ON STOCHASTIC MODELS DESCRIBING THE MOTIONS OF RANDOMLY FORCED LINEAR VISCOELASTIC FLUIDS

P.A. RAZAFIMANDIMBY

Abstract. This paper is devoted to the analysis of a stochastic equation describing the motions of a large class of incompressible linear viscoelastic fluids in two-dimensional subject to periodic boundary condition and driven by random external forces. To do so we distinguish two cases, and for each case a global existence result of probabilistic weak solution for is expounded in this paper. We also prove that under suitable hypotheses on the external random forces the solution turns out to be unique. As concrete examples, we consider the stochastic equations for the Maxwell and Oldroyd fluids that are of great importance in the investigation towards the understanding of the elastic turbulence.

1. Introduction

The study of turbulent flows has attracted many prominent researchers from different fields of contemporary sciences for ages. For indepth coverage of the deep and fascinating investigations undertaken in this field, the abundant wealth of results obtained and remarkable advances achieved we refer to the monographs [19, 32, 34] and references therein. Recent study, see for instance [7], has showed that the non-Newtonian elastic turbulence can be well understood on basis of known viscoelastic models such as the Oldroyd fluids or the Maxwell fluids. Indeed, by computational investigations of the two-dimensional periodic Oldroyd-B model the authors in [7] found that there is a considerable agreement between their numerical results and the experimental observations of elastic turbulence.

The irregular or random nature of all turbulent flows makes any deterministic approach to turbulence problems impossible. For this reason the idea of introducing a noise term for modeling random influences acting on any evolutive fluid has now become widely recognized. Such approach in the mathematical investigation for the understanding of the Newtonian turbulence phenomenon was pioneered by Bensoussan and Temam in [6] where they studied the Stochastic Navier-Stokes Equation(SNSE). Since then stochastic models of fluid dynamics have been the object of intense investigations which have generated several important results. We refer to [5], [10], [11] , [14], [18], [33], [49], just to cite a few. Similar investigations for Non-Newtonian elastic fluids have almost not been undertaken except in very few works; we refer for instance to [21],[23], [24], [30], [35], [48] for some example of the computational studies of stochastic models of polymeric fluids and to [9], [22], [25], [26] for their mathematical analysis. It should be noted that the study of stochastic model for viscoelastic fluids is relevant not only for the analytical approach to turbulent flows but also for practical needs related to the Physics of the corresponding

2000 *Mathematics Subject Classification.* 60H15, 60H30, 76D05,35R15, 35Q10.

Key words and phrases. Stochastic models; Polymeric fluids; Probabilistic weak solution; Viscoelastic fluid; Elastic turbulence ; Oldroyd fluids; Maxwell fluids.

fluids (see [32] for example). It is also very important for the study of the dynamical behavior of the fluids (see for instance [35] for the case of polymeric fluids).

Motivated by the facts cited in the two preceding paragraphs we propose in the present paper to analyze the following problem which is subject to the periodic boundary condition:

$$\begin{cases} du + (u.\nabla)udt + \nabla\mathfrak{P}dt = \operatorname{div}\sigma dt + F(u,t)dt + G(u,t)dW, \\ \operatorname{div} u = 0, \\ \int_D u(x)dx = 0, \\ u_{|t=0} = u_0, \end{cases} \tag{1}$$

$t \in [0,T]$, $T \in (0,\infty]$. This system describes the motion of a large class of incompressible linear viscoelastic fluids driven by random external forces and filling a periodic square $D = [0,L]^2 \subset \mathbb{R}^2$, $L > 0$. Here $u, \mathfrak{P}, W$ represent respectively a random periodic in space random velocity with period L in each direction, a random scalar pressure and a $\mathbb{R}^m$ -valued standard Wiener process, $m \in \{1,2,3....\}$. The tensor $\sigma = (\sigma_{ij})$ is the deviator of the stress tensor of the fluid, we assume throughout that it is a traceless tensor ($\operatorname{tr}\sigma = 0$). In this work we should distinguish the case

$$\sigma = \mathbf{K}\mathbf{D}, \tag{2}$$

and

$$\sigma = 2\nu\mathbf{D} + \mathbf{K}\mathbf{D}, \tag{3}$$

where

$$\mathbf{D} = (1/2)(\nabla u + \nabla^t u),$$

and the operator $\mathbf{K}$ is a continuous mapping satisfying some hypotheses (see (11)-(13)). Note that the problem we consider here is physically meaningful and of great importance for the applied sciences . Indeed thanks to the monographs [8], [35] and the papers [3], [7], and [48] for instance, the system (1) can be taken as a relevant model of turbulent polymeric fluid.

The mathematical works on some linear viscoelastic fluids undertaken by the Soviet mathematician Oskolkov in [36, 37, 38] and by Ladyzhenskaya in [29] have influenced the emergence of the paper [28] where a global solvability result of the deterministic counterpart of the system {(1),(2)} (resp. {(1),(3)}) subject to the periodic boundary condition (resp. nonslip boundary condition) was given. To the best of our knowledge similar investigations for the two general stochastic models {(1),(2)} and {(1),(3)} have not been undertaken yet. The purpose of this paper is to prove that under suitable conditions on $\mathbf{K}$, F and G each of our stochastic models is well-posed (see Theorems 3.3, 3.4, 4.2 and 4.3). In view of the technical difficulties involved , we provide full details of the proof of our results. Due to nontrivial difficulties that arise from the nature of the nonlinearities involved in (1) other mathematical issues such as existence, uniqueness of the invariant measure and its ergodicity are beyond of the scope of this work; we leave these questions for future investigation.

The layout of this paper is as follows. In addition to the current introduction this article consists of three other sections. In Section 2 we give some notations, necessary backgrounds of probabilistic or analytical nature. Section 3 is devoted to the detailed analysis of the problem {(1),(2)}. We prove the existence and pathwise uniqueness of its probabilistic weak solution which yields the existence of a unique probabilistic strong solution. In the very same section we consider the stochastic equations for randomly forced generalized Maxwell fluids as a concrete example. In Section 4 we only state the main theorems related to {(1),(3)} and apply the obtained results to the stochastic model for the generalized viscoelastic Oldroyd fluids; we refer to the previous section for the details of the proofs.

2. Preliminaries-Notations

This section is devoted to the presentation of notations and auxiliary results needed in the work. Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^2$, let $1 \leq p \leq \infty$ and let k be a nonnegative integer. We consider the well-known Lebesgue and Sobolev spaces $L^p(\mathcal{O})$ and $H^k(\mathcal{O})$, respectively. We refer to [1] for detailed information on Sobolev spaces. Let L be a nonnegative number and $D = [0, L]^2$ be a periodic box of side length L. We denote by $H^k(D)$ the spaces consisting of those functions u that are in $H^k_{Loc}(\mathbb{R}^2)$ and that are periodic with period L:

$$u(x + Lr_i) = u(x), \ i = 1, 2,$$

where $\{r_1, r_2\}$ represents the canonical basis of $\mathbb{R}^2$. Here the space $H^k_{Loc}(\mathbb{R}^2)$ is the space of functions u such that $u_{|\mathcal{O}}$ is an element of the Sobolev space $H^k(\mathcal{O})$ for every bounded set $\mathcal{O} \subset \mathbb{R}^2$. For functions v of zero space average , that is

$$\int_D v dx = 0,$$

the following Poincaré's inequality holds

$$|v| \leq \mathcal{P}||v|| \ \forall v \in H^1(D), \tag{4}$$

where $|.|_{sc}$ denotes the L^2-norm, $\mathcal{P} > 0$ is the Poincaré's constant and $||.||_{sc}$ denotes the semi-norm generated by the scalar product

$$((u, v)) = \int_D \nabla u.\nabla v dx = \sum_{i=1}^{2} \int_D \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx,$$

∇ is the gradient operator. From now we denote by $H^1_0(D)$ the space

$$H^1_0(D) := \left\{ u : u \in H^1(D) \text{ and } \int_D u dx = 0 \right\}. \tag{5}$$

Thanks to (4), we can endow $H^1_0(D)$ with the norm $||.||_{sc}$. Besides Poincaré's inequality we also have

$$c|u|_{\mathbb{H}^1(D)} \leq |\operatorname{curl} u| \leq c||u||, \tag{6}$$

which holds for any divergence free fields. For $\beta \in \mathbb{R}$ we can define the space $H^\beta(D)$ via their expansion in Fourier series so that we also have the space

$$H^\beta_0(D) := \left\{ u : u \in H^\beta(D) \text{ and } \int_D u dx = 0 \right\}. \tag{7}$$

We refer to [47] (see also [12], [19]) for more details about these spaces.We proceed with the definitions of additional spaces frequently used in this paper.
In what follows we set

$$X^{\otimes M} = \underbrace{X \times \cdots \times X}_{M \text{ times}},$$

and

$$\mathbb{X} = X \times X,$$

for any Banach space X and any positive integer M. If $|.|_X$ is the norm on X, then

$$|u|^2_{X^{\otimes M}} = \sum_{i=1}^{M} |u_i|^2_X.$$

We introduce the spaces

$$\begin{aligned}\mathcal{V} &= \left\{ u \in [\mathcal{C}^\infty_{per}(D)]^{\otimes 2} : \operatorname{div} u = 0 \text{ and } \int_D u dx = 0 \right\}\\ \mathbb{V} &= \text{ closure of } \mathcal{V} \text{ in } \mathbb{H}^1_0(D)\\ \mathbb{H} &= \text{ closure of } \mathcal{V} \text{ in } \mathbb{L}^2(D),\end{aligned}$$

where $\mathcal{C}^\infty_{per}(D)$ denotes the space of infinitely differentiable periodic function with period L.
We denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and the norm induced by the inner product and the norm in $\mathbb{L}^2(D)$ on $\mathbb{H}$, respectively. Thanks to Poincaré's inequality (4), we can endow $\mathbb{V}$ with the norm $||.||$, which is defined by

$$||u||^2 = \sum_{i=1}^{2} ||u_i||^2_{sc}.$$

From now on, we identify the space $\mathbb{H}$ with its dual space $\mathbb{H}^\star$ via the Riesz representation, and we have the Gelfand triple

$$\mathbb{V} \subset \mathbb{H} \subset \mathbb{V}^\star, \tag{8}$$

where each space is dense in the next one and the inclusions are continuous. It follows that we can make the identification

$$(v, w) =< v, w >,$$

for any $v \in \mathbb{H}$ and $w \in \mathbb{V}$. Here $< .,. >$ denotes the duality product $\mathbb{V}^*, \mathbb{V}$.
Next we define some probabilistic evolution spaces necessary throughout the paper. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a given stochastic basis; that is, $(\Omega, \mathcal{F}, P)$ is complete probability space and $(\mathcal{F}_t)_{0\le t\le T}$ is an increasing sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains every P-null subset of Ω. For any real Banach space $(X, |.|_X)$, for any $r, p \ge 1$ we denote by $L^p(\Omega, \mathcal{F}, P; L^r(0, T; X))$ the space of processes $u = u(\omega, t)$ with values in X defined on $\Omega \times [0, T]$ such that

1. u is measurable with respect to (ω, t) and for each t, $\omega \mapsto u(\omega, t)$ is $\mathcal{F}^t$-measurable.
2. $u(\omega, t) \in X$ for almost all (ω, t) and

$$||u||_{L^p(\Omega,\mathcal{F},P;L^r(0,T;X))} = \left(E \left(\int_0^T ||u||^r_X dt \right)^{\frac{p}{r}} \right)^{\frac{1}{p}} < \infty$$

where E denotes the mathematical expectation with respect to the probability measure P.

When $r = \infty$, we write

$$||u||_{L^p(\Omega,\mathcal{F},P;L^\infty(0,T;X))} = \left(E \sup_{0\le t\le T} ||u||^p_X \right)^{\frac{1}{p}} < \infty.$$

For $p \geq 1$, we also consider the space $L^p(0,T;X)$ of X-valued measurable functions u defined on $[0,T]$ such that

$$||u||_{L^p(0,T;X)} = \left(\int_0^T ||u||_X^p dt \right)^{\frac{1}{p}} < \infty$$

Let W be a standard Wiener process defined on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ and taking its values in $\mathbb{R}^m$. Given a measurable and $\mathcal{F}_t$-adapted $X^{\otimes m}$-valued process f such that

$$E \int_0^T |f(t)|^2_{X^{\otimes m}} dt < \infty,$$

the process

$$I(f)(t) = \int_0^t f(s)dW(s), \;\; 0 \leq t \leq T,$$

is well defined and is a continuous martingale. Moreover it satisfies

$$EI(f)(t) = 0, \;\; 0 \leq t \leq T, \tag{9}$$

$$E|I(f)(t)|^2_X = E \int_0^t |f(s)|^2_X ds, \;\; 0 \leq t \leq T. \tag{10}$$

We refer to [20, 27](see also [13])for further reading on probability theory and stochastic calculus.

Let X be a separable complete metric space and $\mathcal{B}(X)$ its Borel σ-field. A family Π_k of probability measures on $(X, \mathcal{B}(X))$ is relatively compact if every sequence of elements of Π_k contains a subsequence Π_{k_j} which converges weakly to a probability measure Π, that is, for any ϕ bounded and continuous function on Ω,

$$\int \phi(s)d\Pi_{k_j} \longrightarrow \int \phi(s)d\Pi.$$

The family Π_k is said to be tight if for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \Omega$ such that $P(K_\varepsilon) \geq 1 - \varepsilon$, for every $P \in \Pi_k$.
We have the well-known result.

Theorem 2.1 (Prokhorov)**.** *Assume that X is a Polish space, then the family Π_k is relatively compact if and only if it is tight.*

We shall use the following useful theorem due to Skorokhod.

Theorem 2.2 (Skorokhod)**.** *For any sequence of probability measures Π_k on Ω which converges to a probability measure Π, there exists a probability space $(\Omega', \mathcal{F}', P')$ and random variables X_k, X with values in Ω such that the probability law of X_k(resp. X) is Π_k(resp. Π) and $\lim_{k\longrightarrow\infty} X_k = X$ P'-a.s.*

We refer to [13] for the proofs of the two last results.

The following result is very important in Section 3.2 Part 2 where we prove a probabilistic compactness result, its proof can be found in [44].

Theorem 2.3. *Let X, B, Y three Banach spaces such that the following embedding are continuous*

$$X \subset B \subset Y.$$

Moreover, assume that the embedding $X \subset B$ is compact, then the set $\mathfrak{F}$ consisting of functions $v \in L^q(0,T;B) \cap L^1_{loc}(0,T;X)$, $1 \le q \le \infty$ such that

$$\sup_{0\le h\le 1} \int_{t_1}^{t_2} |v(t+h) - v(t)|_Y dt \to 0, \ as\, h \to 0,$$

for any $0 < t_1 < t_2 < T$ is compact in $L^p(0,T;B)$ for any $p < q$.

Throughout the symbol $\sigma : \sigma'$ denotes the summation

$$\sigma : \sigma' = \mathrm{tr}(\sigma\sigma') = \sum_{i,k=1}^{2} \sigma_{ik}\sigma'_{ki}.$$

We assume that $\mathbf{K}$ is a symmetric tensor valued continuous mapping which satisfies the following

- $\mathbf{K}$ is bounded, that is
$$E\int_0^T |\mathbf{K}\mathbf{D}|^2 dt \le CE\int_0^T |\mathbf{D}|^2 dt. \tag{11}$$
- For any $\mathbf{D}_1$ and $\mathbf{D}_2$ we have
$$0 \le E\int_{[0,T]\times D} (\mathbf{K}\mathbf{D}_1 : \mathbf{D}_1) dxdt, \tag{12}$$
and
$$0 \le E\int_{D\times[0,T]} (\mathbf{K}\mathbf{D}_1 - \mathbf{K}\mathbf{D}_2 : \mathbf{D}_1 - \mathbf{D}_2) dxdt, \tag{13}$$

Remark 2.4. The hypotheses (12) seem to be artificial but in accordance to Chapters 2-3 of [2] it has physical meanings. Indeed we have that

$$-(\mathrm{div}\,\sigma, u) = \int_D \sigma : \nabla u = \int_D \sigma : \mathbf{D},$$

and this means that the dissipation of energy is positive. This remark was already stated in Section 1 (Introduction) of [28]. The assumption (13) is a mathematical assumption which allows as to prove the well-posedness of the models we treat. They are satisfied at least for general viscoelastic flows generated by the linear rheological equations of the type

$$\sigma = \int_0^t \mathbf{K}(t-\tau)\mathbf{D}(x,\tau)d\tau.$$

We refer to Chapter 5, Section 5.2 of [8] for some examples of these linear viscoelastic fluids. We also notice that when $\mathbf{K}$ is linear then the hypotheses (13) and (12) are equivalent.

3. Analysis of the stochastic equation of the type {(1),(2)}

In this section we investigate the stochastic equations {(1),(2)}. The first section is devoted to the statement of the main results which is going to be proved in the second subsection.

3.1. **Hypotheses and statement of the main results.** For this section we suppose that

(HYP 1) the mapping F induces a nonlinear operator from $\mathbb{H} \times [0,T]$ into $\mathbb{V}$ which is assumed to be measurable (resp. continuous) with respect to its second (resp. first) variable. We require that there exists constant $C_F > 0$ such that for almost all $t \in [0,T]$ and for each $u \in \mathbb{H}$

$$||F(u,t)|| \leq C_F(1+|u|), \tag{14}$$

(HYP 2) the $\mathbb{V}^{\otimes m}$-valued function G defined on $\mathbb{H} \times [0,T]$ is measurable (resp. continuous) with respect to its second (resp. first) argument, and it verifies

$$|G(u,t)|_{\mathbb{V}^{\otimes m}} \leq C_G(1+|u|), \tag{15}$$

for almost everywhere $t \in [0,T]$ and for any $u \in \mathbb{H}$.

(HYP 3) We assume as well that there exist two positive constants C'_F and C'_G such that

$$||F(u,t) - F(v,t)|| \leq C'_F|u-v|, \tag{16}$$

$$|G(u,t) - G(v,t)|_{\mathbb{V}^{\otimes m}} \leq C'_G|u-v|, \tag{17}$$

for any $u, v \in \mathbb{H}$.

(HYP 4) In addition to (11)-(13) we assume furthermore that

$$0 \leq -(\operatorname{curl}\operatorname{div}(\mathbf{KD}), \operatorname{curl} u). \tag{18}$$

Remark 3.1. For a vector $u \in \mathbb{R}^2$, the operator curl is defined by

$$\operatorname{curl} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

The divergence of a tensor field $\mathbf{D}$ is defined using the recursive relation

$$\operatorname{div}(\mathbf{D}).\mathbf{c} = \operatorname{div}(\mathbf{c}.\mathbf{D}),\ \operatorname{div} v = \operatorname{tr}(\nabla v)$$

where $\mathbf{c}$ is an arbitrary constant vector, v is a vector field, and $\operatorname{tr}(\mathbf{D})$ denotes the trace of $\mathbf{D}$.
Karazeeva remarked in Section 5.2 of [28] that when $\mathbf{K}$ and $\partial d \backslash \partial x_k$, $k = 1, 2$, commute then (18) is a consequence of (12). The condition (18) is met when $\mathbf{K}$ is given by the second equation in Remark 2.4.

We introduce the concept of the solution of the problem {(1),(2)}.

Definition 3.2. By a probabilistic weak solution of the problem {(1),(2)}, we mean a system

$$(\Omega, \mathcal{F}, P, \mathcal{F}^t, W, u),$$

where

1. $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\mathcal{F}^t$ is a filtration on $(\Omega, \mathcal{F}, P)$,
2. $W(t)$ is an m-dimensional $\mathcal{F}^t$ standard Wiener process,
3. $u \in L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;\mathbb{V})) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;\mathbb{H}))$,
4. For almost all t, $u(t)$ is $\mathcal{F}^t$-measurable,
5. P-a.s the following integral equation of Ito type holds

$$\begin{aligned}(u(t) - u(0), \phi) + \int_0^t \int_D (\mathbf{KD} : \nabla\phi) dx ds - \int_0^t ((u.\nabla\phi), u) ds \\ = \int_0^t (F(u(s), s), \phi) ds + \int_0^t (G(u(s), s), \phi) dW(s)\end{aligned} \tag{19}$$

for any $t \in [0,T]$ and $\phi \in \mathcal{V}$.

We have

Theorem 3.3. *If $u_0 \in \mathbb{V}$ and if the hypotheses (HYP 1),(HYP 2) and (HYP 4) hold then the problem $\{(1),(2)\}$ has a solution in the sense of the above definition. Moreover, almost surely the paths of the solution are $\mathbb{V}$(resp. $\mathbb{H}$)-valued weakly (resp. strongly) continuous.*

Theorem 3.4. *Assume that (HYP 1)-(HYP 4) hold and let u_1 and u_2 be two probabilistic weak solutions of $\{(1),(2)\}$ starting with the same initial condition and defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}^t, P)$ with the same Winer process W. If we set $v = u_1 - u_2$, then we have $v = 0$ almost surely.*

3.2. **Proof of Theorems 3.3 and 3.4.** This subsection is devoted to the proof of the existence and uniqueness results stated in the preceding subsection. We split the proof into parts. The proof of the existence theorem is inspired by the works [28], [5] (see also [18]). Throughout this subsection C will designate a positive constant which depends only on the data (u_0, T, C_F, C_G).

Part 1. **The Approximate solution and some a priori estimates**

In this part we derive crucial a priori estimates from the Galerkin approximation. They will serve as a toolkit for the proof of the Theorem 3.3.
The operator $-\Delta$ is a self-adjoint and positive definite on $\mathbb{H}$, and its inverse is completely continuous. Therefore $\mathbb{H}$ has a complete orthonormal basis consisting of the eigenfunctions $(e_i)_{i\geq 1} \in [C^\infty(D)]^{\otimes 2}$ of $-\Delta$. The family $(e_i)_{i\geq 1} \in [C^\infty(D)]^{\otimes 2}$ forms an orthogonal basis in $\mathbb{V}$. We now introduce the Galerkin approximation for the problem (1)-(2). We consider the subset $\mathbb{H}_N = \mathrm{Span}(e_1, \dots, e_N) \subset \mathbb{H}$ and we look for a finite-dimensional approximation of a solution of our problem as a vector $u^N \in \mathbb{H}_N$ that can be written as:

$$u^N(t) = \sum_{i=1}^{N} c_{iN}(t) e_i(x). \tag{20}$$

We set

$$\mathbf{D}^N = (1/2)(\nabla u^N + \nabla^t u^N).$$

Let us consider a complete probabilistic system $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{P}, \bar{\mathcal{F}}^t, \bar{W})$ such that the filtration $\{\bar{\mathcal{F}}_t\}$ satisfies the usual condition and $\bar{W}$ is an m-dimensional standard Wiener process taking values in $\mathbb{R}^m$. We require u^N to satisfy the following system

$$\begin{aligned} d(u^N, e_i) + \left(\int_D (\mathbf{KD}^N : \nabla e_i) dx\right) dt - ((u^N.\nabla) e_i, u^N) dt = ((F(u^N, t), e_i) dt \\ + (G(u^N, t), e_i) d\bar{W}, \end{aligned} \tag{21}$$

$i \in \{1, ..., N\}$. Here u_0^N is the orthogonal projection of u_0 onto the space $\mathbb{H}_N$,

$$u_0^N \to u_0 \text{ strongly in } \mathbb{H}$$

as $N \to \infty$. The sequence of continuous functions u^N exists at least on a short (possibly random) interval $(0, T_N)$. Indeed the coefficients C_{iN} satisfies

$$dC_{iN} + \sum_{k,j=1}^{N} \left(\sum_{l=1}^{2} \int_D \frac{\partial e_j}{\partial x_l} e_k \frac{\partial e_i}{\partial x_l} \right) C_{kN}(t) C_{jN}(t) dt + \sum_{k=1}^{N} (\mathbf{K}(C_{kN} \nabla e_k), e_i) dt$$
$$= (F(u^N, t), e_i) dt + (G(u^N, t), e_i) d\bar{W},$$

which is a system of stochastic ordinary differential equations with continuous coefficients. Thanks to the existence theorem stated in page 59 of [45] (see also [27] Theorem 4.22, page 323) we infer the existence of continuous functions C_{iN} on $(0, T_N)$. Global existence will follow from a priori estimates for u^N. We denote by $\bar{E}$ the mathematical expectation with respect to $\bar{P}$.

Lemma 3.5. *We have*

$$\bar{E} \sup_{0 \le t \le T} |u^N(t)|^p < C, \tag{22}$$

for any $2 \le p < \infty$ and $1 \le N < \infty$.

Proof. Thanks to Ito's formula we derive from (21) that

$$d|u^N|^2 + 2 \left(\int_D (\mathbf{K}\mathbf{D}^N : \mathbf{D}^N) dx \right) dt = 2((F(t, u^N), u^N)) dt + \sum_{i=1}^{N} (G(t, u^N), e_i)^2 dt$$
$$+ 2(G(t, u^N), u^N) d\bar{W}, \tag{23}$$

where we have used the fact that $((u.\nabla)v, w) = -((u.\nabla w), v)$ for any $u, v, w \in \mathbb{V}$. Thanks to (12) we get

$$d|u^N|^2 \le 2|(F(u^N, t)||u^N| dt + \sum_{i=1}^{N} (G(t, u^N), e_i)^2 dt + 2(G(t, u^N), u^N) d\bar{W}. \tag{24}$$

More generally we have

$$d|u^N|^p \le p|(F(u^N, t)||u^N|^{p-1} dt + (1/2)(p(p-1)) \sum_{i=1}^{N} |u^N|^{p-2} (G(t, u^N), e_i)^2 dt$$
$$+ p|u^N|^{p-2} (G(t, u^N), u^N) d\bar{W}, \tag{25}$$

for all $2 \le p < \infty$. For any integer $M \ge 1$ we introduce the stopping time

$$\tau_M = \begin{cases} \inf \{ t \ge 0; |u^N(t)| \ge M \} \\ T \text{ if } \{ t \ge 0; |u^N(t)| \ge M \} = \emptyset. \end{cases}$$

Owing to Schwarz's inequality and the assumptions (14)-(15) we have that

$$\sup_{0 \le s \le t \wedge \tau_M} |u^N(s)|^p \le |u_0^N|^p + p C_F \int_0^{t \wedge \tau_M} (1 + |u^N|) |u^N(s)|^{p-1} ds$$
$$+ (1/2) p(p-1) \sum_{i=1}^{N} \int_0^{t \wedge \tau_M} |u^N|^{p-2} (G(u^N, t), e_i)^2 ds \tag{26}$$
$$+ p \sup_{0 \le s \le t \wedge \tau_M} \left| \int_0^s |u^N|^{p-2} (G(s, u^N), u^N) d\bar{W} \right|.$$

Since

$$\sum_{i=1}^{N}(G(t,u^N),e_i)^2 \leq |G(u^N,t)|^2_{\mathbb{H}^{\otimes m}},$$

we derive from (26) and (15) that

$$\begin{aligned}\sup_{0\leq s\leq t\wedge\tau_M}|u^N(s)|^p \leq pC\int_0^{t\wedge\tau_M}|u^N(s)|^p ds + C_G^2 p(p-1)\int_0^{t\wedge\tau_M}|u^N|^{p-2}(1+|u^N|^2)ds\\ +|u_0^N|^p + p\sup_{0\leq s\leq t\wedge\tau_M}\left|\int_0^s |u^N|^{p-2}(G(s,u^N),u^N)d\bar{W}\right| + CT.\end{aligned}$$

Using Hölder's inequality and taking the mathematical expectation in both sides of this estimates yield

$$\begin{aligned}\bar{E}\sup_{0\leq s\leq t\wedge\tau_M}|u^N(s)|^p \leq &\bar{E}|u_0^N|^p + C\bar{E}\int_0^{t\wedge\tau_M}|u^N(s)|^p ds\\ &+ p\bar{E}\sup_{0\leq s\leq t\wedge\tau_M}\left|\int_0^s |u^N|^{p-2}(G(s,u^N),u^N)d\bar{W}\right| + pC_F,\end{aligned}\tag{27}$$

Let us set

$$\gamma^N = \bar{E}\sup_{0\leq s\leq t\wedge\tau_M}\left|\int_0^s |u^N|^{p-2}(G(s,u^N),u^N)d\bar{W}\right|.$$

By Burkhölder-Davis-Gundy's inequality we obtain

$$p\gamma^N \leq pC\bar{E}\left(\int_0^{t\wedge\tau_m}|u^N|^{2p-4}(G(s,u^N),u^N)^2 ds\right)^{1/2},\tag{28}$$

$$p\gamma^N \leq pC\bar{E}\sup_{0\leq s\leq t\wedge\tau_M}|u^N|^{p/2}\left(\int_0^{t\wedge\tau_M}|u^N|^{p-4}(G(s,u^N),u^N)^2 ds\right)^{p/2},$$

which with the assumption (15) implies that

$$p\gamma^N \leq (1/2)\bar{E}\sup_{0\leq s\leq t\wedge\tau_M}|u^N|^p + (1/2)C\bar{E}\int_0^{t\wedge\tau_M}|u^N|^{p-2}(1+|u^N|)^2 ds\tag{29}$$

Out of this and (27) we infer that

$$\bar{E}\sup_{0\leq s\leq t\wedge\tau_M}|u^N(s)|^p \leq \bar{E}|u_0^N|^p + C(p,C_F,C_G)\bar{E}\int_0^{t\wedge\tau_M}|u^N|^p ds.$$

This estimate implies that

$$h(t) \leq \bar{E}|u_0^N|^p + \int_0^t h(s)ds,$$

where $h(t) = \bar{E}\sup_{0\leq s\leq t\wedge\tau_M}|u^N(s)|^p$. Now, Gronwall's Lemma implies that

$$\bar{E}\sup_{0\leq s\leq t\wedge\tau_M}|u^N(s)|^p \leq C(p,u_0,C_F,C_G,T),\ t\in(0,T)\tag{30}$$

It remains to prove that $T_N = T$, to do so we must prove that $\tau_M \nearrow T$ almost surely as $M\to\infty$. This is classic but we prefer to give the details. From the continuity of

u^N we infer that $u^N(\tau_M) \geq M$. For any $t \in (0,T)$, $\bar{E}(1_{\tau_M<t}) = \bar{P}(\tau_M < t)$. We also have that

$$\bar{E} \sup_{0\leq s\leq t\wedge\tau_M} |u^N(s)|^2 \geq \bar{E}\left(\sup_{0\leq s\leq \tau_M} (|u^N(s)|^2 1_{\tau_M<t})\right), \\ \geq M^2\bar{P}(\tau_M < t),\ t \in (0,T).$$

We infer from this, (30) and the monotonicity of τ_M that $\tau_M \nearrow T$ a.s. as was required. Since the constant C in (30) is independent of N and M, Fatou's Theorem complete the proof of the lemma. □

The estimate of Lemma 3.5 is not sufficient to pass to the limit in the nonlinear term. We still need to derive some additional crucial but nontrivial inequalities.

Lemma 3.6. *We have*

$$\bar{E} \sup_{0\leq t\leq T} ||u^N(t)||^p \leq C, \tag{31}$$

for any $2 \leq p < \infty$ *and* $1 \leq N < \infty$.

Proof. Let P^N be the orthogonal projection of $\mathbb{V}^*$ onto the span $\{e_1, ..., e_N\}$ that is

$$P^N h = \sum_{j=1}^{N} \langle h, e_j\rangle e_j.$$

Because $P^N u^N = u^N$, we can rewrite the equation (21) in the following form which should be understood as the equality between random variables with values in $\mathbb{V}^*$

$$du^N - P^N(\mathrm{div}(\mathbf{KD}^N))dt + P^N(u^N.\nabla u^N)dt = P^N F(u^N,t)dt + P^N G(u^N,t)d\bar{W}. \tag{32}$$

Applying the operator curl ($= \nabla\wedge$) to both sides of this equation implies

$$d\zeta^N - \nabla\wedge(P^N(\mathrm{div}(\mathbf{KD}^N)))dt + \nabla\wedge(P^N(u^N.\nabla u^N))dt = \nabla\wedge(P^N F(u^N,t))dt \\ +\nabla\wedge(P^N G(u^N,t))d\bar{W},$$

where $\zeta^N = \nabla\wedge u^N$. Thanks to the regularity of the e_i-s, the function ζ^N is periodic at the boundary of the square D. Ito's formula for the function $|\zeta^N|^2$ implies that

$$d|\zeta^N|^2 - 2(\nabla\wedge(P^N(\mathrm{div}(\mathbf{KD}^N))), \zeta^N)dt - 2(\nabla\wedge(P^N F(u^N,t)), \zeta^N)dt \\ = |\nabla\wedge(P^N G(u^N,t))|^2 dt + 2(\nabla\wedge(P^N G(u^N,t)), \zeta^N)d\bar{W}.$$

where we have used the fact that

$$2 < \nabla\wedge(P^N(u^N.\nabla u^N)), \zeta^N >= 0$$

in the periodic boundary condition setting. More generally

$$d|\zeta^N|^p - p|\zeta^N|^{p-2}(\nabla\wedge(P^N(\mathrm{div}(\mathbf{KD}^N))), \zeta^N)dt - p|\zeta^N|^{p-2}(\nabla\wedge(P^N F(u^N,t)), \zeta^N)dt \\ = (1/2)p(p-1)|\zeta^N|^{p-2}|\nabla\wedge(P^N G(u^N,t))|^2 dt + p|\zeta^N|^{p-2}(\nabla\wedge(P^N G(u^N,t)), \zeta^N)d\bar{W},$$

for $2 \leq p < \infty$. We use the divergence freeness of u^N, the periodicity of ζ^N and the identities

$$\mathrm{curl}(\mathrm{curl}\, v) = -\Delta v + \nabla(\mathrm{div}\, v), \\ (\mathrm{curl}\, v, \phi) = (v, \mathrm{curl}\, \phi) + \int_{\partial D}(v\times\mathbf{n})\phi dx, \\ P^N\Delta u^N = \Delta(P^N u^N) = \Delta u^N,$$

to reach

$$(\nabla\wedge(P^N(\mathrm{div}(\mathbf{KD}^N))),\zeta^N)=(\mathrm{div}(\mathbf{KD}^N),\nabla\wedge\zeta^N).$$

By utilizing Schwarz's inequality and (18), we derive from the last estimate that

$$\begin{aligned}|\zeta^N(t)|^p\leq|\zeta_0^N|^p+(1/2)p(p-1)\int_0^t|\zeta^N|^{p-2}|\nabla\wedge(P^NG(u^N,t))|^2ds\\ +p\left|\int_0^t|\zeta^N|^{p-2}(\nabla\wedge(P^NG(u^N,t)),\zeta^N)d\bar{W}\right|\\ +p\int_0^t|\zeta^N|^{p-1}|\nabla\wedge(P^NF(u^N,t))|ds\end{aligned}$$

Thanks to the estimates (6), (14) and (15) we deduce from the above estimate that

$$\begin{aligned}\bar{E}\sup_{0\leq s\leq t}|\zeta^N(s)|^p\leq p\bar{E}\sup_{0\leq s\leq t}\left|\int_0^s|\zeta^N|^{p-2}(\nabla\wedge(P^NG(u^N,t)),\zeta^N)d\bar{W}\right|\\ +\bar{E}|\zeta_0^N|^p+p\bar{E}\int_0^t|\zeta^N|^p|ds+CT\end{aligned}\tag{33}$$

Let us set

$$\Gamma^N=p\bar{E}\sup_{0\leq s\leq t}\left|\int_0^s|\zeta^N|^{p-2}(\nabla\wedge(P^NG(u^N,t)),\zeta^N)d\bar{W}\right|.$$

By using Burkhölder-Davis-Gundy's inequlity and Schwarz's inequality we obtain

$$p\Gamma^N\leq\bar{E}\left(\int_0^s|\zeta^N|^{2p-4}|\nabla\wedge(P^NG(u^N,t))|^2\,|\zeta^N|^2ds\right)^{1/2}.$$

We derive from this, the estimates (6) (this is allowed since $P^NG(u^N,t))\mathbb{H}_N$) and (15) that

$$p\Gamma^N\leq(1/2)\bar{E}\sup_{0\leq s\leq t}|\zeta^N|^p+C\bar{E}\int_0^t|\zeta^N|^pds$$

From this, (33) and Gronwall's lemma we deduce that

$$\bar{E}\sup_{0\leq s\leq t}|\zeta^N(s)|^p\leq C.$$

Owing to (6) the proof of the lemma is finished. □

The following result is central in the proof of the forthcoming tightness property of the Galerkin solution.

Lemma 3.7. *For any* $0\leq\delta<1$ *we have*

$$\bar{E}\sup_{|\theta|\leq\delta}\int_0^{T-\delta}|u^N(s+\theta)-u^N(s)|^2_{\mathbb{V}^*}\leq C\delta.\tag{34}$$

Proof. We can rewrite the equation (21) in an integral form as the equality between random variables with values in $\mathbb{V}^*$

$$\begin{aligned}u^N-\int_0^tP^N(\mathrm{div}(\mathbf{KD}^N))ds+\int_0^tP^N(u^N.\nabla u^N)ds=u_0^N+\int_0^tP^NF(u^N,t)ds\\ +\int_0^NP^NG(u^N,t)d\bar{W}.\end{aligned}\tag{35}$$

By using the triangle inequlity for the norm $|.|_{\mathbb{V}^*}$, we deduce from (35) that

$$|u^N(t+\theta)-u^N(t)|^2_{\mathbb{V}^*}\le C\theta\int_t^{t+\theta}|\operatorname{div}(\mathbf{K}\mathbf{D}^N)|^2_{\mathbb{V}^*}ds+C\theta\int_t^{t+\theta}|(u^N.\nabla u^N)|^2_{\mathbb{V}^*}ds$$
$$+C\theta\int_t^{t+\theta}|F(u^N,s)|^2ds+C\left|\int_t^{t+\theta}P^NG(u^N,s)d\bar{W}\right|^2,$$

for any $0\le\theta\le\delta$. This estimate, the continuity of div as linear operator along with (11), (14), Lemmas 3.5 and 3.6 implies that

$$\bar{E}\sup_{0\le\theta\le\delta}\int_0^{T-\delta}|u^N(t+\theta)-u^N(t)|^2_{\mathbb{V}^*}dt\le C\delta+C\delta\bar{E}\int_0^{T-\delta}\int_t^{t+\delta}|(u^N.\nabla u^N)|^2_{\mathbb{V}^*}dsdt$$
$$+C\int_0^{T-\delta}\bar{E}\sup_{0\le\theta\le\delta}\left|\int_t^{t+\theta}P^NG(u^N,s)d\bar{W}\right|^2dt. \tag{36}$$

By making use of Martingale inequality, (15) and Lemma 3.5 we have that

$$\bar{E}\sup_{0\le\theta\le\delta}\int_0^{T-\delta}|u^N(t+\theta)-u^N(t)|^2_{\mathbb{V}^*}dt\le C\delta\bar{E}\int_0^{T-\delta}\int_t^{t+\delta}|(u^N.\nabla u^N)|^2_{\mathbb{V}^*}dsdt$$
$$+C\delta+C\delta^2.$$

By the well-known inequality

$$|(u^N.\nabla u^N)|^2_{\mathbb{V}^*}\le C_B|u^N|^2||u^N||^2, \tag{37}$$

which holds in the 2-dimensional case, we obtain that

$$\bar{E}\sup_{0\le\theta\le\delta}\int_0^{T-\delta}|u^N(t+\theta)-u^N(t)|^2_{\mathbb{V}^*}dt\le C\delta.$$

To complete the proof we use the same argument for negative values of θ. ☐

Part 2. Tightness Property and Application of Prokhorov's and Skorohod's Theorems

We denote by $\mathfrak{Z}$ the following subset of $L^2(0,T;\mathbb{H})$:

$$\mathfrak{Z}=\left\{z\in L^\infty(0,T;\mathbb{V});\ \sup_{|\theta|\le\mu_M}\int_0^{T-\mu_M}|z(t+\theta)-z(t)|^2_{\mathbb{V}^*}\le C\nu_M\right\},$$

for any sequences ν_M,μ_M such that $\nu_M,\mu_M\to 0$ as $M\to\infty$ and $\sum_{M\ge 0}\mu_M/\nu_M<\infty$. The following result is a particular case of Theorem 2.3 (see also Proposition 3.1 in page 45 of [4] for a similar result).

Lemma 3.8. *The set $\mathfrak{Z}$ is compact in $L^2(0,T;\mathbb{H})$.*

Next we consider the space $\mathfrak{S}=C(0,T;\mathbb{R}^m)\times L^2(0,T;\mathbb{H})$ endowed with its Borel σ-algebra $\mathcal{B}(\mathfrak{S})$ and the family of probability measures Π^N on $\mathfrak{S}$, which is the probability measure induced by the following mapping:

$$\phi:\omega\mapsto(\bar{W}(\omega,.),u^N(\omega,.)).$$

That is, for any $A\in\mathcal{B}(\mathfrak{S})$, $\Pi^N(A)=\bar{P}(\phi^{-1}(A))$.
We have the following theorem.

Lemma 3.9. *The family* $(\Pi^N)_{N\geq 1}$ *is tight in* $\mathfrak{S}$.

Proof. For any $\varepsilon > 0$ and $M \geq 1$, we claim that there exists a compact subset $\mathfrak{K}_\varepsilon$ of $\mathfrak{S}$ such that $\Pi^N(\mathfrak{K}_\varepsilon) \geq 1-\varepsilon$. To back our claim we define the sets

$$\mathfrak{W}_\varepsilon = \left\{ W : \sup_{\substack{t,s\in[0,T]\\ |t-s|<\frac{T}{2^M}}} 2^{\frac{M}{8}} |W(t)-W(s)| \leq J_\varepsilon, \forall M \right\}$$

and

$$\mathfrak{Z}_\varepsilon = \Big\{ z; \sup_{0\leq t\leq T} |z(t)|^2 \leq K_\varepsilon, \quad \sup_{0\leq t\leq T} ||z(t)||^2 \leq L_\varepsilon,$$
$$\sup_{|\theta|\leq\mu_M} \int_0^{T-\mu_M} |z(t+\theta)-z(t)|^2_{\mathbb{V}^*} dt \leq R_\varepsilon \nu_M \Big\}$$

where $J_\epsilon, K_\epsilon, L_\epsilon, R_\epsilon$ are positive constants to be fixed in the course of the proof. The sequences ν_M and μ_M are chosen so that they are independent of ε, $\nu_M, \mu_M \to 0$ as $M \to \infty$ and $\sum_M \frac{\mu_M}{\nu_M} < \infty$. It is clear by Ascoli-Arzela's Theorem that $\mathfrak{W}_\varepsilon$ is a compact subset of $C(0,T;\mathbb{R}^m)$, and by Lemma 3.8 $\mathfrak{Z}_\varepsilon$ is a compact subset of $L^2(0,T;\mathbb{H})$. We have to show that $\mathfrak{P}_\varepsilon = \Pi^N\left((\bar{W},u^N) \notin \mathfrak{W}_\varepsilon \times \mathfrak{Z}_\varepsilon\right) < \varepsilon$. Indeed, we have

$$\begin{aligned}\mathfrak{P}_\varepsilon \leq & \bar{P}\left[\bigcup_{M=1}^{\infty}\bigcup_{j=1}^{2^M}\left(\sup_{t,s\in I_j}|\bar{W}(t)-\bar{W}(s)| \geq J_\varepsilon \frac{1}{2^{\frac{M}{8}}}\right)\right] + \bar{P}\left(\sup_{0\leq t\leq T}|u^N(t)|^2 \geq K_\varepsilon\right)\\ &+ \bar{P}\left(\bigcup_M \left\{\sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M}|u^N(t+\theta)-u^N(t)|^2_{\mathbb{V}^*}dt \geq R_\varepsilon\nu_M\right\}\right)\\ &+ \bar{P}\left(\sup_{0\leq t\leq T}||u^N(t)||^2 \geq L_\varepsilon\right),\end{aligned}$$

where $\{I_j : 1\leq j\leq 2^M\}$ is a family of intervals of length $\frac{T}{2^M}$ which forms a partition of the interval $[0,T]$. It is well known that for any Wiener process B

$$\bar{E}|B(t)-B(s)|^{2m} = C_m|t-s|^m \quad \text{for any } m\geq 1,$$

where C_m is a constant depending only on m. From this and the Markov's Inequality

$$\bar{P}(\omega : \zeta(w) \geq \alpha) \leq \frac{1}{\alpha^k}\bar{E}(|\zeta(\omega)|^k)$$

where $\zeta(\omega)$ is a random variable on $(\bar{\Omega},\bar{\mathcal{F}},\bar{P})$ and positive numbers k and α, we obtain

$$\begin{aligned}\mathfrak{P}_\varepsilon \leq & \sum_{M=1}^{\infty}\sum_{j=1}^{2^M} C_m\left(2^{\frac{M}{8}}\right)^{2m}\frac{1}{J_\varepsilon^{2m}}\left(\frac{T}{2^M}\right)^m + \frac{1}{K_\varepsilon}\bar{E}\sup_{t\leq T}|u^N(t)|^2 + \frac{1}{L_\varepsilon}\bar{E}\sup_{0\leq t\leq T}||u^N(t)||^2\\ &+ \sum_M \frac{1}{R_\varepsilon\nu_M}\bar{E}\sup_{|\theta|\leq\mu_M}\int_0^{T-\mu_M}|u^N(t+\theta)-u^N(t)|^2_{\mathbb{V}^*}dt.\end{aligned}$$

Owing to the lemmas 3.5, 3.7 and by choosing $m = 2$, we have

$$\begin{aligned}\mathfrak{P}_\varepsilon &\leq \frac{C_2 T^2}{J_\varepsilon^4}\sum_{M=1}^{\infty} 2^{-\frac{1}{2}M} + C(\frac{1}{K_\varepsilon} + \frac{1}{L_\varepsilon} + \frac{1}{R_\varepsilon}\sum_M \frac{\mu_M}{\nu_M}) \\ &\leq \frac{C_2 T^2}{J_\varepsilon^4}(1+\sqrt{2}) + C(\frac{1}{K_\varepsilon} + \frac{1}{L_\varepsilon} + \frac{1}{R_\varepsilon}\sum_M \frac{\mu_M}{\nu_M}).\end{aligned}$$

A convenient choice of J_ε, K_ε, L_ε, R_ε completes the proof of the claim, and hence the proof of the lemma. □

Now it follows by Prokhorov's Theorem that the family $(\Pi^N)_{N\geq 1}$ is relatively compact in the set of probability measures on $\mathfrak{S}$ equipped with the weak convergence topology. Then, we can extract a subsequence Π^{N_μ} that weakly converges to a probability measure Π. By Skorohod's Theorem, there exists a probability space $(\Omega, \mathcal{F}, P)$ and random variables (W^{N_μ}, u^{N_μ}) and (W, u) on $(\Omega, \mathcal{F}, P)$ with values in $\mathfrak{S}$ such that

$$\begin{aligned}W^{N_\mu} &\to W \text{ in } C(0,T;\mathbb{R}^m)\ P-a.s, \\ u^{N_\mu} &\to u \text{ in } L^2(0,T;\mathbb{H})\ P-a.s.\end{aligned}$$

Moreover, the probability law of (W^{N_μ}, u^{N_μ}) is Π^{N_μ} and that of (W, u) is Π.
For the filtration $\mathcal{F}^t$, it is enough to choose $\sigma(W(s), u(s) : 0 \leq s \leq t)$.
By the same argument as in [4] (Section 3.3 page 49) we can prove that the limit process W is a standard m-dimensional Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}^t\}_{0\leq t\leq T}, P)$.

Theorem 3.10. *The pair u^{N_μ}, W^{N_μ} satisfies the equation*

$$\begin{aligned}&(u^{N_\mu}(t), e_i) + \int_0^t \int_D (\boldsymbol{KD} : \nabla e_i) dx ds + \int_0^t ((u^{N_\mu}.\nabla e_i), u^{N_\mu}) ds \\ &= (u_0^{N_\mu}, e_i) + \int_0^t ((F(u^{N_\mu}(s), s), e_i) ds + \int_0^t (G(u^{N_\mu}(s), s), e_i) dW^{N_\mu}.\end{aligned} \tag{38}$$

for almost all $\omega \in \Omega$, for any $t \in [0, T]$ and $1 \leq i \leq N_\mu$.

Proof. The proof follows the same lines as in [5] (Section 4.3.4 page 282) , so we omit it. □

Part 3. Passage to the limit

To back our goal we need to pass to the limit in the terms of the estimate (38). From the tightness property we have

$$u^{N_\mu} \to u \text{ in } L^2(0,T;\mathbb{H}) \text{ P-a.s}, \tag{39}$$

as $N_\mu \to \infty$. Since u^{N_μ} agrees with (38), then it verifies the same estimates as u^N. In particular the estimate

$$E \sup_{0\leq t\leq T} \left|u^{N_\mu}\right|^p \leq C,$$

for $p \geq 2$ implies that the norm $|u^{N_\mu}|_{L^2(0,T;\mathbb{H})}$ is uniformly integrable with respect to the probability measure. Therefore, we can deduce from Vitali's Theorem that

$$u^{N_\mu} \to u \text{ in } L^2(\Omega, \mathcal{F}, P, L^2(0,T;\mathbb{H})), \tag{40}$$

as $N_\mu \to \infty$. It is readily seen that

$$(u^{N_\mu}, e_i)_{\mathbb{V}} \to (u, e_i)_{\mathbb{V}} \text{ weakly in } L^2(\Omega, \mathcal{F}, P; L^2(0,T)).$$

Thanks to the convergence (40) and the continuity of $\mathbf{K}$ we see that

$$\int_D (\mathbf{K}\mathbf{D}^{N_\mu} : \nabla e_i)dx \to \int_D \mathbf{K}\mathbf{D}\nabla e_i dx \text{ strongly in } L^2(\Omega, \mathcal{F}, P; L^2(0,T)),$$

as $N_\mu \to \infty$. Let χ be an element of $L^\infty(\Omega \times [0,T], dP \otimes dt)$. Thanks to (40) we can prove by arguing as in [31] that

$$E\int_0^T ((u^{N_\mu}.\nabla e_i), u^{N_\mu})\chi dt \to E\int_0^T ((u.\nabla e_i), u)\chi dt, \tag{41}$$

as $N_\mu \to \infty$. The dense injection

$$L^\infty(\Omega \times [0,T], dP \otimes dt) \subset L^2(\Omega \times [0,T], dP \otimes dt),$$

together with the relation (41) show that

$$((u^{N_\mu}.\nabla)e_i, u^{N_\mu}) \rightharpoonup ((u.\nabla)e_i, u) \text{ weakly in } L^2(\Omega, \mathcal{F}, P; L^2(0,T)), \tag{42}$$

as $N_\mu \to \infty$.
It follows from the continuity of F, (40) and Vitali's Theorem that

$$P^{N_\mu}F(u^{N_\mu}(,),.) \to F(u(.),.) \text{ strongly in } L^2(\Omega, \mathcal{F}, P; L^2(0,T;\mathbb{V})), \tag{43}$$

as $N_\mu \to \infty$. This implies in particular that

$$(F(u^{N_\mu}(,),.), e_i) \to ((F(u(.),.), e_i) \text{ strongly in } L^2(\Omega, \mathcal{F}, P; L^2(0,T)),$$

as $N_\mu \to \infty$. We can use the argument in Section 4.3.5 of [5] (see also [41])to prove that

$$\int_0^t (G(u^{N_\mu}, s), e_i)dW^{N_\mu} \rightharpoonup \int_0^t (G(u,s), e_i)dW \text{ weakly in } L^2(\Omega, \mathcal{F}, P; L^2(0,T)), \tag{44}$$

for any $t \in (0,T)$and as $N_\mu \to \infty$.

Combining all these results and passing to the limit in (38), we see that u satisfies the equation (19) which holds for almost all $\omega \in \Omega$, for all $t \in [0,T]$. This proves the first part of Theorem 3.3. By arguing as in [39] (Chapter 2, Lemma 1.2) we get the continuity result stated in Theorem 3.3.

Part 4. Proof of the uniqueness of the solution

Let u_1 and u_2 be two probabilistic weak solutions of {(1),(2)} starting with the same initial condition and defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}^t, P)$ with the same Wiener process W. Set $v = u_1 - u_2$ and

$$\mathbf{D}_v = (1/2)(\nabla v + \nabla^t v),$$
$$\mathbf{D}_i = (1/2)(\nabla u_i + \nabla^t u_i), \quad i = 1, 2.$$

It can be shown that the process v satisfies the equation

$$dv(t) - \mathbb{P}\operatorname{div}(\mathbf{K}\mathbf{D}_1 - \mathbf{K}\mathbf{D}_2)dt + \mathbb{P}((u_1.\nabla)u_1 - (u_2.\nabla)u_2)dt$$
$$= (F(u_1(t),t) - F(u_2(t),t))dt + G(u_1(t),t) - G(u_2(t),t)dW,$$

where $\mathbb{P}$ is the projector from $\mathbb{L}^2(D)$ onto $\mathbb{H}$. Thanks to Ito's formula for $|v|^2$ we have

$$|v(t)|^2 + 2\int_0^t \int_D (\mathbf{KD}_1 - \mathbf{KD}_2 : \mathbf{D}_v)dxdt + 2\int_0^t ((v.\nabla)u_1, v)ds$$
$$= \int_0^t \left(2(F(u_1(s), s) - F(u_2(s), s), v(s)) + |G(u_1(s), s) - G(u_2(s), s)|^2\right) ds$$
$$+2\int_0^t (G(u_1(s), s) - G(u_2(s), s), v(s))dW,$$

Setting $\sigma(t) = \exp(\int_0^t -\eta||u_1(s)||^2 ds)$, $\forall \eta > 0$, we have that

$$\sigma(t)|v(t)|^2 + 2\int_0^t \sigma(s)\int_D (\mathbf{KD}_1 - \mathbf{KD}_2 : \mathbf{D}_v)dxdt + 2\int_0^t \sigma(s)((v.\nabla)u_1, v)ds$$
$$= \int_0^t \sigma(s)\left(2(F(u_1(s), s) - F(u_2(s), s), v(s)) + |G(u_1(s), s) - G(u_2(s), s)|^2\right) ds$$
$$+2\int_0^t \sigma(s)(G(u_1(s), s) - G(u_2(s), s), v(s))dW$$
$$-\eta\int_0^t \sigma(s)||u_1(s)||^2|v(s)|^2 ds,$$

By the assumptions on $\mathbf{K}$, F, G and (37) we have

$$E\sigma(t)|v(t)|^2 \le 2C_B E\int_0^t \sigma(s)|v(s)|^2||u_1(s)||ds + 2C'_F E\int_0^t \sigma(s)|v(s)|^2 ds$$
$$+C'^2_G E\int_0^t \sigma(s)|v(s)|^2 ds - \eta E\int_0^t \sigma(s)||u_1(s)||^2|v(s)|^2 ds,$$

which implies

$$E\sigma(t)|v(t)|^2 \le C_B^2 E\int_0^t \sigma(s)|v(s)|^2||u_1(s)||^2 ds + CE\int_0^t \sigma(s)|v(s)|^2 ds$$
$$-\eta E\int_0^t \sigma(s)||u_1(s)||^2|v(s)|^2 ds$$

By choosing $\eta = C_B^2$ and by making use of Gronwall's lemma we have

$$E\sigma(t)|v(t)|^2 = 0, \tag{45}$$

for any $t \ge 0$. Since $0 \le \sigma(t) < \infty$, then this completes the proof of the Theorem 3.4.

3.3. **Example: The stochastic equation for the Maxwell fluids.** The motion of a randomly forced Maxwell fluids is given by the system $\{(1)\text{-}(2)\}$. The tensor σ for the Maxwell fluids is given by

$$\left(1 + \sum_{l=1}^{L} \lambda_l \frac{\partial^l}{\partial t^l}\right)\sigma = 2\mu\left(1 + \sum_{l=1}^{L-1} k_l \mu^{-1}\frac{\partial^l}{\partial t^l}\right)\mathbf{D},\ L =, 2, 3...., \tag{46}$$

where $\lambda_l > 0$ and $k_l > 0$ represent the relaxation and retardation times, respectively. Considering the polynomials

$$P_m(p) = \mu + \sum_{i=1}^{L}(k_l - \nu\lambda_l)p^l,$$

$$Q(p) = 1 + \sum_{l-1}^{L}\lambda_l p^l.$$

It is shown in [28] that the operator $\mathbf{K}$ for the Maxwell fluids is given by

$$\mathbf{KD} = \sum_{l=1}^{L}\int_0^t \beta_l^{(m)} e^{-\alpha_l(t-\tau)}\mathbf{D}(x,\tau)d\tau, \tag{47}$$

where

$$\beta_l^{(m)} = P_m(-\alpha_l)[Q'(-\alpha_l)]^{-1}$$

is assumed to be positive. Here the numbers $-\alpha_l$ designates the roots of the polynomial Q. The result in [28] shows that $\mathbf{K}$ satisfies (11)-(13) and (18). Hence the results in Theorems 3.3, 3.4 applied to the stochastic equations $\{(1)$- $(2), (47)\}$ for the Maxwell fluids provided that the assumptions (HYP 1)-(HYP 4) hold.

4. Stochastic equation of type {(1),(3)}

This section is devoted to the investigation of $\{(1),(3)\}$. We omit all the details of the proofs since they can be derived from similar idea used in the previous section. We state our main results in the first subsection and we give a concrete example in the second subsection. For this section we suppose that

(AF) the mapping F induces a nonlinear operator from $\mathbb{H} \times [0,T]$ into $\mathbb{H}$ which is assumed to be measurable (resp. continuous) with respect to its second (resp. first) variable. We require that for almost all $t \in [0,T]$ and for each u
$$|F(u,t)| \le C_F(1 + |u|). \tag{48}$$

(AG) The $\mathbb{H}^{\otimes m}$-valued function G defined on $\mathbb{H} \times [0,T]$ is measurable (resp. continuous) with respect to its second (resp. first) argument, and it verifies
$$|G(u,t)|_{\mathbb{H}^{\otimes m}} \le C_G(1 + |u|), \tag{49}$$
for all $t \in [0,T]$ and for any $u \in \mathbb{H}$.

(ASFG) We assume as well that
$$|F(u,t) - F(v,t)| \le C'_F|u - v|, \tag{50}$$
$$|G(u,t) - G(v,t)|_{\mathbb{H}^{\otimes m}} \le C'_G|u - v|, \tag{51}$$
for any $u, v \in \mathbb{H}$.

4.1. Statement of the main results.

We introduce the concept of the solution of the problem $\{(1),(3)\}$.

Definition 4.1. By a probabilistic weak solution of the problem $\{(1),(3)\}$, we mean a system

$$(\Omega, \mathcal{F}, P, \mathcal{F}^t, W, u),$$

where

(1) $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\mathcal{F}^t$ is a filtration on $(\Omega, \mathcal{F}, P)$,

(2) $W(t)$ is an m-dimensional $\mathcal{F}^t$ standard Wiener process,
(3) $u \in L^p(\Omega, \mathcal{F}, P; L^2(0,T;\mathbb{V})) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;\mathbb{H}))\ \ \forall\ 2 \leq p < \infty$,
(4) For almost all t, $u(t)$ is $\mathcal{F}^t$-measurable,
(5) P-a.s the following integral equation of Ito type holds

$$
\begin{aligned}
&(u(t) - u(0), \phi) + \nu \int_0^t ((u,\phi))ds + \int_0^t \int_D \mathbf{KD}\nabla\phi dxds + \int_0^t ((u.\nabla\phi), u)ds \\
&= \int_0^t (F(u(s),s),\phi)ds + \int_0^t (G(u(s),s),\phi)dW(s)
\end{aligned}
\tag{52}
$$

for any $t \in [0,T]$ and $\phi \in \mathcal{V}$.

We have

Theorem 4.2. *If $u_0 \in \mathbb{H}$ and if the hypotheses (AF)-(AG) hold then the problem* $\{(1),(3)\}$ *has a solution in the sense of the above definition. Moreover u is strongly (resp. weakly) continuous in $\mathbb{H}$ (resp. $\mathbb{V}$) with probability one.*

Proof. The proof follows from the Galerkin method and the compactness method, the procedure is very similar to the proof of Theorem 3.3 and it is even easier. We just formally derive the crucial estimates.
The application of Ito's formula for $|u|^2$ yields

$$
\begin{aligned}
|u|^2 + 2\nu \int_0^t ||u||^2 ds + 2\int_0^t \int_D (\mathbf{KD} : \mathbf{D})dxdt \leq 2\int_0^t (F(u,t),u)ds + \int_0^t |G(u,t)|^2 ds \\
+|u_0|^2 + 2\int_0^t (G(u,t),u)dW.
\end{aligned}
\tag{53}
$$

More generally

$$
\begin{aligned}
|u|^p + p\nu \int_0^t |u|^{p-2}||u||^2 ds + p\int_0^t |u|^{p-2} \int_D (\mathbf{KD} : \mathbf{D})dxdt - p\int_0^t |u|^{p-2}(F(u,t),u)ds \\
\leq (1/2)p(p-1)\int_0^t |u|^{p-2}|G(u,t)|^2 ds + |u_0|^p + p\int_0^t |u|^{p-2}(G(u,t),u)dW,
\end{aligned}
$$

for any $2 \leq p < \infty$. Thanks to the assumptions on $\mathbf{K}$, F, G we obtain that

$$
|u|^p + p\nu \int_0^t |u|^{p-2}||u||^2 ds \leq |u_0|^p + C\int_0^t |u|^p ds + p\int_0^t |u|^{p-2}(G(u,t),u)dW,
$$

Standard arguments of Martingale inequality and Gronwall's inequality yields

$$
E \sup_{0\leq t\leq T} |u(t)|^p \leq C. \tag{54}
$$

Coming back to (53) we can show that

$$
E\left(\int_0^T ||u(s)||^2 ds\right)^{p/2}. \tag{55}
$$

□

We also have the uniqueness result whose proof follows from similar arguments used in Theorem 3.4.

Theorem 4.3. *Assume that (AF)-(ASFG) hold and let u_1 and u_2 be two probabilistic weak solutions of* $\{(1),(3)\}$ *starting with the same initial condition and defined on the same stochastic basis* $(\Omega, \mathcal{F}, \mathcal{F}^t, P)$. *If we set* $v = u_1 - u_2$, *then we have* $v = 0$ *almost surely.*

4.2. **Application to the Oldroyd fluids.** The tensor σ for the Oldroyd fluids is given by

$$\left(1 + \sum_{l=1}^{L} \lambda_l \frac{\partial^l}{\partial t^l}\right) \sigma = 2\mu \left(1 + \sum_{l=1}^{L} k_l \mu^{-1} \frac{\partial^l}{\partial t^l}\right) \mathbf{D}, \;\; L =, 2, 3...., \tag{56}$$

where $\lambda_l > 0$ and $k_l > 0$ represent the relaxation and retardation times, respectively. Let

$$P_o(p) = \mu - \nu + \sum_{i=1}^{L} (k_l - \nu\lambda_l) p^l,$$

and

$$\beta_l^{(o)} = P_o(-\alpha_l)[Q'(-\alpha_l)]^{-1}.$$

The latter quantity is assumed to be positive. It is shown in [28] that the operator $\mathbf{K}$ for the Oldroyd fluids is given by

$$\mathbf{KD} = \sum_{l=1}^{L} \int_0^t \beta_l^{(o)} e^{-\alpha_l (t-\tau)} \mathbf{D}(x, \tau) d\tau,$$

and that $\mathbf{K}$ satisfies the assumption (11)-(13). Therefore the Theorems 4.2, 4.3 hold for the Oldroyd fluid provided that the assumptions on F and G (see (AF)-(ASFG)) are valid.

Remark 4.4. The Theorem (4.2) (resp. (3.3)) holds true for those viscoelastic fluids which do not satisfy the assumption (ASFG) (resp. (HYP 3)). One example we can consider is the third order fluids whose tensor is given by

$$\sigma = 2\nu \mathbf{D} + \mu \mathbf{D}^3.$$

Acknowledgment

The research of the author is supported by the University of Pretoria and the National Research Foundation South Africa.

References

[1] R. A. Adams. *Sobolev Spaces.* Academic Press, 1975.

[2] G. Astarita and G. Marucci. *Principles of Non-Newtonian Fluid Mechanics.* McGraw-Hill, London, 1974

[3] T. Becker and B. Eckhardt. Turbulence in a Maxwell fluid. *Z. Phys. B,* 101:461–468, 1996.

[4] A. Bensoussan. Some existence results for stochastic partial differential equations. In *Partial Differential Equations and Applications(Trento 1990)*, volume 268 of *Pitman Res. Notes Math. Ser.*, pages 37–53. Longman Scientific and Technical, Harlow, UK, 1992.

[5] A. Bensoussan. Stochastic Navier-Stokes Equations. *Acta Applicandae Mathematicae*, 38:267–304, 1995.

[6] A. Bensoussan and R. Temam. Equations Stochastiques du Type Navier-Stokes. *Journal of Functional Analysis*, 13:195–222, 1973.

[7] S. Berti, A. Bistagnino, G. Boffeta, A. Celani, and S. Musacchio. Two-dimensional elastic turbulence. *Physical Review E* 77(055306):1–4, 2008.

[8] R.B. Bird, R.C. Armstrong and O. Hassanger. *Dynamics of Polymeric Fluids,* Vol 1, John Wiley & Sons, New York, 1987.
[9] A.Bonito, P. Clément, and M. Picasso Mathematical analysis of a simplified Hookean dumbbells model arising from viscoelastic flows. *J. Evol. Equ.*, 6:381–398, 2006.
[10] M. Capiński and N.J. Cutland Navier-Stokes equation with multiplicative noise. *Nonlinearity,* 6:71–77, 1993.
[11] M. Capiński and N.J. Cutland Statistical solutions of stochastic Navier-Stokes equations. *Indiana Univ. Math. J.* 43(3):927–940, 1994.
[12] P. Constantin and C. Foias. *Navier-Stokes Equations.* Chicago Lectures in Mathematics. The University of Chicago Press, Chicago, 1988.
[13] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions.* Cambridge University Press, 1992.
[14] G. Da Prato and A. Debussche 2D stochastic Navier-Stokes equations with a time-periodic forcing term. *J. Dynam. Differential Equations* 20(2):301–335, 2008.
[15] G. Deugoue and M. Sango, On the stochastic 3D Navier-Stokes-alpha model of fluids turbulence. *Abstract and Applied Analysis* Volume 2009, Article ID 723236, 27 pages.
[16] G. Deugoue and M. Sango On the Strong Solution for the 3D Stochastic Leray-Alpha Model *Boundary Value Problems,* Volume 2010, Article ID 723018, 31 pages.
[17] B. Ferrario and F. Flandoli. On a stochastic version of Prouse model in fluid dynamics. *Stochastic Processes and their Applications,* 118:762–789, 2008.
[18] F. Flandoli and D Gatarek. Martingale and stationary solutions for stochastic Navier-Stokes equations. *Probab. Theory Relat. Fields*, 102:367–391, 1995.
[19] C. Foias, O. Manley, R. Rosa, and R. Temam. Navier-Stokes Equations and Turbulence, Volume 83 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, 2001.
[20] I. I. Gikhman and A. V. Skorohod. *Stochastic Differential Equations.* Springer-Verlag, 1972.
[21] M.A. Hulsen, A.P.G. van Heel, and B.H.A.A. van den Brule. Simulation of viscoelastic flows using Brownian configuration fields. *J. Non-Newtonian Fluid Mech.*, 70:79–101, 1997.
[22] B. Jourdain and T. Lelièvre Mathematical analysis of a stochastic differential equation arising in the micro-macro modelling of polymeric fluids. *Probabilistic Methods in Fluids Proceedings of the Swansea 2002 Workshop.* Davies, I.M., Jacob, N., Truman, A., Hassan, O., Morgan, K.,Weatherill, N.P. (eds.),World Scientific, pp. 205–223, 2003.
[23] B. Jourdain, T. Lelièvre, and C. Le Bris. Numerical analysis of micro-macro simulations of polymeric fluid flows: a simple case. *Math. Models Methods in Appl. Sci.* 12:1205–1243, 2002.
[24] B. Jourdain, C. Le Bris, and T. Lelièvre. On a variance reduction technique for micromacro simulations of polymeric fluids. *J. Non-Newtonian Fluid Mech.* , 122:91–106, 2004.
[25] B. Jourdain, T. Lelièvre, and C. Le Bris. Existence of solution for a micro-macro model of polymeric fluid: the FENE model. *J. Funct. Anal.* 209:162–193, 2004.
[26] B. Jourdain, C. Le Bris, T. Lelièvre, and F. Otto. Long-Time Asymptotics of a Multiscale Model for Polymeric Fluid Flows. *Arch. Rational Mech. Anal.*, 181:97–148, 2006.
[27] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus.* GTM. Springer-Verlag, 1987.
[28] N.A. Karazeeva. Solvability of initial boundary value problems for equations describing motions of linear viscoelastic fluids. *Journal of Applied Mathematics,* 1:59-80, 2005.
[29] O. Ladyzhenskaya. On global existence of weak solutions to some 2-dimensional initial-boundary value probelms for Maxwell fluids. *Applicable Analysis,* 65(3):251–255, 1997.
[30] T. Li, E. Vanden-Eijnden, P. Zhang, and W.E. Stochastic models for polymeric fluids at small Deborah number. *J. Non-Newtonian Fluid Mech.*, 121:117–125, 2004.
[31] J.L. Lions *Quelques méthodes de résolution des problèmes aus limites non linéaires.* Études Mathématiques. Dunod, 1969.
[32] W.D. McComb *The Physics of Fuid Turbulence*, Volume 25 of *Oxford Engineering Science Series.* Clarendon Press, Oxford, 1990.
[33] R. Mikulevicius and B.L. Rozovskii. Stochastic Navier-Stokes Equations and Turbulent Flows. *SIAM J. Math. Anal.*, 35(5):1250-1310, 2004.
[34] A. S. Monin and A. M. Yaglom. *Statistical fluid mechanics: mechanics of turbulence.* Volume II. Translated from the 1965 Russian original. Dover Publications, Inc., Mineola, NY, 2007.

[35] H.C. Öttinger. *Stochastic processes in polymeric fluids*. Springer-Verlag, Berlin, 1996.

[36] A. P. Oskolkov. Certain model nonstationary systems in the theory of non-Newtonian fluids. Tr. Mat. Inst. Steklova, 127:32–57, 1975.

[37] A. P. Oskolkov. On the theory of Maxwell fluids. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. inst. Steklov. (LOMI),* 101:119–127, 1981.

[38] A. P. Oskolkov. initial-boundary value problems for equations of motion of kelvin-Voight fluids and Oldroyd fluids. *Tr. Mat. Inst. Steklova,* 179:126–164, 1988.

[39] E. Pardoux. *Equations aux dérivées partielles stochastiques monotones.* Thèse de Doctorat, Université Paris-Sud, 1975.

[40] C. Prévôt and M. Röckner *A concise course on stochastic partial differential equations.* Volume 1905 of *Lecture Notes in Mathematics*, Springer, Berlin, 2007.

[41] P.A. Razafimandimby and M. Sango. Weak solution of a Stochastic model for two-dimensional second grade fluids *To appear in Boundary Value Problems*

[42] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1999.

[43] M. Sango. Magnetohydrodynamic turbulent flows : Existence results. To appear in Physica D, 2010.

[44] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Annali Mat. Pura Appl.*, 146(4):65–96, 1987.

[45] A. V. Skorokhod. *Studies in the Theory of Random Processes.* Addison-Wesley Publishing Company, 1965.

[46] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis.* North-Holland, 1979.

[47] R. Temam. *Navier-Stokes Equations and Nonlinear Functional Analysis.* SIAM, 1995.

[48] D. Vincenzi, S. Jin, E. Bodenschatz, and L.R. Collins. Stretching of polymers in Isotropic Turbulence: A Statistical Closure. *Physical Review Letters,* 98(024503):1–4, 2007.

[49] M. Viot. *Sur les solutions faibles des Equations aux dérivées partielles stochastiques non linéaires.* Thèse de Doctorat d'état, Université Paris-Sud, 1975.

Department of Mathematics and Applied Mathematics, University of Pretoria, South Africa

E-mail address, P.A. Razafimandimby: paulrazafi@gmail.com