Chapter 4

Isotropic flat shell elements

In this chapter, flat shell elements are formulated through the assembly of membrane and plate elements. The exact solution of a shell approximated by flat facets compared to the exact solution of a truly curved shell may reveal considerable differences in the distribution of bending moments, shearing forces, etc. However, for 'simple' elements the discretization error is approximately of the same order and excellent results can be obtained with the flat shell approximation [39]. Apart from being easy to define geometrically, flat shell elements will always converge to the correct deep shell solution in the limit of mesh refinement [46].

4.1 Plate formulation

For the plate component of the flat shell element the shear deformable formulation of Mindlin is employed. The final formulation is modified to include the assumed strain interpolation of Bathe and Dvorkin [47]. [1]

4.1.1 Mindlin plates: Bending theory and variational formulation

In this section the treatments of Hinton and Huang [48] and Papadopoulos and Taylor [49] are followed closely, albeit with different notations. However, the same may be found in the standard works of, for instance Hughes [38], Zienkiewicz and Taylor [39] and Bathe [50].

The simplest plate formulation which accounts for the effect of shear deformation, is presented. The transverse shear is assumed constant throughout the thickness. The assumptions of the first order Mindlin theory are

$$\sigma_{33} = 0 \tag{4.1}$$

$$u_1 = x_3\psi_1(x_1, x_2)$$

[1] From now on, the drilling degree of freedom ψ, introduced in Chapter 2, is denoted ψ_3 for reasons of clarity.

$$\begin{aligned} u_2 &= x_3\psi_2(x_1, x_2) \\ u_3 &= u_3(x_1, x_2) \end{aligned} \tag{4.2}$$

where u_1, u_2 and u_3 are the displacements components in the x_1, x_2 and x_3 directions respectively, u_3 is the lateral displacement and ψ_1 and ψ_2 are the normal rotations in the x_{13} and x_{23} planes respectively (See Figure 4.1). The element is assumed to be flat, with thickness t. Flatness of the plate is not a necessary assumption, but merely simplifies the required notation and implementation. The element area is denoted Ω.

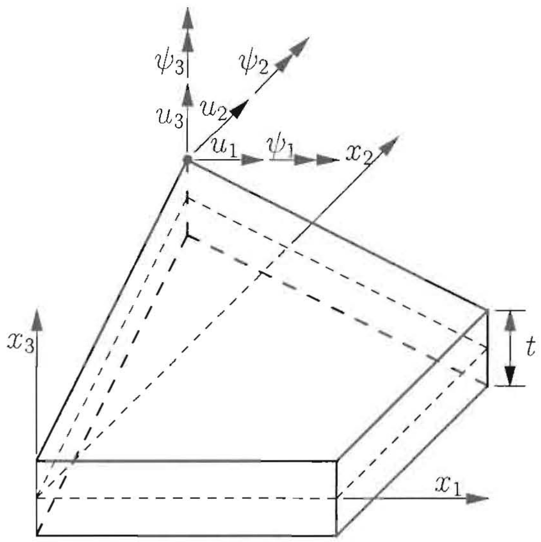

Figure 4.1: Four-node shell element

(4.1) is obviously inconsistent with three-dimensional elasticity. However, the transverse normal stress may be neglected for plates where the thickness is small compared with the other dimensions. Moreover, when a linear or constant through-the-thickness displacement assumption is made, as is customary in the shear-deformable plate theories, limited locking occurs due to the Poisson effect, when σ_{33} is restrained. (4.2) implies that straight normals to the reference surface, $x_3 = 0$, remain straight, but do not necessarily remain normal to the plate after deformation (Figure 4.2). Also, the transverse displacement u_3 is constant through the thickness.

The displacement field assumed in (4.2) yields in-plane strains of the form

$$\begin{aligned} \epsilon_{11} &= x_3\psi_{1,1} \\ \epsilon_{22} &= x_3\psi_{2,2} \\ \gamma_{12} &= x_3(\psi_{1,2} + \psi_{2,1}) \end{aligned} \tag{4.3}$$

where

$$\psi_{1,1} = \frac{\partial \psi_1}{\partial x_1}, \text{ etc.} \tag{4.4}$$

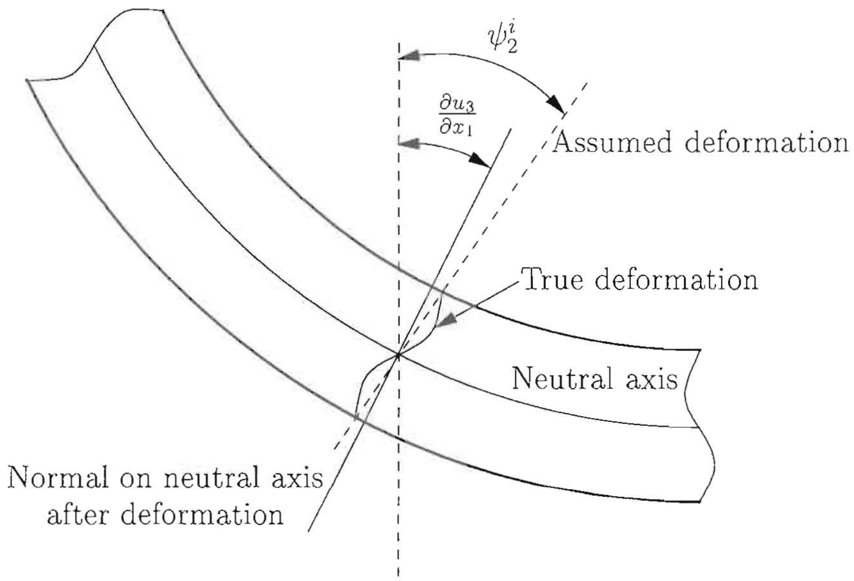

Figure 4.2: Mindlin theory

The transverse shear strains are obtained as

$$\begin{aligned} \gamma_{13} &= u_{3,1} + \psi_2 \\ \gamma_{23} &= u_{3,2} + \psi_1 \end{aligned} \tag{4.5}$$

For plane stress and linear isotropic elasticity the forgoing strain field defines the in-plane stresses as

$$\sigma_{11} = \frac{E}{1-\nu^2}[\epsilon_{11} + \nu\epsilon_{22}] \tag{4.6}$$

$$\sigma_{22} = \frac{E}{1-\nu^2}[\epsilon_{22} + \nu\epsilon_{11}] \tag{4.7}$$

$$\sigma_{12} = \sigma_{21} = G\gamma_{12} \tag{4.8}$$

where E is Young's modulus and ν is Poisson's ratio. Similarly, the out-of-plane stresses are given by

$$\sigma_{13} = \sigma_{31} = G\gamma_{13} \tag{4.9}$$

$$\sigma_{23} = \sigma_{32} = G\gamma_{23} \tag{4.10}$$

where

$$G = \frac{E}{2(1+\nu)} \tag{4.11}$$

Integrating the in-plane stresses, which vary linearly along the plate thickness, gives stress resultants of the form

$$M_{11} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{11} x_3 \, \mathrm{d}x_3 \tag{4.12}$$

$$M_{22} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{22} x_3 \, \mathrm{d}x_3 \tag{4.13}$$

$$M_{12} = M_{21} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{12} x_3 \, \mathrm{d}x_3 \tag{4.14}$$

Introducing matrix notation, the forgoing are written as

$$\boldsymbol{M} = \begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} \tag{4.15}$$

and the curvatures $\boldsymbol{\kappa}$ as

$$\boldsymbol{\kappa} = \begin{bmatrix} \psi_{1,1} \\ \psi_{2,2} \\ \psi_{1,2} + \psi_{2,1} \end{bmatrix} \tag{4.16}$$

It follows that the moment-curvature relation may be expressed as

$$\boldsymbol{M} = \boldsymbol{D}_b \boldsymbol{\kappa} \tag{4.17}$$

where

$$\boldsymbol{D}_b = \frac{Et^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{4.18}$$

Similarly, the out-of-plane stresses, when integrated along the thickness, give transverse shear forces

$$Q_{13} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{13} \, \mathrm{d}x_3 \tag{4.19}$$

$$Q_{23} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{23} \, \mathrm{d}x_3 \tag{4.20}$$

which, using matrix notation, results in

$$\boldsymbol{Q} = \boldsymbol{D}_s \boldsymbol{\gamma} \tag{4.21}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} Q_{13} \\ Q_{23} \end{bmatrix} \tag{4.22}$$

$$\gamma = \begin{bmatrix} u_{3,1} + \psi_2 \\ u_{3,2} + \psi_1 \end{bmatrix} \tag{4.23}$$

$$\boldsymbol{D}_s = Gt \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.24}$$

Summation convention is implied over x_1, x_2 and x_3 for Latin indices and over x_1 and x_2 for Greek indices, so that the local equilibrium equations may be appropriately integrated through the thickness to deduce the plate equilibrium equations

$$\begin{aligned} M_{\alpha\beta,\beta} - S_\alpha &= 0 \\ S_{\alpha,\alpha} + p &= 0 \end{aligned} \tag{4.25}$$

where p denotes the transverse surface loading. The first equation relates the bending moments to the shear forces, whereas the second is a statement of transverse force equilibrium.

In the limiting case where $t \to 0$ the Kirchhoff hypothesis of zero transverse shear strains must hold. Therefore

$$\begin{aligned} u_{3,1} + \psi_2 &= 0 \\ u_{3,2} + \psi_1 &= 0 \end{aligned} \tag{4.26}$$

(4.5) imply that the transverse shear strain remains constant through the element thickness. This is inconsistent with classical theory, where the corresponding transverse shear stress varies quadratically. Also, the transverse shear strain on the plate surface is required to be zero. Consequently, a temporary modification to the displacement field is made, namely

$$u_1 = x_3\psi_1 + (x_3^3 + \beta x_3)\phi(x_1, x_2) \tag{4.27}$$

Imposing the constraint

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} (x_3^3 + \beta x_3)x_3 \, \mathrm{d}x_3 = 0 \tag{4.28}$$

and setting $\gamma_{13} = 0$ on the plate faces results in

$$\gamma_{13} = \left[1 - \frac{5}{3t^2}\left(3x_3^2 - \frac{3t^2}{20}\right)\right](\psi_2 + u_{3,1}) \tag{4.29}$$

Moreover, substituting (4.29) into (4.21) leads to

$$\begin{aligned} Q_{13} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} G\gamma_{13}\, \mathrm{d}x_3 \\ &= G(\psi_2 + u_{3,1}) \int_{-\frac{t}{2}}^{\frac{t}{2}} \left[1 - \frac{5}{3t^2}\left(3x_3^2 - \frac{3t^2}{20}\right)\right] \mathrm{d}x_3 \\ &= \frac{5}{6}G(\psi_2 + u_{3,1}) \end{aligned} \tag{4.30}$$

Therefore, for consistency reasons, a 'shear correction' term is introduced as

$$k = \frac{6}{5} \tag{4.31}$$

into (4.21), which now becomes

$$\boldsymbol{Q} = \bar{\boldsymbol{D}}_s \boldsymbol{\gamma} \tag{4.32}$$

where

$$\bar{\boldsymbol{D}}_s = \frac{\boldsymbol{D}_s}{k} \tag{4.33}$$

The total plate energy, based on potential energy for bending and shear, is written as

$$\Pi(\boldsymbol{\kappa}, \boldsymbol{\gamma}) = \frac{1}{2}\int_\Omega \boldsymbol{\kappa}^T \boldsymbol{D}_b \boldsymbol{\kappa}\; d\Omega + \frac{1}{2}\int_\Omega \boldsymbol{\gamma}^T \bar{\boldsymbol{D}}_s \boldsymbol{\gamma}\; d\Omega - \Pi_{ext} \tag{4.34}$$

where Π_{ext} is the potential energy of the applied loads. The thin plate Kirchhoff conditions of (4.26) should be satisfied in the finite element interpolation.

4.1.2 Finite element interpolation

The displacement in the reference surface of the element is defined by

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sum_{i=1}^{4} N_i^e(\xi, \eta)\boldsymbol{u}_i \tag{4.35}$$

where $N_i^e(\xi, \eta)$ are the isoparametric shape functions

$$N_i^e(\xi, \eta) = \frac{1}{4}(1 + \xi_i\xi)(1 + \eta_i\eta) \quad i = 1, 2, 3, 4 \tag{4.36}$$

The sectional (normal) rotations are interpolated as

$$\psi_1 = N_i^e(\xi, \eta)\psi_1^i \tag{4.37}$$

$$\psi_2 = N_i^e(\xi, \eta)\psi_2^i \tag{4.38}$$

and the transverse mid-surface displacements are interpolated as

$$u_3 = N_i^e(\xi, \eta) u_3^i \tag{4.39}$$

where u_3^i, ψ_1^i and ψ_2^i are the nodal point values of the variables u_3, ψ_1 and ψ_2 respectively. The curvature-displacement relations are now written as

$$\boldsymbol{\kappa} = \sum_{i=1}^{4} \boldsymbol{B}_{bi} \boldsymbol{q}_i \tag{4.40}$$

The element curvature-displacement matrix is given in Appendix A. The unknowns at node i are

$$\boldsymbol{q}_i = \begin{bmatrix} u_3^i \\ \psi_1^i \\ \psi_2^i \end{bmatrix} \tag{4.41}$$

The shear strain-displacement relations are written as

$$\boldsymbol{\gamma} = \sum_{i=1}^{4} \boldsymbol{B}_{si} \boldsymbol{q}_i \tag{4.42}$$

The element shear strain-displacement matrix is given in Appendix A.

4.1.3 Assumed strain interpolations

The stationary condition of (4.34) directly results in the plate force-displacement relationship

$$\boldsymbol{K}^e \boldsymbol{q} = \boldsymbol{r} \tag{4.43}$$

where

$$\boldsymbol{K}^e = (\boldsymbol{K}_b + \bar{\boldsymbol{K}}_s) \tag{4.44}$$

with

$$\boldsymbol{K}_b = \int_\Omega \boldsymbol{\kappa}^T \boldsymbol{D}_b \boldsymbol{\kappa} \; \mathrm{d}\Omega \tag{4.45}$$

$$\bar{\boldsymbol{K}}_s = \int_\Omega \boldsymbol{\gamma}^T \bar{\boldsymbol{D}}_s \boldsymbol{\gamma} \; \mathrm{d}\Omega \tag{4.46}$$

Subscripts b and s indicate bending and shear respectively. For elements with 4 nodes, the expression for $\boldsymbol{K}_b$ is problem free, at least in terms of locking. The employed interpolation field of (4.36) in $\boldsymbol{K}_s$ results in severe locking when full integration is used.

One solution that overcomes the locking phenomena, while ensuring that the final element formulation is rank sufficient, is to incorporate the substitute assumed strain interpolation field of Bathe and Dvorkin [7, 47].

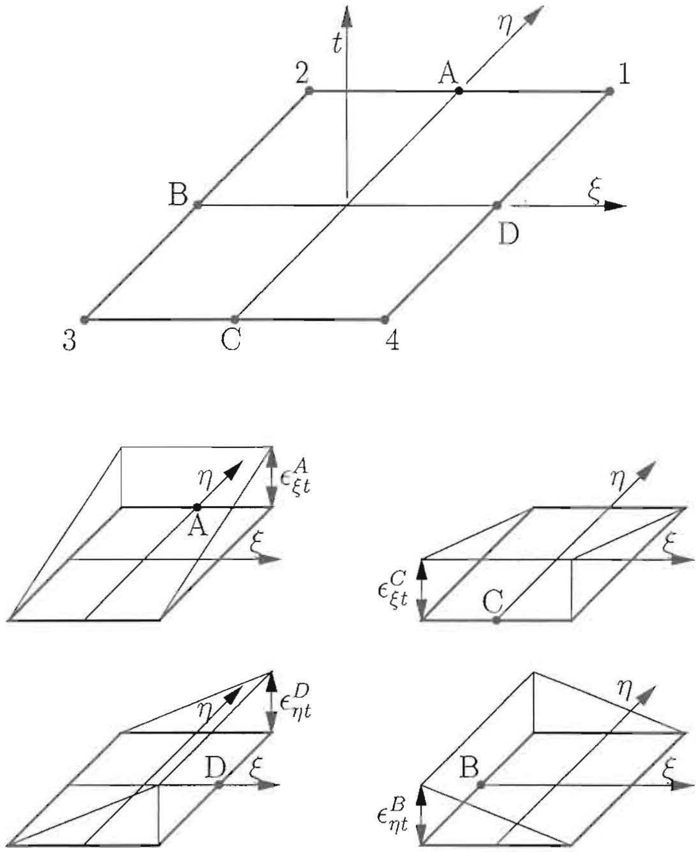

Figure 4.3: Interpolation functions for the transverse shear strains

Depicted in Figure 4.3, the assumed interpolation field of Bathe and Dvorkin is written as

$$\bar{\epsilon}_{\xi t} = \frac{1}{2}(1+\eta)\bar{\epsilon}_{\xi t}^{A} + \frac{1}{2}(1-\eta)\bar{\epsilon}_{\xi t}^{C} \tag{4.47}$$

$$\bar{\epsilon}_{\eta t} = \frac{1}{2}(1+\xi)\bar{\epsilon}_{\eta t}^{D} + \frac{1}{2}(1-\xi)\bar{\epsilon}_{\eta t}^{B} \tag{4.48}$$

where the superscripts A through D designate the sampling points for calculating the co-variant shear strains. The shear strain components in the Cartesian coordinate system, $\bar{\epsilon}_{13}$ and $\bar{\epsilon}_{23}$, are obtained [51] using a transformation which in the case of a flat plate element reduces to the standard (2×2) Jacobian, $\boldsymbol{J}$

$$\left\{ \begin{array}{c} \bar{\epsilon}_{\xi t} \\ \bar{\epsilon}_{\eta t} \end{array} \right\} = \frac{t}{2} \left[\begin{array}{cc} x_{1,\xi} & x_{2,\xi} \\ x_{1,\eta} & x_{2,\eta} \end{array} \right] \left\{ \begin{array}{c} \bar{\epsilon}_{13} \\ \bar{\epsilon}_{23} \end{array} \right\} = \frac{t}{2} \boldsymbol{J} \bar{\boldsymbol{\epsilon}}_s \tag{4.49}$$

Therefore, the substitute shear strains $\bar{\boldsymbol{\epsilon}}_s$ are expressed as

$$\bar{\gamma} = \bar{\boldsymbol{B}}_s \boldsymbol{q} \tag{4.50}$$

while the associated transverse stiffness-displacement relationship becomes

$$\bar{\boldsymbol{K}}_s = \int_\Omega \bar{\boldsymbol{B}}_s^T \bar{\boldsymbol{D}}_s \bar{\boldsymbol{B}}_s \, \mathrm{d}\Omega \tag{4.51}$$

The element stiffness-displacement relationship now becomes

$$(\boldsymbol{K}_b + \bar{\boldsymbol{K}}_s)\boldsymbol{q} = \boldsymbol{r} \tag{4.52}$$

which is the final element formulation. The assumed strain interpolation satisfies the Kirchoff conditions of zero transverse shear strains in the thin plate limit, while locking is also adequately prevented.

4.2 Shell formulation

4.2.1 Element formulation

Flat shell elements are simpler than generally curved shell elements, both in terms of formulation and computer implementation. As the element Jacobian matrix is constant through the thickness, analytical through-the-thickness integration is easily performed.

The element force-displacement relationship of the 8β(M) and 9β(M) membrane families is defined by (2.80), and for the 8β(D) and 9β(D) membrane families by (2.84). These relationships are repeated here using a different notation to distinguish between the membrane and plate components. The two different force-displacement relationships for the membrane families are rewritten in a universal form to clarify the notation

$$\boldsymbol{K}_m \boldsymbol{q}_m = \boldsymbol{r}_m \tag{4.53}$$

where

$$\boldsymbol{K}_m = \boldsymbol{K} + \frac{\gamma}{\Omega} \boldsymbol{h}\boldsymbol{h}^T \tag{4.54}$$

for the mixed formulation, and

$$\boldsymbol{K}_m = \boldsymbol{K} + \boldsymbol{P}_\gamma \tag{4.55}$$

for the displacement formulation.

$\boldsymbol{K}_m$ denotes the membrane stiffness matrix, $\boldsymbol{q}_m$ the element displacements and $\boldsymbol{r}_m$ the element body force vector. The unknown nodal displacements $\boldsymbol{q}_m$ and the specified consistent nodal loads $\boldsymbol{r}_m$ are defined by

$$\boldsymbol{q}_m = [u_1^i \;\; u_2^i \;\; \psi_3^i]^T \tag{4.56}$$
$$\boldsymbol{r}_m = [U_1^i \;\; U_2^i \;\; M_3^i]^T \tag{4.57}$$

where ψ_3^i is the in-plane rotation and M_3^i the in-plane nodal moment.

Similarly, the Mindlin plate force-displacement relationship (see (4.52)) is rewritten as

$$(\boldsymbol{K}_b + \bar{\boldsymbol{K}}_s)\boldsymbol{q}_p = \boldsymbol{r}_p \tag{4.58}$$

The displacements $\boldsymbol{q}_p$, and the specified consistent nodal loads $\boldsymbol{r}_p$, are respectively defined by

$$\boldsymbol{q}_p = [u_3^i \;\; \psi_1^i \;\; \psi_2^i]^T \tag{4.59}$$
$$\boldsymbol{r}_p = [U_3^i \;\; M_1^i \;\; M_2^i]^T \tag{4.60}$$

Through assembly of the membrane and plate elements, the flat shell element stiffness matrix $\boldsymbol{K}^e$ is obtained in a local element coordinate system as

$$\boldsymbol{K}^e = \begin{bmatrix} \boldsymbol{K}_m & 0 \\ \text{symm} & (\boldsymbol{K}_b + \bar{\boldsymbol{K}}_s) \end{bmatrix} \tag{4.61}$$

The local shell force-displacement relationship is given by

$$\boldsymbol{K}^e\boldsymbol{q}^e = \boldsymbol{r}^e \tag{4.62}$$

where the shell nodal displacements and loads for node i respectively are

$$\boldsymbol{q}_i^e = [u_1^i \;\; u_2^i \;\; u_3^i \;\; \psi_1^i \;\; \psi_2^i \;\; \psi_3^i]^T \tag{4.63}$$
$$\boldsymbol{r}_i^e = [U_1^i \;\; U_2^i \;\; U_3^i \;\; M_1^i \;\; M_2^i \;\; M_3^i]^T \tag{4.64}$$

4.2.2 A general warped configuration

The warp correction employed in this study is the so-called 'rigid link' correction suggested by Taylor [6], which is depicted in Figure 4.4. Simple kinematic nodal relationships are used to evaluate the warp effect.

For elements with true rotational degrees of freedom the rotations about the local x_3-axes in the warped and projected planes may be taken as equal. Assuming reasonably small warp, the effect of the drilling degree on the out-of-plane bending rotations is neglected. The strain-displacement modification presented by Taylor is therefore extended by addition of the final row and column as follows [52]

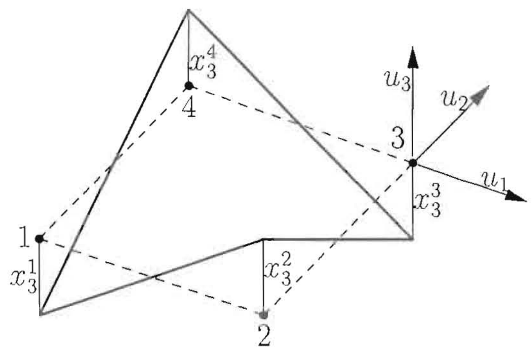

Figure 4.4: Warped and projected quadrilateral shell element

$$\begin{Bmatrix} \bar{u}_1^i \\ \bar{u}_2^i \\ \bar{u}_3^i \\ \bar{\psi}_1^i \\ \bar{\psi}_2^i \\ \bar{\psi}_3^i \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -x_3^i & 0 & 0 & 1 & 0 & 0 \\ 0 & x_3^i & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1^i \\ u_2^i \\ u_3^i \\ \psi_1^i \\ \psi_2^i \\ \psi_3^i \end{Bmatrix} \tag{4.65}$$

where x_3^i defines the warp at each node and bared quantities (for example $\bar{u}_1^i$) act on the flat projection.

This correction is much simpler than for instance the correction presented by Robinson [53].