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Advanced Low Order Orthotropic Finite Element Formulations

by

Susanna Elizabeth Geyer

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Supervisor:
Prof. Albert A. Groenwold

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Abstract

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Author: Susanna Elizabeth Geyer

Degree: M.Eng (Mechanical)

Department: Mechanical Engineering

Supervisor: Prof. Albert A. Groenwold

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In this study advanced low order finite elements for the linear analysis and ultimately, the global optimization of orthotropic shells structures, are presented. Low order quadrilaterals are attractive in optimization, since they result in low connectively of the structural stiffness matrix, and hence, reduced computational effort. However, standard 4-node quadrilaterals are notorious for their low accuracy.

Both drilling degrees of freedom and assumed stress interpolations have the potential to improve the modeling capabilities of low order quadrilateral finite elements. Therefore, it seems desirable to formulate low order elements with both an assumed stress interpolation field and drilling degrees of freedom, on condition that the elements are rank sufficient and invariant.

Firstly, a variational basis for the formulation of two families of assumed stress membrane finite elements with drilling degrees of freedom, is presented. This formulation depends on the formulation of Hughes and Brezzi, and is derived using the unified formulation presented by Di and Ramm. The recent stress mode classification method presented by Feng *et al.* is used to derive the stress interpolation matrices. The families, denoted $8\beta(\mathrm{M})$ and $8\beta(\mathrm{D})$, are rank sufficient, invariant, and free of locking. The membrane locking correction suggested by Taylor ensures that the consistent nodal loads of both families are identical to those of a quadrilateral 4-node membrane finite element with two translational degrees of freedom per node.

Secondly, the rectangular assumed strain plate element presented by Bathe and Dvorkin is combined with the above mentioned membrane families to form flat shell finite elements. The strain-displacement measures of these elements are modified on the element level to incorporate the effect of element warp.

Thirdly, the constitutive relationship of the flat shell elements is extended to include symmetric orthotropy. In opposition to the general trend to employ quadratic or even cubic elements for orthotropic analyses, it is shown that the simpler 4-node assumed stress families with drilling degrees of freedom presented herein are highly accurate and effective.

Finally, the influence of the stability parameter γ, the integration scheme order and the effect of the membrane locking correction are evaluated. The numerical value of the parameter γ is shown to be irrelevant in the patch test. The effect of the previously proposed membrane-bending locking correction when included in in-plane analysis is demonstrated.

The elements have been incorporated in the EDSAP/CALSAP finite element infrastructure.

Opsomming

Titel: Gevorderde Lae-Orde Ortotropiese Eindige Element Formulerings

Outeur: Susanna Elizabeth Geyer

Graad: M.Ing (Meganies)

Departement: Meganiese Ingenieurswese

Sudieleier: Prof. Albert A. Groenwold

Sleutelwoorde: Boor-vryheidsgraad, Aangenome-spanning, Membraan, Plat dop, Eindige-element, Ortotropie

In hierdie studie word gevorderde lae-orde eindige-elemente vir die lineêre analise van ortotropiese dop-strukture ontwikkel. Die uiteindelike doel van hierdie elemente is die globale optimering van ortotropiese dop-strukture. Lae-orde vierhoekige elemente is aantreklik in optimering, omdat dit lei tot lae koppeling in die styfheidsmatriks. Dit lei weer tot verminderde berekeningstyd. Vier-node vierhoekige elemente is egter berug vanweë hulle lae akkuraatheid.

Beide boor-vryheidsgrade ('drilling degrees of freedom') en aangenome-spanningsinterpolasies ('assumed stress interpolations') het die potensiaal om die modelleringseienskappe van lae-orde vierhoekige elemente te verbeter. Daarom is dit wenslik om lae-orde elemente te formuleer met beide 'n aangenome-spanningsinterpolasieveld en boor-vryheidsgrade, op voorwaarde dat die elemente se rang voldoende is en dat die elemente invariant is.

Eerstens word 'n variasionele basis vir die formulering van twee aangenome-spanning membraan eindige-element families met boor-vryheidsgrade aangebied. Dit is gebaseer op die formulasie van Hughes en Brezzi en is afgelei deur gebruik te maak van die genormaliseerde formulasie van Di en Ramm. Die spanningsmode klassifikasie van Feng *et al.* is gebruik vir die afleiding van die spanningsinterpolasie-matrikse. Die families, genoem $8\beta(\mathrm{M})$ en $8\beta(\mathrm{D})$, se rang is voldoende en is invariant. Hierdie families toon ook geen sluitingsgedrag nie. Die membraan-sluitingskorreksie wat voorgestel is deur Taylor verseker dat die nodale kragte in dié families ooreenstem met 'n vier-node vierhoekige membraan eindige-element wat twee verplasings-vryheidsgrade per node besit.

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Tweedens word die reghoekige aangenome-vervorming plaat element van Bathe en Dvorkin gekombineer met bogenoemde membraan element families om plat dop eindige-elemente te vorm. Die vervorming-verplasing verwantskap van hierdie elemente word op die element vlak gemanipuleer om die effek van element uit-vlak distorsie te akkomodeer.

Derdens word die materiaalverwantskap van die plat dop elemente uitgebrei om simmetriese ortotropie in te sluit. In teenstelling met die algemene gebruik om kwadratiese of kubiese elemente vir ortotropiese analises te gebruik, word die eenvoudiger vier-node aangenome-spanning families met boor-vryheidsgrade hier voorgestel. Hierdie elemente lewer baie goeie resultate en is effektief.

Laastens word die invloed van die stabiliteitsparameter γ, die integrasieskema-orde en die effek van die membraan-sluitingskorreksie geëvalueer. Daar word getoon dat die numeriese waarde van die parameter γ irrelevant is in die laptoets ('patch test'). Die effek van die voorheen voorgestelde membraan-sluitingskorreksie wanneer dit ingesluit word in in-vlak analises word gedemonstreer.

Die elemente is geakkomodeer in die EDSAP/CALSAP eindige-element infrastruktuur.

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