Chapter 3

Numerical results: Isotropic membrane elements

In this chapter numerical results are presented for the isotropic membrane elements developed in Chapter 2.

In the following,

- Q4 denotes a standard 4-node bilinear membrane element with 8 degrees of freedom.
- QC9D refers to an implementation by Groenwold and Stander [25] of the Ibrahimbegovic *et al.* membrane finite element with drilling degrees of freedom [23]. The element is based on the variational formulation due to Hughes and Brezzi [5], and the formulation includes a hierarchical bubble shape function. This element has 12 degrees of freedom.
- 5β denotes an implementation of the 5β family due to Di and Ramm [34].
- 8β(M) and 8β(D) denote the two families proposed herein. An asterisk ($*$) indicates that the membrane locking correction is not included. (The formulation by Sze and Ghali [37] is similar to the 8β(D)$*$-EP element presented herein.)
- 9β(M) and 9β(D) denote the two families proposed herein with 9 interpolating parameters.

The penalty stiffness terms $(\gamma/\Omega^e)\boldsymbol{h}^e\boldsymbol{h}^{eT}$ and $\boldsymbol{P}^e_\gamma$ (in (2.80) and (2.84) respectively), which relate the in-plane rotations to the in-plane translations, respectively are integrated using a 2×2 Gaussian quadrature, and a 1 point quadrature. All other integrals are evaluated using 3×3 Gaussian quadrature.

Alternatively, $\boldsymbol{G}^{eT}\boldsymbol{H}^{e-1}\boldsymbol{G}^e$ in (2.81) may be integrated by the 5-point rule presented in [1, 25]. However, reduced integration in mixed/hybrid finite elements is in general not advantageous.

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3.1 Element rank

Eigenvalue analyses of the different 8β(M)-NT and 9β(M)-NT formulations are presented in Table 3.1.

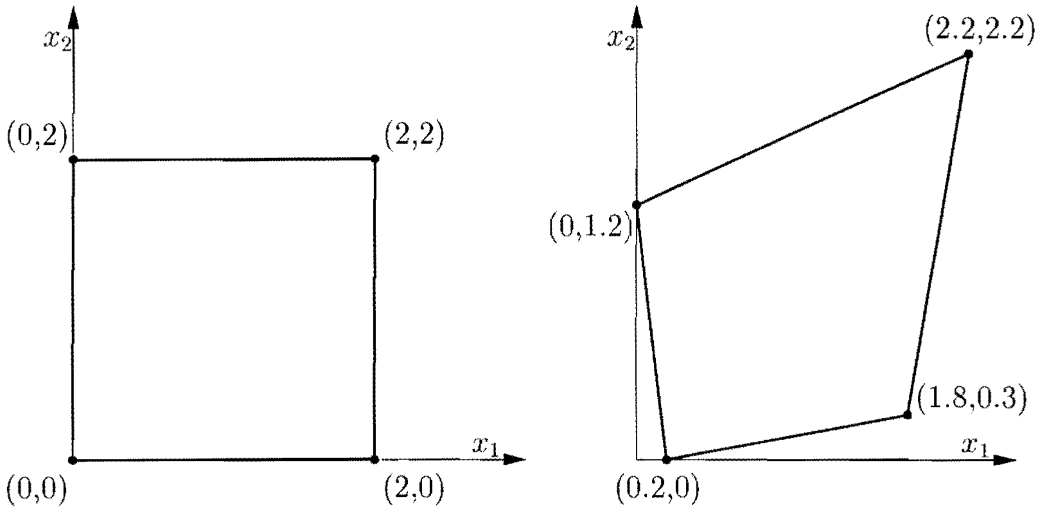

Figure 3.1: Regular and distorted element geometries for eigenvalue analysis

λ_i	P_{h3}	P^i_{h3}	P_{h4}	P^i_{h4}
1	0.14286E+01	0.14286E+01	0.14286E+01	0.14286E+01
2	0.76923E+00	0.76923E+00	0.76923E+00	0.76923E+00
3	0.76923E+00	0.76923E+00	0.76923E+00	0.76923E+00
4	0.57692E+00	0.57692E+00	0.57692E+00	0.57692E+00
5	0.39461E+00	0.50514E+00	0.39461E+00	0.50514E+00
6	0.39461E+00	0.50514E+00	0.39461E+00	0.50514E+00
7	0.80219E-01	0.84600E-01	0.80219E-01	0.84600E-01
8	0.80219E-01	0.84600E-01	0.80219E-01	0.84600E-01
9	0.44444E-01	0.44444E-01	0.44444E-01	0.44444E-01
10	0.63228E-16	0.42331E-16	0.41967E-16	0.37303E-16
11	0.95029E-17	0.13642E-17	0.35205E-16	-0.12714E-16
12	-0.37504E-16	-0.26489E-16	-0.14526E-16	-0.74542E-16

Table 3.1: Eigenvalues of square 8β(M)-NT and 9β(M)-NT elements (plane stress, $|\boldsymbol{J}| = 1$, $E = 1$, $\nu = 0.3$)

The different stress modes used in the elements are given in (2.93) through (2.96). The table reveals that all the formulations are of adequate rank, while rigid body modes are captured adequately. Using the distorted element geometry depicted in Figure 3.1 it is determined that the elements are invariant (not shown in tabulated form).

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3.2 Membrane patch tests

A necessary and sufficient requirement for convergence of finite elements is that the patch tests suggested by Irons [43] is passed [41]. This also applies for mixed formulations [44], while a necessary condition for non-singularity is

$$m \geq n \tag{3.1}$$

where m is the number of degrees of freedom in the primary (displacement) variables, and n is the number of degrees of freedom in the constraint variables of the mixed formulation. The 8β(M) and 9β(M) families satisfy (3.1). For all the patch tests performed in this study, the mesh depicted in Figure 3.2 is used.

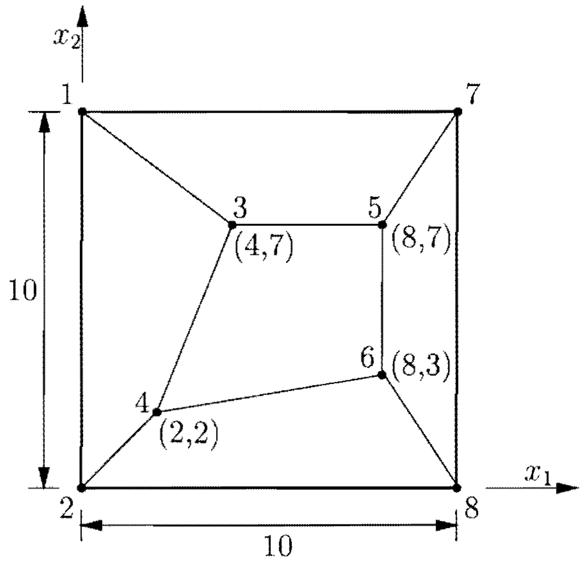

Figure 3.2: Mesh used in patch tests

3.2.1 Constant extension and constant shear patch tests

The 8β and 9β families pass these patch tests, with and without the membrane locking correction. When the membrane locking correction is excluded, constant nodal moments are required. The patch tests are passed for any $\gamma > 0$. See Figure 3.3 for boundary conditions.

3.2.2 Modified shear patch test

This new test illustrates the capabilities of membrane finite elements with drilling degrees of freedom (Figure 3.4).

For a square membrane subjected to complementary shear, the three degrees of freedom at a single node are constrained. The correct displacements and stresses are obtained, while no additional devices are required.

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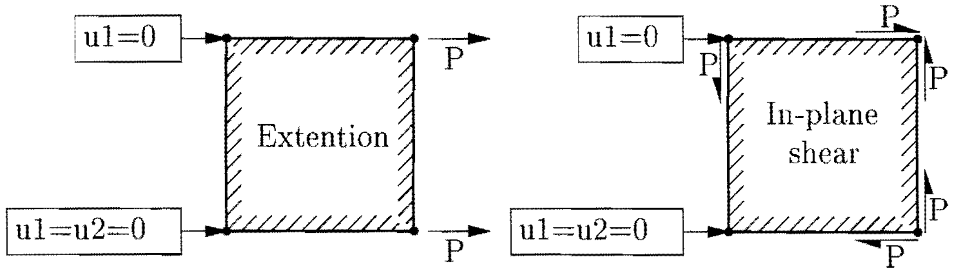

Figure 3.3: Constant extension patch test and constant shear patch test

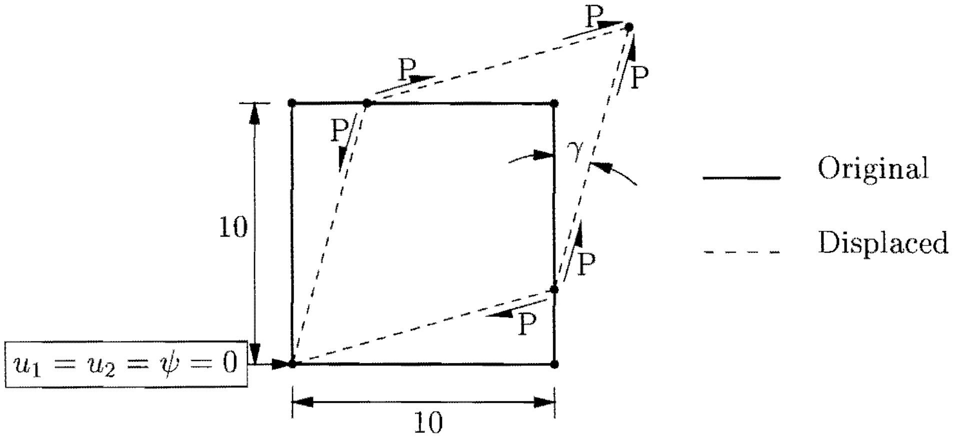

Figure 3.4: Modified constant shear patch test

3.3 Taylor's patch test and Ramm's cantilever beam

The geometry is depicted in Figure 3.5, and tabulated numerical results are presented in Table 3.2.

Load case 1 once again represents a patch test. Obviously, all the elements studied pass this test. Load case 2 represents bending behavior. In general, the NT-formulation in each family is the most accurate. When the locking correction is applied the error is smaller than without this correction.

In load cases 3 and 4 the effect of element distortion is examined. For both these load cases the error is smaller when the locking correction is not applied. For the displacements the $8\beta*$ formulation is by far the most accurate. For both load cases the 9β(M)$*$-NC formulation performs very good for the stress predictions.

The results obtained with the 8β and 9β families are almost identical. In general, the new families are more accurate than the QC9D element, but not quite as accurate as the 5β family.

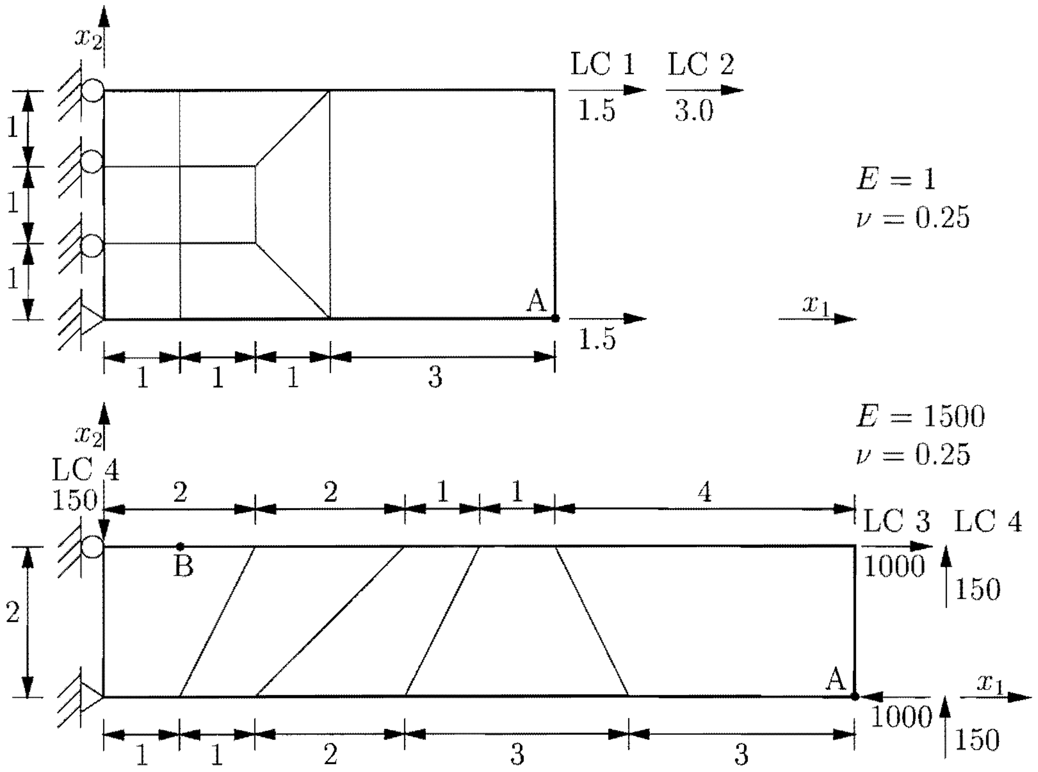

Figure 3.5: Taylor's patch test and Ramm's cantilever beam

3.4 Cook's membrane

This popular test problem (See Figure 3.6) has been used by many authors. The center displacement u_{2_C} for the various elements studied is tabulated in Table 3.3, while the stresses $(\sigma_{\min})_B$ and $(\sigma_{\max})_A$ are presented in Table 3.4.

From Table 3.3 it is clear that the NT-formulation outperforms all the other elements. The elements perform better without the locking correction.

Table 3.4 illustrates that the $8\beta*$ formulation performs better for the maximum stress, while the 9β families outperform the other elements for the minimum stress.

Once again, the displacement results obtained with the 8β and 9β families are almost identical, and in general more accurate than the QC9D results.

Table 3.5 shows that small values of γ lead to higher displacement accuracy, since the rotational field is only weakly enforced. However, the usual choice of $\gamma = G$ results in good accuracy. $\gamma = 0$ results in the most accurate formulation. Due to the redundant constraints in the problem no rank deficiencies arise. However, $\gamma = 0$ is not suitable for general problems.

Table 3.6 reveals that with the 5-point and 8-point integration schemes, the accuracy increases in general. Both the integration schemes improve the performance of the elements when the membrane locking correction is used, due to the introduction of a 'soft' higher order deformation mode.

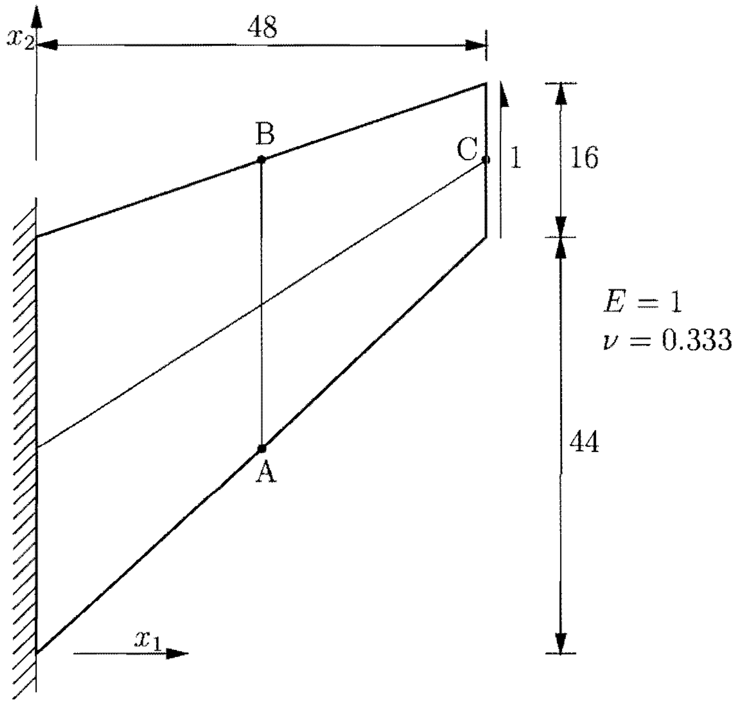

Figure 3.6: Cook's membrane

3.5 Thick walled cylinder

This problem is used to assess the performance of the elements studied for nearly incompressible materials. Depicted in Figure 3.7, numerical results for this problem are tabulated in Table 3.7.

The analytical solution is given by

$$u(r) = \frac{(1+\nu)pR_1^2}{E(R_2^2 - R_1^2)}\left[\frac{R_2^2}{r} + (1-2\nu)r\right] \tag{3.2}$$

where p represents the pressure on the inner surface, R_1 denotes the inner radius and R_2 the outer radius.

MacNeal and Harder [45] calculated the following values:

- $u_{1_A} = 5.0399 \times 10^{-3}$ for $\nu_1 = 0.49$,
- $u_{1_A} = 5.0602 \times 10^{-3}$ for $\nu_2 = 0.499$, and
- $u_{1_A} = 5.0623 \times 10^{-3}$ for $\nu_3 = 0.4999$.

Table 3.7 reveals that the QC9D element becomes over stiff as $\nu \to 0.5$, while the 5β, 8β and 9β families do not reveal this locking like behavior.

Note that the NC-formulations are the most accurate. In general, the 5β family slightly outperforms the other families.

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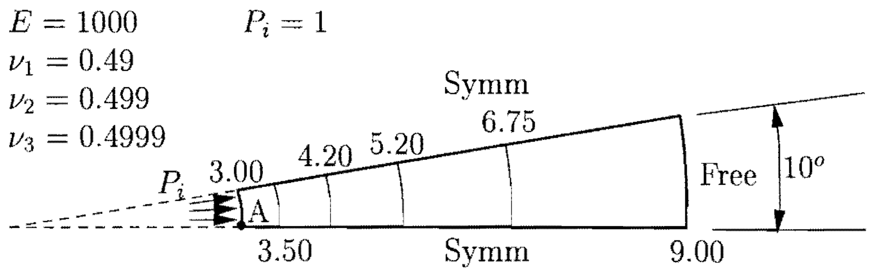

Figure 3.7: Thick walled cylinder

3.6 Cook's beam

The geometry is depicted in Figure 3.8. Numerical results for the displacements are given in Table 3.8, and the stress results are presented in Table 3.9.

For the displacements the $8\beta*$ and $9\beta*$ formulations outperform the other elements. For the stress evaluation the 5β family is by far the most accurate. For the irregular mesh the 8β-OC and $8\beta*$-OC formulations are very accurate. The 8β and 9β formulations are accurate for $-\sigma_{11}$. The results obtained with the 8β and 9β families are almost identical.

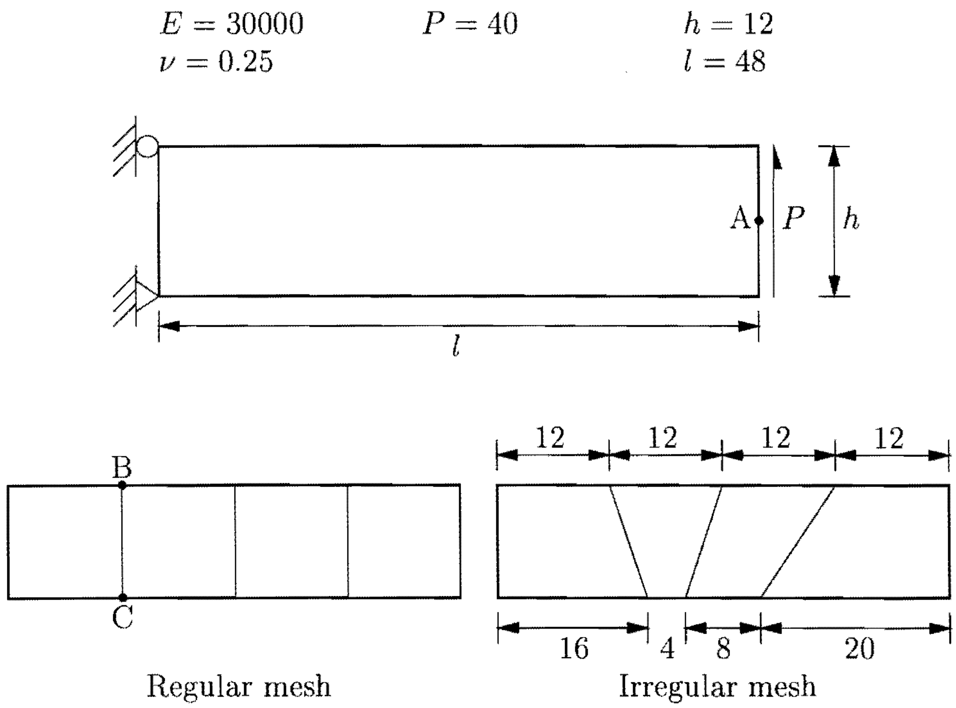

Figure 3.8: Cook's beam

3.7 Higher order patch test

Load case 1 (a unit couple applied at the free ends) represents a higher order patch test (Taylor *et al.* [41]). Load case 2 was presented by Ibrahimbegovic *et al.* [23] (See Figure 3.9). The center displacements and the tip rotations are given in Table 3.10.

For the center displacements of load case 1 with the regular mesh, the 5β family is very accurate. The EP-, OC- and NT-formulations for all the families are also accurate. Note that the 8β(M)-NC element and the OC-formulations for the irregular mesh are very accurate. For the tip rotations of Load case 1, the EP-, OC- and NT-formulations are very accurate for the regular mesh. The NT-formulation is also accurate for the irregular mesh.

The EP-, OC- and NT-formulations are also accurate for the center displacements of load case 2 for the regular mesh. For the irregular mesh the OC-formulations without the locking correction are the most accurate. For the tip rotations for load case 2 the NC- and PH-formulations outperform the other elements. QC9D is the most accurate for the irregular mesh.

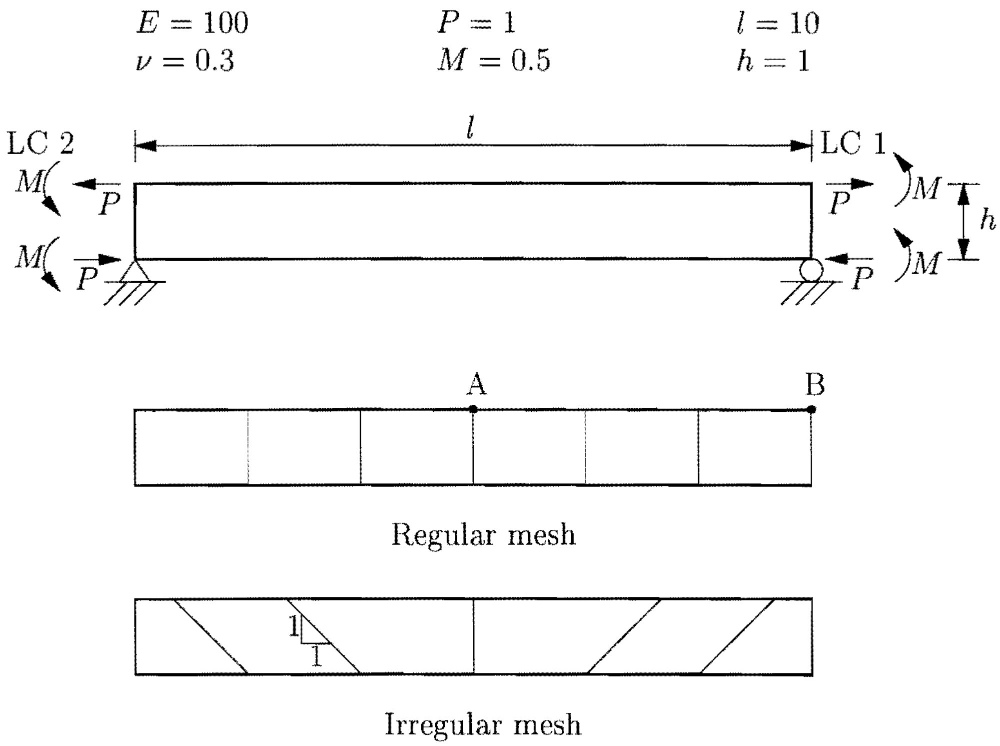

Figure 3.9: Higher order patch test

	Case 1	Case 2	Case 3		Case 4	
Element	u_{1_A}	$-u_{2_A}$	u_{2_A}	$-\sigma_{x_B}$	u_{2_A}	$-\sigma_{x_B}$
Q4	6.000	14.90	44.60	1724	49.54	2395
QC9D	6.000	16.78	81.86	2541	84.59	3453
5β-NC	6.000	17.50	77.54	2775	82.02	3829
5β-EP	6.000	17.28	96.18	3014	98.19	4137
5β-OC	6.000	17.18	93.13	2446	95.06	3349
5β-NT	6.000	18.33	97.33	3019	98.91	4148
5β-PH	6.000	17.86	96.33	2982	98.38	4095
8β(M)$*$-NC	6.000	16.60	98.34	3074	99.40	4151
8β(M)$*$-EP	6.000	16.41	98.97	3015	100.1	4148
8β(M)$*$-OC	6.000	16.63	101.0	2614	102.1	3609
8β(M)$*$-NT	6.000	17.60	99.28	3017	100.3	4141
8β(M)$*$-PH	6.000	17.20	97.37	2887	98.34	3990
8β(M)-NC	6.000	17.09	84.17	2732	87.26	3572
8β(M)-EP	6.000	16.86	84.82	2882	87.96	3824
8β(M)-OC	6.000	16.91	84.49	2456	87.23	3296
8β(M)-NT	6.000	17.73	85.71	2878	89.13	3844
8β(M)-PH	6.000	17.40	89.57	2409	92.30	3174
8β(D)$*$-NC	6.000	16.60	98.38	3055	99.45	4131
8β(D)$*$-EP	6.000	16.41	98.85	2998	99.96	4131
8β(D)$*$-OC	6.000	16.63	101.0	2598	102.0	3592
8β(D)$*$-NT	6.000	17.60	99.21	3004	100.2	4129
8β(D)$*$-PH	6.000	17.20	97.46	2873	98.43	3975
8β(D)-NC	6.000	17.10	84.22	2726	87.31	3565
8β(D)-EP	6.000	16.87	84.86	2881	88.00	3822
8β(D)-OC	6.000	16.92	84.50	2456	87.24	3295
8β(D)-NT	6.000	17.74	85.75	2878	89.18	3843
8β(D)-PH	6.000	17.41	89.67	2415	92.42	3178
9β(M)$*$-NC	6.000	16.60	98.27	3006	99.27	4030
9β(M)$*$-EP	6.000	16.41	98.88	2922	99.79	3960
9β(M)$*$-OC	6.000	16.62	98.95	2906	99.79	3948
9β(M)$*$-NT	6.000	17.60	99.06	2888	99.90	3907
9β(M)$*$-PH	6.000	17.19	97.03	2819	97.92	3863
9β(M)-NC	6.000	17.09	83.83	2551	86.77	3292
9β(M)-EP	6.000	16.86	83.75	2494	86.51	3234
9β(M)-OC	6.000	16.91	83.93	2518	86.80	3294
9β(M)-NT	6.000	17.73	84.23	2440	87.17	3169
9β(M)-PH	6.000	17.40	88.98	2194	91.61	2883
9β(D)$*$-NC	6.000	16.60	98.31	2985	99.32	4007
9β(D)$*$-EP	6.000	16.41	98.76	2899	99.67	3936
Analytical	6.000	18.00	100.0	3000	102.0	4050

Table 3.2: Taylor's patch test and Ramm's cantilever beam: Numerical results

	Case					
	1	2	Case 3		Case 4	
Element	u_{1_A}	$-u_{2_A}$	u_{2_A}	$-\sigma_{x_B}$	u_{2_A}	$-\sigma_{x_B}$
9β(D)∗-OC	6.000	16.63	98.86	2887	99.69	3928
9β(D)∗-NT	6.000	17.59	98.98	2868	99.82	3886
9β(D)∗-PH	6.000	17.19	97.13	2804	98.02	3846
9β(D)-NC	6.000	17.10	83.87	2544	86.81	3284
9β(D)-EP	6.000	16.86	83.77	2492	86.53	3230
9β(D)-OC	6.000	16.91	83.95	2518	86.83	3292
9β(D)-NT	6.000	17.74	84.26	2439	87.19	3166
9β(D)-PH	6.000	17.40	89.08	2200	91.72	2887
Analytical	6.000	18.00	100.0	3000	102.0	4050

Table 3.2: Taylor's patch test and Ramm's cantilever beam: Numerical results (continued)

Element	2×2	4×4	8×8	16×16	32×32
Q4	11.80	18.29	22.08	23.43	23.82
QC9D	19.27	22.61	23.55	23.83	23.92
5β-NC	17.76	21.94	23.38	23.80	23.92
5β-EP	21.13	23.02	23.69	23.88	23.94
5β-OC	21.04	23.02	23.69	23.88	23.94
5β-NT	21.54	23.05	23.69	23.88	23.94
5β-PH	21.13	23.02	23.69	23.88	23.94
8β(M)∗-NC	21.83	23.15	23.64	23.85	23.92
8β(M)∗-EP	22.18	23.28	23.69	23.86	23.92
8β(M)∗-OC	22.38	23.30	23.69	23.86	23.92
8β(M)∗-NT	22.65	23.32	23.69	23.86	23.92
8β(M)∗-PH	21.85	23.19	23.66	23.85	23.93
8β(M)-NC	19.60	22.63	23.56	23.83	23.92
8β(M)-EP	20.11	22.71	23.58	23.84	23.92
8β(M)-OC	20.18	22.73	23.58	23.84	23.92
8β(M)-NT	20.42	22.74	23.58	23.84	23.92
8β(M)-PH	19.59	22.65	23.57	23.84	23.92
8β(D)∗-NC	21.81	23.15	23.64	23.85	23.92
8β(D)∗-EP	22.21	23.29	23.69	23.86	23.92
8β(D)∗-OC	22.41	23.30	23.69	23.86	23.92
8β(D)∗-NT	22.67	23.32	23.69	23.86	23.92
8β(D)∗-PH	21.84	23.19	23.66	23.85	23.93
8β(D)-NC	19.59	22.63	23.56	23.83	23.92
8β(D)-EP	20.12	22.72	23.58	23.84	23.92
Best known			23.90		

Table 3.3: Cook's membrane: Center displacement u_{2_C}

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Element	2 × 2	4 × 4	8 × 8	16 × 16	32 × 32
8β(D)-OC	20.19	22.73	23.58	23.84	23.92
8β(D)-NT	20.42	22.74	23.58	23.84	23.92
8β(D)-PH	19.60	22.65	23.57	23.84	23.92
9β(M)∗-NC	21.83	23.15	23.64	23.85	23.92
9β(M)∗-EP	22.10	23.28	23.69	23.86	23.92
9β(M)∗-OC	22.28	23.29	23.69	23.86	23.92
9β(M)∗-NT	22.59	23.31	23.69	23.86	23.92
9β(M)∗-PH	21.81	23.19	23.66	23.85	23.93
9β(M)-NC	19.51	22.63	23.56	23.83	23.92
9β(M)-EP	19.77	22.70	23.58	23.84	23.92
9β(M)-OC	19.86	22.71	23.58	23.84	23.92
9β(M)-NT	20.09	22.73	23.58	23.84	23.92
9β(M)-PH	19.52	22.65	23.57	23.84	23.92
9β(D)∗-NC	21.81	23.15	23.64	23.85	23.92
9β(D)∗-EP	22.13	23.28	23.69	23.86	23.92
9β(D)∗-OC	22.30	23.29	23.69	23.86	23.92
9β(D)∗-NT	22.61	23.31	23.69	23.86	23.92
9β(D)∗-PH	21.81	23.19	23.66	23.85	23.93
9β(D)-NC	19.50	22.63	23.56	23.83	23.92
9β(D)-EP	19.78	22.71	23.58	23.84	23.92
9β(D)-OC	19.88	22.72	23.58	23.84	23.92
9β(D)-NT	20.11	22.73	23.58	23.84	23.92
9β(D)-PH	19.53	22.65	23.57	23.84	23.92
Best known			23.90		

Table 3.3: Cook's membrane: Center displacement u_{2_C} (continued)

	2 × 2 mesh		4 × 4 mesh		8 × 8 mesh	
Element	$(\sigma_{\max})_A$	$(\sigma_{\min})_B$	$(\sigma_{\max})_A$	$(\sigma_{\min})_B$	$(\sigma_{\max})_A$	$(\sigma_{\min})_B$
Q4	0.1278	-0.0908	0.1905	-0.1508	0.2251	-0.1866
QC9D	0.1839	-0.1616	0.2241	-0.1805	0.2323	-0.2013
5β-NC	0.1657	-0.1614	0.2201	-0.1832	0.2334	-0.1973
5β-EP	0.1855	-0.1564	0.2241	-0.1857	0.2345	-0.1986
5β-OC	0.1776	-0.1688	0.2225	-0.1855	0.2343	-0.1987
5β-NT	0.1721	-0.1775	0.2215	-0.1835	0.2344	-0.1982
5β-PH	0.1884	-0.1557	0.2250	-0.1851	0.2347	-0.1984
8β(M)∗-NC	0.1675	-0.2042	0.2122	-0.1877	0.2266	-0.1981
8β(M)∗-EP	0.2052	-0.2262	0.2340	-0.2025	0.2357	-0.2020
8β(M)∗-OC	0.1927	-0.2261	0.2316	-0.2036	0.2355	-0.2020
Best known	0.2360	-0.2010	0.2360	-0.2010	0.2360	-0.2010

Table 3.4: Cook's membrane: Stress analysis

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	2 × 2 mesh		4 × 4 mesh		8 × 8 mesh	
Element	$(\sigma_{\max})_A$	$(\sigma_{\min})_B$	$(\sigma_{\max})_A$	$(\sigma_{\min})_B$	$(\sigma_{\max})_A$	$(\sigma_{\min})_B$
8β(M)*-NT	0.1798	-0.2420	0.2293	-0.2043	0.2352	-0.2020
8β(M)*-PH	0.2147	-0.2315	0.2333	-0.1971	0.2357	-0.2021
8β(M)-NC	0.1564	-0.1569	0.2092	-0.1736	0.2269	-0.1955
8β(M)-EP	0.1813	-0.1645	0.2264	-0.1822	0.2355	-0.1989
8β(M)-OC	0.1765	-0.1842	0.2246	-0.1809	0.2352	-0.1989
8β(M)-NT	0.1601	-0.2066	0.2212	-0.1801	0.2345	-0.1990
8β(M)-PH	0.1856	-0.1777	0.2281	-0.1813	0.2358	-0.1996
8β(D)*-NC	0.1679	-0.2033	0.2122	-0.1877	0.2266	-0.1981
8β(D)*-EP	0.2058	-0.2257	0.2340	-0.2025	0.2357	-0.2020
8β(D)*-OC	0.1933	-0.2257	0.2316	-0.2037	0.2355	-0.2020
8β(D)*-NT	0.1804	-0.2417	0.2292	-0.2043	0.2352	-0.2020
8β(D)*-PH	0.2148	-0.2307	0.2332	-0.1971	0.2357	-0.2021
8β(D)-NC	0.1565	-0.1562	0.2092	-0.1736	0.2269	-0.1955
8β(D)-EP	0.1810	-0.1646	0.2264	-0.1822	0.2355	-0.1989
8β(D)-OC	0.1763	-0.1842	0.2245	-0.1809	0.2352	-0.1989
8β(D)-NT	0.1600	-0.2062	0.2212	-0.1801	0.2345	-0.1990
8β(D)-PH	0.1853	-0.1775	0.2280	-0.1812	0.2357	-0.1996
9β(M)*-NC	0.1639	-0.1991	0.2123	-0.1884	0.2266	-0.1981
9β(M)*-EP	0.1964	-0.2069	0.2317	-0.2014	0.2354	-0.2019
9β(M)*-OC	0.1876	-0.2121	0.2297	-0.2022	0.2351	-0.2019
9β(M)*-NT	0.1700	-0.2300	0.2263	-0.2030	0.2347	-0.2020
9β(M)*-PH	0.1946	-0.2261	0.2312	-0.1978	0.2350	-0.2021
9β(M)-NC	0.1606	-0.1685	0.2108	-0.1754	0.2270	-0.1960
9β(M)-EP	0.1850	-0.1795	0.2302	-0.1831	0.2352	-0.1995
9β(M)-OC	0.1766	-0.1886	0.2278	-0.1831	0.2349	-0.1996
9β(M)-NT	0.1629	-0.2089	0.2240	-0.1839	0.2342	-0.1999
9β(M)-PH	0.1868	-0.1928	0.2286	-0.1834	0.2352	-0.2001
9β(D)*-NC	0.1641	-0.1983	0.2124	-0.1884	0.2266	-0.1981
9β(D)*-EP	0.1967	-0.2066	0.2317	-0.2015	0.2354	-0.2019
9β(D)*-OC	0.1879	-0.2118	0.2296	-0.2022	0.2351	-0.2019
9β(D)*-NT	0.1703	-0.2297	0.2263	-0.2031	0.2347	-0.2020
9β(D)*-PH	0.1946	-0.2256	0.2312	-0.1978	0.2350	-0.2021
9β(D)-NC	0.1607	-0.1678	0.2108	-0.1754	0.2270	-0.1960
9β(D)-EP	0.1850	-0.1793	0.2302	-0.1831	0.2352	-0.1995
9β(D)-OC	0.1766	-0.1883	0.2277	-0.1831	0.2349	-0.1996
9β(D)-NT	0.1631	-0.2071	0.2240	-0.1839	0.2342	-0.1999
9β(D)-PH	0.1867	-0.1926	0.2285	-0.1834	0.2352	-0.2001
Best known	0.2360	-0.2010	0.2360	-0.2010	0.2360	-0.2010

Table 3.4: Cook's membrane: Stress analysis (continued)

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γ	2×2
$G \times 0$	20.73
$G \times 10^{-3}$	20.73
$G \times 10^{-2}$	20.73
$G \times 10^{-2}$	20.67
$G \times 10^{0}$	20.42
$G \times 10^{1}$	20.05
$G \times 10^{2}$	19.94
$G \times 10^{3}$	19.93
Best known	23.90

Table 3.5: Cook's membrane: Influence of γ for the 2×2 mesh

Element	2 × 2	4 × 4	8 × 8	16 × 16	32 × 32
5 point integration					
8β(D)$*$-NC	21.93	23.15	23.64	23.85	23.92
8β(D)$*$-EP	22.28	23.29	23.69	23.86	23.92
8β(D)$*$-OC	22.02	23.23	23.68	23.86	23.92
8β(D)$*$-NT	22.31	23.25	23.68	23.86	23.92
8β(D)$*$-PH	21.68	23.16	23.66	23.85	23.93
8β(D)-NC	20.61	22.68	23.56	23.84	23.92
8β(D)-EP	20.47	22.73	23.58	23.84	23.92
8β(D)-OC	20.45	22.70	23.58	23.84	23.92
8β(D)-NT	20.68	22.72	23.58	23.84	23.92
8β(D)-PH	20.36	22.67	23.58	23.85	23.93
8 point integration					
8β(D)$*$-NC	21.93	23.15	23.64	23.85	23.92
8β(D)$*$-EP	22.28	23.29	23.69	23.86	23.92
8β(D)$*$-OC	22.10	23.26	23.68	23.86	23.92
8β(D)$*$-NT	22.40	23.28	23.69	23.86	23.92
8β(D)$*$-PH	21.76	23.17	23.66	23.85	23.93
8β(D)-NC	20.60	22.68	23.56	23.84	23.92
8β(D)-EP	20.47	22.73	23.58	23.84	23.92
8β(D)-OC	20.48	22.72	23.58	23.84	23.92
8β(D)-NT	20.72	22.74	23.58	23.84	23.92
8β(D)-PH	20.39	22.68	23.58	23.85	23.93
Full integration					
8β(D)$*$-NC	21.81	23.15	23.64	23.85	23.92
8β(D)$*$-EP	22.21	23.29	23.69	23.86	23.92
8β(D)$*$-OC	22.41	23.30	23.69	23.86	23.92
8β(D)$*$-NT	22.67	23.32	23.69	23.86	23.92
8β(D)$*$-PH	21.84	23.19	23.66	23.85	23.93
8β(D)-NC	19.59	22.63	23.56	23.83	23.92
8β(D)-EP	20.12	22.72	23.58	23.84	23.92
8β(D)-OC	20.19	22.73	23.58	23.84	23.92
8β(D)-NT	20.42	22.74	23.58	23.84	23.92
8β(D)-PH	19.60	22.65	23.57	23.84	23.92
Best known			23.90		

Table 3.6: Cook's membrane: Effect of integration scheme order

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Element	$\nu_1 = .49$	$\nu_2 = .499$	$\nu_3 = .4999$
Q4	4.277E-03	1.821E-03	2.694E-04
QC9D	4.848E-03	3.832E-03	1.226E-03
5β-NC	5.035E-03	5.055E-03	5.057E-03
5β-EP	4.996E-03	5.015E-03	5.017E-03
5β-OC	4.996E-03	5.014E-03	5.016E-03
5β-NT	4.996E-03	5.015E-03	5.017E-03
5β-PH	4.996E-03	5.015E-03	5.017E-03
8β(M)$*$-NC	5.003E-03	5.022E-03	5.024E-03
8β(M)$*$-EP	4.990E-03	5.008E-03	5.010E-03
8β(M)$*$-OC	4.990E-03	5.007E-03	5.009E-03
8β(M)$*$-NT	4.990E-03	5.008E-03	5.010E-03
8β(M)$*$-PH	4.915E-03	4.866E-03	4.853E-03
8β(M)-NC	5.003E-03	5.022E-03	5.024E-03
8β(M)-EP	4.990E-03	5.008E-03	5.010E-03
8β(M)-OC	4.990E-03	5.007E-03	5.009E-03
8β(M)-NT	4.990E-03	5.008E-03	5.010E-03
8β(M)-PH	4.915E-03	4.865E-03	4.852E-03
8β(D)$*$-NC	5.003E-03	5.022E-03	5.024E-03
8β(D)$*$-EP	4.990E-03	5.008E-03	5.010E-03
8β(D)$*$-OC	4.990E-03	5.007E-03	5.009E-03
8β(D)$*$-NT	4.990E-03	5.008E-03	5.010E-03
8β(D)$*$-PH	4.915E-03	4.866E-03	4.853E-03
8β(D)-NC	5.003E-03	5.022E-03	5.024E-03
8β(D)-EP	4.990E-03	5.008E-03	5.010E-03
8β(D)-OC	4.990E-03	5.007E-03	5.009E-03
8β(D)-NT	4.990E-03	5.008E-03	5.010E-03
8β(D)-PH	4.915E-03	4.865E-03	4.852E-03
9β(M)$*$-NC	5.003E-03	5.022E-03	5.024E-03
9β(M)$*$-EP	4.990E-03	5.008E-03	5.009E-03
9β(M)$*$-OC	4.989E-03	5.007E-03	5.009E-03
9β(M)$*$-NT	4.990E-03	5.007E-03	5.009E-03
9β(M)$*$-PH	4.915E-03	4.865E-03	4.852E-03
9β(M)-NC	5.003E-03	5.022E-03	5.024E-03
9β(M)-EP	4.990E-03	5.008E-03	5.009E-03
9β(M)-OC	4.989E-03	5.007E-03	5.009E-03
9β(M)-NT	4.990E-03	5.007E-03	5.009E-03
9β(M)-PH	4.915E-03	4.865E-03	4.852E-03
9β(D)$*$-NC	5.003E-03	5.022E-03	5.024E-03
9β(D)$*$-EP	4.990E-03	5.008E-03	5.009E-03
9β(D)$*$-OC	4.989E-03	5.007E-03	5.009E-03
9β(D)$*$-NT	4.990E-03	5.007E-03	5.009E-03
Analytical	5.040E-03	5.060E-03	5.062E-03

Table 3.7: Thick-walled cylinder: Radial displacement

Element	$\nu_1 = .49$	$\nu_2 = .499$	$\nu_3 = .4999$
9β(D)$*$-PH	4.915E-03	4.865E-03	4.852E-03
9β(D)-NC	5.003E-03	5.022E-03	5.024E-03
9β(D)-EP	4.990E-03	5.008E-03	5.009E-03
9β(D)-OC	4.989E-03	5.007E-03	5.009E-03
9β(D)-NT	4.990E-03	5.007E-03	5.009E-03
9β(D)-PH	4.915E-03	4.865E-03	4.852E-03
Analytical	5.040E-03	5.060E-03	5.062E-03

Table 3.7: Thick-walled cylinder: Radial displacement (continued)

Element	1×4	2×8	4×16	Irregular mesh
Q4	0.2434	0.3161	0.3446	0.2108
QC9D	0.3426	0.3490	0.3536	0.3206
5β-NC	0.3505	0.3516	0.3544	0.3218
5β-EP	0.3505	0.3516	0.3544	0.3483
5β-OC	0.3505	0.3516	0.3544	0.3425
5β-NT	0.3505	0.3516	0.3544	0.3497
5β-PH	0.3505	0.3516	0.3544	0.3483
8β(M)$*$-NC	0.3465	0.3521	0.3548	0.3440
8β(M)$*$-EP	0.3465	0.3521	0.3548	0.3455
8β(M)$*$-OC	0.3465	0.3521	0.3548	0.3452
8β(M)$*$-NT	0.3465	0.3521	0.3548	0.3473
8β(M)$*$-PH	0.3465	0.3521	0.3548	0.3445
8β(M)-NC	0.3456	0.3497	0.3538	0.3264
8β(M)-EP	0.3456	0.3497	0.3538	0.3289
8β(M)-OC	0.3456	0.3497	0.3538	0.3298
8β(M)-NT	0.3456	0.3497	0.3538	0.3317
8β(M)-PH	0.3456	0.3497	0.3538	0.3308
8β(D)$*$-NC	0.3465	0.3521	0.3548	0.3440
8β(D)$*$-EP	0.3465	0.3521	0.3548	0.3456
8β(D)$*$-OC	0.3465	0.3521	0.3548	0.3492
8β(D)$*$-NT	0.3465	0.3521	0.3548	0.3474
8β(D)$*$-PH	0.3465	0.3521	0.3548	0.3445
8β(D)-NC	0.3456	0.3497	0.3538	0.3264
8β(D)-EP	0.3456	0.3497	0.3538	0.3289
8β(D)-OC	0.3456	0.3497	0.3538	0.3299
8β(D)-NT	0.3456	0.3497	0.3538	0.3318
8β(D)-PH	0.3456	0.3497	0.3538	0.3308
9β(M)$*$-NC	0.3465	0.3521	0.3548	0.3437
Analytical			0.3553	

Table 3.8: Cook's beam: Tip displacement u_{2_A}

Element	1 × 4	2 × 8	4 × 16	Irregular mesh
9β(M)∗-EP	0.3465	0.3521	0.3548	0.3450
9β(M)∗-OC	0.3465	0.3521	0.3548	0.3461
9β(M)∗-NT	0.3465	0.3521	0.3548	0.3466
9β(M)∗-PH	0.3465	0.3521	0.3548	0.3442
9β(M)-NC	0.3456	0.3497	0.3538	0.3257
9β(M)-EP	0.3456	0.3497	0.3538	0.3273
9β(M)-OC	0.3456	0.3497	0.3538	0.3291
9β(M)-NT	0.3456	0.3497	0.3538	0.3293
9β(M)-PH	0.3456	0.3497	0.3538	0.3304
9β(D)∗-NC	0.3465	0.3521	0.3548	0.3437
9β(D)∗-EP	0.3465	0.3521	0.3548	0.3450
9β(D)∗-OC	0.3465	0.3521	0.3548	0.3461
9β(D)∗-NT	0.3465	0.3521	0.3548	0.3466
9β(D)∗-PH	0.3465	0.3521	0.3548	0.3442
9β(D)-NC	0.3456	0.3497	0.3538	0.3257
9β(D)-EP	0.3456	0.3497	0.3538	0.3273
9β(D)-OC	0.3456	0.3497	0.3538	0.3291
9β(D)-NT	0.3456	0.3497	0.3538	0.3293
9β(D)-PH	0.3456	0.3497	0.3538	0.3305
Analytical			0.3553	

Table 3.8: Cook's beam: Tip displacement u_{2_A} (continued)

Element	1 × 4 mesh		2 × 8 mesh		Irregular mesh	
	$(-\sigma_{11})_B$	$(\sigma_{12})_C$	$(-\sigma_{11})_B$	$(\sigma_{12})_C$	$(-\sigma_{11})_B$	$(\sigma_{12})_C$
Q4	44.08	1.068	55.70	2.029	38.65	-
QC9D	63.06	2.044	61.87	2.797	56.33	-
5β-NC	60.00	3.333	60.05	3.333	54.72	-
5β-EP	60.00	3.333	60.05	3.333	58.28	-
5β-OC	60.00	3.333	60.05	3.333	55.62	-
5β-NT	60.00	3.333	60.05	3.333	58.79	-
5β-PH	60.00	3.333	60.05	3.333	60.25	-
8β(M)∗-NC	61.63	2.865	61.20	1.035	62.05	-
8β(M)∗-EP	61.63	2.865	61.20	1.035	63.55	-
8β(M)∗-OC	61.63	2.865	61.20	1.035	60.43	-
8β(M)∗-NT	61.63	2.865	61.20	1.035	63.50	-
8β(M)∗-PH	61.63	2.865	61.20	1.035	63.99	-
8β(M)-NC	60.12	2.805	59.78	3.194	58.85	-
8β(M)-EP	60.12	2.805	59.78	3.194	61.89	-
Analytical	60.00	5.000	60.00	5.000	60.00	5.000

Table 3.9: Cook's beam: Stress analysis

	1 × 4 mesh		2 × 8 mesh		Irregular mesh	
Element	$(-\sigma_{11})_B$	$(\sigma_{12})_C$	$(-\sigma_{11})_B$	$(\sigma_{12})_C$	$(-\sigma_{11})_B$	$(\sigma_{12})_C$
8β(M)-OC	60.12	2.805	59.78	3.194	59.40	-
8β(M)-NT	60.12	2.805	59.78	3.194	61.14	-
8β(M)-PH	60.12	2.805	59.78	3.194	57.53	-
8β(D)*-NC	61.63	2.865	61.20	1.035	62.02	-
8β(D)*-EP	61.63	2.865	61.20	1.035	63.53	-
8β(D)*-OC	61.63	2.865	61.20	1.035	60.41	-
8β(D)*-NT	61.63	2.865	61.20	1.035	63.50	-
8β(D)*-PH	61.63	2.865	61.20	1.035	63.94	-
8β(D)-NC	60.12	2.805	59.78	3.194	58.82	-
8β(D)-EP	60.12	2.805	59.78	3.194	61.88	-
8β(D)-OC	60.12	2.805	59.78	3.194	59.39	-
8β(D)-NT	60.12	2.805	59.78	3.194	61.14	-
8β(D)-PH	60.12	2.805	59.78	3.194	57.52	-
9β(M)*-NC	61.63	2.865	61.20	1.035	62.18	-
9β(M)*-EP	61.63	2.865	61.20	1.035	62.72	-
9β(M)*-OC	61.63	2.865	61.20	1.035	63.44	-
9β(M)*-NT	61.63	2.865	61.20	1.035	62.17	-
9β(M)*-PH	61.63	2.865	61.20	1.035	63.53	-
9β(M)-NC	60.12	2.805	59.78	3.194	57.31	-
9β(M)-EP	60.12	2.805	59.78	3.194	57.74	-
9β(M)-OC	60.12	2.805	59.78	3.194	58.99	-
9β(M)-NT	60.12	2.805	59.78	3.194	56.19	-
9β(M)-PH	60.12	2.805	59.78	3.194	55.71	-
9β(D)*-NC	61.63	2.865	61.20	1.035	62.15	-
9β(D)*-EP	61.63	2.865	61.20	1.035	62.69	-
9β(D)*-OC	61.63	2.865	61.20	1.035	63.43	-
9β(D)*-NT	61.63	2.865	61.20	1.035	62.50	-
9β(D)*-PH	61.63	2.865	61.20	1.035	63.48	-
9β(D)-NC	60.12	2.805	59.78	3.194	57.29	-
9β(D)-EP	60.12	2.805	59.78	3.194	57.73	-
9β(D)-OC	60.12	2.805	59.78	3.194	58.99	-
9β(D)-NT	60.12	2.805	59.78	3.194	56.18	-
9β(D)-PH	60.12	2.805	59.78	3.194	55.70	-
Analytical	60.00	5.000	60.00	5.000	60.00	5.000

Table 3.9: Cook's beam: Stress analysis (continued)

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	Center displacement u_{2_A}				Tip rotation ψ_B			
	Regular mesh		Irregular mesh		Regular mesh		Irregular mesh	
Element	LC 1	LC 2	LC 1	LC 2	LC 1	LC 2	LC 1	LC 2
Q4	0.6921	-	0.2749	-	-	-	-	-
QC9D	1.462	1.472	1.181	1.207	0.5801	0.6385	0.5243	0.6801
5β-NC	1.500	-	0.6987	-	-	-	-	-
5β-EP	1.500	-	1.405	-	-	-	-	-
5β-OC	1.500	-	1.389	-	-	-	-	-
5β-NT	1.500	-	1.418	-	-	-	-	-
5β-PH	1.500	-	1.393	-	-	-	-	-
8β(M)$*$-NC	1.461	1.470	2.021	2.004	0.5765	0.6377	0.8855	1.019
8β(M)$*$-EP	1.483	1.493	1.485	1.488	0.5876	0.6455	0.5930	0.7540
8β(M)$*$-OC	1.483	1.493	1.489	1.493	0.5876	0.6455	0.5943	0.7632
8β(M)$*$-NT	1.483	1.493	1.481	1.488	0.5876	0.6455	0.6079	0.7900
8β(M)$*$-PH	1.461	1.470	1.447	1.436	0.5765	0.6377	0.6160	0.8747
8β(M)-NC	1.461	1.470	1.500	1.530	0.5765	0.6377	0.7346	0.9267
8β(M)-EP	1.483	1.493	1.232	1.254	0.5876	0.6455	0.5734	0.7623
8β(M)-OC	1.483	1.493	1.233	1.257	0.5876	0.6455	0.5956	0.7971
8β(M)-NT	1.483	1.493	1.245	1.266	0.5876	0.6455	0.6068	0.8368
8β(M)-PH	1.461	1.470	1.273	1.317	0.5765	0.6377	0.6066	0.9481
8β(D)$*$-NC	1.461	1.470	2.011	1.991	0.5765	0.6377	0.8702	0.9925
8β(D)$*$-EP	1.483	1.493	1.483	1.485	0.5876	0.6455	0.5889	0.7374
8β(D)$*$-OC	1.483	1.493	1.487	1.491	0.5876	0.6455	0.5902	0.7450
8β(D)$*$-NT	1.483	1.493	1.480	1.485	0.5876	0.6455	0.6034	0.7704
8β(D)$*$-PH	1.461	1.470	1.445	1.432	0.5765	0.6377	0.6084	0.8428
8β(D)-NC	1.461	1.470	1.488	1.512	0.5765	0.6377	0.7150	0.8895
8β(D)-EP	1.483	1.493	1.229	1.250	0.5876	0.6455	0.5689	0.7421
8β(D)-OC	1.483	1.493	1.230	1.253	0.5876	0.6455	0.5907	0.7741
8β(D)-NT	1.483	1.493	1.243	1.262	0.5876	0.6455	0.6009	0.8106
8β(D)-PH	1.461	1.470	1.271	1.311	0.5765	0.6377	0.5959	0.9037
9β(M)$*$-NC	1.461	1.470	1.995	1.979	0.5765	0.6377	0.8129	1.007
9β(M)$*$-EP	1.483	1.493	1.480	1.483	0.5876	0.6455	0.5917	0.7534
9β(M)$*$-OC	1.483	1.493	1.482	1.484	0.5876	0.6455	0.5937	0.7622
9β(M)$*$-NT	1.483	1.493	1.473	1.481	0.5876	0.6455	0.6063	0.7886
9β(M)$*$-PH	1.461	1.470	1.434	1.426	0.5765	0.6377	0.6054	0.8542
9β(M)-NC	1.461	1.470	1.484	1.518	0.5765	0.6377	0.7272	0.9187
9β(M)-EP	1.483	1.493	1.228	1.253	0.5876	0.6455	0.5706	0.7616
9β(M)-OC	1.483	1.493	1.230	1.254	0.5876	0.6455	0.5949	0.7979
9β(M)-NT	1.483	1.493	1.243	1.267	0.5876	0.6455	0.6063	0.8365
9β(M)-PH	1.461	1.470	1.265	1.307	0.5765	0.6377	0.5978	0.9273
9β(D)$*$-NC	1.461	1.470	1.985	1.967	0.5765	0.6377	0.8580	0.9807
9β(D)$*$-EP	1.483	1.493	1.478	1.480	0.5876	0.6455	0.5877	0.7369
Analytical	1.500				0.600			

Table 3.10: Higher order patch test: Numerical results

	Center displacement u_{2_A}				Tip rotation ψ_B			
	Regular mesh		Irregular mesh		Regular mesh		Irregular mesh	
Element	LC 1	LC 2	LC 1	LC 2	LC 1	LC 2	LC 1	LC 2
9β(D)$*$-OC	1.483	1.493	1.480	1.482	0.5876	0.6455	0.5898	0.7442
9β(D)$*$-NT	1.483	1.493	1.472	1.479	0.5876	0.6455	0.6019	0.7691
9β(D)$*$-PH	1.461	1.470	1.433	1.422	0.5765	0.6377	0.5988	0.8251
9β(D)-NC	1.461	1.470	1.472	1.500	0.5765	0.6377	0.7079	0.8821
9β(D)-EP	1.483	1.493	1.226	1.248	0.5876	0.6455	0.5660	0.7414
9β(D)-OC	1.483	1.493	1.228	1.250	0.5876	0.6455	0.5901	0.7752
9β(D)-NT	1.483	1.493	1.241	1.262	0.5876	0.6455	0.6003	0.8103
9β(D)-PH	1.461	1.470	1.264	1.301	0.5765	0.6377	0.5881	0.8862
Analytical	1.500				0.600			

Table 3.10: Higher order patch test: Numerical results (continued)