Chapter 2

Assumed stress membranes with drilling d.o.f.

2.1 Introduction

2.1.1 Summary of recent research

Both finite elements with drilling degrees of freedom and mixed/hybrid assumed stress formulations are currently research topics of note. Drilling degrees of freedom are for obvious reasons highly desirable when modeling, for instance, folded plates and beam-shell (or membrane) intersections [9].

Classical attempts to develop membrane elements with rotational degrees of freedom were unsuccessful [5]. Compilations of these early efforts are presented by Frey [10] and Bergan and Felippa [11]. The papers of Bergan and Felippa and Allman [12] presented fresh approaches to the formulation of membrane elements with rotational degrees of freedom [13]. The key to their success was the use of a quadratic displacement function for the normal component of displacement rather than the cubic functions employed in earlier works.

Allman with his simple, but powerful formulation, introduced the term 'vertex rotation' [12]. In this formulation, the vertex rotations are related to the derivatives computed at the element nodes. The vertex rotations introduced by Allman in the constant strain triangle dramatically improved the in-plane behavior of his element. Cook presented a quadrilateral element with drilling degrees of freedom, derived from the Allman triangle [14]. A similar formulation was presented by Allman [15].

Since these attempts, many papers on the subject have appeared, notably those by Jetteur, Jaamei and Frey [10, 16, 17, 18] and by Taylor and Simo *et al.* [6, 19, 20]. However, these elements all suffered from the serious drawback that they are rank deficient. To address this deficiency, Hughes and Brezzi [5] presented a rigorous framework wherein elements with independently interpolated rotation fields could be formulated. Utilizing the formulation of Reissner [21], they argue that formulations employing 'convenient' displacement, rotation and stress interpolations are doomed to failure. Instead, they propose a modified variational principle based on the Euler-Lagrange equations presented by Reissner. However, they improved the stability properties in the discrete approximations.

Finite element interpolations employing the formulation of Hughes and Brezzi were finally presented by Hughes *et al.* [22] and Ibrahimbegovic *et al.* [23, 24]. Since then, the developments in membrane finite elements with drilling degrees of freedom has been numerous.

Previously, Groenwold and Stander applied the 5-point quadrature presented by Dovey [1] to drilling degree of freedom membranes, which improved the element behavior through the introduction of a 'soft' higher order deformation mode [3, 25].

The developments in mixed/hybrid membrane finite elements has been equally important during recent years. Since the assumed stress hybrid finite element presented by Pian [26], numerous formulations have been proposed. A compilation is presented by Pian [27]. The biggest difficulty in deriving hybrid finite elements seems to be the lack of a rational methodology for deriving stress terms [8]. Many approaches were made to address this deficiency, e.g. see [28, 29].

It is recognized that the number of stress modes m in the assumed stress field should satisfy

$$m \geq n - r \tag{2.1}$$

with n the total number of nodal displacements, and r the number of rigid body modes in an element. If (2.1) is not satisfied, rank deficiencies arise, viz. the element stiffness matrix rank is less than the total degree of deformation modes. Furthermore, the equality represented by (2.1) is optimal, since $m > n-r$ increases the element stiffness [30]. Therefore, assumed stress formulations should not only satisfy the requirements of rank sufficiency and invariance, but preferably also the equality condition represented by (2.1). Feng *et al.* [8] present a brief compilation of studies dealing with criteria for stability and convergence. Amongst others, notable contributions are those by Brezzi [31] and Babuska [32], who present necessary and sufficient conditions. Feng *et al.* propose a classification method which also proves that kinematic modes can exist even if $m > n - r$, and show that the m modes are to be chosen from m different stress groups.

The limiting principle of Fraeijs de Veubeke [33] states that a complete but unconstrained assumed stress field becomes identical to the corresponding assumed displacement element. This has lead to the introduction of additional incompatible displacements in numerous formulations. Di and Ramm [34] have chosen not to introduce incompatible modes, but present a rigorous unified formulation to propose stress interpolations.

Previously, a mixed/hybrid assumed stress membrane finite element with drilling degrees of freedom has been presented by Aminpour [35, 36]. However, this element is rank deficient (by one). The framework presented by Hughes and Brezzi [5] can however be used to overcome this drawback.

Sze and Ghali [37] presented a rank sufficient formulation using only 8 interpolating stress modes, denoted HQ8*, which is one less than the equality expressed in (2.1). They used four zero energy modes. One is the equal-rotations mode and the other three are the rigid-body modes. The equal-rotations mode, known as an hourglass mode, is stabilized by a quadratic stress mode. This important contribution probably represents the first ranks sufficient assumed stress membrane finite element with drilling degrees of freedom.

The element presented by Sze and Ghali does not include a locking correction to overcome membrane locking when the element is used as the membrane component of a flat shell finite element. In addition, the interpolation field in the element is not necessarily optimally constrained.

2.1.2 This study

In this study, a variational basis for the formulation of two families of assumed stress membrane finite elements with drilling degrees of freedom is presented, depending on the formulation of Hughes and Brezzi. The families are derived using the unified formulation presented by Di and Ramm [34]. The recent stress mode classification method presented by Feng *et al.* [8] is used to derive the stress interpolation matrices. Both families, denoted 8β(M) and 8β(D), are rank sufficient, invariant, and free of locking. The membrane locking correction suggested by Taylor [6] is used to ensure that the consistent nodal loads in both families are identical to those of a quadrilateral 4-node membrane finite element with two translational degrees of freedom per node.

2.2 A framework for independently interpolated rotation fields

In this section, a rigorous framework for the formulation of independently interpolated rotation fields is presented. The formulation of Hughes and Brezzi [5] is closely followed. The interpolation fields proposed by Ibrahimbegovic *et al.* are presented in Section 2.2.2.

2.2.1 Variational formulation

Let $\Omega \subset \mathbb{R}^d$ be an open set with a piecewise smooth boundary. $d \geq 2$ denotes the number of spatial dimensions. The stress tensor, $\boldsymbol{\sigma}$ (do not assume symmetry), the displacement vector, $\boldsymbol{u}$, and the skew-symmetric rotational tensor, $\boldsymbol{\psi}$, are taken as dependent variables.

The Dirichlet boundary value problem is the focus for this framework. More complicated boundary conditions provide no essential difficulties and may be handled by standard means (see, e.g., [38]). The Euclidean decomposition of a second-rank tensor is used, e.g.,

$$\boldsymbol{\sigma} = \text{symm } \boldsymbol{\sigma} + \text{skew } \boldsymbol{\sigma} \tag{2.2}$$

where

$$\text{symm } \boldsymbol{\sigma} = \frac{1}{2}(\boldsymbol{\sigma} + \boldsymbol{\sigma}^T) \tag{2.3}$$

$$\text{skew } \boldsymbol{\sigma} = \frac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^T) \tag{2.4}$$

The boundary value problem

Given $\boldsymbol{f}$, the body force vector, find $\boldsymbol{u}$, $\boldsymbol{\psi}$ and $\boldsymbol{\sigma}$, such that:
For all $\boldsymbol{x} \in \Omega$

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f} &= \mathbf{0} && (2.5)\\ \operatorname{skew} \boldsymbol{\sigma} &= \mathbf{0} && (2.6)\\ \boldsymbol{\psi} &= \operatorname{skew} \boldsymbol{\nabla u} && (2.7)\\ \operatorname{symm} \boldsymbol{\sigma} &= \boldsymbol{C} \cdot \operatorname{symm} \boldsymbol{\nabla u} && (2.8) \end{aligned}$$

and on the boundary $\Gamma = \partial\Omega$

$$\boldsymbol{u} = \mathbf{0} \tag{2.9}$$

where (2.5) through (2.9) are, respectively, the equilibrium equations, the symmetry conditions for stress, the definition of rotation in terms of displacement gradients, the constitutive equations and the displacement boundary condition.

The elastic moduli, $\boldsymbol{C} = \{C_{ijkl}\}$, $1 \leq i, j, k, l \leq d$, are assumed to satisfy the following conditions:

$$\begin{aligned} C_{ijkl} &= C_{klij} && (2.10)\\ C_{ijkl} &= C_{jikl} = C_{ijlk} && (2.11)\\ C_{ijkl}\epsilon_{ij}\epsilon_{kl} &> 0 \;\; \forall \; \epsilon_{ij} = \epsilon_{ji} \neq 0 && (2.12) \end{aligned}$$

where (2.10) through (2.12) are referred to as, respectively, the major symmetry, the minor symmetries, and positive-definiteness.

For an isotropic material and plane stress, the constitutive modulus tensor $\boldsymbol{C} = \{C_{ijkl}\}$ has the form

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \qquad i, j, k, l \in \{1, 2\} \tag{2.13}$$

where

$$\begin{aligned} \lambda &= \frac{\nu E}{(1 - \nu^2)} && (2.14)\\ \mu &= \frac{E}{2\,(1 + \nu)} && (2.15) \end{aligned}$$

where E and ν are Young's modulus and Poisson's ratio, respectively. λ and μ are the Lamé parameters and δ_{ij} is the Kronecker delta.

Variational form of the boundary value problem

Let $L_2(\Omega)$ denote the space of square-integrable functions on Ω, and let $H^1(\Omega)$ denote the space of functions in $L_2(\Omega)$ with generalized derivatives also in $L_2(\Omega)$. $H_0^1(\Omega)$ is the subset of $H^1(\Omega)$ whose members satisfy zero boundary conditions. The spaces relevant to the boundary value problem are:

$$\begin{aligned} V &= \left\{ \boldsymbol{v} | \boldsymbol{v} \in \left(H_0^1(\Omega)\right)^d \right\} & (2.16) \\ W &= \left\{ \boldsymbol{\omega} | \boldsymbol{\omega} \in \left(L_2(\Omega)\right)^d, \text{symm } \boldsymbol{\omega} = \boldsymbol{0} \right\} & (2.17) \\ T &= \left\{ \boldsymbol{\tau} | \boldsymbol{\tau} \in \left(L_2(\Omega)\right)^d \right\} & (2.18) \end{aligned}$$

where V is the space of trail displacements, W of trail rotations, and T of trail stresses. Consider the following functional [21]:

$$\Pi = V \times W \times T \rightarrow \mathbb{R} \tag{2.19}$$

$$\Pi(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}) = -\frac{1}{2} \int_\Omega \text{symm } \boldsymbol{\tau} \cdot \boldsymbol{C}^{-1} \cdot \text{symm } \boldsymbol{\tau} \text{ d}\Omega + \int_\Omega \boldsymbol{\tau}^T \cdot (\nabla \boldsymbol{v} - \boldsymbol{\omega}) \text{ d}\Omega - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f} \text{ d}\Omega \tag{2.20}$$

The stationary condition and integration-by-parts reveals that the Euler-Lagrange equations emanating from Π correspond to the equations of the boundary value problem (i.e. (2.5)-(2.8)), viz.

$$\begin{aligned} 0 &= \delta\Pi(\boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{\sigma})(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}) \\ &= -\int_\Omega \text{symm } \boldsymbol{\tau} \cdot \boldsymbol{C}^{-1} \cdot \text{symm } \boldsymbol{\sigma} \text{ d}\Omega + \int_\Omega \boldsymbol{\tau}^T \cdot (\nabla \boldsymbol{u} - \boldsymbol{\psi}) \text{ d}\Omega \\ &\quad + \int_\Omega \boldsymbol{\sigma}^T \cdot (\nabla \boldsymbol{v} - \boldsymbol{\omega}) \text{ d}\Omega - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f} \text{ d}\Omega & (2.21) \\ &= -\int_\Omega \text{symm } \boldsymbol{\tau} \cdot (\boldsymbol{C}^{-1} \cdot \text{symm } \boldsymbol{\sigma} - \text{symm} \nabla \boldsymbol{u}) \text{ d}\Omega \\ &\quad - \int_\Omega \text{skew } \boldsymbol{\tau} \cdot (\text{skew} \nabla \boldsymbol{u} - \boldsymbol{\psi}) \text{ d}\Omega - \int_\Omega \boldsymbol{v} \cdot (\text{div } \boldsymbol{\sigma} + \boldsymbol{f}) \text{ d}\Omega \\ &\quad + \int_\Omega \boldsymbol{\omega} \cdot \text{skew } \boldsymbol{\sigma} \text{ d}\Omega \quad \forall \{\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}\} \in V \times W \times T & (2.22) \end{aligned}$$

So that there is no confusion with the index-free notation, note that:

$$\boldsymbol{\sigma}^T \cdot (\nabla \boldsymbol{v} - \boldsymbol{\omega}) \equiv \sigma_{ij}(v_{i,j} - \omega_{ij}) \tag{2.23}$$

where

$$\nabla \boldsymbol{v} = [v_{i,j}] \tag{2.24}$$

From (2.22) observe that $\boldsymbol{\omega}$ plays the role of a Lagrange multiplier that enforces the symmetry of the stress.

Mathematical theory of the continuous case

Let $U \equiv V \times W$. The following mapping needs to be introduced:

$$a : T \times T \rightarrow \mathbb{R} \tag{2.25}$$

$$b : T \times U \rightarrow \mathbb{R} \tag{2.26}$$

$$f : U \rightarrow \mathbb{R} \tag{2.27}$$

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = -\int_{\Omega} \text{symm}\ \boldsymbol{\sigma} \cdot \boldsymbol{C}^{-1} \cdot \text{symm}\ \boldsymbol{\tau}\ \mathrm{d}\Omega \tag{2.28}$$

$$b(\boldsymbol{\tau}, \{\boldsymbol{v}, \boldsymbol{\omega}\}) = (\boldsymbol{\tau}^T, \nabla \boldsymbol{v} - \boldsymbol{\omega}) \equiv \int_{\Omega} \boldsymbol{\tau}^T \cdot (\nabla \boldsymbol{v} - \boldsymbol{\omega})\ \mathrm{d}\Omega \tag{2.29}$$

$$f(\{\boldsymbol{v}, \boldsymbol{\omega}\}) = \int_{\Omega} \boldsymbol{v} \cdot \boldsymbol{f}\ \mathrm{d}\Omega \tag{2.30}$$

Note that (2.28) and (2.29) are bilinear forms. (2.28) is symmetric, and $\boldsymbol{f}$ is continuous. The variational form of the boundary value problem,(2.22), can now be rewritten as follows:

Problem (M)
Find $\{\boldsymbol{u}, \boldsymbol{\psi}\} \in U$ and $\boldsymbol{\sigma} \in T$ such that

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \{\boldsymbol{u}, \boldsymbol{\psi}\}) = 0 \quad \forall \quad \boldsymbol{\tau} \in T \tag{2.31}$$

$$b(\boldsymbol{\sigma}, \{\boldsymbol{v}, \boldsymbol{\omega}\}) = f(\{\boldsymbol{v}, \boldsymbol{\omega}\}) \quad \forall \quad \{\boldsymbol{v}, \boldsymbol{\omega}\} \in U \tag{2.32}$$

The discrete problem

Let V^h, W^h and T^h be finite dimensional subspaces of V, W and T, respectively. The superscript 'h' denotes dependence upon a mesh parameter. V^h, W^h and T^h are typical finite element spaces involving piecewise polynomial interpolations. The standard way of developing a discrete approximation is to pose (2.31) and (2.32) in terms of the finite dimensional subspaces.

Problem (M^h)
Find $\{\boldsymbol{u}^h, \boldsymbol{\psi}^h\} \in U^h \equiv V^h \times W^h$ and $\boldsymbol{\sigma}^h \in T^h$ such that

$$a(\boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) + b(\boldsymbol{\tau}^h, \{\boldsymbol{u}^h, \boldsymbol{\psi}^h\}) = 0 \quad \forall \quad \boldsymbol{\tau}^h \in T^h \tag{2.33}$$

$$b(\boldsymbol{\sigma}^h, \{\boldsymbol{v}^h, \boldsymbol{\omega}^h\}) = f(\{\boldsymbol{v}^h, \boldsymbol{\omega}^h\}) \quad \forall \quad \{\boldsymbol{v}^h, \boldsymbol{\omega}^h\} \in U^h \tag{2.34}$$

Problem (M^h) has a unique solution $\{\boldsymbol{u}^h, \boldsymbol{\psi}^h\} \in U^h$, $\boldsymbol{\sigma}^h \in T^h$. A proof is presented by Hughes and Brezzi [5].

A modified variational formulation

The ellipticity of the continuous problem is not inherited by the discrete problem for convenient finite element spaces. In order to improve upon the ellipticity of the standard mixed formulation, consider the following functional:

$$\Pi_\gamma : V \times W \times T \to \mathbb{R} \tag{2.35}$$

$$\begin{aligned} \Pi_\gamma(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}) &= \Sigma_\gamma\left(\Pi(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau})\right) \\ &\equiv \Pi(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}) - \frac{1}{2}\gamma^{-1} \int_\Omega |\text{skew } \boldsymbol{\tau}|^2 \, d\Omega \end{aligned} \tag{2.36}$$

This functional gives rise to a system of variational equations formally equivalent to those of Π. This may be seen as follows:

$$\begin{aligned} 0 &= \delta\Pi(\boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{\sigma})(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}) \\ &= -\int_\Omega \text{symm } \boldsymbol{\tau} \cdot (\boldsymbol{C}^{-1} \cdot \text{symm } \boldsymbol{\sigma} - \text{symm}\nabla \boldsymbol{u}) \, d\Omega \\ &\quad -\int_\Omega \text{skew } \boldsymbol{\tau} \cdot (\text{skew}\nabla \boldsymbol{u} - \boldsymbol{\psi} - \gamma^{-1}\text{skew } \boldsymbol{\sigma}) \, d\Omega - \int_\Omega \boldsymbol{v} \cdot (\text{div } \boldsymbol{\sigma} + \boldsymbol{f}) \, d\Omega \\ &\quad + \int_\Omega \boldsymbol{\omega} \cdot \text{skew } \boldsymbol{\sigma} \, d\Omega \quad \forall \ \{\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}\} \in V \times W \times T \end{aligned} \tag{2.37}$$

Observe that skew $\boldsymbol{\tau} = \boldsymbol{0}$. Thus the Euler-Lagrange equations of the continuous problem are unchanged. Nevertheless, the consequences of the additional term are significant in the context of approximate solutions. This may be seen more clearly by writing (2.37) in the standard format of a mixed problem.

Problem (M_γ)

Find $\{\boldsymbol{u}, \boldsymbol{\psi}\} \in U$ and $\boldsymbol{\sigma} \in T$ such that

$$a_\gamma(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \{\boldsymbol{u}, \boldsymbol{\psi}\}) = 0 \quad \forall \quad \boldsymbol{\tau} \in T \tag{2.38}$$

$$b(\boldsymbol{\sigma}, \{\boldsymbol{v}, \boldsymbol{\omega}\}) = f(\{\boldsymbol{v}, \boldsymbol{\omega}\}) \quad \forall \quad \{\boldsymbol{v}, \boldsymbol{\omega}\} \in U \tag{2.39}$$

where

$$a_\gamma(\boldsymbol{\sigma}, \boldsymbol{\tau}) = a(\boldsymbol{\sigma}, \boldsymbol{\tau}) - \gamma^{-1}(\text{skew } \boldsymbol{\sigma}, \text{skew } \boldsymbol{\tau}) \tag{2.40}$$

The finite dimensional counterpart of *Problem (M_γ)* is given by:

Problem (M_γ^h)

Find $\{\boldsymbol{u}^h, \boldsymbol{\psi}^h\} \in U^h$ and $\boldsymbol{\sigma}^h \in T^h$ such that

$$a_\gamma(\boldsymbol{\sigma}^h, \boldsymbol{\tau}^h) + b(\boldsymbol{\tau}^h, \{\boldsymbol{u}^h, \boldsymbol{\psi}^h\}) = 0 \quad \forall \quad \boldsymbol{\tau}^h \in T^h \tag{2.41}$$

$$b(\boldsymbol{\sigma}^h, \{\boldsymbol{v}^h, \boldsymbol{\omega}^h\}) = f(\{\boldsymbol{v}^h, \boldsymbol{\omega}^h\}) \quad \forall \quad \{\boldsymbol{v}^h, \boldsymbol{\omega}^h\} \in U^h \tag{2.42}$$

Various special cases of the previous variational formulation can be developed by eliminating fields through the use of Euler-Lagrange equations. The symmetrical components of stress can be eliminated by way of the constitutive equation. Define the functional π_γ by

$$\begin{aligned} \pi_\gamma(\boldsymbol{v}, \boldsymbol{\omega}, \text{skew } \boldsymbol{\tau}) &= \Pi_\gamma(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{C} \cdot \text{symm}\nabla\boldsymbol{v} + \text{skew } \boldsymbol{\tau}) \\ &= \frac{1}{2}\int_\Omega \text{symm}\nabla\boldsymbol{v} \cdot \boldsymbol{C} \cdot \text{symm}\nabla\boldsymbol{v} \text{ d}\Omega \\ &\quad + \int_\Omega \text{skew } \boldsymbol{\tau}^T \cdot (\text{skew}\nabla\boldsymbol{v} - \boldsymbol{\omega}) \text{ d}\Omega \\ &\quad - \frac{1}{2}\gamma^{-1}\int_\Omega |\text{skew } \boldsymbol{\tau}|^2 \text{ d}\Omega - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f} \text{ d}\Omega \end{aligned} \tag{2.43}$$

Displacement-type modified variational formulations

From the practical point of view, the most interesting formulation is one based entirely on kinematic variables, namely, displacement and rotation. To this end, the modified variational formulations permit the elimination of skew $\boldsymbol{\sigma}$ by way of the following Euler-Lagrange equation

$$\gamma^{-1}\text{skew } \boldsymbol{\sigma} = \text{skew}\nabla\boldsymbol{u} - \boldsymbol{\psi} \tag{2.44}$$

The following functional is derived by employing (2.44) in (2.43)

$$\tilde{\pi}_\gamma : V \times W \to \mathbb{R} \tag{2.45}$$

$$\begin{aligned} \tilde{\pi}_\gamma(\boldsymbol{v}, \boldsymbol{\omega}) &= \pi_\gamma(\boldsymbol{v}, \boldsymbol{\omega}, \gamma(\text{skew}\nabla\boldsymbol{v} - \boldsymbol{\omega})) \\ &= \frac{1}{2}\int_\Omega \text{symm}\nabla\boldsymbol{v} \cdot \boldsymbol{C} \cdot \text{symm}\nabla\boldsymbol{v} \text{ d}\Omega \\ &\quad + \frac{1}{2}\gamma\int_\Omega |\text{skew}\nabla\boldsymbol{v} - \boldsymbol{\omega}|^2 \text{ d}\Omega - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f} \text{ d}\Omega \end{aligned} \tag{2.46}$$

Since this is the simplest formulation within this framework, it is the one most likely to be used by program developers [5]. Indeed, this formulation was used by Ibrahimbegovic *et al.* in 1990 [23].

The variational equation emanating from (2.46) is

$$\begin{aligned} 0 &= \delta\tilde{\pi}(\boldsymbol{u},\boldsymbol{\psi})\cdot(\boldsymbol{v},\boldsymbol{\omega}) \\ &= \int_\Omega \text{symm}\nabla\boldsymbol{v}\cdot\boldsymbol{C}\cdot\text{symm}\nabla\boldsymbol{u}\ \mathrm{d}\Omega \\ &\quad + \int_\Omega (\text{skew}\nabla\boldsymbol{v}-\boldsymbol{\omega})^T\cdot(\gamma(\text{skew}\nabla\boldsymbol{u}-\boldsymbol{\psi}))\ \mathrm{d}\Omega - \int_\Omega \boldsymbol{v}\cdot\boldsymbol{f}\ \mathrm{d}\Omega \qquad (2.47) \\ &= -\int_\Omega \boldsymbol{v}\cdot\text{div}\ [\boldsymbol{C}\cdot\text{symm}\nabla\boldsymbol{u}+\gamma(\text{skew}\nabla\boldsymbol{u}-\boldsymbol{\psi})+\boldsymbol{f}]\ \mathrm{d}\Omega \\ &\quad - \int_\Omega \boldsymbol{\omega}^T\cdot(\gamma(\text{skew}\nabla\boldsymbol{u}-\boldsymbol{\psi}))\ \mathrm{d}\Omega \qquad (2.48) \end{aligned}$$

The last term in (2.48) asserts that the skew-symmetric stresses are zero, and the first term express equilibrium in terms of the symmetric stresses. In the corresponding discrete case, the skew-symmetric stresses will not be in general identically zero and thus will play a role in the equilibrium conditions. The mathematical formulation of the variational problem is

Problem (D_γ)
Find $\{\boldsymbol{u},\boldsymbol{\psi}\}\in U$ such that

$$B_\gamma(\boldsymbol{u},\boldsymbol{\psi};\boldsymbol{v},\boldsymbol{\omega}\}) = f(\{\boldsymbol{v},\boldsymbol{\omega}\}) \quad \forall \quad \{\boldsymbol{v},\boldsymbol{\omega}\}\in U \qquad (2.49)$$

where

$$\begin{aligned} B_\gamma(\boldsymbol{u},\boldsymbol{\psi};\boldsymbol{v},\boldsymbol{\omega}\}) &= \int_\Omega \text{symm}\nabla\boldsymbol{v}\cdot\boldsymbol{C}\cdot\text{symm}\nabla\boldsymbol{u}\ \mathrm{d}\Omega \\ &\quad + \int_\Omega (\text{skew}\nabla\boldsymbol{v}-\boldsymbol{\omega})^T\cdot(\gamma(\text{skew}\nabla\boldsymbol{u}-\boldsymbol{\psi}))\ \mathrm{d}\Omega \qquad (2.50) \end{aligned}$$

is a symmetrical bilinear form. The corresponding discrete problem is:

Problem (D^h_γ)
Find $\{\boldsymbol{u}^h,\boldsymbol{\psi}^h\}\in U^h$ such that

$$B_\gamma(\boldsymbol{u}^h,\boldsymbol{\psi}^h;\boldsymbol{v}^h,\boldsymbol{\omega}^h\}) = f(\{\boldsymbol{v}^h,\boldsymbol{\omega}^h\}) \quad \forall \quad \{\boldsymbol{v}^h,\boldsymbol{\omega}^h\}\in U^h \qquad (2.51)$$

Generalization

Hu-Washizu variational formulations are frequently used as a basis for finite element discretizations. A Hu-Washizu-type variational formulation accounting for rotations and non-symmetric stress tensors derives from the following functional:

$$\begin{aligned} H(\boldsymbol{v},\boldsymbol{\omega},\text{skew}\boldsymbol{\tau},\text{symm}\boldsymbol{\tau},\epsilon) &= \frac{1}{2}\int_\Omega \epsilon\cdot\boldsymbol{C}\cdot\epsilon\ \mathrm{d}\Omega + \int_\Omega \text{symm}\boldsymbol{\tau}\cdot(\text{symm}\nabla\boldsymbol{v}-\epsilon)\ \mathrm{d}\Omega \\ &\quad + \frac{1}{2}\int_\Omega \text{skew}\boldsymbol{\tau}^T\cdot(\text{skew}\nabla\boldsymbol{v}-\boldsymbol{\omega})\ \mathrm{d}\Omega - \int_\Omega \boldsymbol{v}\cdot\boldsymbol{f}\ \mathrm{d}\Omega \ (2.52) \end{aligned}$$

where $\epsilon = \epsilon^T$.

2.2.2 Finite element interpolations by Ibrahimbegovic *et al.*

The rotational and translational interpolations of the formulation of Hughes and Brezzi [5] are addressed in detail in the papers of Hughes *et al.* [22] and Ibrahimbegovic *et al.* [23]. Here, the formulation of Ibrahimbegovic *et al.* [23] is followed closely.

The independent rotation field is interpolated as a standard bilinear field over each element. Accordingly

$$\psi^h = \sum_e \sum_{i=1}^{4} N_i^e(\xi,\eta)\psi_i \tag{2.53}$$

where (e.g., see [39])

$$N_i^e(\xi,\eta) = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \quad i = 1,2,3,4 \tag{2.54}$$

The in-plane displacement approximation is taken as an Allman-type interpolation field

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \boldsymbol{u}^h \\ &= \sum_e \sum_{i=1}^{4} N_i^e(\xi,\eta)\boldsymbol{u}_i + \frac{l_{jk}}{8}\sum_e \sum_{i=5}^{8} NS_i^e(\xi,\eta)(\psi_k - \psi_j)\boldsymbol{n}_{jk} \\ &\quad + \sum_e NB_9^e(\xi,\eta)\Delta\boldsymbol{u}_9 \end{aligned} \tag{2.55}$$

l_{jk} and $\boldsymbol{n}_{jk}$ denote the length and the outward unit normal vector on the element side associated with the corner nodes j and k.

$$\boldsymbol{n}_{jk} = \left\{ \begin{array}{c} n_1 \\ n_2 \end{array} \right\} = \left\{ \begin{array}{c} \cos\alpha_{jk} \\ \sin\alpha_{jk} \end{array} \right\} \tag{2.56}$$

and

$$l_{jk} = ((x_{k1}-x_{j1})^2 + (x_{k2}-x_{j2})^2)^{1/2} \tag{2.57}$$

The indices in the above are explicitly given in Appendix A.

In (2.55) the following Serendipity shape functions defined by Zienkiewicz and Taylor [39] are used.

$$NS_i^e(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta_i\eta); \quad i = 5,7 \tag{2.58}$$

$$NS_i^e(\xi,\eta) = \frac{1}{2}(1+\xi_i\xi)(1-\eta^2); \quad i = 6,8 \tag{2.59}$$

To reflect the superior performance of the 9-node Lagrangian element over that of the 8-node Serendipity element, a hierarchical bubble function interpolation is added in (2.55) where

$$NB_9^e(\xi,\eta) = (1-\xi^2)(1-\eta^2) \tag{2.60}$$

The terms in the element stiffness matrix arising from this interpolation may be eliminated at the element level by static condensation [40].

2.2.3 On the numerical value of γ

For isotropic elasticity and Dirichlet boundary value problems, Ibrahimbegovic *et al.* take γ equal to the value of the shear modulus [23]. The choice of $\gamma = G$ was suggested by Hughes *et al.* [22]. Numerical studies by Ibrahimbegovic *et al.* [23] show that their element formulation is insensitive to the value of γ used, at least for several orders of magnitude which bound the shear modulus. This was however shown for one particular problem only. Results by Groenwold and Stander [25] indicated that there may be a more pronounced sensitivity to the value of γ for certain examples. For some problems, therefore, enforcement of the rotational field by sufficiently large values of γ is crucial [25].

Notwithstanding the undesirability of having a problem dependent parameter in the formulation, both the shear and extension patch tests (Figure 3.3) are passed for any positive value of γ. As the patch test is a necessary and sufficient condition for convergence (see [41]), the numerical value of γ becomes irrelevant in the limit of mesh refinement[1].

2.3 Assumed stress membrane element with drilling degrees of freedom formulation

2.3.1 Variational formulation

In this study, the formulation presented by Hughes and Brezzi (see (2.43)) is extended through the addition of the term

$$\int_\Omega \text{symm } \boldsymbol{\tau}^T \cdot (\text{symm}\boldsymbol{\nabla v} - \boldsymbol{\epsilon}) \; \mathrm{d}\Omega \tag{2.61}$$

where $\boldsymbol{\tau}^T$ represents a Lagrangian multiplier. The following Hu-Washizu like functional is obtained

Problem (M_c)

$$\begin{aligned} \boldsymbol{\Pi}_\gamma(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}) \;=\; & \frac{1}{2}\int_\Omega (\text{symm}\boldsymbol{\nabla v})^T \cdot \boldsymbol{C} \cdot \text{symm}\boldsymbol{\nabla v} \; \mathrm{d}\Omega + \int_\Omega \text{symm } \boldsymbol{\tau}^T \cdot (\text{symm}\boldsymbol{\nabla v} - \boldsymbol{\epsilon}) \; \mathrm{d}\Omega \\ & + \int_\Omega \text{skew } \boldsymbol{\tau}^T \cdot (\text{skew}\boldsymbol{\nabla v} - \boldsymbol{\omega}) \; \mathrm{d}\Omega - \frac{1}{2}\gamma^{-1} \int_\Omega [\text{skew } \boldsymbol{\tau}]^2 \; \mathrm{d}\Omega \\ & - \int_\Omega \boldsymbol{v}^T \cdot \boldsymbol{f} \; \mathrm{d}\Omega \end{aligned} \tag{2.62}$$

Substituting the constitutive relationship $\boldsymbol{\epsilon} = \boldsymbol{C}^{-1} \cdot \text{symm } \boldsymbol{\tau}$, *Problem ($M_c$)* can be rewritten to obtain

$$\begin{aligned} \boldsymbol{\Pi}_\gamma(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}) \;=\; & \int_\Omega \text{symm } \boldsymbol{\tau}^T \cdot \text{symm}\boldsymbol{\nabla v} \; \mathrm{d}\Omega - \frac{1}{2}\int_\Omega \text{symm } \boldsymbol{\tau}^T \cdot \boldsymbol{C}^{-1} \cdot \text{symm } \boldsymbol{\tau} \; \mathrm{d}\Omega \\ & + \int_\Omega \text{skew } \boldsymbol{\tau}^T \cdot (\text{skew}\boldsymbol{\nabla v} - \boldsymbol{\omega}) \; \mathrm{d}\Omega - \frac{1}{2}\gamma^{-1} \int_\Omega [\text{skew } \boldsymbol{\tau}]^2 \; \mathrm{d}\Omega \end{aligned}$$

[1] The effect of γ is extensively demonstrated in Chapters 3, 5, and 7

$$- \int_\Omega \boldsymbol{v}^T \cdot \boldsymbol{f} \; \mathrm{d}\Omega \tag{2.63}$$

The variational equation which results from variations on (2.63) is

$$\begin{aligned} 0 \;=\; \delta \boldsymbol{\Pi}_\gamma(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}) \;=\; & \int_\Omega \text{symm } \boldsymbol{\sigma}^T \cdot \text{symm} \boldsymbol{\nabla v} \; \mathrm{d}\Omega + \int_\Omega \text{symm } \boldsymbol{\tau}^T \cdot \text{symm} \boldsymbol{\nabla u} \; \mathrm{d}\Omega \\ & - \int_\Omega \text{symm } \boldsymbol{\tau}^T \cdot \boldsymbol{C}^{-1} \cdot \text{symm } \boldsymbol{\sigma} \; \mathrm{d}\Omega + \int_\Omega \text{skew } \boldsymbol{\tau}^T \cdot (\text{skew} \boldsymbol{\nabla u} - \boldsymbol{\psi}) \; \mathrm{d}\Omega \\ & + \int_\Omega (\text{skew} \boldsymbol{\nabla v}^T \cdot \text{skew } \boldsymbol{\sigma} - \boldsymbol{\omega}^T \cdot \text{skew } \boldsymbol{\sigma}) \; \mathrm{d}\Omega - \gamma^{-1} \int_\Omega \text{skew } \boldsymbol{\tau}^T \cdot \text{skew } \boldsymbol{\sigma} \; \mathrm{d}\Omega \\ & - \int_\Omega \boldsymbol{u}^T \cdot \boldsymbol{f} \; \mathrm{d}\Omega \end{aligned} \tag{2.64}$$

Furthermore, it is possible to eliminate the skew-symmetric part of the stress tensor by substituting

$$\gamma^{-1} \text{skew } \boldsymbol{\sigma} \;=\; \text{skew} \boldsymbol{\nabla u} - \boldsymbol{\psi} \tag{2.65}$$

into *Problem (M_c)* to obtain

Problem (D_c)

$$\begin{aligned} \boldsymbol{\Pi}_\gamma(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}) \;=\; & \int_\Omega \text{symm } \boldsymbol{\tau}^T \cdot \text{symm} \boldsymbol{\nabla v} \; \mathrm{d}\Omega - \frac{1}{2} \int_\Omega \text{symm } \boldsymbol{\tau}^T \cdot \boldsymbol{C}^{-1} \cdot \text{symm } \boldsymbol{\tau} \; \mathrm{d}\Omega \\ & + \frac{1}{2} \gamma \int_\Omega [\text{skew} \boldsymbol{\nabla v} - \boldsymbol{\omega}]^2 \; \mathrm{d}\Omega - \int_\Omega \boldsymbol{v}^T \cdot \boldsymbol{f} \; \mathrm{d}\Omega \end{aligned} \tag{2.66}$$

which is now similar to the generalization presented by Hughes and Brezzi (see [5]). The corresponding variational equation becomes

$$\begin{aligned} 0 \;=\; \delta \boldsymbol{\Pi}_\gamma(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\tau}) \;=\; & \int_\Omega \text{symm } \boldsymbol{\sigma}^T \cdot \text{symm} \boldsymbol{\nabla v} \; \mathrm{d}\Omega \\ & + \int_\Omega \text{symm } \boldsymbol{\tau}^T \cdot \text{symm} \boldsymbol{\nabla u} \; \mathrm{d}\Omega - \int_\Omega \text{symm } \boldsymbol{\tau}^T \cdot \boldsymbol{C}^{-1} \cdot \text{symm } \boldsymbol{\sigma} \; \mathrm{d}\Omega \\ & + \gamma \int_\Omega (\text{skew} \boldsymbol{\nabla v} - \boldsymbol{\omega})^T \cdot (\text{skew} \boldsymbol{\nabla u} - \boldsymbol{\psi}) \; \mathrm{d}\Omega - \int_\Omega \boldsymbol{u}^T \cdot \boldsymbol{f} \; \mathrm{d}\Omega \end{aligned} \tag{2.67}$$

2.3.2 Finite element interpolation

The discrete version of *Problem (M_c)* is obtained as

Problem (M_c^h)

$$\begin{aligned} 0 \;=\; & \int_{\Omega^h} (\text{symm } \boldsymbol{\sigma}^h)^T \cdot \text{symm} \boldsymbol{\nabla v}^h \; \mathrm{d}\Omega + \int_{\Omega^h} (\text{symm } \boldsymbol{\tau}^h)^T \cdot \text{symm} \boldsymbol{\nabla u}^h \; \mathrm{d}\Omega \\ & - \int_{\Omega^h} (\text{symm } \boldsymbol{\tau}^h)^T \cdot \boldsymbol{C}^{-1} \cdot \text{symm } \boldsymbol{\sigma}^h \; \mathrm{d}\Omega + \int_{\Omega^h} (\text{skew } \boldsymbol{\tau}^h)^T \cdot (\text{skew} \boldsymbol{\nabla u}^h - \boldsymbol{\psi}^h) \; \mathrm{d}\Omega \\ & + \int_{\Omega^h} \left((\text{skew} \boldsymbol{\nabla v}^h)^T \cdot \text{skew } \boldsymbol{\sigma}^h - (\boldsymbol{\omega}^h)^T \cdot \text{skew } \boldsymbol{\sigma}^h \right) \; \mathrm{d}\Omega \\ & - \gamma^{-1} \int_{\Omega^h} (\text{skew } \boldsymbol{\tau}^h)^T \cdot \text{skew } \boldsymbol{\sigma}^h \; \mathrm{d}\Omega - \int_{\Omega^h} (\boldsymbol{u}^h)^T \cdot \boldsymbol{f} \; \mathrm{d}\Omega \end{aligned} \tag{2.68}$$

It is required that the three distinct independent interpolation fields arising from the translations, rotations, and the enhanced stresses are interpolated. The rotational and translational interpolations were addressed in detail in the paper of Ibrahimbegovic *et al.* [23] (see Section 2.2.2). However, the newly introduced assumed stress field is presented in more detail in the following.

The independent rotation field is interpolated as in Section 2.2.2. The in-plane displacement approximation is taken as an Allman-type interpolation field

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \boldsymbol{u}^h = \sum_e \sum_{i=1}^{4} N_i^e(\xi,\eta)\boldsymbol{u}_i + \frac{l_{jk}}{8} \sum_e \sum_{i=5}^{8} NS_i^e(\xi,\eta)(\psi_k - \psi_j)\boldsymbol{n}_{jk} \tag{2.69}$$

with NS_i the Serendipity shape functions. In accordance with the limiting principle of Fraeijs de Veubeke [33], the hierarchical bubble shape function is not included. l_{jk} and $\boldsymbol{n}_{jk}$ denote the length and the outward unit normal vector on the element side associated with the corner nodes j and k (Figure 2.1).

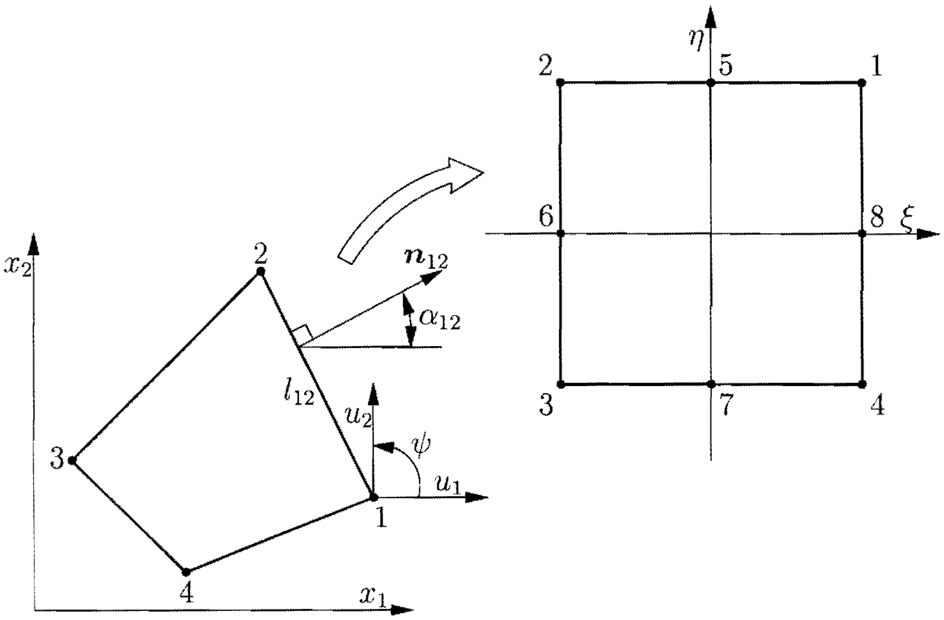

Figure 2.1: Membrane finite element

The skew-symmetric stress field is chosen constant over the element, i.e.

$$\text{skew}\, \boldsymbol{\tau}^h = \sum_e \boldsymbol{\tau}_0^e \tag{2.70}$$

Using matrix notation, symm$\boldsymbol{\nabla u}^e$ and skew$\boldsymbol{\nabla u}^e$ are respectively given by

$$\text{symm}\boldsymbol{\nabla u}^e = \boldsymbol{B}_i^e \boldsymbol{u}_i + \boldsymbol{G}_{\psi i}^e \boldsymbol{\psi}_i \tag{2.71}$$

and

$$\text{skew}\boldsymbol{\nabla u}^e = \boldsymbol{b}_i^e \boldsymbol{u}_i + \boldsymbol{g}_i^e \boldsymbol{\psi}_i \tag{2.72}$$

The operators arising from this interpolation are summarized in Appendix A.

For the assumed stress field, the global stresses are directly interpolated by the stress parameters β_i, i.e.

$$\text{symm}\ \boldsymbol{\sigma}^h = \sum_e \boldsymbol{P}^e \boldsymbol{\beta}^e \tag{2.73}$$

where $\boldsymbol{P}^e$ is the interpolation matrix in terms of the local coordinates and $\boldsymbol{\beta}^e$ is the stress parameter vector. Equations (2.73) represent an unconstrained interpolation field, which is not necessarily optimal. Constraints may be enforced by a suitable transformation matrix $\boldsymbol{A}^e$, such that

$$\text{symm}\ \boldsymbol{\sigma}^h = \sum_e \boldsymbol{A}^e \boldsymbol{P}^e \boldsymbol{\beta}^e \tag{2.74}$$

Various forms for $\boldsymbol{A}^e$ were presented by Di and Ramm [34], and are applied to the new families of elements in sections to follow.

The body force vector is given by

$$\boldsymbol{r} = \int_\Omega \boldsymbol{N}^T \boldsymbol{f}\ \mathrm{d}\Omega \tag{2.75}$$

In matrix notation, the stationary conditions result in

$$\begin{bmatrix} 0 & \boldsymbol{G}^{eT} & \boldsymbol{h}^e \\ \boldsymbol{h}^{eT} & 0 & -\gamma^{-1}\Omega^e \\ \boldsymbol{G}^e & -\boldsymbol{H}^e & 0 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{\beta} \\ \boldsymbol{\tau}_0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{r} \\ 0 \\ 0 \end{bmatrix} \tag{2.76}$$

with

$$\boldsymbol{G}^e = \int_\Omega \boldsymbol{P}^{eT} \cdot [\boldsymbol{B}^e\ \boldsymbol{G}^e]\ \mathrm{d}\Omega \tag{2.77}$$

$$\boldsymbol{H}^e = \int_\Omega \boldsymbol{P}^{eT} \cdot \boldsymbol{C}^{-1} \cdot \boldsymbol{P}^e\ \mathrm{d}\Omega \tag{2.78}$$

$$\boldsymbol{h}^e = \int_\Omega [\boldsymbol{b}^e\ \boldsymbol{g}^e]^T\ \mathrm{d}\Omega \tag{2.79}$$

where $\boldsymbol{C}^{-1}$ denotes the elastic compliance matrix, and where $\boldsymbol{P}^e$ may be replaced by $\boldsymbol{A}^e\boldsymbol{P}^e$. The force-displacement relationship is defined by

$$\left[\boldsymbol{K}^e + \frac{\gamma}{\Omega^e}\boldsymbol{h}^e\boldsymbol{h}^{eT}\right] \boldsymbol{q} = \boldsymbol{r} \tag{2.80}$$

with

$$\boldsymbol{K}^e = \boldsymbol{G}^{eT}\boldsymbol{H}^{e-1}\boldsymbol{G}^e \tag{2.81}$$

Finally, stress recovery is obtained through

$$\boldsymbol{\beta} = \boldsymbol{H}^{e-1}\boldsymbol{G}^e\boldsymbol{q} \tag{2.82}$$

Similarly to the foregoing, the discrete version of *Problem* (D_c) yields

Problem (D_c^h)

$$\begin{aligned} 0 \;=\; & \int_{\Omega^h} (\text{symm}\ \boldsymbol{\sigma}^h)^T \cdot \text{symm}\boldsymbol{\nabla}\boldsymbol{v}^h\ \mathrm{d}\Omega + \int_{\Omega^h} (\text{symm}\ \boldsymbol{\tau}^h)^T \cdot \text{symm}\boldsymbol{\nabla}\boldsymbol{u}^h\ \mathrm{d}\Omega \\ & - \int_{\Omega^h} (\text{symm}\ \boldsymbol{\tau}^h)^T \cdot \boldsymbol{C}^{-1} \cdot \text{symm}\ \boldsymbol{\sigma}^h\ \mathrm{d}\Omega \\ & + \gamma \int_{\Omega^h} (\text{skew}\boldsymbol{\nabla}\boldsymbol{v}^h - \boldsymbol{\omega}^h)^T \cdot (\text{skew}\boldsymbol{\nabla}\boldsymbol{u}^h - \boldsymbol{\psi}^h)\ \mathrm{d}\Omega - \int_{\Omega^h} (\boldsymbol{u}^h)^T \cdot \boldsymbol{f}\ \mathrm{d}\Omega \end{aligned} \tag{2.83}$$

which directly results in

$$\left[\boldsymbol{K}^e + \boldsymbol{P}_\gamma^e\right] \boldsymbol{q} = \boldsymbol{r} \tag{2.84}$$

with

$$\boldsymbol{P}_\gamma^e \;=\; \gamma \int_\Omega \left\{ \begin{array}{c} \boldsymbol{b}^e \\ \boldsymbol{g}^e \end{array} \right\} [\boldsymbol{b}^e\ \ \boldsymbol{g}^e]\, \mathrm{d}\Omega \tag{2.85}$$

The parameter γ in the foregoing formulations is problem dependent, since it is part of a penalty term. The effect of γ is studied in Chapters 3, 5, and 7 to come.

2.3.3 Developing and constraining the assumed stress field

The stress field assumed in (2.73) may, without loss of generality, be expressed as

$$\text{symm}\ \boldsymbol{\sigma}^e = \boldsymbol{P}\boldsymbol{\beta} = \text{symm}\ \boldsymbol{\sigma}_c^e + \text{symm}\ \boldsymbol{\sigma}_h^e = [\boldsymbol{I}_c\ \boldsymbol{P}_h] \left\{ \begin{array}{c} \boldsymbol{\beta}_c \\ \boldsymbol{\beta}_h \end{array} \right\} \tag{2.86}$$

where the superscript e is dropped on $\boldsymbol{P}_\alpha$ for reasons of clarity. In (2.86), $\boldsymbol{I}_c$ allows for the accommodation of constant stress states. The higher order stress field is represented by

$$\text{symm}\ \boldsymbol{\sigma}_h^e = \boldsymbol{P}_h\boldsymbol{\beta}_h = \boldsymbol{P}_{h2}\boldsymbol{\beta}_{h2} + \boldsymbol{P}_{h3}\boldsymbol{\beta}_{h3} \tag{2.87}$$

where $\boldsymbol{P}_{h2}\boldsymbol{\beta}_{h2}$ and $\boldsymbol{P}_{h3}\boldsymbol{\beta}_{h3}$ are introduced for reasons of clarity. Therefore,

$$\text{symm}\ \boldsymbol{\sigma}^e = \boldsymbol{P}\boldsymbol{\beta} = \text{symm}\ \boldsymbol{\sigma}_c^e + \text{symm}\ \boldsymbol{\sigma}_h^e = [\boldsymbol{I}_c\ \boldsymbol{P}_{h2}\ \boldsymbol{P}_{h3}] \left\{ \begin{array}{c} \boldsymbol{\beta}_c \\ \boldsymbol{\beta}_{h2} \\ \boldsymbol{\beta}_{h3} \end{array} \right\} \tag{2.88}$$

Furthermore, the classification of Feng *et al.* [8] is now extended, and written as

$$\boldsymbol{I}_c\boldsymbol{\beta}_c \;=\; [\{\sigma_1\}\{\sigma_2\}\{\sigma_3\}] \left\{ \begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \end{array} \right\} \tag{2.89}$$

with $\{\sigma_1\}$ through $\{\sigma_3\}$ presented in Appendix B, and representing the constant stress capability of the formulation. Various possibilities exist for $\boldsymbol{P}_{h2}$ (e.g. see [8]), but the obvious choice is the linear capability, given for instance by

$$\boldsymbol{P}_{h2}\boldsymbol{\beta}_{h2} \;=\; [\{\sigma_5\}\{\sigma_6\}] \left\{ \begin{array}{c} \beta_4 \\ \beta_5 \end{array} \right\} \tag{2.90}$$

with $\{\sigma_5\}$ and $\{\sigma_6\}$ again given in Appendix B. (2.89) combined with (2.90) yields the usual formulation for a 5-parameter stress field, as is also for instance used by Di and Ramm [34], for their 5β elements. The additional terms required for the finite element with drilling degrees of freedom are chosen as

$$\boldsymbol{P}_{h3}\boldsymbol{\beta}_{h3} = [\{\sigma_{13}\}\{\sigma_{12}\}\{\sigma_{23}\}]\left\{\begin{array}{c}\beta_6\\ \beta_7\\ \beta_8\end{array}\right\} \tag{2.91}$$

viz.

$$\boldsymbol{P}_{h3} = \begin{bmatrix} -\xi & 0 & \eta^2 \\ 0 & -\eta & -\xi^2 \\ \eta & \xi & 0 \end{bmatrix} \tag{2.92}$$

This formulation is similar to the unconstrained field used by Sze and Ghali [37]. A different, invariant possibility is

$$\boldsymbol{P}^i_{h3} = [\{\sigma_9\}\{\sigma_8\}\{\sigma_{23}\}] \tag{2.93}$$

When using 9 interpolating stress modes, (i.e. $m = n - r = 12 - 3 = 9$), the stress modes may be selected as

$$\boldsymbol{P}_{h4}\boldsymbol{\beta}_{h4} = [\{\sigma_{13}\}\{\sigma_{12}\}\{\sigma_{17}\}\{\sigma_{15}\}]\left\{\begin{array}{c}\beta_6\\ \beta_7\\ \beta_8\\ \beta_9\end{array}\right\} \tag{2.94}$$

viz.

$$\boldsymbol{P}_{h4} = \begin{bmatrix} -\xi & 0 & \eta^2 & 0 \\ 0 & -\eta & 0 & \xi^2 \\ \eta & \xi & 0 & 0 \end{bmatrix} \tag{2.95}$$

This formulation is similar to the formulation presented by Aminpour [35]. A different possibility is given by

$$\boldsymbol{P}^i_{h4} = [\{\sigma_9\}\{\sigma_8\}\{\sigma_{17}\}\{\sigma_{15}\}] \tag{2.96}$$

(Here, it is chosen to retain $\boldsymbol{P}_{h2}$ unmodified, which is not a requirement.) $\boldsymbol{P}_{h3}$ is then used instead of $\boldsymbol{P}_{h3}$. As stated previously, constraints may be enforced through a suitable transformation matrix $\boldsymbol{A}$, such that symm $\boldsymbol{\sigma}^e = \boldsymbol{A}^e\boldsymbol{P}^e\boldsymbol{\beta}$. Various forms for $\boldsymbol{A}^e$ were presented by Di and Ramm [34], and are applied in Table 2.1 to the 8β and 9β families, while the 5β family is also given for reasons of completeness. In the table, $|\boldsymbol{J}|$ indicates the determinant of the Jacobian $\boldsymbol{J}$, and g the determinant of the metric tensor. The transformation operators $\boldsymbol{T}_0$, $\boldsymbol{T}$ and $\boldsymbol{Q}$ are given in Appendix C.

The following notation is used:

- NC - The stresses are associated with the strain derived from the displacements and are not subjected to any constraint.

- EP - Pian and Sumihara [28] have developed a rational approach for the assumed stress element in which the equilibrium equations in a weak form related to the internal displacement field are used as a constraint condition; it serves as a pre-treatment for the initial assumed stress trial. With this method, an appropriate perturbation of element geometry is often needed to obtain sufficient constraints.

- OC - The higher order stress is selected to be orthogonal to the constant part in a weak sense [42].
- NT - The initial stress is decomposed into a constant and a higher order part, and then the higher order part is defined independently so that the constant part of the initial stress can be preserved. Following this approach the transformation for the higher order part of the initial stress defined in isoparametric space is normalized.
- PH - The physical components of the higher order stress part are first interpolated in isoparametric space and then converted to their contravariant components. Finally, the latter are transformed to the global system using the transformation matrix.

No	Element	Higher order stress
1	5β-NC	$\boldsymbol{P}_h = \boldsymbol{P}_{h2}$
2	5β-EP	$\boldsymbol{P}_h = \boldsymbol{T}_0\boldsymbol{P}_{h2}$
3	5β-OC	$\boldsymbol{P}_h = \frac{1}{\lvert\boldsymbol{J}\rvert}\boldsymbol{T}_0\boldsymbol{P}_{h2}$
4	5β-NT	$\boldsymbol{P}_h = g\boldsymbol{T}\boldsymbol{P}_{h2}$
5	5β-PH	$\boldsymbol{P}_h = \boldsymbol{T}\boldsymbol{Q}\boldsymbol{P}_{h2}$
6	8β(M)-NC and 8β(D)-NC	$\boldsymbol{P}_h = \boldsymbol{P}_{h2} + \boldsymbol{P}_{h3}$
7	8β(M)-EP and 8β(D)-EP	$\boldsymbol{P}_h = \boldsymbol{T}_0\boldsymbol{P}_{h2} + \boldsymbol{T}_0\boldsymbol{P}_{h3}$
8	8β(M)-OC and 8β(D)-OC	$\boldsymbol{P}_h = \frac{1}{\lvert\boldsymbol{J}\rvert}\boldsymbol{T}_0\boldsymbol{P}_{h2} + \frac{1}{\lvert\boldsymbol{J}\rvert}\boldsymbol{T}_0\boldsymbol{P}_{h3}$
9	8β(M)-NT and 8β(D)-NT	$\boldsymbol{P}_h = g\boldsymbol{T}\boldsymbol{P}_{h2} + g\boldsymbol{T}\boldsymbol{P}_{h3}$
10	8β(M)-PH and 8β(D)-PH	$\boldsymbol{P}_h = \boldsymbol{T}\boldsymbol{Q}\boldsymbol{P}_{h2} + \boldsymbol{T}\boldsymbol{Q}\boldsymbol{P}_{h3}$
11	9β(M)-NC and 9β(D)-NC	$\boldsymbol{P}_h = \boldsymbol{P}_{h2} + \boldsymbol{P}_{h4}$
12	9β(M)-EP and 9β(D)-EP	$\boldsymbol{P}_h = \boldsymbol{T}_0\boldsymbol{P}_{h2} + \boldsymbol{T}_0\boldsymbol{P}_{h4}$
13	9β(M)-OC and 9β(D)-OC	$\boldsymbol{P}_h = \frac{1}{\lvert\boldsymbol{J}\rvert}\boldsymbol{T}_0\boldsymbol{P}_{h2} + \frac{1}{\lvert\boldsymbol{J}\rvert}\boldsymbol{T}_0\boldsymbol{P}_{h4}$
14	9β(M)-NT and 9β(D)-NT	$\boldsymbol{P}_h = g\boldsymbol{T}\boldsymbol{P}_{h2} + g\boldsymbol{T}\boldsymbol{P}_{h4}$
15	9β(M)-PH and 9β(D)-PH	$\boldsymbol{P}_h = \boldsymbol{T}\boldsymbol{Q}\boldsymbol{P}_{h2} + \boldsymbol{T}\boldsymbol{Q}\boldsymbol{P}_{h4}$

Table 2.1: Unified formulation for the 5β, 8β and 9β families

2.4 Membrane locking correction

Flat shell elements assembled from membrane elements with in plane drilling degrees of freedom suffer undesirable membrane-bending interactions associated with the drilling degrees of freedom [6].

Mechanistically, the locking phenomena may be described as follows [6]: Flat quadrilateral shell elements approximate curved shell geometries with the possibilities of kinks between adjacent elements. In these situations the continuity of the three rotation parameters for the shell result in a situation where non-zero drilling degrees of freedom in one element leads to non-zero bending degrees of freedom in the adjacent element (and 'vice-versa'). Accordingly, the elements will exhibit a membrane-bending locking performance, unless the drilling degree of freedom part of the membrane strains may assume a zero value over the element.

For the assumed displacement field of the 8β(M), 8β(D), 9β(M) and 9β(D) elements (see (2.69)) zero strains are not possible for non-zero rotations [6]. An exception is the special case of identical rotations at opposite nodes. One such case is for example, reflected in:

$$l_{ij}\cos\alpha_{ij} NS^e_{l,x_2} = l_{ik}\cos\alpha_{ik} NS^e_{m,x_2} \tag{2.97}$$

Taylor [6] presented a correction which alleviates the membrane-bending locking. The correction, which is based on a three field formulation (displacement, strain and stress), is repeated here, albeit with a slightly different notation.

Using matrix notation, $\text{symm}\boldsymbol{\nabla}\boldsymbol{u}^e$ for the 8β(M), 8β(D), 9β(M) and 9β(D) elements is given by

$$\text{symm}\boldsymbol{\nabla}\boldsymbol{u}^e = \boldsymbol{B}^e_i\boldsymbol{u}_i + \boldsymbol{G}^e_{\psi i}\boldsymbol{\psi}_i \quad i = 1,2,3,4 \tag{2.98}$$

where $\boldsymbol{u}_i$ and $\boldsymbol{\psi}_i$ are nodal values of displacement and rotation respectively and summation is implied.

In the following, the 8β(M), 8β(D), 9β(M) and 9β(D) elements with the interpolation given in (2.98) are now denoted 8β(M)$*$, 8β(D)$*$, 9β(M)$*$ and 9β(D)$*$. Here, the asterisk ($*$) indicates that the membrane locking correction, (which is described in the following), is not performed. For the 8β(M), 8β(D), 9β(M) and 9β(D) elements, the modified strain relationship proposed by Taylor [6] is used. This relationship is given by

$$\text{symm}\boldsymbol{\nabla}\boldsymbol{u}^e = \boldsymbol{B}^e_i\boldsymbol{u}_i + \boldsymbol{G}^e_{\psi i}\boldsymbol{\psi}_i + \text{symm}\boldsymbol{\nabla}\boldsymbol{u}^e_0 \tag{2.99}$$

This modified strain relation is required to satisfy a requirement that the drilling parameter part can be inextensible. Accordingly, it is desired that

$$\boldsymbol{G}^e_{\psi i}\boldsymbol{\psi}_i + \text{symm}\boldsymbol{\nabla}\boldsymbol{u}^e_0 = 0 \tag{2.100}$$

for rotational fields which are inextensible. Unless the drilling degrees of freedom are eliminated completely it is only possible to satisfy (2.100) in a weak sense. A suitable weak form may be constructed by augmenting the usual potential energy of each element for a shell by the term

$$\int_{\Omega^e} \bar{\boldsymbol{\sigma}}^T\left(\boldsymbol{G}^e_{\psi i}\boldsymbol{\psi}_i + \text{symm}\boldsymbol{\nabla}\boldsymbol{u}^e_0\right)\,\mathrm{d}\Omega^e = 0 \tag{2.101}$$

where Ω^e is the surface region of the shell. Both $\bar{\boldsymbol{\sigma}}^T$ and $\text{symm}\boldsymbol{\nabla}\boldsymbol{u}^e_0$ are assumed constant over each element. Performing the variation with respect to $\bar{\boldsymbol{\sigma}}^T$ leads to

$$\text{symm}\boldsymbol{\nabla}\boldsymbol{u}^e_0 = -\frac{1}{\Omega^e}\int_{\Omega^e}\boldsymbol{G}^e_{\psi i}\boldsymbol{\psi}_i\,\mathrm{d}\Omega^e \tag{2.102}$$

and, therefore, the modified strain relationship

$$\mathrm{symm}\boldsymbol{\nabla}\boldsymbol{u}^e = \boldsymbol{B}_i^e\boldsymbol{u}_i + \left(\boldsymbol{G}_{\psi i}^e - \frac{1}{\Omega}\int_\Omega \boldsymbol{G}_{\psi i}^e \,\mathrm{d}\Omega\right)\boldsymbol{\psi}_i \tag{2.103}$$

which is the final result presented by Taylor [6].