On some Optimisation models in a Fuzzy-Stochastic environment☆,☆☆

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Abstract

This paper is on Fuzzy Stochastic Optimisation, an area that is quickly coming to the forefront of mathematical programming under uncertainty. An even stronger motivating factor for the growing interest in this area can be found in the ubiquitous nature of decision problems involving hybrid undeterminacy. More precisely, we consider a range of situations in which random factors and fuzzy information co-occur in an optimisation setting. Related hybrid optimisation models are discussed and converted into deterministic terms through appropriate tools like probabilistic set, uncertain probability, and fuzzy random variable, making good use of uncertainty principles. We also discuss ways to deal with the resulting problems. Numerical examples carried out using class optimisation software demonstrate the efficiency of the proposed approaches. We shall end this article by pointing out some of the challenges that currently occupy researchers in this emerging field.

Keywords: Fuzziness, randomness, optimisation, probability set, uncertain probability, fuzzy random variable

1. Introduction

As a result of the development in computational resources and scientific computing techniques, sophisticated optimisation models (Arnuautu et al., 2005; Pachter and Sturmfels, 2007; Zhou et al., 1996) can now be solved efficiently. Yet many optimisation applications are affected by uncertainty in input data or in model relationships.

In this connection the noted economist Shackle is quoted as saying (Shackle, 1961):

> "In a predestinate world, decision would be illusory; in a world of perfect foreknowledge, empty; in a world without order, powerless. Our intuitive attitude to life implies non-

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illusory, non-empty, non-powerless decision... Since decision in this sense excludes both perfect foresight and anarchy in nature, it must be defined as choice in the face of bounded uncertainty."

Zadeh's incompatibility principle (Dubois and Prade, 1980) also points to the importance of pondering uncertainty quantification in complex systems.

Uncertainty can be described in several ways, depending on the information at hand. Among mathematical tools for coping with uncertainty we mention worst case scenario analysis, evidence theory, probability theory, fuzzy set theory, etc. Accessible accounts of these tools may be found in Sakawa (1993), Shafer (1976), Shiryaer (1996), and in references therein.

While research has progressed at a steady pace in the fields of stochastic optimisation (Kall, 1976; Kall and Wallace, 1994; Schultz and Tiedemann, 2006; Vajda, 1972; Wagner, 2008) and fuzzy mathematical programming (Bhaskar et al., 2009; Lai and Hwang, 1992; Luhandjula, 1989; Zimmerman, 1976), the past decade, in particular, has witnessed a developing interest in situations where fuzziness and randomness are under one "roof" in an optimisation framework (Liu, 2001; Luhandjula, 2004; Luhandjula and Gupta, 1996; Nanda et al., 2006; Van Hop, 2007a,b). This interest has been motivated by the need for basing many human decisions on information which is both fuzzily imprecise and probabilistically uncertain (Katagiri et al., 2005; Zmeškal, 2001).

The multidisciplinary research field (Luhandjula, 2006) that emerged as a result of this interest lies at the boundary of stochastic optimisation and fuzzy mathematical programming. An interested reader is referred to Luhandjula (2004, 2006), where the spectrum of research activities in this field and the richness of ideas in development of theory, algorithms and applications in fuzzy stochastic optimisation are described.

The objective of this paper is threefold. Firstly it aims to discuss some modelling issues in the area of optimisation under hybrid uncertainty. Secondly it seeks to explore properties of probabilistic sets, uncertain probabilities and fuzzy random variables in a way to convert hybrid uncertain optimisation models into their deterministic counterparts. Finally, it gives some hints for procedures to solve the resulting problems.

We focus on three chosen models in the rich array of situations where randomness and fuzziness are under one roof in an optimisation setting. Namely, a flexible program under randomness (P_1); a mathematical program with random variables having fuzzy parameters (P_2); and an optimisation problem with fuzzy random variable coefficients (P_3). The reason for this choice are as follows. Firstly, many real-life problems may be cast into these forms. Secondly, these models have not received the attention they deserve.

Model (P_1) has been considered only for the discrete case (Luhandjula, 1983). Moreover, the problem solving approach used is questionable as it is based on the controversial notion of crisp probability of a fuzzy event. Model (P_2) is scarcely seen in the literature due to the fact that the notion of uncertain probability has been introduced only recently (Buckley, 2004; Buckley and Eslami, 2003, 2004). Model (P_3) has been approached through the use of measures like the fuzzy expected value or the fuzzy variance that summarises some of the characteristics of involved fuzzy random variables. These approaches have the inconvenience of not using exhaustively the information contained in these fuzzy random variables.

The outline of the paper is as follows. Section 2 provides all the necessary "ingredients" that are used in subsequent developments, including the formulation of uncertain optimisation problems. Section 3 is devoted to methodological approaches for converting uncertain optimisation problems under scrutiny into deterministic terms along with convincing arguments as to why the proposed approaches are appropriate. Numerical examples are also supplied for the sake of illustration. We shall end this paper with some concluding remarks, along with some challenges in the emerging field of fuzzy stochastic optimisation.

2. Background and problem formulation

2.1. Mathematical tools for combining fuzziness and randomness

To deal mathematically with situations where fuzziness and randomness are part of the state of affairs, finer webs of ideas from probability and fuzzy set theories have been pieced together, culminating in such notions as: probability of a fuzzy event (Zadeh, 1968), linguistic probabilities (Dubois and Prade, 1980), random fuzzy variable (Liu, 2007), fuzzy random variable (Kwakernaak, 1978, 1979), probabilistic set (Hirota, 1981) and uncertain probabilities (Buckley, 2004; Buckley and Eslami, 2003, 2004). In what follows, we shall briefly discuss the last three notions that are needed in the sequel. Before proceeding further we need to lay out the notion of a fuzzy number, which is central in subsequent developments.

2.1.1. Fuzzy number

A fuzzy number is a normal and convex fuzzy set of $\mathbb{R}$. A fuzzy number is well suited to represent a vague datum (Dubois and Prade, 1980). For instance, the vague datum: *"close to m"* can be represented by the fuzzy number whose membership function μ is given in Figure 1. In the case of Figure 1, we have a triangular fuzzy number denoted by $\langle \underline{m}/m/\bar{m} \rangle$.

2.1.2. Probabilistic set

A helpful tool for quantifying and analysing situations where fuzziness and randomness are part of the state of affairs is that of a probabilistic set (Dubois and Prade, 1980). Given a probability space

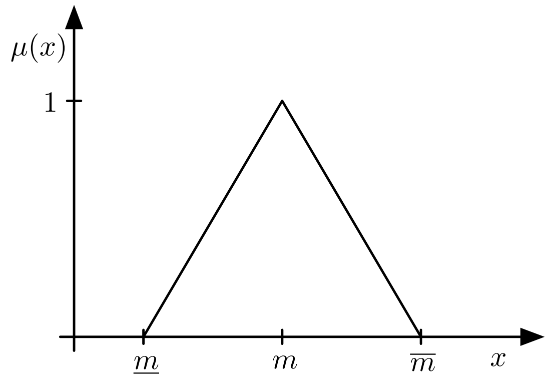

Figure 1: Membership function of the vague datum: *"close to m"*

$(\Omega, \mathscr{F}, P)$, a probability set B on X is defined as a fuzzy set of $X \times \Omega$ whose membership function $B(x, \cdot)$ is measurable.

The n-dimensional distribution function of a probabilistic set B for any elements $x_1, x_2, \ldots, x_n$ of X is given by:

$$F_{B(x_1)B(x_2)\ldots B(x_n)}(z_1, \ldots, z_n) = P\left(B(x_1, \omega) < z_1, \ldots, B(x_n, \omega) < z_n\right).$$

Set theoretic operations have been extended to probabilistic sets in a straightforward way. Moreover, a moment analysis can be performed on probabilistic sets (Hirota, 1981), which is extremely advantageous for applications.

2.1.3. Uncertain probabilities

An uncertain probability (Buckley and Eslami, 2003, 2004) is a probability distribution in which some parameters are not known with precision and are modelised by fuzzy numbers. Using fuzzy arithmetic, basic laws of uncertain probabilities have been obtained (Buckley, 2004).

To put things in context, consider a continuous random variable X with probability density function $f(x, \theta)$, where θ is a parameter describing the density function. If θ can be generated as a fuzzy number $\tilde{\theta}$, then X has density $f(x, \tilde{\theta})$ and the probability of the event: "X is between c and d" is a fuzzy set whose α-cut is defined as follows:

$$\left[\widetilde{P}(c \leq X \leq d)\right]^{\alpha} = \left\{ \int_{c}^{d} f(x, \theta) dx | \theta \in \tilde{\theta}^{\alpha}; \int_{-\infty}^{\infty} f(x, \theta) dx = 1 \right\}$$

The first two moments of X are defined through its α-cuts as follows:

$$\tilde{m}_X^{\alpha}(\theta) = \left\{ \int_{-\infty}^{\infty} x f(x,\theta) dx | \theta \in \tilde{\theta}^{\alpha}; m_X(\theta) \in \tilde{m}_X^{\alpha}(\theta); \int_{-\infty}^{\infty} f(x,\theta) dx = 1 \right\}$$

$$\tilde{\sigma}_X^{2\alpha}(\theta) = \left\{ \int_{-\infty}^{\infty} \left(x - m_X(\theta)\right)^2 f(x,\theta) dx | \theta \in \tilde{\theta}^{\alpha}; m_X(\theta) \in \tilde{m}_X^{\alpha}(\theta); \sigma_X^2(\theta) \in \tilde{\sigma}_X^{2\alpha}(\theta); \int_{-\infty}^{\infty} f(x,\theta) dx = 1 \right\}$$

Random variables $X_i, i = 1, 2, \ldots, n$ with fuzzy parameters and having joint density function $f(x_1, \ldots, x_n; \tilde{\theta})$ and marginal density function $f(x_i, \tilde{\theta})$ are said to be independent if, for $\alpha \in (0, 1]$ and for all $\theta \in \tilde{\theta}$,

$$f(x_1, x_2, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i, \theta).$$

An interested reader is advised to consult Buckley and Eslami (2003, 2004) and Buckley (2004) for more details on discrete and continuous uncertain probabilities.

2.1.4. Fuzzy random variable

A fuzzy random variable (frv) on a probability space $(\Omega, \mathscr{F}, P)$ is a fuzzy-valued function:

$$X : \Omega \to \mathscr{F}_0(\mathbb{R})$$
$$\omega \rightsquigarrow X_\omega$$

such that for every Borel set $\boldsymbol{A}$ of $\mathbb{R}$ and for every $\alpha \in (0, 1], (X^{\alpha})^{-1}(\boldsymbol{A}) \in \mathscr{F}$. Here $\mathscr{F}_0(\mathbb{R})$ and X^{α} stand respectively for the set of fuzzy numbers and the set-valued function:

$$X^{\alpha} : \Omega \to 2^{\mathbb{R}}$$
$$\omega \rightsquigarrow X_\omega^{\alpha} = \left\{x \in \mathbb{R} | X_\omega(x) \geq d\right\}.$$

As an example of frv, consider an opinion poll in which individuals are asked to characterise weather in Europe for the following summer. To this end, categories such as "warm", "almost cold", "very windy", etc. might be used. There is some stochastic variation due to the difficulty in predicting, with precision, the occurrence of the above mentioned categories of weather. Fuzziness comes into play because the above categories are subjective and only partially quantifiable.

Zadeh's decomposition principle (Dubois and Prade, 1980) extends canonically to frvs, i.e. if X is an frv, then:

$$X = \bigcup_{\alpha \in (0,1]} \alpha X^{\alpha}$$

A question that immediately arises is: what does X^α look like? A superb reply to this question is given in (Colubi et al., 2001). It tells us that, for a frv X, $X^\alpha(\alpha \in (0,1])$ are random intervals. Limit theorems have also been obtained for frvs (Kruse and Meyer, 1987).

It is also worth mentioning that under mild assumptions, frvs possess the Radon-Nikodym property, i.e. any P-continuous fuzzy measure of bounded variation has a Radon-Nikodym derivative which is an integrably bounded frv (Bán, 1990).

A considerable body of literature has grown out of the concept of frv in a wide range of fields. An interested reader may consult Colubi et al. (2001) and Kratschmer (2001) for details on these matters.

The following partial order has been defined on the set of fuzzy random variables (Colubi et al., 2001). Given two fuzzy random variables X and Y on $(\Omega, \mathscr{F}, P)$, we have:

$$X \leq Y \iff X^\alpha \leq Y^\alpha \qquad \forall \alpha \in (0,1].$$

2.2. Problem formulation

The problem we address in this paper is how to let randomness and fuzziness be taken into account in optimisation problems, along with ways in which a decision must be reached in such a mixed environment. We shall focus on three hybrid problems mentioned in the Introduction. For a state of the art in Fuzzy Stochastic Optimisation, the reader is referred to Mohan and Nguyen (1997) and Luhandjula (2004, 2006).

We shall restrict ourselves to linear optimisation problems so that the main ideas are illustrated in a simpler context. Our point of departure is the standard linear program:

$$(P) \quad \begin{cases} \min cx \\ Ax \leq b \\ x \geq 0 \end{cases}$$

where A, b and c are respectively a $m \times n$, a $m \times 1$ and a $1 \times n$ matrix and $x \in \mathbb{R}^n$.

A first extension of (P) to be considered in this paper is obtained by assuming that the components of A, b and c are no longer deterministic but random variables with known distribution. Moreover, the objective and constraints are not strict imperatives. Some leeway may be accepted in their fulfilment. This leads to the following linear program under hybrid uncertainty:

$$(P_1) \quad \begin{cases} \tilde{\min}\ \bar{c}x \\ \bar{A}_i x \lesssim \bar{b}_i ; i = 1, \ldots, m \\ x \in X = \{x \in \mathbb{R}^n | x \geq 0\} \end{cases}$$

where "$\sim$" means flexible and " $^-$ " means that the datum is random. This model has been addressed in the literature (Luhandjula, 1983) for the case involving random variables with discrete distributions. Here the model is extended to incorporate continuous random variables.

Another direction toward the generalisation of (P) is that of supposing that input data in the objective function are random while those in the constraints are random variables with fuzzy parameters (see § 2.1.3). The corresponding problem reads:

$$(P_2) \quad \begin{cases} \min \ \bar{c}x \\ A_i^\star x \leq b_i^\star; i = 1, \ldots, m \\ x \in X = \{x \in \mathbb{R}^n | x \geq 0\} \end{cases}$$

where "$\star$" means the datum is random with some fuzzy parameters. Model (P_2) has many applications. As a matter of fact, in many real-life problems that may be cast into an optimisation framework, experts who provide data may feel more comfortable in coupling their vague perception with hard statistical data. To this end, they may prefer to represent them in the form of random variables with vague parameters. As an example, consider a portfolio selection problem where, due to stock experts' judgements and investors' different opinions, the security returns are modelled as random variables with imprecise parameters.

This model has not received attention in the literature. This is due to the lack of tools for coping with probability distributions involving fuzzy parameters. These tools appeared only recently (Buckley, 2004; Buckley and Eslami, 2003, 2004).

The third generalisation of (P) that is also discussed in this paper has the form:

$$(P_3) \quad \begin{cases} \min \ c^{\star\star}x \\ A_i^{\star\star}x \boxed{\leq} b_i^{\star\star}; i = 1, \ldots, m \\ x \in X = \{x \in \mathbb{R}^n | x \geq 0\} \end{cases}$$

where components of the vectors $A_i^{\star\star}(i = 1, \ldots, m)$, $c^{\star\star}$ as well as $b_i^{\star\star}(i = 1, \ldots, m)$ are fuzzy random variables on a probabilistic space $(\Omega, \mathscr{F}, P)$, and '$\boxed{\leq}$' means the inequality is between fuzzy random variables. It is worth mentioning that (P_3) includes purely stochastic and purely fuzzy linear programs as special cases.

Existing approaches (Luhandjula, 2004; Van Hop, 2007a,b) consider some measures like the fuzzy expected value, the fuzzy variance or some indices that summarise some of the characteristics of the involved fuzzy random variables. These approaches have the drawback of not exploiting all the information contained in these fuzzy random variables.

3. Satisfying solutions to $(\mathbf{P_1})$–$(\mathbf{P_3})$

Mathematical programs (P_1)–(P_3) share the common feature of being tainted with twofold imprecision. This turns out to be a source of substantive difficulties. As a matter of fact, for these optimisation problems, both the notion of *optimum optimorum* and the *pure rationality* principle no longer apply.

One has therefore to resort to Simon's bounded rationality principle (Simon, 1945) and seek for a satisfying solution (in some sense) instead of an optimal one. In what follows we provide for each of the three optimisation problems:

- a possibilistic-probabilistic transformation that converts the problem into either a deterministic or a purely stochastic or a purely fuzzy problem;
- an algorithmic approach for efficiently solving the resulting problem; and
- a numerical example for the sake of illustration.

3.1. Satisfying solution to $(\mathbf{P_1})$

Here we opt for the symmetrical approach of considering objective and constraints as the same concept. After having put the objective function of (P_1) in the following constraint form:

$$cx \lesssim c_0$$

where c_0 is a threshold fixed by the decision-maker, the problem (P_1) reads merely:

$$\text{Find } x \in X \text{ such that:}$$
$$\bar{A}_i x \lesssim \bar{b}_i; i = 0, 1, 2, \ldots, m \tag{1}$$

where $\bar{A}_0 = \bar{c}$ and $\bar{b}_0 = c_0$.

Each constraint i of system (1) is then represented as a probabilistic set on $(\Omega, \mathscr{F}, P)$ with membership function $\mu_i(x, \omega)$. The interpretation that goes with this construction is to view $\mu_i(x, \omega)$ as the level to which the constraint:

$$\bar{A}_i(\omega)x \leq \bar{b}_i(\omega); (\omega \in \Omega)$$

is satisfied.

The following simple kind of piecewise linear function may be used for $\mu_i(x, \omega)$:

$$\mu_i(x,\omega) = \begin{cases} 1 & \text{if } \bar{A}_i(\omega)x \leq \bar{b}_i(\omega) \\ 1 - \frac{\bar{A}_i(x,\omega)x - \bar{b}_i(\omega)}{d_i} & \text{if } \bar{b}_i(\omega) < \bar{A}_i(\omega)x \leq \bar{b}_i(\omega) + d_i \\ 0 & \text{if } \bar{A}_i(\omega)x > \bar{b}_i(\omega) + d_i \end{cases}$$

where $d_i > 0$ is a constant chosen by the decision-maker for a permitted violation of constraint i. Other kinds of membership functions, more appropriate to the situation at hand, may be used instead of the above piecewise linear functions.

According to Bellman-Zadeh's confluence principle (Bellman and Zadeh, 1970), a decision in a fuzzy environment is an option that is at the intersection of fuzzy goals and fuzzy constraints. Therefore, a satisfying solution to (P_1) should be a solution of the following stochastic optimisation problem:

$$(P_1)' \quad \begin{cases} \max \ \mu_D(x,\omega) \\ x \in X \cap \ \text{Support} \ \ \mu_D \end{cases}$$

where

$$\mu_D(x,\omega) = \min_{i=0,1,\dots,m} \mu_i(x,\omega).$$

When we now perform the probabilistic transformation consisting of taking the expectation of $\mu_D(x,\omega)$, we get the following deterministic problem:

$$(P_1)'' \quad \begin{cases} \max E\mu_D(x,\omega) \\ x \in X \cap \ \text{Support} \ \mu_D \end{cases}$$

To handle $(P_1)''$, we need an analytical expression of the distribution of $\mu_D(x,\omega)$ denoted by $F_{\mu_D(x,\omega)}$. Such an expression is given below, depending on the operator chosen for the meaning of the connective: "and".

Theorem 1.

(a) *If*

$$\mu_D(x,\omega) = \min_{i=0,1,\dots,m} \mu_i(x,\omega)$$

then

$$F_{\mu_D(x,\omega)}(z) = \sum_{i=0}^{m} F_{\mu_i(x,\omega)}(z) - \sum_{j<k} F_{\mu_j(x,\omega)\mu_k(x,\omega)}(z,z) + \sum_{i<j<k} F_{\mu_i(x,\omega)\mu_j(x,\omega)\mu_k(x,\omega)}(z,z,z)$$
$$+ \dots (-1)^{m+2} F_{\mu_0(x,\omega)\mu_1(x,\omega)\dots\mu_m(x,\omega)}(z,\dots,z) \quad (2)$$

(b) *If*

$$\mu_D(x,\omega) = \gamma \min_i \mu_i(x,\omega) + (1-\gamma) \min\left(1, \sum_i \mu_i(x,\omega)\right)$$

where γ stands for a coefficient of compensation ranging between 0 and 1, then

$$F_{\mu_D(x,\omega)}(z) = L^{-1}\Big(g_1(s).g_2(s)\Big) \tag{3}$$

where

$$g_1(s) = \int_0^\infty e^{-\left(\frac{s}{\gamma}\right)z} F_{\min_i \mu_i(x,\omega)}(z)dz$$
$$g_2(s) = \int_0^\infty e^{-\left(\frac{s}{1-\gamma}\right)z} F_{\min\left(1,\sum_i \mu_i(x,\omega)\right)}(z)dz$$

and L^{-1} stands for the inverse of the Laplace transform.

The proof of this theorem is given in the Appendix.

3.1.1. Finding a satisfying solution of (P_1)

The above result may be exploited to find a solution of (P_1). To this end one may proceed as follows.

Step 1: Put the objective function in the form of a flexible constraints.

Step 2: Consider the system (1) and define the membership function $\mu_i(x, \omega)$ for each i.

Step 3: Find $F_{\mu_D(x,\omega)}$ analytically using Theorem 1. Alternatively, use a hybrid procedure integrating a simulation technique and a neural network to approximate $F_{\mu_D(x,\omega)}$.

Step 4: Solve $(P_1)''$ using a genetic algorithm.

Step 5: Print the solution obtained in Step 4.

Step 6: Stop.

3.1.2. Numerical example

Consider the following flexible linear program with random coefficients.

$$(P_1)''' \quad \begin{cases} \tilde{\min} \; -3x_1 - 1.5x_2 - x_3 \\ \text{subject to} \\ 8x_1 + 6x_2 + x_3 \lesssim \bar{b}_1 \\ 4x_1 + 2x_2 + 1.5x_3 \lesssim \bar{b}_2 \\ 2x_1 + 1.5x_2 + 0.5x_3 \lesssim \bar{b}_3 \\ x_i \geq 0; i = 1, 2, 3 \end{cases}$$

where $\bar{b}_1$, $\bar{b}_2$ and $\bar{b}_3$ are $N(48,5)$, $N20,3)$ and $N(8,1)$ random variables respectively. Solving the corresponding expected value model with crisp objective function and crisp constraints relationships yields, as optimal value of the objective function, -16.11. We then take $\bar{b}_0$ as a $N(-16.11,1)$ random variable. It is worth noticing that the standard deviation of $\bar{b}_0$ has been taken as the minimum of standard deviations of $\bar{b}_1$, $\bar{b}_2$ and $\bar{b}_3$. $(P_1)'''$ now merely reads:

$$\text{Find } x \in X = \{x \in \mathbb{R}^3 | x \geq 0\} \text{ such that:}$$
$$A_0(x) = -3x_1 - 1.5x_2 - x_3 \lesssim \bar{b}_0$$
$$A_1(x) = 8x_1 + 6x_2 + x_3 \lesssim \bar{b}_1$$
$$A_2(x) = 4x_1 + 2x_2 + 1.5x_3 \lesssim \bar{b}_2$$
$$A_3(x) = 2x_1 + 1.5x_2 + 0.5x_3 \lesssim \bar{b}_3$$

Using piecewise linear functions defined in §3.1.1 to represent constraints $A_i(x) \lesssim \bar{b}_i (i = 0,1,2,3)$ (with $d_0 = 5$; $d_1 = 5$; $d_2 = 3$; $d_3 = 1$) and discretising $\bar{b}_i (i = 0,1,2,3)$ with 5 equidistant partitions as indicated in Table 1, produces the following counterpart of $(P_1)''$:

$$(P_1)^{IV} \quad \begin{cases} \max \sum_{r_i} \left(\prod_{i \in \{0,1,2,3\}} p^{r_i} \right) \mu_D^{r_0,r_1,r_2,r_3}(x,\omega) \\ \text{subject to} \\ A_i x \leq \bar{b}_i^{r_i} + d_i; i = 0,1,2,3 \\ x_i \geq 0; i = 1,2,3 \end{cases}$$

where p^{r_i} denotes the probability that the r_i^{th} realisation of the right-hand side of constraint i is evaluated, and $\mu_D^{r_0,r_1,r_2,r_3}(x,\omega)$ denotes the unique instance of $\mu_D(x,\omega)$ when the r_0^{th} realization of $\bar{b}_0$, the r_1^{th} realisation of $\bar{b}_1$, the r_2^{th} realisation of $\bar{b}_2$ and the r_3^{th} of $\bar{b}_3$ are concerned.

Implementing $(P_1)^{IV}$ in *GAMS* as a mixed integer nonlinear problem and solving with *BARON* version 8.1.4 solver produces the solution: $x_1 = 4.51$, $x_2 = 0$, $x_3 = 0.430$ which is a satisfying solution to $(P_1)'''$.

The approach described here departs strongly from the one described in Luhandjula (1983). As a matter of fact, it is based on the Bellman-Zadeh confluence principle (Bellman and Zadeh, 1970) and permits the integration of continuous random variables.

3.2. Satisfying solution to $(\mathbf{P_2})$

3.2.1. Analysis

To convert (P_2) in deterministic form, we use a fuzzified version of the well-known chance-constrained programming approach (Kall, 1976). In this case, a deterministic counterpart of (P_2) is obtained

Table 1: Discrete approximations for right-hand side values, $\bar{b}_i|i>0$

i		**Realisation of $\mathbf{r_i}$**				
		1	2	3	4	5
0	p^{r_0}	0.017	0.220	0.524	0.223	0.016
	$\bar{b}_0^{r_0}$	-18.987	-17.547	-16.107	-14.667	-13.227
1	p^{r_1}	0.013	0.201	0.536	0.235	0.015
	$\bar{b}_1^{r_1}$	33.105	40.477	47.849	55.222	62.594
2	p^{r_2}	0.014	0.252	0.566	0.163	0.005
	$\bar{b}_2^{r_2}$	10.900	15.721	20.542	25.363	30.184
3	p^{r_3}	0.005	0.121	0.495	0.343	0.036
	$\bar{b}_1^{r_3}$	4.674	6.135	7.597	9.058	10.519

through the following problem:

$$(P_2)' \quad \begin{cases} \min \ cx \\ \tilde{P}\left(\sum\limits_{j=1}^{n} a_{ij}^{\star} x_j \leq b_i^{\star}\right) \geq \tilde{\delta}_i; i = 1,\ldots,m \\ x \in X = \{x \in \mathbb{R}^n | x \geq 0\} \end{cases}$$

where $\tilde{P}$ stands for uncertain probability, $\tilde{\delta}_i (i = 1,\ldots,m)$ are fuzzy numbers used as thresholds and $c = E(\bar{c})$. We shall consider three cases.

Case 1: $b_i^{\star}(i = 1,\ldots,m)$ are real numbers denoted merely by $b_i(i = 1,\ldots,m)$. This means that for all i, $b_i^{\star}$ is regarded as a (one-valued) random variable having a set of parameters that consists of a (crisp) singleton set. We further assume that, for all (i,j), $a_{ij}^{\star}$ is a normally distributed random variable with fuzzy number mean $\tilde{m}_{ij}$ and fuzzy number variance $\tilde{\sigma}_{ij}^2$. Then $\tilde{\mu}_i = \sum\limits_{j=1}^{n} a_{ij}^{\star} x_j$ is also a random variable whose mean and variance are fuzzy numbers $\tilde{m}_{\mu_i}$ and $\tilde{\sigma}_{\mu_i}^2$ respectively (Buckley, 2004). As $\tilde{\delta}_i$, $\tilde{m}_{\mu_i}$ and $\tilde{\sigma}_{\mu_i}^2$ are fuzzy numbers, their α-cuts are real intervals. Let us denote them as follows:

$$\begin{aligned} \tilde{\delta}_i^{\alpha} &= \left[\delta_i^{\alpha L}, \delta_i^{\alpha U}\right], \\ \tilde{m}_{\mu_i}^{\alpha} &= \left[m_{\mu_i}^{\alpha L}, m_{\mu_i}^{\alpha U}\right], \text{ and} \\ \tilde{\sigma}_{\mu_i}^{2\alpha} &= \left[\sigma_{\mu_i}^{2\alpha L}, \sigma_{\mu_i}^{2\alpha U}\right]. \end{aligned}$$

The following result provides a deterministic counterpart of $(P_2)'$.

Theorem 2.

Under the above assumptions on $b_i^{\star}$ and $a_{ij}^{\star}$ and if $a_{ij}^{\star}(i = 1,\ldots,n)$ are independent, then $(P_2)'$

is equivalent to the following mathematical program:

$$(P_2)'' \quad \begin{cases} \min \ cx \\ \Phi\left(\frac{b_i - m^{\alpha U}_{\mu_i}}{\sigma^{\alpha U}_{\mu_i}}\right) \geq \delta^{\alpha U}_i \quad \forall \alpha \in (0,1] \ ; \ i = 1, \ldots, m \\ x \in X = \{x \in \mathbb{R}^n | x \geq 0\} \end{cases}$$

where Φ *is the cumulative distribution of the normal 0-1.*

Proof

Let

$$\Delta_1 = \left\{ (m_{\mu_i}, \sigma_{\mu_i}) \left| m_{\mu_i} \in \tilde{m}^{\alpha}_{\mu_i}, \sigma^2_{\mu_i} \in \tilde{\sigma}^{2\alpha}_{\mu_i} \text{ and } \frac{1}{\sigma_{\mu_i}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(\mu_i - m_{\mu_i})^2}{2\sigma^2_{\mu_i}}} d_{\mu_i} = 1 \right. \right\}$$

We have:

$$\left[\tilde{P}(\tilde{\mu}_i \leq b_i)\right]^{\alpha} = \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b_i} e^{-\frac{(\mu_i - m_{\mu_i})^2}{2\sigma^2_{\mu_i}}} d_{\mu_i} \right| (m_{\mu_i}, \sigma_{\mu_i}) \in \Delta_1 \bigg\}.$$

By the change of variable:

$$z = \frac{\mu_i - m_{\mu_i}}{\sigma_{\mu_i}},$$

we obtain

$$\begin{aligned} \left[\tilde{P}(\tilde{\mu}_i \leq b_i)\right]^{\alpha} &= \left\{ \frac{1}{\sigma_{\mu_i}\sqrt{2\pi}} \int_{-\infty}^{\frac{b_i - m_{\mu_i}}{\sigma_{\mu_i}}} e^{-\frac{z^2}{2}} dz \right| (m_{\mu_i}, \sigma_{\mu_i}) \in \Delta_1 \bigg\} \\ &= \left\{ \Phi\left(\frac{b_i - m_{\mu_i}}{\sigma_{\mu_i}}\right) \right| m_{\mu_i} \in \tilde{m}^{\alpha}_{\mu_i}, \sigma^2_{\mu_i} \in \sigma^{2\alpha}_{\mu_i} \bigg\}. \end{aligned}$$

Since Φ is an increasing and continuous function, we have:

$$\left[\tilde{P}(\tilde{\mu}_i \leq b_i)\right]^{\alpha} = \left[\Phi\left(\frac{b_i - m^{\alpha U}_{\mu_i}}{\sigma^{\alpha U}_{\mu_i}}\right), \Phi\left(\frac{b_i - m^{\alpha L}_{\mu_i}}{\sigma^{\alpha L}_{\mu_i}}\right) \right].$$

Making use of the facts that:

$$\tilde{\delta}^{\alpha}_i = \left[\delta^{\alpha L}_i, \delta^{\alpha U}_i\right]$$

and for two fuzzy numbers $\tilde{A}$ and $\tilde{B}$,

$$\tilde{A} \leq \tilde{B} \iff A^{\alpha} \leq B^{\alpha} \quad \forall \alpha \in (0,1],$$

the constraints of $(P_2)'$ read:

$$\Phi\left(\frac{b_i - m^{\alpha U}_{\mu_i}}{\sigma^{\alpha U}_{\mu_i}}\right) \geq \delta^{\alpha U}_i \qquad \forall \alpha \in (0,1] \ ; \ i = 1, \ldots, m$$

$$x \in \boldsymbol{X}.$$

$\square$

Therefore $(P_2)'$ and $(P_2)''$ are equivalent.

Case 2: $a^\star_{ij}(i = 1, \ldots, m; j = 1, \ldots, n$ are real numbers).

We further assume that, for all i, $b^\star_i$ is a normally distributed random variable whose mean and variance are $\tilde{m}_{b_i}$ and $\tilde{\sigma}^2_{b_i}$ respectively. The α-cuts of $\tilde{m}_{b_i}$ and $\tilde{\sigma}^2_{b_i}$ are, respectively,

$$\tilde{m}^\alpha_{b_i} = \left[m^{\alpha L}_{b_i}, m^{\alpha U}_{b_i}\right]$$

and

$$\tilde{\sigma}^{2\alpha}_{b_i} = \left[\sigma^{2\alpha L}_{b_i}, \sigma^{2\alpha U}_{b_i}\right].$$

The following result provides a crisp counterpart of (P_2) through $(P_2)'$.

Theorem 3. *Under the above assumptions on $a^\star_{ij}$ and $b^\star_i$ and if in addition $b^\star_i(i = 1, \ldots, m)$ are independent, then $(P_2)'$ is equivalent to the following mathematical program:*

$$(P_2)''' \quad \begin{cases} \min \; cx \\ \Phi\left(\frac{\sum_{j=1}^m a_{ij}x_j - m^{\alpha U}_{b_i}}{\sigma^{2\alpha U}_{b_i}}\right) \leq 1 - \delta^{\alpha U}_i \quad \forall \alpha \in (0,1] \; ; \; i = 1, \ldots, m \\ x \in X = \{x \in \mathbb{R}^n | x \geq 0\} \end{cases}$$

The proof of this theorem is similar to that of Theorem 2 and is omitted for the sake of space.

Case 3: General case

Here we assume that both $a^\star_{ij}$ and $b^\star_i$ are random variables with fuzzy parameters. Put

$$\xi^\star_i(x) = \sum_{j=1}^n a^\star_{ij}x_j - b^\star_i.$$

Then $\xi^\star_i(x)$ is also normally distributed with fuzzy means $\tilde{m}_{\xi_i(x)}$ and fuzzy variance $\tilde{\sigma}^2_{\xi_i(x)}$ (Buckley, 2004). We are now in a position to state and prove the following result that gives a crisp counterpart of (P_2) (through $(P_2)'$) for the general case.

Theorem 4.

If the above mentioned assumptions on $a^\star_{ij}$ and $b^\star_i$ are satisfied and if $a^\star_{ij}$ and $b^\star_i$ are independent then $(P_2)'$ is equivalent to the program:

$$(P_2)^{IV} \begin{cases} \min cx \\ \Phi\left(\frac{-m^{\alpha U}_{\xi_i(x)}}{\sigma^{\alpha U}_{\xi_i(x)}}\right) \geq \delta^{\alpha U}_i \quad \forall \alpha \in (0,1] \; ; \; i = 1, \ldots, m) \\ x \in \boldsymbol{X} \end{cases}$$

Proof

We have that:

$$\begin{aligned}
\left[\tilde{P}\left(\sum a_{ij}^\star x_j \le b_i^\star\right)\right]^\alpha &= \left[\tilde{P}\left(\xi_i^\star(x) \le 0\right)\right]^\alpha \\
&= \left\{P\left(\xi_i(x) \le 0\right) \middle| m_{\xi_i(x)} \in \tilde{m}_{\xi_i(x)}^\alpha ; \sigma_{\xi_i(x)}^2 \in \tilde{\sigma}_{\xi_i(x)}^{2\alpha}\right\} \\
&= \left\{P\left(\frac{\xi_i(x) - m_{\xi_i(x)}}{\sigma_{\xi_i(x)}}\right) \le \frac{-m_{\xi_i(x)}}{\sigma_{\xi_i(x)}} \middle| m_{\xi_i(x)} \in \tilde{m}_{\xi_i(x)}^\alpha ; \sigma_{\xi_i(x)}^2 \in \tilde{\sigma}_{\xi_i(x)}^{2\alpha}\right\} \\
&= \left\{\Phi\left(\frac{-m_{\xi_i(x)}}{\sigma_{\xi_i(x)}}\right) \middle| m_{\xi_i(x)} \in \tilde{m}_{\xi_i(x)}^\alpha ; \sigma_{\xi_i(x)}^2 \in \tilde{\sigma}_{\xi_i(x)}^{2\alpha}\right\} \\
&= \left[\Phi\left(\frac{-m_{\xi_i(x)}^{\alpha U}}{\sigma_{\xi_i(x)}^{\alpha U}}\right), \Phi\left(\frac{-m_{\xi_i(x)}^{\alpha L}}{\sigma_{\xi_i(x)}^{\alpha L}}\right)\right].
\end{aligned}$$

So

$$\left[\tilde{P}\left(\sum \tilde{a}_{ij} x_j \le \tilde{b}_i\right)\right]^\alpha \ge \tilde{\delta}_i^\alpha$$

is equivalent to

$$\Phi\left(\frac{-m_{\xi_i(x)}^{\alpha U}}{\sigma_{\xi_i(x)}^{\alpha U}}\right) \ge \delta_i^{\alpha U}.$$

□

This establishes equivalence between $(P_2)'$ and $(P_2)^{IV}$.

It is worth mentioning that the results for the above three cases are consistent with those obtained for corresponding stochastic programming models, when random variables with fuzzy parameters become usual random variables (Kall and Wallace, 1994).

3.2.2. Algorithm for finding a satisfying solution to (P_2)

The following procedure, which can be implemented effortlessly, can produce a satisfying solution to (P_2). The basic idea behind this procedure is to choose the appropriate deterministic counterpart of (P_2) (see Theorems 1, 2 and 3) and to apply a cutting-plane method described in Luhandjula et al. (1992), called Cut-Algo, to find a solution to the resulting semi-infinite program. The steps of the procedure are as follows:

Step 1: Read data of (P_2).

Step 2: Check whether the assumptions of case 1 are met. If the above mentioned assumptions are not met, go to Step 4.

Step 3: Solve $(P_2)''$ using Cut-Algo. Go to Step 8.

Step 4: Check whether the assumptions of case 2 are met. If not, go to Step 6.

Step 5: Solve $(P_2)'''$ using Cut-Algo. Go to Step 8.

Step 6: Check whether the assumptions of case 3 are met. If not, go to Step 9

Step 7: Solve $(P_2)^{IV}$ using Cut-Algo Go to Step 8.

Step 8: Print the solution obtained.

Step 9: Stop

3.2.3. Numerical example

Consider the linear program:

$$(P_2)^V \quad \begin{cases} \min\ 2x_1 + 3x_2 \\ 2x_1 + 2x_2 \leq \tilde{b} \\ 3x_1 - x_2 \geq 4 \\ x_1, x_2 \geq 0 \end{cases}$$

where $\tilde{b}$ is a normally distributed random variable with mean and variance triangular fuzzy numbers $\tilde{m}_b = \langle 5, 6, 7 \rangle$ and $\tilde{\sigma}_b^2 = \langle 3/4/5 \rangle$ respectively. If we take as threshold the triangular number $\tilde{0.3} = \langle 0.2/0.3/0.4 \rangle$, we have the following counterpart of $(P_2)^V$:

$$(P_2)^{VI} \quad \begin{cases} \min\ 2x_1 + 3x_2 \\ \tilde{P}\left(2x_1 + 2x_2 \leq \tilde{b}\right) \geq \tilde{0.3} \\ 3x - x_2 \geq 4 \\ x_1, x_2 \geq 0 \end{cases}$$

After transformations, this program reads:

$$(P_2)^{VII} \quad \begin{cases} \min\ 2x_1 + 3x_2 \\ \frac{2x_1 + 2x_2 - 5 - \alpha}{\sqrt{3 + \alpha}} \geq \Phi(0.4 - 0.1\alpha); \quad \forall \alpha \in (0, 1] \\ 3x_1 - x_2 \geq 4 \\ x_1, x_2 \geq 0 \end{cases}$$

This is a semi-infinite mathematical program, that may be solved using Cut-Algo. For a fixed value of α, say $\alpha = 0.4$, we get the linear program:

$$(P_2)^{VIII} \quad \begin{cases} \min\ 2x_1 + 3x_2 \\ 2x_1 + 2x_2 - 5.4 \leq \sqrt{3.4}(0.643) \\ 3x_1 - x_2 \geq 4 \\ x_1, x_2 \geq 0 \end{cases}$$

that yields, using again *GAMS* and solving with *iLog*'s *CPLEX* version 11.0.1 solver, the solution $x_1 = 1.333^{\cdot}$, $x_2 = 0$.

3.3. Satisfying solution to $(\mathbf{P_3})$

3.3.1. Analysis

Consider (P_3) and assume we have obtained the fuzzy expectation of $c^{\star\star}$ and we have defuzzified this fuzzy expectation to obtain a deterministic vector, say d. Using the partial order defined in the set of fuzzy random variables (See § 2.1.4), (P_3) can be written as follows:

$$(P_3)' \quad \begin{cases} \min\ dx \\ \sum_{j=1}^{n} \left(a_{ij}^{\star\star}\right)^{\alpha} x_j \leq (b_i^{\star\star})^{\alpha} \quad \forall \alpha \in (0,1] \\ x_j \geq 0; i = 1, \ldots, m \end{cases}$$

Moreover, it is known (Kruse and Meyer, 1987) that:

$$\left(a_{ij}^{\star\star}\right)^{\alpha}(\omega) = \left[a_{ij}^{\alpha L}(\omega), a_{ij}^{\alpha U}(\omega)\right] \ ; \ \omega \in \Omega$$
$$(b_i^{\star\star})^{\alpha}(\omega) = \left[b_i^{\alpha L}(\omega), b_i^{\alpha U}(\omega)\right] \ ; \ \omega \in \Omega$$

where $a_{ij}^{\alpha L}(\omega)$, $a_{ij}^{\alpha U}(\omega)$, $b_i^{\alpha L}(\omega)$ and $b_i^{\alpha U}(\omega)$ are random variables. Therefore $(P_3)'$ is equivalent to the program:

$$(P_3)'' \quad \begin{cases} \min\ cx \\ \sum_{j=1}^{n} a_{ij}^{\alpha U}(\omega) x_j \leq b_i^{\alpha L}(\omega) \quad \forall \alpha \in (0,1] \ ; \ i = 1, \ldots, m \\ x_j \geq 0 \end{cases}$$

$(P_3)''$ is a semi-infinite mathematical program with random data. This program may be solved using the sample average approximation approach described in Luhandjula (2007).

3.3.2. Finding a satisfying solution to (P_3)

Step 1: Read data of (P_3).

Step 2: Compute α-level cuts of $a_{ij}^{\star\star}$ and $b_i^{\star\star}$.

Step 3: Write the program $(P_3)''$.

Step 4: Solve $(P_3)''$ using SAA-cutAlgo.

Step 5: Print the solution.

Step 6: Stop.

This method is effective in that the whole distributions of the involved fuzzy random variables are considered instead of sticking on only a few values like fuzzy expectations, fuzzy variances or fuzzy indices.

3.3.3. Numerical example

Consider the linear program:

$$(P_5)\quad \begin{cases} \min \sum\limits_{j=1}^{6} c_j^{\star\star} x_j \\ \text{subject to} \\ \sum\limits_{j=1}^{6} a_{ij}^{\star\star} x_j \geq b_i^{\star\star} \\ |x_i| \leq 10; i = 1, \ldots, 6 \end{cases}$$

where $a_{1j}^{\star\star}$ and $b_1^{\star\star}$ are fuzzy random variables, the α-levels of which are:

$$a_{11}^{\alpha L} = \alpha; a_{12}^{\alpha L} = \zeta_2 \alpha^2; a_{13}^{\alpha L} = \zeta_3^2 \alpha^3;$$
$$a_{14}^{\alpha L} = \zeta_4^3 \alpha^4; a_{15}^{\alpha L} = (1 - \zeta_5); a_{16}^{\alpha L} = (1 - \zeta_6)\alpha;$$
$$b_1^{\alpha U} = 2e^{\alpha}; V_p(c_1^{\star\star}) = 1; V_p(c_2^{\star\star}) = \frac{1}{\zeta_2}$$
$$V_p(c_3^{\star\star}) = \frac{1}{2\zeta_3}; V_p(c_4^{\star\star}) = \frac{1}{3\zeta_4}; V_p(c_5^{\star\star}) = \frac{1}{4\zeta_5}; V_p(c_6^{\star\star}) = \frac{1}{5\zeta_6}$$

and $\zeta_2, \zeta_3, \ldots, \zeta_6$ are random variables whose distributions are as follows:

$$\zeta_2 \rightsquigarrow U(3,4); \zeta_3 \rightsquigarrow N(5,1), \zeta_4 \rightsquigarrow Exp(6);$$
$$\zeta_5 \rightsquigarrow N(4,1); \zeta_6 \rightsquigarrow Exp(5).$$

where $U(a,b)$ means a uniform distribution between a and b, $N(c,d)$ means a normal distribution with mean c and standard deviation d, $Exp(e)$ means an exponential distribution with $\lambda = e$, and $V_p(k^{\star\star})$ is the most possible value of $E(k^{\star\star})$. Using the general procedure described in § 3.3.2 we obtain the results indicated in Table 2. Since $|\nu_{9000} - \nu_{13000}|$ is too small, we stop the procedure and take as satisfying solution $x_{13000}^{\#} = (-9.99, -8.28, -1.56, 0.58, 0.07, -9.84)$.

Table 2: results for the numerical example (P_5)

N	**Optimal solution** $\mathbf{x}_{\mathbf{N}}^{\#}$	**Optimal value of the objective function** $\nu_{\mathbf{N}}^{\#}$
1000	(-9.89, -1.42, 6.47, 0.24, -5.30, 9.75)	-41.38
5000	(-9.74, -7.54, -2.93, 0.71, 8.87, -9.17)	-36.67
9000	(-9.95, -8.28, -1.31, 0.38, 1.66, -9.18)	-22.00
13000	(-9.99, -8.28, -1.56, 0.58, 0.07, -9.84)	-22.47

4. Concluding remarks

Today, managers have to take increasingly complex decisions in the face of uncertainty and conflicting information. Poor decision-making may have a significantly adverse impact on the viability of an organisation. The hard modelling approaches of (hard) Operations Research rarely permit the gain in generality and in insight that results in pondering different kinds of irreducible imprecision in a decision-making process.

The topic of the present paper is a specific type of research in optimisation under hybrid uncertainty that began in Luhandjula (1983) and Liu (2001), and continued in Luhandjula (2004, 2006), Luhandjula and Gupta (1996), Nanda et al. (2006), and Van Hop (2007a,b). This line of inquiry has found many applications in different disciplines, such as economics (Lai and Hwang, 1993; Zmeškal, 2005), industry, marketing (Weber and Sun, 2000), and water resource management (Ben Abdelaziz et al., 2004), to mention just a few.

In this paper, we have brought together modelling ideas and mathematical techniques, ranging from simple optimisation methods to sample average approximation techniques via cutting plane methods, in a way that captures fuzziness and randomness in an optimisation setting.

Our efforts were expressed in terms of fuzzy set theory and hybrid concepts such as probabilistic sets, fuzzy random variables and uncertain probabilities. We showed that our optimisation models under uncertainty can be stated more palatably. We also described powerful algorithms to solve them.

We have also provided arguments that show why the described uncertainty reduction approaches are appropriate.

Counterparts of fuzzy stochastic optimisation problems discussed here share the common property of having infinitely many constraints. This highlights the relationships between uncertainty and infinity in an optimisation framework (Luhandjula et al., 1992). Moving between these two extremes is still a subtle and poorly understood business that needs to be clarified.

This work departs from other research in fuzzy stochastic optimisation in several respects, including the way in which randomness and fuzziness enter into the optimisation model, the nature of the distribution of involved imprecise variables, and the epistemological and methodological choices in the transformation process.

A comparative investigation between existing approaches would be of great benefit. From such a comparative investigation, a Decision Support System (DSS) for helping capturing fuzziness and randomness in an optimisation setting would emerge. The advantage of using the DSS paradigm in this context lies in the fact that such an approach departs from the most frequently used reductionist approach based on reliance on one technique when confronted with an ill-structured complex problem. It would also be helpful to develop software for automating all computations in the procedures described in this paper.

To avoid difficulties unrelated to our subject, we have restricted ourselves to linear fuzzy stochastic models. Nevertheless, ideas discussed here may be generalised to nonlinear optimisation models under randomness and fuzziness.

The papers Liu and Liu (2003, 2005) are tremendous resources for someone who wants to embark in the field of nonlinear fuzzy stochastic optimisation.

Another model that is of undeniable importance in practical applications is that of fuzzy random multiobjective optimisation. It would be interesting if methods discussed in this paper could be tailored to be suitable for the multiobjective case. The method thus obtained should then be compared with those in the literature (Ammar, 2009; Hasuike and Ishii, 2009; Katagiri et al., 2008; Li et al., 2006; Peña et al., 2009).

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Appendix: Proof of Theorem 1

(a) Let $z \in \mathbb{R}$. Bearing in mind that:

$$\begin{aligned} & P\left[\max\left[\mu_0(x,\omega), \mu_1(x,\omega), \ldots, \mu_m(x,\omega)\right] < z\right] \\ =& P\left[(\mu_0(x,\omega) < z) \cap (\mu_1(x,\omega) < z) \cap \ldots \cap (\mu_m(x,\omega) < z)\right] \end{aligned}$$

and making use of the fact that:

$$\begin{aligned} & \min\left[\mu_0(x,\omega), \mu_1(x,\omega), \ldots, \mu_m(x,\omega)\right] \\ =& -\max\left[-\mu_0(x,\omega), -\mu_1(x,\omega), \ldots, -\mu_m(x,\omega)\right], \end{aligned}$$

we may write:

$$\begin{aligned} & P\left[\min\left[\mu_0(x,\omega), \mu_1(x,\omega), \ldots, \mu_m(x,\omega)\right] < z\right] \\ =& P\left[-\max\left[-\mu_0(x,\omega), -\mu_1(x,\omega), \ldots, -\mu_m(x,\omega)\right] < z\right] \\ =& P\left[\max\left[-\mu_0(x,\omega), -\mu_1(x,\omega), \ldots, -\mu_m(x,\omega)\right] > -z\right] \\ =& 1 - P\left[(-\mu_0(x,\omega) \leq -z) \cap (-\mu_1(x,\omega) \leq -z) \cap \ldots \cap (-\mu_m(x,\omega) \leq -z)\right] \\ =& 1 - P\left[(\mu_0(x,\omega) \geq z) \cap (\mu_1(x,\omega) \geq z) \cap \ldots \cap (\mu_m(x,\omega) \geq z)\right] \\ =& P\left[(\mu_0(x,\omega) < z) \cup (\mu_1(x,\omega) < z) \cup \ldots \cup (\mu_m(x,\omega) < z)\right] \end{aligned}$$

$$= \sum_{i=0}^{m} P\left[\mu_i(x,\omega) < z\right] - \sum_{0 \le j < k} P\left[(\mu_j(x,\omega) < z) \cap (\mu_k(x,\omega) < z)\right]$$

$$+ \ldots + (-1)^{m+2} P\left[(\mu_0(x,\omega) < z) \cap (\mu_1(x,\omega) < z) \cap \ldots \cap (\mu_m(x,\omega) < z)\right].$$

Hence the distribution function of $\mu_D(x,\omega) = \min_{i=0,1,\ldots,m} \mu_i(x,\omega)$ takes the form of (2) and we are done.

(b) Let now

$$g_1(s) = \int_0^\infty e^{-\left(\frac{s}{\gamma}\right)t} F_{\mu_D(x,\omega)}(t)dt$$

and

$$g_2(s) = \int_0^\infty e^{-\left(\frac{s}{1-\gamma}\right)t} F_{\min\left(1, \sum_{i=0}^{m} \mu_i(x,\omega)\right)}(t)dt.$$

Then $g_1(s)$ and $g_2(s)$ are Laplace's transforms of $\gamma \min_i \mu_i(x,\omega)$ and $(1-\gamma)\min\left(1, \sum_{i=0}^{m} \mu_i(x,\omega)\right)$ respectively. Invoking the Convolution Theorem (Kreider et al., 1966), we have that:

$$\mathscr{L}\left[F_{\gamma \min_i \mu_i(x,\omega)} * F_{(1-\gamma)\min\left(1, \sum_{i=0}^{m} \mu_i(x,\omega)\right)}\right] = g_1(s)g_2(s).$$

That is,

$$\mathscr{L}\left[F_{\gamma \min_i \mu_i(x,\omega)} * F_{(1-\gamma)\min\left(1, \sum_{i=0}^{m} \mu_i(x,\omega)\right)}\right] = \mathscr{L}^{-1}\Big(g_1(x).g_2(x)\Big).$$

Making use of the fact that the first member of this equality is equal to

$$F_{\gamma \min_i \mu_i(x,\omega) + (1-\gamma)\min\left(1, \sum_{i=0}^{m} \mu_i(x,\omega)\right)}$$

we are done. Note that '$*$' denotes the *product of convolution* and should not be confused with the '$\star$' used in the paper to denote uncertain probabilities.

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