THE CONVERGENCE SPACE OF MINIMAL USCO MAPPINGS

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Abstract. A convergence structure generalizing the order convergence structure on the set of Hausdorff continuous interval functions is defined on the set of minimal usco maps. The properties of the obtained convergence space are investigated and essential links with the pointwise convergence and the order convergence are revealed. The convergence structure can be extended to a uniform convergence structure so that the convergence space is complete. The important issue of the denseness of the subset of all continuous functions is also addressed.

1. Introduction

The aim of this paper, a first in the literature, is to define a topological structure on the space $\mathcal{M}(X,Y)$ of *minimal usco* maps of X into Y, where X, Y are topological spaces. We recall that such maps $f \in \mathcal{M}(X,Y)$ are *set-valued*, thus we have $f(x) \subseteq Y$, for $x \in X$. In the usual case of spaces of point valued functions, there has for longer been a large variety of useful topological structures on such spaces. Yet, in spite of the enormous development in set-valued analysis during the last few decades, there have only been few topological structures established on spaces of set-valued functions. And as seen next, there is at the present stage a manifest need for such topological structures. This paper, therefore, introduces a *convergence structure* on the space $\mathcal{M}(X,Y)$ of minimal usco maps, and does so with the following three features : (i) it brings forth many new useful properties of the spaces $\mathcal{M}(X,Y)$, (ii) it allows natural embeddings into $\mathcal{M}(X,Y)$ of classical spaces of point valued functions, among them the space $C(X,Y)$ of continuous functions from X to Y, (iii) when $Y = \mathbb{R}$ then $\mathcal{M}(X,Y)$ becomes the space $\mathbb{H}(X,\mathbb{R})$ of Hausdorff continuous interval valued maps, and remarkably, the convergence structure defined in this paper on $\mathcal{M}(X,Y)$ becomes the well known order convergence on $\mathbb{H}(X,\mathbb{R})$, see [4], thus $\mathcal{M}(X,Y)$ is a natural generalization of the space $\mathbb{H}(X,\mathbb{R})$. Here it should be mentioned that the space $\mathbb{H}(X,\mathbb{R})$ has recently proved to play important role in several directions, among them, Approximation Theory, regularity of solutions for very large classes of nonlinear systems of PDEs, and as containing the Dedekind order completion of the space $C(X,\mathbb{R})$ of real valued continuous functions on X. The latter result solves a long outstanding basic problem, see [1, 2, 3]. It can therefore be expected that the convergence structure on the space $\mathcal{M}(X,Y)$ of minimal usco maps introduced and studied in this paper can further facilitate

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the development of the study of set valued functions. Also, it can further improve the regularity of solutions for the mentioned very large classes of nonlinear systems of PDEs.

The paper is organized as follows.

For completeness of the exposition we recall in the next section the definitions of usco and minimal usco maps and give some of their basic properties. We also define the notion of a quasiminimal usco map which is quite important for the topic.

In Section 3 we define convergence of filters on the space $\mathcal{M}(X,Y)$ and we prove it satisfies the axioms of a convergence structure. Some basic properties together with characterizations of the convergent sequences and nets are also presented. It is shown that in general the convergence is not topological. Hence $\mathcal{M}(X,Y)$ is a convergence space but not a topological space.

The relationship of the convergence in $\mathcal{M}(X,Y)$ and the pointwise convergence is studied in Section 4. It is shown through examples that in general neither convergence implies the other. Nevertheless a strong connection exists. In particular, for X a Baire space and Y a metric space any filter convergent in $\mathcal{M}(X,Y)$ converges pointwise on a residual subset of X.

In Section 5 we consider the special case when the target space Y is the real line. Then $\mathcal{M}(X,\mathbb{R})$ can be ordered similarly to the way interval functions are ordered and we show that the convergence in $\mathcal{M}(X,\mathbb{R})$ is equivalent to the order convergence. Hence $\mathcal{M}(X,\mathbb{R})$ is isomorphic to the convergence space of Hausdorff continuous functions on X equipped with the order convergence structure.

Section 6 contains the definition of a uniform convergence structure on $\mathcal{M}(X,Y)$ for the case when X is a Baire space and Y a metric space. We show that this uniform convergence structure induces our convergence structure and that $\mathcal{M}(X,Y)$ is complete.

In Section 7 we consider the set $\mathcal{C}(X,Y)$ of all continuous functions. The concept of minimal usco generalizes the concept of continuity while retaining some of its essential properties. It is interesting from both theoretical and practical points of view when the set $\mathcal{C}(X,Y)$ is dense in $\mathcal{M}(X,Y)$. We give a partial answer formulating some open questions as well.

2. Usco and minimal usco maps

Let X and Y be topological spaces. A set-valued map $g : X \to Y$ is called *upper semicontinuous compact valued* (shortly *usco*) if

- $g(x)$ is a nonempty compact subset of Y for each $x \in X$;
- $\{x \in X : g(x) \subset U\}$ is open in X for each open subset U of Y.

We will always assume that the range space Y is Hausdorff. For the domain space X we require no separation axioms.

A set-valued map $g : X \to Y$ is canonically identified with its *graph*, i.e. with the set

$$\{(x,y) \in X \times Y : y \in g(x)\}.$$

Using this identification we will consider unions, intersections and inclusions of set-valued mappings. Hence, for example, if $g : X \to Y$ and $h : X \to Y$ are two set-valued mappings, then $g \subset h$ means that the graph of g is a subset of the graph of h, i.e., $g(x) \subset h(x)$ for each $x \in X$.

If $g : X \to Y$ is a set-valued mapping and $A \subset X$ we use, following the standard convention, the symbol $g(A)$ to denote $\bigcup\{g(x) : x \in A\}$.

Some basic properties of usco maps needed in the sequel are given in the next two lemmata. The proofs are easy exercises on usco maps.

Lemma 1.

(i) *Let $g_j : X \to Y$, $j = 1, \dots, n$, be usco maps. Then $g_1 \cup \dots \cup g_n$ is usco as well.*

(ii) *Let $\mathcal{G}$ be a family of usco maps from X to Y such that for each finite subfamily $\mathcal{K} \subset \mathcal{G}$ the intersection $\bigcap \mathcal{K}$ is a nonempty-valued mapping. Then $\bigcap \mathcal{G}$ is usco.*

Lemma 2. *Let $g : X \to Y$ be a usco map.*

(i) *Let $h \subset g$ be a set-valued mapping. Suppose there is an open set $U \subset X$ such that*
- *$h(x) = g(x)$ for $x \in X \setminus U$;*
- *$h|_U : U \to Y$ is usco.*

Then h is usco.

(ii) *Let $U \subset X$ be open and $F \subset Y$ be closed. Then the mapping $h : X \to Y$ defined by*

$$h(x) = \begin{cases} g(x) \cap F, & x \in U, \\ g(x), & x \in X \setminus U, \end{cases}$$

is usco provided it is nonempty-valued.

A usco map $g : X \to Y$ is called *minimal* if it is minimal with respect to inclusion, i.e., if $g = h$ whenever $h : X \to Y$ is usco satisfying $h \subset g$. It is a well-know consequence of Zorn's lemma (and of Lemma 1(ii)) that for each usco map $g : X \to Y$ there is a minimal usco $h \subset g$, [6].

The following characterization of minimal usco maps can be found in [7, Lemma 3.1.2].

Lemma 3. *Let $g : X \to Y$ be a usco map. The following assertions are equivalent.*

(i) *g is minimal.*

(ii) *Whenever $V \subset X$ and $U \subset Y$ are open sets such that $g(V) \cap U \neq \emptyset$, there is a nonempty open set $W \subset V$ with $g(W) \subset U$.*

We will denote by $\mathcal{M}(X,Y)$ the set of all minimal usco maps from X to Y. The minimal usco maps generalize the concept of continuous function and retain some of its properties. For example, a minimal usco map is completely determined by its values on a dense subset of the domain as stated in the next lemma. The proof is trivial and hence it is omitted.

Lemma 4. *Let f and g be usco mappings from X to Y such that f is minimal. If $f \not\subset g$, then there is a nonempty open set $U \subset X$ such that $f(U) \cap g(U) = \emptyset$.*

In particular, if $f, g \in \mathcal{M}(X,Y)$ are such that there exists a dense subset D of X such that $f(x) \cap g(x) \neq \emptyset$ for each $x \in D$, then $f = g$.

Let g be an usco map from X to Y. We associate with g the following subset of $\mathcal{M}(X,Y)$:

$$[g] = \{f \in \mathcal{M}(X,Y) : f \subset g\} \tag{1}$$

By the above the set $[g]$ is not empty. If g is a minimal usco map then we have $[g] = \{g\}$. On the other hand, if $[g]$ contains only one element, it need not be minimal. Since such usco maps will be important for us, we call them *quasiminimal.*

The following lemmata present some properties of quasiminimal usco maps which we will need in the sequel.

Lemma 5. *Let g be an usco map from X to Y. If there exists a dense subset D of X such that $g(x)$ is a singleton for all $x \in D$ then g is quasiminimal.*

The proof is an easy consequence of Lemma 4.

Lemma 6. *Let g_1 and g_2 be quasiminimal usco maps with $[g_1] = [g_2]$. Then $g_1 \cup g_2$ is quasiminimal, too. (And $[g_1 \cup g_2] = [g_1]$.)*

Proof. The map $g_1 \cup g_2$ is usco by Lemma 1. Let f be the unique element of $[g_1]$. Then clearly $f \in [g_1 \cup g_2]$. We will show it is the unique element with the latter property.

Let $h \in [g_1 \cup g_2]$. If $h \cap g_1$ is nonempty-valued, then it is usco by Lemma 1. Then it follows from the fact that $[g_1] = \{f\}$ that $f \subset h \cap g_1$. By the minimality of h we get $f = h$.

Therefore, if $h \neq f$, there is $x \in X$ such that $h(x) \cap g_1(x) = \emptyset$. Using the Hausdorff property of Y we get an open set $V \subset X$ containing x such that $h(V) \cap g_1(V) = \emptyset$. Define a map $\tilde{h} : X \to Y$ by the formula

$$\tilde{h}(x) = \begin{cases} g_2(x), & x \in X \setminus V, \\ h(x), & x \in V. \end{cases}$$

Then $\tilde{h} \subset g_2$ and by Lemma 2 it is usco. Hence $f \subset \tilde{h}$, a contradiction. □

Lemma 7. *Let X be a Baire topological space, Y a metrizable space and $g : X \to Y$ a quasiminimal usco mapping. Then $g(x)$ is a singleton for all x in a residual subset of X.*

The proof can be done by a minor modification of the proof of [7, Proposition 3.1.4].

3. Convergence structure on $\mathcal{M}(X, Y)$

In this section we will define convergence of filters on $\mathcal{M}(X, Y)$ and show that it defines a convergence structure.

Definition 8. A filter $\mathcal{F}$ on $\mathcal{M}(X, Y)$ *converges to* $f \in \mathcal{M}(X, Y)$ and we write $\mathcal{F} \to f$ if

$$\{f\} = \bigcap \{[g] : g \text{ is a usco map}, [g] \in \mathcal{F}\}.$$

Remarks 9.

1. If $\mathcal{F} \to f$, then there is at least one usco map $g : X \to Y$ with $[g] \in \mathcal{F}$. If the filter $\mathcal{F}$ is such that there exists an usco map $g : X \to Y$ with $[g] \in \mathcal{F}$ it is called *usco-bounded*. Since this is the only concept of boundedness considered in the paper we call the usco-bounded filters simply bounded.
2. If g_1 and g_2 are two usco maps such that both $[g_1]$ and $[g_2]$ belong to a filter $\mathcal{F}$, then $[g_1] \cap [g_2] \in \mathcal{F}$ as well. Thus $g_1 \cap g_2$ is nonempty-valued and therefore it is a usco map by Lemma 1. Further, clearly $[g_1 \cap g_2] = [g_1] \cap [g_2]$. It follows that the family

$$\{[g] : g \text{ is a usco map}, [g] \in \mathcal{F}\} \qquad (2)$$

is closed to finite intersections and hence it is a filter base provided it is nonempty. For any bounded filter $\mathcal{F}$ on $\mathcal{M}(X,Y)$ we denote by $\mathcal{G}_{\mathcal{F}}$ the filter which is generated by the family (2). Obviously, $\mathcal{G}_{\mathcal{F}}$ is coarser then $\mathcal{F}$ and we have from the definition that

$$\mathcal{F} \to f \iff \mathcal{G}_{\mathcal{F}} \to f \tag{3}$$

3. Let $\mathcal{F}$ be a bounded filter. Set
$$g_{\mathcal{F}} = \bigcap \{g : g \text{ is a usco map}, [g] \in \mathcal{F}\}.$$
Then $g_{\mathcal{F}}$ is a usco map (by the previous remark and Lemma 1) and we have
$$\bigcap_{F \in \mathcal{G}_{\mathcal{F}}} F = [g_{\mathcal{F}}].$$
Further, $\mathcal{F} \to f$ if and only if $[g_{\mathcal{F}}] = \{f\}$.
4. It is obvious from the definition that one filter cannot converge to more than one element of $\mathcal{M}(X,Y)$.
5. If the domain space X is a singleton, then $\mathcal{M}(X,Y)$ can be canonically identified with Y. Then a filter $\mathcal{F}$ on Y converges to $y \in Y$ in $\mathcal{M}(X,Y)$ if and only if it contains a compact subset of Y and converges to y in the topology of Y.

According to Definition 8 with every point $f \in \mathcal{M}(X,Y)$ we associate a set of filters $\lambda(f)$ which converge to f. The mapping λ from $\mathcal{M}(X,Y)$ into the power set of the set of filters on $\mathcal{M}(X,Y)$ is called a convergence structure and $(\mathcal{M}(X,Y),\lambda)$ is called a convergence space if the following conditions are satisfied for all $f \in \mathcal{M}(X,Y)$, see [5]:

(4) • $\langle f\rangle \in \lambda(f)$, where $\langle f\rangle$ denotes the filter generated by $\{\{f\}\}$.

(5) • If $\mathcal{F}_1, \mathcal{F}_2 \in \lambda(f)$ then $\mathcal{F}_1 \cap \mathcal{F}_2 \in \lambda(f)$.

(6) • If $\mathcal{F}_1 \in \lambda(f)$ then $\mathcal{F}_2 \in \lambda(f)$ for all filters $\mathcal{F}_2$ on $\mathcal{M}(X,Y)$ which are finer than $\mathcal{F}_1$.

Theorem 10. *The mapping λ is a convergence structure on $\mathcal{M}(X,Y)$.*

Proof. We need to show that for every $f \in \mathcal{M}(X,Y)$ conditions (4)–(6) are satisfied. Conditions (4) and (6) follow immediately from Definition 8. We will show that condition (5) also holds. Let $\mathcal{F}_1, \mathcal{F}_2 \in \lambda(f)$. We define the following set of usco mappings:
$$\Phi = \{g^{(1)} \cup g^{(2)} : g^{(1)}, g^{(2)} \text{ are usco maps}, [g^{(1)}] \in \mathcal{F}_1, [g^{(2)}] \in \mathcal{F}_2\}.$$
By Lemma 1 the family Φ consists of usco maps. As
$$[g^{(1)} \cup g^{(2)}] \supset [g^{(1)}] \cup [g^{(2)}],$$
we get $\{[h] : h \in \Phi\} \subset \mathcal{F}_1 \cap \mathcal{F}_2$. Set $g = \bigcap \Phi$. It is easy to check that
$$g = g_{\mathcal{F}_1} \cup g_{\mathcal{F}_2}.$$
As $[g_{\mathcal{F}_1}] = [g_{\mathcal{F}_2}] = \{f\}$, it follows by Lemma 6 that $[g] = \{f\}$ as well. From the inclusion $[g_{\mathcal{F}_1 \cap \mathcal{F}_2}] \subseteq [g]$ it follows that $[g_{\mathcal{F}_1 \cap \mathcal{F}_2}] = \{f\}$. Hence $\mathcal{F}_1 \cap \mathcal{F}_2 \to f$, which completes the proof. □

Let $(f_\nu)_{\nu\in I}$ be a net in $\mathcal{M}(X,Y)$ indexed by a directed set I. Following the general theory of convergence spaces the net $(f_\nu)_{\nu\in I}$ converges to $f\in\mathcal{M}(X,Y)$ if the filter generated by $\{\{f_\nu : \nu\ge\nu_0\} : \nu_0\in I\}$, converges to f in the convergence structure λ. In particular, a sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f\in\mathcal{M}(X,Y)$ if its Fréchet filter, that is, the filter generated by $\{\{f_m : m\ge n\} : n\in\mathbb{N}\}$, converges to f in the convergence structure λ. The following theorem gives an alternative characterization of the convergent nets and sequences.

Theorem 11.

a) *A net $(f_\nu)_{\nu\in I}$ converges to $f\in\mathcal{M}(X,Y)$ if and only if there is some $\nu_0\in I$ and usco mappings g_ν, $\nu\ge\nu_0$ such that*
  (i) *$f_\nu\subset g_\nu$ for $\nu\ge\nu_0$;*
  (ii) *$g_\nu\subset g_{\nu'}$ for $\nu\ge\nu'\ge\nu_0$;*
  (iii) *f is the unique minimal usco contained in $\bigcap\limits_{\nu\ge\nu_0} g_\nu$.*

b) *A sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f\in\mathcal{M}(X,Y)$ if and only if there exists a sequence of usco maps $(g_n)_{n\in\mathbb{N}}$ such that*
  (i) *$f_n\subset g_n$ for $n\in\mathbb{N}$;*
  (ii) *$g_m\subset g_n$ for each $m\ge n$;*
  (iii) *f is the unique minimal usco contained in $\bigcap\limits_{n\in\mathbb{N}} g_n$.*

Proof. a) Let $(f_\nu)_{\nu\in I}$ converge to $f\in\mathcal{M}(X,Y)$. Denote by $\mathcal{F}$ the filter generated by $\{\{f_\nu : \nu\ge\nu_0\} : \nu_0\in I\}$. For $\nu\in I$ set

$$g_\nu=\bigcap\{g : g \text{ is a usco map}, f_{\nu'}\subset g \text{ for } \nu'\ge\nu\}.$$

As $\mathcal{F}\to f$, there is a usco map g such that $[g]\in\mathcal{F}$. Then there is some ν_0 such that $f_\nu\subset g$ for $\nu\ge\nu_0$. Therefore g_{ν_0} is a well defined usco map (by Lemma 1). Hence g_ν is a well defined usco for each $\nu\ge\nu_0$. The conditions (i) and (ii) are satisfied by the definition. To see that the condition (iii) is satisfied too, it suffices to observe that for any usco map g with $[g]\in\mathcal{F}$ there is $\nu\ge\nu_0$ with $g_\nu\subset g$.

Assume now that there exists a net of usco maps $(g_\nu)_{n\ge\nu_0}$ satisfying conditions (i), (ii) and (iii). It follows from (i) and (ii) that $[g_\nu]\in\mathcal{F}$ for each $n\ge n_0$. Thus, due to (iii), $\mathcal{F}\to f$.

b) Suppose that $f_n\to f$. It follows from (1) that there is $n_0\in\mathbb{N}$ and usco maps g_n, $n\ge n_0$, satisfying a), b) and c). For $n<n_0$ we take

$$g_n=f_n\cup\ldots\cup f_{n_0-1}\cup g_{n_0}.$$

Then the usco maps g_n, $n\in\mathbb{N}$, fulfil the conditions (i), (ii) and (iii).

The inverse implication follows from that in a). □

Remark 12. The preceding theorem indicates a relation of the convergence on $\mathcal{M}(X,Y)$ to the order convergence on a lattice. We will examine the relationship in more detail in Section 5.

The preceeding theorem enables us to show that the convergence on $\mathcal{M}(X,Y)$ is not in general generated by a topology.

Example 13. *If $X=Y=[0,1]$, then the convergence in $\mathcal{M}(X,Y)$ is not generated by any topology.*

Proof. Let q_n, $n \in \mathbb{N}$ be an enumeration of rational numbers from $[0,1]$. We define continuous functions $f_n : [0,1] \to [0,1]$ by the formula

$$f_n(x) = \begin{cases} 1 - n|x - q_n|, & x \in (q_n - \frac{1}{n}, q_n + \frac{1}{n}) \cap [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Then the sequence f_n does not converge in $\mathcal{M}(X,Y)$. Indeed, if $\mathcal{F}$ is the Fréchet filter of this sequence, it is easy to check that

$$g_{\mathcal{F}}(x) = [0,1], \qquad x \in [0,1],$$

which is obviously not quasiminimal.

On the other hand, if a subsequence q_{n_k} converges to some $q \in [0,1]$, the sequence f_{n_k} converges to 0. Indeed, if we set

$$g_k(x) = \begin{cases} \{f_{n_l}(x) : l \geq k\}, & x \in [0,1] \setminus \{q\}, \\ [0,1], & x = q, \end{cases}$$

we get a decreasing sequence of usco maps such that $g_k \supset f_{n_k}$ for each k. Further,

$$\bigcap_{k \in \mathbb{N}} g_k(x) = \begin{cases} [0,1], & x = q, \\ \{0\}, & x \in [0,1] \setminus \{q\}. \end{cases}$$

This usco map is clearly quasiminimal and the only minimal usco contained in it is the constant zero function. Thus f_{n_k} converges to 0 by Theorem 11.

As each subsequence of q_n has a further convergent subsequence, we get that each subsequence of f_n has a further subsequence converging to 0. If the convergence were a topological one, it would imply that f_n converge to 0 as well. But it is not the case by the first paragraph, hence the convergence is not a topological one. □

Theorem 14. *The convergence given in Definition 8 is stable with respect to restrictions to open sets, that is, if a filter $\mathcal{F}$ on $\mathcal{M}(X,Y)$ converges to $f \in \mathcal{M}(X,Y)$ then for any open subset D of X the filter $\mathcal{F}|_D$ generated by the restrictions $\{F|_D : F \in \mathcal{F}\}$, where $F|_D = \{h|_D : h \in F\} \subset \mathcal{M}(D,Y)$, converges on D to the restriction of the limit $f|_D$.*

Proof. The restriction of a minimal usco to an open set is also a minimal usco, [10, Lemma 2]. Therefore $\mathcal{F}|_D$ is a filter on $\mathcal{M}(D,Y)$ and $f|_D \in \mathcal{M}(D,Y)$.

Set

$$\Phi = \{g|_D : g \text{ is a usco map}, [g] \in \mathcal{F}\}.$$

Then Φ is a family of usco maps from D to Y. Further,

$$[\Phi] = \{[h] : h \in \Phi\} \subset \mathcal{F}|_D.$$

Indeed, we have

$$[g]|_D = \{h|_D : h \in [g]\} \subset [g|_D]$$

for any usco map g. Obviously $f|_D \in \bigcap[\Phi]$. It remains to show that $\bigcap[\Phi]$ has no more elements. Let $h \in \bigcap[\Phi]$ be any element. Set

$$\psi(x) = \begin{cases} h(x), & x \in D, \\ g_{\mathcal{F}}(x), & x \in X \setminus D. \end{cases}$$

Then ψ is usco by Lemma 2. Further, $\psi \subset g_{\mathcal{F}}$, and hence $f \subset \psi$. It follows that $f|_D \subset h$, hence $f|_D = h$. This completes the proof. □

4. Relationship to pointwise convergence

In this section we give some relations of the convergence in $\mathcal{M}(X,Y)$ to the pointwise convergence.

Theorem 15. *Let X be a Baire space, f_n be a bounded sequence in $\mathcal{M}(X,Y)$ and $\varphi : X \to Y$ be a quasiminimal usco with $[\varphi] = \{f\}$. Suppose that for each $x \in X$ the sequence of compact sets $f_n(x)$ cumulates at $\varphi(x)$, i.e. for each open set $U \subset Y$ containing $\varphi(x)$ there is some $n_0 \in \mathbb{N}$ such that $f_n(x) \subset U$ for $n \geq n_0$. Then the sequence f_n converges to f in $\mathcal{M}(X,Y)$.*

Proof. Let g_n be the intersection of all usco maps containing f_k for $k \geq n$ and let g be the intersection of all g_n's. As the sequence f_n is bounded, g_n's and g are well-defined usco maps. By Theorem 11 it suffices to prove that $[g] = \{f\}$.

Choose $h \in [g]$ arbitrary. Suppose that $h \neq f$. Then there is some $x_0 \in X$ such that $h(x_0) \cap \varphi(x_0) = \emptyset$. Indeed, otherwise $h \cap \varphi$ would be a usco map contained in h and hence we would have $h \cap \varphi = h$, i.e. $h \subset \varphi$. But then necessarily $h = f$.

As Y is Hausdorff, there are disjoint open sets $V_1 \supset \varphi(x_0)$ and $V_2 \supset h(x_0)$. Further, there is an open neighborhood U of x_0 such that $\varphi(U) \subset V_1$ and $h(U) \subset V_2$. For each $n \in \mathbb{N}$ set

$$F_n = \{x \in X : (\forall k \geq n)(f_k(x) \setminus V_2 \neq \emptyset)\}.$$

Then each F_n is a closed subset of X. Moreover, the sets F_n cover U. Indeed, if $x \in U$, then $\varphi(x) \subset V_1$. Hence there is some n such that for each $k \geq n$ we have $f_k(x) \subset V_1$. Thus $x \in F_n$. As X is a Baire space, U is non-meager and hence there is some $n \in \mathbb{N}$ such that $F_n \cap U$ has nonempty interior. It means that there is a nonempty open set $W \subset U$ such that $f_k(x) \setminus V_2 \neq \emptyset$ whenever $k \geq n$ and $x \in W$. Hence, by the minimality of f_k we get (using Lemma 3) that $f_k(W) \cap V_2 = \emptyset$ for each $k \geq n$. Therefore, if we define

$$\widetilde{g}_n(x) = \begin{cases} g_n(x) \setminus V_2, & x \in W, \\ g_n(x), & x \in X \setminus W, \end{cases}$$

we get a usco map containing f_k for $k \geq n$. Thus $\widetilde{g}_n = g_n$. As $g \subset g_n$, we get that $g(x) \cap V_2 = \emptyset$ for each $x \in W$, which is a contradiction with the assumption $h \in [g]$. This completes the proof. $\square$

As a corollary we get the following assertions on sequences of continuous functions.

Corollary 16. *Let X be a Baire space, f_n be a sequence of continuous functions bounded in $\mathcal{M}(X,Y)$.*

(i) *If f_n pointwise converges to a continuous function f, then f_n converges to f in $\mathcal{M}(X,Y)$.*
(ii) *If $f \in \mathcal{M}(X,Y)$ is such that the sequence $f_n(x)$ converges to an element of $f(x)$ for each $x \in X$, then f_n converges to f in $\mathcal{M}(X,Y)$.*

The following example shows that all assumptions in Theorem 15 are needed. Namely, one can drop neither the assumption that the sequence is bounded (even if X is compact), nor the assumption that X is a Baire space (even if Y is compact and hence all filters are bounded). Further, Theorem 15 is not true for nets, even if both X and Y are compact.

Example 17.

1. *There is a sequence of continuous functions* $f_n : [0,1] \to \mathbb{R}$ *pointwise converging to* 0 *which is unbounded in* $\mathcal{M}([0,1],\mathbb{R})$.
2. *There is a sequence of continuous functions* $f_n : \mathbb{Q} \to [0,1]$ *pointwise converging to* 0 *which is not convergent in* $\mathcal{M}(\mathbb{Q},[0,1])$.
3. *There is a net of continuous functions* $f_\nu : [0,1] \to [0,1]$ *pointwise converging to* 0 *which is not convergent in* $\mathcal{M}([0,1],[0,1])$.

Proof. 1. It is sufficient to take the sequence

$$f_n(x) = \begin{cases} n(1-2n|x-\frac{1}{2n}|), & x \in [0,\frac{1}{n}],\\ 0 & \text{otherwise.}\end{cases}$$

2. Let q_n, $n \in \mathbb{N}$, be an enumeration of $\mathbb{Q}$. For each $n \in \mathbb{N}$ choose a continuous function $f_n : \mathbb{Q} \to [0,1]$ such that

$$f_n(x) = \begin{cases} 0 & x \in \{q_1,\dots,q_n\},\\ 1 & x \in \mathbb{Q} \setminus \bigcup_{k=1}^n (q_k - \frac{1}{n^2}, q_k + \frac{1}{n^2}).\end{cases}$$

Then f_n pointwise converge to 0. Further, fix $n \in \mathbb{N}$ and a usco map $g : \mathbb{Q} \to [0,1]$ containing f_m for $m \geq n$. Then $0 \in g(x)$ for each $x \in \mathbb{Q}$. Moreover, $1 \in g(x)$ for each

$$x \in \bigcup_{m\geq n}\left(\mathbb{Q} \setminus \bigcup_{k=1}^m \left(q_k - \frac{1}{m^2}, q_k + \frac{1}{m^2}\right)\right) = \mathbb{Q} \setminus \bigcap_{m\geq n}\bigcup_{k=1}^m \left(q_k - \frac{1}{m^2}, q_k + \frac{1}{m^2}\right).$$

As $\bigcap_{m\geq n}\bigcup_{k=1}^m [q_k - \frac{1}{m^2}, q_k + \frac{1}{m^2}]$ is a closed subset of $\mathbb{R}$ of Lebesgue measure zero, its complement is an open dense set. Therefore $1 \in g(x)$ for all x from a dense subset of $\mathbb{Q}$, and thus for all $x \in \mathbb{Q}$.

It follows that $\{0,1\} \subset g(x)$ for each $x \in \mathbb{Q}$ and hence the sequence f_n is not convergent in $\mathcal{M}(\mathbb{Q},[0,1])$ by Theorem 11.

3. For any nonempty finite set $A \subset [0,1]$ choose a continuous function $f_A : [0,1] \to [0,1]$ such that

$$f_A(x) = \begin{cases} 0 & x \in A,\\ 1 & x \in [0,1] \setminus \bigcup_{a\in A}\left(a - \frac{1}{|A|^2}, a + \frac{1}{|A|^2}\right),\end{cases}$$

where $|A|$ denotes the cardinality of A. If we consider finite subsets of $[0,1]$ ordered by inclusion, the net f_A pointwise converges to 0. Moreover, the net f_A is not convergent in $\mathcal{M}([0,1],[0,1])$. Indeed, let $B \subset [0,1]$ be a nonempty finite set and $g : [0,1] \to [0,1]$ be a usco map containing f_A for each $A \supset B$. Then $0 \in g(x)$ for each $x \in [0,1]$. Further, $1 \in g(x)$ for each

$$x \in \bigcup_{A\supset B}\left([0,1] \setminus \bigcup_{a\in A}\left(a - \frac{1}{|A|^2}, a + \frac{1}{|A|^2}\right)\right)$$
$$= [0,1] \setminus \bigcap_{A\supset B}\bigcup_{a\in A}\left(a - \frac{1}{|A|^2}, a + \frac{1}{|A|^2}\right)$$

Again, $\bigcap_{A\supset B}\bigcup_{a\in A}(a - \frac{1}{|A|^2}, a + \frac{1}{|A|^2})$ is a set of Lebesgue measure zero, hence the complement is dense in $[0,1]$. Therefore, $1 \in g(x)$ for all x from a dense subset of

$[0,1]$ and thus for all $x \in [0,1]$. It follows that $\{0,1\} \subset g(x)$ for all $x \in [0,1]$ and so the net f_A does not converge by Theorem 11. □

In the next theorem we give a characterization of the convergence for the case when X is a Baire space and Y a metric space. To formulate it we need the following natural notation.

Let $F \subset \mathcal{M}(X,Y)$. For a given $x \in X$, $F(x)$ is the set

$$F(x) = \bigcup \{f(x) : f \in F\}$$

Let $\mathcal{F}$ be a filter on $\mathcal{M}(X,Y)$. Then for every $x \in X$

$$\{F(x) : F \in \mathcal{F}\}$$

is a filter base on Y. Denote by $\mathcal{F}(x)$ the filter it generates.

Recall that $\mathcal{G}_{\mathcal{F}}$ denotes the filter generated by the family

$$\{[g] : g \text{ is a usco map}, [g] \in \mathcal{F}\}$$

(see Remark 9.2).

Theorem 18. *Let X be a Baire space and Y be a metric space with metric ρ. Let $\mathcal{F}$ be a filter on $\mathcal{M}(X,Y)$. Then $\mathcal{F}$ converges to $f \in \mathcal{M}(X,Y)$ if and only if there exists a dense (equivalently residual) subset D of X such that for every $x \in D$*

(i) *$f(x)$ is a singleton, that is, $f(x) \in Y$;*
(ii) *$\mathcal{G}_{\mathcal{F}}(x)$ converges to $f(x)$ with respect to the metric ρ.*

Proof. As $\mathcal{F} \to f$, the usco map $g_{\mathcal{F}}$ is quasiminimal and $[g_{\mathcal{F}}] = \{f\}$. By Lemma 7 there is a residual (and hence dense) set $D \subset X$ such that $g_{\mathcal{F}}(x) = f(x)$ is a singleton for each $x \in D$.

Fix $x \in D$. By the previous paragraph we have

$$\{f(x)\} = \bigcap \{g(x) : g \text{ is a usco map}, [g] \in \mathcal{F}\}.$$

If g is a usco map such that $[g] \in \mathcal{F}$, then

$$g(x) \supset \bigcup \{h(x) : h \in [g]\} = [g](x) \in \mathcal{G}_{\mathcal{F}}(x).$$

As these $g(x)$'s are compact subsets of Y belonging to $\mathcal{G}_{\mathcal{F}}(x)$ and their intersection is just $\{f(x)\}$, we get $\mathcal{G}_{\mathcal{F}}(x) \to f(x)$.

Conversely suppose that $D \subset X$ is dense and for each $x \in D$ the conditions (i) and (ii) hold. It follows from the condition (ii) that $\mathcal{F}$ is bounded in $\mathcal{M}(X,Y)$ and hence the mapping $g_{\mathcal{F}}$ is well-defined. It follows easily from (i) and (ii) that $g_{\mathcal{F}}(x) = f(x)$ for $x \in D$. Therefore $g_{\mathcal{F}}$ is quasiminimal (Lemma 5) and $[g_{\mathcal{F}}] = \{f\}$ (Lemma 4), i.e. $\mathcal{F} \to f$. This completes the proof. □

5. Convergence in $\mathcal{M}(X,\mathbb{R})$ and the order convergence

In this section we show that the convergence in $\mathcal{M}(X,\mathbb{R})$ is equivalent to the order convergence with respect to the natural partial order on $\mathcal{M}(X,\mathbb{R})$. Before giving the definitions and stating the equivalence we show a natural correspondence of the space $\mathcal{M}(X,\mathbb{R})$ and the space $\mathbb{H}(X,\mathbb{R})$ of Hausdorff continuous functions (see [12, 1]).

We start by the following obvious lemma.

Lemma 19.

- *Let $g : X \to \mathbb{R}$ be a usco map. Then the map $x \mapsto \max g(x)$ is upper semicontinuous on X and the map $x \mapsto \min g(x)$ is lower semicontinuous on X.*
- *Let $f_1 : X \to \mathbb{R}$ be a lower semicontinuous function and $f_2 : X \to \mathbb{R}$ be an upper semicontinuous function such that $f_1 \leq f_2$ on X. Then the set-valued map $x \mapsto [f_1(x), f_2(x)]$ is usco.*

Let $f \in \mathcal{M}(X, \mathbb{R})$. We define the following two real functions on X:

$$\begin{aligned} \underline{f}(x) &= \min f(x) \\ \overline{f}(x) &= \max f(x) \end{aligned}$$

By the previous lemma $\underline{f}$ is lower semicontinuous and $\overline{f}$ is upper semicontinuous. Further, we define a map $f_C : X \to \mathbb{R}$ by

$$f_C(x) = [\underline{f}(x), \overline{f}(x)].$$

By Lemma 19 it is a usco map. Moreover, we have the following.

Lemma 20.

(i) *If $f, g \in \mathcal{M}(X, \mathbb{R})$ are distinct, then $f_C(x) \cap g_C(x) = \emptyset$ for some $x \in X$.*
(ii) *For each $f \in \mathcal{M}(X, \mathbb{R})$ we have $[f_C] = \{f\}$.*
(iii) *For each $f \in \mathcal{M}(X, \mathbb{R})$ the usco map f_C is minimal within the convex valued (i.e., interval-valued) usco maps.*

Proof. (i) Let f and g be distinct elements of $\mathcal{M}(X, \mathbb{R})$. Then there is some $x_0 \in X$ with $f(x_0) \cap g(x_0) = \emptyset$ (otherwise $f \cap g$ would be a usco contained both in f and g and hence we would have $f = g = f \cap g$). Let $a = \max g(x_0)$ and $b = \max f(x_0)$. Then $a \neq b$, we can suppose without loss of generality that $a < b$. Choose some $c \in (a, b)$. As g is usco, there is an open neighborhood U of x_0 such that $g(U) \subset (-\infty, c)$. We have $f(U) \cap (c, +\infty) \neq \emptyset$, and hence by Lemma 3 there is a nonempty open set $V \subset U$ with $f(V) \subset (c, +\infty)$. It follows that $f_C(x) \cap g_C(x) = \emptyset$ for any $x \in V$.

(ii) This follows immediately from (i).

(iii) Let $g \subset f_C$ be an interval-valued usco. By (ii) we have $f \in [g]$. Hence it follows from the definition of f_C that $f_C \subset g$. □

It is easy to check that the minimal interval-valued usco maps are exactly the Hausdorff continuous functions in the sense of [12]. Hence, due to the previous lemma the mapping $f \mapsto f_C$ is a bijection of $\mathcal{M}(X, \mathbb{R})$ onto $\mathbb{H}(X, \mathbb{R})$. On the set $\mathbb{H}(X, \mathbb{R})$ there is a natural partial order (see [1]). We define a partial order on $\mathcal{M}(X, \mathbb{R})$ using the correspondence $f \mapsto f_C$:

For $f, g \in \mathcal{M}(X, \mathbb{R})$ we have

$$f \leq g \iff \underline{f}(x) \leq \underline{g}(x),\ \overline{f}(x) \leq \overline{g}(x),\ x \in X. \tag{7}$$

Using the minimality of f and g it is easy to see that either one of the inequalities on the right hand side above will suffice, that is, we have

$$f \leq g \iff \underline{f}(x) \leq \underline{g}(x),\ x \in X \iff \overline{f}(x) \leq \overline{g}(x),\ x \in X. \tag{8}$$

Indeed, let $\underline{f} \leq \underline{g}$ on X. The function $h(x) = \min\{\overline{f}(x), \overline{g}(x)\}$ is upper semicontinuous and clearly

$$\underline{f} \leq h \leq \overline{f}$$

on X. Hence the map $x \mapsto [\underline{f}(x), h(x)]$ is an interval-valued usco (see Lemma 19) contained in f_C. Then it is equal to f_C by Lemma 20. Hence $h = \overline{f}$, which means

$\overline{f} \le \overline{g}$ on X. This proves one implication, the inverse one can be proved in the same way.

Since the mapping $f \mapsto f_C$ is an order isomorphism of $\mathcal{M}(X,\mathbb{R})$ onto $\mathbb{H}(X,\mathbb{R})$ the set $\mathcal{M}(X,\mathbb{R})$ has the same order properties as $\mathbb{H}(X,\mathbb{R})$. For example, since $\mathbb{H}(X,\mathbb{R})$ is Dedekind order complete, see [1], $\mathcal{M}(X,\mathbb{R})$ is also Dedekind order complete. In particular this implies that $\mathcal{M}(X,\mathbb{R})$ is a lattice.

The following theorem shows an essential similarity between the the mappings in $\mathcal{M}(X,\mathbb{R})$ and the usual continuous real valued functions on X. It follows from the respective statement for Hausdorff continuous functions, see [1, Theorem 4].

Theorem 21. *Let $f, g \in \mathcal{M}(X,\mathbb{R})$ and let D be a dense subset of X. Then*

$$f|_D \le g|_D \Longrightarrow f \le g$$

Next we will establish a link between the order convergence on $\mathcal{M}(X,\mathbb{R})$ with respect to the order (7) and the convergence structure λ. Let us recall the definition for order convergence of filters. For a filter $\mathcal{F}$ on $\mathcal{M}(X,\mathbb{R})$ we consider the set of lower bounds

$$\mathcal{F}^- = \{\phi \in \mathcal{M}(X,\mathbb{R}) : \exists F \in \mathcal{F} : \phi \le h \text{ for all } h \in F\}$$

and the set of upper bounds

$$\mathcal{F}^+ = \{\psi \in \mathcal{M}(X,\mathbb{R}) : \exists F \in \mathcal{F} : \psi \ge h \text{ for all } h \in F\}.$$

We say that the filter $\mathcal{F}$ order converges to $f \in \mathcal{M}(X,\mathbb{R})$ if $\mathcal{F}^- \neq \emptyset$, $\mathcal{F}^+ \neq \emptyset$ and

$$f = \sup \mathcal{F}^- = \inf \mathcal{F}^+. \tag{9}$$

Remark 22. Let us notice that $\phi \in \mathcal{F}^-$ and $\psi \in \mathcal{F}^+$ if and only if the order interval $[\phi, \psi]$ belongs to $\mathcal{F}$. Hence a filter $\mathcal{F}$ order converges to f if and only if

$$\{f\} = \bigcap \{[\psi, \phi] : \psi, \phi \in \mathcal{M}(X,\mathbb{R}), [\psi, \phi] \in \mathcal{F}\}$$

Theorem 23. *A filter $\mathcal{F}$ on $\mathcal{M}(X,\mathbb{R})$ order converges to $f \in \mathcal{M}(X,\mathbb{R})$ iff $\mathcal{F} \in \lambda(f)$.*

Proof. Let $\mathcal{F}$ order converge to f. For arbitrary $\phi \in \mathcal{F}^-$ and $\psi \in \mathcal{F}^+$ we have $\phi \le \psi$. Hence the usco map $h_{\psi,\phi} : x \mapsto [\underline{\phi}(x), \overline{\psi}(x)]$ is well defined on X and $[h_{\psi,\phi}] \in \mathcal{F}$. Due to (9) we have that f is the only minimal usco contained in the map

$$\varphi = \bigcap \{h_{\phi,\psi} : \phi \in \mathcal{F}^-, \psi \in \mathcal{F}^+\} : x \mapsto \bigcap_{\phi \in \mathcal{F}^-,\, \psi \in \mathcal{F}^+} [\underline{\phi}(x), \overline{\psi}(x)],\ x \in X.$$

Since $g_{\mathcal{F}} \subset \varphi$ the map $g_{\mathcal{F}}$ is quasiminimal and contains f. Therefore $\mathcal{F} \in \lambda(f)$.

For the inverse implication assume that $\mathcal{F} \in \lambda(f)$. It is easy to see that

$$\underline{g_{\mathcal{F}}}(x) = \sup\{\underline{g}(x) : g \text{ is a usco map}, [g] \in \mathcal{F}\}. \tag{10}$$

For a given usco map g, denote by α_g the unique minimal usco contained in the map $x \mapsto [\underline{g}(x), \underline{g}^*(x)]$, where $\underline{g}^*$ is the upper semicontinuous envelope of $\underline{g}$, i.e. $\underline{g}^*$ is the pointwise infimum of all upper semicontinuous functions greater then $\underline{g}$. Clearly α_g is a lower bound of $[g]$. Hence the set $\mathcal{F}^-$ is not empty. Furthermore $f = \alpha_{g_{\mathcal{F}}}$. Then it follows from (10) that

$$f = \sup\{\alpha_g : g \text{ is a usco map}, [g] \in \mathcal{F}\} \le \sup \mathcal{F}^-. \tag{11}$$

In a similar way we prove that $\mathcal{F}^+$ is not empty and that

$$f \geq \inf \mathcal{F}^+. \tag{12}$$

Using that $\sup \mathcal{F}^- \leq \inf \mathcal{F}^+$ and the inequalities (11) and (12) we obtain (9). Hence $\mathcal{F}$ order converges to f. □

Remark 24. Let us note that the infimum and the supremum in (9) are not the pointwise ones. More precisely, we have that f is the unique minimal usco map contained in the quasiminimal usco map

$$x \mapsto [\sup_{\phi \in \mathcal{F}^-} \underline{\phi}(x), \inf_{\psi \in \mathcal{F}^+} \overline{\psi}(x)] \ , \quad x \in X.$$

Furthermore, if X is a Baire space there exists a residual subset D of X such that for all $x \in D$ the value of f is a singleton and

$$f(x) = \sup_{\phi \in \mathcal{F}^-} \underline{\phi}(x) = \inf_{\psi \in \mathcal{F}^+} \overline{\psi}(x)$$

Remark 25. The concept of order convergence is better known in the context of sequences, [9]. Let us recall that a sequence (f_n) on $\mathcal{M}(X, \mathbb{R})$ order converges to $f \in \mathcal{M}(X, \mathbb{R})$ if there exist an increasing sequence (α_n) and a decreasing sequence (β_n) on $\mathcal{M}(X, \mathbb{R})$ such that

$$\alpha_n \leq f_n \leq \beta_n, \tag{13}$$

$$\sup \alpha_n = \inf \beta_n = f. \tag{14}$$

Using the Dedekind completeness of $\mathcal{M}(X, \mathbb{R})$ it is easy to see that the order convergence of filters given through (9) induces the order convergence of sequences defined above. Therefore, the class of order convergent sequences coincides with the class of convergence sequences in λ.

Remark 26. It was shown in [4] that the sequential order convergence on $\mathbb{H}(X, \mathbb{R})$ cannot be induced by topology. Using that the mapping $f \mapsto f_C$ is an order isomorphism from $\mathcal{M}(X, \mathbb{R})$ to $\mathbb{H}(X, \mathbb{R})$, this also holds true for the order convergence on $\mathcal{M}(X, \mathbb{R})$. Since the convergence structure λ induces the sequential order convergence on $\mathcal{M}(X, \mathbb{R})$, see Theorem 23 and Remark 25, the convergence structure λ on $\mathcal{M}(X, \mathbb{R})$ is not topological.

6. Uniform convergence structure on $\mathcal{M}(X, Y)$.

In this section we assume that X is a Baire space and Y is a metric space with a metric ρ. In this case a usco mapping $f : X \to Y$ is quasiminimal if and only if it is singlevalued at points of a dense (equivalently residual) set (Lemma 5 and 7). We will need the following lemma on product mappings.

Lemma 27. *Let f and g be usco mappings from X to Y and $f \times g : X \to Y \times Y$ be defined by $(f \times g)(x) = f(x) \times g(x)$. Then the following is true.*

(i) *$f \times g$ is usco.*
(ii) *If f and g are quasiminimal then $f \times g$ is quasiminimal as well.*
(iii) *$f \times g$ need not be minimal even if f and g are minimal.*

Proof. The assertion (i) is well-known and easy to see. To show (ii) it is enough to notice that $f \times g$ is singlevalued at points of a residual set whenever both f and g have that property.

To show (iii) set $X = [0,\omega]$, $Y = [0,1]$ and define

$$f(x) = g(x) = \begin{cases} \{0\}, & x < \omega \text{ even}, \\ \{1\}, & x < \omega \text{ odd}, \\ \{0,1\}, & x = \omega. \end{cases}$$

Then f and g are minimal but $f \times g$ is not minimal. □

In particular, if $f, g \in \mathcal{M}(X,Y)$ the product mapping $f \times g$ is quasiminimal. So we can define a mapping $\chi : \mathcal{M}(X,Y) \times \mathcal{M}(X,Y) \to \mathcal{M}(X, Y \times Y)$ by the formula

$$\{\chi(f,g)\} = [f \times g].$$

Now we are ready to define a uniform convergence structure on $\mathcal{M}(X,Y)$. Let us recall that such a uniform convergence structure is a collection Υ of filters on $\mathcal{M}(X,Y) \times \mathcal{M}(X,Y)$ satisfying the following conditions (see [5]):

(15) • $\langle f \rangle \times \langle f \rangle \in \Upsilon$ for all $f \in \mathcal{M}(X,Y)$.

(16) • $\mathcal{U} \cap \mathcal{V} \in \Upsilon$ whenever $\mathcal{U}, \mathcal{V} \in \Upsilon$.

(17) • If $\mathcal{U} \in \Upsilon$, then $\mathcal{V} \in \Upsilon$ for each filter $\mathcal{V}$ on $\mathcal{M}(X,Y) \times \mathcal{M}(X,Y)$ such that $\mathcal{V} \supseteq \mathcal{U}$.

(18) • If $\mathcal{U} \in \Upsilon$ then $\mathcal{U}^{-1} \in \Upsilon$.

(19) • For all $\mathcal{U}, \mathcal{V} \in \Upsilon$ one has $\mathcal{U} \circ \mathcal{V} \in \Upsilon$ whenever the composition $\mathcal{U} \circ \mathcal{V}$ exists.

Recall that $\langle f \rangle$ denotes the filter generated by $\{\{f\}\}$ and that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on $\mathcal{M}(X,Y)$, $\mathcal{F}_1 \times \mathcal{F}_2$ denotes the filter on $\mathcal{M}(X,Y) \times \mathcal{M}(X,Y)$ which is generated by $\{F_1 \times F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$.

In (18) above we use the common notation: If U is a subset of $\mathcal{M}(X,Y) \times \mathcal{M}(X,Y)$ then

$$U^{-1} = \{(f,g) : (g,f) \in U\}.$$

For any filter $\mathcal{U}$ on $\mathcal{M}(X,Y) \times \mathcal{M}(X,Y)$ we have $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$. The operation composition used in (19) is defined as follows. For any two subsets U and V of $\mathcal{M}(X,Y) \times \mathcal{M}(X,Y)$

$$U \circ V = \{(f,g) \in \mathcal{M}(X,Y) \times \mathcal{M}(X,Y) : \exists h \in \mathcal{M}(X,Y) : (f,h) \in V,\ (h,g) \in U\}.$$

If $\mathcal{U}$ and $\mathcal{V}$ are filters on $\mathcal{M}(X,Y) \times \mathcal{M}(X,Y)$ and $U \circ V \neq \emptyset$ for all $U \in \mathcal{U}$ and all $V \in \mathcal{V}$ then the filter generated by $\{U \circ V : U \in \mathcal{U},\ V \in \mathcal{V}\}$ is denoted by $\mathcal{U} \circ \mathcal{V}$ and called the composition filter of $\mathcal{U}$ and $\mathcal{V}$. In this case one says that the composition $\mathcal{U} \circ \mathcal{V}$ exists.

Denote by Δ the diagonal in $Y \times Y$ and set

$$\mathcal{D} = \{\phi \in \mathcal{M}(X, Y \times Y) : \phi(X) \subset \Delta\}.$$

Let Υ be the family of all filters $\mathcal{U}$ on $\mathcal{M}(X,Y) \times \mathcal{M}(X,Y)$ such that

$$[g_{\chi(\mathcal{U})}] \subset \mathcal{D}.$$

Let us remark that by $\chi(\mathcal{U})$ we denote the filter generated by $\{\chi(U) : U \in \mathcal{U}\}$. Further recall that $g_{\chi(\mathcal{U})}$ is the intersection of all usco maps g such that $[g] \in \chi(\mathcal{U})$ (see Remark 9.3). It follows that $\chi(\mathcal{U})$ is bounded whenever $\mathcal{U} \in \Upsilon$.

We will show that Υ is a uniform convergence structure on $\mathcal{M}(X,Y)$ which induces the convergence structure defined in Definition 8. To do this we need two lemmata.

Lemma 28. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be filters on $\mathcal{M}(X,Y)$.*

(i) *The filter $\chi(\mathcal{F}_1 \times \mathcal{F}_2)$ is bounded in $\mathcal{M}(X, Y \times Y)$ if and only if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are bounded in $\mathcal{M}(X,Y)$.*

(ii) $\mathcal{G}_{\chi(\mathcal{F}_1\times\mathcal{F}_2)} \subset \chi(\mathcal{G}_{\mathcal{F}_1} \times \mathcal{G}_{\mathcal{F}_2})$.

Proof. (i) Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are bounded in $\mathcal{M}(X,Y)$. Then there are usco maps g_1 and g_2 such that $[g_1] \in \mathcal{F}_1$ and $[g_2] \in \mathcal{F}_2$. Then

$$[g_1 \times g_2] \supset \chi([g_1] \times [g_2]) \in \chi(\mathcal{F}_1 \times \mathcal{F}_2),$$

hence $\chi(\mathcal{F}_1 \times \mathcal{F}_2)$ is bounded, too.

Conversely, let $\chi(\mathcal{F}_1 \times \mathcal{F}_2)$ be bounded. Then there is a usco map ϕ with $[\phi] \in \chi(\mathcal{F}_1 \times \mathcal{F}_2)$. Denote by p_1 and p_2 the projections of $Y \times Y$ onto the first and second coordinates, respectively. Then $g_j = p_j \circ \phi$ is a usco mapping from X to Y for $j = 1,2$. Moreover, $\phi \subset g_1 \times g_2$. Hence $[g_1 \times g_2] \in \chi(\mathcal{F}_1 \times \mathcal{F}_2)$. It means that there are $F_j \in \mathcal{F}_j$ for $j = 1,2$ such that $\chi(F_1 \times F_2) \subset [g_1 \times g_2]$. We claim that $F_1 \times F_2 \subset [g_1] \times [g_2]$. Let $(h_1,h_2) \in F_1 \times F_2$. By Lemma 7 there is a dense subset D of X such that for each $x \in D$ both $h_1(x)$ and $h_2(x)$ are singletons. Hence, for each $x \in D$ we have

$$h_1(x) \times h_2(x) = \chi(h_1,h_2)(x) \subset g_1(x) \times g_2(x),$$

so $h_j(x) \subset g_j(x)$ for $j = 1,2$. It follows from Lemma 4 that $f_j \subset g_j$ for $j = 1,2$.

Hence, for $i = 1,2$ we have $F_j \subset [g_j]$. Denote by f_j the set-valued mapping obtained as the closure of $\bigcup F_j$ in $X \times Y$. By [7, Lemma 3.1.1] it is usco. Now, clearly $F_j \subset [f_j]$ and so $\mathcal{F}_j$ is bounded.

(ii) Let $U \in \mathcal{G}_{\chi(\mathcal{F}_1\times\mathcal{F}_2)}$. Then there is a usco mapping ϕ with $[\phi] \in \chi(\mathcal{F}_1 \times \mathcal{F}_2)$ such that $[\phi] \subset U$. Further, there are $F_j \in \mathcal{F}_j$ for $j = 1,2$, such that $\chi(F_1 \times F_2) \subset [\phi]$.

Denote by f_j the set-valued mapping obtained as the closure of $\bigcup F_j$ in $X \times Y$. In the same way as in the proof of (i) we can show that f_j is a usco map. As $F_j \subset [f_j]$, we have $[f_j] \in \mathcal{F}_j$. Therefore we will be done if we show that

$$\chi([f_1] \times [f_2]) \subset [\phi].$$

Let $h \in \chi([f_1] \times [f_2])$. Then h is a minimal usco and $h \subset f_1 \times f_2$. Suppose that $h \not\subset \phi$. Choose $x_0 \in X$ and $(y_0,z_0) \in h(x_0) \setminus \phi(x_0)$. Let V_1 and V_2 be disjoint open subset of $Y \times Y$ with $\phi(x_0) \subset V_1$ and $(y_0,z_0) \in V_2$. Choose W_1, W_2 open subsets in Y such that $(y_0,z_0) \in W_1 \times W_2 \subset V_2$.

As ϕ is usco, there is U_0, a neighborhood of x_0 such that $\phi(U_0) \subset V_1$. As h is minimal, there is (by Lemma 3) a nonempty open set $U_1 \subset U_0$ with $h(U_1) \subset W_1 \times W_2$. Choose some $x_1 \in U_1$ and $(y_1,z_1) \in h(x_1)$. Then $y_1 \in f_1(x_1)$. By the definition of f_1 there is some $g_1 \in F_1$, $x_2 \in U_1$ and $y_2 \in g_1(x_2) \cap W_1$. As g_1 is minimal, there is (again by Lemma 3) a nonempty open set $U_2 \subset U_1$ with $g_1(U_2) \subset W_1$. Similarly there is some $g_2 \in F_2$ and a nonempty open set $U_3 \subset U_2$ such that $g_2(U_3) \subset W_2$. Thus $(g_1 \times g_2)(U_3) \subset W_1 \times W_2$, so $(g_1 \times g_2)(U_3) \cap \phi(U_3) = \emptyset$. Therefore $\chi(g_1,g_2) \notin [\phi]$, a contradiction. □

Lemma 29. *Let $f \in \mathcal{M}(X,Y)$ and $\mathcal{F}$ be a filter on $\mathcal{M}(X,Y)$. Then $\langle f \rangle \times \mathcal{F} \in \Upsilon$ if and only if $\mathcal{F} \to f$ in $\mathcal{M}(X,Y)$.*

Proof. Suppose that $\mathcal{F} \to f$. Then $\mathcal{F}$ is bounded. Moreover, $\langle f\rangle$ is also bounded, hence $\chi(\langle f\rangle \times \mathcal{F})$ is bounded as well by Lemma 28.

Moreover, if g is a usco map such that $[g] \in \mathcal{F}$, then $[f \times g] \in \chi(\langle f\rangle \times \mathcal{F})$. Thus

$$g_{\chi(\langle f\rangle\times\mathcal{F})} \subset \bigcap\{f \times g : g \text{ is usco}, [g] \in \mathcal{F}\} = f \times g_{\mathcal{F}}.$$

As $[g_{\mathcal{F}}] = \{f\}$, we have

$$[g_{\chi(\langle f\rangle\times\mathcal{F})}] = [f \times f].$$

As the diagonal Δ is closed in $Y \times Y$ and $(f \times f)(x) \cap \Delta \neq \emptyset$ for each $x \in D$, the mapping $x \mapsto (f \times f)(x) \cap \Delta$ is usco (by Lemma 2). Hence $\chi(f, f)(x) \subset \Delta$ for each $x \in D$, i.e. $\chi(f, f) \in \mathcal{D}$. This completes the proof that $\chi(\langle f\rangle \times \mathcal{F})$ belongs to Υ.

Conversely, suppose that $\langle f\rangle \times \mathcal{F}$ belongs to Υ. Then the filter $\chi(\langle f\rangle \times \mathcal{F})$ is bounded, and hence $\mathcal{F}$ is bounded as well by Lemma 28. We have

$$[g_{\chi(\langle f\rangle\times\mathcal{F})}] \subset \mathcal{D}.$$

Moreover, by Lemma 28(ii) we get

$$\begin{aligned}[g_{\chi(\langle f\rangle\times\mathcal{F})}] &\supset \bigcap\{\chi(\{f\} \times [g]) : g \text{ is usco}, [g] \in \mathcal{F}\} \\ &\supset \chi\left(\bigcap\{\{f\} \times [g] : g \text{ is usco}, [g] \in \mathcal{F}\}\right) = \chi(\{f\} \times [g_{\mathcal{F}}]),\end{aligned}$$

hence

$$\chi(\{f\} \times [g_{\mathcal{F}}]) \subset \mathcal{D}.$$

If $h \in [g_{\mathcal{F}}]$ is different from f, then $f(x) \cap h(x) = \emptyset$ for some $x \in X$ (by Lemma 4). But this implies that $\chi(f, h) \notin \mathcal{D}$, a contradiction. Hence $[g_{\mathcal{F}}] = \{f\}$, i.e. $\mathcal{F} \to f$. □

Theorem 30. *The collection of filters Υ is a uniform convergence structure inducing the convergence structure on $\mathcal{M}(X, Y)$.*

Proof. To prove that Υ is a uniform convergence structure we need to show that Υ satisfies the properties (15)–(19). The fact that Υ generates the convergence structure on $\mathcal{M}(X, Y)$ then follows immediately from Lemma 29.

The properties (17) and (18) are obvious. The property (15) follows immediately from Lemma 29 as $\langle f\rangle \to f$.

Let us show the property (16). Let $\mathcal{U}$ and $\mathcal{V}$ belong to Υ. First we show that

$$\chi(\mathcal{U} \cap \mathcal{V}) = \chi(\mathcal{U}) \cap \chi(\mathcal{V}).$$

Indeed, the inclusion $\subset$ is obvious. To prove the inverse let us choose an element S in the set on the right-hand side. Then there are $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $\chi(U) \subset S$ and $\chi(V) \subset S$. Then $U \cup V \in \mathcal{U} \cap \mathcal{V}$ and

$$\chi(U \cup V) = \chi(U) \cup \chi(V) \subset S,$$

hence $S \in \chi(\mathcal{U} \cap \mathcal{V})$. Now, in the same way as in the proof of Theorem 10 one can easily show that $\chi(\mathcal{U} \cap \mathcal{V})$ is bounded and, moreover,

$$g_{\chi(\mathcal{U}\cap\mathcal{V})} \subset g_{\chi(\mathcal{U})} \cup g_{\chi(\mathcal{V})}.$$

Therefore it is enough to prove the following claim:

$$\phi, \psi : X \to Y \times Y \text{ usco maps}, [\phi] \cup [\psi] \subset \mathcal{D} \Rightarrow [\phi \cup \psi] \subset \mathcal{D}. \tag{20}$$

Suppose that $h \in [\phi\cup\psi]\setminus\mathcal{D}$. Choose $x_0 \in X$ and $(y_1, y_2) \in h(x_0)$ such that $y_1 \neq y_2$. Find disjoint open sets $V_1, V_2 \subset Y$ such that $y_1 \in V_1$ and $y_2 \in V_2$. By Lemma 3 there is a nonempty open set $U_0 \subset X$ such that $h(U_0) \subset V_1 \times V_2$.

We claim that there is a nonempty open set $U_1 \subset U_0$ such that either $h|_{U_1} \subset \phi|_{U_1}$ or $h|_{U_1} \subset \psi|_{U_1}$. Indeed, suppose that $h|_{U_0} \not\subset \phi|_{U_0}$. As $h|_{U_0}$ is minimal ([10, Lemma 2]) by Lemma 4 we get a nonempty open set $U_1 \subset U_0$ with $\phi(U_1) \cap h(U_1) = \emptyset$. As $h \subset \phi \cup \psi$, it follows $h|_{U_1} \subset \psi|_{U_1}$ which proves our claim.

So suppose, say, that $h|_{U_1} \subset \phi|_{U_1}$. Define a mapping $\widetilde{\phi}$ by

$$\widetilde{\phi}(x) = \begin{cases} \phi(x), & x \in X \setminus U_1, \\ h(x), & x \in U_1. \end{cases}$$

By Lemma 2 it is a usco map. Further, $[\widetilde{\phi}] \subset [\phi]$ and $[\widetilde{\phi}] \cap \mathcal{D} = \emptyset$ (as $\widetilde{\phi}(U_1) \subset V_1 \times V_2$), a contradiction completing the proof of (20).

It remains to prove the condition (19). Let $\mathcal{U}$ and $\mathcal{V}$ be elements of Υ such that $\mathcal{U} \circ \mathcal{V}$ exists.

First let us show that $\chi(\mathcal{U} \circ \mathcal{V})$ is bounded. We know that both $\chi(\mathcal{U})$ and $\chi(\mathcal{V})$ are bounded, and hence $\chi(\mathcal{U} \cap \mathcal{V})$ is bounded as well (by the already proved condition (16)). Hence there is a usco map ϕ such that $[\phi] \in \chi(\mathcal{U} \cap \mathcal{V})$. Let $\alpha = p_1 \circ \phi$ and $\alpha = p_2 \circ \phi$ (where p_1 and p_2 are projections of $Y \times Y$, see the proof of Lemma 28(i)). Then α and β are usco maps and $[\alpha \times \beta] \in \chi(\mathcal{U} \cap \mathcal{V})$. Hence there is some $U \in \mathcal{U} \cap \mathcal{V}$ such that $\chi(U) \subset [\alpha \times \beta]$. We will show that $\chi(U \circ U) \subset [\alpha \times \beta]$ as well.

Let $(f, g) \in U \circ U$. Then there is $h \in \mathcal{M}(X, Y)$ such that $(f, h) \in U$ and $(h, g) \in U$. Thus both $\chi(f, h)$ and $\chi(h, g)$ are contained in $\alpha \times \beta$. By Lemma 7 there is a dense set $D \subset X$ such that for each $x \in D$ all the values $f(x)$, $h(x)$ and $g(x)$ are singletons. Hence for $x \in D$ we have $f(x) \subset \alpha(x)$ and $g(x) \subset \beta(x)$. In particular, $\chi(f, g)(x) = f(x) \times g(x) \subset (\alpha \times \beta)(x)$ for $x \in D$. Therefore $\chi(f, g) \subset \alpha \times \beta$ by Lemma 4.

This completes the proof that $\chi(U \circ U) \subset [\alpha \times \beta]$ and hence $\chi(\mathcal{U} \circ \mathcal{V})$ is bounded.

To finish the proof that $\mathcal{U} \circ \mathcal{V}$ belongs to Υ choose $\alpha \in \mathcal{M}(X, Y \times Y) \setminus \mathcal{D}$ arbitrary. Find $x_0 \in X$ and distinct points $y_0, z_0 \in Y$ such that $(y_0, z_0) \in \alpha(x_0)$. Let $c > 0$ be such that $c < \rho(y_0, z_0)$. By the minimality of α and Lemma 3 there is a nonempty open set $U_0 \subset X$ such that

$$\alpha(U_0) \subset \{(y, z) \in Y \times Y : \rho(y, z) > c\}.$$

By the already proved condition (16) we know that $[g_{\chi(\mathcal{U} \cap \mathcal{V})}] \subset \mathcal{D}$. Thus there is some $x_1 \in U_0$ such that

$$g_{\chi(\mathcal{U} \cap \mathcal{V})}(x_1) \cap \{(y, z) \in Y \times Y : \rho(y, z) \geq \tfrac{c}{2}\} = \emptyset.$$

Indeed, otherwise

$$h(x) = \begin{cases} g_{\chi(\mathcal{U} \cap \mathcal{V})}(x) \cap \{(y, z) \in Y \times Y : \rho(y, z) \geq \frac{c}{2}\}, & x \in U_0, \\ g_{\chi(\mathcal{U} \cap \mathcal{V})}(x), & x \in X \setminus U_0, \end{cases}$$

would be a usco mapping (by Lemma 2) contained in $g_{\chi(\mathcal{U} \cap \mathcal{V})}$ but not containing any element of $\mathcal{D}$, a contradiction. Now, by the definition of $g_{\chi(\mathcal{U} \cap \mathcal{V})}$ there is some usco map ϕ with $[\phi] \in \chi(\mathcal{U} \cap \mathcal{V})$ such that

$$\phi(x_1) \cap \{(y, z) \in Y \times Y : \rho(y, z) \geq \tfrac{c}{2}\} = \emptyset.$$

As ϕ is usco, there is an open set U_1 with $x_1 \in U_1 \subset U_0$ such that

$$\phi(U_1) \cap \{(y, z) \in Y \times Y : \rho(y, z) \geq \tfrac{c}{2}\} = \emptyset.$$

There is some $M \in \mathcal{U} \cap \mathcal{V}$ such that $\chi(M) \subset [\phi]$. Then we have

$$\chi(M \circ M) \subset [\phi \star \phi], \tag{21}$$

where $\phi \star \phi$ is the usco mapping defined by

$$(\phi \star \phi)(x) = \phi(x) \circ \phi(x).$$

Let us show first that $\phi \star \phi$ is a usco mapping. We will use [7, Lemma 3.1.1]. Let x_τ be a net in X converging to some $x \in X$ and let $(y_\tau, z_\tau) \in (\phi \star \phi)(x_\tau)$. For each τ there is some $u_\tau \in Y$ such that $(y_\tau, u_\tau) \in \phi(x_\tau)$ and $(u_\tau, z_\tau) \in \phi(x_\tau)$. As ϕ is usco, there is a subnet (y_ν, u_ν) of (y_τ, u_τ) converging to some $(y, u) \in \phi(x)$. Using once more that ϕ is usco, we obtain a subnet (u_μ, z_μ) of (u_ν, z_ν) converging to some $(u, z) \in \phi(x)$ (note that u_ν converges to u, and hence the limit of u_μ is also u). Then (x_μ, z_μ) converges to (x, z) and $(x, z) \in \phi(x) \circ \phi(x)$.

Let us proceed to the proof of (21). Pick $(f, g) \in M \circ M$. Then there is $h \in \mathcal{M}(X, Y)$ such that both (f, h) and (h, g) belong to M. Hence both $\chi(f, h)$ and $\chi(h, g)$ are contained in ϕ. By Lemma 7 there is a dense set $D \subset X$ such that all the mappings f, g, h are singlevalued on D. Let $x \in D$. Then $f(x) \times h(x) \subset \phi(x)$ and $h(x) \times g(x) \subset \phi(x)$, hence

$$f(x) \times g(x) \subset \phi(x) \circ \phi(x) = (\phi \star \phi)(x).$$

Therefore $\chi(f, g)(x) \subset (\phi \star \phi)(x)$ for each $x \in D$. It follows from Lemma 4 that $\chi(f, g) \subset \phi \star \phi$ which completes the proof of (21).

Hence $\phi \star \phi$ is a usco map and $[\phi \star \phi] \in \chi(\mathcal{U} \circ \mathcal{V})$. Let $x \in U_1$ and $(y, z) \in (\phi \star \phi)(x)$. Then there is $u \in Y$ such that both (y, u) and (u, z) belong to $\phi(x)$. Then

$$\rho(y, z) \le \rho(y, u) + \rho(u, z) < c.$$

Thus $(\phi \star \phi)(U_1) \cap \alpha(U_1) = \emptyset$. Therefore $\alpha \notin [\phi \star \phi]$ which completes the proof. □

An important question associated with uniform convergence spaces is their completeness, that is, the convergence of Cauchy filters. Let us recall that a filter $\mathcal{F}$ on $\mathcal{M}(X, Y)$ is called Cauchy if $\mathcal{F} \times \mathcal{F} \in \Upsilon$.

Theorem 31. *The uniform convergence space* $(\mathcal{M}(X, Y), \Upsilon)$ *is complete.*

Proof. Assume that the filter $\mathcal{F}$ on $\mathcal{M}(X, Y)$ is Cauchy, that is, $\mathcal{F} \times \mathcal{F} \in \Upsilon$. Then $\mathcal{F}$ is bounded by Lemma 28(i). Hence the usco map $g_{\mathcal{F}}$ is well defined and nonempty valued on X. Moreover, by Lemma 28(ii) we have

$$\begin{aligned} \mathcal{D} \supset [g_{\chi(\mathcal{F} \times \mathcal{F})}] &\supset \bigcap \{\chi([g] \times [g]) : g \text{ is usco}, [g] \in \mathcal{F}\} \\ &\supset \chi\left(\bigcap \{[g] \times [g] : g \text{ is usco}, [g] \in \mathcal{F}\}\right) = \chi([g_{\mathcal{F}}] \times [g_{\mathcal{F}}]). \end{aligned}$$

If f_1, f_2 are two different elements of $[g_{\mathcal{F}}]$, by Lemma 4 there is some $x \in X$ with $f_1(x) \cap f_2(x) = \emptyset$, hence $\chi(f_1, f_2) \notin \mathcal{D}$. So $g_{\mathcal{F}}$ is quasiminimal and hence $\mathcal{F}$ converges to the unique element of $[g_{\mathcal{F}}]$. □

Remarks 32.

1. Notice that the definition of the uniform convergence structure Υ depends only on the topology of Y, i.e. is the same for all equivalent metrics on Y.
2. If X is a singleton, then both $\mathcal{M}(X, Y) \times \mathcal{M}(X, Y)$ and $\mathcal{M}(X, Y \times Y)$ can be canonically identified with $Y \times Y$. In this case Υ consist of those filters $\mathcal{U}$ on $Y \times Y$ such that there is a compact set $K \subset Y$ such that the filter generated by the neighborhoods of the diagonal in $K \times K$ is contained in $\mathcal{U}$. In particular, if Y is compact, Υ coincide with the (unique) uniformity on Y.

7. The subspace of continuous functions

The space of minimal usco maps $\mathcal{M}(X,Y)$ contains a natural subspace $\mathcal{C}(X,Y)$ consisting of continuous functions from X to Y. In the previous section we have shown that the convergence space $\mathcal{M}(X,Y)$ is complete for the natural uniform convergence structure whenever X is a Baire space and Y is a metric space. Therefore the closure of $\mathcal{C}(X,Y)$ in $\mathcal{M}(X,Y)$ could be viewed as a completion of $\mathcal{C}(X,Y)$. In this section we study the question when $\mathcal{C}(X,Y)$ is dense in $\mathcal{M}(X,Y)$.

Let us recall the definition of a closed subset of a convergence space and related notions. A subset A of a convergence space is *closed* if $f \in A$ whenever there is a filter $\mathcal{F}$ converging to f and satisfying $A \in \mathcal{F}$. The *closure* of a set A is the smallest closed set containing A. And a set is *dense* if its closure is the whole space.

First we note that $\mathcal{C}(X,Y)$ is not always dense in $\mathcal{M}(X,Y)$.

Example 33. *$\mathcal{C}(\mathbb{R},\{0,1\})$ is a proper closed subset of $\mathcal{M}(\mathbb{R},\{0,1\})$.*

Proof. The mapping g defined by

$$g(x) = \begin{cases} \{0\}, & x<0, \\ \{0,1\}, & x=0, \\ \{1\}, & x>0, \end{cases}$$

belongs to $\mathcal{M}(\mathbb{R},\{0,1\}) \setminus \mathcal{C}(\mathbb{R},\{0,1\})$.

Further, let us show that $\mathcal{C}(\mathbb{R},\{0,1\})$ is closed. Let $\mathcal{F}$ be a filter on $\mathcal{M}(\mathbb{R},\{0,1\})$ converging to some $f \in \mathcal{M}(\mathbb{R},\{0,1\})$ satisfying $\mathcal{C}(\mathbb{R},\{0,1\}) \in \mathcal{F}$. Let g be a usco map such that $[g] \in \mathcal{F}$. Then $[g]$ contains an element of $\mathcal{C}(\mathbb{R},\{0,1\})$. As $\mathcal{C}(\mathbb{R},\{0,1\})$ has only two elements (constant function 0 and constant function 1), there is one of them contained in $[g]$ for every g satisfying $[g] \in \mathcal{F}$. Therefore $[g_{\mathcal{F}}]$ contains an element of $\mathcal{C}(\mathbb{R},\{0,1\})$, which implies $f \in \mathcal{C}(\mathbb{R},\{0,1\})$. □

This example shows that in order to have $\mathcal{C}(X,Y)$ dense in $\mathcal{M}(X,Y)$, some assumptions on Y are needed. A natural assumption of this kind is that Y is a convex subset of a normed linear space. A partial positive result is the following one.

Theorem 34. *Let X be a Baire metric space and Y be a closed convex subset of a Banach space. Then $\mathcal{C}(X,Y)$ is dense in $\mathcal{M}(X,Y)$.*

Proof. Let $g \in \mathcal{M}(X,Y)$. It follows for example from [8] that g has a selection of the first Baire class, i.e., there is a (single-valued) function $f : X \to Y$ which is of the first Baire class (i.e., the pointwise limit of a sequence of continuous functions) such that $f(x) \in g(x)$ for each $x \in X$.

By [13, Theorem 1.2] there is a usco map $h : X \to Y$ and a sequence of continuous functions $f_n : X \to Y$ which pointwise converges to f and $f_n \subset h$ for each n. (If Y is a closed convex subset of $\mathbb{R}^d$, the proof is easier and can be found in [11, Theorem 3.3].)

Note that the sequence f_n is bounded in $\mathcal{M}(X,Y)$. It follows from Corollary 16 that the sequence f_n converges to g in $\mathcal{M}(X,Y)$. This completes the proof. □

We do not know whether the result on density is valid in more general situations. Let us formulate some of these problems.

Question 35. *Let X be a Baire metric space and Y a convex subset of a normed linear space. Is $\mathcal{C}(X,Y)$ dense in $\mathcal{M}(X,Y)$?*

Note that in this situation every $g \in \mathcal{M}(X, Y)$ has a selection of the first Baire class (this follows for example from [14, Theorem 2.2]) and Corollary 16 could be applied as well. The missing ingredient is the analogue of [13, Theorem 1.1]. It seems to be unknown whether such an analogue holds.

Another problem is whether we can drop the assumption of metrizability of X.

Question 36. *Let X be a Baire topological space and Y a convex subset of a normed linear space. Is $\mathcal{C}(X, Y)$ dense in $\mathcal{M}(X, Y)$?*

In this case sequences are not enough as we can see from the following example. However, to prove the density of $\mathcal{C}(X, Y)$ we are not obliged to use sequences. Nets or filters are allowed as well. But then some other technics should be used, as Theorem 15 (and Corollary 16) is true only for sequences (due to Example 17).

Example 37. *There is a compact Hausdorff space X and a proper subset $A \subset \mathcal{M}(X, [0, 1])$ which contains $\mathcal{C}(X, [0, 1])$ and is closed to taking limits of sequences. Moreover, in this case $\mathcal{C}(X, [0, 1])$ is dense in $\mathcal{M}(X, [0, 1])$.*

Proof. Let X be the ordinal interval $[0, \omega_1]$ and

$$A = \{g \in \mathcal{M}(X, [0, 1]) : g(\omega_1) \text{ is a singleton}\}.$$

Then clearly $A \supset \mathcal{C}(X, [0, 1])$. Further, A is a proper subset of $\mathcal{M}(X, [0, 1])$ as the mapping $g : [0, \omega_1] \to [0, 1]$ defined by

$$g(\alpha) = \begin{cases} \{0\}, & \alpha \text{ odd non-limit ordinal}, \\ \{1\}, & \alpha \text{ even non-limit ordinal}, \\ \{0, 1\}, & \alpha \text{ limit ordinal}, \end{cases}$$

is minimal usco and does not belong to A.

Next we shall show that A is closed to limits of sequences. Let f_n be a sequence from A converging to some $f \in \mathcal{M}(X, [0, 1])$. It follows from Lemma 7 and Theorem 18 that there is a residual subset D of X such that for all $x \in D$ the values $f(x)$ and all $f_n(x)$'s are singletons and, moreover, $f_n(x) \to f(x)$ in the topology of $[0, 1]$. Note that the set D must contain all the isolated ordinals.

Further note, that for any $h \in A$ there is $\alpha < \omega_1$ such that $h(x) = h(\omega_1)$ for all $x \in [\alpha, \omega_1]$. Hence, to each f_n we can associate such an α_n. Let α be the supremum of α_n's. Then for each isolated ordinal $x \in [\alpha, \omega_1]$ we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(\omega_1).$$

Therefore f assumes for each isolated $x \in [\alpha, \omega_1]$ the same singleton value, and hence $f(\omega_1)$ is a singleton as well. This completes the proof that A is closed to taking limits of sequences.

Now we will prove that $\mathcal{C}(X, [0, 1])$ is dense in $\mathcal{M}(X, [0, 1])$.

Take any $g \in \mathcal{M}(X, [0, 1])$. For each $\alpha < \omega_1$ define the mapping

$$g_\alpha(x) = \begin{cases} g(x), & x \in [0, \alpha], \\ \{0\}, & x \in (\alpha, \omega_1]. \end{cases}$$

Then each g_α is a minimal usco belonging to A. Moreover, the net g_α converges to g. To see this we use Theorem 11. We define usco maps h_α by the formula

$$h_\alpha(x) = \begin{cases} g(x), & x \in [0, \alpha], \\ g(x) \cup \{0\}, & x \in (\alpha, \omega_1]. \end{cases}$$

Then $g_\alpha \subset h_\alpha$ and $h_\beta \subset h_\alpha$ for each $\alpha \leq \beta < \omega_1$. Further, the intersection of all h_α's is the usco map

$$h(x) = \begin{cases} g(x), & x \in [0, \omega_1), \\ g(x) \cup \{0\}, & x = \omega_1. \end{cases}$$

It is clear that h is quasiminimal and $[h] = \{g\}$.

This shows that A is dense in $\mathcal{M}([0,\alpha],[0,1])$. We conclude by showing that $\mathcal{C}(X,[0,1])$ is dense in A. Let $g \in A$. Then there is $\alpha < \omega_1$ such that $g(x) = g(\omega_1)$ for each $x \in (\alpha, \omega_1]$. Then $g|_{[0,\alpha]}$ belongs to $\mathcal{M}([0,\alpha],[0,1])$. As $[0,\alpha]$ is a metrizable compact space, there is (by the proof of Theorem 34) a sequence of continuous functions $f_n : [0,\alpha] \to [0,1]$ converging to $g|_{[0,\alpha]}$ in $\mathcal{M}([0,\alpha],[0,1])$. Extend the functions f_n to functions $h_n : X \to [0,1]$ by defining $h(x) = g(\omega_1)$ for $x > \alpha$. Then h_n are continuous and clearly converge to g in $\mathcal{M}(X,[0,1])$. This completes the proof. □

In fact, although the assumption that the domain space X is Baire is quite natural, we do not know the answer to the following question.

Question 38. *Is there a topological space X and a convex subset Y of a normed linear space such that $\mathcal{C}(X,Y)$ is not dense in $\mathcal{M}(X,Y)$?*

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