

On the interlacing of zeros of linear combinations of Jacobi polynomials from different sequences

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Abstract

We investigate the interlacing of the zeros of linear combinations $p_n + aq_m$ with the zeros of the components p_n and q_m , where $\{p_n\}_{n=0}^{\infty}$ and $\{q_m\}_{m=0}^{\infty}$ are different sequences of Jacobi polynomials. The results we prove hold when p_n and q_m are Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and $P_m^{(\alpha',\beta')}(x)$ for certain values of α' and β' with $m = n$ or $m = n-1$. Numerical counterexamples are given in situations where interlacing fails to occur. We also show that the zeros of the linear combination $p_n + aq_m$ interlace with the zeros of some Jacobi polynomials besides the components of the linear combination.

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1 Introduction

Let $\{p_n\}_{n=1}^\infty$ be a sequence of polynomials with p_n of degree n that is orthogonal with respect to a positive Borel measure μ . The sequence is unique up to normalisation and it is well known that for each $n \in \mathbb{N}$, the zeros of p_n are real and simple and lie in the convex hull of the support of μ . Moreover, if $x_1 < x_2 < \dots < x_n$ are the zeros of p_n and $y_1 < y_2 < \dots < y_{n-1}$ are the zeros of p_{n-1} , then

$$x_1 < y_1 < x_2 < y_2 < \dots < x_{n-1} < y_{n-1} < x_n, \quad (1)$$

a property called the interlacing of the zeros.

As a partial converse, Wendroff (cf. [12]) proved that, given any n real distinct points $x_1 < x_2 < \dots < x_n$ and any $n - 1$ real distinct points $y_1 < y_2 < \dots < y_{n-1}$, such that (1) holds, the polynomials $p_n(x) = \prod_{i=1}^n (x - x_i)$

and $p_{n-1} = \prod_{i=1}^{n-1} (x - y_i)$ can be embedded in a sequence of (monic)orthogonal polynomials. However, it is not difficult to show (cf. [5]) that if the zeros of polynomials of successive degree in an infinite sequence satisfy the interlacing property, this by no means ensures the orthogonality of the sequence with respect to some positive Borel measure.

Nevertheless, the interlacing property of zeros is important in many other contexts. Examples where the interlacing of zeros plays a role include the convergence of numerical quadrature formulae (cf. [10]), the approximation of zeros by fixed point iteration techniques (cf. [11]), the completeness of the set of eigenfunctions to a self-adjoint Sturm-Liouville eigenvalue problem (cf. [2]) and the uniform convergence of derivatives arising in extended Lagrange interpolation (cf. [4]).

In this paper, we focus on the interlacing of the zeros of linear combinations of Jacobi polynomials of the form $p_n + \nu q_m$, where $p_n = P_n^{(\alpha, \beta)}$ and $q_m = P_m^{(\alpha', \beta')}$, $(\alpha, \beta) \neq (\alpha', \beta')$, with the zeros of the component polynomials p_n and q_m when $m = n$ and $m = n - 1$. We will also examine when interlacing takes place between the zeros of the linear combination and other selected Jacobi polynomials that are different from the component polynomials p_n and q_m .

Our method of proof makes extensive use of the interlacing property of the zeros of $P_n^{(\alpha, \beta)}$ and $P_m^{(\alpha', \beta')}$ for $m = n$ and $m = n - 1$ and $(\alpha', \beta') = (\alpha \pm t, \beta \pm k)$, $0 < t \leq 2$ and $0 < k \leq 2$, proved in [7]. These results proved

a conjecture posed by R. Askey in 1989 (cf. [1, p.29]) and also showed that the interlacing property of the zeros is retained not only for integer changes of the parameter β as Askey conjectured, but also for continuous variation of both the parameters α and β within a specified range.

2 Interlacing of the zeros of linear combinations of different Jacobi polynomials polynomials with the component polynomials

Interlacing properties of orthogonal polynomials can often be derived from the following simple result that has been proved in several contexts, for example, in dealing with polynomials associated with sequences of power moment functions [8, p. 117] and when considering quasi-orthogonality [3, Theorem 3].

Lemma 2.1 *Let $\{p_n\}$ and $\{q_n\}$ be two sequences of polynomials that are orthogonal with respect to positive Borel measures μ_1 and μ_2 , $\mu_1 \neq \mu_2$.*

- (a) *Assume that the zeros of p_n interlace with the zeros of q_n .*
 - (i) *The zeros of $E_n = p_n + \nu q_n$, $\nu \neq 0$ are all real and simple and interlace with the zeros of p_n and q_n .*
 - (ii) *If $\nu_1 \neq \nu_2$ are any two real numbers, then the zeros of $p_n(x) + \nu_1 q_n(x)$ and those of $p_n(x) + \nu_2 q_n(x)$ interlace.*
- (b) *Assume that the zeros of p_n and q_{n-1} interlace.*
 - (i) *The zeros of $F_n = p_n + \kappa q_{n-1}$, $\kappa \neq 0$ are all real, simple and interlace with the zeros of p_n and q_{n-1} .*
 - (ii) *Let κ_1 and κ_2 be two real numbers such that $\kappa_1 \neq \kappa_2$. Then the zeros of $p_n(x) + \kappa_1 q_{n-1}(x)$ and those of $p_n(x) + \kappa_2 q_{n-1}(x)$ interlace.*

Note that Lemma 2.1 also holds if the constants ν and κ in the linear combinations depend on n .

Corollary 2.2 *Let $\alpha > 1$, $\beta > -1$ and $\nu \neq 0$. Let*

$$E_n^{(\alpha, \beta, k, t)}(x) = P_n^{(\alpha, \beta)}(x) + \nu P_n^{(\alpha - k, \beta + t)}(x) \text{ for } t, k \in (0, 2].$$

The zeros of $E_n^{(\alpha, \beta, k, t)}(x)$ are real, simple and interlace with the zeros of $P_n^{(\alpha, \beta)}(x)$ as well as those of $P_n^{(\alpha - k, \beta + t)}(x)$.

Proof. It was shown in [7, Theorem 2.6], that the zeros of $P_n^{(\alpha,\beta)}$ interlace with the zeros of $P_n^{(\alpha-k,\beta+t)}$ for $t, k \in (0, 2]$. The result then follows from Lemma 2.1(a). \square

Remark The condition $\alpha > 1$ is necessary to ensure the orthogonality of $P_n^{(\alpha-2,\beta)}$ when $\beta > -1$. We note that analogous interlacing results will follow for the zeros of the linear combination

$$P_n^{(\alpha,\beta)}(x) + \nu P_n^{(\alpha+k,\beta-t)}(x), \quad \nu \neq 0$$

and those of $P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha+k,\beta-t)}(x)$ respectively, where $t, k \in (0, 2]$, by replacing α with $\alpha + k$ and β with $\beta - t$ in Corollary 2.2.

It is interesting to note that in the case of linear combinations of Jacobi polynomials of degree n , the zeros of $E_n^{(\alpha,\beta,k,t)}(x)$ do not necessarily interlace with the zeros of either $P_{n-1}^{(\alpha,\beta)}(x)$ or $P_{n-1}^{(\alpha-k,\beta+t)}(x)$. Indeed, even in the simplest case when $t = k = 1$ and $n = 6$, $\alpha = 2.3$, $\beta = 3.2$, $\nu = 3$, the zeros of $E_6^{(2.3,3.2,1,1)}(x)$ are given by

$$x_1 = -0.666, x_2 = -0.347, x_3 = 0.0014, x_4 = 0.341, x_5 = 0.6359, x_6 = 0.8571$$

while those of $P_5^{(2.3,3.2)}$ are

$$x = -0.684915, x = -0.328066, x = 0.0711339, x = 0.457021 \text{ and } x = 0.775148$$

and those of $P_5^{(2.3-1,3.2+1)}$ are

$$x = -0.571753, x = -0.177335, x = 0.22934, x = 0.592974, x = 0.862258.$$

Figures 1 and 2 show the zeros of these polynomials.

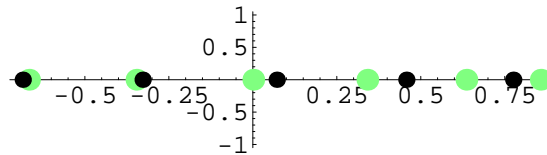
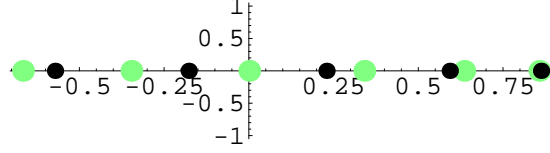


Figure 1: The zeros of $E_6^{(2.3,3.2,1,1)}$ are given by the larger green dots, while those of $P_5^{(2.3,3.2)}$ are smaller and black

Figure 2: The larger green dots represent the zeros of $E_6^{(2.3,3.2,1,1)}$ while the black dots are the zeros of $P_5^{(2.3-1,3.2+1)}$



Lemma 2.1(a) requires that the zeros of p_n and q_n are interlacing. We showed in [7] that the zeros of Jacobi polynomials of the same degree do not interlace when both the parameters α and β are increased simultaneously. Using this, it is not difficult to construct examples with $p_n = P_n^{(\alpha,\beta)}$ and $q_n = P_n^{(\alpha+k,\beta+t)}$ where the zeros of $p_n + \nu q_n$ and p_n or q_n do not interlace. For example, Figure 3 shows the zeros of the linear combination $P_n^{(\alpha,\beta)} + \nu P_n^{(\alpha+k,\beta+t)}$ and the component polynomial $P_n^{(\alpha,\beta)}$ for $n = 4$, $\alpha = 1.266$, $\beta = 1.85$, $\nu = 4.76$, $k = t = 0.5$.

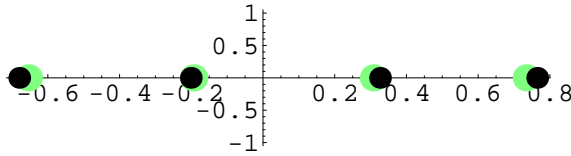


Figure 3: The zeros of $P_4^{(1.266,1.85)} + 4.76P_4^{(1.266+0.5,1.85+0.5)}$ are represented by the larger dots in green and those of $P_4^{(1.266,1.85)}$ are the smaller black dots.

The assumption made in Lemma 2.1(b) that the zeros of p_n and q_{n-1} interlace, is satisfied when $p_n = P_n^{(\alpha,\beta)}$ and $q_{n-1} = P_{n-1}^{(\alpha+t,\beta+k)}$ with $0 \leq t \leq 2$ and $0 \leq k \leq 2$ (cf. [7, Theorem 2.3]).

Corollary 2.3 *Let $\alpha > -1$, $\beta > -1$, $t, k \in [0, 2]$ and $F_n^{(\alpha,\beta,t,k)}(x) = P_n^{(\alpha,\beta)}(x) + \mu P_{n-1}^{(\alpha+t,\beta+k)}(x)$. Then the zeros of $P_n^{(\alpha,\beta)}(x)$ and the zeros of $P_{n-1}^{(\alpha+t,\beta+k)}(x)$ interlace with the zeros of $F_n^{(\alpha,\beta,t,k)}(x)$.*

Proof. It was shown in [7, Theorem 2.3], that the zeros of $P_n^{(\alpha,\beta)}$ interlace with the zeros of $P_{n-1}^{(\alpha+t,\beta+k)}$ and the result follows as an immediate consequence of Lemma 2.1(b). \square

In general, the zeros of $F_n^{(\alpha,\beta,t,k)}(x)$ do not interlace with the zeros of $P_{n-1}^{(\alpha,\beta)}(x)$. Indeed, if $n = 7$, $\alpha = 2.3$, $\beta = 3.2$ and $\nu = 1.2$, $t = 1.7$, $k = 1$ then the zeros of $F_n^{(\alpha,\beta,t,k)}(x)$ are

$$\begin{aligned} x_1 = -1.56243, \quad x_2 = -0.690889, \quad x_3 = -0.403688, \quad x_4 = -0.0921743, \\ x_5 = 0.21912, \quad x_6 = 0.507874, \quad x_7 = 0.756135 \end{aligned}$$

while those of $P_{n-1}^{(\alpha,\beta)}(x)$ are

$$\begin{aligned} y_1 = -0.748195, \quad y_2 = -0.454948, \quad y_3 = -0.112964, \\ y_4 = 0.239816, \quad y_5 = 0.563443, \quad y_6 = 0.821418. \end{aligned}$$

Remark We note that similar results have been obtained for the zeros of linear combinations of Laguerre polynomials and their components (cf. [6]) and that the one parameter family of classical orthogonal polynomials, the Gegenbauer polynomials, are a special case of the Jacobi polynomials where $\alpha = \beta$.

3 Interlacing of the zeros of linear combinations of different Jacobi polynomials with other Jacobi polynomials

The results that we prove in this section on the interlacing of the zeros of linear combinations $p_n + \nu q_n$, where $p_n = P_n^{(\alpha,\beta)}$ and $q_n = P_n^{(\alpha',\beta')}$, with Jacobi polynomials other than the components of the linear combination, are of independent interest and will also have specific application in a subsequent paper where we consider interlacing properties of the zeros of the linear combination $p_n + \nu q_n$ with those of $p_{n+1} + \nu q_{n+1}$.

Our method of proof makes extensive use of the relationship between ${}_2F_1$ and Jacobi polynomials (cf. [9, p.254]), as well as the contiguous function relations of the hypergeometric polynomials.

Theorem 3.1 *Let $\beta > 0$, $0 < r < \frac{\alpha + n}{1 + \beta + n}$ and let*

$$E_n^{(\alpha,\beta,1,1)} = P_n^{(\alpha,\beta)} + rP_n^{(\alpha-1,\beta+1)}.$$

a) *If $\alpha > -1$ then $E_n^{(\alpha,\beta,1,1)}$ and $P_{n+1}^{(\alpha,\beta-1)}$ have interlacing zeros.*

b) *If $\alpha > 0$ then $E_n^{(\alpha,\beta,1,1)}$ and $P_{n+1}^{(\alpha,\beta)}$ have interlacing zeros.*

Proof. a) The connection between Jacobi and hypergeometric polynomials, together with the contiguous relation (cf. [9, p.71, eqn.1]), yields

$$(1 + \alpha + \beta + 2n)P_n^{(\alpha, \beta)}(x) = (1 + \alpha + \beta + n)P_n^{(\alpha, \beta+1)}(x) + (\alpha + n)P_{n-1}^{(\alpha, \beta+1)}(x),$$

while the contiguous relation (cf. [9, p.71, eqn 13]) may be written as

$$\begin{aligned} (1 + \alpha + \beta + 2n)P_n^{(\alpha-1, \beta+1)}(x) \\ = (1 + \alpha + \beta + n)P_n^{(\alpha, \beta+1)}(x) - (\beta + n + 1)P_{n-1}^{(\alpha, \beta+1)}(x). \end{aligned}$$

Since $E_n^{(\alpha, \beta, 1, 1)} = P_n^{(\alpha, \beta)} + rP_n^{(\alpha-1, \beta+1)}$,

$$\begin{aligned} (1 + \alpha + \beta + 2n)E_n^{(\alpha, \beta, 1, 1)}(x) \\ = (1 + \alpha + \beta + n)(1 + r)P_n^{(\alpha, \beta+1)}(x) + [\alpha + n - r(\beta + n + 1)]P_{n-1}^{(\alpha, \beta+1)}(x). \end{aligned} \quad (2)$$

On the other hand, from ([9, p.71, eqn.6])

$$\begin{aligned} 2(1 + \alpha + \beta + 2n)(n + 1)P_{n+1}^{(\alpha, \beta-1)}(x) \\ = (1 + \alpha + \beta + n)[1 + \alpha - \beta + 2n + (1 + \alpha + \beta + 2n)x]P_n^{(\alpha, \beta+1)}(x) \\ - 2\beta(\alpha + n)P_{n-1}^{(\alpha, \beta+1)}(x). \end{aligned} \quad (3)$$

Thus

$$\begin{aligned} E_n^{(\alpha, \beta, 1, 1)}(x) \\ = -\frac{2(1 + \alpha + \beta + 2n)(n + 1)[\alpha + n - r(\beta + n + 1)]}{2\beta(\alpha + n)(1 + \alpha + \beta + 2n)}P_{n+1}^{(\alpha, \beta-1)}(x) \\ + \frac{(1 + \alpha + \beta + n)}{2\beta(\alpha + n)(1 + \alpha + \beta + 2n)}A_n^{(\alpha, \beta, r)}P_n^{(\alpha, \beta+1)}(x) \end{aligned} \quad (4)$$

where

$$A_n^{(\alpha, \beta, r)} = 2\beta(\alpha + n)(1 + r) + [\alpha + n - r(\beta + n + 1)][1 + \alpha + \beta + 2n + (1 + \alpha + \beta + 2n)x]$$

changes sign only if $x = -1 - \frac{2\beta r}{\alpha + n - r(\beta + n + 1)}$. It is clear that if $\alpha > -1$, $\beta > 0$ and $0 < r < \frac{\alpha + n}{1 + \beta + n}$, the coefficient of $P_n^{(\alpha, \beta+1)}$ in (4) does not change sign on $(-1, 1)$. Evaluating (4) at consecutive zeros x_i and x_{i+1} , $i = 1, \dots, n$, of $P_{n+1}^{(\alpha, \beta-1)}(x)$, one obtains $E_n^{(\alpha, \beta, 1, 1)}(x_i)E_n^{(\alpha, \beta, 1, 1)}(x_{i+1}) < 0$ since $P_{n+1}^{(\alpha, \beta-1)}$ and $P_n^{(\alpha, \beta+1)}$ have interlacing zeros (cf. [7, Theorem 2.3]).

b) From equation (cf. [9, p.72, ex.23]) one obtains

$$\begin{aligned} & 2(1 + \alpha + \beta + n)(n + 1)P_{n+1}^{(\alpha, \beta)}(x) \\ &= [2 + \alpha + \beta + 2n + (2 + \alpha + \beta + 2n)x](\alpha + n)P_n^{(\alpha-1, \beta+1)}(x) \\ & \quad + (\beta + n + 1)[- \alpha - \beta + (2 + \alpha + \beta + 2n)x]P_n^{(\alpha, \beta)}(x). \end{aligned}$$

Replacing $P_n^{(\alpha-1, \beta+1)}$ by $\frac{1}{r}[E_n^{(\alpha, \beta, 1, 1)} - P_n^{(\alpha, \beta)}]$, we have

$$\begin{aligned} & 2r(1 + \alpha + \beta + n)(n + 1)P_{n+1}^{(\alpha, \beta)}(x) \\ &= (2 + \alpha + \beta + 2n)(1 + x)(\alpha + n)E_n^{(\alpha, \beta, 1, 1)}(x) \\ & \quad + \{r(\beta + n + 1)[- \alpha - \beta + (2 + \alpha + \beta + 2n)x] \\ & \quad - (2 + \alpha + \beta + 2n)(1 + x)(\alpha + n)\}P_n^{(\alpha, \beta)}(x). \end{aligned} \quad (5)$$

For $\alpha > 0$, $\beta > 0$ and $0 < r < \frac{\alpha + n}{1 + \beta + n}$, the coefficient of $P_n^{(\alpha, \beta)}$ in this equation does not change sign on $(-1, 1)$ and since $P_{n+1}^{(\alpha, \beta)}$ and $P_n^{(\alpha, \beta)}$ have interlacing zeros, we deduce the result by evaluating (5) at consecutive zeros of $P_{n+1}^{(\alpha, \beta)}(x)$. □

Theorem 3.2 *Let $E_n^{\alpha, \beta, 0, 1} = P_n^{(\alpha, \beta)} + rP_n^{(\alpha, \beta+1)}$. If $\alpha > 0$, $\beta > -1$ and $r > \frac{n + 1}{\beta + n + 1}$, the zeros of $E_n^{\alpha, \beta, 0, 1}$ and those of $P_{n+1}^{(\alpha-1, \beta+1)}$ interlace.*

Proof. Shifting c to $c - 1$ in [9, p.72] and using [9, eq.13, p.71] we obtain

$$\begin{aligned} & 2(n + 1)P_{n+1}^{(\alpha-1, \beta+1)}(x) \\ &= (\alpha + \beta + (2 + \alpha + \beta + 2n)x)P_n^{(\alpha, \beta+1)}(x) - 2(\beta + n + 1)P_n^{(\alpha, \beta)}(x). \end{aligned}$$

$$\text{Since } P_n^{(\alpha, \beta)}(x) = E_n^{(\alpha, \beta, 0, 1)}(x) - rP_n^{(\alpha, \beta+1)}(x),$$

$$\begin{aligned} & 2(n + 1)P_{n+1}^{(\alpha-1, \beta+1)}(x) \\ &= [\alpha + \beta + 2r(\beta + n + 1) + (2 + \alpha + \beta + 2n)x]P_n^{(\alpha, \beta+1)}(x) \\ & \quad - 2(\beta + n + 1)E_n^{(\alpha, \beta, 0, 1)}(x). \end{aligned} \quad (6)$$

The coefficient of $P_n^{(\alpha, \beta+1)}$ is zero only if $x = -1 - \frac{2r(\beta + n + 1) - 2(n + 1)}{\alpha + \beta + 2 + 2n}$ and therefore the coefficient does not change sign on $(-1, 1)$ when

$$r > \frac{n + 1}{\beta + n + 1}.$$

Evaluating (6) at consecutive zeros x_i and x_{i+1} , $i = 1, 2, \dots, n$, of $P_{n+1}^{(\alpha-1, \beta+1)}$ we obtain $E_n^{(\alpha, \beta, 0, 1)}(x_i)E_n^{(\alpha, \beta, 0, 1)}(x_{i+1}) < 0$ since the zeros of $P_{n+1}^{(\alpha-1, \beta+1)}$ and $P_n^{(\alpha, \beta+1)}$ interlace (cf. [7, Theorem 2.2]). We deduce that there is at least one zero of $E_n^{(\alpha, \beta, 0, 1)}$ between any two consecutive zeros of $P_{n+1}^{(\alpha-1, \beta+1)}$ and the result follows. \square

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