Appendix A

Element operators

A.1 Membrane element operators

In (2.71), the operators $\boldsymbol{B}_i$ and $\boldsymbol{G}_{\psi i}$ are given as

$$\boldsymbol{B}_i = \begin{bmatrix} N_{i,1} & 0 \\ 0 & N_{i,2} \\ N_{i,2} & N_{i,1} \end{bmatrix} \; ; \; i = 1,2,3,4 \tag{A.1}$$

with N_i, $i = 1,2,3,4$, the Lagrangian interpolation functions. The strain operator associated with the drilling rotation is defined by

$$\boldsymbol{G}_{\psi i} = \frac{1}{8} \begin{bmatrix} (l_{ij}\cos\alpha_{ij}N_{l,1} - l_{ik}\cos\alpha_{ik}N_{m,1}) \\ (l_{ij}\sin\alpha_{ij}N_{l,2} - l_{ik}\sin\alpha_{ik}N_{m,2}) \\ \left\{ \begin{array}{c} l_{ij}\cos\alpha_{ij}N_{l,2} - l_{ik}\cos\alpha_{ik}N_{m,2} \\ + \\ l_{ij}\sin\alpha_{ij}N_{l,1} - l_{ik}\sin\alpha_{ik}N_{m,1} \end{array} \right\} \end{bmatrix} \tag{A.2}$$

where l_{jk} represent the lengths of sides jk and, using a FORTRAN pseudo language,

$$i = 1,2,3,4; \; m = i + 4; \; l = m - 1 + 4\ \text{int}(1/i);$$

$$k = \text{mod}(m,4) + 1; \; j = l - 4 \tag{A.3}$$

The functions N_i, $i = 5,6,7,8$ are serendipity mid-side interpolation functions.
The operators associated with the penalty stiffness $(\gamma/\Omega^e)\boldsymbol{h}^e\boldsymbol{h}^{eT}$ and $\boldsymbol{P}^e_\gamma$ are

$$\boldsymbol{b}_i = < -\frac{1}{2}N_{i,2} \;\; \frac{1}{2}N_{i,1} > \; ; \;\; i = 1,2,3,4 \tag{A.4}$$

and

$$\begin{aligned} g_i = & -\frac{1}{16}(l_{ij}\cos\alpha_{ij}N_{l,2} - l_{ik}\cos\alpha_{ik}N_{m,2}) \\ & +\frac{1}{16}(l_{ij}\sin\alpha_{ij}N_{l,1} - l_{ik}\sin\alpha_{ik}N_{m,1}) - N_i \; ; \;\; i = 1,2,3,4 \end{aligned} \tag{A.5}$$

with indices j, k, l, m again defined by (A.3). In (2.69), a FORTRAN-like definition of adjacent corner nodes is also employed:

$$j = i - 4 \; ; \quad k = \mathrm{mod}(i, 4) + 1 \tag{A.6}$$

A.2 Plate element operators

In (4.40) the element curvature-displacement matrix is given by [38]

$$\boldsymbol{B}_{bi} = \begin{bmatrix} 0 & N_{i,1} & 0 \\ 0 & 0 & N_{i,2} \\ 0 & N_{i,2} & N_{i,1} \end{bmatrix} \tag{A.7}$$

and in (4.42) the element shear strain-displacement matrix is given by [38]

$$\boldsymbol{B}_{si} = \begin{bmatrix} N_{i,1} & -N_i & 0 \\ N_{i,2} & 0 & -N_i \end{bmatrix} \tag{A.8}$$

Appendix B

Classification of stress modes

After Feng *et al.*, the following constant and linear stress modes are defined

$$[\{\sigma_1\}\{\sigma_2\}\{\sigma_3\}\{\sigma_4\}\{\sigma_6\}\{\sigma_8\}\{\sigma_5\}\{\sigma_7\}\{\sigma_9\}] = \begin{bmatrix} 1 & 0 & 0 & \xi & 0 & 0 & \eta & 0 & 0 \\ 0 & 1 & 0 & 0 & \xi & 0 & 0 & \eta & 0 \\ 0 & 0 & 1 & 0 & 0 & \xi & 0 & 0 & \eta \end{bmatrix}$$

i.e. $\{\sigma_1\} = \{1\ 0\ 0\}^T$, etc. Four alternative stress modes are defined as

$$[\{\sigma_{10}\}\{\sigma_{11}\}\{\sigma_{12}\}\{\sigma_{13}\}] = \begin{bmatrix} 1 & 1 & 0 & -\xi \\ 1 & -1 & -\eta & 0 \\ 0 & 0 & \xi & \eta \end{bmatrix}$$

while the higher order terms are here defined as

$$[\{\sigma_{14}\}\{\sigma_{15}\}\{\sigma_{16}\}\cdots\{\sigma_{24}\}] = \begin{bmatrix} \xi^2 & 0 & 0 & \eta^2 & 0 & 0 & \xi\eta & 0 & 0 & \eta^2 & \xi^2 \\ 0 & \xi^2 & 0 & 0 & \eta^2 & 0 & 0 & \xi\eta & 0 & -\xi^2 & -\eta^2 \\ 0 & 0 & \xi^2 & 0 & 0 & \eta^2 & 0 & 0 & \xi\eta & 0 & 0 \end{bmatrix}$$

For the 5β family, modes 20 through 24 belong to the zero-energy stress mode. For the 8β(M), 8β(D), 9β(M) and 9β(D) families however, these modes contribute energy.

Appendix C

Constraining the assumed stress field

The transformation operators $\boldsymbol{T}_0$, $\boldsymbol{T}$ and $\boldsymbol{Q}$ are given below:

$$\boldsymbol{T}_0 = \begin{bmatrix} a_1^2 & a_3^2 & 2a_1a_3 \\ b_1^2 & b_3^2 & 2b_1b_3 \\ a_1b_1 & a_3b_3 & a_1b_3 + a_3b_1 \end{bmatrix} \tag{C.1}$$

with the parameters a_i and b_i defined by

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} \tag{C.2}$$

Also,

$$\boldsymbol{T} = \begin{bmatrix} J_{11}^2 & J_{21}^2 & 2J_{11}J_{21} \\ J_{12}^2 & J_{22}^2 & 2J_{12}J_{22} \\ J_{11}J_{12} & J_{21}J_{22} & J_{11}J_{22} + J_{12}J_{21} \end{bmatrix} \tag{C.3}$$

with

$$J_{ki} = \frac{\partial x^i}{\partial \xi^k} = t^k \quad (i, k = 1, 2) \tag{C.4}$$

Finally,

$$\boldsymbol{Q} = \frac{1}{|\boldsymbol{J}|} \begin{bmatrix} \rho & 0 & 0 \\ 0 & 1/\rho & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \rho = \sqrt{\frac{J_{21}^2 + J_{22}^2}{J_{11}^2 + J_{12}^2}} \tag{C.5}$$

Appendix D

Reduced integration

D.1 Derivation of numerical integration schemes[1]

Consider the area integral given by

$$I = \int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)\ \mathrm{d}\xi\mathrm{d}\eta \tag{D.1}$$

where $F(\xi,\eta)$ is any polynomial function of ξ and η. Any polynomial expression of two variables can be expressed in the form

$$F(\xi,\eta) = \sum_{i,j} A_{ij}\xi^i\eta^i \tag{D.2}$$

No limits are placed on the summation indices i and j as any arbitrary polynomial is being considered.

Let any N-point rule be written as

$$I^* = \sum_{n=1}^{N} W_n F(\xi,\eta) \tag{D.3}$$

where I^* represents the numerical approximation to I. Integration point n is given by (ξ_n,η_n) and the associated weight is given as W_n.

Each term of (D.2) may be trivially integrated as follows

$$\int_{-1}^{1}\int_{-1}^{1} A_{ij}\xi^i\eta^i\ \mathrm{d}\xi\mathrm{d}\eta = \begin{cases} \frac{2^2 A_{ij}}{(i+1)(j+1)} & i,j \text{ both even} \\ 0 & \text{otherwise} \end{cases} \tag{D.4}$$

Application of the quadrature rule of (D.3) to the function $F(\xi,\eta)$ in the form of (D.2) gives the following result which is expressed in terms of the coefficients A_{ij} as

$$I^* = A_{00}\sum_{n=1}^{N} W_n + A_{10}\sum_{n=1}^{N} W_n\xi_n + A_{01}\sum_{n=1}^{N} W_n\eta_n + A_{20}\sum_{n=1}^{N} W_n\xi_n^2 + \ldots \tag{D.5}$$

Two points are to be noted:

- Symmetry of the rule in each coordinate implies that the coefficients corresponding to all odd powers will vanish in (D.5). This of course corresponds to the vanishing of the integral of odd powers over this region.
- Symmetry with respect to both coordinates is required to ensure invariance of the rule.

Equating the coefficients of A_{ij} between (D.4) and (D.5) gives a series of equations in the weights W_n and the coordinates ξ_n and η_n. Evidently the number of equations that are satisfied for a particular set of weights and coordinates indicate which polynomial terms are integrated exactly by that particular rule. Also, the degree to which each remaining equation is not satisfied gives the error in that polynomial term. Each equation has the form

$$\sum_{n=1}^{N} W_n\xi_n^i\eta_n^j = \frac{2^2}{(i+1)(j+1)} \tag{D.6}$$

for the coefficient A_{ij}. Clearly all equations containing odd values for either i or j are satisfied identically for symmetric rules.

The maximum number of equations needed for (D.6) is determined by the order of the function $F(\xi,\eta)$ which is to be integrated. If the maximum number of equations possible are satisfied for a particular configuration, then an optimal scheme for that configuration is obtained. However, if less than the maximum number are satisfied a less efficient rule is obtained, but freedom is available for arbitrary selection of some values of weights or coordinates.

D.2 A 5-point integration scheme

The first 5-point integration scheme presented by Dovey [1] is employed to selected problems. (See Figure D.1).

Due to symmetry, the weights W_α are identical. The rule is indicated by

$$I^* = W_0F(0,0) + W_\alpha F(\pm\alpha,\pm\alpha) \tag{D.7}$$

The second term in (D.7) indicates four points when all combinations of positive and negative signs are taken.

Employing (D.6) we obtain the first four equations for the appropriate terms A_{ij} as

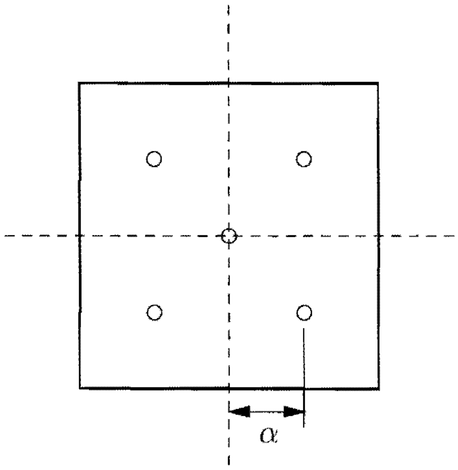

Figure D.1: 5-point integration scheme

$$\begin{array}{llcl} A_{00} & : W_0 + 4W_\alpha & = & 4 \\ A_{20}, A_{02} & : 4W_\alpha \alpha^2 & = & \frac{4}{3} \\ A_{22} & : 4W_\alpha \alpha^4 & = & \frac{4}{9} \\ A_{40}, A_{04} & : 4W_\alpha \alpha^4 & = & \frac{4}{5} \end{array} \tag{D.8}$$

The last two of these equations are directly inconsistent and so the last is discarded. Also, however, the first three are inconsistent if the center point is retained.

Solving (D.8) leads to

$$\begin{aligned} W_0 &= 0 \\ W_\alpha &= 1 \\ \alpha &= \frac{1}{\sqrt{3}} \end{aligned} \tag{D.9}$$

which is the 2×2 Gaussian product rule. The leading error term is defined by the last of (D.8) and gives the error $(I^* - I)$, corresponding to the fourth power terms ξ^4 and η^4 as

$$E_{40} = \left(4W_\alpha \alpha^4 - \frac{4}{5}\right) A_{40} = E_{04} = \left(4W_\alpha \alpha^4 - \frac{4}{5}\right) A_{04} \tag{D.10}$$

However, the center point may be retained by selecting the value of W_0, computing W_α and α from the first two relationships in (D.8). This implies an error in the A_{22} term. The scheme is now defined by

$$W_\alpha = 1 - \frac{W_0}{4} \tag{D.11}$$

$$\alpha = \left(\frac{1}{3W_\alpha}\right)^{\frac{1}{2}} \tag{D.12}$$

The scheme only has physical meaning while $0 \leq W_0 \leq 4$. The error in the A_{22} term is minimized as $W_0 \rightarrow 0$. In practice this implies that the 5-point scheme converges to the 2×2 Gaussian scheme as $W_0 \rightarrow 0$.

D.3 An 8-point integration scheme

The 8-point rule is depicted in Figure D.2.

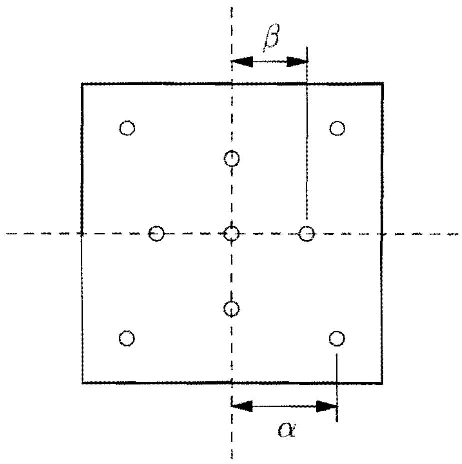

Figure D.2: 8-point integration scheme

This rule was previously employed for membrane elements with in-plane rotational degrees of freedom by Stander and Wilson [62] in the QC9(8) element and also by Ibraimbegovic *et al.* [23] in their drilling degree of freedom membrane element. The rule is described by

$$I^* = W_\alpha F(\pm\alpha, \pm\alpha) + W_\beta[F(\pm\beta, 0) + F(0, \pm\beta)] \tag{D.13}$$

Due to symmetry, the weights W_α are identical as are the weights W_β. The governing equations are given by

$$\begin{array}{lcrcl} A_{00} & : & 4W_\alpha + 4W_\beta & = & 4 \\ A_{20}, A_{02} & : & 4W_\alpha\alpha^2 + 2W_\beta\beta^2 & = & \frac{4}{3} \\ A_{22} & : & 4W_\alpha\alpha^4 & = & \frac{4}{9} \\ A_{40}, A_{04} & : & 4W_\alpha\alpha^4 + 2W_\beta\beta^4 & = & \frac{4}{5} \end{array} \tag{D.14}$$

All four equations may be satisfied and the solution is

$$\begin{aligned} \alpha &= \sqrt{\frac{7}{9}} \\ W_\alpha &= \frac{9}{49} \end{aligned}$$

$$\begin{aligned} \beta &= \sqrt{\frac{7}{15}} \\ W_\beta &= \frac{40}{49} \end{aligned} \tag{D.15}$$

This rule gives the same order of accuracy as the 3×3 Gaussian rule. A scheme of lower accuracy is defined by

$$W_\alpha = 1 - W_\beta \tag{D.16}$$

$$\alpha = \left(\frac{1}{9W_\alpha}\right)^{\frac{1}{4}} \tag{D.17}$$

$$\beta = \left(\frac{\frac{2}{3} - 2W_\alpha \alpha^2}{W_\beta}\right)^{\frac{1}{4}} \tag{D.18}$$

The bounds for W_β are $0 \leq W_\beta \leq 1$. In [23, 62] the typical choice of $W_0 = 0.01$ was employed.