Bimatrix variate beta type IV distribution:

relation to Wilks's statistic and bimatrix variate Kummer-beta type IV distribution

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Abstract-In this paper the bimatrix variate beta type IV distribution is derived from independent Wishart distributed matrix variables. We explore specific properties of this distribution which is then used to derive the exact expressions of the densities of the product and ratio of two dependent Wilks's statistics and to define the bimatrix Kummer-beta type IV distribution.

Keywords and phrases-bimatrix variate beta type IV distribution; bimatrix variate Kummer-beta type IV distribution; hypergeometric function of matrix argument; invariant polynomials; moment generating function; Meijer's G-function; Wilks's statistic; Wishart distribution.

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1 Introduction

Consider three independent Wishart matrix variables $\boldsymbol{S}_i \sim W_p(n_i, \boldsymbol{\Sigma})$, $i = 1, 2$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$ and define the transformation

$$\boldsymbol{X}_i = \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}} \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}, \tag{1}$$

where $\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{B}^{\frac{1}{2}} = \boldsymbol{B}$ and $\boldsymbol{0} < \boldsymbol{X}_i < \boldsymbol{I}_p$, $i = 1, 2$. In this paper we derive the distribution of $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ and refer to it as the *bimatrix variate beta type IV* distribution. Specific properties are explored and then used as the basis to derive two further important results.

Firstly, it follows from (1) that $\Lambda_1 = \left|\dfrac{\boldsymbol{S}_1}{\boldsymbol{S}_1 + \boldsymbol{B}}\right| = |\boldsymbol{X}_1|$ and $\Lambda_2 = \left|\dfrac{\boldsymbol{S}_2}{\boldsymbol{S}_2 + \boldsymbol{B}}\right| = |\boldsymbol{X}_2|$, thus $Z_1 = |\boldsymbol{X}_1\boldsymbol{X}_2| = \Lambda_1\Lambda_2$ and $Z_2 = \left|\dfrac{\boldsymbol{X}_1}{\boldsymbol{X}_2}\right| = \dfrac{\Lambda_1}{\Lambda_2}$ are the product and ratio of two *dependent* Wilks's statistics. We derive $E\left(|\boldsymbol{X}_1|^{h_1}|\boldsymbol{X}_2|^{h_2}\right)$ for the bimatrix variate beta type IV distribution and use it to obtain exact expressions for the densities of Z_1 and Z_2 in terms of Meijer's G-function. The Wilks's statistic is widely used for various statistical tests in multivariate analysis. Exact expressions have been derived for the distribution of the Wilks's statistic (Mathai and Rathie, 1969) and also for the product and ratio of two *independent* Wilks's statistics (Kshirsagar, 1972; Pham-Gia, 2008).

Secondly, the moment generating function of the bimatrix variate beta type IV distribution is derived and then applied to define the *bimatrix variate Kummer-beta type IV* distribution. This is an extension of the *bimatrix variate beta type IV* distribution. In the bivariate case it is an extension of the distribution proposed independently by Libby and Novick (1982), Jones (2001) and Olkin and Liu (2003). Gupta et al. (2001) defined the matrix variate Kummer-Dirichlet type I and type II distributions which reduce

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under certain conditions to the matrix variate Kummer-beta distribution (Nagar and Gupta, 2002) and the matrix variate Kummer-gamma distribution (Nagar and Cardeño, 2001).

The distribution of $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ is derived in Section 2 by using the transformation in (1). In Section 3 the product moment of this distribution is applied to obtain the exact expressions for the densities of $Z_1 = |\boldsymbol{X}_1\boldsymbol{X}_2|$ and $Z_2 = \left|\frac{\boldsymbol{X}_1}{\boldsymbol{X}_2}\right|$. The moment generating function of the bimatrix beta type IV distribution is derived in Section 4. This result is used to define the *bimatrix variate Kummer-beta type IV* distribution. In Section 5 the form of the densities of Z_1 and Z_2 is illustrated, as well as the effect of the parameters on the conditional moments for the bivariate beta type IV distribution. The form of the density of the *bivariate Kummer-beta type IV* distribution also receives attention.

2 Bimatrix variate beta type IV distribution

In this section the bimatrix variate beta type IV distribution is derived from a transformation of independent Wishart distributed random matrices.

Theorem 1

Let $\boldsymbol{S}_1 \sim W_p(n_1, \boldsymbol{\Sigma})$, $\boldsymbol{S}_2 \sim W_p(n_2, \boldsymbol{\Sigma})$ and $\boldsymbol{B} \sim W_p(m, \boldsymbol{\Sigma})$ be independently distributed. Define

$$\boldsymbol{X}_i = \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}} \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}, \; i = 1, 2, \tag{2}$$

where $\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{B}^{\frac{1}{2}} = \boldsymbol{B}$ and $\boldsymbol{0} < \boldsymbol{X}_i < \boldsymbol{I}_p$, $i = 1, 2$.

Furthermore, $n_i > (p-1)$, $i = 1, 2$ *and* $m > (p-1)$.

The joint density of $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ *is*

$$\left\{\beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2}\right)\right\}^{-1} \left\{\prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)}\right\}$$
$$\cdot |\boldsymbol{I}_p - \boldsymbol{X}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)} |\boldsymbol{I}_p - \boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)} \tag{3}$$

where $\beta_p\left(\frac{n_1}{2}, \frac{n_2}{2}; \frac{m}{2}\right) = \dfrac{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}\right)}$, *and* $\Gamma_p(\cdot)$ *is the multivariate gamma function.*

The density in (3) *is that of the bimatrix variate beta type IV distribution and is denoted as* $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}\left(\frac{n_1}{2}, \frac{n_2}{2}; \frac{m}{2}\right)$.

Proof:

The joint density of $(\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{B})$ is given by

$$K\left\{\prod_{i=1}^{2}\left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}_i\right)|\boldsymbol{S}_i|^{\frac{1}{2}(n_i-p-1)}\right]\right\}\left[\operatorname{etr}\left(-\tfrac{1}{2}\boldsymbol{\Sigma}^{-1}\boldsymbol{B}\right)|\boldsymbol{B}|^{\frac{1}{2}(m-p-1)}\right] \tag{4}$$

where $K^{-1} = 2^{\frac{1}{2}(n_1+n_2+m)p}\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)|\boldsymbol{\Sigma}|^{\frac{1}{2}(n_1+n_2+m)}$.

Making the transformations

$$\boldsymbol{X}_i = \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}} \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}} \left(\boldsymbol{I}_p + \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}\right)^{-\frac{1}{2}}, \; i = 1, 2,$$

and

$$\boldsymbol{Z}_i = \boldsymbol{B}^{-\frac{1}{2}}\boldsymbol{S}_i\boldsymbol{B}^{-\frac{1}{2}}, \; i = 1, 2,$$

gives $\boldsymbol{X}_i = (\boldsymbol{I}_p + \boldsymbol{Z}_i)^{-1} \boldsymbol{Z}_i$ since $\boldsymbol{Z}_i$ commutes with any rational function of $\boldsymbol{Z}_i$. The Jacobian of the transformation (see Gupta and Nagar, 2000) is

$$\begin{aligned} J(\boldsymbol{S}_1, \boldsymbol{S}_2 \to \boldsymbol{X}_1, \boldsymbol{X}_2) &= \left\{ \prod_{i=1}^{2} J(\boldsymbol{S}_i \to \boldsymbol{Z}_i) J(\boldsymbol{Z}_i \to \boldsymbol{X}_i) \right\} \\ &= |\boldsymbol{B}|^{(p+1)} \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-(p+1)}. \end{aligned}$$

Substituting this in (4) gives the joint density of $(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{B})$ as

$$\begin{aligned} f(\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{B}) = \; & K \left\{ \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \right\} \left\{ \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \right\} \\ & \cdot |\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m) - \frac{1}{2}(p+1)} \operatorname{etr}\left\{ -\tfrac{1}{2} \boldsymbol{B}^{\frac{1}{2}} \boldsymbol{\Sigma}^{-1} \boldsymbol{B}^{\frac{1}{2}} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right] \right\}. \end{aligned} \tag{5}$$

Making the transformation $\boldsymbol{X}_i \to \boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'$, $i = 1, 2$ and $\boldsymbol{B} \to \boldsymbol{H}\boldsymbol{B}\boldsymbol{H}'$ where $\boldsymbol{H}$ $(p \times p)$ is orthogonal (Díaz-García and Gutiérrez-Jáimez, 2006) and substituting in (5) gives

$$\begin{aligned} & f(\boldsymbol{H}\boldsymbol{X}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{X}_2\boldsymbol{H}', \boldsymbol{H}\boldsymbol{B}\boldsymbol{H}') \\ &= K \left\{ \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \right\} \left\{ \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \right\} \\ & \quad \cdot |\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m) - \frac{1}{2}(p+1)} \operatorname{etr}\left\{ -\tfrac{1}{2} \left(\boldsymbol{H}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{H}' \right) \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{H}\boldsymbol{B}^{\frac{1}{2}}\boldsymbol{H}' \right) \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{H}\boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \boldsymbol{H}' \right] \right\} \end{aligned}$$

We consider the symmetrized density function of $(\boldsymbol{X}_1, \boldsymbol{X}_2)$ defined by Greenacre (1973), that is $f_S(\boldsymbol{X}_1, \boldsymbol{X}_2) = \int_{\boldsymbol{B}>\boldsymbol{0}} \int_{O(p)} f(\boldsymbol{H}\boldsymbol{X}_1\boldsymbol{H}', \boldsymbol{H}\boldsymbol{X}_2\boldsymbol{H}', \boldsymbol{H}\boldsymbol{B}\boldsymbol{H}') \mathrm{d}\boldsymbol{H} \mathrm{d}\boldsymbol{B}$ where $\boldsymbol{H}$ $(p \times p)$ is orthogonal and $\mathrm{d}\boldsymbol{H}$ is the normalized invariant measure on $O(p)$. Note that $\mathrm{d}\boldsymbol{B} = \mathrm{d}\boldsymbol{H}\boldsymbol{B}\boldsymbol{H}'$ (Díaz-García and Gutiérrez-Jáimez, 2006).

Then

$$\begin{aligned} & f_S(\boldsymbol{X}_1, \boldsymbol{X}_2) \\ &= K \left\{ \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \right\} \left\{ \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \right\} \\ & \quad \cdot \int_{\boldsymbol{B}>\boldsymbol{0}} |\boldsymbol{B}|^{\frac{1}{2}(n_1+n_2+m) - \frac{1}{2}(p+1)} \int_{O(p)} \operatorname{etr}\left\{ -\tfrac{1}{2} \boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\boldsymbol{B}^{\frac{1}{2}} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right] \boldsymbol{B}^{\frac{1}{2}} \right\} \mathrm{d}\boldsymbol{H} \mathrm{d}\boldsymbol{B}. \end{aligned} \tag{6}$$

It follows from Ehlers, Bekker and Roux (2009, Lemma 5, page 114) that

$$\begin{aligned} & \int_{O(p)} \operatorname{etr}\left\{ -\tfrac{1}{2} \boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H}\boldsymbol{B}^{\frac{1}{2}} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right] \boldsymbol{B}^{\frac{1}{2}} \right\} \mathrm{d}\boldsymbol{H} \\ &= \int_{O(p)} \operatorname{etr}\left\{ -\tfrac{1}{2} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right]^{\frac{1}{2}} \boldsymbol{H}'\boldsymbol{\Sigma}^{-1}\boldsymbol{H} \left[\boldsymbol{I}_p + \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \right]^{\frac{1}{2}} \boldsymbol{B} \right\} \mathrm{d}\boldsymbol{H}. \end{aligned}$$

Form the above, changing the order of integration in (6) and integrating with respect to $\boldsymbol{B}$ as well as using Gupta and Nagar (2000, Equation 1.4.6, page 19) gives

$$\begin{aligned} f_S(\boldsymbol{X}_1, \boldsymbol{X}_2) = \; & \left\{ \beta_p\left(\tfrac{n_1}{2}, \tfrac{n_2}{2}; \tfrac{m}{2} \right) \right\}^{-1} \left\{ \prod_{i=1}^{2} |\boldsymbol{X}_i|^{\frac{1}{2}n_i - \frac{1}{2}(p+1)} \right\} \left\{ \prod_{i=1}^{2} |\boldsymbol{I}_p - \boldsymbol{X}_i|^{-\frac{1}{2}n_i - \frac{1}{2}(p+1)} \right\} \\ & \cdot \int_{O(p)} \left| \boldsymbol{I}_p + \boldsymbol{H} \sum_{i=1}^{2} \boldsymbol{X}_i (\boldsymbol{I}_p - \boldsymbol{X}_i)^{-1} \boldsymbol{H}' \right|^{-\frac{1}{2}(n_1+n_2+m)} \mathrm{d}\boldsymbol{H}. \end{aligned} \tag{7}$$

Since $f_S(\boldsymbol{X}_1,\boldsymbol{X}_2)=\int_{O(p)} f(\boldsymbol{H}\boldsymbol{X}_1\boldsymbol{H}',\boldsymbol{H}\boldsymbol{X}_2\boldsymbol{H}')\mathrm{d}\boldsymbol{H}$, it follows from (7) that

$$
\begin{aligned}
& f(\boldsymbol{H}\boldsymbol{X}_1\boldsymbol{H}',\boldsymbol{H}\boldsymbol{X}_2\boldsymbol{H}') \\
&= \left\{\beta_p\left(\tfrac{n_1}{2},\tfrac{n_2}{2};\tfrac{m}{2}\right)\right\}^{-1}\left\{\prod_{i=1}^{2}|\boldsymbol{X}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\right\}\left\{\prod_{i=1}^{2}|\boldsymbol{I}_p-\boldsymbol{X}_i|^{-\frac{1}{2}n_i-\frac{1}{2}(p+1)}\right\} \\
&\quad\cdot\left|\boldsymbol{I}_p+\boldsymbol{H}\sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p-\boldsymbol{X}_i)^{-1}\boldsymbol{H}\right|^{-\frac{1}{2}(n_1+n_2+m)}.
\end{aligned}
$$

Making the transformation $\boldsymbol{H}\boldsymbol{X}_i\boldsymbol{H}'\rightarrow\boldsymbol{X}_i,\ i=1,2$ and rewriting $\left|\boldsymbol{I}_p+\sum_{i=1}^{2}\boldsymbol{X}_i(\boldsymbol{I}_p-\boldsymbol{X}_i)^{-1}\right|$ as

$$
\begin{aligned}
&\left|\boldsymbol{I}_p+\boldsymbol{X}_2(\boldsymbol{I}_p-\boldsymbol{X}_2)^{-1}-\boldsymbol{X}_2(\boldsymbol{I}_p-\boldsymbol{X}_2)^{-1}\boldsymbol{X}_1\right||\boldsymbol{I}_p-\boldsymbol{X}_1|^{-1} \\
&=\left|\boldsymbol{I}_p-\boldsymbol{X}_2(\boldsymbol{I}_p-\boldsymbol{X}_2)^{-1}\boldsymbol{X}_1+\boldsymbol{X}_2(\boldsymbol{I}_p-\boldsymbol{X}_2)^{-1}\boldsymbol{X}_1\boldsymbol{X}_2\right||\boldsymbol{I}_p-\boldsymbol{X}_1|^{-1}|\boldsymbol{I}_p-\boldsymbol{X}_2|^{-1} \\
&=|\boldsymbol{I}_p-\boldsymbol{X}_1\boldsymbol{X}_2||\boldsymbol{I}_p-\boldsymbol{X}_1|^{-1}|\boldsymbol{I}_p-\boldsymbol{X}_2|^{-1}
\end{aligned}
$$

gives the result. ■

Remarks

1. If $(\boldsymbol{X}_1,\boldsymbol{X}_2)\sim BB_p^{IV}\left(\frac{n_1}{2},\frac{n_2}{2};\frac{m}{2}\right)$ with density function given in (3), then
   (a) the marginal density functions are matrix variate beta type I, $\boldsymbol{X}_i\sim B_p^I\left(\frac{n_i}{2};\frac{m}{2}\right),\ i=1,2$;
   (b) the conditional density of $\boldsymbol{X}_2|\boldsymbol{X}_1$ is given by

$$
\left\{\beta_p\left(\tfrac{n_2}{2};\tfrac{n_1+m}{2}\right)\right\}^{-1}|\boldsymbol{I}_p-\boldsymbol{X}_1|^{\frac{1}{2}n_2}|\boldsymbol{X}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)},\quad \boldsymbol{0}<\boldsymbol{X}_2<\boldsymbol{I}_p. \tag{8}
$$

2. The matrix variate Dirichlet type IV, denoted as $(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r)\sim D_p^{IV}\left(\frac{n_1}{2},\ldots,\frac{n_r}{2};\frac{m}{2}\right)$, results by extending (2) to r independent Wishart matrix variables, $\boldsymbol{S}_i\sim W_p(n_i,\boldsymbol{\Sigma})$, $i=1,\ldots,r$, all independent of $\boldsymbol{B}\sim W_p(m,\boldsymbol{\Sigma})$. The joint density of $(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r)$ is given by

$$
\begin{aligned}
& f(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r) \\
&= \left\{\beta_p\left(\tfrac{n_1}{2},\ldots,\tfrac{n_r}{2};\tfrac{m}{2}\right)\right\}^{-1}\left\{\prod_{i=1}^{r}|\boldsymbol{X}_i|^{\frac{1}{2}n_i-\frac{1}{2}(p+1)}\right\} \\
&\quad\cdot\left\{\prod_{i=1}^{r}|\boldsymbol{I}_p-\boldsymbol{X}_i|^{-\frac{1}{2}n_i-\frac{1}{2}(p+1)}\right\}\left|\boldsymbol{I}_p+\sum_{i=1}^{r}\boldsymbol{X}_i(\boldsymbol{I}_p-\boldsymbol{X}_i)^{-1}\right|^{-\frac{1}{2}(n_1+\cdots+n_r+m)}
\end{aligned}
$$

   where $\boldsymbol{0}<\boldsymbol{X}_i<\boldsymbol{I}_p,\ i=1,\ldots,r.$

3 Distribution of product and ratio of dependent Wilks's statistics

In this section we derive the exact expressions for the density functions of $Z_1=|\boldsymbol{X}_1\boldsymbol{X}_2|=\Lambda_1\Lambda_2$ and $Z_2=\left|\frac{\boldsymbol{X}_1}{\boldsymbol{X}_2}\right|=\frac{\Lambda_1}{\Lambda_2}$, the product and ratio of two dependent Wilks's statistics, in terms of Meijer's G-function. The proofs are based on the Mellin transforms of the density functions of Z_1 and Z_2, and their inverse Mellin transforms.

Lemma 1

If $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}\left(\frac{n_1}{2}, \frac{n_2}{2}; \frac{m}{2}\right)$ *with density function given in* (3), *then*

$$E\left[|\boldsymbol{X}_1|^{h_1}|\boldsymbol{X}_2|^{h_2}\right] = \frac{\beta_p\left(\frac{n_1}{2}+h_1; \frac{n_2+m}{2}\right)\beta_p\left(\frac{n_2}{2}+h_2; \frac{n_1+m}{2}\right)}{\beta_p\left(\frac{n_1}{2}, \frac{n_2}{2}; \frac{m}{2}\right)} \cdot {}_3F_2\left(\frac{n_1}{2}+h_1, \frac{n_2}{2}+h_2, \frac{n_1+n_2+m}{2}; \frac{n_1+n_2+m}{2}+h_1, \frac{n_1+n_2+m}{2}+h_2; \boldsymbol{I}_p\right) \quad (9)$$

where ${}_3F_2(\cdot)$ *is a hypergeometric function of matrix argument, (see Muirhead, 1982, Definition 7.3.1, page 258).*

Proof:

Using Gupta and Nagar (2000, Equations 1.6.6 and 1.6.8, page 36) the result in (9) follows. ■

Theorem 2

Let $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}\left(\frac{n_1}{2}, \frac{n_2}{2}; \frac{m}{2}\right)$ *and* $Z_1 = |\boldsymbol{X}_1\boldsymbol{X}_2| = \Lambda_1\Lambda_2$.
The density function of Z_1*, the product of two dependent Wilks statistics, is given by*

$$\frac{\Gamma_p\left(\frac{n_1+m}{2}\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\boldsymbol{I}_p)}{k!}\Gamma_p\left(\frac{n_1+n_2+m}{2}, \kappa\right) G_{2p,2p}^{2p,0}\left(z_1\Big|_{b_1,\ldots,b_{2p}}^{a_1,\ldots,a_{2p}}\right), \quad 0 < z_1 < 1 \quad (10)$$

where $a_j = \begin{cases} \frac{n_1+n_2+m}{2} - 1 + k_{(j+1)/2} - \frac{1}{4}(j-1) & \text{for } j = 1,3,5,\ldots,2p-1 \\ \frac{n_1+n_2+m}{2} - 1 + k_{j/2} - \frac{1}{4}(j-2) & \text{for } j = 2,4,6,\ldots,2p, \end{cases}$

and $b_j = \begin{cases} \frac{n_1}{2} - 1 + k_{(j+1)/2} - \frac{1}{4}(j-1) & \text{for } j = 1,3,5,\ldots,2p-1 \\ \frac{n_2}{2} - 1 + k_{j/2} - \frac{1}{4}(j-2) & \text{for } j = 2,4,6,\ldots,2p. \end{cases}$

Furthermore $C_\kappa(\boldsymbol{I}_p)$ *is the zonal polynomial of* $\boldsymbol{I}_p$ *corresponding to* κ *and*
$\Gamma_p(a,\kappa) = \pi^{\frac{1}{4}p(1-p)}\prod_{j=1}^{p}\Gamma\left[a + k_j - \frac{1}{2}(j-1)\right]$ *(see Gupta and Nagar, 2000).*

Proof:

It follows from (9) that

$$E\left(Z_1^{h-1}\right) = \frac{\Gamma_p\left(\frac{n_1+m}{2}\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\boldsymbol{I}_p)}{k!}\Gamma_p\left(\frac{n_1+n_2+m}{2}, \kappa\right)\frac{\prod_{j=1}^{2p}\Gamma(b_j+h)}{\prod_{j=1}^{2p}\Gamma(a_j+h)},$$

where a_j and b_j are as defined above.

Using Meijer's G-function (Mathai, 1993, Definition 2.1, page 60) and the inverse Mellin transform (Mathai, 1993, Definition 1.8, page 23), the desired result (10) follows. ■

Theorem 3

Let $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_p^{IV}\left(\frac{n_1}{2}, \frac{n_2}{2}; \frac{m}{2}\right)$ *and* $Z_2 = \left|\frac{\boldsymbol{X}_1}{\boldsymbol{X}_2}\right| = \frac{\Lambda_1}{\Lambda_2}$.
The density function of Z_2*, the ratio of two dependent Wilks's statistics, is given by*

$$\frac{\Gamma_p\left(\frac{n_1+m}{2}\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\boldsymbol{I}_p)}{k!}\Gamma_p\left(\frac{n_1+n_2+m}{2}, \kappa\right) G_{2p,2p}^{p,p}\left(z_2\Big|_{b_1,\ldots,b_{2p}}^{a_1,\ldots,a_{2p}}\right) \quad \text{for } z_2 > 0 \quad (11)$$

where $a_j = \begin{cases} -\frac{n_2}{2} - k_j + \frac{1}{2}(j-1) & \text{for } j = 1,2,\ldots,p \\ \frac{n_1+n_2+m}{2} - 1 + k_{j-p} - \frac{1}{2}(j-p-1) & \text{for } j = p+1, p+2,\ldots,2p, \end{cases}$

and $b_j = \begin{cases} \frac{n_1}{2} - 1 + k_j - \frac{1}{2}(j-1) & \text{for } j = 1,2,\ldots,p \\ -\frac{n_1+n_2+m}{2} - k_{j-p} + \frac{1}{2}(j-p-1) & \text{for } j = p+1, p+2,\ldots,2p. \end{cases}$

Furthermore $C_\kappa(\boldsymbol{I}_p)$ *is the zonal polynomial of* $\boldsymbol{I}_p$ *corresponding to* κ *and* $\Gamma_p(a,\kappa)=\pi^{\frac{1}{4}p(1-p)}\prod_{j=1}^{p}\Gamma\left[a+k_j-\frac{1}{2}(j-1)\right]$ *(see Gupta and Nagar, 2000).*

Proof:
It follows from (9) that

$$E\left(Z_2^{h-1}\right)=\frac{\Gamma_p\left(\frac{n_1+m}{2}\right)\Gamma_p\left(\frac{n_2+m}{2}\right)}{\Gamma_p\left(\frac{n_1}{2}\right)\Gamma_p\left(\frac{n_2}{2}\right)\Gamma_p\left(\frac{m}{2}\right)}\cdot\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\boldsymbol{I}_p)}{k!}\frac{\Gamma_p\left(\frac{n_1+n_2+m}{2},\kappa\right)\Gamma_p\left(\frac{n_1}{2}+h-1,\kappa\right)\Gamma_p\left(\frac{n_2}{2}-h+1,\kappa\right)}{\Gamma_p\left(\frac{n_1+n_2+m}{2}+h-1,\kappa\right)\Gamma_p\left(\frac{n_1+n_2+m}{2}-h+1,\kappa\right)}.$$

where a_j and b_j are defined above.

Using Meijer's G-function, the inverse Mellin transform and Mathai (1993, Equation 2.2.4, page 72), the required result (11) follows. ■

Remark
In the case when $p=1$, the results in (10) and (11) simplifies to the densities of the product and ratio of correlated beta type I random variables obtained by Nagar et al. (2009).

4 Bimatrix variate Kummer-beta type IV distribution

In this section we obtain the moment generating function of the bimatrix variate beta type IV distribution and define the bimatrix variate Kummer-beta type IV distribution.

Lemma 2
Suppose that $\boldsymbol{X}=[\boldsymbol{X}_1:\boldsymbol{X}_2]$ *and* $\boldsymbol{T}=[\boldsymbol{T}_1:\boldsymbol{T}_2]$. *Then the moment generating function of the bimatrix variate beta type IV distribution given in* (3) *is given by*

$$\begin{aligned}\mathcal{M}(\boldsymbol{T}) &= \left\{\beta_p\left(\tfrac{n_1}{2},\tfrac{n_2}{2};\tfrac{m}{2}\right)\right\}^{-1}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{z=0}^{\infty}\sum_{\kappa}\sum_{\tau}\sum_{\zeta}\sum_{\delta\in\kappa\cdot\tau}\sum_{\gamma\in\kappa\cdot\zeta}\frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{k!t!z!}\\ &\cdot\frac{g_{\kappa,\tau}^{\delta}\left(\frac{n_2}{2}\right)_\delta\beta_p\left(\frac{n_2}{2},\frac{n_1+m}{2}\right)}{\left(\frac{n_1+n_2+m}{2}\right)_\delta}\frac{f_{\kappa,\zeta}^{\gamma}\left(\frac{n_1}{2}\right)_\gamma\beta_p\left(\frac{n_1}{2},\frac{n_2+m}{2}\right)}{\left(\frac{n_1+n_2+m}{2}\right)_\gamma}\\ &\cdot\frac{C_\zeta(\boldsymbol{T}_1)C_\tau(\boldsymbol{T}_2)C_\delta(\boldsymbol{I}_p)C_\gamma(\boldsymbol{I}_p)}{C_\kappa(\boldsymbol{I}_p)C_\tau(\boldsymbol{I}_p)C_\zeta(\boldsymbol{I}_p)}.\end{aligned}\qquad(12)$$

Proof:
By definition, the moment generating function of $\boldsymbol{X}$ is given by

$$\begin{aligned}&\mathcal{M}(\boldsymbol{T})\\ =\;& E\left[\mathrm{etr}(\boldsymbol{T}\boldsymbol{X})\right]\\ =\;& \left\{\beta_p\left(\tfrac{n_1}{2},\tfrac{n_2}{2};\tfrac{m}{2}\right)\right\}^{-1}\int_{\boldsymbol{0}<\boldsymbol{X}_1<\boldsymbol{I}_p}\int_{\boldsymbol{0}<\boldsymbol{X}_2<\boldsymbol{I}_p}|\boldsymbol{X}_1|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}|\boldsymbol{X}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\\ &\cdot|\boldsymbol{I}_p-\boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)}\mathrm{etr}(\boldsymbol{T}_1\boldsymbol{X}_1)\mathrm{etr}(\boldsymbol{T}_2\boldsymbol{X}_2)d\boldsymbol{X}_2d\boldsymbol{X}_1\\ =\;& \left\{\beta_p\left(\tfrac{n_1}{2},\tfrac{n_2}{2};\tfrac{m}{2}\right)\right\}^{-1}\int_{\boldsymbol{X}_1}|\boldsymbol{X}_1|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\mathrm{etr}(\boldsymbol{T}_1\boldsymbol{X}_1)A(\boldsymbol{X}_1)d\boldsymbol{X}_1,\end{aligned}$$

(13)

$$
\begin{aligned}
&A(\boldsymbol{X}_1)\\
=\ & \int_{\boldsymbol{X}_2} |\boldsymbol{X}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)}\mathrm{etr}(\boldsymbol{T}_2\boldsymbol{X}_2)|\boldsymbol{I}_p-\boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)}\mathrm{d}\boldsymbol{X}_2\\
=\ & \sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{k!t!}\int_{\boldsymbol{X}_2}|\boldsymbol{X}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)}C_\kappa(\boldsymbol{X}_1\boldsymbol{X}_2)C_\tau(\boldsymbol{T}_2\boldsymbol{X}_2)\mathrm{d}\boldsymbol{X}_2\\
=\ & \sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{k!t!}\\
&\cdot\int_{\boldsymbol{X}_2}|\boldsymbol{X}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)}C_\tau(\boldsymbol{T}_2\boldsymbol{X}_2)\int_{O(p)}C_\kappa(\boldsymbol{H}'\boldsymbol{X}_1\boldsymbol{H}\boldsymbol{X}_2)\mathrm{d}\boldsymbol{H}\mathrm{d}\boldsymbol{X}_2.
\end{aligned}
$$

It follows from Muirhead (1982, Theorem 7.2.5, page 243 and Theorem 7.2.10, page 254) and Chikuse (1981, Equation 2.7) that

$$
\begin{aligned}
&A(\boldsymbol{X}_1)\\
=\ & \sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{k!t!}\frac{C_\kappa(\boldsymbol{X}_1)}{C_\kappa(\boldsymbol{I}_p)}\int_{\boldsymbol{X}_2}|\boldsymbol{X}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)}C_\kappa(\boldsymbol{X}_2)C_\tau(\boldsymbol{T}_2\boldsymbol{X}_2)\mathrm{d}\boldsymbol{X}_2\\
=\ & \sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{k!t!}\frac{C_\kappa(\boldsymbol{X}_1)C_\tau(\boldsymbol{T}_2)}{C_\kappa(\boldsymbol{I}_p)C_\tau(\boldsymbol{I}_p)}\int_{\boldsymbol{X}_2}|\boldsymbol{X}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)}C_\kappa(\boldsymbol{X}_2)C_\tau(\boldsymbol{X}_2)\mathrm{d}\boldsymbol{X}_2\\
=\ & \sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{k!t!}\frac{C_\kappa(\boldsymbol{X}_1)C_\tau(\boldsymbol{T}_2)}{C_\kappa(\boldsymbol{I}_p)C_\tau(\boldsymbol{I}_p)}\\
&\cdot\sum_{\delta\in\kappa\cdot\tau}g^{\delta}_{\kappa,\tau}\int_{\boldsymbol{X}_2}|\boldsymbol{X}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)}C_\delta(\boldsymbol{X}_2)\mathrm{d}\boldsymbol{X}_2\\
=\ & \sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{\tau}\sum_{\delta\in\kappa\cdot\tau}\frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{k!t!}\frac{g^{\delta}_{\kappa,\tau}\left(\frac{n_2}{2}\right)_\delta\beta_p\left(\frac{n_2}{2},\frac{n_1+m}{2}\right)}{\left(\frac{n_1+n_2+m}{2}\right)_\delta}\frac{C_\kappa(\boldsymbol{X}_1)C_\tau(\boldsymbol{T}_2)C_\delta(\boldsymbol{I}_p)}{C_\kappa(\boldsymbol{I}_p)C_\tau(\boldsymbol{I}_p)}
\end{aligned}
\tag{14}
$$

Substituting (14) in (13) and continuing in the same way, the result follows. ■

<u>Definition</u>

The $p\times p$ symmetric positive definite random matrices $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ on the unit p-sphere are said to have the bimatrix variate Kummer-beta type IV distribution with parameters n_1, n_2, m and $\boldsymbol{\Psi}$, denoted by $(\boldsymbol{X}_1,\boldsymbol{X}_2)\sim BKB^{IV}_p\left(\frac{n_1}{2},\frac{n_2}{2};\frac{m}{2};\boldsymbol{\Psi}\right)$ if their joint density is given by

$$
\begin{aligned}
&K|\boldsymbol{X}_1|^{\frac{1}{2}n_1-\frac{1}{2}(p+1)}|\boldsymbol{X}_2|^{\frac{1}{2}n_2-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_1|^{\frac{1}{2}(n_2+m)-\frac{1}{2}(p+1)}\\
&\cdot|\boldsymbol{I}_p-\boldsymbol{X}_2|^{\frac{1}{2}(n_1+m)-\frac{1}{2}(p+1)}|\boldsymbol{I}_p-\boldsymbol{X}_1\boldsymbol{X}_2|^{-\frac{1}{2}(n_1+n_2+m)}\mathrm{etr}\left[-\boldsymbol{\Psi}\left(\boldsymbol{X}_1+\boldsymbol{X}_2\right)\right]
\end{aligned}
$$

where

$$
\begin{aligned}
K^{-1} &= \mathcal{M}(-\boldsymbol{\Psi}:-\boldsymbol{\Psi})\\
&= \left\{\beta_p\left(\frac{n_1}{2},\frac{n_2}{2};\frac{m}{2}\right)\right\}^{-1}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{z=0}^{\infty}\sum_{\kappa}\sum_{\tau}\sum_{\zeta}\sum_{\delta\in\kappa\cdot\tau}\sum_{\gamma\in\kappa\cdot\zeta}\frac{\left(\frac{n_1+n_2+m}{2}\right)_\kappa}{k!t!z!}\\
&\cdot\frac{g^{\delta}_{\kappa,\tau}\left(\frac{n_2}{2}\right)_\delta\beta_p\left(\frac{n_2}{2},\frac{n_1+m}{2}\right)}{\left(\frac{n_1+n_2+m}{2}\right)_\delta}\frac{f^{\gamma}_{\kappa,\zeta}\left(\frac{n_1}{2}\right)_\gamma\beta_p\left(\frac{n_1}{2},\frac{n_2+m}{2}\right)}{\left(\frac{n_1+n_2+m}{2}\right)_\gamma}\\
&\cdot\frac{C_\zeta\left(-\boldsymbol{\Psi}\right)C_\tau(-\boldsymbol{\Psi})C_\delta(\boldsymbol{I}_p)C_\gamma(\boldsymbol{I}_p)}{C_\kappa(\boldsymbol{I}_p)C_\tau(\boldsymbol{I}_p)C_\zeta\left(\boldsymbol{I}_p\right)}.
\end{aligned}
$$

The expression for K follows directly from (12).

5 Shape analysis

In Section 5.1 we illustrate the effect of the parameters on the shape of the densities of $Z_1 = |\boldsymbol{X}_1\boldsymbol{X}_2|$ and $Z_2 = \left|\frac{\boldsymbol{X}_1}{\boldsymbol{X}_2}\right|$. The effect of the parameters on the conditional moments of the bivariate beta type IV variables as well as on the form of the bivariate Kummer-beta type IV distribution are illustrated in Sections 5.2 and 5.3 respectively. Only the two cases $p = 2$ and $p = 1$ are studied, for products and ratios of dependent Wilks's statistics, and conditional densities respectively. For the Kummer-beta type IV distribution, again only the unidimensional case is studied. The cases$p > 2$ are much more complicated to deal with numerically.

5.1 Distributions of product and ratio of dependent Wilks's statistics

We consider the *bimatrix* case with $p = 2$ where the density function of $Z_1 = |\boldsymbol{X}_1\boldsymbol{X}_2| = \Lambda_1\Lambda_2$ (see (10)) simplifies to

$$f(z_1) = \frac{\Gamma_2\left(\frac{n_1+m}{2}\right)\Gamma_2\left(\frac{n_2+m}{2}\right)}{\Gamma_2\left(\frac{n_1}{2}\right)\Gamma_2\left(\frac{n_2}{2}\right)\Gamma_2\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\boldsymbol{I}_2)}{k!}\Gamma_2\left(\frac{n_1+n_2+m}{2},\kappa\right)G_{4,4}^{4,0}\left(z_1\middle|_{b_1,b_2,b_3,b_4}^{a_1,a_2,a_3,a_4}\right),\quad 0 < z_1 < 1 \quad (15)$$

where
$a_1 = \frac{n_1+n_2+m}{2} - 1 + k_1$, $a_2 = \frac{n_1+n_2+m}{2} - 1 + k_1$, $a_3 = \frac{n_1+n_2+m}{2} - 1 + k_2 - \frac{1}{2}$ and $a_4 = \frac{n_1+n_2+m}{2} - 1 + k_2 - \frac{1}{2}$;
$b_1 = \frac{n_1}{2} - 1 + k_1$, $b_2 = \frac{n_2}{2} - 1 + k_1$, $b_3 = \frac{n_1}{2} - 1 + k_2 - \frac{1}{2}$ and $b_4 = \frac{n_2}{2} - 1 + k_2 - \frac{1}{2}$.

Figure 1 illustrates the shape of $f(z_1)$ for increasing values of n_2 and m. We note that as n_2 increases the density shifts towards larger values of z_1. The opposite happens when m increases.

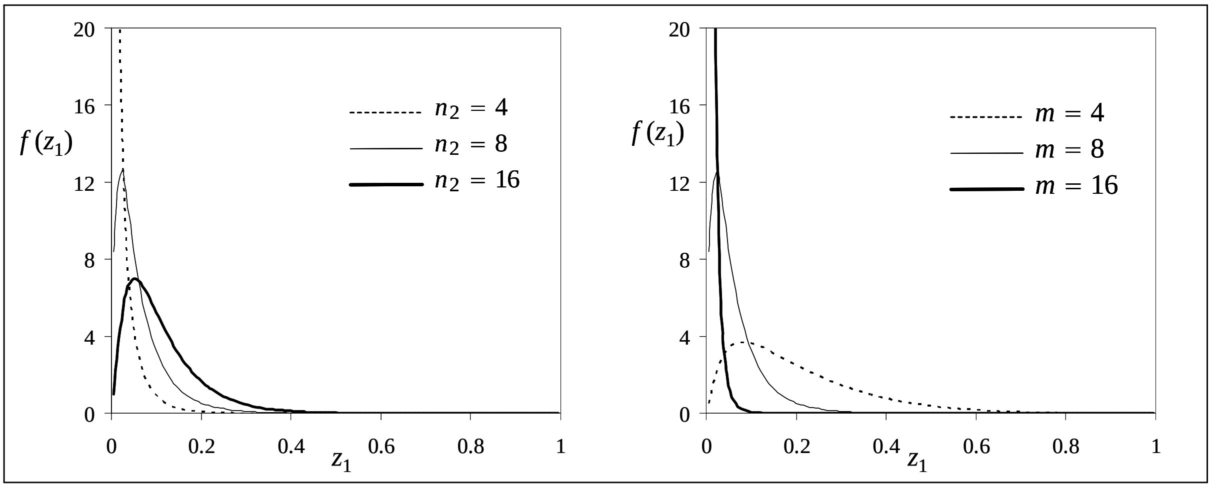

Figure 1: Density function of $Z_1 = |\boldsymbol{X}_1\boldsymbol{X}_2| = \Lambda_1\Lambda_2$ for
(i) increasing values of n_2, $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_2^{IV}\left(4, \frac{n_2}{2}; 4\right)$ and
(ii) increasing values of m, $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_2^{IV}\left(4, 4; \frac{m}{2}\right)$.

Similarly for $p = 2$, the density function of $Z_2 = \frac{\Lambda_1}{\Lambda_2}$ given in (11) simplifies to

$$f(z_2) = \frac{\Gamma_2\left(\frac{n_1+m}{2}\right)\Gamma_2\left(\frac{n_2+m}{2}\right)}{\Gamma_2\left(\frac{n_1}{2}\right)\Gamma_2\left(\frac{n_2}{2}\right)\Gamma_2\left(\frac{m}{2}\right)}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\boldsymbol{I}_2)}{k!}\Gamma_2\left(\frac{n_1+n_2+m}{2},\kappa\right)G_{4,4}^{2,2}\left(z_2\middle|_{b_1,b_2,b_3,b_4}^{a_1,a_2,a_3,a_4}\right),\quad z_2 > 0 \quad (16)$$

where
$a_1 = -\frac{n_2}{2} - k_1$, $a_2 = -\frac{n_2}{2} - k_2 + \frac{1}{2}$, $a_3 = \frac{n_1+n_2+m}{2} - 1 + k_1$ and $a_4 = \frac{n_1+n_2+m}{2} - 1 + k_2 - \frac{1}{2}$;
$b_1 = \frac{n_1}{2} - 1 + k_1$, $b_2 = \frac{n_1}{2} - 1 + k_2 - \frac{1}{2}$, $b_3 = -\frac{n_1+n_2+m}{2} - k_1$ and $b_4 = -\frac{n_1+n_2+m}{2} - k_2 + \frac{1}{2}$.

Figure 2 shows that graphs of $f(z_2)$ for increasing values of n_2 and m. As n_2 increases the density shifts towards smaller values of z_2 and the spread of the density decreases. The parameter n_1 has the opposite effect. As m increases the density shifts towards smaller values of z_2 and the spread of the density increases.

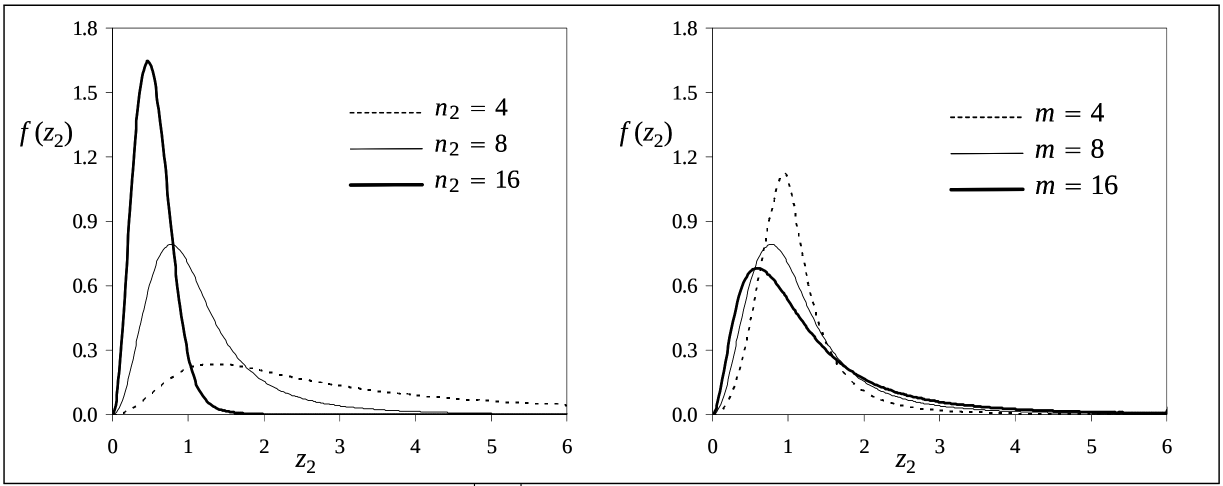

Figure 2: Density function of $Z_2 = \left|\frac{\boldsymbol{X}_1}{\boldsymbol{X}_2}\right| = \frac{\Lambda_1}{\Lambda_2}$ for
(i) increasing values of n_2, $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_2^{IV}\left(4, \frac{n_2}{2}; 4\right)$ and
(ii) increasing values of m, $(\boldsymbol{X}_1, \boldsymbol{X}_2) \sim BB_2^{IV}\left(4, 4; \frac{m}{2}\right)$.

Nagar et al. (2009) illustrated the effect of the parameters on the density of the product of *correlated* beta variables, i.e. $Z_1 = X_1X_2$. There are some algorithms available for calculating such functions as (15) and (16) and facilitating the use of these distributions (see Gutierrez et al. (2000) and Koev and Edelman (2006)). There are also mathematical packages, such as Maple and Mathematica for computing and drawing densities in terms of Meijer's G-function. In this paper we used simulations to give graphical representations of the densities of Z_1 and Z_2.

5.2 Effect of parameters on the conditional moments of the bivariate beta type IV distribution

In the case where $p = 1$ the conditional density of X_2 given X_1 in (8) is given by

$$f(x_2|x_1) = \left\{\beta\left(\tfrac{n_2}{2}; \tfrac{n_1+m}{2}\right)\right\}^{-1}(1-x_1)^{\frac{1}{2}n_2}\, x_2^{\frac{1}{2}n_2-1}(1-x_2)^{\frac{1}{2}(n_1+m)-1}(1-x_1x_2)^{-\frac{1}{2}(n_1+n_2+m)}, \quad 0 < x_2 < 1. \tag{17}$$

In Figure 3 the graphs of $E(X_2|X_1 = x_1)$ and $var(X_2|X_1 = x_1)$ show the underlying structure of (17) for increasing values of m. As m increases whilst all the other parameters are held constant, $E(X_2|X_1 = x_1)$ becomes smaller. Similar graphs for increasing values of n_2 are given in Figure 4. The conditional expected value, $E(X_2|X_1 = x_1)$, increases as n_2 increases. The parameter n_1 has the same effect as m on $E(X_2|X_1 = x_1)$ and $var(X_2|X_1 = x_1)$ as can be seen from (17). A detailed discussion of the bivariate case is given by Olkin and Liu (2003).

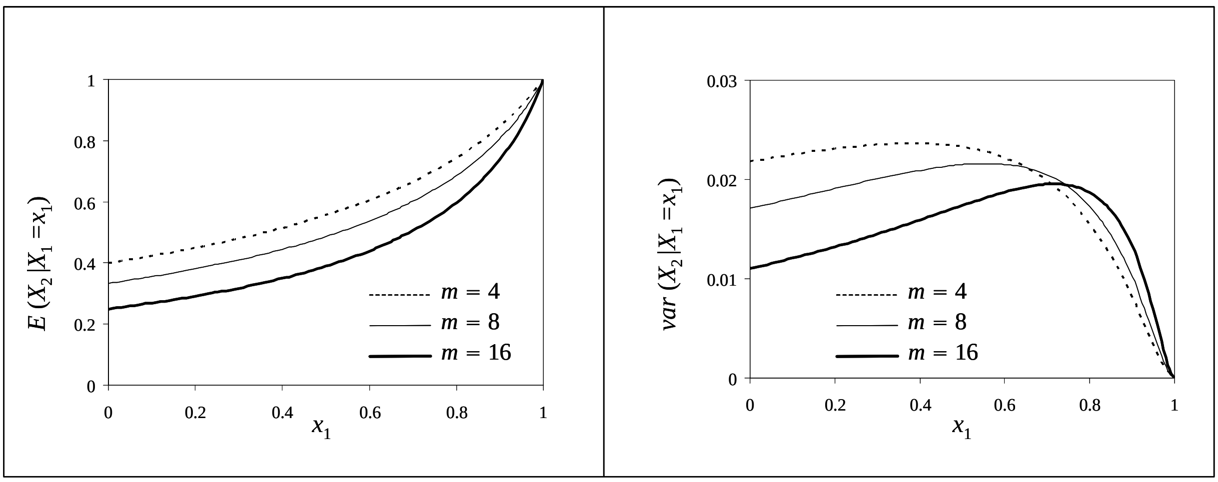

Figure 3: Effect of m on $E(X_2|X_1 = x_1)$ and $var(X_2|X_1 = x_1)$, $(X_1, X_2) \sim BB^{IV}\left(4, 4; \frac{m}{2}\right)$

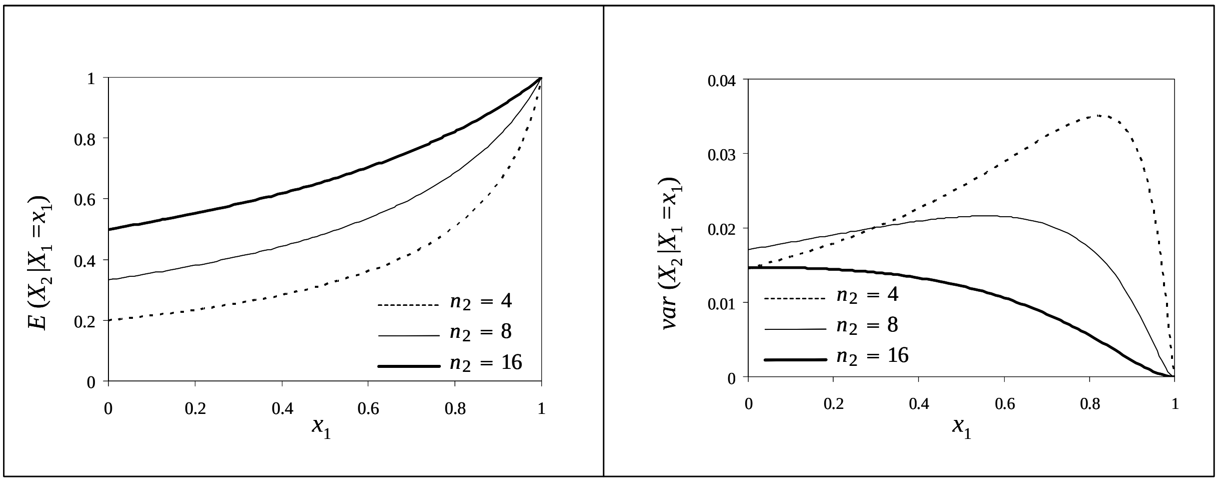

Figure 4: Effect of n_2 on $E(X_2|X_1 = x_1)$ and $var(X_2|X_1 = x_1)$, $(X_1, X_2) \sim BB^{IV}\left(4, \frac{n_2}{2}; 4\right)$

5.3 Bivariate Kummer-beta type IV distribution

In the case where $p = 1$ the density of the *bivariate Kummer-beta type IV distribution* is given by

$$f(x_1, x_2) = K x_1^{\frac{1}{2}n_1 - 1} x_2^{\frac{1}{2}n_2 - 1} (1 - x_1)^{\frac{1}{2}(n_2+m)-1} (1 - x_2)^{\frac{1}{2}(n_1+m)-1} (1 - x_1x_2)^{-\frac{1}{2}(n_1+n_2+m)} \exp[-\psi(x_1 + x_2)],$$

where $0 < x_1 < 1$ and $0 < x_2 < 1$. The normalizing constant is

$$\begin{aligned} K^{-1} &= \frac{\Gamma\left(\frac{n_1+m}{2}\right)\Gamma\left(\frac{n_2+m}{2}\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}\right)} \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\Gamma\left(\frac{n_1}{2}+k\right)\Gamma\left(\frac{n_2}{2}+k\right)}{\Gamma\left(\frac{n_1+n_2+m}{2}+k\right)} \\ &\quad \cdot {}_1F_1\left(\frac{n_1}{2}+k; \frac{n_1+n_2+m}{2}+k; -\psi\right) {}_1F_1\left(\frac{n_2}{2}+k; \frac{n_1+n_2+m}{2}+k; -\psi\right) \end{aligned}$$

where ${}_1F_1(\cdot)$ is the confluent hypergeometric function with scalar argument. The distribution is denoted as $(X_1, X_2) \sim BKB^{IV}\left(\frac{n_1}{2}, \frac{n_2}{2}; \frac{m}{2}; \psi\right)$. Figure 5 illustrates the effect of the parameter ψ on this density function. Note that if $\psi = 0$ the density simplifies to the bivariate beta type IV (Jones, 2001).

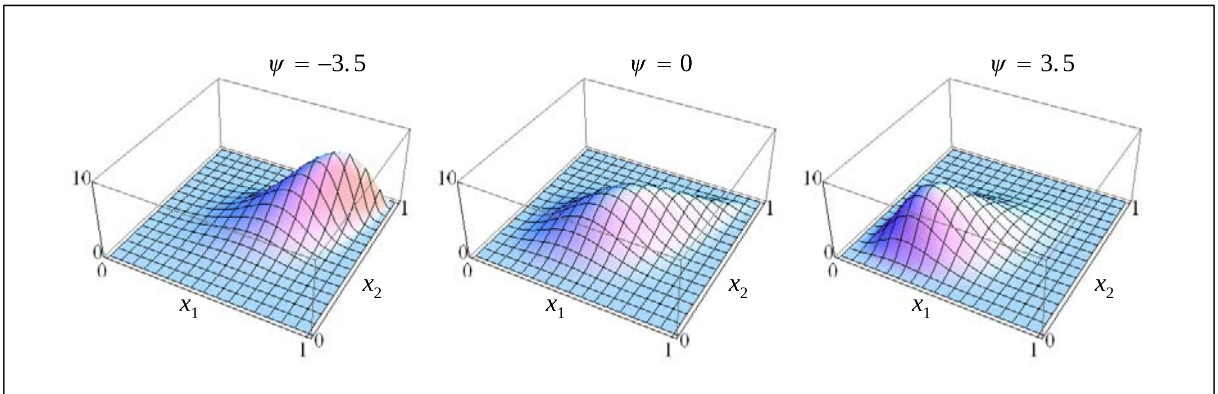

Figure 5: Effect of ψ on $f(x_1, x_2)$, $(X_1, X_2) \sim BKB^{IV}(4, 4; 4; \psi)$

6 Conclusion

In this paper we introduced the bimatrix variate beta type IV distribution and used it to derive the exact expressions of the densities of the product and ratio of two dependent Wilks's statistics. We also defined the bimatrix variate Kummer-beta type IV distribution which followed from the moment generating function of the bimatrix variate beta type IV distribution. The effect of the parameters on the shape of the densities were also illustrated. The availability of closed form expressions for the densities of the product and ratio of two dependent Wilks's statistics and the newly proposed bimatrix variate Kummer-beta type IV distribution should stimulate further research and applications.

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