APPENDIX 1. FLAC MODELS AND DERIVATIONS

A1.1 Applied models for FLAC code

A1.1.1 Model for the vertical stress comparison between the FLAC *ubiquitous joints* model and the theoretical development in Jaeger and Cook (1979)

```
title
Compressive strength of a shale specimen with a plane of
weakness
g 5 10
set mess off
def hsol
  loop k (0,18)
    beta=90.0*(18.0-k)/18.0
    alfa=90-beta
    command
      mo null
      mo ubi
      pro den 2700 bulk 4.5e9 she 2.3e9 fric 19 co 1.4e5 ten
3.5e5
      pro jco 1e5 jfric 8 jang alfa jten 1e6
      fix y j 1
      fix y j 11
      ini yvel -1e-7 j 11
      ini yvel 1e-7 j 1
      set st_damp comb
      step 4000
      print beta
      print sigmav
      print anal
    end_command
  end_loop
end
;
def sigmav
  sum=0.0
  loop i (1,igp)
    sum=sum+yforce(i,jgp)
  end_loop
  sigmav=sum/(x(igp,jgp)-x(1,jgp))
end

def ve
  ve=(ydisp(3,1)-ydisp(3,11))/(y(3,11)-y(3,1))
end
```

```
;
def anal
  mc=cohesion(1,1)
  mfi=friction(1,1)*degrad
  jc=jcohesion(1,1)
  jfi=jfriction(1,1)*degrad
  sm=2.0*mc*cos(mfi)/(1.0-sin(mfi))
  if beta=90*int(beta/90) then
    sj=-1
  else
    divsj=((1.0-
tan(jfi)*tan(beta*degrad))*sin(2.0*beta*degrad))
    if divsj=0.0 then
      sj=-1
    else
      sj=2.0*jc/divsj
    end_if
  end_if
  if sj<0 then
    anal=sm
  else
    anal=min(sj,sm)
  end_if
end

hist nstep 100
hist unbal
hist sigmav
hist anal
hist beta
hist ve
hist yv i 1 j 1
hsol
save UCT.sav
plot hold grid
plot hold his 2 3 cross vs 4 begin 4000 skip 40
return
```

A1.1.2 Model for homogeneous sandstone profile with undulated ground surface – 15^0 inclination at the limb’s surface

```
g 250,100
m m
prop s=5.2e9 b=5.9e9 d=2600 fri=21 coh=1e10 ten=1e10
def mon
        rj=1.0/jzones
        sum=0.0
        loop i (130,235)
                y_change=-1.1*sin(igp*degrad)
                y(i,1)=y(i-1,1)+0.9*y_change
```

```
                sum=sum+(y(i-1,1)-y(1,1))
                y(i,1)=y(i,1)-0.2*sum/i
                loop j (2,jgp-1)
                        y(i,j)=y(i,1) + (y(i,jgp)-
y(i,1))*(j-1)*rj
                end_loop
         end_loop
end
mon
fix x i=1
fix x i=251
fix x y j=1
hist unbal
set large
solve
;
;
title
k=2.0; 15 deg incl.
prop s=5.2e9 b=5.9e9 ten=5.5e6 coh=7e5 fri=21 d=2600
def k0_set
        loop i (1,izones)
                loop j (1,jzones)
                        sxx(i,j)=2.0*syy(i,j)
                end_loop
        end_loop
end
k0_set
;
set grav=9.81
ini xdis=0 ydis=0
solve
save k15-sst.sav
```

A1.1.3 FLAC model for 2m thick embedded shale layer at 30m depth and 5^0 inclination at the limb’s surface

```
g 250,100
m m j 1 70
m u j 71 72
m m j 73 100
prop s=5.2e9 b=5.9e9 d=2600 fri=21 coh=1e10 ten=1e10 j=1,70
prop s=2.3e9 b=4.5e9 d=2700 fri=14 coh=1e10 ten=1e10 j=71,72
prop ja=0 jc=1e10 jf=8 jt=1e10 j=71,72
prop s=5.2e9 b=5.9e9 d=2600 fri=21 coh=1e10 ten=1e10
j=73,100
def mon
        rj=1.0/jzones
        sum=0.0
        loop i (130,235)
```

```
                y_change=-1.1*sin(igp*degrad)
                y(i,1)=y(i-1,1)+0.3*y_change
                sum=sum+(y(i-1,1)-y(1,1))
                y(i,1)=y(i,1)-0.2*sum/i
                loop j (2,jgp-1)
                        y(i,j)=y(i,1) + (y(i,jgp)-
y(i,1))*(j-1)*rj
                end_loop
         end_loop
end
mon
set grav=9.81
fix x i=1
fix x i=251
fix x y j=1
hist ydis i=76 j=100
solve
;
;
title
k=2.0; 2m shale; 05 deg
prop s=5.2e9 b=5.9e9 ten=5.5e6 coh=7e5 fri=21 d=2600   j=1
70  ;SST
prop s=2.3e9 b=4.5e9 ten=3.5e6 coh=4.4e5 fri=14 d=2700
j=71,72 ;Shale
prop ja=0 jc=1e5 jf=8 jt=1e6 i=36 120 j=71,72
prop s=5.2e9 b=5.9e9 ten=5.5e6 coh=7e5 fri=21 d=2600
j=73,100;SST
def k0_set
        loop i (1,izones)
                loop j (1,jzones)
                        sxx(i,j)=2.0*syy(i,j)
                end_loop
        end_loop
end
k0_set
ini xdis=0 ydis=0
set large
solve
```

A1.2 Stress analysis

Many authors, such as Singh (1979) and Feda (1992), discuss shear stress as the only stress that triggers slope failure. Kulhawy et al. (1973) seem to be the first to mention the difference between the Mohr-Coulomb shear failure criterion and the stress failure criterion developed authors. Using finite element analysis, these calculated the safety factor based on stress level after the assumption that the rock is brought to failure by increasing the value of one of the principal stresses σ_1, while holding the other, σ_3, constant.

Figure A1.1a shows a flaw or a micro fracture in a two-dimensional Cartesian coordinate system. Let us assume that a pair of stresses acts on the flaw (presented by horizontal and vertical stress components). Their result is the stress normal to the flaw's plane. Since the flaw is not collinear with one of the principal stress directions, there is some shear stress at the flaw's plane as well. These stresses in the virgin stress conditions are in equilibrium at the flaw, so there is no flaw extension, propagation or coalescence with the neighbouring ones. These stress conditions are known as a virgin stress state and could be denoted as virgin horizontal (σ_{XX}^{V}), virgin vertical (σ_{YY}^{V}) and virgin shear (σ_{XY}^{V}) stress components. In this case we can denote the stress normal to the flaw's plane as σ_{N}^{V}.

mining activities (Figure A1.1b) bring about a change in the stress state, known as "resultant stress state". These stresses could be denoted as resultant horizontal stress (σ_{XX}^{R}), resultant vertical stress (σ_{YY}^{R}), and resultant shear stress (σ_{XY}^{R}). The first two resultant stress components (horizontal and vertical) will form a new resultant state, normal to stress of the flaw's plane (σ_{N}^{R}).

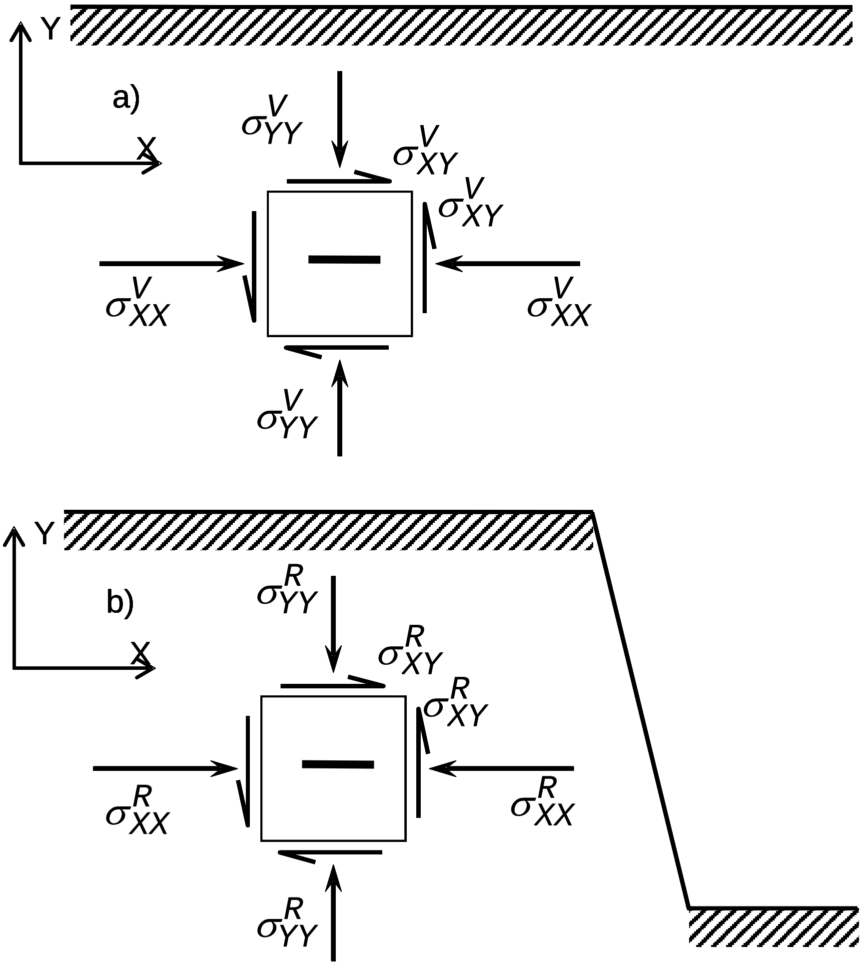

Figure A1.1

Stress state in the infinitesimal flaw, in virgin conditions, and b) after excavation

Therefore, we can say that any possible changes in the flaw pattern (extension, new flaw propagation, coalescence between microcracks in the rock or even failure) will result from the difference between those two loading conditions. Hence, it follows that:

$$\Delta\sigma_{XX} = \sigma^R_{XX} - \sigma^V_{XX} \tag{A1.1}$$

$$\Delta\sigma_{YY} = \sigma^R_{YY} - \sigma^V_{YY} \tag{A1.2}$$

$$\Delta\sigma_{XY} = \sigma_{XY}^{R} - \sigma_{XY}^{V} \tag{A1.3}$$

where $\Delta\sigma_{XX}$, $\Delta\sigma_{YY}$ and $\Delta\sigma_{XY}$ are the stress differences for the horizontal, vertical, and shear stress components. Olson (1993) uses this principle to calculate stress changes caused by tectonic irregularities.

One can easily see that the stress difference in Equations A1.1 – A1.3 has a negative sign in the case when the material relaxes from the virgin stress state or a positive sign in the case of increased loading when using rock mechanics sign conversion.

Normal force to the failure plane is in use as a basal element for the limit equilibrium methods in slope stability analyses. As mentioned in Chapter 1, observed failure planes with embedded anisotropic weaker layers are mainly parallel to the sedimentation.

According to the type of horizontal and vertical stresses (virgin and resultant), the normal to the failure plane induced stress (after Equations A1.1-A1.3) can be a combination of two compressive stresses, a combination of two tensile stresses or a combination of tensile and compressive stress. Figure A1.2 shows the case of biaxial tension applied to the material plane of weakness.

In matrix notation, stress transforms as follows

$$[\sigma]' = [\lambda][\sigma][\lambda]^T \tag{A1.4}$$

where $[\lambda]$ is directional *cosine* matrix. For the angle α shown in Figure A1.2

$$[\lambda]=\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \tag{A1.5}$$

Applying Equation A1.4, we could write

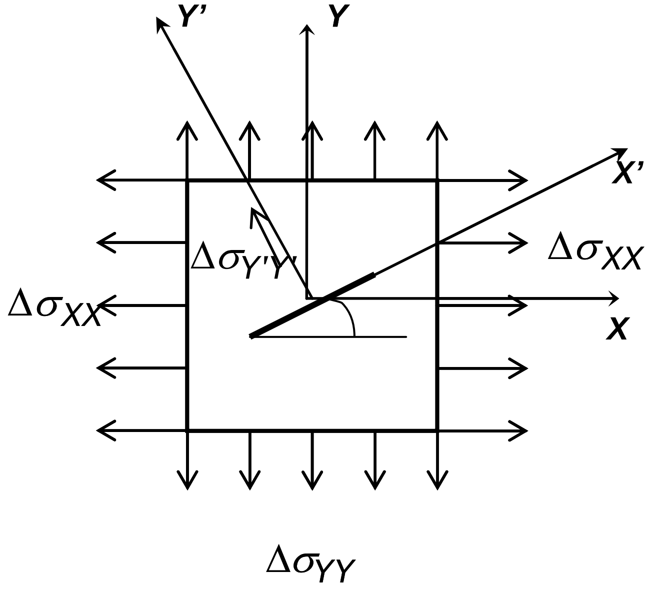

Figure A1.2

Stress state applied to the plane of weakness and remote biaxial loading

$$\begin{aligned}\sigma'_{11} &= \lambda_{11}\lambda_{11}\sigma_{11} + \lambda_{11}\lambda_{12}\sigma_{12} + \lambda_{12}\lambda_{11}\sigma_{21} + \lambda_{12}\lambda_{12}\sigma_{22} \\ &= \sigma_{11}\cos^2\alpha + \sigma_{12}\sin\alpha\cos\alpha + \sigma_{21}\sin\alpha\cos\alpha + \sigma_{22}\sin^2\alpha\end{aligned} \tag{A1.6}$$

$$\begin{aligned}\sigma'_{12} &= \lambda_{11}\lambda_{21}\sigma_{11} + \lambda_{11}\lambda_{22}\sigma_{12} + \lambda_{12}\lambda_{21}\sigma_{21} + \lambda_{12}\lambda_{22}\sigma_{22} \\ &= -\sigma_{11}\sin\alpha\cos\alpha + \sigma_{12}\cos^2\alpha - \sigma_{21}\sin^2\alpha + \sigma_{22}\sin\alpha\cos\alpha \\ &= \sigma'_{21}\end{aligned} \tag{A1.7}$$

$$\begin{aligned}\sigma'_{22} &= \lambda_{21}\lambda_{21}\sigma_{11} + \lambda_{21}\lambda_{22}\sigma_{12} + \lambda_{22}\lambda_{21}\sigma_{21} + \lambda_{22}\lambda_{22}\sigma_{22} \\ &= \sigma_{11}\sin^2\alpha - \sigma_{12}\sin\alpha\cos\alpha - \sigma_{21}\sin\alpha\cos\alpha + \sigma_{22}\cos^2\alpha\end{aligned} \tag{A1.8}$$

Hence, for the new set of the co-ordinate system ($X'Y'$), the normal stress difference $\Delta\sigma_{Y'Y'} = \Delta\sigma_N$ will have the form of

$$\Delta\sigma_{Y'Y'} = \Delta\sigma_N = \Delta\sigma_{XX}\,\boldsymbol{sin}^2\alpha - 2\Delta\sigma_{XY}\,\boldsymbol{sin}\,\alpha\,\boldsymbol{cos}\,\alpha + \Delta\sigma_{YY}\,\boldsymbol{cos}^2\alpha \tag{A1.9}$$

where $\Delta\sigma_{XX}$, $\Delta\sigma_{YY}$ and $\Delta\sigma_{XY}$ are the stress differences between the stress state after slope excavation and the virgin stress state for the horizontal, vertical and shear stress components respectively. See Equations A1.1 to A1.3.

A1.3 Equations in Chapter 4

<u>A1.3.1 Equation 4.16</u>

$$K_I^D = -2\sigma_t\sqrt{\frac{1}{W}\boldsymbol{tan}\frac{\pi c}{W}}\int_a^c \boldsymbol{cos}\frac{\pi\xi}{W}\left[\left(\boldsymbol{sin}\frac{\pi c}{W}\right)^2 - \left(\boldsymbol{sin}\frac{\pi\xi}{W}\right)^2\right]^{-\frac{1}{2}} d\xi \tag{A1.10}$$

Replace $\boldsymbol{sin}\frac{\pi\xi}{W} = u$. Then $du = \frac{\pi}{W}\boldsymbol{cos}\frac{\pi\xi}{W}d\xi$ and Equation A1.10 has the form:

$$K_I^D = -\frac{2\sigma_t W\sqrt{\frac{1}{W}\boldsymbol{tan}\frac{\pi c}{W}}}{\pi}\int_{\boldsymbol{sin}\frac{\pi}{W}a}^{\boldsymbol{sin}\frac{\pi}{W}c} \frac{du}{\sqrt{\left(\boldsymbol{sin}\frac{\pi c}{W}\right)^2 - u^2}} = \tag{A1.11}$$

$$= -\frac{2\sigma_t W}{\pi}\sqrt{\frac{1}{W}\tan\frac{\pi c}{W}}\,\arcsin\left.\frac{u}{\sin\frac{\pi c}{W}}\right|_{\sin\frac{\pi a}{W}}^{\sin\frac{\pi c}{W}} = \tag{A1.12}$$

$$= -\frac{2\sigma_t W}{\pi}\sqrt{\frac{1}{W}\tan\frac{\pi c}{W}}\left[\arcsin 1 - \arcsin\frac{\sin\left(\frac{\pi a}{W}\right)}{\sin\left(\frac{\pi c}{W}\right)}\right] = \tag{A1.13}$$

$$= -\frac{2\sigma_t W}{\pi}\sqrt{\frac{1}{W}\tan\frac{\pi c}{W}}\left[\frac{\pi}{2} - \arcsin\frac{\sin\left(\frac{\pi a}{W}\right)}{\sin\left(\frac{\pi c}{W}\right)}\right] = \tag{A1.14}$$

$$= -\sigma_t\sqrt{W\tan\frac{\pi c}{W}}\left[1 - \frac{2}{\pi}\arcsin\frac{\sin\left(\frac{\pi a}{W}\right)}{\sin\left(\frac{\pi c}{W}\right)}\right] \tag{A1.15}$$

A1.3.2 Equation 4.19

$$\Delta\sigma_N\sqrt{W\tan\frac{\pi c}{W}} - \sigma_t\sqrt{W\tan\frac{\pi c}{W}}\left[1 - \frac{2}{\pi}\arcsin\frac{\sin\left(\frac{\pi a}{W}\right)}{\sin\left(\frac{\pi c}{W}\right)}\right] = 0 \tag{A1.16}$$

Dividing both sides on Equation A1.16 by $\sqrt{W\tan\frac{\pi c}{W}}$:

$$\Delta\sigma_N - \sigma_t\left[1 - \frac{2}{\pi}\arcsin\frac{\sin\left(\frac{\pi a}{W}\right)}{\sin\left(\frac{\pi c}{W}\right)}\right] = 0 \tag{A1.17}$$

$$\frac{2\sigma_t}{\pi}\,\mathbf{arcsin}\frac{\mathbf{sin}\left(\dfrac{\pi a}{W}\right)}{\mathbf{sin}\left(\dfrac{\pi c}{W}\right)} = \sigma_t - \Delta\sigma_N \tag{A1.18}$$

$$\mathbf{arcsin}\frac{\mathbf{sin}\left(\dfrac{\pi a}{W}\right)}{\mathbf{sin}\left(\dfrac{\pi c}{W}\right)} = \frac{\pi(\sigma_t - \Delta\sigma_N)}{2\sigma_t} \tag{A1.19}$$

$$\frac{\mathbf{sin}\left(\dfrac{\pi a}{W}\right)}{\mathbf{sin}\left(\dfrac{\pi c}{W}\right)} = \mathbf{sin}\frac{\pi(\sigma_t - \Delta\sigma_N)}{2\sigma_t} \tag{A1.20}$$

$$\mathbf{sin}\left(\frac{\pi c}{W}\right) = \frac{\mathbf{sin}\left(\dfrac{\pi a}{W}\right)}{\mathbf{sin}\left(0.5\pi - \dfrac{0.5\pi\Delta\sigma_N}{\sigma_t}\right)} \tag{A1.21}$$

$$\frac{\pi c}{W} = \mathbf{arcsin}\frac{\mathbf{sin}\left(\dfrac{\pi a}{W}\right)}{\mathbf{cos}\left(\dfrac{0.5\pi\Delta\sigma_N}{\sigma_t}\right)} \tag{A1.22}$$

$$c = \frac{W}{\pi}\,\mathbf{arcsin}\frac{\mathbf{sin}\left(\dfrac{\pi a}{W}\right)}{\mathbf{cos}\left(\dfrac{0.5\pi\Delta\sigma_N}{\sigma_t}\right)} \tag{A1.23}$$

Replacing *c* with $c = a + l$

$$l = \frac{W}{\pi} \arcsin \frac{\sin\left(\frac{\pi a}{W}\right)}{\cos\left(\frac{0.5\pi\Delta\sigma_N}{\sigma_t}\right)} - a \qquad \text{(A1.24)}$$

A1.3.3 Equation 4.21

$$l_c = \frac{W}{\pi} \arcsin \frac{\sin(\pi\, a / W)}{\cos(0.5\pi\, \Delta\sigma_N / \sigma_t)} - a \qquad \text{(A1.25)}$$

$$\frac{\pi(l_c + a)}{W} = \arcsin \frac{\sin(\pi\, a / W)}{\cos(0.5\pi\, \Delta\sigma_N / \sigma_t)} \qquad \text{(A1.26)}$$

$$\sin[\pi(l_c + a)/W] = \frac{\sin(\pi\, a / W)}{\cos(0.5\pi\, \Delta\sigma_N / \sigma_t)} \qquad \text{(A1.27)}$$

$$\cos(0.5\pi\, \Delta\sigma_N / \sigma_t) = \frac{\sin(\pi\, a / W)}{\sin[\pi(l_c + a)/W]} \qquad \text{(A1.28)}$$

$$0.5\pi\, \Delta\sigma_N / \sigma_t = \arccos \frac{\sin(\pi\, a / W)}{\sin[\pi(l_c + a)/W]} \qquad \text{(A1.29)}$$

$$\Delta\sigma_N^P = \frac{2\sigma_t}{\pi} \arccos \frac{\sin(\pi\, a / W)}{\sin[\pi(l_c + a)/W]} \qquad \text{(A1.30)}$$

A1.3.4 Equation 4.24

Substituting Equation 4.19 into Equation 4.23:

$$2a + \frac{2W}{\pi}\,\boldsymbol{arcsin}\frac{\boldsymbol{sin}(\pi a / W)}{\boldsymbol{cos}(0.5\pi\Delta\sigma_N / \sigma_t)} - 2a = W \qquad \text{(A1.31)}$$

and

$$\frac{2}{\pi}\,\boldsymbol{arcsin}\frac{\boldsymbol{sin}(\pi a / W)}{\boldsymbol{cos}(0.5\pi\Delta\sigma_N / \sigma_t)} = 1 \qquad \text{(A1.32)}$$

$$\frac{\boldsymbol{sin}(\pi a / W)}{\boldsymbol{cos}(0.5\pi\Delta\sigma_N / \sigma_t)} = \boldsymbol{sin}\left(\frac{\pi}{2}\right) = 1 \qquad \text{(A1.33)}$$

$$\boldsymbol{sin}(\pi a / W) = \boldsymbol{cos}\left(\frac{\pi}{2}\frac{\Delta\sigma_N}{\sigma_t}\right) \qquad \text{(A1.34)}$$

We can replace $\boldsymbol{sin}\left(\frac{\pi a}{W}\right)$ at the left hand side of the Equation A1.34 with a $\boldsymbol{cos}(f)$, which is shown in Equations A1.35a and A1.35b below.

$$\boldsymbol{sin}\left(\frac{\pi a}{W}\right) = \begin{cases} -\boldsymbol{cos}\left(\frac{\pi}{2} + \frac{\pi a}{W}\right) & \text{(A1.35a)} \\ \text{or} & \\ \boldsymbol{cos}\left(\frac{\pi}{2} - \frac{\pi a}{W}\right) & \text{(A1.35b)} \end{cases}$$

Let us first combine Equations A1.34 and A1.35a. Then we have:

$$-\boldsymbol{cos}\left(\frac{\pi}{2} + \frac{\pi a}{W}\right) = \boldsymbol{cos}\left(\frac{\pi\Delta\sigma_N}{2\sigma_t}\right) \qquad \text{(A1.36)}$$

As the $\cos\upsilon = \cos(-\upsilon)$, then the Equation A1.36 has four possible solutions. If the both arguments are with the same sign ("+" or "-"), then after applying the inverse cosine function the Equation A1.36 can be written as:

$$-\left(\frac{\pi}{2} + \frac{\pi\, a}{W}\right) = \frac{\pi}{2}\frac{\Delta\sigma_N}{\sigma_t} \tag{A1.37}$$

Dividing both sides of Equation A1.37 by π/2 we obtain with further manipulation:

$$-\frac{W + 2a}{W} = \frac{\Delta\sigma_N}{\sigma_t} \tag{A1.38}$$

It is seen that Equation A1.38 is an impossible solution because the left-hand side of the equation always will have negative value (W and a are real positive numbers), while the right-hand side always will be positive.

If we assume that both arguments of cosine functions in Equation A1.36 are with opposite signs, then we will have:

$$\frac{\pi}{2} + \frac{\pi a}{W} = \frac{\pi}{2}\frac{\Delta\sigma_N}{\sigma_t}. \tag{A1.39}$$

and similar to Equation A1.38 we can write:

$$\frac{W + 2a}{W} = \frac{\Delta\sigma_N}{\sigma_t} \tag{A1.40}$$

It is also seen that Equation A1.40 is impossible solution because left-hand side of the equation always will have value bigger than one, while the right-hand side always will have value lower than one.

Let us now combine Equations A1.34 and A1.35b. Then we have again two options: both cosine arguments are either the same or opposite sign. Let us first consider the case with the same sign arguments. Hence, we will have equation in the form of:

$$\cos\left(\frac{\pi}{2}-\frac{\pi\, a}{W}\right)=\cos\left(\frac{\pi\,\Delta\sigma_N}{2\sigma_t}\right) \quad \text{(A1.41)}$$

Therefore,

$$\frac{\pi}{2}-\frac{\pi\, a}{W}=\frac{\pi}{2}\frac{\Delta\sigma_N}{\sigma_t} \quad \text{(A1.42)}$$

Dividing both sides of Equation A1.40 by π/2, we obtain after further manipulation:

$$\frac{W-2a}{W}=\frac{\Delta\sigma_N}{\sigma_t}. \quad \text{(A1.43)}$$

Equation A1.43 has only meaning if $W-2a>0$ (particularly, when $W>2a$.) and can be used in the further because its both sides are with the same sign. After assuming that $2a$ is smaller than W, both sides of Equation A1.43 are positive and smaller than one.

If we assume that the cosine arguments in Equation A1.41 have different signs, then we can write:

$$\frac{\pi\, a}{W}-\frac{\pi}{2}=\frac{\pi}{2}\frac{\Delta\sigma_N}{\sigma_t} \quad \text{(A1.44)}$$

and

$$\frac{2a-W}{W}=\frac{\Delta\sigma_N}{\sigma_t}. \quad \text{(A1.45)}$$

It can be seen that Equation A1.45 is also possible solution because both sides of the equation are the same sign and smaller than one in cases where $2a > W$. On the other hand the condition $2a > W$ does not comply with Equation 4.23 and Figure 4.9, Chapter 4. Therefore, the only possible solution is the equation A1.43. If 2a = W, then we will have pre-existing tensile fracture propagation and consequently, $\Delta\sigma_N = 0$.

Hence we can write

$$\sigma_N^P = \sigma_t \frac{W - 2a}{W} \qquad \text{(A1.46)}$$

A1.4 Equations in Chapter 5

Equation 5.16

$$\tan\delta = \tan\phi + \frac{cl}{R\cos\delta} \qquad \text{(A1.47)}$$

Multiply both sides of Equation A1.47 by $\cos\delta$, we will have

$$\sin\delta = \tan\phi\cos\delta + \frac{cl}{R} \qquad \text{(A1.48)}$$

Substituting $\cos\delta = \sqrt{1 - \sin^2\delta}$ in Equation A1.48:

$$\sin\delta = \tan\phi\sqrt{1 - \sin^2\delta} + \frac{c\,l}{R} \qquad \text{(A1.49)}$$

If we transfer the coefficient $\frac{c\,l}{R}$ from right-hand side of Equation A1.48 to the left-hand side and square both sides:

$$\sin^2\delta - 2\frac{c\,l}{R}\sin\delta + \left(\frac{c\,l}{R}\right)^2 = \tan^2\phi - \sin^2\delta\tan^2\phi \qquad \text{(A1.50)}$$

Equation A1.50 becomes after simplification:

$$\sin^2\delta\left(1+\tan^2\phi\right)-2\frac{c\,l}{R}\sin\delta+\left(\frac{c\,l}{R}\right)^2-\tan^2\phi=0 \tag{A1.51}$$

This is a quadratic equation in $\sin\delta$ in which there is a real solution only if:

$$4\left(\frac{cl}{R}\right)^2-4\left(1+\tan^2\phi\right)\left(\frac{c^2l^2}{R^2}-\tan^2\phi\right)\geq 0\,. \qquad \text{A1.52)}$$

Equation A1.51 simplifies to

$$\sec^2\phi-\left(\frac{c\,l}{R}\right)^2\geq 0 \tag{A1.53}$$

where

$$\sec\phi\geq\frac{cl}{R} \tag{A1.54}$$

and

$$\phi\geq\sec^{-1}\left(\frac{cl}{R}\right) \tag{A1.55}$$