compactification of the set of naturals $\beta\mathbf{N}$. Since f^β is 0 on the corona $\beta\mathbf{N}\backslash\mathbf{N}$, its restriction to the subspace $\mathbf{N}$, a bounded sequence, must vanish at infinity by the the continuity of f^β.

We shall prove this rigourously:

(i) *If $f^\beta \in C(\beta\mathbf{N})$ vanishes on the corona $\beta\mathbf{N}\backslash\mathbf{N}$, then, for all $\epsilon > 0$, the restriction of f^β to $\mathbf{N}$, $f^\beta|_{\mathbf{N}}$, cannot be larger than ϵ for all $n \in \mathbf{N}$.*

Suppose on the contrary that there exists an $\epsilon_0 > 0$ such that $f^\beta|_{\mathbf{N}}(n) \geq \epsilon_0$ for all $n \in \mathbf{N}$. Since $\mathbf{N}$ is dense in $\beta\mathbf{N}$, for every point p in the corona $\beta\mathbf{N}\backslash\mathbf{N}$, there exists a net, (n_α), in $\mathbf{N}$ such that $n_\alpha \to p$. By the continuity of f^β, $f^\beta(n_\alpha) \to f^\beta(p) = 0$. This is a contradiction since the numerical net $f^\beta(n_\alpha) = f^\beta|_{\mathbf{N}}(n_\alpha)$ stays away from $B(0, \epsilon_0)$.

(ii) For every $\epsilon > 0$, the restriction $f^\beta|_{\mathbf{N}}$ is greater than ϵ for all but a finite number of elements in $\mathbf{N}$: $f^\beta|_{\mathbf{N}} \in C_0(\mathbf{N})$.

Assume on the contrary that there exists an $\epsilon_0 > 0$ such that the restriction $f^\beta|_{\mathbf{N}}$ is greater than ϵ_0 for an infinite subset $W \subset \mathbf{N}$. The subset W is therefore a cofinal subset of $\mathbf{N}$: for each $n \in \mathbf{N}$, there exists a $w \in W | w \geq n$. We arrive at a contradiction as follows:

(a) There exists a free ultra filter, A^p, on $\mathbf{N}$ containing W. A filter is free if its core, $\bigcap_{F \in A^p} F$, is the empty set.

Consider the filter $\mathcal{F}$ on $\mathbf{N}$ which has as its subbasis, $\mathcal{S}$, the set of all cofinite subsets of $\mathbf{N}$ along with W. Since W is cofinal, $\mathcal{S}$ has the finite intersection property (F.I.P) and hence is a filter subbasis: it generates the filter base $\mathcal{B}$ of all sets which are intersection sets of finitely many members of $\mathcal{S}$; the filter $\mathcal{F}$ is the set of all subsets of $\mathbf{N}$ which superset any member of $\mathcal{B}$.

Therefore, the filter $\mathcal{F}$ is contained in an ultra-filter A^p of $\mathbf{N}$ [Chapter 5, theorem 5.28 [29]]. The ultra-filter A^p is free : $\bigcap_{F \in A^p}\{F\} \subset \bigcap_{F \ is \ cofinite}\{F\} = \emptyset$.

(b) *The free ultra filter, A^p, converges to a point p in the corona $\beta\mathbf{N}\backslash\mathbf{N}$. The image filter $f^\beta[\mathcal{F}]$ converges to 0.*

Since $\mathbf{N}$, a completely regular space, has the discrete topology on it, there is no distinction between filters consisting of zero-sets, *z-filters*, and the set-theoretic notion of a filter. By the construction of the Stone-Cĕch compactification of $\mathbf{N}$, the free z-ultra filter A^p converges in the compact space $\beta\mathbf{N}$ to a point, p, in the corona $\beta\mathbf{N}\backslash\mathbf{N}$ [Chapter 6, Theorem 6.5 [31]].

Therefore, the image filter $f^\beta[\mathcal{F}]$ converges to $f^\beta(p) = 0$ [Chapter 7, theorem 7.1(8) [29]], where the image filter $f^\beta[\mathcal{F}]$ is the filter on $\mathbf{C}$ which has as its subbasis the set of all sets of the form $f^\beta[F]$ where $F \in A^p$.

(c) *The ultra-filter A^p on $\mathbf{N}$ is void of any subsets of $\mathbf{N}\backslash W$.*

This follows from the fact that W belongs to A^p along with the fact that $\emptyset \notin A^p$. In fact, every $F \in A^p$ meets W, and therefore for all $F \in A^p$, $f^\beta[F]$ cannot be a subset of $B(0, \frac{\epsilon_0}{2})$. This contradicts statement (b).

Therefore, the restriction $f^\beta|_{\mathbf{N}}$ is a sequence that converges to 0.

Q.E.D

Example 6 (Commutative Origin Of the Corona C*-algebra) *In the notation of Theorem 8, let the locally compact σ-compact Hausdorff space X be the compact space $\beta\mathbf{N}$, the Stone-Cĕch compactification of the set of naturals $\mathbf{N}$ with the discrete topology. Let the closed set Y be the corona $\beta\mathbf{N}\backslash\mathbf{N}$ [Example 5].*

By theorem 8 and Proposition 1,

$$C_0(\beta\mathbf{N})/C_0(\mathbf{N}) \cong C_0(\beta\mathbf{N}\backslash\mathbf{N}) = C(\beta\mathbf{N}\backslash\mathbf{N}) \tag{2.16}$$

which is equivalent to :

$$M(C_0(\mathbf{N}))/C_0(\mathbf{N}) \cong C_0(\beta\mathbf{N}\backslash\mathbf{N}) = C(\beta\mathbf{N}\backslash\mathbf{N}) \tag{2.17}$$

where $M(C_0(\mathbf{N}))$ is the Double Centralizer Algebra of the C-algebra $C_0(\mathbf{N})$ [Example 4, Appendix A.2]. Since $C_0(\beta\mathbf{N}\backslash\mathbf{N})$ has orthogonal local units because $\beta\mathbf{N}\backslash\mathbf{N}$ is sub-Stonean, equation (2.17) amounts to the statement that the quotient C*-algebra $M(C_0(\mathbf{N}))/C_0(\mathbf{N})$ possesses orthogonal local units. [Chapter 2.5.2, Theorem 7].*

The quotient C-algebra $M(C_0(\mathbf{N}))/C_0(\mathbf{N})$ formed by factoring out the C*-algebra $C_0(\mathbf{N})$ from the Double Centralizer Algebra of $C_0(\mathbf{N})$, $M(C_0(\mathbf{N}))$, is called the corona C*-algebra of the C*-algebra $C_0(\mathbf{N})$. Recall that the C*-algebra $C_0(\mathbf{N})$ is embedded as a closed 2-sided ideal of $M(C_0(\mathbf{N}))$.*

2.5.3 Application of Non-Commutative Topology : SAW*-algebra, Corona C*-algebra

In this section we generalize the C*-algebraic properties of $C_0(X)$ for a sub-Stonean X, to define a new class of C*-algebras, SAW*-algebras. The following definition of a SAW*-algebra is a generalization of Theorem 5 (Chapter 2.5.2):

Definition 4 (SAW*-algebra: Orthogonal Local Units) *A C*-algebra A is a SAW*-algebra if for any two positive orthogonal elements [Chapter 1.3.1, Definition 1] x and y in A^+, there is a positive element e in A^+ such that $ex = x$ and $ey = 0$. Taking adjoints, we have $xe = x$ and $ye = 0$.*

Applying the condition to the orthogonal pair of positive elements e, y, we have a positive element d in A^+ such that $dy = yd = y$ and $de = ed = 0$.

We call the elements e and d an orthogonal pair of local units for x and y. We say e is a local unit for x with respect to y and d is a local unit for y with respect to x.

Example 7 (SAW* - algebra) *Let the C*-algebra A be the C*-algebra $C_0(X)$ where X is sub-Stonean. Then by the construction used in the proof of Theorem 5 (iii) (Chapter 2.5.2), $C_0(X)$ is a SAW*-algebra.*

As another generalization of Example 6 (Chapter 2.5.2) and Example 1 (Chapter 2.5.1), we have the concept of the corona of a general C*-algebra:

Definition 5 (The Corona) *Let A be a non-unital, σ - unital C*-algebra. Then the corona of A is defined to be the quotient C*-algebra $M(A)/A$ formed by factoring out the C*-algebra A from the Double Centralizer Algebra of A, $M(A)$. Recall that the C*-algebra A is embedded as a closed 2-sided ideal of $M(A)$ [Theorem 11, Chapter 1.2.3]. We shall denote the corona of A as $C(A)$.*

2.5.3.1 Local Corona Properties are Global Multiplier Properties

Often, some of the n-ary properties $P(m_1, \ldots, m_n)$ which hold for the elements $m_1, \ldots, m_n$ in $M(A)$ that are invariant under perturbations by elements from the C*-algebra A taken as a 2-sided ideal of $M(A)$, are best characterized as another n-ary property $Q(c_1, \ldots, c_n)$ in the corona, $C(A)$, of A, where $(c_1, \ldots, c_n)$ is a *fixed* n-tuple of elements in $C(A)$. Hence P is a *global* property in $M(A)$ yet Q is a *local* property of $C(A)$. We however relax the condition on Q by allowing Q to be a more simpler property than P. Therefore, the n-ary property Q which holds for certain elements of the corona $C(A)$ translates into a global property of $M(A)$ of which the original C*-algebra A is a closed 2-sided ideal : the elements in the C*-algebra $M(A)$ for which property P is true, belong to one of the elements of the corona for which Q is true: local properties in the corona $C(A)$ are ideal-perturbed-invariant global properties for $M(A)$.

Example 8 (Property of Vanishing on the Corona) *Let the C*-algebra A be* $C_0(\mathbf{N})$ *where the locally compact Hausdorff space* $\mathbf{N}$ *is the set of naturals with the discrete topology. Then* $M(C_0(\mathbf{N}))$ *is* $C(\beta\mathbf{N})$*.*

Let $P(m)$ *be the property on* $M(C_0(\mathbf{N})) = C(\beta\mathbf{N})$ *where* $P(m)$ *is true if and only if the function* $m \in C(\beta\mathbf{N})$ *vanishes on the corona of* $\mathbf{N}$, $\beta\mathbf{N}\backslash\mathbf{N}$*. That is* m *belongs to the 2-sided closed ideal* $I = C_0(\beta\mathbf{N}\backslash Y) = C_0(\mathbf{N})$ *[Proposition 1, Chapter 2.5.2]. Trivially* P *is invariant under perturbations from the ideal* I*.*

Defining the property $Q(c)$ *to be the property on the corona,* $C(A)$*, of* A *as* $Q(c)$ *is true if and only if* $c = 0$*, is a characterization of the property* P*.* Q *is a local property : the coset* $c \in C(A)$ *such that* $Q(c)$ *is true is the fixed zero element, which contains all the elements of the C*-algebra* $A = C_0(\mathbf{N})$*.*

We now introduce the concept of the index of a bounded operator on a Hilbert space $\mathbf{H}$ to give us a second example of a global phenomena from the corona:

Definition 6 (Index of a Bounded operator) *Let* T *be a bounded operator on a Hilbert space* $\mathbf{H}$*. Then we define the index,* $ind(T)$*, of the bounded operator* T *to be the number*

$$dim(ker(T)) - dim(coker(T))$$

where $dim(ker(T))$ *is the dimension of the kernal of the operator* T *and* $dim(coker(T))$ *is the dimension of the complementary subspace of* $Ran(T)$, $\mathbf{H}/Ran(T)$*, where* $Ran(T)$ *is the range space of the operator* T*. We shall call it the defect of* T*.*

Example 9 *Let* T *be the left shift operator of Example 3, Chapter 1.3.2, on the Hilbert space* $\mathbf{H} = l^2$ *of all square summable sequences. Then* $ind(T) = 1$ *since* $dim(ker(T)) = 1$ *and* $dim(coker(T)) = 0$*.*

We can view the concept of numbers as the first level of mathematical abstraction. The concept of a function embodies the notion of mapping numbers into a number : functions are therefore the second level of mathematical abstraction. Operators on function spaces map functions into functions and are therefore the third level of abstraction. Now, the concept of the index is a meaningful way of linking the third level of abstraction back down to the first level of abstraction. To see this, we introduce the concept of a Noether operator:

Definition 7 (Noether Operator) *A bounded operator* T *on a Hilbert space* $\mathbf{H}$ *is Noether if and only if :*

(i) T *is normally solvable :* $Ran(T)$ *is closed .*

(ii) The dimension of the kernal of T*,* $dim(ker(T))$*, is finite.*

(iii) The dimension of the cokernal of T, $dim(coker(T))$, is finite.

The first condition (i) is required to express the relation

$$dim(ker(T)) - dim(coker(T))$$

in a more symmetrical form:

$$dim(ker(T)) - dim(ker(T^*)).$$

For any bounded operator T, $\overline{Ran(T)} = ker(T^*)^\perp$. Since T is normally solvable, $Ran(T) = ker(T^*)^\perp$. Consequently, $Ran(T)^\perp = ker(T^*)^{\perp\perp} = ker(T^*)$. Therefore, $\mathbf{H}/Ran(T) \cong ker(T^*)$.

Conditions (ii) and (iii) ensure that the index, $ind(T)$ is a finite integer. In fact, conditions (ii) and (iii) imply condition (i): condition (i) is not necessary [Chapter VII, Section 2.4, Proposition 2.4.1 [25]].

The following theorem shows that the concept of the index provides a meaningful link between the Noether operators and the integer numbers:

Theorem 9 *Let $\mathcal{F}$ denote the set of all the Noether operators of $B(\mathbf{H})$. Then, as a subspace topology of the normed space $B(\mathbf{H})$, the set $\mathcal{F}$ can be written as a disjoint union of its connected components.*

The index map $i : \mathcal{F} \to \mathbf{Z} \mid T \mapsto ind(T)$ where $\mathbf{Z}$ is the set of all integers with the discrete topology, is locally constant (and hence continuous): for every $T \in \mathcal{F}$ there exists a ball B about T of the subspace topology such that $i[B]$ is some constant integer [Part I.5, Section C, Theorem 2 [26]]. Therefore, the index is constant on the connected components of $\mathcal{F}$. In fact, distinct components have distinct indexes.

We are now ready to give our second example of a global phenomena from the corona:

Example 10 (Property of a Finite Index) *Let the C^*-algebra A be the non-unital C^*-algebra $K(\mathbf{H})$ where $\mathbf{H}$ is some separable infinite dimensional Hilbert space $\mathbf{H}$. Then $M(A)$ is $B(\mathbf{H})$ [Chapter 1.2.3, Example 33].*

Let $P(m)$ be the property on $M(K(\mathbf{H})) = B(\mathbf{H})$ where $P(m)$ is true if and only if the index, $ind(m)$, of the operator $m \in B(\mathbf{H})$ is finite. Then $P(m)$ is true if and only if $m \in B(\mathbf{H})$ is Noether : $m \in B(\mathbf{H})$ is Noether if and only if conditions (ii) and (iii) of Definition 7 holds. Now P is a property invariant under perturbations by elements in the ideal $A = K(\mathbf{H})$ [Chapter VII, Theorem 2.6.3 [25]].

Defining the property $Q(c)$ to be the property on the corona, $C(A)$, of A as $Q(c)$ is true if and only if c is invertible, characterizes the property P [Chapter

VII, Remark 2.6.4 [25]). The elements of the coset $c \in C(A)$ such that $Q(c)$ are exactly the elements of $M(A) = B(\mathbf{H})$ for which property P holds. Note that the property $P(m)$ does not hold for any element in the C-algebra $A = K(\mathbf{H})$ taken as a 2-sided ideal of $M(A)$.*

Let us now define an n-ary property, $P(m_1, \ldots, m_n)$ on $M(A)$ as follows:

$P(m_1, \ldots, m_n)$ is true if and only if $m_1 + A, \ldots, m_n + A$ are elements of $C(A)$ which are n-zero divisors with the additional condition that there exists elements $a_1, \ldots, a_n$ in the C-algebra A which we identify as a 2-sided ideal of $M(A)$ such that $m_1 - a_1, \ldots, m_n - a_n$ are n-zero divisors themselves.*

Note that this n-ary property P is equivalent to saying that the property of n-zero divisors can be lifted from the corona $C(A)$ onto $M(A)$. Defining the characterization of P in $C(A)$ as the n-ary property $Q(c_1, \ldots, c_n)$ on $C(A)$ where $Q(c_1, \ldots, c_n)$ is true if and only if $c_1, \ldots, c_n$ are elements in $C(A)$ which are n-zero divisors with the additional condition it contains elements $m_1, \ldots, m_n$ in the C*-algebra $M(A)$ which are n-zero divisors themselves. Like P, this n-ary property Q on $C(A)$ is not a property on $C(A)$ since it involves elements in $M(A)$. Further the property Q is not invariant under perturbations from *all* elements of the C*-algebra A taken as a 2-sided ideal of $M(A)$. Nonetheless, there are enough clues to suggest that such a lifting cannot be ruled out completely. Indeed, in the next section we shall lift the property of n-zero divisors from the corona $C(A)$ onto $M(A)$. To this end, we need the following two statements.

2.5.3.2 Two Fundamental Results For Lifting the Property of n-zero divisors from the corona $C(A)$ onto $M(A)$.

The first fundamental result is a generalization of Chapter 2.5.2, Example 6 where the corona of the non-unital σ-unital C*-algebra $C_0(\mathbf{N})$ turned out to be a SAW*-algebra.

Theorem 10 *(Theorem 13 [4]) For each non - unital, σ - unital C*-algebra A, its corona $C(A)$ is a SAW*-algebra.*

The second fundamental result involves the local units of a SAW*-algebra.

Proposition 2 (Existence Of Local Units in the Unit Sphere) *Let A be a SAW*-algebra. Then for any pair of positive orthogonal elements a and b in A^+, there is a positive element e, a local unit of a in A^+, such that $ae = ea = a$ and $eb = be = 0$.*

If the local unit e for a does not have a norm of exactly one, we can replace it with another local unit e' for a which does have a norm equal to 1.

In the case where A has an identity, 1, the element $1 - e$ is positive: $e \leq 1$. Further, the norm of $1 - e$ is exactly one.

Proof

(i) The norm of any local unit exceeds 1.

This follows from the following string of inequalities $\| a \| = \| ae \| \leq \| a \| \| e \|$. Dividing out by $\| a \|$, we conclude that $\| e \| \geq 1$.

(ii) For any local unit e for a, there exists a local unit e′ for a whose norm is less than or equal to 1.

Consider the commutative C*-algebra generated by the elements a and e: $C^*(a,e)$: a and e commute. Therefore $C^*(a,e)$ is a $C_0(X)$ [Gelfand Naimark Theorem I, Theorem 2, Chapter 1.1]. Taking a and e as functions in $C_0(X)$, since $ae = a$, the function e is 1 on the complement $X \backslash Z_a$ of Z_a, the zero-set of a. Since e vanishes at infinity, there exists a compact set $K \subset X$ such that $e(x) \leq 1$ for all $x \in X \backslash K$, where $X \backslash Z_a \subset K$.
Then arguing as in Appendix A.3, Theorem 3, Step 1, there exists a continuous function $\Delta_K \in C_0(\Omega)$ acting as a continuous approximant of the characteristic function on the compact set K [Chapter 7, Theorem 7.14, [21]]:

$$\Delta_K(x) = \begin{cases} 1 & : \quad x \in K \\ 0 & : \quad x \notin O \supset K \end{cases}$$

where O is a proper open set containing the compact set K.

Then the meet function $e' = \Delta_K \bigwedge e$ is a positive valued function with norm less than or equal to one such that $e'a = ae' = a$ and $e'b = be' = 0$ since $eb = be = 0$.

In the case where A has an identity 1, consider the commutative C*-algebra $C^*(e',1)$ generated by e' and 1. Then $C^*(e',1)$is a $C(K)$. The identity element 1 is the constant 1 function, K_1, of $C(K)$ and e' is a positive valued continuous function whose norm is equal to 1. Hence $K_1 - e'$ is a positive valued function : $e' \leq 1$.

Finally, since $1 - e'$ is a local identity to b, it has a norm of exactly one.

Q.E.D

2.6 Lifting n-zero divisors : Corona C*-algebra

Analogous to the lifting of n-zero divisors in Von Neumann C*-algebras, we exploit the SAW*-algebra algebraic property of possessing orthogonal local units to lift n-zero divisors in the corona of a non-unital σ-unital C*-algebra.

Theorem 1 (Lifting n-zero Divisors: Corona C*-algebra) *(Lemma 6.1 [6]) Let A be a non-unital σ - unital C*-algebra. Then the corona $C(A) = M(A)/A$ is a SAW*-algebra [Theorem 10, Chapter 2.5.3.2] where A is a 2-sided ideal I of $M(A)$. If $x_1, \ldots, x_n$ are elements of $M(A) - A$ with $\prod_{1\le i\le n} x_i \in I = A$, then there exist elements $h_1, \ldots, h_n$ in the ideal $I = A$ with $\prod_{1\le i\le n}(x_i - h_i) = 0$.*

Proof. [Proof by Induction] The induction step for n = 2 is true [Chapter 2.2, Theorem 1]. Suppose that the theorem holds for the case $n = n$. We now show that it also holds for the case of $n + 1$.

Just as in case of lifting n-zero divisors in Von Neumann C*-algebras [Chapter 2.4.3, Theorem 4], we 'split' the original product $\prod_{1\le i\le n+1} x_i$ which resides in the ideal A into two terms closely related to $\left(\prod_{1\le i\le n} x_i\right)$ and x_{n+1} which also reside in the ideal A in order to invoke the induction hypothesis. More precisely, there are positive elements a and b in $M(A)$ such that $\left(\prod_{1\le i\le n-1} x_i\right)(x_n(1-a))$ and $(1 - b)x_{n+1}$ reside in the ideal A where $\pi(a)\pi(b) = 0$. We do this by exploiting the fact that $C(A)$ is a SAW*-algebra [Chapter 2.5.3.2, Theorem 10].

Step 1. *Let y denote $\left(\prod_{1\le i\le n} x_i\right)$. The elements $\pi(y^*y)$ and $\pi(x_{n+1}x_{n+1}^*)$ are a pair of positive orthogonal elements of $C(A)$. Therefore, there is an orthogonal pair of local units which we write as $\pi(d)$ and $\pi(e)$, where the elements d and e are positive elements of $M(A)$*

By hypothesis, $\prod_{1\le i\le n+1} x_i \in A$. Equivalently,

$$\pi(yx_{n+1}) = 0.$$

Now,

$$\pi(y^*)\pi(yx_{n+1})\pi(x_{n+1}^*) = 0.$$

Hence

$$\pi(y^*y)\pi(x_{n+1}x_{n+1}^*) = 0.$$

Since $C(A)$ is a SAW*-algebra, there is an orthogonal pair of local units for the pair of positive orthogonal elements, $\pi(y^*y)$ and $\pi(x_{n+1}x_{n+1}^*)$, respectively. We write this orthogonal pair of local units as $\pi(d)$ and $\pi(e)$, where the elements d and e are positive elements of $M(A)$ [Chapter 1.3.4, Proposition 1].

Step 2. *There is a positive element* a *in* $M(A)$ *such that* $y(1-a) \in A$ *or* $\left(\prod_{1\leq i\leq n-1} x_i\right)(x_n(1-a)) \in A$.

Identify $C(A) = M(A)/A$ with a norm-closed *-subalgebra of $B(\mathbf{H})$ [Chapter 1.2.2, Gelfand Naimark Theorem II]. Then by the Polar Decomposition theorem [Chapter 1.3.2, Theorem 2] in $B(\mathbf{H})$ applied to the element $\pi(y) \in B(\mathbf{H})$, we have

$$\pi(y) = u|\pi(y)| = u\sqrt{\pi(y^*y)}$$

Hence,

$$\begin{aligned} \pi(y) &= u\sqrt{\pi(y^*y)\pi(d)} \\ &= u\sqrt{\pi(y^*y)}\sqrt{\pi(d)} \qquad (2.18) \\ &= \pi(y)\sqrt{\pi(d)} \end{aligned}$$

where equation (2.18) follows from Corollary 3, Chapter 1.3.1.

Therefore,

$$\begin{aligned} \pi(y)[1_{C(A)} - \sqrt{\pi(d)}] &= \pi(y)[\pi(1_{M(A)}) - \sqrt{\pi(d)}] \\ &= \pi(y)[\pi(1_{M(A)}) - \pi(\sqrt{d})] \qquad (2.19) \\ &= 0 \end{aligned}$$

where $1_{C(A)}, 1_{M(A)}$ are the identities of $C(A)$ and $M(A)$ respectively and equation (2.19) follows from Chapter 1.3.4, Proposition 2. Since d is a positive element of $M(A)$, the element $a = \sqrt{d}$ is a well defined element of $M(A)$. We therefore have,

$$y(1_{M(A)} - a) \in I = A$$

or equivalently,

$$x_1 \cdots x_{n-1}\Big(x_n(1_{M(A)} - a)\Big) \in I = A.$$

Step 3. *There is a positive element* b *in* $M(A)$ *such that* $(1-b)x_{n+1} \in A$.

We repeat the construction of Step 2 for the element $\pi(x^*_{n+1})$:

$$\begin{aligned} \pi(x^*_{n+1}) &= v\sqrt{\pi(x_{n+1}x^*_{n+1})} = v\sqrt{\pi(x_{n+1}x^*_{n+1})\pi(e)} \\ &= v\sqrt{\pi(x_{n+1}x^*_{n+1})}\sqrt{\pi(e)} = \pi(x^*_{n+1})\sqrt{\pi(e)} \\ &= \pi(x^*_{n+1})\pi(\sqrt{e}) \end{aligned}$$

so that

$$\begin{aligned}\pi(x_{n+1}^*) - \pi(x_{n+1}^*)\pi(\sqrt{e}) &= \pi(x_{n+1}^*)\Big(\pi(1_{M(A)}) - \pi(\sqrt{e})\Big)\\ &= 0.\end{aligned}$$

Since e is a positive element of $M(A)$, the element $b = \sqrt{e}$ is a well defined element of $M(A)$. We therefore have:

$$x_{n+1}^*(1_{M(A)} - b) \in I = A$$

On taking adjoints, we have

$$\Big(1_{M(A)} - b\Big)x_{n+1} \in I = A.$$

Step 4. *We show that* $\pi(a)\pi(b) = 0$.

Recalling that $a = \sqrt{d}$ and $b = \sqrt{e}$, note that

$$\begin{aligned}\pi(a)\pi(b) &= \pi(\sqrt{d})\pi(\sqrt{e}) = \sqrt{\pi(d)}\sqrt{\pi(e)}\\ &= \sqrt{\pi(d)\pi(e)} = 0.\end{aligned}$$

since $\pi(d)\pi(e) = 0$.

Step 5. *We can assume without loss of generality that the positive elements a and b are orthogonal :* $ab = 0$

From step 4, $\pi(a)$ and $\pi(b)$ are positive zero divisors of $C(A)$ where $a, b \in M(A)^+$. Therefore, we can perturb the positive elements a and b by elements f and g of the ideal A so that $a-f$ and $b-g$ are still positive with $\pi(a) = \pi(a-f)$ and $\pi(b) = \pi(b-g)$ [Chapter 2.2, Corollary 1]. We can take $a-f$ as a and $b-g$ as b to force $ab = 0$.

Step 6. *Invoke the induction hypothesis on* $\Big(\prod_{1\le i\le n-1} x_i\Big)(x_n(1-a))$.

By the induction hypothesis on $\Big(\prod_{1\le i\le n-1} x_i\Big)(x_n(1-a))$, there exists elements $\{h_1, \ldots, h_n\}$ in the ideal A such that:

$$\Big(\prod_{1\le i\le n-1}(x_i - h_i)\Big)(x_n(1-a) - h_n) = 0. \tag{2.20}$$

Defining $h_{n+1} = (1-b)x_{n+1}$ we have the element h_{n+1} in the ideal A [Step 3] and these are the required ideal perturbations:

$$
\begin{aligned}
\prod_{1\le i\le n+1}(x_i - a_i) &= \Big(\prod_{1\le i\le n-1}(x_i - h_i)\Big)(x_n - h_n)(x_{n+1} - h_{n+1}) \\
&= \Big(\prod_{1\le i\le n-1}(x_i - h_i)\Big)(x_n - h_n)(bx_{n+1}) \\
&= \Big(\prod_{1\le i\le n-1}(x_i - h_i)\Big)(x_n(1-a) + x_n a - h_n)(bx_{n+1}) \\
&= \Big(\prod_{1\le i\le n-1}(x_i - h_i)\Big)(x_n(1-a) - h_n + x_n a)(bx_{n+1}) \\
&= \Big(\prod_{1\le i\le n-1}(x_i - h_i)\Big)\Big((x_n(1-a) - h_n)(bx_{n+1}) + (x_n a)(bx_{n+1})\Big) \\
&= \Big(\Big[\prod_{1\le i\le n-1}(x_i - h_i)\Big](x_n(1-a) - h_n)\Big)(bx_{n+1}) + \ldots \\
&\qquad \ldots + \Big(\prod_{1\le i\le n-1}(x_i - h_i)\Big)(x_n a)(bx_{n+1}) \\
&= \Big(\Big[\prod_{1\le i\le n-1}(x_i - h_i)\Big](x_n(1-a) - h_n)\Big)(bx_{n+1}) + 0 \qquad (2.21) \\
&= 0 + 0 = 0.
\end{aligned}
$$

where equation (2.21) follows from $ab = 0$ and the last equation from the induction hypothesis (2.20).

2.7 Lifting n-zero divisors : The General Case

In the case of lifting n-zero divisors in the corona of a non-unital σ-unital C*-algebra, the ideal was a closed essential 2-sided ideal. Our first step in carrying over this lifting in the corona to the general case of any C*-algebra is to construct closed essential 2-sided ideals from the given closed 2-sided ideal.

2.7.1 Essential Ideals From Ideals : Annihilators

The crux of the construction rests on the ring-theoretic concept of a 2-sided annihilator of a set in the C*-algebra A:

Definition 1 (Annihilator of a Set) *For every subset B of a C*-algebra A, we define the 2-sided annihilator $B^{\perp}$ of B as the set $\{x \in A | xB = Bx = 0\}$.*

Therefore, the 2-sided annihilator $B^{\perp}$ is the intersection of the 1-sided left and right annihilators, $ann_R(B) = \{x \in A | Bx = 0\}$ and $ann_L(B) = \{x \in A | xB = 0\}$.

As a ring-theoretic tool, (right) annihilators characterize maximal right ideals of the ring : maximal right ideals are right annihilators of certain singleton sets. From the perspective of the Jacobson radical which is the intersection of all the maximal right ideals of the C*-algebra A taken as a ring, annihilators are redundant: the Jacobson radical is always the trivial 0-ideal [Chapter I.9, Corollary I.9.13 [12]]. However, in the study of SAW*-algebras, 2-sided annihilators clarifies the relations between SAW*-algebras and other closely related C*-algebras: Rickart algebras and AW*-algebras [Proposition 1, [4]]. Here, 2-sided annihilators enable us to construct from a given closed ideal of the C*-algebra A, a closed essential ideal of A.

Proposition 1 (Constructing Closed Essential Ideals from Closed Ideals) *Let I be a closed 2-sided ideal. Then the 2-sided annihilator, $I^{\perp}$, of I, is a closed 2-sided ideal which is disjoint from the ideal I : $I^{\perp} \bigcap I = 0$.*

The sum of the two disjoint ideals, $I + I^{\perp}$, is a closed 2-sided essential ideal.

Proof. *(i) The 2-sided annihilator $I^{\perp}$ is a closed 2-sided ideal of A*

Firstly, the 2-sided annihilator $I^{\perp}$ is a 2-sided ideal: I is a left and right ideal; hence $ann_R(I)$ and $ann_L(I)$ are right and left ideals of A [Chapter 13, Lemma 13.1(b) [28]]; consequently, $I^{\perp} = ann_R(I) \cap ann_L(I)$ is a 2-sided ideal.

We are now left with showing that the 2-sided annihilator $I^{\perp}$ is closed. Let (x_n) be a sequence in $I^{\perp}$ that converges to some x in the norm topology of A. Then, for each $i \in I$,

$$x_n i \to xi$$

by the joint continuity of the product in A. Since $x_n i = 0$ for each $i \in I$, $xi = 0$ for each $i \in I$; that is, the limit point x belongs to $I^\perp$.

(ii) The 2-sided annihilator, $I^\perp$, is disjoint from the ideal I : $I^\perp \bigcap I = 0$

Consider the element $i \in I \bigcap I^\perp$. The adjoint i^* also belongs to $I \bigcap I^\perp$ since all closed 2-sided ideals are *-closed [Chapter 1.2.4 ,Proposition 4]. Consequently, $ii^* = 0$ since $i \in I^\perp$ and $i^* \in I$. But $(\| i \|)^2 = (\| i^* \|)^2 = \| ii^* \| = 0$. Therefore $i = 0$.

(iii) The sum $I + I^\perp$ is a closed 2-sided ideal of A.

Trivially, $I + I^\perp$ is an ideal of A. It remains to show that it is closed. This we do by showing that it is complete. The crux of the proof is the following inequality, which is an approximate version of the Pythagorean Theorem in Hilbert Spaces. We shall call it the pseudo-Pythagorean inequality:

Pseudo-Pythagorean Inequality By (ii), each element in $I + I^\perp$ can be uniquely written as the sum $a + b^\perp$ where $a \in I$ and $b^\perp \in I^\perp$. Then

$$\frac{1}{2}(\| a \| + \| b^\perp \|) \leq \| a + b^\perp \| \leq \| a \| + \| b^\perp \| \tag{2.22}$$

Proof. The latter inequality of (2.22) is merely the triangle inequality. We prove the first inequality of (2.22). First note that

$$\| a + b^\perp \| = \sup_{x \in S} \| x(a + b^\perp) \| \tag{2.23}$$

where S is the unit sphere in A, $a \in I$ and $a^\perp \in I^\perp$ [Chapter 1.2.3.1 ,Lemma 3].

Now consider $\widehat{a^*} = \frac{a^*}{\|a^*\|} = \frac{a^*}{\|a\|} \in S$, the unit vector of the adjoint of $a \in I$. Since I is *-closed, $a^* \in I$ and hence $\widehat{a^*}$ is also in I. Invoking equation (2.23) and the fact that $b^\perp$ annihilates $\widehat{a^*}$, we have:

$$\| a + b^\perp \| \geq \frac{\| a^* a \|}{\| a \|} = \| a \| \tag{2.24}$$

Similarly,

$$\| a + b^\perp \| \geq \| b \| . \tag{2.25}$$

Adding equations (2.24) and (2.25) completes the proof of the pseudo-Pythagorean Inequality.

We are now left with showing that $I + I^\perp$ is complete. Let (x_n) be a Cauchy sequence in $I + I^\perp$. Each x_n can be written uniquely as the sum $a_n + b_n^\perp$ where $a_n \in I$ and $b_n^\perp \in I^\perp$.

(a) The induced "component" sequences (a_n) *and* $(b_n^{\perp})$ *are also Cauchy sequences in* I *and* $I^{\perp}$*, respectively.*

Since $\lim_{m,n\to\infty} \| x_n - x_m \| = 0$, it follows from the pseudo-Pythagorean Inequality that:

$$\begin{aligned} 0 &= \lim_{m,n\to\infty} \| (a_n - a_m) + (b_n^{\perp} - b_m^{\perp}) \| \\ &\geq \frac{1}{2}(\lim_{m,n\to\infty} \| (a_n - a_m) \| + \| (b_n^{\perp} - b_m^{\perp}) \|) \\ &\geq 0. \end{aligned}$$

Therefore by a squeeze play,

$$\lim_{m,n\to\infty} \| (a_n - a_m) \| + \| (b_n^{\perp} + b_m^{\perp}) \| = 0.$$

Since we have the sum of two positive valued sequences being a null sequence,

$$\lim_{m,n\to\infty} \| (a_n - a_m) \| = 0 \text{ and } \lim_{m,n\to\infty} \| [b_n^{\perp} - b_m^{\perp}] \| = 0.$$

(b) The "component" Cauchy sequences (a_n) *and* (b_n) *converge to points* $a \in I$ *and* $b \in I^{\perp}$*, respectively. This follows from the fact that completeness is closed hereditary :* I *and* $I^{\perp}$ *are closed in the complete space* A*.*

Therefore the Cauchy sequence (x_n) *converges to the point* $a + b \in I + I^{\perp}$ *by the joint continuity of the sum.*

(iv) The sum ideal $I + I^{\perp}$ *is essentially faithful with respect to the* A*, taken as an over-ring of* $I + I^{\perp}$*.*

Suppose the element $a \in A$ annihilates $I + I^{\perp}$: $a(I + I^{\perp}) = 0$. Then $a \in I^{\perp}$:

Firstly, $ai + aj^{\perp} = 0$ for all $i, j^{\perp} \in I,\ I^{\perp}$, respectively. Therefore, $ai = -aj^{\perp} \in I^{\perp}$ for all $i, j^{\perp} \in I,\ I^{\perp}$, respectively. But $ai \in I$ for each $i \in I$. Hence $ai = 0$ for each $i \in I$ since $I \bigcap I^{\perp} = 0$ [(ii)]. Equivalently, $a \in I^{\perp}$

Symmetrically, since $aj^{\perp} = -ai \in I$ for all $j^{\perp} \in I^{\perp}$, it follows that $a \in (I^{\perp})^{\perp}$. Therefore $a = 0$ since $I^{\perp} \bigcap (I^{\perp})^{\perp} = 0$.

Q.E.D

2.7.2 Lifting n-zero divisors

We now prove the lifting of n-zero divisors affirmatively in the general C*-algebra. The proof essentially is a reduction of the problem of lifting n-zero divisors in the general C*-algebra into the problem of lifting n-zero divisors in the corona C*-algebra of a non-unital σ-unital C*-algebra, which was proved affirmatively [Chapter 2.6, Theorem 1].

Theorem 1 (Lifting n-zero Divisors: General C*-algebra) *Let A be a C*-algebra, I a closed 2 -sided ideal in A. If $x_1, \ \dots \ , x_n$ are elements of A with $\prod_{1 \leq i \leq n} x_i \in I$, then there exist $a_1, \ \dots \ , a_n$ in I with $\prod_{1 \leq i \leq n}(x_i - a_i) = 0$.*

Proof. In anticipation of the Lifting of n-zero Divisors: Corona C*-algebra Theorem [Chapter 2.6, Theorem 1], we need to assume that the C*-algebra A is both non-unital and σ - unital.

A **is non-unital.** *As it turns out, we can assume without loss of generality that the C*-algebra A is non-unital. As far as lifting is concerned, it is equivalent to replace A with the non-unital stable algebra $A \bigodot K(\mathbf{H})$.* In the case that A has an identity, we shall see that replacing A with the non-unital stable algebra $A \bigodot K(\mathbf{H})$ will keep the validity of the proof intact. We have a pathological case if A does have an identity [Chapter 1.2.3.5, Theorem 8].

Therefore in the case A has an identity, we shall take the actual element $x \in A$ as $diag[x, 0] \in A \bigodot K(\mathbf{H})$. Furthermore, since A is a closed 2-sided ideal of $A \bigodot K(\mathbf{H})$ [Chapter 1.1 ,Theorem 4], the ideal $I \subset A$ will be a closed 2-sided ideal of $A \bigodot K(\mathbf{H})$ and we shall view I as such.

A **[resp. $A \bigodot K(\mathbf{H})$ if A has an identity] is σ-unital.** *We can assume without loss of generality that the C*-algebra A [resp. $A \bigodot K(\mathbf{H})$ if A has an identity] is σ-unital: it suffices to show that A [resp. $A \bigodot K(\mathbf{H})$ if A has an identity] is separable, since all separable C*-algebras are σ-unital [Chapter 3.13, Proposition 13.1 [8]].*

Consider $x_1, \ \dots \ , x_n \in A$ [resp. $diag[x_1, 0], \ \dots \ , diag[x_n, 0] \in A \bigodot K(\mathbf{H})$ if A has an identity] as elements of the C*-subalgebra, B, generated by $\{x_1, \ \dots \ , x_n\}$ [resp. $\{diag[x_1, 0], \ \dots \ , diag[x_n, 0]\}$ if A has an identity]. Then B is the closure in A [resp. $A \bigodot K(\mathbf{H})$ if A has an identity] of the countable set, $\mathbf{Q} \times \mathbf{Q}[x_1, \ \dots \ , x_n, x_1^*, \dots, x_n^*]$ [resp. $\mathbf{Q} \times \mathbf{Q}[diag[x_1, 0], \ \dots \ , diag[x_n, 0],$ $diag[x_1^*, 0], \dots, diag[x_n^*, 0]]$, of all polynomials in $\{x_1, \ \dots \ , x_n, x_1^*, \dots, x_n^*\}$ [resp. $\{diag[x_1, 0], \ \dots \ , diag[x_n, 0], diag[x_1^*, 0], \dots, diag[x_n^*, 0]\}$ if A has an identity], whose coefficients are complex numbers with rational real and imaginary parts.

The C*-algebra B is therefore a separable normed space and is therefore σ-unital.

So in the case A [resp. $A \bigodot K(\mathbf{H})$ if A has an identity] is not σ-unital, we take A [resp. $A \bigodot K(\mathbf{H})$ if A has an identity] as B and the closed 2-sided ideal I as $B \bigcap I$: $B \bigcap I$ absorbs products taken from the C*-algebra B ($i \in B \bigcap I$ implies $bi, ib \in B \bigcap I$ for every $b \in B$) by virtue of the fact that each element of $B \bigcap I$ is a member of both the C*-subalgebra B and the ideal I.

We can further assume without loss of generality that the ideal I is not disjoint from the C*-subalgebra B : $B \bigcap I \neq \{0\}$. In the case that the ideal I is disjoint from the C*-subalgebra B, the condition $\prod_{1 \leq i \leq n} x_i \in I$ of the hypothesis of the theorem, amounts to the pathological condition $\prod_{1 \leq i \leq n} x_i \in B \cap I = \{0\}$ since the C*-subalgebra B, is closed with respect to multiplication.

We can now lift n-zero divisors in the general C*-algebra:

Step 1. We construct a closed essential ideal $I + I^{\perp}$ of A [resp. $A \bigodot K(\mathbf{H})$ in the case A has an identity : we construct the 2-sided annihilator $I^{\perp}$ (Definition 1, Chapter 2.8.1) in $A \bigodot K(\mathbf{H})$] from I [Proposition 1, Chapter 2.8.1]. This then allows us to embed A isometrically *-isomorphically in the Double Centralizer Algebra $M(I + I^{\perp})$ [Theorem 11, Chapter 1.2.3].

Step 2. Treating the elements $x_1, \ldots, x_n$ in A as elements of $M(I + I^{\perp})$, we can invoke Theorem 1 [Chapter 2.7] since the C*-algebra $I + I^{\perp}$ is σ- unital and non-unital : $I + I^{\perp}$ is σ-unital since it is a C*-subalgebra of a σ-unital C*-algebra A [resp. $A \bigodot K(\mathbf{H})$ if A has an identity]; $I + I^{\perp}$ is non-unital since it is an essential ideal of a non-unital C*-algebra A [resp. $A \bigodot K(\mathbf{H})$ if A has an identity]:

Proposition 2 *If I is an essential two sided ideal of a non-unital C*-algebra A, then I does not have an identity element itself.*

Proof. Assume on the contrary that there exists an element $e \in I$ such that $ex = xe = x$ for all $x \in I$. Then there exists a $y \in A$ such that $ey \neq y$ or $ye \neq y$ since A does not have an identity. The restrictions of the right shift maps R_y and R_{ey} if $ey \neq y$ (or the left shift maps L_y and L_{ye} in the case $ye \neq y$) are identical on I contradicting the essential faithfulness of I with respect to A.

Q.E.D

Therefore there exists elements $h_1, \ldots, h_n$ in $I + I^{\perp}$ such that $\Pi_{1 \leq i \leq n}(x_i - h_i) = 0$ [Theorem 1, Chapter 2.7].

Step 3. Let us decompose h_i uniquely into the sum $a_i + b_i$, where $a_i \in I$, $b_i \in I^{\perp}$: $I \bigcap I^{\perp} = 0$. Then the zero product $\Pi_{1 \leq i \leq n}(x_i - h_i)$ can be written as

$$\Pi_{1 \leq i \leq n}([x_i - a_i] - b_i) = \left(\Pi_{1 \leq i \leq n}(x_i - a_i)\right) + B = 0$$

where B is a sum of products, each product containing at least one of the factors $b_i, i = 1, \ldots, n$.

Step 4. Let us note that

$$\Pi_{1\leq i\leq n}(x_i - h_i) = \Big(\Pi_{1\leq i\leq n}(x_i - a_i)\Big) + B = 0$$

is a decomposition of the zero product $\Pi_{1\leq i\leq n}(x_i - h_i)$ in $I + I^{\perp}$ since $\Pi_{1\leq i\leq n}(x_i - a_i) \in I$ and $B \in I^{\perp}$.

Firstly, $B \in I^{\perp}$ since each of the $b_i \in I^{\perp}$. Secondly, $\Pi_{1\leq i\leq n}(x_i - a_i) \in I$: if π denotes the *-homomorphism $\pi : A \to A/I \mid a \mapsto a + I$, then

$$\pi(\Pi_{1\leq i\leq n}(x_i - a_i)) = \Pi_{1\leq i\leq n}\pi(x_i - a_i) = \Pi_{1\leq i\leq n}\pi(x_i) = \pi(\Pi_{1\leq i\leq n}(x_i)) = 0$$

since by assumption $\Pi_{1\leq i\leq n}x_i \in I$.

Step 5. By the uniqueness of the decomposition $0 = 0 + 0$ in $I + I^{\perp}$, we conclude that both B and more importantly $\Pi_{1\leq i\leq n}(x_i - a_i)$ are 0.

Therefore, the dumping of all the elements which are not from the original C*-algebra A into the term B of Step 3, effectively allows us replace A with $A \bigodot K(\mathbf{H})$ as far as lifting n-zero divisors is concerned. In short, we can assume without loss of generality that A does not have an identity.

This theme will recur again in the lifting of nilpotent elements in next chapter, Chapter 3.

Q.E.D

Chapter 3

Lifting: Nilpotent Elements

3.1 Lifting nilpotent elements : Statement of Problem

In this section, the property we want to lift is the ring theoretic (algebraic) property of a nilpotent element.

Definition 1 (Property of Nilpotent Element) *Let x be an element in the C*-algebra A taken as a ring. Let the property $P(x)$ be the ring-theoretic property that the element x is a nilpotent element. An element x in the ring A is a nilpotent element if there exists an $n \in \mathbf{N}^+$ such that $x^n = 0$. The smallest n such that $x^n = 0$ is called the degree of nilpotency.*

Example 1 *Let A be the C*-algebra $M_3(\mathbf{C})$, the set of all 3×3 matrices with entries taken from the complex number field $\mathbf{C}$. Then the matrices*

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are nilpotent matrices with degree of nilpotency 3 and 2 respectively.

Taking $M_3(\mathbf{C})$ as the set of all bounded operators on the three dimensional Hilbert space $\mathbf{H} = \mathbf{C}^3$, the degree of nilpotency is bounded from above by $dim(\mathbf{H}) = 3$. [Chapter 8, Corollary 8.8 [33]]. Further, for any nilpotent operator $n \in M_3(\mathbf{C})$, there exists a basis of $\mathbf{C}^3$ such that the matrix of n, with respect to the new basis, is a matrix where all the entries on and below the diagonal are 0's [Chapter 8, Lemma 8.25 [33]].

In the C*-algebra $M_n(\mathbf{C})$, the nilpotent elements play a critical role in the decomposition of an arbitrary element of $M_n(\mathbf{C})$: any matrix in $M_n(\mathbf{C})$ is a *direct sum* of finitely many matrices of the form scalar multiple of the identity matrix, e, plus a nilpotent matrix:

Theorem 1 *(Chapter 8, Theorem 8.22 [33]) Consider the C*-algebra $M_n(\mathbf{C})$ as the set of all operators on the n-dimensional Hilbert space $\mathbf{H} = \mathbf{C}^n$ over the complex field. Let x be an arbitrary element of $M_n(\mathbf{C})$. Then there exists a decomposition of $\mathbf{C}^n$ into m x-invariant subspaces: $\mathbf{C}^n = U_1 \oplus \ldots \oplus U_m$, such that the restriction $(x - \lambda_j e)|_{U_j}$ is nilpotent for some $\lambda_j \in \mathbf{C}$ where $j = 1, \ldots, m$.*

In total contrast to the C*-algebra of example 1, the commutative C*-algebra $C_0(X)$ has only the 0-function as the only nilpotent element.

Let A denote a C*-algebra, I a closed 2-sided ideal in A, A/I the quotient C*-algebra and $\pi : A \to A/I$ the quotient map [Chapter 1.2.4, Proposition 4]. Taking the C*-algebra A/I as a ring and a non-zero element $x+I$ in the quotient C*-algebra A/I, the property P of the element $x+I$ being nilpotent, $P(x+I)$ is true, lifts when there exists an element a in the ideal I such that the perturbed element $x - a$ in the finer C*-algebra A is also nilpotent : $P(x-a)$ is true. In fact, the degree of nilpotency is preserved : if n is the degree of nilpotency of the element $x + I$ in A/I, then m, the degree of nilpotency of the element $(x - a)$ in A, equals n : the equation $\pi[(x-a)^m] = (x+I)^m = 0$ implies that $n < m$; if on the contrary, $m < n$, then by the same equation $(x+I)^m = 0$; this contradicts the minimality of n, the degree of nilpotency for the element $x + I$.

We state the lifting problem of nilpotent elements which we shall prove affirmatively later on, as follows:

Theorem 2 (Lifting Nilpotent Elements) *Let A be a C*-algebra, I a closed 2 -sided ideal in A. If x is an element of A such that $x^n \in I$ where n is the degree of nilpotency of the element $x + I$ in A/I, then there exists an a in I where $(x - a)$ is a nilpotent element in the finer C*-algebra A with degree of nilpotency n.*

In the previous chapter we lifted n-zero divisors in a general C*-algebra A [Chapter 2.7.2, Theorem 1]. Setting all the $x_i = x$ in Chapter 2.7.2, Theorem 1, we conclude that there are elements $a_1, \ldots, a_n$ in I such that $\prod_{1 \leq i \leq n}(x - a_i) = 0$. However, this is not good enough for lifting nilpotent elements. We need a single perturbation: $a_i = a \in I$ for $i = 1, \ldots, n$, which does the job. We need to do more work.

We first consider some pathological cases. Consider the trivial nilpotent element, the zero-element of A/I which has degree of nilpotency 1. $x + I$ is the 0-element if and only if $x \in I$. Then, trivially picking $x \in I$ as the ideal element, the perturbed element $x - x = 0 \in A$ is nilpotent with degree of nilpotency 1.

3.2 Lifting nilpotent elements: Degree of nilpotency 2

Here we shall lift nilpotent elements with degree of nilpotency 2. Note that the problem of lifting a positive element of degree of nilpotency 2, reduces to the trivial problem of lifting the trivial nilpotent element, the zero-element of A/I which has degree of nilpotency 1. Suppose $x + I$ is a positive nilpotent element with degree of nilpotency 2. Then x is positive, and so is x^2. But the degree of nilpotency is 2: $x^2 \in I$. Therefore, by the functional calculus on self-adjoint elements applied to the C*-algebra I, so also is its square root $x : x \in I$.

Suppose $x + I \in A/I$ is a nilpotent element with degree of nilpotency 2. Then the element $x + I$ is a special type of zero divisor : set the element $y + I$ to be $x + I$ in the statement of the Lifting Zero Divisors Theorem [Chapter 2.2, Theorem 1], as an additional condition on the zero divisor $x + I$. As the Lifting Zero Divisors Theorem stands, the required perturbations to lift this specialized zero divisor does not necessarily force $a = b$ since $a = u|x|^{\frac{1}{2}}a_1$ and $b = a_1|x^*|^{\frac{1}{2}}u$. The proof of the lifting of nilpotent elements with degree of nilpotency 2, follows a similar route as the proof used in lifting self-adjoint zero divisors [Chapter 2.2, Corollary 1].

Theorem 3 (Lifting Nilpotent Elements with degree of Nilpotency 2) *(Proposition 2.8 [3]) Let A be a general C*-algebra and I be a closed 2-sided ideal of A. If $x \in A$ with $x^2 \in I$, then there exists $a \in I$ with $(x - a)^2 = 0$.*

Proof. Consider A as a uniformly closed self-adjoint subalgebra of operators on its universal Hilbert space $\mathbf{H}$. Let $x = u|x|$ and $x^* = u^*|x^*|$ denote the polar decompositions in $B(\mathbf{H})$ of the elements x and x^*, respectively [Chapter 1.3.2, Theorem 4]. Then, as in Step 1 of the proof of lifting zero divisors [Chapter 2.2, Theorem 1], consider the decomposition of the difference $|x| - |x^*|$, which is a self-adjoint element in A, into its unique *orthogonal* positive parts [Chapter 1.3.1, Corollary 1]:

$$|x| - |x^*| = (|x| - |x^*|)_+ - (|x| - |x^*|)_-$$

where $(|x| - |x^*|)_+, (|x| - |x^*|)_- \geq 0$ and $(|x| - |x^*|)_+ \cdot (|x| - |x^*|)_- = (|x| - |x^*|)_- \cdot (|x| - |x^*|)_+ = 0$: the positive parts $(|x| - |x^*|)_+$ and $(|x| - |x^*|)_-$ commute and we shall denote them as d_1 and d_2, respectively.

Similarly to the proof used in lifting self-adjoint zero divisors [Chapter 2.2, Corollary 1], we define the required ideal perturbation a as the element $x - x_1$ where x_1 are defined as:

$$x_1 = (d_2)^{\frac{1}{3}} u|x|^{\frac{1}{3}} (d_1)^{\frac{1}{3}}$$

Note that the positive elements d_1 and d_2 generate a commutative C*-algebra, C*(d_1, d_2), of A where the elements $(d_1)^{\frac{1}{3}}$ and $(d_2)^{\frac{1}{3}}$ are orthogonal well defined elements in C*(d_1, d_2). Therefore, $x_1^2 = 0$. We are therefore left with showing that:

a $\in$ **I**: We show that $\pi(a) = \pi(x - x_1) = \pi(x) - \pi(x_1) = 0$ by showing that $\pi(x) = \pi(x_1)$.

Firstly, by the Quotient Mapping Of the Functional Calculus For Self Adjoint Elements [Chapter 1.3.4 ,Proposition 2] and its corollary [Chapter 1.3.4, Corollary 5]:

$$\pi(x_1) = [\Big(\pi(|x|) - \pi(|x^*|)\Big)_-]^{\frac{1}{3}}\pi(u|x|^{\frac{1}{3}})[\Big(\pi(|x|) - \pi(|x^*|)\Big)_+]^{\frac{1}{3}} \tag{3.1}$$

Secondly, $|x||x^*| \in I$ since [cf. Chapter 2.2, Theorem 1, equation (2.2)]:

$$|x||x^*| = \lim_{n\to\infty} a_n = \lim_{n\to\infty}(\frac{1}{n} + |x|)^{-1}(x^*x)(xx^*)(\frac{1}{n} + |x^*|)^{-1} \tag{3.2}$$

Therefore the positive elements $\pi(|x|)$ and $\pi(|x^*|)$ are orthogonal. Hence, $\pi(|x|) - \pi(|x^*|)$ is the unique orthogonal decomposition of $\pi(|x|) - \pi(|x^*|)$: $\Big(\pi(|x|) - \pi(|x^*|)\Big)_+ = \pi(|x|)$ and $\Big(\pi(|x|) - \pi(|x^*|)\Big)_- = \pi(|x^*|)$. Equation (3.1) then becomes:

$$\begin{aligned}\pi(x_1) &= [\pi(|x^*|)]^{\frac{1}{3}}\pi(u|x|^{\frac{1}{3}})[\pi(|x|)]^{\frac{1}{3}} &(3.3)\\ &= [\pi(|x^*|^{\frac{1}{3}})]\pi(u|x|^{\frac{1}{3}})[\pi(|x|^{\frac{1}{3}})] &(3.4)\\ &= \pi\Big(|x^*|^{\frac{1}{3}}(u|x|^{\frac{1}{3}})|x|^{\frac{1}{3}}\Big) &(3.5)\end{aligned}$$

We are done once we show that $|x^*|^{\frac{1}{3}}u = u|x|^{\frac{1}{3}}$. This follows from the onto *-isomorphism $\Phi : C^*(|x|) \to C^*(|x^*|)||x| \mapsto u|x|u^*$ which is a well defined definition since $C^*(|x|)$ has the set of all polynomial expressions in $|x|$ as a dense subset.

Q.E.D

3.3 Lifting nilpotent elements: Preliminary Results For the General Case

So far we have lifted nilpotent elements whose degree of nilpotency is 1 or 2. To lift nilpotent elements of any degree of nilpotency, we shall need the following additional propositions:

The proof of the first proposition is due to Professor Stroh.

Proposition 1 *Let a, b and c be positive elements of the C^*-algebra A. Then $a(b+c) = 0$ if and only if $ab = 0 = bc$.*

We only prove that $a(b+c) = 0$ implies $ab = 0 = ac$ since the other direction is trivial.

Step 1. $(b+c)$ is a 2-sided annihilator of $\sqrt{a}$.

Firstly, $b+c$ is a 2-sided annihilator of a since $a(b+c) = 0 = 0^* = (b+c)a$. Therefore, $b+c$ will be a 2 sided annihilator of all powers of a, a^n, where $n \in \mathbf{N}$.

Secondly, consider the commutative C*-algebra, $C^*(a, b+c)$ generated by the commuting positive elements a and $b+c$. The commutative C*-algebra $C^*(a)$ generated by a, is included in $C^*(a, b+c)$. Now, by the Stone Weierstrass theorem on $C(\sigma_A(a))$, since the square root function vanishes at 0, it is the uniform limit of a sequence of polynomials which do not have constant terms. Hence, $\sqrt{a}$ is the limit of a sequence of terms which are polynomials without constant terms in a which therefore reside in $A \subset A_e$. [Local Representation Theory I, Theorem 1, Chapter 1.3]

Finally, by the joint continuity of the product in A, $(b+c)\sqrt{a}$ is the limit of a zero sequence in A since $(b+c)$ annihilates all the polynomial without constant terms in a.

Step 2. $(\sqrt{a})b(\sqrt{a}) = 0 = (\sqrt{a})c(\sqrt{a})$.

From step 1, $(b+c)\sqrt{a} = 0 \Rightarrow b\sqrt{a} + c\sqrt{a} = 0$. Then left multiplying by $\sqrt{a}$, we have

$$(\sqrt{a})b(\sqrt{a}) + (\sqrt{a})c(\sqrt{a}) = 0.$$

This is the sum of two positive elements since the summands are of the form x^*x for some element x in A :

$$(\sqrt{a})b(\sqrt{a}) = \sqrt{a}\sqrt{b}\sqrt{b}\sqrt{a} = (\sqrt{b}\sqrt{a})^*\sqrt{b}\sqrt{a}$$

and similarly for $\sqrt{a}c\sqrt{a}$. Therefore, each of the summands are zero [Chapter VI, Corollary 7.10 [9]].

Step 3. $\sqrt{a}\sqrt{b} = 0 = \sqrt{a}\sqrt{c}$.

We only prove the first equality : $\sqrt{a}\sqrt{b} = 0$. The other inequality is proved in an identical fashion.

We shall consider A as a uniformly closed self-adjoint subalgebra of operators on its universal Hilbert space $\mathbf{H}$ [Gelfand Naimark Theorem II, Chapter 1.2.2]. From step 2, we have

$$(\sqrt{b}\sqrt{a})^*\sqrt{b}\sqrt{a} = 0$$

which is of the form $T^*T = 0$ where $T \in B(\mathbf{H})$ is the representation of the element $\sqrt{b}\sqrt{a}$. Therefore, $T = 0 = T^*$ [Polar Decomposition Theorem, Theorem 2, Chapter 1.3.2].

Step 4. $ab = 0 = ac$.

We only prove the first equality : $ab = 0$. The other inequality is proved in an identical fashion.

This follows from left and right multiplying the equation $\sqrt{a}\sqrt{b} = 0$ of Step 3 by $\sqrt{a}$ and $\sqrt{b}$, respectively.

Q.E.D

We now extend the above proposition by allowing a just to be self adjoint:

Corollary 1 (Extension Of Proposition 1) *Let a be a self-adjoint element. Let b and c be positive elements of the C^*-algebra A. Then $a(b + c) = 0$ if and only if $ab = 0 = bc$.*

Proof. Note that if $a(b + c) = 0$ then $a^2(b + c) = 0$. The element a^2 is positive since it is the square of a self-adjoint element : the associated quadratic form associated with the operator a^2 is always positive.

Then invoking Proposition 1, we conclude that

$$a^2b = a^2c = 0$$

Hence

$$ba^2b = baab = (ab)^*ab = 0$$

from which we infer that $ab = 0$ [see Step 3 of the proof of Proposition 1].

Q.E.D

The second proposition which we shall need states that if a normed space is short of being complete, a Banach space, by a finite dimensional subspace, it is of no relevance : the space will be a Banach space:

Proposition 2 *(Chapter VI Remark 3.11 [9]) If a normed linear space X has a complete linear subspace Y of finite dimension codimension in X, then X is complete and X is naturally isomorphic (as a locally convex space) with* $Y \bigoplus \mathbf{C}^n$.

3.4 Lifting nilpotent elements: The Corona

3.4.1 The General Overview

Let us recall the general approach in lifting the property of n-zero divisors in a general C*-algebra. Firstly, by the Von Neumann Lifting Lemma (Chapter 2.4.3, Lemma 3), we lifted the property of n-zero divisors in a Von Neumann C*algebra. Secondly, we lifted the property of n-zero divisors in the corona of a non-unital σ-unital C*-algebra A. The corona is a SAW*-algebra which has the property of possessing orthogonal local units : a very good approximation of the Von Neumann Lifting Lemma. We then lifted the property of n-zero divisors in the general C*-algebra A by embedding effectively reducing A into a non-unital σ-unital C*-algebra A.

Just as in the case of lifting the property of n-zero divisors, in the case of lifting the property of a nilpotent element, it is again by virtue of the Von Neumann Lifting Lemma that we can lift the property of a nilpotent element in a Von Neumann C*-algebra. In fact, one mimicks the proof of Theorem 2.4, The Structure Theorem For Polynomially Compact Operators, of C. Olsen's paper, *A Structure Theorem For Polynomially Compact Operators, Amer. J. Math. 93 (1971), p 686 - 698.* Buoyed by the success of the approach taken in lifting the property of n-zero divisors, we shall therefore assume the same general approach in lifting the property of a nilpotent element.

In this section we shall lift the property of nilpotent elements in the corona, $C(A) = M(A)/A$, of a non-unital σ - unital C*-algebra A as follows: firstly, we establish a triangular form [Chapter 3.4.2, Theorem 4] for the nilpotent element x of the corona $C(A)$ relative to a finite commutative set of elements, $\{e_0, \ldots, e_n\}$, where n is the degree of nilpotency of x; this commutative set, $\{e_0, \ldots, e_n\}$, resides in the unit sphere of the corona $C(A)$, except the element e_0 which is 0, and generates orthogonal elements in a SAW*-algebra way : e_k is a "local unit" of e_{k-1} for $k = 1, \ldots, n$; secondly, we lift the finite family $\{e_0, \ldots, e_n\} \subset C(A)$ as the finite family $\{d_0, \ldots, d_n\} \subset M(A)$ such that the norm is preserved : the d_j's reside on the unit sphere of $M(A)$, and only one of the properties defining the set $\{e_0, \ldots, e_n\} \subset C(A)$ [Chapter 3.4.2, Theorem 4] is lifted [Chapter 3.4.3]; thirdly, with the aid of the functional calculus, we construct an element in the finer C*-algebra $M(A)$ using the finite family $\{d_0, \ldots, d_n\} \subset M(A)$ which will lift the property of being nilpotent [Chapter 3.4.4, Theorem 6].

3.4.2 Triangular Form: The Corona

We now define and establish the existence of a triangular form for the nilpotent element x of the corona $C(A)$ relative to a finite commutative set of elements, $\{e_0, \ldots, e_n\}$, where n is the degree of nilpotency of x.

Theorem 4 (Triangular Form For Nilpotent Element) *(Lemma 6.3 [6]) Let $C(A) = M(A)/A$ be the corona of a non-unital σ -unital C*-algebra A. Let x be a nilpotent element of the corona $C(A)$ with n as the degree of nilpotency: $x^n = 0$ for some $n \in \mathbf{N}^+$.*

Then there are positive elements $e_0, \ldots, e_n$ of the corona $C(A)$ dominated by the identity element $1_{C(A)}$: $0 \leq e_k \leq 1_{C(A)}$, such that:

$$\begin{array}{lll} (a) & (1_{C(A)} - e_k)x^{n-k} = 0 & 0 \leq k \leq n \\ (b) & (1_{C(A)} - e_{k-1})xe_k = 0 & 1 \leq k \leq n \\ (c) & (e_k e_{k-1}) = e_{k-1} & 1 \leq k \leq n \end{array}$$

where we define $e_0 = 0$ and $e_1 = 1_{C(A)}$. We call the above a triangular form for the nilpotent element x of the corona $C(A)$ relative to a finite commutative set of elements, $\{e_0, \ldots, e_n\}$

It comes to no surprise that the above theorem which holds for the corona, which is a SAW*-algebra, is a generalization of a triangular form for a nilpotent element in a Von Neumann C*-algebra. In the case of a Von Neumann C*-algebra, a triangular form for a nilpotent element occurs naturally:

Example 2 [Triangular Form For Nilpotent Element in Von Neumann C*-algebra] *Let A be a Von Neumann C*-algebra. Let x denote a nilpotent element in A. Then we have*

$$1 = x^0, x, x^2, \ldots, x^k, \ldots, x^n = 0.$$

Consequently, identifying x with a bounded operator on some Hilbert space $\mathbf{H}$ [Definition 4, Chapter 2.4.1] we have

$$\mathbf{H} \supseteq \overline{Ran(x)} \supseteq \overline{Ran(x^2)} \ldots \supseteq \overline{Ran(x^k)} \ldots \supseteq \overline{Ran(x^n)} = \{0\} \tag{3.6}$$

We then define the positive element e_k as the range projection of x^{n-k} : $e_k = \overline{Ran(x^{n-k})}$. Therefore each e_k is a positive operator residing on the unit sphere of A and is bounded by $1_{\mathbf{H}}$.

It is easy to verify the conditions (a) - (c). We start with condition (c) which follows from the following inequality which is by virtue of equation (3.6),

$$1_{\mathbf{H}} = e_n \geq e_{n-1} \geq e_{n-2}, \ldots, \geq e_k \geq e_{k-1}, \ldots, \geq e_0 = 0.$$

For condition (a), we simply note that $1_{\mathbf{H}} - e_k$ *is the projection onto* $\overline{Ran(x^{n-k})}^{\perp} = Ran(x^{n-k})^{\perp}$ *by the joint continuity of the inner product.*

For condition (b), since e_{k-1} *is the projection onto* $\overline{Ran(x^{n-(k-1)})}$, $1_{\mathbf{H}} - e_{k-1}$ *is the projection onto* $\overline{Ran(x^{n-(k-1)})}^{\perp}$. *Now* $Ran(e_k) = \overline{Ran(x^{n-k})}$ *so that* $Ran(xe_k) \subset \overline{Ran(x^{n-k+1})} = Ran(e_{k-1})$ *since* x *is a bounded operator.*

2.4.2.1 Corollaries of the Triangular Form for Nilpotent Elements

Before we prove the above theorem, we first show that the commutativity of the set $e_0, \ldots, e_n$ follows from condition (c) and the fact that the set $e_0, \ldots, e_n$ resides in the unit sphere of the corona is a corollary of Theorem 4:

Corollary 2 *The set* $\{e_0, \ldots, e_n\}$ *of Chapter 3.4.2, Theorem 4 is a commutative set : the elements of the set commute.*

Proof. It suffices to show that $e_j e_k = e_k$ for all $j \geq k$ which is an extension of condition (c) since $e_j e_k = e_k e_j = e_k$ for all $j > k$ on taking adjoints : e_k is positive. We prove this by induction on m where $j = k + m$.

The induction step of $m = 1$ results from taking adjoints of condition (c) : e_k, e_j is positive.

Now assume that the statement is true for $m = m$: $e_k = e_{k+m} e_k$. Consider the case of $m + 1$. Therefore,

$$\begin{aligned} e_{k+(m+1)} e_k &= e_{(k+m)+1} e_k \\ &= e_{(k+m)+1} e_{k+m} e_k \\ &= e_{k+m} e_k \\ &= e_k. \end{aligned}$$

Q.E.D

Corollary 3 *The subset* $\{e_1, \ldots, e_n\}$ *of Chapter 3.4.2, Theorem 4, is a subset of the unit sphere of the corona. Further,* e_k *is not invertible for* $k = 0, \ldots, n-1$.

Proof. Firstly, $x^{n-k} \neq 0$ for $k = 1, \ldots, n$ since n is the degree of nilpotency. Therefore, the elements $1 - e_k$ for $k = 1, \ldots, n$ are not invertible [condition (a)].

Now $\| e_k \| \leq \| 1 \| = 1$ since $0 \leq e_k \leq 1_{C(A)}$ [Equation (1.11), Chapter 1.1]. If $\| e_k \|$ is strictly less than 1, this would force $1 - e_k$ to be invertible [Chapter V, Proposition 6.2, [9]]. Therefore $\| e_k \| = 1$ for $k = 1, \ldots, n$: the elements reside on the unit sphere of the C*-algebra A.

Finally, by definition $e_0 = 0$, hence it is not invertible. In the construction of the element e_k for $k = 1, \ldots, n-1$, the e_k's were local units of some positive element x with respect to another strictly positive element $y \neq 0 : e_k y = 0$. This contradicts the fact that $e_k = 1_{C(A)}$.

Q.E.D

2.4.2.2 Proof of The Triangular Form For Nilpotent Elements

We shall prove Theorem 4, via a proof by induction on m, where $m < n$ is the index such that $e_0, \ldots, e_m$ has already been constructed to satisfy conditions (a) - (c).

Consider the induction step of $m = 1$. By definition, $e_0 = 0$. We use the fact that $C(A)$ is a SAW*-algebra [Chapter 2.5.3.2, Theorem 10] to construct the element e_1 which satisfies the conditions (a) - (c) for $k = 1$: condition (c) is trivial since $e_0 = 0$; *if x and x^{n-1} were positive elements*, conditions (a) and (b) amount to saying that e_1 is a local unit for x^{n-1} with respect to x, since $x^n = xx^{n-1} = 0$ [Chapter 2.5.3, Definition 4]. We now prove that conditions (a) and (b) holds for non positive x and x^{n-1}:

Step 1. *There exists a positive element e_1' in the corona $C(A)$ which is a local unit for $|(x^{n-1})^*|^2$ with respect to $|x|^2$.*

(i) Firstly, the elements $|x|^2$ and $|(x^{n-1})^*|^2$ are a pair of positive orthogonal elements in the corona $C(A)$.

This follows from the fact that, $|x|^2|(x^{n-1})^*|^2 = x^*xx^{n-1}(x^{n-1})^* = 0$ since $x^n = 0$.

(ii) Secondly, we use the fact that the corona $C(A)$ is a SAW*-algebra to assume the existence of a positive element e_1' in the corona $C(A)$ such that $e_1'|x|^2 = |x|^2e_1' = 0$ and $e_1'|(x^{n-1})^*|^2 = |(x^{n-1})^*|^2e_1' = |(x^{n-1})^*|^2$: e_1' is a local unit for the element $|(x^{n-1})^*|^2$ with respect to $|x|^2$. Further the element e_1' resides in the unit sphere of the corona $C(A)$ [Chapter 2.5.3.2, Proposition 2].

Step 2. *Define $e_1 = \sqrt{e_1'}$. Then $xe_1 = 0$ and $(x^{n-1})^*e_1 = (x^{n-1})^*$. Further e_1 resides in the unit sphere of the corona $C(A)$.*

Consider the corona $C(A)$ as a uniformly closed self-adjoint subalgebra of operators on its universal Hilbert space $\mathbf{H}$. Let $x = u|x|$, $x^{n-1} = v|x^{n-1}|$ and $(x^{n-1})^* = v^*|(x^{n-1})^*|$ denote the polar decompositions in $B(\mathbf{H})$ of the elements x, x^{n-1} and $(x^{n-1})^*$, respectively, where $|x| = \sqrt{x^*x}$ and $|(x^{n-1})^*| = x^{n-1}(x^{n-1})^*$ [Chapter 1.3.2, Theorem 2].

Then, by Chapter 1.3.1, Corollary 3:

$$\begin{aligned}(x^{n-1})^* &= v^*|(x^{n-1})^*| = v^*\sqrt{|(x^{n-1})^*|^2} \\ &= v^*\sqrt{|(x^{n-1})^*|^2 e_1'} = v^*\sqrt{|(x^{n-1})^*|^2}\sqrt{e_1'} \\ &= v^*|(x^{n-1})^*|e_1 = (x^{n-1})^* e_1. \qquad (3.7)\end{aligned}$$

and

$$\begin{aligned}xe_1 &= u|x|e_1 = u\sqrt{|x|^2}e_1 \\ &= u\sqrt{|x|^2}\sqrt{e_1'} = u\sqrt{|x^2|e_1'} \\ &= u0 = 0.\end{aligned}$$

Taking the adjoint of equation (3.7) completes the proof that conditions (a) and (b) are satisfied.

Applying the functional calculus to the positive element e_1' [Chapter 1.3.1, Theorem 1], taking e_1' as a function in $C(\sigma(e_1'))$, there exists an $\tau \in \sigma(e_1')$ such that $e_1'(\tau) = 1$. Consequently, $\sqrt{e_1'}(\tau) = 1$ and hence $\| e_1 \| = 1$. This completes the induction step.

Consider the case of $m+1$. By the induction hypothesis, there exists $e_0, \ldots, e_m$ which satisfy the conditions (a) - (c). We need to construct the element e_{m+1} that also satisfies conditions (a) - (c).

Step 3. *There exists a positive element e_1' in the corona $C(A)$ which is a local unit for the element $|(1_{C(A)} - e_m)x|^2$ with respect to $|(x^{n-(m+1)})^*|^2 + e_m^2$. Further the element e_1' resides in the unit sphere of the corona $C(A)$.*

(i) Firstly, the elements $|(1_{C(A)} - e_m)x|$ and $|(x^{n-(m+1)})^*|^2 + e_m^2$ are a pair of positive orthogonal elements in the corona $C(A)$.

Invoking condition (a) for the case $k = m$, we have

$$(1_{C(A)} - e_m)x^{n-m} = (1_{C(A)} - e_m)xx^{n-(m+1)} = 0.$$

Consequently,

$$((1_{C(A)} - e_m)x)^*(1_{C(A)} - e_m)xx^{n-(m+1)}(x^{n-(m+1)})^* = 0.$$

or

$$|(1_{C(A)} - e_m)x|^2|(x^{n-(m+1)})^*|^2 = 0. \qquad (3.8)$$

Now, invoking condition (b) for the case $k = m$, we have

$$\Big((1_{C(A)} - e_m)x\Big)^*(1_{C(A)} - e_m)xe_m = 0$$

or

$$|(1_{C(A)} - e_m)x|^2 e_m = 0.$$

and therefore

$$|(1_{C(A)} - e_m)x|^2 e_m^2 = 0. \tag{3.9}$$

Combining equations (3.8) and (3.9) , we have

$$|(1_{C(A)} - e_m)x|^2(|(x^{n-(m+1)})^*|^2 + e_m^2) = 0. \tag{3.10}$$

(ii) Secondly, we use the fact that the corona $C(A)$ is a SAW*-algebra to assume the existence of a positive element e'_{m+1} in the corona $C(A)$ such that

$$e'_{m+1}|(1_{C(A)} - e_m)x|^2 = |(1_{C(A)} - e_m)x|^2 e'_{m+1} = |(1_{C(A)} - e_m)x|^2 \tag{3.11}$$

and

$$e'_{m+1}(|(x^{n-(m+1)})^*|^2 + e_m^2) = (|(x^{n-(m+1)})^*|^2 + e_m^2)e'_{m+1} = 0. \tag{3.12}$$

That is, e'_{m+1} is a local unit for the element $|(1_{C(A)} - e_m)x|^2$ with respect to $|(x^{n-(m+1)})^*|^2 + e_m^2$. Further the element e'_{m+1} resides in the unit sphere of the corona $C(A)$ [Chapter 2.5.3.2, Proposition 2].

Step 4. *By equation (3.12), it follows from Chapter 3.3, Proposition 1 that,*

$$(|(x^{n-(m+1)})^*|^2)e'_{m+1} = e'_{m+1}(|(x^{n-(m+1)})^*|^2) = 0 \tag{3.13}$$

and

$$e'_{m+1}(e_m^2) = (e_m^2)e'_{m+1} = 0 \tag{3.14}$$

In the next step we shall see that equation (3.13) and (3.14) takes care of condition (a) and (c), while equation (3.11) takes care of condition (b).

Step 5. *Define* $e_{m+1} = 1_{C(A)} - \sqrt{e'_{m+1}}$. *Then* $\sqrt{e'_{m+1}}(x^{n-(m+1)}) = 0$ *and* $(1_{C(A)} - e_m)x\sqrt{e'_{m+1}} = (1_{C(A)} - e_m)x$ *ensures conditions (a) and (b) are satisfied. Further,* $\sqrt{e'_{m+1}}e_m = 0$ *ensures condition (c) is satisfied. Further* e_{m+1} *resides in the unit sphere of the corona* $C(A)$.

Consider the corona $C(A)$ as a uniformly closed self-adjoint subalgebra of operators on its universal Hilbert space $\mathbf{H}$. Let $(1_{C(A)} - e_m)x = U|(1_{C(A)} - e_m)x|$

and $(x^{n-(m+1)})^* = V|(x^{n-(m+1)})^*|$ denote the polar decompositions in $B(\mathbf{H})$ of the elements $(1_{C(A)} - e_m)x$ and $(x^{n-(m+1)})^*$, respectively.

Then, by equation (3.11) and Chapter 1.3.1, Corollary 3:

$$\begin{aligned}
(1_{C(A)} - e_m)x &= U|(1_{C(A)} - e_m)x| = U\sqrt{|(1_{C(A)} - e_m)x|^2} \\
&= U\sqrt{|(1_{C(A)} - e_m)x|^2 e'_{m+1}} \\
&= U\sqrt{|(1_{C(A)} - e_m)x|^2}\sqrt{e'_{m+1}} \\
&= U|(1_{C(A)} - e_m)x|\sqrt{e'_{m+1}} \\
&= (1_{C(A)} - e_m)x\sqrt{e'_{m+1}}
\end{aligned}$$

and similarly from equation (3.13):

$$\begin{aligned}
(x^{n-(m+1)})^*\sqrt{e'_{m+1}} &= V|(x^{n-(m+1)})^*|\sqrt{e'_{m+1}} \\
&= V\sqrt{|(x^{n-(m+1)})^*|^2}\sqrt{e'_{m+1}} \\
&= V\sqrt{|(x^{n-(m+1)})^*|^2 e'_{m+1}} \\
&= V0 \\
&= 0 \qquad (3.15)
\end{aligned}$$

and from equation (3.14):

$$\sqrt{(e_m^2)e'_{m+1}} = e_m\sqrt{e'_{m+1}} = \sqrt{0} = 0.$$

Taking the adjoint of equation (3.15) we see that $e_{m+1}(x^{n-(m+1)}) = 0$.

Applying the functional calculus to the positive element $\sqrt{e'_{m+1}}$ just as in the case of the induction step, we conclude that $\| \sqrt{e'_{m+1}} \| = 1$. Then $\| e_{m+1} \| = 1$ from Chapter 2.5.3.2, Proposition 2.

Q.E.D

3.4.3 Lifting the Triangular Form : The Double Centralizer Algebra.

We are now ready to lift the finite commutative set $\{e_0, \ldots, e_n\} \subset C(A)$ of Chapter 3.4.2, Theorem 4 as the set $\{d_0, \ldots, d_n\} \subset M(A)$ such that the norm is preserved : the d_j's reside on the unit sphere of $M(A)$, and only one of the properties defining the set $\{e_0, \ldots, e_n\} \subset C(A)$, property (c) [Chapter 3.4.2, Theorem 4], is lifted. We shall prove this is in a more general context in the category of a general C*-algebra:

Theorem 5 (Lifting Triangular Form) *(Lemma 6.5 [6]) Let A be a general C*-algebra and B a C*-algebra with an identity, 1_B. If $\pi : A \to B$ is a surjective morphism between C*-algebras A and B, and $(e_n)_{n=1}^{\infty}$ is an infinite sequence of positive elements in B such that $0 \le e_n \le 1_B$ and $e_n e_{n+1} = e_n$ for all n, then there exists a sequence $(d_n)_{n=1}^{\infty}$ in A such that*

$$0 \le d_n \le 1, \quad d_n d_{n+1} = d_n, \quad \pi(d_n) = e_n$$

for all $n = 1, 2, \ldots,$

We can assume without loss of generality that A has an identity, 1_A, and that $\pi(1_A) = 1_B$. In the case A does not have an identity, replace A with A_e and take B as B_e, respectively [Unitization Theorem, Theorem 1, Chapter 1.1]. Then the surjective *-homomorphism map $\overline{\pi} : A_e \to B_e | (a, \lambda) \mapsto (\pi(a), \lambda)$ extends the map π and maps the identity of A into the identity of B_e which we take as B. Further, $\| \overline{\pi} \| \le 1$ [Chapter VI.3, Proposition 3.7 [9]]. We take π as $\overline{\pi}$.

We can assume without loss of generality that π is the canonical quotient map $A \to A/I | a \mapsto a + I$ where $I = ker(\pi)$ is the closed 2-sided ideal in A. This follows from the fact that the onto *-isomorphism map $A/I \to B | a+I \mapsto \pi(a)$ is an isometry [Chapter VI.3, Corollary 3.9 [9]]. We therefore identify B with A/I.

Proof. It suffices to prove the above for the case of $n = 1, 2$.

(i) Rephrase the condition

$$e_n e_{n+1} = e_n \textit{ for all } n = 1, 2, 3, \ldots$$

as a positive zero divisor condition

$$e_n(1_B - e_{n+1}) = 0 \textit{ for all } n = 1, 2, 3, \ldots$$

We lift the property of a positive zero divisor in B: there exists a positive zero divisor $d_1 \in A$ such that

$$\pi(d_1) = e_1 \text{ and } d_1 \overline{d_2} = 0$$

for a non-zero positive element $\overline{d_2} \in A$ such that $\pi(\overline{d_2}) = 1 - e_2$ [Chapter 2.2, Corollary 1].

(ii) The norm is preserved: $\| d_1 \| = \| \overline{d_2} \| = 1$, once we assume the hypotheses of Theorem 4, Chapter 3.4.2:

(a) $\| d_1 \|, \| \overline{d_2} \| \leq 1$: It suffices to show that the positive elements $d_1 \leq 1_A$ and $\overline{d_2} \leq 1_A$ [Equation (1.11), Chapter 1.1]. Since $\pi(d_1) = e_1 \leq 1$ and $\pi(1_A) = 1_B$,

$$\pi(d_1) \leq \pi(1_A) \text{ iff } \pi(1_A - d_1) \geq 0$$

Since the positive cone of A is mapped onto the positive cone of A/I, we conclude, $1_A - d_1 \geq 0$. The proof for $\overline{d_2}$ is similar.

(b) If $\| d_1 \| < 1$, then $1_A - d_1$ is invertible [Chapter V.6, Proposition 6.2 [9]]. Consequently, $1_B - e_1 = \pi(1_A - d_1)$ is also invertible which contradicts the Corollary 3, Chapter 3.4.2. Similarly, if $\| \overline{d_2} \| < 1$ then $1 - \overline{d_2}$ is invertible and hence e_2 will be invertible, again contradicting Corollary 3, Chapter 3.4.2.

(iii) The construction of $d_2 \in A$ proceeds over two steps. The construction for the other d_k for $k = 3, \ldots$ is done in an identical fashion.

Before we start the construction, we need the concept of a *hereditary C*-subalgebra.*

Definition 2 (Hereditary C*-subalgebra) *A hereditary C*-subalgebra B of a C*-algebra A is a C*-subalgebra with the condition that if $0 \leq x \leq y$, $y \in B^+$, then $x \in B^+$ whenever $x \in A^+$.*

Trivially A and $\{0\}$ are hereditary C*-subalgebras. A hereditary C*-subalgebra generated by a set $S \subset A$ is the smallest hereditary C*-subalgebra containing the set S. In particular, the hereditary C*-subalgebra generated by the singleton set $S = \{a\}$, where a is a *positive* element, is of the form $(aAa)^-$, the closure of the subalgebra $aAa = \{axa | x \in A\}$ [Chapter 3.2, Corollary 3.2.4 [13]]. We shall call this the hereditary C*-subalgebra generated by the positive element a.

With our d_1 defined, we construct our d_2, over two steps, from the positive element $\overline{d_2}$ such that $\pi(d_2) = e_2$, d_1 annihilates $1 - d_2$ and there exists a d_3 such that $d_3 d_2 = d_2$. All of the above will occur in the context of the C*-algebra $\overline{A_2} = A_2 + \mathbf{C}1_A$ which we can consider as the unitization C*-algebra [Chapter VI.3, Proposition 3.10 [9]] of the hereditary C*-subalgebra, A_2 of A, generated by the positive element $\overline{d_2}$. The reason is that all the zero divisors e_n and $1 - e_n$ for $n = 1, 2, 3, \ldots$ are trapped in $\pi(\overline{A_2})$ [Step 1], with the fact that $\pi(\overline{A_2})$ being a proper hereditary C*-subalgebra of B forcing $d_n d_{n+1} = d_n$ [Step 2(a)]. The mechanism of Step 2(a) allows us to proceed inductively [see (iv)]:

Step 1. $1_B - e_j \in \pi(A_2)$ for $j \geq 2$.

Firstly, the element $1_B - e_2 \in \pi(A_2)$ since $1_B - e_2 = \pi(\overline{d_2})$ and A_2 is the hereditary C*-subalgebra containing $\overline{d_2}$. Now $\pi(A_2) = \Big((1_B - e_2)\pi(A)(1_B - e_2)\Big)^- = \Big((1_B - e_2)B(1_B - e_2)\Big)^-$ so that $1_B - e_3 \in \pi(A_2)$:

$$(1_B - e_3) = (1_B - e_2)(1_B - e_3)(1_B - e_2)$$

since e_2 annihilates $1_B - e_3$.

Note that $\pi(A_2)$ is a hereditary C*-subalgebra containing $1_B - e_3$. Therefore $\pi(A_2)$ will contain the hereditary C*-subalgebra generated by $1_B - e_3$, $\Big((1_B - e_3)B(1_B - e_3)\Big)^-$. Then, by an identical argument, $1_B - e_4 \in \pi(A_2)$. Inductively, $1_B - e_j \in \pi(A_2)$ for $j \geq 2$.

Step 2. Construct a $d_2 \in \overline{A_2}$ such that $\pi(d_2) = e_2$ and $1_A - d_2 \in A_2$ where d_1 annihilates A_2.

(a) The positive zero divisor e_2 with respect to $1_B - e_3$ belongs to $\pi(\overline{A_2})$.

Consider the unitization C*-subalgebra $\overline{A_2}$. Then $1_A \in \overline{A_2}$ so that $1_B \in \pi(\overline{A_2})$. Then the positive zero divisors e_2 belongs to $\pi(\overline{A_2})$ on rewriting e_2 as $e_2 = 1_B - (1_B - e_2)$.

Trivially, from step 1, $1_B - e_3 \in \pi(A_2) \subset \pi(\overline{A_2})$.

We lift the property of a positive zero divisor in B: there exists a positive zero divisor $d_2 \in \overline{A_2}$ such that

$$\pi(d_2) = e_2 \text{ and } d_2\overline{d_3} = 0$$

for a non-zero positive element $\overline{d_3} \in \overline{A_2}$ such that $\pi(\overline{d_3}) = 1_B - e_3$ [Chapter 2.2, Corollary 1]. As before [see part (i)], $\| d_2 \|=\| \overline{d_3} \|= 1$.

(b) Once we show that $1_A - d_2 \in A_2$, then d_1 annihilates $1_A - d_2$: $A_2 = (\overline{d_2}A\overline{d_2})^-$ and $d_1\overline{d_2} = 0$.

Firstly, since $d_2 \in \overline{A_2}$, $d_2 = \lambda 1_A + a$ for some $a \in A_2$. Consequently, we need to show that $\lambda = 1$. It suffices to show that $\pi(A_2)$ is a proper hereditary C*-subalgebra of B or equivalently that $\pi(A_2)$ is devoid of the identity 1_B:

(α) *$\pi(A_2)$ is a proper hereditary C*-subalgebra of B if and only if $\pi(A_2)$ is devoid of the identity 1_B.*

If L is a closed left ideal of C*-algebra B, then the map $L \mapsto L \bigcap L^*$ is a bijection from the set of closed left ideals onto the set of hereditary C*-subalgebras of B [Chapter 3.2, Theorem 3.2.1(1) [13]]. Hence a proper hereditary C*-subalgebra corresponds to a proper closed left ideal which is devoid of 1.

(β) *If $\pi(A_2)$ is devoid of the identity 1_B, then $\lambda = 1$.*

Note that

$$\pi(1_A - d_2) \in \pi(A_2)$$

since $\pi(1_A - d_2) = 1_B - e_2 \in \pi(A_2)$. Therefore,

$$\pi(1_A - d_2) = 1_B - \lambda 1_B - \pi(a) \in \pi(A_2)$$

Now if $1_B \notin \pi(A_2)$ then $\lambda = 1$ since $\pi(a) \in \pi(A_2)$ and $1_B \notin \pi(A_2)$.

(γ) *$\pi(A_2)$ is a proper hereditary C*-subalgebra of B*

Firstly, $\pi(A_2) = (\pi(\overline{d_2})B\pi(\overline{d_2}))^-$ which is the smallest hereditary C*-subalgebra of B containing the positive element $\pi(\overline{d_2}) = 1_B - e_2$, since $A_2 = \overline{\overline{d_2}A\overline{d_2}}$.

Secondly, the left ideal $B(1_B - e_2)$ is a proper left ideal : since $1_B - e_2$ is not invertible [Corollary 3, Chapter 3.4.2], it is not *left* invertible since it is self-adjoint : left and right inverses coincide. Therefore, there exists a maximal left ideal I where $I \supset B(1_B - e_2)$ [Chapter V.5, Proposition 5.11 [9]] and is norm closed [Chapter V.6, Proposition 6.6 [9]] : all left ideals are trivially modular in a unitary C*-algebra.

Thirdly, $1_B - e_2 \in I$ since $(1_B - e_2) = 1_B(1_B - e_2) \in B(1_B - e_2)$. Therefore the proper hereditary C*-subalgebra $I \bigcap I^* \supset A_2$ which corresponds uniquely to some hereditary C*-subalgebra that contains $1_B - e_2$ and is devoid of 1_B and contains $\pi(A_2)$.

(iv) Finally, we construct d_3 in an identical fashion [Step 2] from the C*-subalgebra, $\overline{A_3} = A_3 + \mathbf{C}1_A$ where $A_3 \subset A_2$ is the hereditary C*-algebra generated by the positive element $\overline{d_3}$ [Step 2, (i)] : by step 1, we can rewrite e_3 as e_4, d_3 as d_4 and $\overline{d_3}$ as $\overline{d_4}$ in Step 2(a); also by step 1, we can take Step 2(b) for granted since $\pi(A_3) \subset \pi(A_2)$ where $\pi(A_2)$ is a proper hereditary C*-subalgebra; then, $1_A - d_3 \in A_3$ and $d_2 d_3 = d_2$.

We therefore proceed inductively to construct all the other $d_k's$ all within the environment of $\overline{A_2}$.

Q.E.D

3.4.4 Lifting Nilpotent Elements : The Corona.

In Chapter 3.4.2, given a fixed nilpotent element x of the corona $C(A)$ with degree of nilpotency n, we showed the existence of a special commutative set of $n+1$ positive elements, $\{e_0, \ldots, e_n\}$ of the unit sphere which we called a triangular form. This triangular form of the corona was then "lifted" in Chapter 3.4.3, from the corona $C(A)$ onto the multiplier algebra $M(A)$ as the set of $n+1$ positive elements $\{d_0, \ldots, d_n\}$ of the unit sphere of $M(A)$ which preserved property (c) of Theorem 4, Chapter 3.4.2. In this section, we show that with the help of the Functional Calculus for Normal Elements [Chapter 1.3.1, Theorem 1], these are the only ingredients needed to construct a nilpotent element y of the multiplier algebra that will lift the property of a nilpotent element of the corona $C(A)$ onto the multiplier algebra $M(A)$: we shall only work in the environment of the C*-subalgebra generated by $\{e_0, \ldots, e_n\}$ and $\{d_0, \ldots, d_n\}$. Formally:-

Theorem 6 *Let A be a non-unital, σ - unital C*-algebra. Let $\pi : M(A) \rightarrow C(A)$ be the quotient map onto the corona C*-algebra of A. If $x \in C(A)$ is a nilpotent element with n as the degree of nilpotency: $x^n = 0$, then there exists a $y \in M(A)$ such that $\pi(y) = x$ and $y^n = 0$.*

Proof.

Step 1 Pick from $C[0,1]$, two functions f and g such that $fg = f$. We enforce the additional conditions:-

(i) $f(0) = g(0) = 0$.

(ii) $f(1) = g(1) = 1$.

Note. In anticipation of the Functional Calculus for Normal Elements, the terms $f(e_k), g(e_k), f(d_k)$ and $g(d_k)$ for $k = 0, \ldots, n$ need to be well defined. Now the norm of the positive elements e_k, d_k is exactly 1. Therefore, their spectrums are a subset of the positive real line interval $[0,1]$: the spectral radius and the norm of e_k coincide [Chapter VI, Proposition 3.6 [9]]. We let the functions f, g in the terms $f(e_k), g(e_k), f(d_k)$ and $g(d_k)$ for $k = 0, \ldots, n$ denote the restriction of $f, g \in C[0,1]$ to the spectrums of e_k, d_k for $k = 0, \ldots, n$. Then the terms $f(e_k), g(e_k), f(d_k)$ and $g(d_k)$ for $k = 0, \ldots, n$ are well defined elements in the C*-subalgebra generated by the sets $e_0, \ldots, e_n$ and $d_0, \ldots, d_n$. The additional condition (i) is required in case the C*-subalgebra generated by $\{e_0, \ldots, e_n\}$ and $\{d_0, \ldots, d_n\}$ does not contain the identity.

Step 2. Let p_n, q_m be polynomial functions on $C[0,1]$ of degree n and m respectively which do not have constant terms : $p_n = \Sigma_{i=1}^{n} a_i x^i$ and $q_m = \Sigma_{j=1}^{m} b_j x^j$. Then $q_m(e_{k-1}) x p_n(e_k) = x p_n((\Sigma_{j=1}^{m} b_j) e_k)$.

Firstly, e_{k-1} is a left identity of $xp_n(e_k)$:

$$e_{k-1}xp_n(e_k) = xp_n(e_k).$$

To see this, first note that e_{k-1} is a left identity of $x{e_k}^j$ for all $j \geq 1$: right multiply the equation $e_{k-1}xe_k = xe_k$ [Chapter 3.4.2, Theorem 4(b)] by e_{k-1}^{j-1}. Then,

$$\begin{aligned} e_{k-1}xp_n(e_k) &= e_{k-1}x\Big(\Sigma_{i=1}^n a_i e_k^i\Big) \\ &= \Sigma_{i=1}^n\Big(a_i e_{k-1}xe_k^i\Big) = \Sigma_{i=1}^n\Big(a_i xe_k^i\Big) \\ &= xp_n(e_k). \end{aligned}$$

Consequently, e_{k-1}^s is a left identity of $xp_n(e_k)$ for all $s \geq 1$ where $s \in \mathbf{N}$. Therefore,

$$\begin{aligned} q_m(e_{k-1})xp_n(e_k) &= \Big(\Sigma_{j=1}^m b_j e_{k-1}^j\Big)xp_n(e_k) \\ &= \Sigma_{j=1}^m\Big(b_j e_{k-1}^j xp_n(e_k)\Big) = \Sigma_{j=1}^m b_j xp_n(e_k) \\ &= \Sigma_{i=1}^n a_i(\Sigma_{j=1}^m b_j)xe_k^i \\ &= xp_n((\Sigma_{j=1}^m b_j)e_k). \end{aligned}$$

Step 3. $f(e_{k-1})$ is a left identity of $xg(e_k)$: $f(e_{k-1})xg(e_k) = xg(e_k)$.

By the Weierstrass theorem for $C[0,1]$, the space of all continuous functions on the compact interval $[0,1]$ [Chapter 4, Theorem 4.6.1 [20]], f and g are, respectively, the uniform limits of the sequences of Bernstein polynomials

$$(B_n f)(x) = \Sigma_{k=0}^n x^k(1-x)^{n-k}f(k/n)$$

and

$$(B_n g)(x) = \Sigma_{k=0}^n x^k(1-x)^{n-k}g(k/n).$$

Consequently, since $f(1) = 1$, $(B_n f)(1) = f(1) = 1$: that is, f is the uniform limit of a sequence of polynomials which evaluates to 1 at 1 : the sum of the coefficients is 1; similarly, $f(0) = 0$ implies that the Bernstein polynomials do not have any constant terms.

Hence, by the joint continuity of the product in $C[0,1]$,

$$\begin{aligned} f(e_{k-1})xg(e_k) &= \lim_{n\to\infty} B_n f(e_{k-1})x \lim_{n\to\infty}(B_n g)(e_k) \\ &= \lim_{n\to\infty}(B_n f)(e_{k-1})x(B_n g)(e_k) \\ &= \lim_{n\to\infty} x(B_n g)(e_k). \end{aligned}$$

The last equality follows from step 2 since the sum of the coefficients of each $B_n f$ is exactly 1. Therefore, $f(e_{k-1})xg(e_k) = xg(e_k)$.

Step 4. $f(e_{k-1})$ is a left identity of $xg(e_{k-1}) : f(e_{k-1})xg(e_{k-1}) = xg(e_k)$.

Using a proof similar to that of step 3, $f(e_{k-1})$ is a left identity of xe_k^j for all $j \geq 1$.

By the Weierstrass theorem for $C[0,1]$ [Chapter 4, Theorem 4.6.1 [20]], we can write g as the limit of a sequence of Bernstein polynomials, $(B_n g)_{n=1}^{\infty}$, where $B_n g$ is an n-th order polynomial without constant terms which we denote as $\Sigma_{i=1}^n c_i x^i$. Since $e_{k-1} = e_k e_{k-1}$,

$$\begin{aligned} f(e_{k-1})x\Big(\Sigma_{i=1}^n c_i x^i\Big)(e_{k-1}) &= f(e_{k-1})x\Big(\Sigma_{i=1}^n c_i x^i\Big)(e_k e_{k-1}) \\ &= f(e_{k-1})x\Sigma_{i=1}^n c_i e_k^i e_{k-1}^i \end{aligned}$$

the last equality following from the fact that e_k and e_{k-1} commutes.

But,

$$f(e_{k-1})x\Sigma_{i=1}^n c_i e_k^i e_{k-1}^i = \Sigma_{i=1}^n c_i f(e_{k-1})xe_k^i e_{k-1}^i = \Sigma_{i=1}^n c_i xe_k^i e_{k-1}^i$$

since $f(e_{k-1})$ is a left identity of xe_k^j for all $j \geq 1$.

Since

$$\Sigma_{i=1}^n xc_i e_k^i e_{k-1}^i = x(B_n g)(e_k e_{k-1}) = x(B_n g)(e_{k-1})$$

we have shown that $f(e_{k-1})$ is a left identity of $xB_n g(e_{k-1})$. Hence on taking limits, we have the desired conclusion.

Step 5. $\Big(g(d_k) - g(d_{k-1})\Big)f(d_{j-1}) = 0$ if $j \leq k$.

For the case of $j < k$, we note that d_k^s and d_{k-1}^s for all $s \geq 1$ are left identities of d_{j-1}. [see Corollary 2, Chapter 3.4.2]. Consequently, d_k, d_{k-1} are left identities of any polynomial expression in d_{j-1}:

$$d_k p_n(d_{j-1}) = p_n(d_{j-1}) = d_{k-1}p_n(d_{j-1})$$

and as in step 2:

$$\begin{aligned} q_m(d_k)p_n(d_{j-1}) &= \Big(\Sigma_{j=1}^m b_j d_k^j p_n(d_{j-1})\Big) \\ &= \Big(\Sigma_{j=1}^m b_j p_n(d_{j-1})\Big) \\ &= p_n((\Sigma_{j=1}^m b_j)d_{j-1}). \end{aligned} \tag{3.16}$$

and likewise,

$$q_m(d_{k-1})p_n(d_{j-1}) = p_n((\Sigma_{j=1}^m b_j)d_{j-1}). \tag{3.17}$$

By the Weierstrass theorem for $C[0,1]$ [Chapter 4, Theorem 4.6.1 [20]], we can write g, f as the limit of sequences of Bernstein polynomials, $(q_m)_{m=1}^\infty, (p_n)_{n=1}^\infty$, respectively, whose coefficients sum up to 1. Then by the joint continuity of the product applied to equations (3.16) and (3.17),

$$g(d_k)f(d_{j-1}) = g(d_{k-1})f(d_{j-1}).$$

For the case of $j = k$,

$$g(d_{k-1})f(d_{j-1}) = g(d_{k-1})f(d_{k-1}) = f(d_{k-1}) \tag{3.18}$$

since $fg = f$. Now [see case of $j < k$]

$$g(d_{k-1})f(d_{j-1}) = g(d_k)f(d_{k-1}) = f(d_{k-1}). \tag{3.19}$$

Step 6. We can now construct a sequence $y_1, \ldots, y_k$ of elements in $M(A)$ such that y_k annihilates y_j whenever $j \leq k$:

$$\begin{aligned}
y_1 &= f(d_0)z(g(d_1) - g(d_0)) \\
y_2 &= f(d_1)z(g(d_2) - g(d_1)) \\
&\vdots \\
y_k &= f(d_{k-1})z(g(d_k) - g(d_{k-1})) \\
&\vdots \\
y_n &= f(d_{n-1})z(g(d_n) - g(d_{n-1}))
\end{aligned}$$

First note that $y_1 = 0$ since $d_0 = 0$ so that $f(d_0) = 0$ since f is the limit of a sequence of polynomials which do not have constant terms; secondly, $y_k y_j = 0$ when $j \leq k$ by Step 5.

Then:

$$\begin{aligned}
\pi(y_1) &= f(e_0)x(g(e_1) - g(e_0)) = x(g(e_1) - g(e_0)) \\
\pi(y_2) &= f(e_1)x(g(e_2) - g(e_1)) = x(g(e_2) - g(e_1)) \\
&\vdots \\
\pi(y_k) &= f(e_{k-1})x(g(e_k) - g(e_{k-1})) = x(g(e_k) - g(e_{k-1})) \\
&\vdots \\
\pi(y_n) &= f(e_{n-1})x(g(e_n) - g(e_{n-1})) = x(g(e_n) - g(e_{n-1}))
\end{aligned}$$

since $\pi\Big(f(d_{k-1})\Big) = f(\pi(d_{k-1})) = f(e_{k-1})$ and $\pi\Big(g(d_{k-1})\Big) = g(\pi(d_{k-1})) = g(e_{k-1})$ [Proposition 2, Chapter 1.3.4] and $f(e_{k-1})$ is a left identity of both $x(g(e_k)$ [Step 3] and $x(g(e_{k-1}))$ [step 4].

Step 7. The required y is precisely $y = \Sigma_{i=1}^{n} y_i$.

By a telescopic summation, $\pi(y) = x(g(e_n) - g(e_0)) = x(g(1_{C(A)}) - g(0_{C(A)})) = x$ since g is the uniform limit of a sequence of polynomials whose coefficient sum up to 1 and do not have constant terms. Further y^n is a sum of products where each product has n factors from the set $\{y_1, \ldots, y_n\}$ and hence must be 0 since y_k annihilates y_j whenever $j \leq k$.

Q.E.D

Example 3 *The following functions of* $C[0,1]$*, the space of all continuous functions on the compact interval* $[0,1]$*, fulfill the conditions of step 1:*

$$f = \begin{cases} 0 & : \ 0 \leq x \leq \frac{1}{2} \\ 2x - 1 & : \ \frac{1}{2} \leq x \leq 1 \end{cases}$$

and

$$g = \begin{cases} 2x & : \ 0 \leq x \leq \frac{1}{2} \\ 1 & : \ \frac{1}{2} \leq x \leq 1 \end{cases}$$

3.5 Lifting nilpotent elements : The General Case

We now prove the lifting of nilpotent elements in the general C*-algebra. Proceeding as in Chapter 2.7.2, Theorem 1, we shall reduce the problem of lifting a nilpotent element in the general C*-algebra into the problem of lifting a nilpotent element in the corona C*-algebra of a non-unital σ-unital C*-algebra, which was proved affirmatively in the previous section [Theorem 6, Chapter 3.4.4].

Theorem 7 *Let A be a C*-algebra, I a closed 2 -sided ideal in A. If x is an element of A such that $x^n \in I$ where n is the degree of nilpotency of the element $x + I$ in A/I, then there exists an a in I where $(x + a)^n = 0$.*

Proof. Just as in Chapter 2.7.2, Theorem 1, we may assume without loss of generality that A is non-unital and σ - unital. In the case A is not σ - unital, we take A as B, the C*-subalgebra generated by x [resp. $diag[x, 0] \in A \bigodot K(\mathbf{H})$ if A has an identity] and consider x [resp. $diag[x, 0] \in A \bigodot K(\mathbf{H})$ if A has an identity] as an element of B. B has a countable fundamental set and is therefore a separable normed space and hence σ-unital. Further, we take the closed two-sided ideal I as the non-trivial closed two-sided ideal $B \cap I$ of B. Proceeding as before:

Step 1. We construct a closed essential ideal $I + I^\perp$ of A from I [Chapter 2.7.1, Proposition 1]. This then allows us to embed A isometrically *-isomorphically in the Double Centralizer Algebra $M(I + I^\perp)$ [Theorem 11, Chapter 1.2.3].

Step 2. Treating the nilpotent element x in A as an element of $M(I + I^\perp)$, we can invoke Chapter 3.4.4, Theorem 6, since the C*-algebra $I + I^\perp$ is σ- unital and non-unital [see Chapter 2.7.2, Theorem 1, Step 2].

Therefore there exists an element h in $I + I^\perp$ such that $(x - h)^n = 0$.

Step 3. Let us decompose h uniquely into the sum $a + b$, where $a \in I$, $b \in I^\perp$: $I \bigcap I^\perp = 0$. Therefore the zero product $(x - h)^n$ can be written as

$$(x - h)^n = ((x - a) - b)^n = (x - a)^n + B = 0$$

where B is a sum of products, each product containing the factor b.

Step 4. Let us note that

$$(x - h)^n = ((x - a) - b)^n = (x - a)^n + B = 0$$

is a decomposition of the zero product $(x - h)^n$ in $I + I^\perp$: $(x - a)^n \in I$ and $B \in I^\perp$.

Firstly, $B \in I^{\perp}$ since $b \in I^{\perp}$. Secondly, $(x-a)^n \in I$: let π denote the *-homomorphism $\pi : A \to A/I \mid a \mapsto a + I$. $(x-a)^n \in I$ since $\pi((x-a)^n) = \Big(\pi(x-a)\Big)^n = \Big(\pi(x)\Big)^n = \pi(x^n) = 0$ since by assumption $x^n \in I$.

Step 5. By the uniqueness of the decomposition $0 = 0 + 0$ in $I + I^{\perp}$, we conclude that both B and more importantly $(x-a)^n$ are 0.

Q.E.D

Chapter 4

Lifting Polynomially Ideal Elements : A Criteria

4.1 Lifting Polynomially Ideal Elements : A Counter Example

In this section, we show that the ring theoretic (algebraic) property of a polynomially ideal element cannot be lifted in the general case.

Definition 1 (Property of Polynomially Ideal Element) *Let x be an element of the C^*-algebra A taken as a ring. Let I be a closed 2-sided ideal in A. Let the property $P(x, I)$ be the ring-theoretic property that the element x is a polynomially ideal element. An element x in the ring A is a polynomially ideal element if there exists a polynomial function $\mathbf{p}(z)$ over the complex number field $\mathbf{C}$, $\mathbf{C}[X]$, such that $\mathbf{p}(x) \in I$.*

Example 1 *Every nilpotent element x is polynomially ideal with respect to any closed 2-sided ideal. Let n be the degree of nilpotency. Then the polynomial function $\mathbf{p}(z) = z^n$ in $\mathbf{C}[X]$ annihilates the nilpotent element : $\mathbf{p}(x) = 0 \in I$.*

Definition 2 (Lifting Polynomially Ideal Element) *Let A be a C^*-algebra, I a closed 2-sided ideal of A. Let x be a polynomially ideal element of the C^*-algebra A taken as a ring. Then there exists a polynomial function $\mathbf{p}(z)$ over the complex number field $\mathbf{C}$, $\mathbf{C}[X]$, such that $\mathbf{p}(x) \in I$. The property $P(x, I)$ of the polynomially ideal element x is lifted precisely when there exists an element a in the ideal I such that $\mathbf{p}(x - a) = 0$.*

Example 2 (Lifting Nilpotent Elements : A special case of Lifting Polynomially Ideal Element) *Every nilpotent element is polynomially ideal. The lifting of nilpotent elements [Theorem 7, Chapter 3.5] is a special case of lifting polynomially ideal elements where the polynomial is of the form of a power function $\mathbf{p}(z) = z^n$ where n is the degree of nilpotency.*

Example 3 (Lifting Polynomially Ideal Element : Polynomially Compact Operators) *Consider the C*-algebra $A = B(\mathbf{H})$. We fix as our closed 2-sided ideal $I = K(\mathbf{H})$. If $x \in A$ such that $\mathbf{p}(x) \in I$ where $\mathbf{p}$ is a complex polynomial function, then $\mathbf{p}(x-a) = 0$ for some a in I [Theorem 2.4, C. Olsen, "A Structure theorem for polynomially compact operators", Amer. J. Math, 93 (1971), pp 686 - 698].*

The proof of the above example rested on the existence of the projection $e \in A = B(\mathbf{H})$ such that $ae \in I$ and $(1-e)b \in I$ whenever $ab \in I$ [Theorem 2.3, C. Olsen, *"A Structure theorem for polynomially compact operators", Amer. J. Math, 93 (1971), pp 686 - 698*]. This property holds when A is a Von Neumann algebra : the Von Neumann Lifting Lemma [Chapter 2.4.3, Lemma 3].

Therefore, the proof of Theorem 2.4, Olsen, *"A Structure theorem for polynomially compact operators", Amer. J. Math, 93 (1971), pp 686 - 698]*, generalizes to any closed ideal I of a Von Neumann algebra [Theorem 4.3 [3]].

Example 4 (Lifting Polynomially Ideal Element : Von Neumann C*-algebra) *Let A be a Von Neumann C*-algebra and I any 2-sided closed ideal of A. Then, we can always lift the property $P(x, I)$ of a polynomially ideal element x of A with respect to any closed 2-sided ideal of the C*-algebra A.*

We now give a counterexample [Example 2.9 [3]] to the lifting of polynomially ideal elements in a general C*-algebra. Topological obstructions are the key to providing the following counterexample.

Step 1. Consider the C*-algebra $A = C[0,1]$. In direct contrast to Example 4, the connectedness of the real line interval $[0,1]$ forces the C*-algebra A to be projectionless [Chapter 2.4.2, Example 2].

The finite subset $S = \{0,1\} \subset [0,1]$ is closed and hence all the functions which vanish on S is a closed ideal of $C[0,1]$ [Example 1, Chapter 1.2.4]. We take as our closed 2-sided ideal $I = \{f \in C[0,1] \mid f(0) = f(1) = 0\}$.

Step 2. Consider the complex polynomial function $\mathbf{p}(z) = z^2 - z$. Define the polynomially ideal element x of the C*-algebra $A = C[0,1]$ to be the identity map $x(t) = t$ for all $t \in [0,1]$: $\mathbf{p}(x)$ is the function $g(t) = t^2 - t$ which vanishes on S; $\mathbf{p}(x) \in I$. We now show that $\mathbf{p}(x-f) \neq 0$ for all ideal perturbations $f \in I$.

Step 3. The difference function $(x-f)$ is a continuous complex valued function on $[0,1]$. Therefore, the *graph* of $x - f$ is a connected path Γ in the complex plane $\mathbf{C}$ which starts at (0, 0) and ends at (1, 1): the continuous image of a connected space is connected. In the complex plane $\mathbf{C}$, connectedness is equivalent to polygonal connectedness: $\Gamma \subset \mathbf{C}$ is polygonally connected if and only if for each distinct pair of points in Γ, there exists a polygonal arc, finite string of line segments joined end-to-end [Chapter II, Theorem 3.7 [22]].

Step 4. If the real part $Re(z)$ of the complex number z is equal to $\frac{1}{2}$, then $\mathbf{p}(z)$ is a real number whose absolute value exceeds $\frac{1}{4}$: if $z = a + bi$ then $\mathbf{p}(z) = (a^2 - a - b^2) + b(2a - 1)i$; if $a = \frac{1}{2}$ then $\mathbf{p}(z) = -\frac{1}{4} - b^2$ where $|-\frac{1}{4} - b^2| = \frac{1}{4} + b^2 \geq \frac{1}{4}$.

Step 5. Finally, since the path Γ must intersect the line $Re(z) = \frac{1}{2}$, there exists a $t \in [0, 1]$ such that $Re\Big((x - f)(t)\Big) = \frac{1}{2}$ so that $\| \mathbf{p}(x - f) \| \geq \frac{1}{4}$. That is $\mathbf{p}(x - f) \neq 0$.

4.2 Lifting Polynomially Ideal Elements : Preliminary Results

In the previous section, we showed that in a general C*-algebra, the lifting of the property of a polynomially ideal element is not always possible. However all is not lost: in this section, we establish a criteria under which the lifting of a polynomially ideal element is possible. More precisely, the lifting of a polynomially ideal element from the quotient C*-algebra A/I onto the finer C*-algebra A occurs exactly when any finite orthogonal family of projections can be lifted from A/I onto A as a finite orthogonal family of projections in A. We shall call this criteria the *Finite Orthogonal Projection Criteria.*

Here we state the preliminary lemmas and propositions used in establishing the Finite Orthogonal Projection Criteria.

4.2.1 Linear Algebra Preliminaries

Definition 3 (The Minimal Polynomial) *Let A be a $p \times p$ matrix. Then there exists a unique monic polynomial $\mathbf{p}$ of smallest degree such that $\mathbf{p}(A) = 0$. This polynomial is called the minimal polynomial of A.*

Definition 4 *An $n \times n$ Jordan Block with diagonal entry equal to λ is a $n \times n$ matrix whose entries on the diagonal are precisely all λ and whose entries immediately above the diagonal are 1:*

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

Definition 5 (Direct Sum) *Let A, B be $k \times k$ and $m \times m$ matrices respectively. The direct sum of A and B, denoted by $A \bigoplus B$ is the $(k+m) \times (k+m)$ matrix :*

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

Direct sum matrices behave like diagonal matrices:

Proposition 1 *Let A_1, A_2 be $k \times k$ matrices. Let B_1, B_2 be $m \times m$ matrices. Let C_i be the direct sum matrix $A_i \bigoplus B_i$ for $i = 1, 2$. Then the matrix product $C_1 * C_2$ is the direct sum of the matrices $A_1 * A_2 \bigoplus B_1 * B_2$. Consequently, if $\mathbf{q}$ is a polynomial over the complex number field: $\mathbf{q} \in \mathbf{C}[X]$ and D is the direct sum matrix $A \bigoplus B$, then $\mathbf{q}(D)$ is the direct sum matrix $\mathbf{q}(A) \bigoplus \mathbf{q}(B)$.*

Theorem 1 *(Chapter 8, Theorem 8.33 [33]) Let A be a $p \times p$ matrix. Let $\mathbf{q}$ be a polynomial over the complex number field: $\mathbf{q} \in \mathbf{C}[X]$. Then $\mathbf{q}(A) = 0$ if and only if the minimal polynomial of A is a divisor $\mathbf{q}$.*

Proposition 2 *For every monic polynomial, $\mathbf{q}$, over the complex number field, there exists a matrix which has $\mathbf{q}$ as its minimal polynomial.*

Proof. First decompose $\mathbf{q}$ as a product of primes in $\mathbf{C}[X]$: $\mathbf{q}(z) = (z - \lambda_1)^{k_1}(z-\lambda_2)^{k_2} \cdots (z-\lambda_n)^{k_n}$, with $\lambda_i \neq \lambda_j$ if $i \neq j$. Let A_i be the $k_i \times k_i$ Jordan Block with diagonal entry equal to λ_i for $i = 1, \ldots, n$. Then the matrix A which is the direct sum of the matrices A_i for $i = 1, \ldots, n$: $A = A_1 \bigoplus A_2 \bigoplus \cdots \bigoplus A_n$ has $\mathbf{q}$ as its minimal polynomial. This follows from theorem 1: let $\mathbf{p} \in \mathbf{C}[X]$ such that $\mathbf{p}(A) = 0$; that is $\mathbf{p}(A_1) \bigoplus \mathbf{p}(A_2) \bigoplus \ldots \mathbf{p}(A_n) = 0$ which happens if and only if $\mathbf{p}(A_i) = 0$ for $i = 1, \ldots, n$; each Jordan Block A_i has $(z - \lambda_i)^{k_i}$ as its minimal polynomial; by theorem 1, $(z - \lambda_i)^{k_i}$ divides $\mathbf{p}$ for all i; hence $\mathbf{q}$ divides $\mathbf{p}$.

Proposition 3 *Let A be the direct sum matrix $A_1 \bigoplus A_2 \bigoplus \cdots \bigoplus A_n$ of Proposition 2. Then for each i, there exists a polynomial $\mathbf{p}_i$ identical to the minimal polynomial $\mathbf{q}(z) = (z - \lambda_1)^{k_1}(z - \lambda_2)^{k_2} \cdots (z - \lambda_n)^{k_n}$ except that the $(z - \lambda_i)^{k_i}$ factor being replaced by a polynomial, $\mathbf{r}(z)$, of degree $k_i - 1$: $\mathbf{r}(z) = a_0 + a_1(z - \lambda_i) + a_2(z - \lambda_i)^2 + \ldots + a_{k_i-1}(z - \lambda_i)^{k_i-1}$ such that $\mathbf{p}_i(A)$ is the direct sum matrix $0 \bigoplus \ldots 0 \bigoplus I_{k_i} \bigoplus 0 \ldots \bigoplus 0$ where 0 is the 0-matrix and I_{k_i} is the identity $k_i \times k_i$ matrix.*

Proof. We prove the theorem for the case of $i = 1$. The other cases are proved identically.

Step 1. Write A_1, the $k_1 \times k_1$ Jordan Block, as the sum $\lambda_1 I_{k_1} + N$ where I_{k_1} is the identity $k_1 \times k_1$ matrix (we shall treat $\lambda_1 I_{k_1}$ as the number λ_1) and the nilpotent matrix, N, of order k_1:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Step 2. Since $\lambda_1 I_{k_1}$ and N commute, by the Binomial theorem for commutative rings, we have $(\lambda_1 I_{k_1}+N)^n = \sum_{r=0}^{n}\binom{n}{r}\lambda^r N^{n-r}$. Since N is nilpotent degree k_1-1 we need only consider powers of N up to k_1-1 . In fact, $N, N^2, \ldots, N^{k_1-1}$ are the matrices:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

respectively.

Step 3. We now show that there exists a polynomial $\mathbf{p}_1$, as defined in the proposition, up to degree $k_1 - 1$ such that $\mathbf{p}_1(A_1) = I_{k_1}$.

Since $\big((\lambda_1 - \lambda_2) + N\big)^{k_2}\big((\lambda_1 - \lambda_3) + N\big)^{k_3}\cdots\big((\lambda_1 - \lambda_n) + N\big)^{k_n}$ is:

$$\left[\prod_{i=2}^{n}(\lambda_1 - \lambda_i)^{k_i}\right] + (S_1^1)N + (S_1^2 + S_2^2)N^2 + \ldots + (S_1^{k_1-1} + \ldots + S_{k_1-1}^{k_1-1})N^{k_1-1}$$

where:

$$S_1^1 = \sum_{i=2}^{n}\left[k_i(\lambda_1 - \lambda_i)^{k_i-1}\prod_{j\neq i}(\lambda_1 - \lambda_j)\right]$$

$$S_1^2 = \sum_{i=2}^{n}[\tbinom{k_i}{2}(\lambda_1 - \lambda_i)^{k_i-2}\prod_{j\neq i}(\lambda_1 - \lambda_j)]$$
$$S_2^2 = \sum_{i=2,\ i\neq j}^{n} k_i(\lambda_1 - \lambda_i)^{k_i-1}k_j(\lambda_1 - \lambda_j)^{k_j-1}\prod_{k\neq i,\ k\neq j}(\lambda_1 - \lambda_k)$$
$$\vdots$$
$$S_{k_1-1}^1 = \sum_{i=2}^{n}[\tbinom{k_i}{k_1-1}(\lambda_1 - \lambda_i)^{k_i-(k_1-1)}\prod_{j\neq i}(\lambda_1 - \lambda_j)]$$
$$S_2^{k_1-1} = \sum_{i=2,i\neq j}^{n}[\tbinom{k_i}{k_1-2}(\lambda_1 - \lambda_i)^{k_i-(k_1-2)}\tbinom{k_j}{1}(\lambda_1 - \lambda_j)^{k_j-1}\prod_{k\neq i,k\neq j}(\lambda_1 - \lambda_k)]$$

$$\vdots$$

$$S_{k_1-1}^{k_1-1} = \sum_{i=2}^{n}\left[(\lambda_1 - \lambda_i)^{k_i}\sum_{2\leq k_{i_j}\leq n}[\prod_{i_j,i_j\neq i,i_j\neq i_{j'}ifj\neq j'}^{j=k_1-1}\tbinom{k_{i_j}}{1}(\lambda_1 - \lambda_{i_j})^{k_{i_j}-1}]\right]$$

it follows that:

$$
\begin{aligned}
\mathbf{p}_1(\lambda_1+N) &= [a_0+a_1N+a_2N^2+\ldots+a_{k_1-1}N^{k_1-1}]\Big((\lambda_1-\lambda_2)-N\Big)^{k_2}\cdots\Big((\lambda_1-\lambda_n)+N\Big)^{k_n}\\
&= (a_0\prod_{i=2}^{n}(\lambda_1-\lambda_i)^{k_i}) + \Big[a_1\prod_{i=2}^{n}(\lambda_1-\lambda_i)^{k_i}+a_0(S_1^1)\Big]N\\
&+ \Big[a_2\prod_{i=2}^{n}(\lambda_1-\lambda_i)^{k_i}+a_1S_1^1+a_0(S_1^2+S_2^2)\Big]N^2+\ldots\\
&+ \Big[a_{k_1-1}\prod_{i=2}^{n}(\lambda_1-\lambda_i)^{k_i}+a_{k_1-2}(S_1^1)+a_{k_1-2}(S_1^2+S_2^2)+a_0(S_1^{k_1-1}+\ldots+S_{k_1-1}^{k_1-1})\Big]N^{k_1-1}
\end{aligned}
$$

Therefore, $\mathbf{p}_1(\lambda_1+N)=I_{k_1-1}$ if and only if:

$a_0\prod_{i=2}^{n}(\lambda_1-\lambda_i)^{k_i}=1$
$a_1\prod_{i=2}^{n}(\lambda_1-\lambda_i)^{k_i}+a_0(S_1^1)=0$
$a_2\prod_{i=2}^{n}(\lambda_1-\lambda_i)^{k_i}+a_1S_1^1+a_0(S_1^2+S_2^2)=0$
$\vdots$
$a_{k_1-1}\prod_{i=2}^{n}(\lambda_1-\lambda_i)^{k_i}+a_{k_1-2}(S_1^1)+a_{k_1-2}(S_1^2+S_2^2)+a_0(S_1^{k_1-1}+\ldots+S_{k_1-1}^{k_1-1})=0$

Since $\prod_{i=2}^{n}(\lambda_1-\lambda_i)^{k_i}\neq 0$ the above system of equations always has solutions for $a_0,\ldots,a_{k_1-1}$. Therefore the polynomial $\mathbf{p}_1$ is well defined.

Step 4. The polynomial $\mathbf{p}_1$ as defined in the step 3 annihilates $A_2,\ldots,A_n$

This follows from the fact that each A_i has $(z-\lambda_i)^{k_i}$ as its minimal polynomial for $i=2,\ldots,n$ [Theorem 1].

Step 5. The proof is now complete since $\mathbf{p}_1(A)=\mathbf{p}_1(A_1)\bigoplus\ldots\bigoplus\mathbf{p}_1(A_n)$ [Proposition 1].

Q.E.D

4.2.2 C*-algebra Preliminaries I: Construction of Positive Invertible Elements

In this section, we give a canonical way of constructing a positive invertible element in a C*-algebra with an identity from a *partition of unity.* We say that a set of finitely many elements is a partition of unity if their sum is the identity.

Proposition 4 (Partition of Unity: Positive Invertible Element) *Let A be a C*-algebra with an identity, 1, and the finite set $\{e_1, e_2, \ldots, e_n\}$ be a partition of unity : $e_1+e_2+\ldots+e_n = 1$. Then the element $s = e_1^*e_1+e_2^*e_2+\ldots+e_n^*e_n$ is an invertible element.*

Proof. Firstly, we take A as a norm closed *-subalgebra of bounded operators on its universal Hilbert space $\mathbf{H}$. We shall now prove the above proposition by contradiction. Suppose on the contrary that the bounded operator s is not invertible.

Step 1. Since the operator s is normal (in fact, positive), the infimum of the norm of the image of the unit sphere of the Hilbert space $\mathbf{H}$ under the operator s is zero : $\inf\{\| s(\eta) \| \,|\eta \in \mathbf{H}, \| \eta \|= 1\} = 0$. [Chapter 2.4, Lemma 2.4.8 [10]]. Therefore, there exists a sequence of vectors $(\eta_m)_{m=1}^{\infty}$, on the unit sphere of the Hilbert space, such that $\| s(\eta_m) \|$ is a null sequence:

$$\| s(\eta_m) \| \to 0 \quad as \quad m \to \infty. \tag{4.1}$$

Step 2. Then,

$$\Big(s(\eta_m), \eta_m\Big) \to 0 \quad as \quad m \to \infty \tag{4.2}$$

where $\Big(\cdot, \cdot\Big)$ denotes the inner product of the Hilbert space $\mathbf{H}$. This follows from the inequality $0 \leq |\Big(s(\eta_m), \eta_m\Big)| \leq \| s(\eta_m) \| \| \eta_m \| = \| s(\eta_m) \|$ [Cauchy Schwartz inequality]. By the squeeze play theorem and equation (2.1), $|\Big(s(\eta_m), \eta_m\Big)| = |\Big(s(\eta_m), \eta_m\Big) - 0| \to 0$ as $m \to \infty$; that is, $\Big(s(\eta_m), \eta_m\Big) \to 0$.

Step 3. But $s = \sum_{i=1}^n e_i^* e_i$. Therefore,

$$\Big(\sum_{i=1}^n e_i^* e_i(\eta_m), \eta_m\Big) = \sum_{i=1}^n \Big(e_i^* e_i(\eta_m), \eta_m\Big) = \sum_{i=1}^n \| e_i(\eta_m) \|^2 \to 0. \tag{4.3}$$

The sequence $(\sum_{i=1}^n \| e_i(\eta_m) \|^2)_{m=1}^{\infty}$ is a sequence where each term is the sum of n positive terms : $\| e_1(\eta_m) \|^2, \ldots, \| e_n(\eta_m) \|^2$. Equivalently, it is the sum of n positive valued sequences $(\| e_1(\eta_m) \|^2)_{m=1}^{\infty}, \ldots, (\| e_n(\eta_m) \|^2)_{m=1}^{\infty}$. Since it is a null sequence, each of these positive valued sequences are null: $\| e_i(\eta_m) \|^2 \to 0$ for $i = 1, \ldots, n$. Therefore, $\| e_i(\eta_m) \| \to 0$ for $i = 1, \ldots, n$.

Therefore, arguing as in step 2,

$$\Big(e_i(\eta_m), \eta_m\Big) \to 0 \text{ as } m \to \infty \text{ for each } i = 1, \ldots, n$$

This contradicts the fact that for each η_m,

$$\sum_{i=1}^{n} \Big(e_i(\eta_m), \eta_m\Big) = 1$$

since $1 = \| \ \eta_m \ \|^2 = \Big(\eta_m, \eta_m\Big) = \Big(\sum_{i=1}^{n} e_i(\eta_m), \eta_m\Big) = \sum_{i=1}^{n} \Big(e_i(\eta_m), \eta_m\Big)$ as $\{e_1, \ldots, e_n\}$ is a partition of unity.

Q.E.D

4.2.3 C*-algebra Preliminaries II: Lifting Positive Invertible Elements

Firstly, we show that the ring theoretic (algebraic) property of an invertible element cannot be lifted in the general case. We now give an easy counterexample:

Example 5 (Lifting Invertible Elements : A Counter Example) *Let A be the C*-algebra $B(\mathbf{H})$ where $\mathbf{H}$ is the infinite dimensional Hilbert space l^2, the sequence space of all square summable sequences and I the closed 2-sided ideal of all the compact operators on l^2.*

Consider the left shift operator $L : l^2 \to l^2 : (x_1, x_2, \ldots, x_n, \ldots) \mapsto (x_2, \ldots, x_n, \ldots)$ [Example 3, Chapter 1.3.2]. Since the range of L is $\mathbf{H}$ and hence trivially closed, L is a Noether operator [Chapter 2.5.3.1, Example 9]. The element $L + K(\mathbf{H})$ is therefore an invertible element of the quotient C-algebra $B(\mathbf{H})/K(\mathbf{H})$ [Chapter VII, Remark 2.6.4 [25]]*

Since the index of the operator is invariant with respect to perturbation by a compact operator [Chapter VII, Theorem 2.6.3 [25]], all the operators which belong to the coset $L + K(\mathbf{H})$ are of index 1. Hence none of them are invertible: all invertible operators have index 0 since both the operator and its adjoint are invertible.

However all is not lost: such a lifting is possible for *positive* invertible elements. We shall however need the following lemma:

Lemma 1 Lifting Commutative Property : Positive Invertible Elements. *Let A be a C*-algebra with an identity 1, I a closed 2-sided ideal in A. Let $x + I$ be a positive invertible element of A/I. Let us denote the inverse element as $y + I$. We can assume without loss of generality that x is positive. Therefore the elements $x + I, y + I$ of the quotient C*-algebra belong to the commutative C*-subalgebra C*(x + I, 1 + I) of the quotient C*-algebra generated by the positive element $x + I$ and the identity $1 + I$.*

Then there exists an element $b \in I$ such that $y - b$ belongs to the commutative C-subalgebra C*(x, 1) of the finer C*-algebra A, generated by the positive element x and the identity 1 : x and $y - b$ commute.*

Proof. Firstly, we assume without loss of generality that x is positive : since $x + I$ is a positive element of the quotient C*-algebra A/I, we can write $x + I$ as $(a + I)(a^* + I) = aa^* + I$ for some $a + I \in A/I$. If x is not positive, we take x as aa^* which is a positive element of A.

Secondly, we take the commutative C*-subalgebra, $C^*(x + I, 1 + I)$, of A/I generated by the positive element $x + I$ and the identity $1 + I$ as the C*-algebra, $C(S)$, of all continuous functions on the compact Hausdorff space $S = \sigma_{A/I}(x + I) \subset \mathbf{R}^+$ [Chapter 1.3.1, Theorem 1].

Thirdly, since the inverse element $y + I$ is a member of $C^*(x + I, 1 + I)$, there exists an $f \in C(S)$ which corresponds uniquely with $y + I : y + I = f(x + I)$. Invoking Tietze's Extension Theorem [Chapter 3, Theorem 3.2.13 [20]], we can extend $f \in C(S)$ to a continuous function $f^e \in C(K)$ where K is $\sigma_A(x)$, the spectrum in the finer C*-algebra A, of the positive element $x \in A$: S is a closed subspace of the compact metric space K. Then by the Stone Weierstrass theorem on $C(K)$, we conclude that there exists a net of polynomials $(\mathbf{p}_i)$ in $C(K)$ which converges uniformly to the continuous function $f^e \in C(K)$.

Fourthly, let $\mathbf{p}_i^r, f$ denote the restrictions of $\mathbf{p}_i, f^e \in C(K)$ to the closed subspace S, respectively. Then the uniform convergence of $\mathbf{p}_i$ in $C(K)$ to $f^e \in C(K)$ implies the uniform convergence of $\mathbf{p}_i^r$ in $C(S)$ to $f^r \in C(S)$. Consequently $y+I$ is the limit of the net of polynomial expressions in $x + I$ [Chapter 1.3.1, Theorem 1(c)]:

$$\mathbf{p}_i^r(x + I) = \mathbf{p}_i(x) + I \to y + I \tag{4.4}$$

since $\mathbf{p}_i^r$ is the same polynomial expression as $\mathbf{p}_i$. Now, the net $\mathbf{p}_i(x)$ of elements in the commutative C*-subalgebra $C^*(x, 1)$ will converge to the element $f^e(x) \in C^*(x, 1)$ [Chapter 1.3.1, Theorem 1].

Finally, $\pi(f^e(x)) = f(\pi(x)) = y + I$ [Chapter 1.3.4, Proposition 2] : there exists an element $b \in I$ such that $y - b = f^e(x)$ and $f^e(x)$ commutes with x since it is the limit of polynomial expressions in x which commute with x [Equation (4.4)].

Q.E.D

Proposition 5 (Lifting Positive Invertible Elements.) *Let A be a C*-algebra with an identity 1, I a closed 2-sided ideal in A. Let $x+I$ be a positive invertible element of A/I. Then there exists a positive invertible element a in the finer C*-algebra A such that $a \in x + I$.*

Proof.

Step 1. Lifting Positive Invertible Element : Commutative C*-algebra. We first show that the above proposition holds if A is a commutative C*-algebra. Since A is a commutative C*-algebra with an identity 1, we identify A with the C*-algebra $C(K)$. Therefore we associate with the closed ideal I, a closed set $S \subset K$ on which the functions of I vanish : $I = \{f \in C(K) \mid f|_K = 0\}$ [Example 34, Chapter 1.2.4.1]. We let $Z(f)$ denote the zero set of f which is the closed set $\{t \in K | f(t) = 0\}$.

Since $x + I$ is a positive element in A/I, there exists an $a \in A$ such that $x + I = a^*a + I$. Let f_{a^*a} denote positive valued function in $C(K)$ which corresponds to the C*-algebra element a^*a and $Z(f_{a^*a})$, the zero set of f_{a^*a} which is the closed set $\{t \in K | f_{a^*a}(t) = 0\}$.

Sub-step 1 : *$Z(f_{a^*a})$ is disjoint from S.*

The quotient C*-algebra element $a^*a + I$ is invertible if and only if there exists a function $f_{a^*a}^{-1}$ of $C(K)$ such that:

$$f_{a^*a}f_{a^*a}^{-1} - K_1 \in I \text{ if and only if } f_{a^*a}f_{a^*a}^{-1}|_S = 1$$

where K_1 is the constant 1 function and $f_{a^*a}f_{a^*a}^{-1}|_S$, the restriction of the function $f_{a^*a}f_{a^*a}^{-1}$ to the closed set S.

Consequently, the zero set of f_{a^*a}, $Z(f_{a^*a})$, is disjoint from the closed subset S.

Sub-step 2 : *The function $K_1 - \Lambda_S$ of $C(K)$ is a positive element of the ideal I. The function Λ_S is a function of $C(K)$ which behaves as a continuous approximant of the discontinuous characteristic function on S separating the disjoint closed sets S and $Z_{f_{a^*a}}$: $\Lambda_S(S) = \{1\}$ and $\Lambda_S(Z_{f_{a^*a}}) = 0$.*

Since K is a compact Hausdorff space, it is normal [Chapter 7, Theorem 7.13 [21]]. Hence, by Urysohn's Lemma for Normal Spaces [Chapter 7, Theorem 7.2, [21]], there exists a continuous approximation of the characteristic function on S: there exists a continuous function Λ_S of $C(K)$, such that $\Lambda_S(K) \subset [0, 1]$ and $\Lambda_S(S) = \{1\}$ and $\Lambda_S(Z_{f_{a^*a}}) = 0$.

Therefore, the function $K_1 - \Lambda_F$ is a *positive* valued continuous function on K which belongs to the ideal I.

Sub-step 3 : *The function $f_{a^*a} + (K_1 - \Lambda_S)$ of $C(K)$ is a strictly positive function in $C(K)$. Hence it is invertible and is mapped under the quotient map to $f_{a^*a} + I$.*

Step 2. Lifting Positive Invertible Element : General C*-algebra. If the general C*-algebra A is not commutative, we take A as the commutative C*-subalgebra $B = C^*(x, 1)$ generated by the positive element x and the identity 1. We then take as our ideal I the closed 2-sided ideal $B \bigcap I$ of the commutative C*-algebra B.

Then we can assume without loss of generality that the closed 2-sided ideal $B \bigcap I$ is not trivial since $B \cap I$ is trivial if and only if the original ideal I is trivial which makes the theorem a pathological case : by Lemma 1, we can assume that the elements $1, x, y$ are all in the C*-algebra $B = C^*(x, 1)$; in particular $1 - xy, 1 - yx \in B$; so if $B \cap I$ is trivial, $1 - xy = 0 = 1 - yx$.

Therefore by Step 1, the proof is complete.

Q.E.D

We now give another proof of Proposition 5 which is due to Professor Stroh:

Let us recall that an element a of a C*-algebra is positive if it is self adjoint and its spectrum $\sigma_A(a)$ in A is a subset of the positive real line $\mathbf{R}^{+0}$. Consequently, a is a positive *invertible* element provided that it is self-adjoint and its spectrum does not contain 0. Since the spectrum is a closed, in fact compact, subset of the positive real line, *the infimum of the spectrum,* $\inf\{\lambda \in \mathbf{R}^+ | \lambda \in \sigma_A(a)\}$ *of a positive invertible element* $a \in A$ *is always strictly positive.*

The following proof relies on shifting the spectrum to stay clear of 0.

(i) *There exists a strictly positive real number* $c \in \mathbf{R}^+$ *such that the element* $x + I - c1_{A/I}$ *of the quotient C*-algebra* A/I *is a positive element.*

Set c to be the infimum of the spectrum of the positive invertible element $x + I$. It suffices to show that the spectrum of $x + I - c1_{A/I}$ is a subset of $\mathbf{R}^+ = \mathbf{R}^{+\mathbf{0}} \setminus \{0\}$ since $x + I - c1_{A/I}$ is evidently self-adjoint.

Now $\sigma_{A/I}(x+I-c1_{A/I}) = \{\lambda - c | \lambda \in \sigma_{A/I}(x+I)\}$ [Spectral Mapping Theorem, Chapter 3, Proposition 3.2.10 [10]] since $x+I-c1_{A/I} = \mathbf{p}(x+I)$ where $\mathbf{p}$ is the single variable polynomial $\mathbf{p}(z) = z - c$. The desired result now follows from the definition of c.

(ii) Write $x + I$ *as the sum* $c1_{A/I} + B$ *for some positive element* $B \in A/I$. *There is a positive element* $b \in A$ *such that* $\pi(b) = B$. *The element* $a = c1_A + b$ *is a positive invertible element of* A *such that* $\pi(a) = x + I$.

From (i), $x + I - c1_{A/I} \geq 0$: hence $x + I = c_{1_{A/I}} + B$ for some positive element $B \in A/I$. Then there is a positive element $b \in A$ such that $\pi(b) = B$ [Chapter 1.3.4, Proposition 1] and by another application of the Spectral Mapping Theorem, $a = c1_A + b$ is positive. Evidently, $\pi(a) = x + I$.

Q.E.D

4.2.4 Lifting Polynomially Ideal Elements : A Criteria

Recall that we stated in the beginning of this section without proof that the lifting of polynomially ideal elements from the quotient C*-algebra A/I onto the finer C*-algebra A occurs exactly when *any* finite orthogonal family of projections (self adjoint idempotents) can be lifted from A/I onto A as a finite orthogonal family of projections in A. We called this criteria the Finite Orthogonal Projection Criteria. We shall prove this criteria in this subsection.

We now define the minimal polynomial of an *algebraic* element of a C*-algebra.

Definition 6 (Minimal Polynomial of Algebraic Element) *Suppose x is an algebraic element of the C*-algebra A. Then there exists a polynomial $\mathbf{q}$ over the complex field such that $\mathbf{q}(x) = 0$. Let $\mathbf{p}$ denote the monic polynomial of the smallest degree of the non empty set of all polynomials which annihilate x: the set contains at least $\mathbf{q}$. We shall call this the minimal polynomial of x.*

Proposition 6 *Let x be an algebraic element. Let $\mathbf{q}$ be a polynomial over the complex number field. Then $\mathbf{q}(x) = 0$ if and only if the minimal polynomial of x is a divisor of the polynomial $\mathbf{q}$.*

Proof. Identical as for Chapter 4.2.1, Theorem 1.

Q.E.D.

Theorem 2 (Finite Orthogonal Projection Criteria) *(Theorem 2 [7]) Let A denote a C*-algebra, I a closed 2-sided ideal in A, A/I the quotient C*-algebra and $\pi : A \to A/I$ the quotient map.Suppose B is an algebraic element of the quotient C*-algebra A/I. Let $\mathbf{p}$ denote the minimal polynomial of B and let it have n distinct complex roots.*

Then there exists an orthogonal family $\mathcal{F} = \{P_1, \ldots, P_n\}$ of n nonzero projections in A/I whose sum is the identity $1_{A/I}$ such that T.F.A.E.:

(i) There is an element b in the finer C-algebra A with $\mathbf{p}(b) = 0$ and $\pi(b) = B$.*

(ii) There is an orthogonal family $\{p_1, \ldots, p_n\}$ of n nonzero projections in A such that $\pi(p_i) = P_i$ for $1 \leq i \leq n$.

Proof. Firstly, write the minimal polynomial $\mathbf{p}$ of B as a product of primes in $\mathbf{C}[X] : \mathbf{p}(z) = (z - \lambda_1)^{k_1}(z - \lambda_2)^{k_2} \cdots (z - \lambda_n)^{k_n}$, with $\lambda_i \neq \lambda_j$ if $i \neq j$. Then let x denote the matrix which has $\mathbf{p}$ as its minimal polynomial as constructed in the proof of Chapter 4.2.1, Proposition 2.

Step 1. There exists a decomposition of the unit $1_{A/I}$ into n 'orthogonal' idempotents $E_1, \ldots, E_n$ of the quotient C*-algebra A/I : $E_1 + \ldots + E_n = 1_{A/I}$ and $E_iE_j = \delta_{ij}E_j$. Further, $E_i(B - \lambda_i)^{k_i}E_i = 0$ for $1 \leq i, j \leq n$.

We prove this over two sub-steps by first showing that the above properties holds in an algebra (over the field $\mathbf{C}$) that is *ring isomorphic* to a subalgebra of A/I: the above properties are defined purely in terms of the ring operations so that they will carry over under a ring isomorphism.

Sub-step 1. *Let $\mathcal{S} \subset A/I$ be the sub-ring of polynomials in the algebraic element $B \in A/I$. Then $\mathcal{S}$ is ring isomorphic to the ring $\mathcal{T}$ of polynomials in the matrix x, which shares the same minimum polynomial as B .*

Proof. Firstly, consider the C*-algebra A/I with the identity $1_{A/I}$ as an unitary over-ring of the complex number field $\mathbf{C}$, taken as a ring : identify $\mathbf{C}$ with the set $\mathbf{C}1_{A/I}$. Then the evaluation at B function,

$$\sigma_B : \mathbf{C}[X] \to \mathcal{S} \subset A/I | \mathbf{q}(X) \mapsto \mathbf{q}(B)$$

is a well-defined onto ring homomorphism from $\mathbf{C}[X] \to S$ [Lemma 16.1 [28]]. Hence, by the Fundamental Homomorphism Theorem,

$$S \text{ is } \textit{ring isomorphic} \text{ to } \mathbf{C}[X]/Ker(\sigma_B).$$

Since $\mathbf{C}[X]$ is a principal ideal domain [Theorem 16.4, [28]], the ideal $Ker(\sigma_B)$ is generated by a single element. By proposition 6, that element is $\mathbf{p}$, the minimal polynomial of B. Hence,

$$\mathcal{S} \text{ is ring isomorphic to } \mathbf{C}[X]/ < \mathbf{p} >$$

where $< \mathbf{p} >$ is the principal ideal generated by the polynomial $\mathbf{p}$.

Secondly, the quotient ring $\mathbf{C}[X]/ < \mathbf{p} >$ consists of elements of the form $\mathbf{r}+ < \mathbf{p} >$ where $\mathbf{r}$ is a polynomial of degree up to one less the degree of $\mathbf{p}$. The map

$$\Phi : \mathbf{C}[X]/ < \mathbf{p} > \quad \to \quad \mathcal{T} \quad | \quad \mathbf{r}+ < \mathbf{p} > \mapsto \mathbf{r}(x)$$

is an onto *ring isomorphism* : $\mathbf{r}_1(x) - \mathbf{r}_2(x) = 0$ if and only if $\mathbf{r}_1 - \mathbf{r}_2$ is a multiple of $\mathbf{p}$; that is, $\mathbf{r}_1+ < \mathbf{p} >= \mathbf{r}_2+ < \mathbf{p} >$.

NOTE : We can assume without loss of generality that B is not in $\mathbf{C} = \mathbf{C}1_{A/I}$: $B \in \mathbf{C}$ is the the trivial case of the theorem since we can always lift the identity element which has $\mathbf{p}(z) = z - 1$ as its minimal polynomial.

Q.E.D.

Sub-step 2. *There exists a decomposition of the identity matrix* $1 \in \mathcal{T}$ *into* n *'orthogonal' idempotents* $E_1, \ldots, E_n$ *in the algebra* $\mathcal{T}$ *of polynomials in the matrix* x *:* $E_1 + \ldots + E_n = 1$ *and* $E_i E_j = \delta_{ij} E_j$*.*

Proof. The matrix x is the direct sum matrix $A_1 \bigoplus A_2 \bigoplus \cdots \bigoplus A_n$, where A_i is the $k_i \times k_i$ Jordan Block with diagonal entry equal to λ_i for $i = 1, \ldots, n$ since $\mathbf{p}(z) = (z - \lambda_1)^{k_1}(z - \lambda_2)^{k_2} \cdots (z - \lambda_n)^{k_n}$, with $\lambda_i \neq \lambda_j$ if $i \neq j$ [Chapter 4.2.1, Proposition 2]. Consequently, for each i, there exists a polynomial $\mathbf{p}_i$ such that $\mathbf{p}_i(x)$ is the direct sum matrix $0 \bigoplus \ldots 0 \bigoplus I_{k_i} \bigoplus 0 \ldots \bigoplus 0$ where 0 is the 0-matrix and I_{k_i} is the identity $k_i \times k_i$ matrix. We let E_i denote the idempotent matrix $\mathbf{p}_i(x)$ which we identify as $\mathbf{p}_i(B)$ under the ring isomorphism σ_B which identifies x with B [Chapter 4.2.1, Proposition 3].

Evidently $E_1 + \ldots + E_n = 1$, $E_i E_j = \delta_{ij} E_j$ and $E_i(x - \lambda_i)^{k_i} E_i = 0$ for $1 \leq i, j \leq n$ [Chapter 4.2.1, Proposition 1]. This completes the proof.

Q.E.D

Although each E_i in the ring $\mathcal{T}$ is a self adjoint idempotent, the corresponding element in the isomorphic ring $\mathcal{S} \subset A/I$, identified under the ring isomorphism σ_B and also ambiguously denoted E_i, need not be self adjoint : the map σ_B is only a ring isomorphism. We nevertheless construct from the E_i's in the quotient C*-algebra A/I, self adjoint idempotents, that is, projections which partition the identity and are mutually orthogonal.

Step 2. There exists a decomposition of the unit $1_{A/I}$ into n orthogonal projections $P_1, \ldots, P_n$ in the quotient C*-algebra A/I : $P_1 + \ldots + P_n = 1_{A/I}$ and $P_i P_j = \delta_{ij} P_j$.

Proof. Define the positive invertible element element S in A/I from the partition of unity $\{E_1, \ldots, E_n\}$ as follows: $S = E_1^* E_1 + \ldots + E_n^* E_n$ [Chapter 4.2.2, Proposition 4]. Its inverse S^{-1} is also a positive invertible element: S^{-1} is self adjoint and that the spectrum $\sigma(S^{-1})$ of S^{-1} consists of the inverse of spectral values of the spectrum of S [Chapter 3, Proposition 3.2.10 [10]]. We then have $\sqrt{S^{-1}} = (\sqrt{S})^{-1}$ [Chapter 1.3.1, Corollary 3].

Defining $P_i = \sqrt{S} E_i (\sqrt{S})^{-1}$ for each i, completes the proof of Step 2:

(i) Each P_i is self-adjoint: First note that $SE_i = (E_1^* E_1 + \ldots + E_n^* E_n) E_i$ is the positive element $E_i^* E_i$. Hence, taking the adjoints,

$$SE_i = E_i^* S.$$

Then, pre- and post- multiplying each term of the equation by $(\sqrt{S})^{-1}$, we have:

$$\sqrt{S} E_i (\sqrt{S})^{-1} = (\sqrt{S})^{-1} E_i^* \sqrt{S} = \left(\sqrt{S} E_i (\sqrt{S})^{-1}\right)^*$$

(ii) Each P_i is idempotent: $P_i^2 = \sqrt{S}E_i(\sqrt{S})^{-1}\sqrt{S}E_i(\sqrt{S})^{-1} = \sqrt{S}E_iE_i(\sqrt{S})^{-1} = P_i$.

(iii)If $i \neq j$ then $P_iP_j = 0$: Follows from the fact $E_iE_j = 0$.

Since $E_1 + \ldots E_n = 1$, first pre- , then followed by post multiplication by $\sqrt{S}$ and $(\sqrt{S})^{-1}$ respectively, shows that the P_i's sum up to 1.

Q.E.D

Step 3. The lifting problem for the algebraic element $B \in A/I$ is equivalent to the lifting problem of the element $\sqrt{S}B(\sqrt{S})^{-1} \in A/I$.

Firstly, $\mathbf{p} \in \mathbf{C}[X]$ is a minimum polynomial of the algebraic element $B \in A/I$ if and only if $\mathbf{p}$ is a minimum polynomial of the element $\sqrt{S}B(\sqrt{S})^{-1} \in A/I$: this follows from the fact that $\mathbf{p}(\sqrt{S}B(\sqrt{S})^{-1}) = \sqrt{S}\mathbf{p}(B)(\sqrt{S})^{-1}$ since $\mathbf{p}$ is a polynomial expression and that $\sqrt{S}$ is invertible.

Secondly, since $S \in A/I$ is a positive invertible element we can lift it from the quotient C*-algebra A/I onto the finer C*-algebra A: there exists a positive invertible element s in A such that $\pi(s) = S$ [Chapter 4.2.3, Proposition 5]. Its inverse s^{-1} is also a positive invertible element. We then have $\sqrt{s^{-1}} = (\sqrt{s})^{-1}$ [Chapter 1.3.1, Corollary 3].

Thirdly, for the self adjoint element $s \in A$, we have $\pi(\sqrt{s}) = \sqrt{S}$, $\pi(s^{-1}) = S^{-1}$ and $\pi((\sqrt{s})^{-1}) = (\sqrt{S})^{-1}$ [Chapter 1.3.4, Proposition 2].

Therefore, the condition that there is an element b in the finer C*-algebra A with $\mathbf{p}(b) = 0$ and $\pi(b) = B$ is equivalent to the condition that there is an element c in the finer C*-algebra A such that $c = \sqrt{s}b(\sqrt{s})^{-1}$ where $\mathbf{p}(c) = 0$ and $\pi(c) = \sqrt{S}B(\sqrt{S})^{-1}$.

Hence, the lifting problem of the algebraic element $B \in A/I$ is equivalent to the lifting problem of the algebraic element $\sqrt{S}B(\sqrt{S})^{-1} \in A/I$, which we shall now pursue.

Step 4. The lifting of the algebraic element $\sqrt{S}B(\sqrt{S})^{-1} \in A/I$ forces the lifting of the orthogonal family of projections $\{P_1, \ldots, P_n\} \subset A/I$ as constructed in Step 2.

Let us assume that the algebraic element $\sqrt{S}B(\sqrt{S})^{-1} \in A/I$ with $\mathbf{p}$ as its minimum polynomial, can be lifted. Then there is an element c in the finer C*-algebra A with $\mathbf{p}(c) = 0$ and $\pi(c) = \sqrt{S}B(\sqrt{S})^{-1}$. *Note that* $\mathbf{p}$ *is also the minimal polynomial for* c : suppose not; then there exists a polynomial $\mathbf{q}$ of strictly smaller degree than $\mathbf{p}$ such that $\mathbf{q}(c) = 0$; since $\pi(\mathbf{q}(c)) = \mathbf{q}(\pi(c))$ for $\mathbf{q}$ is a polynomial, it follows that $\mathbf{q}(\sqrt{S}B(\sqrt{S})^{-1}) = 0$, contradicting the fact that $\mathbf{p}$ is the minimal polynomial of B.

Let the element c of the finer C*-algebra play the role of the algebraic element B of the quotient C*-algebra A/I in Step 1: both have exactly the same minimum polynomial $\mathbf{p}$. Consequently, we construct a partition of unity of the identity $1 \in A$, $\{e_1, \ldots, e_n\}$, where $e_i = \mathbf{p_i}(c)$ for $i = 1, \ldots, n$ with the polynomials $\mathbf{p}_i$ defined as in Step 1.

We then construct *orthogonal projections* p_i out of each e_i for $i = 1, \ldots, n$ by defining p_i as $\sqrt{w}e_i(\sqrt{w})^{-1}$ where $w = e_1^*e_1 + \ldots + e_n^*e_n$ [Step 2]. We then obtain the desired orthogonal family $\{p_1, \ldots, p_n\}$ of n nonzero projections in A such that $\pi(p_i) = P_i$ for $1 \leq i \leq n$:

(i) Firstly, $\pi(e_i) = P_i$ for $1 \leq i \leq n$:

$$\begin{aligned} \pi(e_i) &= \pi(\mathbf{p}_i(c)) = \mathbf{p}_i(\pi(c)) \\ &= \mathbf{p}_i(\sqrt{S}B(\sqrt{S})^{-1}) = \sqrt{S}\mathbf{p}_i(B)(\sqrt{S})^{-1} \\ &= \sqrt{S}E_i(\sqrt{S})^{-1} = P_i \end{aligned}$$

since $\mathbf{p}_i$ is a polynomial for each $i = 1, \ldots, n$.

(ii)Secondly, $\pi(w) = 1$ since $\pi(e_i^*e_i) = P_i$ for each $i = 1, \ldots, n$ and $\{P_i, \ldots, P_n\}$ is a partition of unity: since $c = \sqrt{s}b(\sqrt{s})^{-1}$,

$$\begin{aligned} e_i &= \mathbf{p}_i(c) = \sqrt{s}\mathbf{p}_i(b)(\sqrt{s})^{-1} \\ e_i^* &= \mathbf{p}_i(c) = \sqrt{s}[\mathbf{p}_i(b)]^*(\sqrt{s})^{-1} \end{aligned}$$

so that with $\mathbf{p}_i(\pi(b)) = \mathbf{p}_i(B) = E_i$ and $\pi\Big([\mathbf{p}_i(b)]^*\Big) = [\mathbf{p}_i(\pi(b))]^* = E_i^*$

$$\begin{aligned} e_i^*e_i &= \mathbf{p}_i(c) = \sqrt{s}[\mathbf{p}_i(b)]^*\mathbf{p}_i(b)(\sqrt{s})^{-1}\pi(\mathbf{p}_i(c)) \\ \pi(e_i^*e_i) &= \sqrt{S}(E_i)^*E_i(\sqrt{S})^{-1} \\ &= \sqrt{S}(E_i)^*(\sqrt{S})^{-1}\sqrt{S}E_i(\sqrt{S})^{-1} \\ &= P_i^*P_i = P_i^2 = P_i \end{aligned}$$

(iii) Finally, $\pi(p_i) = \sqrt{\pi(w)}P_i(\sqrt{\pi(w)})^{-1} = P_i$ for $1 \leq i \leq n$

Step 5. Conversely, suppose the orthogonal projections $\{P_1, \ldots, P_n\} \subset A/I$ as constructed in Step 2 can be lifted to an orthogonal family of projections $\{p_1, \ldots, p_n\}$ in A. Then we can lift the algebraic element $\sqrt{S}B(\sqrt{S})^{-1} \in A/I$.

We prove this over the following five sub-steps:

Sub-step 1. *We assume without loss of generality that the orthogonal family of projections $\{p_1, \ldots, p_n\}$ in A sum to 1.*

If the family of orthogonal projections $\{p_1, \ldots, p_n\}$ do not sum up to the identity, we can replace p_1 by the orthogonal projection $1 - (p_2 + \ldots + p_n)$: since the projections $p_2, \ldots, p_n$ are orthogonal, $1 - (p_2 + \ldots + p_n)$ is an orthogonal projection which is orthogonal to $p_2, \ldots, p_n$ [Corollary 2.5.4, 2.5.5. [10]].

Sub-step 2. *We can assume without loss of generality that the element $c \in A$ such that $\pi(c) = \sqrt{S}B(\sqrt{S})^{-1} \in A/I$ commutes with each of the orthogonal projections p_i for $1 \leq i \leq n$ and that $\pi(p_1cp_1 + \ldots + p_ncp_n) = \sqrt{S}B(\sqrt{S})^{-1} \in A/I$*

Lemma 2 *Let c' denote any element in A such that $\pi(c') = \sqrt{S}B(\sqrt{S})^{-1}$. Then $\pi(p_ic'p_j) = 0$ if $i \neq j$.*

Proof A straightforward computation shows that:

$$\begin{aligned} P_i\sqrt{S}B(\sqrt{S})^{-1}P_j &= \sqrt{S}E_i(\sqrt{S})^{-1}\sqrt{S}B(\sqrt{S})^{-1}\sqrt{S}E_j(\sqrt{S})^{-1} \\ &= \sqrt{S}E_iBE_j(\sqrt{S})^{-1} \end{aligned}$$

Since the element $B \in A/I$ can be identified with the direct sum matrix $x = A_1 \bigoplus \ldots \bigoplus A_n$, the idempotents E_i and E_j of A/I with the direct sum matrices $0 \bigoplus \ldots 0 \bigoplus I_{k_i} \bigoplus 0 \bigoplus \ldots \bigoplus 0$ and $0 \bigoplus \ldots 0 \bigoplus I_{k_j} \bigoplus 0 \bigoplus \ldots \bigoplus 0$ [Chapter 4.2.1, Proposition 3], respectively *up to ring isomorphism*, it follows that $E_iBE_j = 0$ if $i \neq j$ [Chapter 4.2.1, Proposition 1].

Q.E.D

Let c' be any element in A such that $\pi(c') = \sqrt{S}B(\sqrt{S})^{-1}$. Since the p_i's sum to 1, $c' = (p_1 + \ldots + p_n)c'(p_1 + \ldots + p_n)$.

Now, by Lemma 2,

$$\sqrt{S}B(\sqrt{S})^{-1} = \pi(c') = \pi(p_1c'p_1 + \ldots + p_nc'p_n)$$

Therefore, if we set c as $p_1c'p_1 + \ldots + p_nc'p_n$, then $\pi(c) = \sqrt{S}B(\sqrt{S})^{-1}$ and c will commute with each of the p_i's : $cp_i = p_ic = p_icp_i$ for all $1 \leq i \leq n$. Further, $p_1cp_1 + \ldots + p_ncp_n = p_1c'p_1 + \ldots + p_nc'p_n = c$

Sub-step 3. *For each $1 \leq i \leq n$, p_iAp_i is a C*-algebra with p_iIp_i as a closed 2-sided ideal. Now recall that we defined the minimal polynomial $\mathbf{p}$ of B as $\mathbf{p}(z) = (z-\lambda_1)^{k_1}(z-\lambda_2)^{k_2}\cdots(z-\lambda_n)^{k_n}$, with $\lambda_i \neq \lambda_j$ if $i \neq j$. There exists an element $c_i \in p_iAp_i$ such that $p_i(c_i-\lambda_i)^{k_i}p_i = 0$ and $\pi(c_i) = \pi(p_icp_i)$for $1 \leq i \leq n$. We can rewrite the equation $p_i(c_i-\lambda_i)^{k_i}p_i = 0$ as $\mathbf{q}_i(c_i) = 0$ where $\mathbf{q}_i \in \mathbf{C}[X]$ is the polynomial $\mathbf{q}_i(z) = (z-\lambda_i)^{k_i} = 0$ for $i = 1, \ldots, n$.*

(α) *Let us first establish that p_iAp_i is a C*-algebra with p_iIp_i as a closed 2-sided ideal for each $1 \leq i \leq n$.*

(i) *p_iAp_i is norm-closed.* Let $(p_ix_np_i)_{n=1}^{\infty}$ be a sequence in p_iAp_i which converges in the norm topology to L. We show $L \in p_iAp_i$:

By the joint continuity of multiplication and idempotency of the fixed element p_i,

$$p_i(p_ix_np_i)p_i = (p_ix_np_i) \to p_iLp_i$$

Since limits are unique, $L = p_iLp_i \in p_iAp_i$.

(ii) *p_iAp_i is *-subalgebra.* Trivially,p_iAp_i is closed under addition; it is closed under multiplication since $p_i^2 = p_i$ and is *-closed since $p_i^* = p_i$.

Therefore, p_iAp_i is a norm closed *-subalgebra of A and hence a C*-algebra.

(iii) *p_iIp_i is norm-closed.* This follows from the argument used in (i) and the fact that I is closed.

(iv) *p_iIp_i is a 2 -sided ideal of p_iAp_i.* We first show p_iIp_i is right multiplication closed: let t be any element of the ideal I; then since I is an ideal:

$$(p_itp_i)(p_iap_i) = p_i(tp_ip_ia)p_i = p_it'p_i$$

where $t' = (tp_ip_ia) \in I$. Closure under left multiplication is proved identically. Since I is a subalgebra, p_iIp_i is a subalgebra of p_iAp_i by a similar argument as in (ii).

(β) *There exists an element $c_i \in p_iAp_i$ such that $p_i(c_i-\lambda_i)^{k_i}p_i = 0$ and $\pi(c_i) = \pi(p_icp_i)$*

(**i**) Firstly, $p_i(c-\lambda_i)^{k_i}p_i = \Big(p_i(c-\lambda_i)p_i\Big)^{k_i}$ and $P_i(c-\lambda_i)^{k_i}P_i = \Big(P_i(c-\lambda_i)P_i\Big)^{k_i}$.

The element $c-\lambda_i$ commutes with each of the p_i for $1 \leq i \leq n$ since c commutes with each p_i for $1 \leq i \leq n$. For notational convenience, let B' denote the element $\sqrt{S}B(\sqrt{S})^{-1}$. Then, $\pi(c-\lambda_i) = B'-\lambda_i$ commutes with $\pi(p_i) = P_i$.

Therefore:

$$\Big(p_i(c-\lambda_i)p_i\Big)^{k_i} = p_i(c-\lambda_i)^{k_i}p_i. \tag{4.5}$$

and,

$$\Big(P_i(B'-\lambda_i)P_i\Big)^{k_i} = P_i(B'-\lambda_i)^{k_i}P_i. \tag{4.6}$$

(ii) Secondly, $P_i(B'-\lambda_i)P_i$ is nilpotent in A/I where k_i is the degree of nilpotency by the identification of B' [1] with the direct sum matrix x of Step 1, sub-step 2:

$$\Big(P_i(B'-\lambda_i)P_i\Big)^{k_i} = P_i(B'-\lambda_i)^{k_i}P_i = 0 \tag{4.7}$$

Applying Step 1 to B', we conclude $E_i(B'-\lambda_i)^{k_i}E_i = 0$. Therefore:

$$\sqrt{S}E_i(\sqrt{S})^{-1}\sqrt{S}(B'-\lambda_i)^{k_i}(\sqrt{S})^{-1}\sqrt{S}E_i(\sqrt{S})^{-1} = P_i(B'-\lambda_i)^{k_i}P_i = 0$$

(iii) Thirdly, we take the C*-algebra A as p_iAp_i and the closed 2-sided ideal I as p_iIp_i with respect to the lifting of the nilpotent element $P_i(B'-\lambda_i)P_i \in A/I$. Consequently, p_i1p_i is the identity of the C*-algebra p_iAp_i where 1 denotes the identity of A.

The element $p_i(c-\lambda_i)p_i \in p_iAp_i$ and $\pi(\Big(p_i(c-\lambda_i)p_i\Big)^{k_i}) = \pi(p_i(c-\lambda_i)^{k_i}p_i) = P_i(B'-\lambda_i)^{k_i}P_i$ which is the zero element of A/I [equation (4.7)]. Therefore, $p_i(c-\lambda_i)^{k_i}p_i \in I$ so that $p_i(c-\lambda_i)^{k_i}p_i \in p_iIp_i$ since p_i is idempotent. Hence, $p_i(c-\lambda_i)^{k_i}p_i + p_iIp_i = \Big(p_i(c-\lambda_i)p_i\Big)^{k_i} + p_iIp_i = 0$ [Equation (4.5)]

(iv) Finally, we can perturb the element $p_i(c-\lambda_i)p_i$ by the ideal element $p_itp_i \in p_iIp_i$ such that

$$\Big(p_i(c-\lambda_i)p_i\Big) - p_itp_i = \Big(p_i((c-t)-\lambda_i)p_i\Big)$$

is a nilpotent element of the C*-algebra p_iAp_i, with degree of nilpotency k_i [Chapter 3.5, Theorem 7]. Denoting $\Big(p_i((c-t)-\lambda_i)p_i\Big)$ as c_i' we conclude that $(c_i')^{k_i} = 0$ and $c_i' \in p_iAp_i$. Hence:

$$\begin{aligned} p_i(c_i')^{k_i}p_i &= 0 && (4.8)\\ \pi(c_i') &= \pi\Big(p_i(c-\lambda_i)p_i\Big) && (4.9)\end{aligned}$$

[1] B' also has $\mathbf{p}$ as its minimal polynomial

Then the desired c_i is defined as $c_i' + \lambda_i$:

$$p_i(c_i - \lambda_i)^{k_i} p_i = 0 \tag{4.10}$$

$$\pi(c_i) = \pi(c_i' + \lambda_i) = \pi(p_i c p_i) \tag{4.11}$$

where (4.10) and (4.11) are direct consequences of (4.8) and (4.9), respectively.

Finally, since we have identified the identity as $p_i 1 p_i$ and $c \in p_i A p_i$ can be rewritten as $p_i c p_i$, equations (4.5) and (4.10) allow us to identify equation (4.10) as $\mathbf{q}_i(c_i) = 0$ where $\mathbf{q}_i \in \mathbf{C}[X]$ is the polynomial $\mathbf{q}_i(z) = (z - \lambda_i)^{k_i} = 0$.

Sub-step 4. *The family $\{c_i | 1 \leq i \leq n\}$ is a family of mutually "orthogonal" elements of the C^*-algebra A: $c_i c_j = 0$ if $i \neq j$. Consequently, for any polynomial $\mathbf{p} \in \mathbf{C}[X]$, $\mathbf{p}(c) = \mathbf{p}(c_1) + \ldots \mathbf{p}(c_n)$.*

Firstly, the "mutual" orthogonality of the c_i's follows immediately from the mutual orthogonality of the family of projections $\{p_1, \ldots, p_n\}$. Secondly, $\mathbf{p}(c) = \mathbf{p}(c_1) + \ldots \mathbf{p}(c_n)$ follows from the observation that $(c_1 + \ldots + c_n)^n = c_1^n + \ldots + c_n^n$ for all $n \in \mathbf{N}$.

Sub-step 5. *By the notation of Chapter 4.2.4, Theorem 2, the lifting of the element $\sqrt{S}B(\sqrt{S})^{-1} \in A/I$ occurs when we define the element b of the C^*-algebra A as $b = c_1 + \ldots + c_n$.*

(i) $\pi(b) = \sqrt{S}B(\sqrt{S})^{-1}$.

Writing the element $c \in A$ as defined in Sub-step 2 as $(p_1 + \ldots + p_n)c(p_1 + \ldots + p_n)$, since $\pi(c_i) = \pi(p_i c p_i)$ for $1 \leq i \leq n$, $\pi(b) = \sqrt{S}B(\sqrt{S})^{-1}$.

(ii) $\mathbf{p}(b) = 0$.

Recall that the minimum polynomial $\mathbf{p}(z) = (z - \lambda_1)^{k_1}(z - \lambda_2)^{k_2} \cdots (z - \lambda_n)^{k_n}$. Then $\mathbf{p}(b) = \mathbf{p}(c_1) + \ldots \mathbf{p}(c_n) = 0 + \ldots + 0 = 0$ by sub-step 4 and sub-step 3.

This completes the proof of Step 5 and hence the equivalence.

Q.E.D

Appendix A

Computing Double Centralizers

A.1 The Hilbert Tensor Product $\mathbf{H} \otimes_h \overline{\mathbf{H}} = K(\mathbf{H})$

We wish to compute the double centralizer algebra of the C*-algebra of all compact operators on an infinite dimensional Hilbert space $\mathbf{H}$. To aid this computation, we firstly show that the C*-algebra of all compact operators, $K(\mathbf{H})$, on a Hilbert space $\mathbf{H}$ is exactly the Hilbert tensor product $\mathbf{H} \bigotimes_h \overline{\mathbf{H}}$.

The elements of the *Hilbert* tensor product $\mathbf{H} \bigotimes_h \overline{\mathbf{H}}$ of the Hilbert spaces $\mathbf{H}$ and $\overline{\mathbf{H}}$ are complex valued functions whose domain is the cartesian product $\mathbf{H} \times \mathbf{H}$. More precisely, the elements are the *square summable* bilinear functionals or forms on the cartesian product $\mathbf{H} \times \mathbf{H}$ and they do form a Hilbert space.

Definition 1 (Square Summable Bilinear Forms) *(Chapter 2, Proposition 2.6.2 [10]) A square summable bounded bilinear functional or form, $\beta(\cdot|\cdot)$, is a bounded bilinear form on the cartesian product $\mathbf{H} \times \mathbf{H}$ such that*

$$\sum_{e_i \in Y_1} \sum_{e'_j \in Y_2} |\beta(e_i|e'_j)|^2 \ < \ \infty$$

where Y_1, Y_2 are orthonormal basis for $\mathbf{H}$.

Since we are working with bilinear forms on the cartesian product $\mathbf{H} \times \mathbf{H}$, we introduce the concept of a conjugate Hilbert space to convert sesqilinear forms into bilinear forms.

Definition 2 (Conjugate Hilbert Space) *Let $\mathbf{H}$ be a Hilbert space. The conjugate Hilbert space $\overline{\mathbf{H}}$ is the same set $\mathbf{H}$, with the same vector addition except that the scalar multiplication $\bar{\cdot}$ is defined as $\lambda \bar{\cdot} x = \overline{\lambda} \cdot x$ where $\cdot$ is the scalar multiplication in $\mathbf{H}$ and the inner product is the conjugate of the inner product of $\mathbf{H}$.*

Example 1 *Let* $\mathbf{H}$ *be the two-dimensional Hilbert space* $\mathbf{C}^2$ *over the field* $\mathbf{C}$*. Then we define the inner product* $(\mathbf{x} \mid \mathbf{y})$ *of the two vectors* $\mathbf{x} = (x_1, x_2)$ *and* $\mathbf{y} = (y_1, y_2)$ *as the complex number* $x_1\overline{y_1} + x_2\overline{y_2}$ *and the scalar multiplication* $\lambda\mathbf{x}$ *as* $(\lambda x_1, \lambda x_2)$*. Consequently, in* $\overline{\mathbf{H}}$*, the inner product of* $\mathbf{x}$ *and* $\mathbf{y}$ *will be* $\overline{x_1}y_1 + \overline{x_2}y_2$ *and* $\lambda\mathbf{x} = (\bar{\lambda}x, \bar{\lambda}x_2)$*.*

Proposition 1 *If* $\beta(\cdot|\cdot)$ *is a bounded sesqilinear form the cartesian product* $\mathbf{H} \times \mathbf{H}$*, then* $\beta(\cdot|\cdot)$ *is a bounded bilinear form or functional on* $\mathbf{H} \times \overline{\mathbf{H}}$ *which as a set is identical to* $\mathbf{H} \times \mathbf{H}$*.*

The above proposition enables us to talk about bounded bilinear forms on $\mathbf{H} \times \mathbf{H}$ in terms of bounded operators on the Hilbert space $\mathbf{H}$.

Proposition 2 (Bounded bilinear forms on $\mathbf{H} \times \mathbf{H}$ are bounded operators on the Hilbert space $\mathbf{H}$.) *The correspondence,* Ψ*, defined by* $\beta(u|v) = (Tu|v)$ *where* $(\cdot|\cdot)$ *is the inner product of the Hilbert space* $\mathbf{H}$ *is an isometric isomorphism between the normed space* $B(\mathbf{H})$ *of all the bounded linear operators* $T : \mathbf{H} \to \mathbf{H}$ *and the normed space* $B(\mathbf{H} \times \mathbf{H})$ *of all the bounded sesquilinear forms* $\beta(u|v)$ *on* $\mathbf{H} \times \mathbf{H}$ *[Chapter 6, Proposition 2.4.3 [25]].*

The correspondence Ψ *also defines an isometric isomorphism between the space* $B(\mathbf{H})$ *of all the bounded linear operators* $T : \overline{\mathbf{H}} \to \mathbf{H}$ *and the space* $B(\overline{\mathbf{H}} \times \overline{\mathbf{H}})$ *of all the bounded bilinear forms* $\beta(u|v)$ *on* $\overline{\mathbf{H}} \times \overline{\mathbf{H}}$*.*

In this section we shall establish the isometric normed space isomorphism between the *square summable* bilinear forms, $\mathbf{H} \bigotimes_h \overline{\mathbf{H}}$, on $\mathbf{H} \times \mathbf{H}$ and the bounded *compact* operators,$K(\mathbf{H})$, on the Hilbert space $\mathbf{H}$. We shall first introduce new terminology:

Definition 3 (Hilbert Schmidt Operator) *A bounded operator* $T : \overline{\mathbf{H}} \to \mathbf{H}$ *is called a Hilbert-Schmidt operator from* $\overline{\mathbf{H}}$ *into* $\mathbf{H}$ *if and only if the uniquely associated bounded bilinear form* $\beta_T(u|v) = (Tu|v)$ *is a square summable bounded bilinear form.*

Our goal is to show that Hilbert Schmidt Operators are precisely the compact operators.

Definition 4 (Rank One Operator) *A rank one operator is an operator whose range is a one dimensional subspace.*

These are the simplest non zero operators since $T \neq 0 \Rightarrow \exists x|Tx = z \neq 0$ *and the vectors* $\{\lambda z\}$ *form a one-dimensional subspace of* $Ran(T)$*.*

There are elements of $\mathbf{H} \bigotimes_h \overline{\mathbf{H}}$ which we shall call *simple tensors* and denote as $x \otimes y$ where $x, y \in \mathbf{H}$ such that:

(i) The closed linear span of all the simple tensors is the Hilbert tensor product $\mathbf{H} \bigotimes_h \overline{\mathbf{H}}$ of all the square summable bilinear forms.

(ii) It assigns the value $(x|u)(y|v)$ to the element $(u, v) \in \mathbf{H} \times \mathbf{H}$.

As we shall see, the simple tensors are precisely the rank one operators on the Hilbert space $\mathbf{H}$.

Proposition 3 *(Proposition 2.6.9. [10]) The rank one operator*

$$T_{x,y} : \overline{\mathbf{H}} \to \mathbf{H} | u \mapsto (x|u)y$$

where $(\cdot|\cdot)$ is the inner product of $\mathbf{H}$ is a Hilbert-Schmidt operator from $\overline{\mathbf{H}}$ into $\mathbf{H}$.

Proof. $T_{x,y} : \overline{\mathbf{H}} \to \mathbf{H}$ is a Hilbert-Schmidt operator from $\overline{\mathbf{H}}$ into $\mathbf{H}$ if and only if the uniquely associated bounded bilinear form $\beta_{T_{x,y}} : \mathbf{H} \times \overline{\mathbf{H}} | (u|v) \mapsto (T_{x,y}(u)|v) = ((x,u)y|v) = (x|u)(y|v)$ is square summable [Definition 3] :

$$\begin{aligned}
\sum_{e_i \in Y_1} \sum_{e'_j \in Y_2} |\beta_{T_{x,y}}(e_i|e'_j)|^2 &= \sum_{e_i \in Y_1} \sum_{e'_j \in Y_2} |(x|e_i)|^2 |(y|e'_j)|^2 \\
&= \sum_{e_i \in Y_1} |(x|e_i)|^2 \sum_{e'_j \in Y_2} |(y|e'_j)|^2 \\
&= \| x \| \| y \| < \infty
\end{aligned}$$

Q.E.D

Definition 5 **(Simple Tensors $x \otimes y$)** *Let $\mathbf{H}$ be a Hilbert space. Given the elements $x, y \in \mathbf{H}$, we define the simple tensor $x \otimes y$ as the square summable bilinear form $\beta_{T_{x,y}}$, defined above. In light of Proposition 2, we identify the rank one operator $T_{x,y}$ with $x \otimes y$. Evidently, condition (ii) is satisfied. Condition (i) is also satisfied by Chapter 2, Theorem 2.6.4 [10].*

Example 2 *Let the Hilbert space $\mathbf{H}$ be as in Example 1. Then let e_1 and e_2 be the ordered basis elements:*

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then we identify the "products" or simple tensors $e_i \otimes e_j$ where $0 \leq i, j \leq 2$ with the bounded operators T_{e_i,e_j}. The operator T_{e_i,e_j} is the elementary 2×2 matrix with 1 in the (j,i) - th entry and 0 everywhere else with respect to the ordered basis (e_1, e_2).

Note that the concepts of bilinear forms, bounded bilinear forms and square summable bilinear forms on the set $\mathbf{C}^2 \times \mathbf{C}^2$ *coincide. Since the action of the bilinear form is uniquely defined by its action on the pairs* $(e_i, e_j) \in \mathbf{C}^2 \times \mathbf{C}^2$ *for* $i, j \leq 2$*, the* 2×2 *matrices uniquely encode any bilinear form on* $\mathbf{C}^2 \times \mathbf{C}^2$ *.*

The following proposition allows us to identify the simple tensors with rank one operators in the category of normed vector spaces.

Proposition 4 *A bounded operator* $T : \overline{\mathbf{H}} \to \mathbf{H}$ *is rank one if and only if it is of the form* $T_{x,y}$ *for some* $x, y \in \mathbf{H}$.

Proof. Since T is a rank one operator, we can choose any non-zero vector y in the range of T as the basis vector of the subspace $Ran(T)$: $Ran(T) = \{\lambda y | \lambda \in \mathbf{C}\}$. Then, for any $x \in \overline{\mathbf{H}}$, $T(x) = \lambda_x y$. The map $\Phi : x \mapsto \lambda_x$ is a bounded linear functional in $\overline{\mathbf{H}}$.

The Riesz representation theorem converts the linear functional Φ into a bilinear form:

$$\textit{There exists a } z_0 \in \mathbf{H} | \Phi(x) = (x|z_0).$$

Hence $T(x) = \lambda_x y = (x|z_0)y = T_{z_0,y} = z_0 \otimes y$.

The map $\Phi : x \otimes y \mapsto T_{x,y}$ is a normed isometric isomorphism since $\| T_{x,y} \| = \sup_{\|z\|=1} T_{x,y}(z) = \| x \| \| y \|$: apply the Cauchy-Schwartz inequality for one inequality and set $z = \frac{x}{\|x\|}$ for the reverse inequality.

Q.E.D

We are now on the verge of showing that the C*-algebra of all compact operators, $K(\mathbf{H})$, are Hilbert Schmidt operators $\mathbf{H} \bigotimes_h \overline{\mathbf{H}}$ in the category of normed vector spaces:

Theorem 1 *The vector space of all compact operators,* $K(\mathbf{H})$*, on a Hilbert space* $\mathbf{H}$*, is the closure of the linear span of all the rank-one projections* $T_{x,y}$ *or* $x \otimes y$*. Equivalently,* $\{T_{x,y} | x, y \in \mathbf{H}\}$ *is a fundamental subset of* $K(\mathbf{H})$*.*

Proof. Let $F(\mathbf{H})$ denote the vector subspace of all finite rank operators. Then $F(\mathbf{H})$ is the linear span of the rank one operators [Chapter 2, Theorem 2.4.6 [13]]. Noting that $K(\mathbf{H})$ is the closure of $F(\mathbf{H})$ [Chapter 2, Theorem 2.4.5 [13]], the proof is completed by proposition 4 which characterizes all rank one operators as the simple tensors.

Q.E.D

Example 3 (Non-closed self adjoint 2-sided ideal) *Although all closed ideals are necessarily self-adjoint [Proposition 4, Chapter 1.2.4], we show that the converse is not true.*

Consider $F(\mathbf{H})$ *as a 2-sided self-adjoint ideal of* $B(\mathbf{H})$*:* $Ran(T^*) = Ran(T^*T)$ *[Chapter 2, Proposition 2.5.13(8) [10]]; the dimension,* $dim(Ran(T))$*, of the range of the operator* T*,* $Ran(T)$*, is finite; hence so is* $dim(Ran(T^*)) = dim(Ran(T^*T))$*.*

If the Hilbert space $\mathbf{H}$ *is a finite dimensional Hilbert space, then* $K(\mathbf{H}) = \overline{F(\mathbf{H})} = F(\mathbf{H})$*. We therefore look for an infinite dimensional Hilbert space where* $K(\mathbf{H}) = \overline{F(\mathbf{H})} \neq F(\mathbf{H})$*.*

Consider the infinite dimensional Hilbert space $\mathbf{H} = L^2([-\pi,\pi], \mathcal{B}([-\pi,\pi]), \mu)$ *[Example 43, Chapter 1.2.4]. The C*-algebra,* $C[-\pi,\pi]$ *with the inner product norm* $(f,g) = \int_{-\pi}^{\pi} f\overline{g}dt$ *is a dense subspace of* $L^2([-\pi,\pi], \mathcal{B}([-\pi,\pi]), \mu)$*. Now consider the compact operator* $T \in B(C[-\pi,\pi])$ *where*

$$T : C[-\pi,\pi] \to C[-\pi,\pi] | x(t) \mapsto \int_0^t x(s)ds$$

Now $Ran(T) \supseteq \mathbf{R}[X]$ *where* $\mathbf{R}[X]$ *is the set of all the polynomials over the real numbers* $\mathbf{R}$*. Consequently,* $\mathbf{R}[X]$ *being an infinite dimensional subspace of the Hilbert space* $\mathbf{H} = L^2([-\pi,\pi], \mathcal{B}([-\pi,\pi]), \mu)$ *, disqualifies* T *from being in* $F(\mathbf{H})$*.*

We finally note that T *has a unique extension to* $\hat{T}$ *as a bounded operator on* $L^2([-\pi,\pi], \mathcal{B}([-\pi,\pi]), \mu)$*. This extension is also compact.*

A.2 Computing The Double Centralizer Algebra of $K(\mathbf{H})$

Here we show that the double centralizer algebra of the C*-algebra $K(\mathbf{H})$, the space of all the compact operators on an infinite dimensional Hilbert space $\mathbf{H}$ is $B(\mathbf{H})$, is the space of all the bounded operators on the Hilbert space $\mathbf{H}$.

To compute the double centralizer algebra of the C*-algebra $K(\mathbf{H})$, we use the fact that the double centralizer C*-algebra $B(\mathbf{H})$ of $K(\mathbf{H})$ is the largest C*-algebra with an identity which contains $K(\mathbf{H})$ as a closed essential ideal [Theorem 11, Chapter 1.2.3] : the C*-algebra $K(\mathbf{H})$ is essentially faithful with respect to the over-ring $B(\mathbf{H})$.

Theorem 2 *(Chapter 3, Example 3.1.2 [13]) The double centralizer C*-algebra of the C*-algebra, $K(\mathbf{H})$, of all the compact operators on an infinite dimensional Hilbert space $\mathbf{H}$ is the C*-algebra, $B(\mathbf{H})$, of all the bounded operators on the Hilbert space $\mathbf{H}$.*

Proof.

Step 1. $K(\mathbf{H})$ is a closed essential ideal of $B(\mathbf{H})$: $K(\mathbf{H})$ is essentially faithful with respect to the over-ring $B(\mathbf{H})$.

Firstly, $K(\mathbf{H})$ is a closed 2-sided ideal of $B(\mathbf{H})$ [Chapter VI Corollary 2.6.3. [25]]. We now show that it is an essential ideal or essentially faithful with respect to the over-ring $B(\mathbf{H})$: $SK(\mathbf{H}) = 0$ implies that $S = 0$ for all $S \in B(\mathbf{H})$.

Consider the Hilbert-Schmidt operator, $T_{x,y}(u) = (x|u)y$,from $\overline{\mathbf{H}}$ into $\mathbf{H}$ where $(\cdot|\cdot)$ is the inner product in $\mathbf{H}$. For each pair of non-zero vectors $x, y \in \mathbf{H}$, $T_{x,y}$ is a rank one operator : therefore it is compact and belongs to $K(\mathbf{H})$.

Suppose the operator S left annihilates $K(\mathbf{H})$. Then, for each pair of non-zero vectors $x, y \in \mathbf{H}$,

$$S \circ T_{x,y} = T_{x,Sy} = 0$$

or equivalently,

$$(x|u)Sy = 0$$

Setting $x = u = y = e_i \in Y_1$ where Y_1 is an orthonormal basis for $\mathbf{H}$, we conclude that

$$S(e_i) = 0 \text{ for all } e_i \in Y_1.$$

Therefore $S = 0$.

Step 2 : $\Psi|^{B(\mathbf{H})}$ is onto.

Recall that the double representation

$$\Psi|^{B(\mathbf{H})} : B(\mathbf{H}) \to M(K(\mathbf{H}))|S \mapsto (L_S|_{K(\mathbf{H})}, R_S|_{K(\mathbf{H})})$$

is an injective *-homomorphism on the over-ring $B(\mathbf{H})$ into the semigroup of double centralizers $M(K(\mathbf{H}))$. To complete the proof we show that $\Psi|^{B(\mathbf{H})}$ is onto. Equivalently:

For an arbitrary double centralizer pair $(L, R) \in M(K(\mathbf{H}))$ where L, R are bounded operators on $K(\mathbf{H})$ viewed as a Banach space [Theorem 9, Chapter 1.2.3], once we find an operator $S \in B(\mathbf{H})$ such that

$$\Psi|^{B(\mathbf{H})}(S) = (L_S|_{K(\mathbf{H})}, R_S|_{K(\mathbf{H})}) = (L, R)$$

we are done. Since two centralizers are equal provided that their first components are equal, it suffices to show:

$$L_S|_{K(\mathbf{H})} \in B(K(\mathbf{H})) = L \in B(K(\mathbf{H})). \tag{A.1}$$

Since the set of all rank one operators $D = \{T_{x,y}|x, y \in \mathbf{H}\}$ is a fundamental subset of $K(\mathbf{H})$ [Theorem 1, Appendix A.1],

$$L_S \in B(K(\mathbf{H})) = L \in B(K(\mathbf{H})) \text{ if and only if } L_S,\ L \text{ agree on } D.$$

Denoting the Hilbert Schmidt operator $T_{x,y}$ as $x \otimes y$, the crux of the choice of the correct bounded operator S lies in the "Parseval identity like" following decomposition of $x \otimes y$:

$$x \otimes y = (e \otimes y)(x \otimes e) \tag{A.2}$$

where e is any vector in $\mathbf{H}$ with norm 1. We prove the above mentioned equality by the following direct computation:

For any $z \in \mathbf{H}$, $x \otimes y(z) = (x, z)y$. Now, $(e \otimes y)(x \otimes e)(z) = e \otimes y\Big((x|z)e\Big) = (x|z)(e|e)y = (x|z)y$ since $\| e \| = (e|e)^2 = 1$.

Since for any $z \in \mathbf{H}$

$$\begin{aligned} L_S(x \otimes y)(z) &= (S \circ (x \otimes y))(z) = (x \otimes Sy)(z) \\ &= (x|z)Sy \end{aligned}$$

and

$$\begin{aligned} L(x \otimes y)(z) &= L(e \otimes y)(x \otimes e)(z) = L(e \otimes y)(x|z)e \\ &= (x|z)L(e \otimes y)(e) \end{aligned}$$

we define $S : \mathbf{H} \to \mathbf{H} | y \mapsto L(e \otimes y)(e)$. S is then a bounded operator on $\mathbf{H}$ whose norm is less than $\| L \|$.

Q.E.D

A.3 Computing The Double Centralizer Algebra of $C_0(\Omega)$

Theorem 3 *(Example 3.1.3 [13]) The double centralizer algebra of the C*-algebra $C_0(\Omega)$, the space of all continuous functions on a locally compact space Ω which vanishes at infinity is $C_b(\Omega)$, the space of all bounded continuous functions on the locally compact space Ω. The space of all complex valued bounded continuous functions, $C_b(\Omega)$, is isometrically *-isomorphic to $C(\beta\Omega)$, the space of all complex valued continuous functions on the Stone-Cěch compactification of Ω.*

To start off, we will need the following lemma:

Lemma 1 *(Chapter VIII, Proposition 1.7 [9]) If A is an abelian C*-algebra, then the double centralizer algebra $M(A)$ is also an abelian C*-algebra, with an identity.*

Proof. Consider an arbitrary double centralizer (L, R) in $M(A)$. Let a denote any fixed element of the C*-algebra A. Then $(L, R) = (L, L)$:

$$L(a)b = L(ab) = L(ba) = (L(b))a = aL(b) = (aR)b = R(a)b$$

for all $b \in A$. Therefore, by the right faithfulness of A [Proposition 1, Chapter 1.2.3], $L(a) = R(a)$.

Now $(L,\ L) \cdot (L',\ L') = (L \circ L',\ L' \circ L)$. Since all double centralizers are of the form $(T,\ T)$, we conclude that $L \circ L' = L' \circ L$. Therefore, $(L,\ L) \cdot (L',\ L') = (L',\ L') \cdot (L,\ L)$

Q.E.D

We now are ready to prove the theorem:

Proof.

Step 1. $C_0(\Omega)$ is a closed essential ideal of $C_b(\Omega)$. $C_0(\Omega)$ is essentially faithful with respect to the over-ring $C_b(\Omega)$.

Firstly, $C_0(\Omega)$ is a closed 2-sided ideal of $B(\mathbf{H})$. We now show that it is an essential ideal or essentially faithful with respect to the over-ring $C_b(\Omega)$: $fC_0(\Omega) = 0$ implies that $f = 0$ for all $f \in C_b(\Omega)$.

Suppose $f \neq 0$. Then $\exists x | f(x) \neq 0$. Since Ω is locally compact, there exists a compact neighbourhood K of x. Further there exists a *proper* open set O such that $K \subset O$: since Ω is open and $K \subset \Omega$, there exists an open set O such that $\overline{O}$ is compact and $K \subset O \subset \overline{O} \subset \Omega$; since Ω is not compact, $\overline{O} \neq \Omega$ [Chapter 7, Proposition 7.22, [21]].

Therefore there exists a continuous function $\Delta_K \in C_0(\Omega)$ acting as a continuous approximant of the characteristic function on the compact set K [Urysohn's Lemma For Locally Compact Spaces : Chapter 7, Theorem 7.14, [21]]:

$$\Delta_K(x) = \begin{cases} 1 & : \quad x \in K \\ 0 & : \quad x \notin O \supset K \end{cases}$$

Consequently, $\Delta_K f \neq 0$.

Step 2 : $\Psi|^{C_b(\Omega)}$ is onto.

Recall that the double representation

$$\Psi|^{C_b(\Omega)} : C_b(\Omega) \to M(C_0(\Omega)) | f \mapsto (L_f|_{C_0(\Omega)}, R_f|_{C_b(\Omega)})$$

is an injective *-homomorphism on the over-ring $C_b(\Omega)$ into the semigroup of double centralizers $M(C_0(\Omega))$. To complete the proof we show that $\Psi|^{C_b(\Omega)}$ is onto. Equivalently:

For every left centralizer L on $C_0(\Omega)$, there exists a multiplication map L_f on $C_0(\Omega)$ with $f \in C_b(\Omega)$ such that

$$L(h) = L_f(h) = fh \text{ for all } h \in C_0(\Omega).$$

Let (e_λ) be an approximate unit for $C_0(\Omega)$. We shall use the approximate unit (e_λ) of functions $e_\lambda \in C_0(\Omega)$ to express L as a multiplication map L_f where $f \in C_b(\Omega)$:

For an arbitrary $h \in C_0(\Omega)$, $\lim_\lambda (e_\lambda h) = h$. Therefore:

$$\begin{aligned} L(h) &= L(\lim_\lambda (e_\lambda h)) \\ &= \lim_\lambda L(e_\lambda h) && \text{(A.3)} \\ &= (\lim_\lambda [L(e_\lambda)])h && \text{(A.4)} \end{aligned}$$

where $L(e_\lambda)$ is a net in $C_0(\Omega)$ and equation (A.3) follows from the continuity of L and (A.4) follows from L being a left centralizer. Equation (A.4) makes sense only if $(\lim_\lambda [L(e_\lambda)])$ exists in $C_0(\Omega)$: the convergence of this net in $C_0(\Omega)$ is with respect to the supremum norm of $C_0(\Omega)$. Denoting this *assumed* limit by f, since uniform convergence implies pointwise convergence,

$$f : \Omega \to \mathbf{C} | x \mapsto \lim_\lambda [L(u_\lambda)(x)] \qquad \text{(A.5)}$$

where $f(x)$ is the point-wise limit of the net $[L(e_\lambda)](x)$ in Ω. We shall take as our required f, f defined point-wise on Ω as in (A.5).

As it turns out, defining f point-wise as in (A.5) is sufficient:

$$
\begin{aligned}
[L_f(h)](x) &= fh(x) = f(x)h(x) = \lim_{\lambda}[L(e_\lambda)(x)]h(x) \\
&= \lim_{\lambda}[L(e_\lambda h)](x) \\
&= \lim_{\lambda} L[(e_\lambda(x)h(x)] && \text{(A.6)} \\
&= L \lim_{\lambda}[(e_\lambda(x)h(x)] && \text{(A.7)} \\
&= L[h(x)] && \text{(A.8)}
\end{aligned}
$$

where equation (A.6) follows from L being a left centralizer on $C_0(\Omega)$, equation (A.7) from the continuity of L and equation (A.8) follows from the fact that $h \in C_0(\Omega)$ and the uniform convergence $\lim_\lambda e_\lambda h = h$, by definition of the approximate identity, implying point-wise convergence $\lim_\lambda e_\lambda(x)h(x) = h(x)$.

So far, we have only assumed that the net $[L(e_\lambda)](x)$ in Ω is a convergent net in Ω. This is equivalent to the fact that f as defined in equation (A.5) exists and is bounded. However, we need to further show that f is continuous : $f \in C_b(\Omega)$.

Case I : L is a positive operator.

A positive operator L on the C*-algebra $C_0(\Omega)$ which we take as an ordered vector space, is an operator with the additional property that $L(h) \geq 0$ whenever $h \geq 0$. Now:

(i) **The net $[L(e_\lambda)](x)$ in Ω is a convergent net in Ω. Equivalently, f exists and is bounded**.

We can take the approximate identity $e_\lambda \in C_0(\Omega)$ of the C*-algebra $C_0(\Omega)$ as an increasing net of positive elements bounded by 1. Since L is a positive operator on the function space $C_0(\Omega)$, $L(e_\lambda) \in C_0(\Omega)$ is also an increasing net of positive elements in $C_0(\Omega)$ bounded by $\| L \|$, where $\| \cdot \|$ is the supremum norm on $C_0(\Omega)$. Therefore $L(u_\lambda)(x)$ is an increasing net in $\mathbf{R}^+$ which is bounded above by $\| L \|$. Therefore, $\lim_\lambda[L(u_\lambda)(x)]$ is well defined, unique and is bounded by $\| L \|$.

(ii) **f is continuous**.

It suffices to show that if x_λ is any net in Ω which converges to x_0, then the image net $f(x_\lambda)$ converges to $f(x_0)$. The crux of the proof is to find a continuous version of f about some neighbourhood of x_0.

Since Ω is locally compact, there exists a compact neighbourhood K of x_0 such that $K \subset K'$ where K' is a compact subset of Ω [see Step 1].

Now, since $(x_\lambda) \to x_0$, there exists a $\lambda_0 | \lambda \geq \lambda_0 \Rightarrow x_\lambda \in K$. Since $L(e_\lambda)$ is an increasing net of continuous functions bounded by $\| L \|$, the restriction of $L(e_\lambda)$ to the compact set J is an increasing net of positive valued functions in the space, $C(J)$, of all continuous functions on the compact space J. Therefore, the net $L(e_\lambda)$ in $C(J)$ converges uniformly to a continuous function f' in $C(J)$ where f' is the restriction of f to the compact space $J : f' = f|_J$ [Dini's Theorem for compact spaces: Chapter 2.2, Theorem 2]. Therefore $f|_J$ belongs to $C(J)$

Then by the continuity of the function $f|_J$:

$$f(x_0) = f|_J(x_0) = \lim_\lambda f|_J(x_\lambda) = \lim_\lambda f(x_\lambda) \text{ for all } \lambda \geq \lambda_0$$

since $x_\lambda \in J$ for all $\lambda \geq \lambda_0$.

Case II : L is an arbitrary operator

Firstly, an arbitrary operator L on the function space $C_0(\Omega)$ is the sum $L_1 - L_2 + i(L_3 - L_4)$ where $L_1, \ldots, L_4$ are positive operators on $C_0(\Omega)$. Consequently, we now bootstrap as follows:

$$\begin{aligned} L(h) &= L_1(h) - L_2(h) + i(L_3(h) - L_4(h)) \\ &= L_{f_1}(h) - L_{f_2}(h) + i(L_{f_3}(h) - L_{f_4}(h)) \\ &= f_1 h - f_2 h + i(f_3 h - f_4 h) \\ &= L_{f_1 + f_2 + i(f_3 - f_4)}(h) \end{aligned}$$

by case I.

Since $f_1, \ldots, f_4$ belong to the space, $C_b(\Omega)$, of all bounded continuous functions on Ω, $f_1 + f_2 + i(f_3 - f_4)$ is also a member of $C_b(\Omega)$.

Step 3. The space of all real [complex] valued bounded continuous functions, $C_b(\Omega)$, is isometrically *-isomorphic to $C(\beta\Omega)$, the space of all real [complex] valued continuous functions on the Stone-Cĕch compactification of Ω.

Case I : Real Valued Bounded Continuous Functions.

Since Ω is a locally compact Hausdorff space, Ω is completely regular [Chapter 3.15, Proposition 3.15 [31]]. Therefore, it has a Stone - Cĕch compactification $\beta\Omega$ such that every f in $C_b(\Omega)$ has an extension to a function f^β in $C(\beta\Omega)$[Chapter 6.5, Theorem 6.5 [31]]. Therefore, the map $\Phi : f \mapsto f^\beta$ is a *ring isomorphism* of $C_b(\Omega)$ onto $C(\beta\Omega)$ [Chapter 6, Remarks 6.6(b) [31]].

We further need to show that Φ is a *-isometry : $\| f^\beta \| = \| f \|$, $\| \cdot \|$ denoting both the supremum norms of the normed *-algebras $C_b(\Omega)$ and $C(\beta\Omega)$.

(i) Φ is an isometry.

Firstly Ω is dense in its Stone - Cěch compactification $\beta\Omega$. Therefore, for each point p in $\beta\Omega$, there exists a net x_α in Ω which converges to p. Since f^β is continuous, $f^\beta(x_\alpha)$ converges in $(\mathbf{R}, |\cdot|)$ to $f^\beta(p)$. But f^β extends f. Therefore, $f(x_\alpha) \to f^\beta(p)$ in $(\mathbf{R}, |\cdot|)$. By the continuity of $|\cdot|$, $|f(x_\alpha)| \to |f^\beta(p)|$. Hence $\| f^\beta \| \leq \| f \|$ since $|f(x_\alpha)| \leq \| f \|$ for all x_α.

The reverse inequality is trivial : $\| f \| \leq \| f^\beta \|$ since f^β extends f.

(ii) Φ preserves the involution.

This follows from $\overline{f}^\beta = \overline{f^\beta}$.

Case II : Complex Valued Bounded Continuous Functions.

Every complex valued function f can uniquely be written as $Re(f) + iIm(f)$, where $Re(f) : \Omega \to \mathbf{R} | x \mapsto Re(f(x))$ and $Im(f) : \Omega \to \mathbf{R} | x \mapsto Im(f(x))$. In fact, the complex valued function f is continuous and bounded if and only if $Re(f)$ and $Im(f)$ are both continuous and bounded. Hence, by a bootstrapping argument using case I, the required extension $f^\beta \in C(\beta\Omega)$ for f will $[Re(f)]^\beta + i[Im(f)]^\beta$. The map $\Phi : f \mapsto f^\beta$ is the required *-isometric isomorphism from $C_b(\Omega)$ onto $C(\beta\Omega)$.

Q.E.D

Example 4 *Consider the C*-algebra* $C_0(\Omega)$*, where* Ω *is the set of all natural numbers,* $\mathbf{N}$*, endowed with the discrete topology. Since points are both closed and open,* N *is trivially a locally compact Hausdorff space.*

Since the topology is discrete, the only compact subsets are the finite subsets. Therefore, $C_0(\Omega)$ *is the space of all null sequences which we shall denote as* c_0*. Therefore, its double centralizer algebra* $M(C_0(\Omega))$ *is* $C_b(\Omega)$*, the space of all bounded sequences,* l^∞*, since all functions are continuous with respect to the discrete topology. Therefore* $M(c_0) = l^\infty$*.*

Appendix B

Anti-Unitization

In this appendix we assume that the C*-algebra A has an identity.

B.1 The Holomorphic Functional Calculus : Every C*-algebra is a Local Banach Algebra

In chapter 1.3.3 we introduced the functional calculus on a normal element, x, of a C*-algebra, A, enabling us to define the element $f(x) \in A$ where f is a continuous function defined on the spectrum, $\sigma_A(x)$, of x in A, where $\sigma_A(x)$ is a compact Hausdorff topological space. We now define the holomorphic functional calculus on an *arbitrary* element, x, of the C*-algebra A enabling us to define the element $f(x) \in A$, where f is an holomorphic or analytic function of a complex variable defined on an open set containing the spectrum of the arbitrary C*-algebra element x.

We shall rephrase the above in terms of a *local Banach algebra* which we define as follows [compare Chapter II, Definition 3.1.1 [27]]:

Definition 1 (Local Banach Algebra) *A local Banach algebra is a Banach algebra with an identity which is closed under the holomorphic functional calculus. That is, for any* $x \in A$ *and any analytic function of a complex variable* $f : U \subset \mathbf{C} \to \mathbf{C}$ *which is analytic on some open set* U *containing the spectrum of* x, $f(x) \in A$.

Theorem 1 (Every C*-algebra is a Local Banach Algebra) *Let* A *be a C*-algebra with an identity. Then* A *is trivially a Banach algebra and is closed under the holomorphic functional calculus. The C*-algebra* A *is therefore a local Banach algebra.*

In order to prove Theorem 1 [Appendix B.1.3], we establish a few preliminary results:

B.1.1 Preliminary Result I : Cauchy Integral Formula for C*-algebra valued analytic functions of a complex variable

Let A be a C*-algebra. Let $\mathbf{f}$ denote a A-valued analytic function of a complex variable : $\mathbf{f}$ is differentiable at each point z_0, in the sense that the limit of the difference quotient exists in the norm topology of A. We take the domain $U \subset \mathbf{C}$ of $\mathbf{f}$ as an open subset of the complex plane. Then the Cauchy Integral Formula for the classical complex valued analytic function of a complex variable carries over faithfully to $\mathbf{f}$ once we define the line integral of $\mathbf{f}$ analogously:

Definition 2 (Line Integral of a C*-algebra valued continuous function) *Let $\mathbf{f}$ denote a A-valued analytic function of a complex variable. We take the domain, U, of $\mathbf{f}$ as an open connected set without loss of generality since every open set is a disjoint union of open connected sets. We further take as our curve C in the domain to be continuously differentiable : the parametrization $z : [a, b] \to U$ where $[a, b]$ is a compact interval of the real line, is continuously differentiable: z' exists at each point of $[a, b]$ (in the one-sided sense at the end-points) and is continuous on $[a, b]$.*

The line integral $\int_C \mathbf{f} dz = \int_a^b \mathbf{f}(z(t))z'(t)dt \in A$ is taken as the norm limit of the Riemann Sums of the form

$$\sum_{i=1}^{n} \mathbf{f}(z(t_i^*))[z(t_i) - z(t_{i-1})] = \sum_{i=1}^{n} \mathbf{f}(z(t_i^*))z'(t_i^*)\Delta t_i \tag{B.1}$$

where $a = t_0 < t_1 < \ldots < t_n = b$, $\Delta t_i = t_i - t_{i-1}$ and $t_i^* \in [t_{i-1}, t_i]$ such that $z'(t_i^*)\Delta t_i = z(t_i) - z(t_{i-1})$.

The limit is taken as the norm of the mesh of the partition, $max\{\Delta t_i | i = 1, \ldots, n\}$, approaches 0.

This limit will exist by the "uniform continuity" of the C-algebra valued continuous function $(\mathbf{f} \circ z)z' : [a, b] \to \mathbf{C} | t \mapsto \mathbf{f}(z(t))z'(t)$: the Riemann sums (B.1) will form a Cauchy sequence in the Banach space A as the norm of the mesh approaches 0.*

The passage of the Cauchy Integral Formula for the classical complex valued analytic function of a complex variable to the C*-algebra valued analytic functions of a complex variable is a result of the Hahn-Banach theorem for the C*-algebra A taken as a Banach space. Let φ be any bounded linear functional on the Banach space A. Then the composition $\varphi \circ \mathbf{f}$ is a classical complex valued analytic function of a complex variable. In particular, by the nature of the Riemann sums defined in equation (B.1),

$$\varphi(\int_C \mathbf{f} dz) = \int_C (\varphi \circ \mathbf{f}) dz \tag{B.2}$$

Applying the Cauchy Integral Formula to the classical complex valued analytic function $\varphi \circ \mathbf{f}$ of a complex variable with the added assumption that the curve C is topologically trivial : it can be continuously deformed to a point in U, as well as being simple and closed : z is 1-1 on $[a, b)$ and $z(a) = z(b)$, we have

$$(\varphi \circ \mathbf{f})(z_0) - \int_C \frac{(\varphi \circ \mathbf{f})(z)}{z - z_0} dz = 0 \tag{B.3}$$

and

$$\int_C (\varphi \circ \mathbf{f}) dz = 0 \tag{B.4}$$

for all $\varphi \in A^*$ where A^* is the continuous dual of the C*-algebra A taken as a Banach space and z_0 is any point in the interior of the topologically trivial, simple closed curve C.

Applying equation (B.2) and the homogeneity of the bounded linear functional φ, we conclude

$$\varphi\Big(\mathbf{f}(z_0) - \int_C \frac{\mathbf{f}(z)}{z - z_0} dz\Big) = 0 \tag{B.5}$$

and

$$\varphi\Big(\int_C \mathbf{f} dz\Big) = 0 \tag{B.6}$$

By the Hahn-Banach Theorem, A^* separates points in A. Therefore,

$$\mathbf{f}(z_0) = \int_C \frac{\mathbf{f}(z)}{z - z_0} dz \tag{B.7}$$

for all z_0 in the interior of C, and

$$\int_C \mathbf{f} dz = 0 \tag{B.8}$$

The above method used in the passage of the Cauchy Integral Formula for the classical complex valued analytic function of a complex variable to the C*-algebra-valued analytic functions of a complex variable, also yields the fact that an analytic C*-algebra valued function of a complex variable, has a power series representation [Chapter 3, Theorem 3.3.1 [10]]:

Theorem 2 (Power Series Representation) *Let* $\mathbf{f}$ *be an analytic C*-algebra valued function of a complex variable with domain* U. *Then* $\mathbf{f}(z)$ *can be represented as a power series* $\sum_{n=0}^{\infty} x_n (z - z_0)^n$ *in the C*-algebra* A, *with coefficients* $x_n \in A$, *for each* z *contained in the largest open disk with prescribed center* $z_0 \in U$. *The radius of this largest disk centered at* $z_0 \in U$ *is* $(\overline{lim} \parallel x_n \parallel^{1/n})^{-1}$.

B.1.2 Preliminary Result II : Analyticity of an important C*-algebra valued function

Consider the C*-algebra valued function

$$\mathbf{f} : U \subset \mathbf{C} \to A | z \mapsto \frac{1}{z1_A - x_0}$$

of a complex variable, where x_0 is a fixed C*-algebra element of A. The element 1_A denotes the identity element of the C*-algebra A and the domain $U \subset \mathbf{C}$ is an open and connected subset of the complex plane which does *not* include the spectrum, $\sigma_A(x_0)$, of the fixed C*-algebra element x_0.

Then using the same technique in passing the Cauchy Integral Theorem for the classical complex valued analytic function of a complex variable to the C*-algebra valued analytic function of a complex variable, we conclude [Chapter 3, Theorem 3.2.3 [10]]

Proposition 1 *The C*-algebra valued function*

$$\mathbf{f} : U \subset \mathbf{C} \to A | z \mapsto \frac{1}{z1_A - x_0}$$

of a complex variable, where x_0 is a fixed C-algebra element of A is an analytic function. The domain $U \subset \mathbf{C}$ is an open and connected subset of the complex plane which does not include the spectrum, $\sigma_A(x_0)$, of the fixed C*-algebra element x_0. Hence the term $\frac{1}{z1_A - x_0}$ is well defined.*

B.1.3 The Holomorphic Functional Calculus

In analogy with equation (B.7), we have the origin of the holomorphic functional calculus in a C*-algebra A:

Proposition 2 *Let x_0 be a fixed element of the C*-algebra A. Let C be a simple closed continuously differentiable curve in the complex plane which has the spectrum $\sigma_A(x_0)$ of the element x_0 in its interior. For, simplicity sake, we take C as the circumference of a circle centered at the origin whose radius is strictly larger than the spectral radius of x_0.*

Then,

$$x_0^n = \frac{1}{2\pi i} \int_C \mathbf{f} dz = \frac{1}{2\pi i} \int_C \frac{z^n}{z1_A - x_0} dz \qquad \text{(B.9)}$$

where n is a positive integer and

$$\mathbf{f} : U \subset \mathbf{C} \to A | z \mapsto \frac{z^n}{z1_A - x_0}. \qquad \text{(B.10)}$$

Proof

(i) The C*-algebra valued function of a complex variable $\mathbf{f}$ defined in equation (B.10), is analytic on the resolvent, $R(x_0)$, of x_0 : $R(x_0) = \mathbf{C}\backslash\sigma_A(x_0)$. We take the domain, U, of $\mathbf{f}$ as the resolvent $R(x_0)$: we can take the resolvent of x_0 as an open and connected set without loss of generality since every open set is a disjoint union of open and connected sets [Proposition 1, Appendix B.1.2].

(ii) We take as our closed curve $C \subset U$ a curve that contains the spectrum of x_0 in its interior : for, simplicity sake, we take C as the circumference of a circle centered at the origin whose radius is strictly larger than the norm, $\| x_0 \|$, of x_0. This is to prevent $C \subset U$ from being topologically trivial, leading to the pathological case of equation (B.8).

(iii) The analytic function $\mathbf{f}$ has a power series representation on C [Theorem 2, Appendix B.1.1] for each $z \in C$ since $|z| \geq \| x_0 \|$:

$$\mathbf{f}(z) = \sum_{k=0}^{\infty} x_0^k z^{n-k-1} \tag{B.11}$$

for all $z \in C \subset \mathbf{C}\backslash\sigma(x_0)$: rewrite $\frac{z^n}{z1_A - x_0} = z^n(z1_A - x_0)^{-1}$ as $z^n z^{-1}(1_A - z^{-1}x_0)^{-1}$ and recall that

$$(1_A - z^{-1}x_0)^{-1} = \sum_{k=0}^{\infty} z^{-k} x_0^k \tag{B.12}$$

for $\| z^{-1}x_0 \| \leq 1$.

(iii) Since the convergence of equation (B.12) occurs in the norm topology of the C*-algebra A and uniformly on the compact interval, $[a, b]$, of parametrization of C, applying a term-by-term integration of equation (B.11):

$$\int_C \mathbf{f} dz = \int_C \sum_{k=0}^{\infty} x_0^k z^{n-k-1} dz = 2\pi i x_0^n \tag{B.13}$$

since $\int_C z^{n-k-1} dz = 0$ unless $k = n$.

Q.E.D

We are now ready to construct the holomorphic functional calculus on the C*-algebra A with an identity: every C*-algebra is a local Banach algebra.

By the linearity of the $\int_C dz$ operator, it follows from equation (B.9) that

$$f(x_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z1_A - x_0}dz \tag{B.14}$$

where f is a polynomial function which is analytic everywhere on the complex plane $\mathbf{C}$.

Now let f be an holomorphic or analytic function of a complex variable defined on an *open disk*, U, containing the spectrum of the fixed C*-algebra element x_0. Then we take our curve $C \subset U$ as a circle centered at the origin containing the spectrum $\sigma_A(x_0)$ of the element x_0 in its interior. Now $f(z)$ has a power series representation $\sum_{n=0}^{\infty} a_n z^n$ centered at the origin. This series converges uniformly on the closed curve C so that integrating term-by-term,

$$\begin{aligned}
\frac{1}{2\pi i}\int_C \frac{f(z)}{z1_A - x_0}dz &= \frac{1}{2\pi i}\int_C \frac{\sum_{n=0}^{\infty} a_n z^n}{z1_A - x_0}dz \\
&= \sum_{n=0}^{\infty} a_n\Big(\frac{1}{2\pi i}\int_C \frac{z^n}{z1_A - x_0}dz\Big) \\
&= \sum_{n=0}^{\infty} a_n x_0^n \qquad \text{(B.15)} \\
&= f(x_0) \qquad \text{(B.16)}
\end{aligned}$$

where equation (B.15) follows from Proposition 2, equation (B.9). We take equation (B.16) as a definition of the element $f(x_0)$ where f be an holomorphic or analytic function of a complex variable defined on an *open disk*, U, containing the spectrum of the fixed C*-algebra element x_0.

B.2 The Spatial Tensor Product $A \otimes M_n(\mathbf{C})$

B.2.1 The Spatial Tensor Product : The General Case

Throughout this section, all C*-algebras are taken as *represented C*-algebras* : C*-algebras of bounded operators acting on a known Hilbert space. In particular, the C*-algebras $M_n(\mathbf{C})$ and A are the C*-algebras of bounded operators on the n-dimensional Hilbert space $\mathbf{C}^n$ and the universal Hilbert space $\mathbf{H}$, respectively : $\Phi : A \to B(\mathbf{H}) | a \mapsto \Phi_a$ denotes the universal representation of the C*-algebra A [Chapter 1.2.2, Gelfand Naimark Theorem II]. From now on the C*-algebra element a will denote the bounded operator Φ_a. We construct the *spatial tensor product* C*-algebra $A \bigotimes M_n(\mathbf{C})$ as a represented C*-algebra, as follows:

Firstly, the Hilbert space in question is the Hilbert tensor product $\mathbf{H} \bigotimes_h \mathbf{C}^n$ of all the bounded square summable bilinear forms on the Cartesian product $\mathbf{H} \times \mathbf{C}^n$ [Chapter 2, Proposition 2.6.2 [10]].

Secondly, we consider the set F of all the tensor products, $a \otimes M$, of the bounded operators $a \in B(\mathbf{H})$ and $M \in M_n(\mathbf{C})$:

$$F = \{a \otimes M | a \in A, M \in M_n(\mathbf{C})\} \tag{B.17}$$

The tensor product $a \otimes M$ is a bounded operator on the Hilbert space $\mathbf{H} \bigotimes_h \mathbf{C}^n$ [Chapter 2, Proposition 2.6.12 [10]]. In fact,

$$\| a \otimes M \| = \| a \| \| M \| \tag{B.18}$$

where $\| M \| = max_{\|\mathbf{v}\|_2 \leq 1} \| M(\mathbf{v}) \|_2$ where $\| \cdot \|_2$ is the usual Euclidean norm on $\mathbf{C}^n$. It suffices to define the action of $a \otimes M$ on the fundamental set of simple tensors of $\mathbf{H} \bigotimes_h \mathbf{C}^n$:

$$a \otimes M : \mathbf{H} \bigotimes_h \mathbf{C}^n \to \mathbf{H} \bigotimes_h \mathbf{C}^n | h \otimes \mathbf{v} \mapsto a(h) \otimes M(\mathbf{v})$$

Finally, the linear span of the set F is a *-subalgebra of the C*-algebra, $B(\mathbf{H} \bigotimes_h \mathbf{C}^n)$, of all the bounded operators on $\mathbf{H} \bigotimes_h \mathbf{C}^n$ where we define multiplication as composition [Chapter 2, Proposition 2.6.12 (15) [10]]. We define the closed linear span of F in $B(\mathbf{H} \bigotimes_h \mathbf{C}^n)$ as the spatial tensor product C*-algebra $A \bigotimes M_n(\mathbf{C})$: F is a fundamental set of $A \bigotimes M_n(\mathbf{C})$. Formally,

Definition 3 (Spatial Tensor Product $A \bigotimes M_n(\mathbf{C})$) *The spatial tensor product, $A \bigotimes M_n(\mathbf{C})$, of the C*-algebras A and $M_n(C)$ is the C*-subalgebra of $B(\mathbf{H} \bigotimes_h \mathbf{C}^n)$, the C*-algebra of all the bounded operators on $\mathbf{H} \bigotimes_h \mathbf{C}^n$, which has the set F as defined in equation (B.17) as its fundamental set.*

B.2.2 The Spatial Tensor Product : Explicit Description

Here we show that the spatial tensor product $A \bigotimes M_n(\mathbf{C})$ is precisely $M_n(A)$, the C*-algebra of all $n \times n$ matrices with entries from A. We do this by first establishing that the represented C*-algebra $A \bigotimes M_n(\mathbf{C})$ (Definition 3) is highly algebraic in nature:

Every element of the spatial tensor product, $A \bigotimes M_n(\mathbf{C}) \subset B(\mathbf{H} \bigotimes_h \mathbf{C}^n)$, is of the form

$$\sum_{1 \le j,k \le n} a_{jk} \otimes E_{jk}$$

where E_{jk} is the elementary $n \times n$ matrix with 1 in the (j, k)-th entry:

Proposition 3 *(Chapter 11.1, Example 11.1.5 [11]) The closed linear span of the fundamental set F in $B(\mathbf{H} \bigotimes_h \mathbf{C}^n)$ coincides with the linear span of the fundamental set F in $B(\mathbf{H} \bigotimes_h \mathbf{C}^n)$. Consequently, every element of the spatial tensor product, $A \bigotimes M_n(\mathbf{C})$, is of the form $\sum_{1 \le j,k \le n} a_{jk} \otimes E_{jk}$ where E_{jk} is the elementary $n \times n$ matrix with 1 in the (j, k)-th entry.*

Proof. A typical element C of the linear span of the fundamental set F is of the form $\sum_{t=1}^{m} a_t \otimes B_t$ where $a_1, \dots, a_m \in A$ and $B_1, \dots, B_m \in M_n(\mathbf{C})$. But, since $M_n(\mathbf{C})$ is spanned by E_{jk} where $1 \le j, k \le n$, we can write C as $\sum_{1 \le j,k \le n} a'_{jk} \otimes E_{jk}$.

Given $C_1 = \sum_{1 \le j,k \le n} a^1_{jk} \otimes E_{jk}$ and $C_1 = \sum_{1 \le j,k \le n} a^2_{jk} \otimes E_{jk}$, the inequality

$$\| a^1_{jk} - a^2_{jk} \| \le \| C_1 - C_2 \| \tag{B.19}$$

will establish the uniqueness of the representation of the element C as the sum $\sum_{1 \le j,k \le n} a'_{jk} \otimes E_{jk}$ and the completeness of the linear span of the fundamental set F.

Q.E.D

Now the Hilbert space in the represented C*-algebra $A \bigotimes M_n(\mathbf{C})$ is the Hilbert tensor product $\mathbf{H} \bigotimes_h \mathbf{C}^n$. We show that this Hilbert space is isomorphic (as a Hilbert space) to the Hilbert direct sum space $\sum_{i=1}^{n} \oplus \mathbf{H}$:

Proposition 4 *The Hilbert direct sum space $\sum_{i=1}^{n} \oplus \mathbf{H}$ is isomorphic to the Hilbert tensor product $\mathbf{H} \bigotimes_h \mathbf{C}^n$.*

Proof. The map $W : \sum_{i=1}^{n} \oplus \mathbf{H} \to \mathbf{H} \bigotimes_h \mathbf{C}^n | (h_1, \dots, h_n) \mapsto h_1 \otimes e_1 + \dots + h_n \otimes e_n$ where $(e_1, \dots, e_n)$ is a basis of $\mathbf{C}^n$, is an onto isometric vector space homomorphism : (the family of simple tensors $(h_i \otimes e_i | i = 1, \dots, n)$ is an orthogonal family : $(h_i \otimes e_i, h_j \otimes e_j) = (h_i, h_j)(e_i, e_j) = \| h_i \|^2 \delta_{ij}$).

Q.E.D

Thus far, we have established that $A \bigotimes M_n(\mathbf{C}) \subset B(\mathbf{H} \bigotimes_h \mathbf{C}^n)$ where intuitively we can regard $B(\mathbf{H} \bigotimes_h \mathbf{C}^n)$ as $B(\sum_{i=1}^n \oplus \mathbf{H})$ by virtue of Proposition 4. We now show that $B(\sum_{i=1}^n \oplus \mathbf{H})$ is the C*-algebra of $n \times n$ matrices with entries in $B(\mathbf{H})$:

Proposition 5 (Matrix Representation of $B(\sum_{i=1}^n \oplus \mathbf{H})$) *(Chapter 11.1, Example 11.1.5 [11]) Consider the Hilbert space, $\sum_{i=1}^n \oplus \mathbf{H}$, the finite Hilbert direct sum space $\mathbf{H} \bigoplus \ldots \bigoplus \mathbf{H}$ (n times) where*

$$\| (h_1, \ldots, h_n) \| = \Big(\| h_1 \|^2 + \ldots + \| h_n \|^2 \Big)^{\frac{1}{2}}. \tag{B.20}$$

Defining the bounded linear operators U_i and V_j for $1 \le i, j \le n$ as

$$U_i : \mathbf{H} \to \sum_{i=1}^n \oplus \mathbf{H} | h \mapsto (h_k)_{k=1}^n, V_j : \sum_{i=1}^n \oplus \mathbf{H} \to \mathbf{H} | (h_k)_{k=1}^n \mapsto h_j \tag{B.21}$$

where $h_k = h$ if and only if $k = i$ and 0 otherwise, straightforward computation shows that every bounded linear operator T on $\sum_{i=1}^n \oplus \mathbf{H}$ has the matrix representation:

$$\begin{pmatrix} T_{11} & \ldots & T_{1n} \\ \vdots & & \vdots \\ T_{n1} & \ldots & T_{nn} \end{pmatrix}$$

where $T_{ij} = U_i T U_j \in B(\mathbf{H})$ for $1 \le i, j \le n$ and we define

$$\begin{pmatrix} T_{11} & \ldots & T_{1n} \\ \vdots & & \vdots \\ T_{n1} & \ldots & T_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} T_{11}(h_1) + \ldots + T_{1n}(h_n) \\ \vdots \\ T_{n1}(h_1) + \ldots + T_{nn}(h_n) \end{pmatrix}$$

Let $M_n(B(\mathbf{H}))$ denote the set of $n \times n$ matrices with entries from $B(\mathbf{H})$. Then defining the adjoint, the product and the norm analogously as in $M_n(\mathbf{C})$, the set $M_n(B(\mathbf{H}))$ is a C*-algebra. The map $\Phi : B(\sum_{i=1}^n \oplus \mathbf{H}) \to M_n(B(\mathbf{H})) : T \mapsto [T_{ij}]_{1 \le i,j \le n}$ is an onto *-isomorphism (or equivalently, an onto isometric *-isomorphism [Chapter VI, Corollary 3.9 [9]]):

$$\sup_{\|v\|=1} \| Tv \| = \sup_{\sum_{j=1}^n \|h_j\|^2 = 1} \sum_i \sum_j \| T_{ij}(h_j) \| \tag{B.22}$$

$$\| Tv \|^2 \le \Big(\sum_i \sum_j \| T_{ij} \|^2 \Big) \| v \|^2 \tag{B.23}$$

where $v = (h_j)_{1 \le j \le n} \in \sum_{i=1}^n \oplus \mathbf{H}$.

Q.E.D

We now rigourously show that the spatial tensor product $A \bigotimes M_n(\mathbf{C}) \subset B(\mathbf{H} \bigotimes_h \mathbf{C}^n)$ is precisely $M_n(A) = B(\sum_{i=1}^n \mathbf{H})$, the C*-algebra of all $n \times n$ matrices with entries from A by showing that the vector space isomorphism between the Hilbert spaces $\sum_{i=1}^n \mathbf{H}$ and $\mathbf{H} \bigotimes_h \mathbf{C}^n$ in Proposition 4 establishes an equivalence of the *-representations [Chapter 1.2.1, remark following Example 5]. This will then allow us to identify each $\sum_{1 \le j,k \le n} a_{jk} \otimes E_{jk} \in A \bigotimes M_n(\mathbf{C})$ with the matrix $[a_{jk}]_{1 \le j,k \le n} \in M_n(A)$.

This follows from the following equations:

$$\begin{aligned}
W^{-1} \sum_{1 \le j,k \le n} a_{jk} \otimes E_{jk} W\Big((h_1, \ldots, h_n)\Big) &= W^{-1} \sum_{1 \le j,k \le n} a_{jk} \otimes E_{jk} \Big((h_1 \otimes e_1 + \ldots + h_n \otimes e_n)\Big) \\
&= W^{-1} \sum_{l=1} \Big(\sum_{1 \le j,k \le n} a_{jk}(h_l) \otimes E_{jk}(e_l)\Big) \\
&= W^{-1} \Big(\sum_{1 \le j,k \le n} a_{jk}(h_k) \otimes E_{jk}(e_k)\Big) \\
&= W^{-1} \Big(\sum_{1 \le j \le n} \Big(\sum_{1 \le k \le n} a_{jk}(h_k)\Big) \otimes e_j \Big) \\
&= [a_{jk}](h_k)_{k=1}^n
\end{aligned}$$

where

$$[a_{jk}](h_k)_{k=1}^n = \begin{pmatrix} T_{11} & \ldots & T_{1n} \\ \vdots & & \vdots \\ T_{n1} & \ldots & T_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} T_{11}(h_1) + \ldots + T_{1n}(h_n) \\ \vdots \\ T_{n1}(h_1) + \ldots + T_{nn}(h_n) \end{pmatrix}$$

B.3 The Normed Inductive Direct Limit $M_\infty(A)$

We shall now construct from the family of unital C*-algebras $\mathcal{A} = \{M_n(A) | n \in \mathbf{N}\}$, a σ-unital normed algebra, $M_\infty(A)$, which does not have an identity : it turns out that we can view $M_\infty(A)$ as the normed *-algebra of all infinite matrices with entries in A where only finitely many of the entries are non-zero. The method of construction is to first cast the family $\mathcal{A}$ as a *normed inductive system* from which we can then construct the *normed inductive limit* $M_\infty(A)$. We shall now first define a prerequisite concept : *direct family of sets.*

Definition 4 (Direct Family Of Sets) *(Chapter 3.21, Definition 1 [34]) Working initially in the category of sets, we first define a direct family of sets $\mathcal{A}$ as a triplet of the following objects:*

(i) A directed partially ordered set $(I, \leq)$ called the carrier of $\mathcal{A}$: for any $i, j \in I$, there exists a $k \in I$ such that $i \leq k$ and $j \leq k$ where $\leq$ is a partial order on I.

(ii) Sets A_i for each $i \in I$.

(iii) Mappings φ_{ij} for all $i \leq j$, where φ_{ij} maps A_i into A_j such that

$$\varphi_{ij}\varphi_{jk} = \varphi_{ik} \text{ if } i \leq j \leq k$$

and φ_{ii} is the identity mapping for all $i \in I$.

Example 1 (Direct Family Of Sets) *Let $\mathcal{A}$ be the family of unital C*-algebras $\mathcal{A} = \{M_n(A) | n \in \mathbf{N}\}$. If we define the map $\varphi_{ij} : M_i(A) \to M_j(A) | T \mapsto diag(T, 0)$ where $diag(T, 0)$ is the direct sum matrix $T \bigoplus 0$ [Chapter 4.2.1, Definition 5] where 0 is the zero $(j-i) \times (j-i)$ - matrix : T is embedded in the upper left hand corner of $M_j(A)$, and the carrier, I, of $\mathcal{A}$ to be the well ordered set $\mathbf{N}$ with the usual ordering, then $\mathcal{A} = \{M_n(A) | n \in \mathbf{N}\}$ becomes a direct family of sets.*

We are now in a position to define an inductive system. In a nutshell, it is a directed family of sets lifted into the category of local Banach algebras [Appendix B.1, Definition 1]:

Definition 5 (Normed Inductive System) *A normed inductive system is a directed family of sets $\mathcal{A}$ with the additional condition that*

(i) Sets A_i for each $i \in I$ are local Banach algebras.

(ii) The mappings φ_{ij} for all $i \leq j$, where φ_{ij} maps A_i into A_j are bounded normed algebra homomorphisms and for each i, $\lim_i \sup_j \| \varphi_{ij} \| < \infty$.

Proposition 6 *The family of unital C*-algebras* $\mathcal{A} = \{M_n(A)|n \in \mathbf{N}\}$ *taken as a directed family of sets [Example 1] is a normed inductive system .*

Proof. Condition (i) is satisfied by virtue of Theorem 1, Appendix B1. Condition (ii) is trivially satisfied since each $\varphi_{ij} : M_i(A) \to M_j(A)|T \mapsto diag[T, 0]$ is an isometry so that $\| \varphi_{ij} \| = 1$ for all i, j.

Q.E.D

Reverting back into the category of sets, we now define the *algebraic direct limit* of a direct family of sets $\mathcal{A}$:

Definition 6 (Algebraic Direct Limit) *Let* $\mathcal{A}$ *be a direct family of sets. We first consider the union set* $\bigcup \mathcal{A} = \bigcup_{i \in I}\{A_i\}$ *where we assume that the sets* A_i *are pairwise disjoint; take the disjoint union otherwise. Define an equivalence relation,* R*, on* $\bigcup \mathcal{A}$ *as follows:*

$$xRy \text{ if and only if } x \in A_i, y \in A_j \text{ for some } i, j \in I \text{ and there exists a } z \in A_k \text{ where } i, j \le k \text{ such that } \varphi_{ik}(x) = \varphi_{jk}(y) = z.$$

Then we define the algebraic direct limit A_∞ *as the set of all equivalence classes* $x/R = \{y|xRy\}$.

Example 2 (Algebraic Direct Limit) *Let* M_∞ *denote the algebraic direct limit of the direct family of sets* $\mathcal{A} = \{M_n(A)|n \in \mathbf{N}\}$*. Since each*

$$\varphi_{ij} : M_i(A) \to M_j(A)|T \mapsto diag[T, 0]$$

is an injective embedding, M_∞ *can be taken as the union* $\bigcup_{n=1}^{\infty}\{M_n(A)\}$.

Therefore, we can regard M_∞ *as the set of all infinite matrices with entries taken from* A *with only finitely many nonzero entries: for each* $T \in M_\infty(A)$ *there exists some* $n \in \mathbf{N}$ *such that* $T \in M_n(A)$.

We can now define the normed inductive limit taking us back into the category of normed spaces since we construct a norm on the algebraic direct limit:

Definition 7 (Normed inductive limit) *(Chapter II.3.3 [27]) Given a normed inductive system,* $(\mathcal{A}, \varphi_{ij})$*, there is a natural seminorm on the algebraic direct limit* A_∞*:*

$$|||x||| = \lim_i \sup_j \| \varphi_{ij}(x) \| \tag{B.24}$$

where $x \in A_i$*. The normed inductive limit is the quotient of the algebraic inductive limit by the elements of semi-norm 0.*

As it turns out, the normed inductive limit of the family of unital C*-algebras $\mathcal{A} = \{M_n(A)|n \in \mathbf{N}\}$ is highly algebraic in nature : the normed inductive limit coincides with the algebraic direct limit of $\mathcal{A}$.

Proposition 7 (The algebraic direct limit is the normed inductive limit) *The algebraic direct limit, M_∞, of the family of unital C*-algebras $\mathcal{A} = \{M_n(A)|n \in \mathbf{N}\}$ [Example 2] is exactly the normed inductive limit.*

*This follows from the fact that each φ_{ij} is an injective *-homomorphism or equivalently, an isometric embedding : $\| \varphi_{ij}(T) \| = \| T \|$ for all the j. Consequently, $||| \cdot |||$ is a norm: $|||T||| = 0$ if and only if $T = 0$. For any $T \in M_\infty$, there exists an $n \in \mathbf{N}$ such that T can be regarded as an element of $M_n(A)$ where the norm of T in M_∞ is precisely the norm in $M_n(A)$ which is a well defined C*-algebra norm [Chapter 1.1, equation (1.4)].*

The vector space operations in $M_\infty(A)$ are defined as follows:

For any $S,T \in M_\infty$, $S \in M_i(A), T \in M_j(A)$ for some $i,j \in \mathbf{N}$. Assuming without loss of generality that $j \geq i$, we consider S as $\varphi_{ij}(x) \in M_j(A)$ and we perform vector addition, $S+T$, and perform scalar multiplication in the context of the vector space $M_j(A)$. These are well defined operations, by virtue of the equivalence relation, R, defined in Definition 6.

Proposition 8 (M_∞ is a σ-unital normed *-algebra without an identity) *The normed vector space $M_\infty(A)$ becomes a normed *-algebra with an isometric involution by virtue of the fact that each $M_n(A)$ is a normed *-algebra:*

*For any $S,T \in M_\infty$, $S \in M_i(A), T \in M_j(A)$ for some $i,j \in \mathbf{N}$. Assuming without loss of generality that $j \geq i$, we define the product, ST, and involution all in the context of the *-algebra $M_j(A)$.*

Now M_∞ does not have an identity : suppose I is an identity of $M_\infty(A)$; then $I \in M_n(A)$ for some $n \in \mathbf{N}$ and if I_{n+1} denotes the identity matrix of $M_{n+1}(A)$, then $II_{n+1} = \varphi_{n,n+1}(I)I_{n+1} = \varphi_{n,n+1}(I) \neq I_{n+1}$

The net $\{I_n\}_{n=1}^\infty$ is an approximate identity of the C-algebra $M_\infty(A)$ and since it is indexed by the countable set $\mathbf{N}$, the C*-algebra $M_\infty(A)$ is σ-unital.*

We shall therefore identify $M_\infty(A)$ with the elements of $B(\sum_{i=1}^\infty \oplus \mathbf{H})$, where $\mathbf{H}$ is the Universal Hilbert space associated with the *-representation of the C*-algebra A, where $x \in M_\infty(A)$ is the infinite matrix with x on the upper left hand corner. We shall now show that $M_\infty(A)$ is far from being a complete *-subalgebra of the C*-algebra $B(\sum_{i=1}^\infty \oplus \mathbf{H})$:

Proposition 9 ($M_\infty(A)$ is not complete) *Consider the Cauchy sequence $(T_n)_{n=1}^{\infty} \in M_\infty(A)$ where T_n is the infinite diagonal matrix with the first n terms of the sequence $(\frac{1}{m}1_A)_{m=1}^{\infty}$ on the top left hand corner, 0 elsewhere, where 1_A is the identity element of A. This Cauchy sequence has no limit in $M_\infty(A)$.*

Proof. Consider the following string of inequalities:

$$\begin{aligned}
\| T_n - T_m \| &= \sup_{\|v\|=1} \| (T_n - T_m)(v) \| \\
&\leq \sum_i \sum_j \| (T_n - T_m)_{ij} \|^2 \| v \|^2 \qquad \text{(B.25)} \\
&= \sum_i \sum_j \| (T_n - T_m)_{ij} \|^2 \\
&< \sum_{k \geq n+1} (\frac{1}{k^2})
\end{aligned}$$

where equation (B.25) follows from equation (B.23). Since the series $\sum_{n=1} \frac{1}{n^2}$ is convergent, the remainder series, $\sum_{k \geq n+1}(\frac{1}{k})$, is a null sequence. Therefore the sequence $(T_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Suppose on the contrary that there exists a $T \in M_\infty(A)$ such that $T_n \to T$. Since $T \in M_\infty(A)$, there exists a $k \in \mathbf{N}$ such that $T \in M_k(A)$. Then,

$$\| T_{n+1} - T \| = \sup_{\|v\|=1} \| T_{n+1} - T \| (v) \geq \frac{1}{k+1} \qquad \text{(B.26)}$$

for all $n \geq k$ since the infinite diagonal matrix $T_{n+1} - T$ has the element $\frac{1}{k+1}1_A$ on the $(k+1)$- th row and $(k+1)$ -th column: the vector $e_{k+1} \in \sum_{1=1}^{\infty} \oplus \mathbf{H}$ which has $h \in \mathbf{H}$ on the $(k+1)$-th entry where $\| h \| = 1$ and 0 elsewhere, belongs to the unit sphere of $\sum_{1=1}^{\infty} \oplus \mathbf{H}$ and $\| (T_{n+1} - T)(e_{k+1}) \| = \frac{1}{k+1}$.

Q.E.D

By a similar application of inequality (B.23) it is trivial to see that the distinct elements in the net of the approximate identity $(I_n)_{n=1}^{\infty}$ are all distance 1 apart:

Proposition 10 *The distinct elements in the net of the approximate identity $(I_n)_{n=1}^{\infty}$ are all distance 1 apart.*

B.4 The stable algebra $A \bigodot K(\mathbf{H})$

The spatial tensor product $A \bigotimes K(\mathbf{H})$ where $K(\mathbf{H})$ is the non-unital C*-algebra of all the compact operators on an infinite dimensional separable Hilbert space, is a C*-algebra that can be identified with the C*-algebra completion of the normed *-algebra $M_\infty(A)$. The C*-algebra completion is the norm completion of $M_\infty(A)$ as a normed *-subalgebra of the C*-algebra $B(\sum_{i=1}^{\infty} \oplus \mathbf{H})$ [Appendix B3, Proposition 9]. This norm completion always yields a C*-algebra [Chapter VI.3, section 3.3 [9]] since the norm of $M_\infty(A)$ satisfies the strong C*-norm condition [Chapter 1.1, equation (1.4)]. We call and denote this C*-algebra completion of $M_\infty(A)$, the stable algebra, $A \bigodot K(\mathbf{H})$. Formally,

Definition 8 (Stable Algebra) *The stable algebra $A \bigodot K(\mathbf{H})$ is the norm completion of $M_\infty(A)$ taken as a normed *-subalgebra of the C*-algebra $B(\sum_{i=1}^{\infty} \oplus \mathbf{H})$.*

In the general case, the C*-completion of a normed space proceeds over three stages. We first complete $M_\infty(A)$ in the category of normed spaces [Chapter 4.1 pp 170 [20]]. We then complete this space as a Banach *-algebra and finally complete the Banach *-algebra as a C*-algebra [Chapter VI.10, [9]]. The third stage is needed since the completion as a Banach *-algebra does not necessarily lead to a symmetric algebra [Chapter 6.33, Example 33.6 [8]]: all C*-algebras are symmetric [Chapter VI.7, Theorem 7.11 [9]].

The identification of the spatial tensor product $A \bigotimes K(\mathbf{H})$ with the stable algebra, $A \bigodot K(\mathbf{H})$ requires the Hilbert space $\mathbf{H}$ to be separable. Intuitively, the separability of $\mathbf{H}$ enables us to construct a countable orthonormal basis and hence regard it as a 'limit' of the sequence $(\mathbf{C}^n)_{n=1}^{\infty}$; hence all finite rank operators on $\mathbf{H}$ can be regarded as a sequential limit of elements taken from $\{M_n(\mathbf{C}) | n \in \mathbf{N}\}$ enabling us to identify $F(\mathbf{H})$ with $M_\infty(A)$; since, every compact operator is a sequential limit of finite rank operators of $F(\mathbf{H})$ [Chapter 2, Theorem 2.4.5 [13]], in taking the spatial tensor product $A \bigotimes K(\mathbf{H})$, this limit process 'transfers' as elements in the norm completion of $M_\infty(A)$.

We now show that $A \bigodot K(\mathbf{H})$ does not have an identity, noting that the approximate identity is a divergent sequence in $A \bigodot K(\mathbf{H})$ [Appendix B3, Proposition 10]:

Proposition 11 ($A \bigodot K(\mathbf{H})$ does not have an identity) *The stable algebra $A \bigodot K(\mathbf{H})$ does not have an identity.*

Proof. Every $T \in A \bigodot K(\mathbf{H})$ has a matrix representation as the square summable operator valued infinite matrix $[T_{ij}]_{1 \leq i,j < \infty}$:

$$\sum_i \sum_j \| T_{ij} \|^2 < \infty \qquad \text{(B.27)}$$

where T_{ij} is defined similarly as in Proposition 5, Appendix B.2.2.

Therefore since the identity matrix has for its matrix representation the non-square summable infinite diagonal matrix with the identity operator $1_{\mathbf{H}}$ on each diagonal entry, it cannot belong to $A \bigodot K(\mathbf{H})$.

Q.E.D

We conclude by proving the anti-unitization theorem, Theorem 4, of Chapter 1.1:

Theorem 3 (Anti-Unitization) *If A is a C^*-algebra with an identity, there exists a C^*-algebra, the stable algebra $A \bigodot K(\mathbf{H})$, such that A embeds by a *-isometric isomorphism into $A \bigodot K(\mathbf{H})$ as a closed 2-sided ideal.*

Proof. The proof is evident by Proposition 11 on noting that the map:

$$\Phi : A \to A \bigodot K(\mathbf{H}) | a \mapsto diag[a, 0]$$

where $diag[a, 0]$ is the infinite operator valued matrix with a on the upper left hand corner entry and 0 elsewhere, is an isometric *-isomorphism.

By the definition of multiplication in $A \bigodot K(\mathbf{H})$ as matrix multiplication of the representative matrices, A is a 2-sided ideal of $A \bigodot K(\mathbf{H})$. Since A is complete and Φ is an isometry, A is a closed 2-sided ideal of $A \bigodot K(\mathbf{H})$.

Q.E.D

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Ideal perturbation of elements in C*-algebras

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Degree: MSc

SUMMARY

The aim of this thesis is to prove the lifting property of zero divisors, n-zero divisors, nilpotent elements and a criteria for the lifting of polynomially ideal elements in C*-algebras. Chapter 1 establishes the foundation on which the machinery to prove the lifting properties stated above rests upon. Chapter 2 proves the lifting of zero divisors in C*-algebras. The generalization of this problem to lifting n-zero divisors in C*-algebras requires the advent of the corona C*-algebra, a result of the school of non-commutative topology. The actual proof reduces the general case to the case of the corona of a non-unital σ-unital C*-algebra. Chapter 3 proves the lifting of the property of a nilpotent element also by a reduction to the case of the corona of a non-unital σ-unital C*-algebra. The case of the corona of a non-unital σ-unital C*-algebra is proved via a lifting of a triangular form in the corona. Finally in Chapter 4, a criterion is established to determine exactly when the property of a polynomially ideal element can be lifted. It is also shown that due to topological obstructions, this is not true in any C*-algebra.