DENSITIES OF COMPOSITE WEIBULLIZED GENERALIZED GAMMA VARIABLES

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Summary: The Weibullized generalized gamma distribution, which is a very flexible distribution due to it richness in parameter structure, is derived by introducing the Weibullizing constant to the generalized gamma distribution. In this paper we first investigate the properties of the Weibullized generalized gamma distribution and then derive densities of composite Weibullized generalized gamma variables.

1. Introduction

If one wants to model a phenomena that takes on positive values, such as income and lifetime, the exponential distribution (with only one parameter) is one of the simplest distributions, followed by the gamma distribution (with two parameters). The *generalized gamma* (GG) distribution is another positively skew distribution and more general with four parameters. We will first look at the generalized gamma distribution and its shape analysis, since this will guide us towards the *Weibullized generalized gamma* (WGG) distribution.

A random variable X is said to have the generalized gamma distribution, denoted by $\mathrm{GG}(\alpha, a, b, m)$, if its probability density is given by

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$$\begin{aligned} f(x) &= Cx^{\alpha-1}e^{bx}(a+x)^{-m}, \quad x \geq 0 & (1.1) \\ \text{where} \quad C^{-1} &= a^{\alpha-m}\Gamma(\alpha)\,\psi(\alpha, \alpha-m+1, ab) & (1.2) \end{aligned}$$

with ψ the confluent hypergeometric function of the second kind (see e.g. Gradshteyn and Ryzhik, 2000) given by

$$\psi(\alpha, \gamma, z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zt}t^{\alpha-1}(1+t)^{\gamma-\alpha-1}\,dt, \quad \text{Re}\alpha > 0, \quad \text{Re}z > 0.$$

The distribution given by (1.1) has been known at least since 1983 when Arnold and Press derived it as the marginal density of the shape parameter of the classical Pareto distribution. It is also worth noting that the generalized gamma distribution fits into the extension of the Pearson system. Since those early days, this distribution has received considerable attention in the literature. From a literature review, it seems as if this distribution (and alternative forms thereof) was derived either as a result, or by generalization of existing probability distributions. It is usually in the Bayesian context that (1.1) appears as a result, see Singh, et al. (1995); Pandey et al. (1996); Armero and Bayarri (1997); Mostert (2000). Also appearing as a result, but not in the Bayesian context, Bekker (1990) derived (1.1) in a characterization result with two conditional gamma distributions. On the other hand, researchers derived the generalized gamma distribution as a generalization of existing distributions and referred to it by different names. See Roux and Bekker (1990); Ng and Kotz (1993); Agarwal and Kalla (1996); Ghitany (1998); Gordy (1998); Agarwal and Al-Saleh (2001); Kalla et al. (2001); Al-Saleh and Agarwal (2002); Nadarajah and Gupta (2007); Al-Saleh and Agarwal (2007).

In Section 2 we will make use of a new approach, the *kernel approach*, to derive the generalized gamma distribution and also investigate the different shape types of this distribution. In Section 3 we introduce the Weibullized generalized gamma distribution, do a shape analysis for the WGG distribution based on the different shape types of the generalized gamma distribution and study the properties of the WGG distribution. Densities of composite Weibullized generalized gamma variables are derived and graphed in Section 4. The huge variation in composite Weibullized generalized gamma densities is illustrated in this section. When fitting a model to data, one usually has an idea of the behaviour of the data, either through experience or by graphing empirical data. This section aims at assisting the researcher in deciding whether the Weibullized generalized gamma distribution is an appropriate model to describe the data. The article is concluded with a discussion in Section 5.

2. The generalized gamma distribution

2.1 The Kernel Approach

Having discussed in the introduction the methods by which the generalized gamma distribution was derived in past research, we would like to introduce a new approach in deriving this density function. With this approach, referred to as the *kernel approach*, a density function is considered a function created from the product of two or more kernel functions. The domain of the new function is given by the intersection of the separate domains of the component functions. The newly created function is multiplied by a normalization constant and thus a density function is defined.

Figure 2.1 illustrates how the generalized gamma distribution is defined by making use of the kernel approach.

Figure 2.1 (a) considers the case where $m \geq 0$ and Figure 2.1 (b) considers the case where $m < 0$.

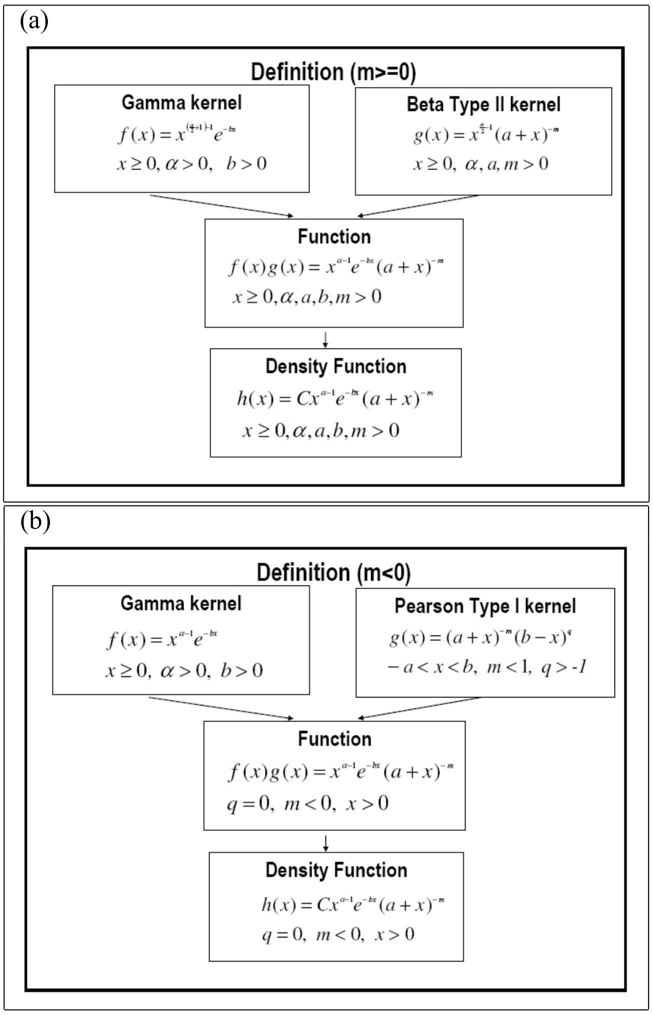


Figure 2.1. Density function (1.1) for (a) $m \geq 0$ and (b) $m < 0$

These two diagrams illustrate the position of the generalized gamma distribution as a *parent* distribution to some of the well-known positive skew distributions.

2.2 Shape analysis

Ng and Kotz (1995) called the specific form of the generalized gamma function where $a = 1$ the Kummer-gamma distribution and investigated the modality of this function. This method is useful, since it illustrates the different "shape types" of the density.

In order to study the modality of the function, the following derivatives were derived:

$$\frac{\partial}{\partial x} \ln\ f(x) = \frac{\alpha - 1}{x} - b + \frac{-m}{1+x} \text{ and } \frac{\partial^2}{\partial x^2} \ln\ f(x) = \frac{1-\alpha}{x^2} + \frac{m}{(1+x)^2}.$$

and the roots of the equation $\frac{\partial}{\partial x} \ln\ f(x) = 0$

$$x = (2b)^{-1} (-m + \alpha - 1 - b) \pm \sqrt{(-m + \alpha - 1 - b)^2 - 4b(1-\alpha)}$$

Ng and Kotz derived results for all cases where $a \leq 1$. These results are summarized in (1)– (4) below and can be verified by investigating the derivatives and roots of $ln\ f(x)$. However, Ng and Kotz did not give the results for the cases where $\alpha > 1$. These cases are investigated in (5).

1. If $\alpha \leq 1$ and $m \geq 0$ the pdf is decreasing and convex with $f(0+) = \infty$ and $f(\infty) = 0$.
2. If $\alpha = 1$ and $0 < -m \leq b$ then the pdf has the same form as in (1).
3. If $\alpha = 1$ and $-m > b > 0$ there is only one positive root implying a single mode. The pdf rises from zero, reaches the single mode at $x = \frac{-m-b}{b}$

and decreases to zero as $x \longrightarrow \infty$.

4. If $\alpha < 1$ and $-m > 0$ a necessary and sufficient condition for the two roots to be positive and distinct is: $(-m + \alpha - 1 - b) > 0$ and $(-m + \alpha - 1 - b)^2 - 4b(1 - \alpha) > 0$, thus implying two turning points. For this case the pdf decreases from ∞ at $x = 0$, reaches the antimode at the smaller of the two roots, then increases until the larger of the two roots and finally decreases to zero as $x \longrightarrow \infty$.

5. For the case where $\alpha > 1$, it will always hold that $\sqrt{(-m + \alpha - 1 - b)^2 - 4b(1 - \alpha)} > (-m + \alpha - 1 - b)$ therefore there will always be only one positive root. Let $a = (-m + \alpha - 1 - b)$ and $c = -4b(1 - \alpha) > 0$

   (i) If $\alpha > 1$ and $m < 0$ then $a > 0$ if $(-m + \alpha - 1) > b$ and $a \pm \sqrt{a^2 + c}$, implies one positive root and one negative root.
   (ii) If $\alpha > 1$ and $m < 0$ then $a < 0$ if $(-m+\alpha-1) < b$ and $a \pm \sqrt{a^2 + c}$, implies one positive root and one negative root.
   (iii) If $\alpha > 1$ and $m > 0$ then $a > 0$ if $(\alpha - 1) < (b+m)$ and $a \pm \sqrt{a^2 + c}$, implies one positive root and one negative root.
   (iv) If $\alpha > 1$ and $m > 0$ then $a < 0$ if $(\alpha - 1) > (b + m)$ and $a \pm \sqrt{a^2 + c}$, implies one positive root and one negative root.

These five cases are illustrated in Figure 2.2. For case 5, option (iii) is illustrated. The parameters chosen for the different cases are summarized below.

Parameters chosen for Figure 2.2

	α	a	b	m
Case 1	0.5	1	2	2
Case 2	1	1	2	-1
Case 3	1	1	2	-3
Case 4	0.1	1	4	-10
Case 5(iii)	3	1	2	3

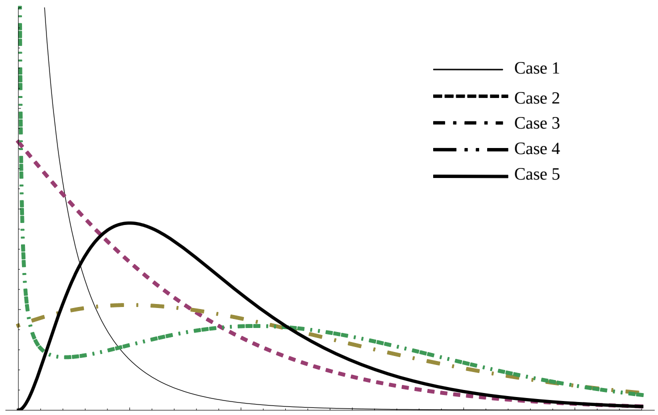


Figure 2.2 Different "shape types" of density (1.1) for specific values of the parameters

The five shape types for the generalized gamma distribution will be used as the basis for investigating the shape types of the Weibullized generalized gamma distribution and also the distributions of composite WGG variables.

3. The Weibullized Generalized Gamma Distribution

3.1 Introducing a Weibullizing parameter to the Generalized Gamma Distribution

The powers of positive variables (such as the exponential and gamma) are sometimes encountered, resulting in the so-called Weibullized distributions (see e.g. Gupta and Nadarajah (2004, p. 118); Malik (1967); McDonald and Xu (1995); Bekker et al. (2000, 2009)).

By defining $Y = X^c$ with c positive, the generalized gamma density can be extended with density

$$f(y) = C^* y^{\frac{\alpha}{c}-1} e^{-by^{\frac{1}{c}}} \left(a + y^{\frac{1}{c}}\right)^{-m}, \quad y \geq 0 \tag{3.1}$$

where $C^{*-1} = ca^{\alpha-m}\Gamma(\alpha)\,\psi(\alpha, \alpha - m + 1, ab)$

This density will from now on be referred to as the *Weibullized generalized gamma distribution* and will be denoted by $WGG(\alpha, a, b, m, c)$.

We use the five shape types of the generalized gamma distribution to illustrate the influence of the parameter c. In Figure 3.1 (a)–(e) the WGG distribution is graphed for $c = 0.3,\ 0.5,\ 1, 5$, and 10. Note that for $c = 1$, we have the GG distribution. The effect is what one would expect from the Weibullizing parameter: The larger c, the more it "pushes" the density function towards the vertical axis. For c less than one, we get a long tail density, the smaller c, the smaller the variance.

3.2 Properties of the Weibullized Generalized Gamma Distribution

In this section we will investigate the properties of the WGG distribution, starting with the distribution function in Theorem 1. In Theorem 2 we derive the moment-generating function and use it to find the r^{th} moment of the WGG distribution. Making use of the distribution function in Theorem 1, we derive the hazard rate function in Theorem 3 and also investigate the influence of the Weibullizing constant c on the hazard rate function.

Theorem 1

Let $Y \sim WGG(\alpha, a, b, m, c)$, then the *distribution function* of Y is given by

$$F(y) = C^{*} a^{-m} b^{-\alpha} c \sum_{r=0}^{\infty} \frac{\Gamma(m+r)}{\Gamma(m)\, r!} \left(\frac{-1}{ab}\right)^{r} \gamma\left(\alpha + r, b y^{\frac{1}{c}}\right) \tag{3.2}$$

with $C^{*-1} = c a^{\alpha-m} \Gamma(\alpha)\, \psi(\alpha, \alpha - m + 1, ab)$ where $\gamma(\alpha; x)$ is the incomplete gamma function given by $\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt$, Re$(\alpha) > 0$.

Proof

By definition the distribution function of Y is given by

$$F(y) = C^{*} \int_0^y t^{\frac{\alpha}{c}-1} e^{-bt^{\frac{1}{c}}} \left(a + t^{\frac{1}{c}}\right)^{-m} dt.$$

By setting $w = bt^{\frac{1}{c}}$, expanding the term $\left(1 + \frac{w}{ab}\right)^{-m}$ as an infinite series and using the incomplete gamma function, proves the theorem.

Theorem 2

Let $Y \sim WGG(\alpha, a, b, m, c)$, then the *moment-generating function* of Y is given by

$$M(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[a^{cr} (\alpha)_{cr} \frac{\psi(\alpha + cr, \alpha + cr - m + 1, ab)}{\psi(\alpha, \alpha - m + 1, ab)}\right]$$

where $(a)_n = a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer symbol.

Proof

By definition the moment-generating function of Y is given by

$$M(t) = C^* \int_0^\infty e^{ty} y^{\frac{\alpha}{c}-1} e^{-by^{\frac{1}{c}}} \left(a + y^{\frac{1}{c}}\right)^{-m} dy.$$

Expanding the term e^{ty} and setting $z = \frac{y^{\frac{1}{c}}}{a}$ proves the theorem.

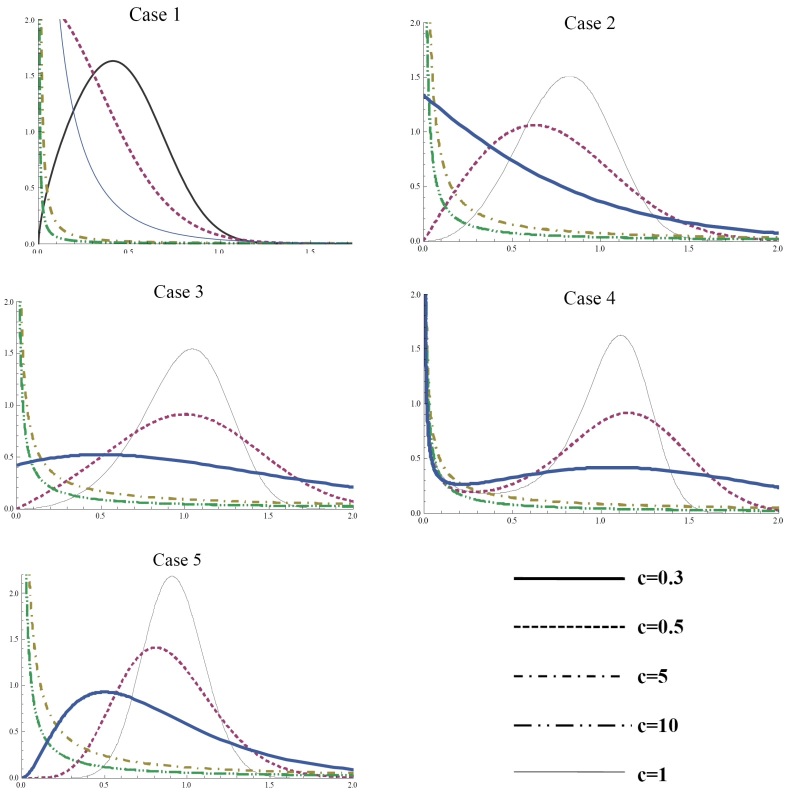


Figure 3.1 The influence of the Weibullizing parameter c

Remark

Using the moment-generating function in Theorem 2 above, the r^{th} moment is given by

$$E\left(Y^{r}\right)=a^{cr}\left(\alpha\right)_{cr}\frac{\psi\left(\alpha+cr,\alpha+cr-m+1,ab\right)}{\psi\left(\alpha,\alpha-m+1,ab\right)}$$

Theorem 3

Let $Y\sim WGG\left(\alpha,a,b,m,c\right)$, then the *hazard rate function* of Y is given by

$$h\left(y\right)=\frac{a^{m}b^{\alpha}y^{\frac{\alpha}{c}-1}e^{-by^{\frac{1}{c}}}\left(a+y^{\frac{1}{c}}\right)^{-m}}{c\left[a^{\alpha}b^{\alpha}\Gamma\left(\alpha\right)\psi\left(\alpha,\alpha-m+1,ab\right)-\sum_{r=0}^{\infty}\frac{\Gamma\left(m+r\right)}{\Gamma\left(m\right)r!}\left(\frac{-1}{ab}\right)^{r}\gamma\left(\alpha+r,by^{\frac{1}{c}}\right)\right]}$$

Proof

Using (3.1) and (3.2) in the definition of the hazard rate function of $Y,\ h\left(y\right)=\frac{f\left(y\right)}{1-F\left(y\right)}$, proves the theorem.

The influence of c on the hazard function is illustrated in Figures 3.2 (a) and (b). Since $h(y)$ is an increasing function for increasing values of Y, the Weibullized generalized gamma density seems plausible as lifetime model in most situations. In Figure 3.2 (a) the first four parameters were fixed at $\alpha=1,\ a=1,\ b=2$ and $m=-1$ and c varied from 0.3, 0.5, 1, 5 and 10. In Figure 3.2(b) m was changed to -3.

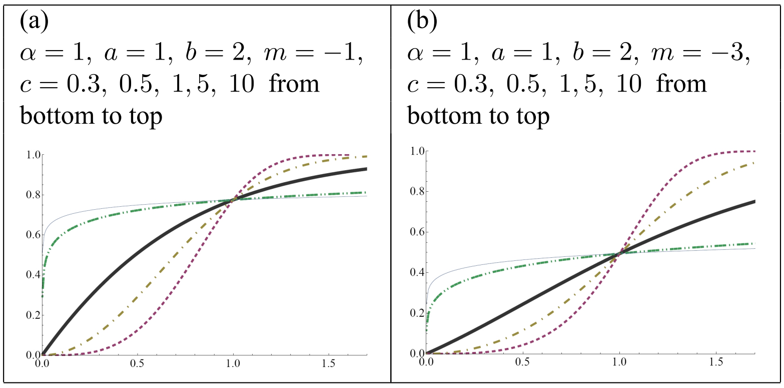


Figure 3.2 Influence of c on the hazard function

4. Densities of composite Weibullized Generalized Gamma Variables

4.1 Derivation of Densities of Composite Weibullized Generalized Gamma Variables

In this paragraph we derive densities of the composites $U = Y_1/\left(Y_1 + Y_2\right)$ and $V = Y_1/Y_2$, where Y_1 and Y_2 are independent Weibullized generalized gamma variables. In Theorem 4(a) we derive the density of $U = Y_1/\left(Y_1 + Y_2\right)$ when $c_1 \neq c_2$ for the special case where $m = -1$ and in Theorem 4(b) we assume that the Weibullizing constant c for both Y_1 and Y_2 are the same.

Theorem 4

Let $Y_i \sim WGG(\alpha_i, a_i, b, m_i, c_i)$ $\quad i = 1, 2$ be independent random variables, the density of $U = \dfrac{Y_1}{Y_1 + Y_2}$ is as follows:

(a) for $c_1 \neq c_2$ and $m = -1$,

$f(u) = A + B$

$$A = C^{**} a_2 c_1 u^{-\frac{\alpha_2}{c_2}-1} (1-u)^{\frac{\alpha_2}{c_2}-1} \cdot \left[\sum_{r=0}^{\infty} \frac{(-b)^r}{r!} u^{\frac{-r}{c_2}} (1-u)^{\frac{r}{c_2}} \Gamma\left(\alpha_1 + \frac{c_1}{c_2}(\alpha_2 + r)\right) \cdot b^{-\alpha_1 - \frac{c_1}{c_2}(\alpha_2+r)} \left\{ a_1 + \left(\alpha_1 + \frac{c_1}{c_2}(\alpha_2 + r)\right) b \right\} \right]$$

$$B = C^{**} c_1 u^{-\frac{\alpha_2}{c_2} - \frac{1}{c_1} - 1} (1-u)^{\frac{\alpha_2}{c_2} + \frac{1}{c_1} - 1} \cdot \left[\sum_{r=0}^{\infty} \frac{(-b)^r}{r!} u^{-\frac{r}{c_2}} (1-u)^{\frac{r}{c_2}} \Gamma\left(\alpha_1 + \frac{c_1}{c_2}(\alpha_2 + r + 1)\right) \cdot b^{-\alpha_1 - \frac{c_1}{c_2}(\alpha_2+r+1)-1} \left\{ a_1 + \left(\alpha_1 + \frac{c_1}{c_2}(\alpha_2 + r + 1)\right) b \right\} \right]$$

with

$$C^{**-1} = c_1 c_2 a_1^{\alpha_1+1} a_2^{\alpha_2+1} \Gamma(\alpha_1) \Gamma(\alpha_2) \psi(\alpha_1, \alpha_1 + 2;\ a_1 b) \cdot \psi(\alpha_2, \alpha_2 + 2;\ a_2 b) \tag{4.1}$$

(b) for $c_1 = c_2 = c > 0$,

$$f(u) = \frac{a_1^{\alpha_2} a_2^{-\alpha_2} c^{-1} u^{-(\alpha_2/c+1)} (1-u)^{\alpha_2/c-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(m_1+m_2)\psi(\alpha_1,\alpha_1-m_1+1,a_1b)\psi(\alpha_2,\alpha_2-m_2+1,a_2b)}$$
$$\cdot \sum_{r=0}^{\infty} \frac{\left[-a_1 b\left(1+\left(\frac{1}{u}-1\right)^{1/c}\right)\right]^r}{r!}$$
$$\cdot \Gamma(\alpha_1+\alpha_2+r)\Gamma(m_1+m_2-\alpha_1-\alpha_2-r)\cdot$$
$$\cdot {}_2F_1\left(m_2, \alpha_1+\alpha_2+r; m_1+m_2; 1-\frac{a_1}{a_2}\left(\frac{1-u}{u}\right)^{1/c}\right),$$
$$0 < u < 1$$
$$m_1 + m_2 > \alpha_1 + \alpha_2 + r \qquad (4.2)$$

with ${}_2F_1(\alpha,\beta;\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt$ the Gauss hypergeometric function.

Proof

(a) Defining $U = \frac{Y_1}{Y_1+Y_2}$ and $W = Y_1 + Y_2$ gives the joint pdf of U and W as follows:

$$f(u,w) = C^*(w)(uw)^{\frac{\alpha_1}{c_1}-1}(w(1-u))^{\frac{\alpha_2}{c_2}-1} e^{-b(uw)^{\frac{1}{c_1}}} e^{-b(w(1-u))^{\frac{1}{c_2}}}$$
$$\cdot\left(a_1 + (uw)^{\frac{1}{c_1}}\right)\left(a_2 + (w(1-u))^{\frac{1}{c_2}}\right)$$

To obtain the density of U, we integrate $f(u,w)$ over w.

Making the transformation $(uw)^{\frac{1}{c_1}} = z$ in the integral expression and expanding the term $e^{-b(w(1-u))^{\frac{1}{c_2}}}$ proves part (a).

(b) The proof is similar to part (a).

Remark

1. By using theorem 4(b), the density of $V = \frac{Y_1}{Y_2} = \left(\frac{U}{1-U}\right)$ can be derived as

$$f(v) = \frac{a_1^{\alpha_2} a_2^{-\alpha_2}}{c\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(m_1+m_2)\psi(\alpha_1,\alpha_1-m_1+1,a_1b)\psi(\alpha_2,\alpha_2-m_2+1,a_2b)}$$

$$\cdot \sum_{r=0}^{\infty} \frac{(-b)^r}{r!} a_1^r v^{-(\alpha_2+r+c)/c} \left(1+v^{\frac{1}{c}}\right)^r$$

$$\cdot \Gamma(\alpha_1+\alpha_2+r)\Gamma(m_1+m_2-\alpha_1-\alpha_2-r)$$

$$\cdot {}_2F_1\left(m_2, \alpha_1+\alpha_2+r; m_1+m_2; 1-\frac{a_1}{a_2}v^{-\frac{1}{c}}\right),$$

$$v > 0$$

$$m_1+m_2 > \alpha_1+\alpha_2+r \tag{4.3}$$

2. Nadarajah and Kotz (2004) derived the density of $U = \frac{X_1^c}{X_1^c + X_2^c}$ with $c > 0$ for X_1 and X_2 independent gamma variables. More recently (2009), Bekker et al. derived the density of U for X_1 and X_2 independent generalized beta-prime variables. Since the gamma distribution and the generalized beta-prime distribution are both incorporated by the generalized gamma distribution given by (1.1), the results in these two papers are special cases of (4.2).

4.2 Shape Analysis of Composite Weibullized Generalized Gamma Distributions

In order to make graphing of (4.2) possible for the case where $c = 1$, one can write the density function of U in the form

$$
\begin{aligned}
f(u) &= C^{**}\int_0^\infty (a_1+uw)^{-m_1}(a_2+w(1-u))^{-m_2} \\
&\quad \cdot \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}u^{\alpha_1-1}(1-u)^{\alpha_2-1}\frac{b^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} \\
&\quad \cdot w^{\alpha_1+\alpha_2-1}e^{-bw}dw \\
&= C^{**}\int_0^\infty (a_1+uw)^{-m_1}(a_2+w(1-u))^{-m_2} \\
&\quad \cdot g(u)\cdot h(w)\,dw \qquad (4.4)
\end{aligned}
$$

where

$$
\begin{aligned}
(C^{**})^{-1} &= a_1^{\alpha_1-m_1}a_2^{\alpha_2-m_2}b^{-(\alpha_1+\alpha_2)}\psi(\alpha_1,\alpha_1-m_1+1;a_1b) \\
&\quad \cdot\psi(\alpha_2,\alpha_2-m_2+1;a_2b)
\end{aligned}
$$

where $g(u)$ is the density function of a beta type I variable with parameters α_1 and α_2 and $h(w)$ is the density function of a gamma variable with parameters b^{-1} and $\alpha_1+\alpha_2$. Using numerical integration, (4.4) is graphed in Figure 4.1 for the five different "shape types" given in Figure 2.2. The choice of parameters for Figure 4.1 is the same as for Figure 2.2.

It is evident from Figure 4.1 that the density of $U = \frac{Y_1}{Y_1+Y_2}$ with $c = 1$ can take a wide variety of shapes including U-shapes, unimodal and bimodal curves. It is also interesting to note that the "matrix" in Figure 4.1 is symmetrical, since $\frac{Y_1}{Y_1+Y_2} = 1 - \frac{Y_2}{Y_1+Y_2}$. Thus, if one wishes to get the mirror image of a density one only needs to switch the role of Y_1 with that of Y_2 and vice versa. This implies that for every positive skew distribution there is a sibling negative skew distribution. This is quite surprising, since none of

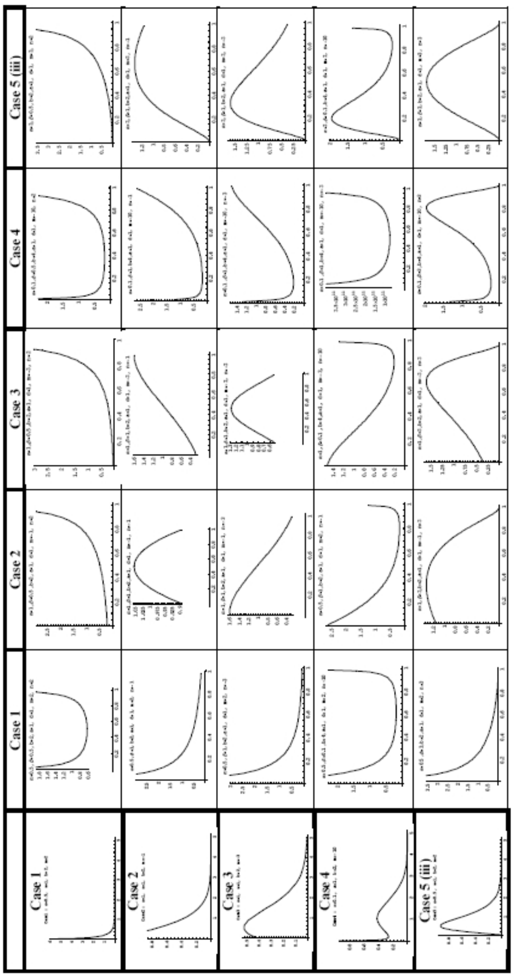


Figure 4.1 Summary of shapes for the pdf of $U = \dfrac{Y_1}{Y_1 + Y_2}$, $c = 1$ with row legend $Y_2 = X_2^1$ and column legend $Y_1 = X_1^1$

the 5 "shape types" were originally negative skew. The symmetry of Figure 4.1 also implies that neither the distribution of Y_1, nor the distribution of Y_2 dominates the shape of the distribution of U.

The density function of $V = \frac{Y_1}{Y_2}$ given in (4.3) can be graphed in a similar way by first writing the density as

$$\begin{aligned} f(v) &= C^{**}\int_0^\infty \left(a_1 + \frac{uv}{1+v}\right)^{-m_1}\left(a_2 + \frac{u}{1+v}\right)^{-m_2} \\ &\quad \cdot \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} v^{\alpha_1-1}(1+v)^{-\alpha_1-\alpha_2} \\ &\quad \cdot \frac{b^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} u^{\alpha_1+\alpha_2-1} e^{-bu} du \\ &= C^{**}\int_0^\infty \left(a_1 + \frac{uv}{1+v}\right)^{-m_1}\left(a_2 + \frac{u}{1+v}\right)^{-m_2} \\ &\quad \cdot g(v) \cdot h(u)\, du \end{aligned} \tag{4.5}$$

where

$$(C^{**})^{-1} = a_1^{\alpha_1-m_1} a_2^{\alpha_2-m_2} b^{-(\alpha_1+\alpha_2)} \psi(\alpha_1, \alpha_1-m_1+1; a_1 b)\, \psi(\alpha_2, \alpha_2-m_2+1; a_2 b)$$

where $g(v)$ is the density function of a beta-prime variable with parameters α_1 and α_2 and $h(u)$ is the density function of a gamma variable with parameters b^{-1} and $\alpha_1+\alpha_2$. Using numerical integration, (4.5) is graphed in Figure 4.2 for the five different "shape types" given in Figure 2.2. The choice of parameters for Figure 4.2 is the same as for Figures 2.2 and 4.1.

From Figure 4.2 the density of $V = \frac{Y_1}{Y_2}$ with $c = 1$ do not take on such a wide variety of shapes as the density of U in Figure 4.1. Also, the "matrix" in Figure 4.2 is not symmetrical. Another interesting feature of the shape of the density (4.5) is that it seems to be dominated by the shape of the density

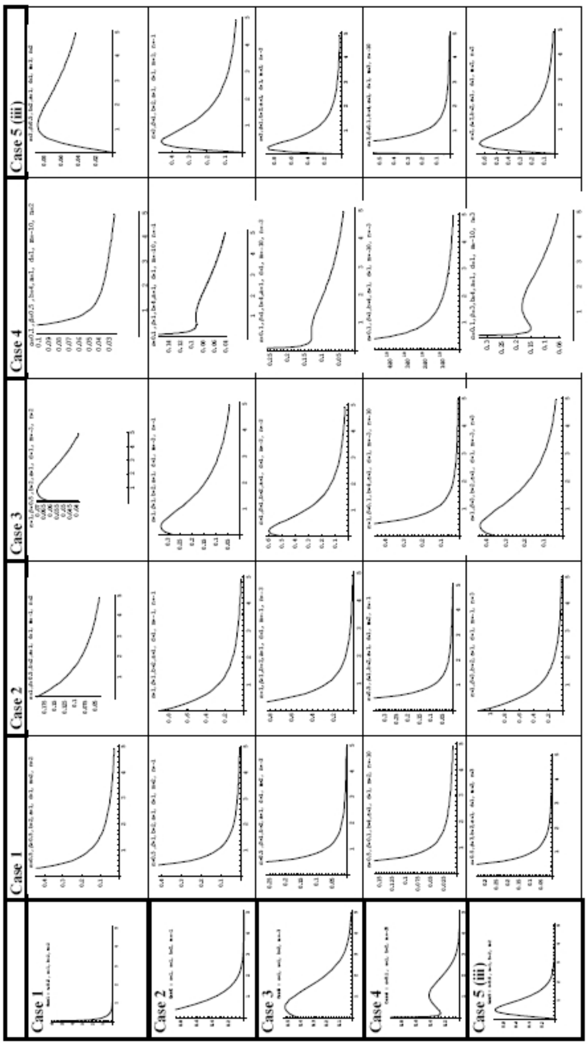


Figure 4.2 Summary of shapes for the pdf of $V = \dfrac{Y_1}{Y_2}$, $\; c = 1$ with row legend $Y_2 = X_2^1$ and column legend $Y_1 = X_1^1$

of the numerator (Y_1) with a few exceptions (especially where the shape of the density of Y_2 is of case 4).

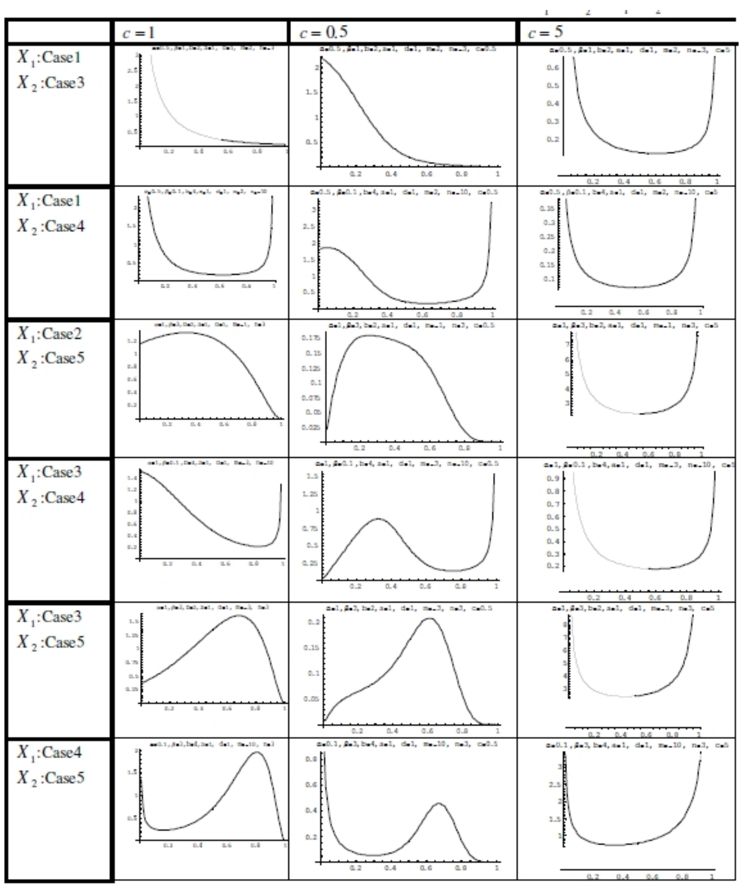


Figure 4.3 The effect of c on the shape of the density function of
$$U = \frac{X_1^c}{X_1^c + X_2^c} = \frac{Y_1}{Y_1 + Y_2}$$

To show the effect of c, in Figure 4.3 the density of $U = \dfrac{Y_1}{Y_1 + Y_2}$ is again graphed for the most interesting combinations in Figure 4.1, but this time not only for $c = 1$, but also for $c = 0.5$ and $c = 5$. To make comparison easier, the case where $c = 1$ is repeated from Figure 4.1. It is interesting to note that for $c = 5$ the density is always in the skewed U-shape, whereas for $c = 0.5$ the density again takes on a very wide variety of shapes which, in some cases, look like the mixture of two distributions.

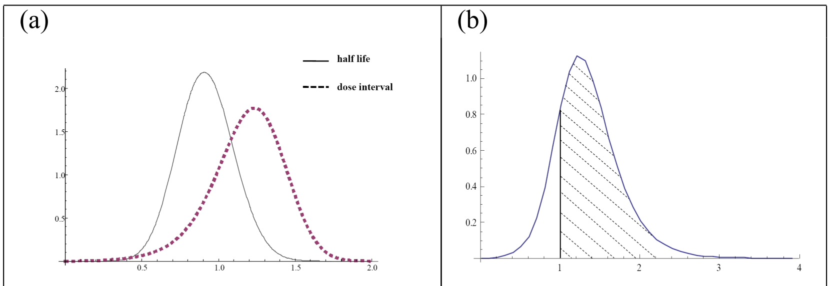
5. Discussion

The quotient of two independent variables has many applications, one of which is the determination of a safe dose interval in pharmacology. When a specific drug is administered to the human body, the concentration of the drug in the blood plasma reaches a certain level. As the drug is biochemically modified by the body, the concentration in the blood plasma decreases and reaches a point where the plasma concentration is half the original value. Pharmacologists refer to this point as the "half-life" of the drug. It is of utmost importance that the dose interval indicated on the label is longer than the half-life of the drug in order to prevent the drug from building up in the body, which may lead to serious toxic effects.

Let variable Y_1 represent the dose interval for a certain drug with half-life that is represented by Y_2 with both variables Weibullized generalized gamma distributed. The safety (θ) of the treatment is the probability that the dose interval exceeds the half-life of the drug and is given by

$$\theta = P\left(Y_1 > Y_2\right) = P\left(\frac{Y_1}{Y_2} > 1\right) = P\left(V > 1\right).$$

This probability should typically be close to one. The WGG distribution is defined over the positive range and can take on a variety of shapes: positive skew, negative skew and also approximately symmetrical. This make the WGG distribution ideal to model both dose interval and half-life. One may expect half-life to follow an approximately symmetrical distribution with a small variance, and the dose interval to follow a slightly negatively skew distribution. Figure 5.1 (a) gives the distribution of dose interval (Y_1) and half-life (Y_2) if we let $Y_1 \sim WGG\,(1,\ 1,\ 2,\ -5,\ 0.3)$ and $Y_2 \sim WGG\,(3,\ 1,\ 2,\ 3,\ 0.3)$. The table gives the means and the variances for both distributions. The distribution of $V = \dfrac{Y_1}{Y_2}$ is graphed in Figure 5.1 (b). The safety of the treatment is given by the shaded area in the graph and can be calculated as $\theta = 0.777$.



	Mean	Variance
Dose interval Y_1	1.186	0.057
Half-life Y_2	0.918	0.033

Safety $\theta = 0.777$

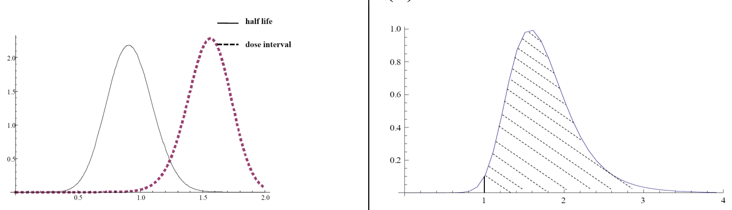
Figure 5.1 Safety of the treatment for dose interval (Y_1) and half-life (Y_2) with $Y_1 \sim WGG\,(1,\ 1,\ 2, -5,\ 0.3)$ and $Y_2 \sim WGG\,(3,\ 1,\ 2,\ 3,\ 0.3)$.

It is typically more desirable for the safety of the treatment to be closer to one. One may improve the safety of the treatment by increasing the dose interval. Keeping half-life the same as above, one can change the distribution of the dose interval to $Y_1 \sim WGG\,(1,\ 1,\ 2, -10,\ 0.3)$.

Similar to Figure 5.1, Figure 5.2 (a) gives the distribution of dose interval (Y_1) and half-life (Y_2) if we change the distribution of dose interval to $Y_1 \sim$

$WGG\,(1,\ 1,\ 2,-10,\ 0.3)$. The safety of the treatment is indicated by the shaded area in Figure 5.2 (b) and can be calculated as $\theta = 0.986$, which is a more desirable situation than the one illustrated in Figure 5.1.

(a) (b)



	Mean	Variance
Dose interval Y_1	1.547	0.032
Half-life Y_2	0.918	0.033

Safety $\theta = 0.986$

Figure 5.2 Safety of the treatment for dose interval (Y_1) and half-life (Y_2) with $Y_1 \sim WGG\,(1,\ 1,\ 2,-10,\ 0.3)$ and $Y_2 \sim WGG\,(3,\ 1,\ 2,\ 3,\ 0.3)$

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