LULU operators for functions of continuous argument

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Abstract

The LULU operators, well known in the nonlinear multiresolution analysis of sequences, are extended to functions defined on a continuous domain, namely, a real interval. We show that the extended operators replicate the essential properties of their discrete counterparts. More precisely, they form a fully ordered semi-group of four elements, preserve the local trend and the total variation.

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1 Introduction

The well known (linear-) Functional Analysis fits in appropriately in the theory of linear smoothers. Typically a smoother is designed to pass sequences that are samplings of functions with low frequencies with minimal distortion (error), but to map high frequencies near to the zero sequence. The least squares norm is appropriate, for various other reasons, but also that such smoothers then map a element x_i onto weighted averages of sequence elements in a "window" around x_i, say $\{x_{i-n}, \ldots, x_i, \ldots, x_{i+n}\}$. The linearity also ensures that sequences of elements that are generated identically, independently distributed from a very general symmetrical distribution e, are rapidly mapped near zero due to the Central Limit Theorem. When we want to smooth a sequence, we can choose to construct a convenient smoother that is a "bandpass" filter, and practically remove high frequencies. The design of such filters is done in the well established theory of Digital Filters. The book by Hamming [2] is well known and instructive. The essential background theory is Fourier Analysis. This works well because the basic trigonometric functions sin and cos are eigensequences of linear operators, and the "transfer function" approximates with eigenvalues near one in the frequencies that are to pass, and eigenvalues near 0 where the frequencies are to be reduced to near zero.

Low pass filters should therefore marginally distort sequences that are samplings of functions that have Fourier expansions that converge fast. This is well known to be related to continuity of lower derivatives. Discontinuities in low order derivatives result in slowly converging Fourier expansions and the digital filters will remove the high frequencies, distorting significantly the sequence of samplings. A typical bad case is isolated impulsive noise added spuriously. This necessitates the presmoothing by nonlinear smoothers, of which the median smoothers, popularised by Tukey, are well known. Since eigensequence analysis is not natural, nor easily justifiable, for nonlinear operators the lack of a theory for analysis and design was considered to be difficult, if not impossible [11]. Design was generally essentially considered to be an art.

Over the last twenty years a theory for Nonlinear (general) Smoothers, that is based on order structure and min/max operations, has been developed and demonstrated to be very consistent and useful, even able to explain most of the "good" behaviour of the (related) median smoothers, as well as their "enigmatic" behaviour. A monograph presenting the so-called *LULU*-theory and its motivation and development appeared in 2005 [7]. The theory is based on compositions of two types of smoothers L_n and U_n. They are Morphological Filters with special properties.

One of the powerful ideas resulting from this theory, was the development of Nonlinear Multiresolution Analysis. This was done using the heuristic ideas from Fourier Analysis and Wavelet Analysis. It resulted eventually in Discrete Pulse Transforms [4]. These transforms may turn out to be as useful for vision as the Fourier Transforms are for hearing [8].

When applications of Wavelet Transforms (and Fourier Transforms) are under discussion it is natural for understanding to consider samplings of "band limited" functions as ideal candidates for such decompositions, both for theoretical derivation and practical applications. For Nonlinear Decomposition there has been a lack of such a relation between the theory of real functions and the theory of the sequences that are samplings of these functions. Generalizing $LULU$-operators and the associated theory and concepts to functions is the first appropriate attempt towards establishing such a link.

Central in the $LULU$-theory for sequences are the class of locally monotone sequences $\mathcal{M}_n$ defined as the sets of all sequences x that have $\{x_i, x_{i+1}, \ldots, x_{i+n+1}\}$ monotone for each index i. We need to establish natural links between these classes and classes of real functions of which the sequences can be considered as samplings. Also natural to $LULU$-theory is the Total Variation as norm [7]. There is a clue to establishing links with standard Real Analysis, as is typically presented by Royden in the first few chapters of his book [9]. Total Variation of functions and local monotonicity are linked to the derivative in this theory. We seek to extend and solidify these links with established Real Analysis of functions. To do this we, look at the basic ideas of the $LULU$-theory for sequences [7]. We start directly with the definitions of the "atoms" involved.

Given a bi-infinite sequence $\xi = (\xi_i)_{i\in\mathbb{Z}}$ and $n \in \mathbb{N}$ the operators L_n and U_n are defined as follows

$$(L_n\xi)_i = \max\{\min\{\xi_{i-n}, ..., \xi_i\}, \min\{\xi_{i-n+1}, ..., \xi_{i+1}\}, ..., \min\{\xi_i, ..., \xi_{i+n}\}\},\ i \in \mathbb{Z},$$
$$(U_n\xi)_i = \min\{\max\{\xi_{i-n}, ..., \xi_i\}, \max\{\xi_{i-n+1}, ..., \xi_{i+1}\}, ..., \max\{\xi_i, ..., \xi_{i+n}\}\},\ i \in \mathbb{Z}.$$

In analogy with the above discrete LULU operators, for a given $\delta > 0$ the basic smoothers L_δ and U_δ in the LULU theory are defined for functions on Ω through the concepts of the so called lower and upper δ-envelopes of these functions. These definitions are given in Section 2, where it is also shown that the operators L_δ and U_δ preserve essential properties of their discrete counterparts. Section 3 deals with the semi-group generated by the operators

L_δ and U_δ via composition. It is shown that L_δ and U_δ are a Matheron pair. Hence they generate through composition a fully ordered four element semi-group, also called a strong LULU structure. In Sections 4 and 5 we discuss the preservation of the trend and the total variation respectively.

2 The basic smoothers L_δ and U_δ

Let $\mathcal{A}(\Omega)$ denote the set of all bounded real functions defined on a real interval $\Omega \subseteq \mathbb{R}$. Let $B_\delta(x)$ denote the closed δ-neighborhood of x in Ω, that is, $B_\delta(x) = \{y \in \Omega : |x - y| \leq \delta\}$. The pair of mappings I, $S : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ defined by

$$I(f)(x) = \sup_{\delta>0} \inf\{f(y) : y \in B_\delta(x)\}, \ x \in \Omega, \tag{1}$$

$$S(f)(x) = \inf_{\delta>0} \sup\{f(y) : y \in B_\delta(x)\}, \ x \in \Omega, \tag{2}$$

are called lower Baire, and upper Baire operators, respectively, [10]. The fixed points of these operators are the lower and the upper semi-continuous functions respectively. Let us recall that a function $f \in \mathcal{A}(\Omega)$ is called *lower semi-continuous* on Ω if for every $x \in \Omega$ and every $m < f(x)$ there exists $\delta > 0$ such that $m < f(y)$ for all $y \in B_\delta(x)$. Similarly, a function $f \in \mathcal{A}(\Omega)$ is called *upper semi-continuous* on Ω if for every $x \in \Omega$ and $m > f(x)$ there exists $\delta > 0$ such that $m > f(y)$ for all $y \in B_\delta(x)$.

Then the lower and upper Baire operators can be defined in the following equivalent way. For every $f \in \mathcal{A}(\Omega)$ the function $I(f)$ is the maximal lower semi-continuous function which is not greater than f. Hence, it is also called lower semi-continuous envelope. In a similar way, $S(f)$ is the smallest upper semi-continuous function which is not less than f and is called the upper semi-continuous envelope of f. In analogy with $I(f)$ and $S(f)$ we call the functions

$$I_\delta(f)(x) = \inf\{f(y) : y \in B_\delta(x)\}, \ x \in \Omega, \tag{3}$$

$$S_\delta(f)(x) = \sup\{f(y) : y \in B_\delta(x)\}, \ x \in \Omega, \tag{4}$$

a lower δ-envelope of f and an upper δ-envelope of f, respectively.

The following operators can be considered as continuous analogues of the discrete $LULU$ operators given in the Introduction:

$$L_\delta = S_{\frac{\delta}{2}} \circ I_{\frac{\delta}{2}} \ , \quad U_\delta = I_{\frac{\delta}{2}} \circ S_{\frac{\delta}{2}} \ . \tag{5}$$

The δ-envelopes and the operators L_δ and U_δ have applications in the Approximation Theory for deriving locally monotone approximation, [1]. However, they also have roots in Mathematical Morphology. In fact, within this theory, I_δ and S_δ are called respectively erosion and dilation with a structural element B_δ, while L_δ and U_δ are called opening and closing with structural element $B_{\frac{\delta}{2}}$, e.g [11], [12]. Below are some morphological properties of L_δ and U_δ which will be useful in what follows:

- increasing
$$f \le g \Longrightarrow L_\delta(f) \le L_\delta(g),\ U_\delta(f) \le U_\delta(g); \tag{6}$$
- L_δ is anti-extensive, U_δ is extensive, that is,
$$L_\delta(f) \le f \le U_\delta(f); \tag{7}$$
- monotonicity with respect to δ
$$0 < \delta_1 \le \delta_2 \ \Longrightarrow (L_{\delta_1}(f) \ge L_{\delta_2}(f), U_{\delta_1}(f) \le U_{\delta_2}(f)); \tag{8}$$
- absorbtion: for every $\delta_1, \delta_2 > 0$ we have
$$L_{\delta_1} \circ L_{\delta_2} = L_{\max\{\delta_1,\delta_2\}} \ , \quad U_{\delta_1} \circ U_{\delta_2} = U_{\max\{\delta_1,\delta_2\}}; \tag{9}$$
- idempotence
$$L_\delta \circ L_\delta = L_\delta \ , \quad U_\delta \circ U_\delta = U_\delta. \tag{10}$$

The following identities, which are instrumental in deriving some of the above properties will be useful in the sequel:

$$I_\delta \circ S_\delta \circ I_\delta = I_\delta \ , \quad S_\delta \circ I_\delta \circ S_\delta = S_\delta. \tag{11}$$

Central to the LULU theory for sequences is the concept of *separator*. This concept is defined in [7] only for operators on sequences due to the context of the book. However, it is meaningful in more general settings. In fact, some of the axioms have been used earlier, e.g. see [11], for functions on arbitrary domains. We give below the definition of separator within our current context.

Definition 1 *An operator $P : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ is called a* **separator** *if P is*

(i) invariant under horizontal and vertical translations;

(ii) positively invariant: $P(\alpha f) = \alpha P(f),\ \alpha \in \mathbb{R},\ \alpha \geq 0,\ f \in \mathcal{A}(\Omega)$;

(iii) idempotent: $P \circ P = P$;

(iv) co-idempotent: $(id - P) \circ (id - P) = id - P$, where id denotes the identity operator on $\mathcal{A}(\Omega)$.

The operators L_δ and U_δ similarly to their discrete counterparts, are separators. Indeed, properties (i)–(iii) are satisfied since these are properties of the morphological opening and closing. We will show that (iv) also holds. The co-idempotence of the operator L_δ is equivalent to $L_\delta \circ (id - L_\delta) = 0$. Using the inequalities (6) and (13) one can easily obtain $L_\delta \circ (id - L_\delta) \geq 0$. Hence, for the co-idempotence of L_δ it remains to show that $L_\delta \circ (id - L_\delta) \leq 0$. Assume the opposite. Namely, there exists a function $f \in \mathcal{A}(\Omega)$ and $x \in \Omega$ such that $(L_\delta \circ (id - L_\delta))(f)(x) > 0$. Let $\varepsilon > 0$ be such that $(L_\delta \circ (id - L_\delta))(f)(x) > \varepsilon > 0$. Using the definition of L_δ the above inequality implies that there exists $y \in B_{\frac{\delta}{2}}(x)$ such that for every $z \in B_{\frac{\delta}{2}}(y)$ we have $(id - L_\delta)(f)(z) > \varepsilon$, or equivalently

$$f(z) > L_\delta(f)(z) + \varepsilon,\ z \in B_{\frac{\delta}{2}}(y). \tag{12}$$

For every $z \in B_{\frac{\delta}{2}}(y)$ we also have $L_\delta(f)(z) \geq I_{\frac{\delta}{2}}(f)(y) = \inf\{f(t) : t \in B_{\frac{\delta}{2}}(y)\}$. Hence there exists $t \in B_{\frac{\delta}{2}}(y)$ such that $f(t) < I_{\frac{\delta}{2}}(f)(y) + \varepsilon \leq L_\delta(f)(z) + \varepsilon$, $z \in B_{\frac{\delta}{2}}(y)$. Taking $z = t$ in the above inequality we obtain $f(t) < L_\delta(f)(t) + \varepsilon$, which contradicts (12). The co-idempotence of U_δ is proved in a similar way. Therefore L_δ and U_δ are separators.

3 The LULU semi-group

In this section we consider the set of the operators L_δ and U_δ and their compositions. For operators on $\mathcal{A}(\Omega)$ we consider the point-wise defined partial order. Namely, for operators P, Q on $\mathcal{A}(\Omega)$ we have

$$P \leq Q \iff P(f) \leq Q(f),\ f \in \mathcal{A}(\Omega).$$

Then the inequalities in (7) can be represented in the form

$$L_\delta \leq id \leq U_\delta. \tag{13}$$

By a well known theorem of Matheron [3], in general, two ordered morphological operators generate via composition a six element semi-group, which is only partially ordered. However, there is a special case when this semi-group collapses to a four element totaly ordered semi-group. In this case the operators are called a Matheron pair, see [8]. We give below the definition of this concept within our current context.

Definition 2 *The operators $P : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ and $Q : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$ are called a Matheron pair if*

(a) $P \leq Q$ and

(b) $Q \circ P \leq P \circ Q$.

Theorem 3 *For any $\delta > 0$ the operators L_δ and U_δ are a Matheron pair.*

Proof. The property (a) follows from (13). We need to show (b), that is, $U_\delta \circ L_\delta \leq L_\delta \circ U_\delta$. Let $f \in \mathcal{A}(\Omega)$ and let $x \in \Omega$. Denote $p = (L_\delta \circ U_\delta)(f)(x) = S_{\frac{\delta}{2}}(I_\delta(S_{\frac{\delta}{2}}(f)))(x)$. Let ε be an arbitrary positive. For every $y \in B_{\frac{\delta}{2}}(x)$ we have

$$I_\delta(S_{\frac{\delta}{2}}(f))(y) \leq p < p + \varepsilon. \tag{14}$$

<u>Case 1.</u> There exists $z \in B_{\frac{\delta}{2}}(x)$ such that $S_{\frac{\delta}{2}}(f)(z) < p + \varepsilon$. Then $f(t) < p + \varepsilon$ for $t \in B_{\frac{\delta}{2}}(z)$, which implies that $I_{\frac{\delta}{2}}(f)(t) < p + \varepsilon$ for $t \in B_\delta(z)$. Hence $S_\delta(I_{\frac{\delta}{2}}(f))(z) \leq p + \varepsilon$. Then $(U_\delta \circ L_\delta)(f)(t) = I_{\frac{\delta}{2}}(S_\delta(I_{\frac{\delta}{2}}(f)))(t) \leq p + \varepsilon$ for $t \in B_{\frac{\delta}{2}}(z)$. Since $x \in B_{\frac{\delta}{2}}(z)$ (see the case assumption), from the above inequality we have $(U_\delta \circ L_\delta)(f)(x) \leq p + \varepsilon$.
<u>Case 2.</u> For every $z \in B_{\frac{\delta}{2}}(x)$ we have $S_{\frac{\delta}{2}}(f)(z) \geq p + \varepsilon$. Denote

$$D = \left\{ z \in \Omega : S_{\frac{\delta}{2}}(f)(z) < p + \varepsilon \right\}.$$

We will show that for every $z \in B_\delta(x)$ we have

$$B_\delta(z) \cap D \neq \emptyset\ . \tag{15}$$

Due to the inequality (14) we have that (15) holds for every $z \in B_{\frac{\delta}{2}}(x)$. Let $z \in B_\delta(x)$ and let $z > x + \frac{\delta}{2}$. This implies that $x + \frac{\delta}{2} \in \Omega$. Using inequality (14) for $y = x + \frac{\delta}{2}$ as well as the case assumption we obtain that the set $\left(x + \frac{\delta}{2}, x + \frac{3\delta}{2}\right] \cap D$ is not empty. Then $B_\delta(z) \cap D \supset \left(x + \frac{\delta}{2}, x + \frac{3\delta}{2}\right] \cap D \neq \emptyset$. For $z < x - \frac{\delta}{2}$ condition (15) is proved in a similar way. Hence (15) holds for all $z \in B_\delta(x)$. Let $z \in B_\delta(x)$ and $v \in B_\delta(y) \cap D$. Since $v \in D$ we have $f(t) < p + \varepsilon$, for $t \in B_{\frac{\delta}{2}}(v)$. Using that $B_{\frac{\delta}{2}}(z) \cap B_{\frac{\delta}{2}}(v) \neq \emptyset$ we obtain that $I_{\frac{\delta}{2}}(f)(z) < p + \varepsilon$, $z \in B_\delta(x)$. Therefore $S_\delta(I_{\frac{\delta}{2}}(f))(x) \leq p + \varepsilon$. Then

$$(U_\delta \circ L_\delta)(f)(x) = I_{\frac{\delta}{2}}(S_\delta(I_{\frac{\delta}{2}}(f)))(x) \leq S_\delta(I_{\frac{\delta}{2}}(f))(x) \leq p + \varepsilon.$$

Combining the results of Case 1 and Case 2 we have $(U_\delta \circ L_\delta)(f)(x) \leq p + \varepsilon$. Since ε is arbitrary this implies that $(U_\delta \circ L_\delta)(f)(x) \leq p = (L_\delta \circ U_\delta)(f)(x)$. ■

The composition table of L_δ and U_δ replicates the composition table of the operators for sequences and is given below:

	L_δ	U_δ	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$
L_δ	L_δ	$L_\delta \circ U_\delta$	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$
U_δ	$U_\delta \circ L_\delta$	U_δ	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$
$U_\delta \circ L_\delta$	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$
$L_\delta \circ U_\delta$	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$	$U_\delta \circ L_\delta$	$L_\delta \circ U_\delta$

The smoothing of functions in $\mathcal{A}(\Omega)$ by the compositions $L_\delta \circ U_\delta$ and $U_\delta \circ L_\delta$ can be described through the concept of local δ-monotonicity.

Definition 4 *Let $\delta > 0$. A function $f \in \mathcal{A}(\Omega)$ is called locally δ-monotone if f is monotone (increasing or decreasing) on any interval $[x, y] \subseteq \Omega$ of length not exceeding δ.*

The following theorem holds, [1].

Theorem 5 *For any given $\delta > 0$ and $f \in \mathcal{A}(\Omega)$ the functions $(L_\delta \circ U_\delta)(f)$ and $(U_\delta \circ L_\delta)(f)$ are both locally δ-monotone.*

4 Trend preservation

Definition 6 *An operator A is called* ***local trend preserving*** *if for every $f \in \mathcal{A}(\Omega)$ and interval $[x_1, x_2] \subset \Omega$ the function $A(f)$ is monotone increasing on $[x_1, x_2]$ whenever f is monotone increasing on $[x_1, x_2]$ and $A(f)$ is monotone decreasing on $[x_1, x_2]$ whenever f is monotone decreasing on $[x_1, x_2]$.*

Definition 7 *An operator A is called* ***fully trend preserving*** *if A and $id - A$ are both local trend preserving.*

If A is a local trend preserving operator then the local trend preserving property of $id - A$ can be equivalently formulated as: if f is monotone (increasing or decreasing) on an interval $[x_1, x_2] \subset \Omega$ then

$$|A(f)(x_1) - A(f)(x_2)| \leq |f(x_1) - f(x_2)|. \tag{16}$$

Remark 8 *Definition 6 and Definition 7 generalize the concepts of* neighbor trend preserving *and* fully trend preserving *for operators on sequences. In the context of sequences the property (16) is called difference reducing, [5, 6, 7].*

Theorem 9 *If the operators A and B are fully (local) trend preserving then so is their composition $A \circ B$.*

The proof is similar to the proof of the respective statement for sequences, see [7, Theorem 6.10] and will be omitted.

We will prove that the operators L_δ, U_δ and their compositions, similar to their discrete counterparts, are all fully trend preserving. To this end, the following technical lemma is useful.

Lemma 10 *Let function $f \in \mathcal{A}(\Omega)$ be given and let $\delta > 0$ be arbitrary.*

a) If f is monotone increasing on the interval $[x_1, x_2] \subseteq \Omega$ then the function $I_\delta(f)$ is monotone increasing on $[x_1 - \delta, x_2 - \delta] \cap \Omega$ and $S_\delta(f)$ is monotone increasing on $[x_1 + \delta, x_2 + \delta] \cap \Omega$.

b) If f is monotone decreasing on the interval $[x_1, x_2] \subseteq \Omega$ then the function $I_\delta(f)$ is monotone decreasing on $[x_1 + \delta, x_2 + \delta] \cap \Omega$ and $S_\delta(f)$ is monotone increasing on $[x_1 - \delta, x_2 - \delta] \cap \Omega$.

Proof. We will prove only a) since b) is proved in a similar way. Let $y_1, y_2 \in [x_1 - \delta, x_2 - \delta] \cap \Omega$ and $y_1 < y_2$. We have

$$I_\delta(f)(y_1) = \inf\{f(x) : x \in [y_1 - \delta, y_1 + \delta] \cap \Omega\} \tag{17}$$

Since f is increasing on $[x_1, x_2]$ and $[y_1+\delta, y_2+\delta] \subset [x_1, x_2]$ we have $f(y_1+\delta) \leq f(x)$ for $x \in [y_1+\delta, y_2+\delta]\cap\Omega$. Therefore enlarging the interval $[y_1-\delta, y_1+\delta]$ to the interval $[y_1 - \delta, y_2 + \delta] = [y_1 - \delta, y_1 + \delta] \cup [y_1 + \delta, y_2 + \delta]$ is not going to change the value of the infimum in (17) above. Using that the infimum of a smaller set is larger we further have

$$\begin{aligned}
I_\delta(f)(y_1) &= \inf\{f(x) : x \in [y_1 - \delta, y_2 + \delta] \cap \Omega\} \\
&\leq \inf\{f(x) : x \in [y_2 - \delta, y_2 + \delta] \cap \Omega\} \\
&= I_\delta(f)(y_2)
\end{aligned}$$

This shows that $I_\delta(f)$ is monotone increasing on $[x_1 - \delta, x_2 - \delta] \cap \Omega$. We prove that $S_\delta(f)$ is monotone increasing on $[x_1+\delta, x_2+\delta]\cap\Omega$ using a similar approach. Let $y_1, y_2 \in [x_1 + \delta, x_2 + \delta] \cap \Omega$ and $y_1 < y_2$. By the monotonicity of f on the interval $[y_1 - \delta, y_2 - \delta] \subset [x_1, x_2]$ we have

$$\begin{aligned}
S_\delta(f)(y_2) &= \sup\{f(x) : x \in [y_2 - \delta, y_2 + \delta] \cap \Omega\} \\
&= \sup\{f(x) : x \in [y_1 - \delta, y_2 + \delta] \cap \Omega\} \\
&\geq \sup\{f(x) : x \in [y_1 - \delta, y_1 + \delta] \cap \Omega\} \\
&= S_\delta(f)(y_1).
\end{aligned}$$

■

Theorem 11 *For an arbitrary $\delta > 0$ the operators L_δ, U_δ and their compositions are all fully trend preserving.*

Proof. We will prove only that L_δ is fully trend preserving since the proof of the statement for U_δ is done in a similar way. Then the fully trend preserving property of the compositions follows from Theorem 9. Therefore it is sufficient to show that if a function $f \in \mathcal{A}(\Omega)$ is monotone increasing or monotone decreasing on an interval $[x_1, x_2]$ then so are the functions $L_\delta(f)$ and $(id - L_\delta)(f)$. Due to the analogy we will only discuss the situation when f is increasing.

Let f be monotone increasing on $[x_1, x_2]$.

A. Proof that $L_\delta(f)$ is monotone increasing on $[x_1, x_2]$.

Applying Lemma 10 a) to the operator $I_{\frac{\delta}{2}}$ we obtain that $I_{\frac{\delta}{2}}(f)$ is monotone increasing on the interval $[x_1 - \frac{\delta}{2}, x_2 - \frac{\delta}{2}] \cap \Omega$.

Case 1. $[x_1 - \frac{\delta}{2}, x_2 - \frac{\delta}{2}] \subset \Omega$

Using again Lemma 10 a) for the operator $S_{\frac{\delta}{2}}$ applied to $I_{\frac{\delta}{2}}(f)$ on the interval $[x_1 - \frac{\delta}{2}, x_2 - \frac{\delta}{2}]$ we obtain that $L_\delta(f) = S_{\frac{\delta}{2}}(I_{\frac{\delta}{2}}(f))$ is monotone increasing on $[x_1, x_2]$.

Case 2. $[x_1 - \frac{\delta}{2}, x_2 - \frac{\delta}{2}] \cap \Omega = \emptyset$

Let a be the left endpoint of the interval Ω. For clarity of the exposition we assume that $a \in \Omega$ but the argument also holds if this is not true. It is easy to see that for any $g \in \mathcal{A}(\Omega)$ the function $S_{\frac{\delta}{2}}(g)$ is monotone increasing on the interval $\left[a, a + \frac{\delta}{2}\right]$. Indeed, for $x \in \left[a, a + \frac{\delta}{2}\right]$ we have

$$S_{\frac{\delta}{2}}(g)(x) = \sup\left\{ g(y) : y \in \left[a, x + \frac{\delta}{2}\right]\right\}.$$

where an increase in x enlarges the interval $\left[a, x + \frac{\delta}{2}\right]$ resulting in a higher value of the supremum. The case assumption implies that $[x_1, x_2] \subset \left[a, a + \frac{\delta}{2}\right]$. Since $L_\delta(f) = S_{\frac{\delta}{2}}(I_{\frac{\delta}{2}}(f))$ is increasing on $\left[a, a + \frac{\delta}{2}\right]$, it is also increasing on the subinterval $[x_1, x_2]$.

Case 3. If neither of the assumptions in Case 1 and Case 2 hold one obtains the monotonicity of $L_\delta(f)$ on $[x_1, x_2]$ by applying Case 1 and Case 2 to suitable subintervals of $[x_1, x_2]$.

B. Proof that $(id - L_\delta)(f)$ is monotone increasing on $[x_1, x_2]$.

Let $y_1, y_2 \in [x_1, x_2]$, $y_1 < y_2$. It follows from Part A of the proof that

$$L_\delta(f)(y_1) \le L_\delta(f)(y_2). \tag{18}$$

Case 1. $L(f)(y_1) = f(y_1)$. Then using that $L_\delta(f)(y_2) \le f(y_2)$ we obtain

$$(id - L_\delta(f))(y_1) = f(y_1) - L_\delta(f)(y_1) = 0 \le f(y_2) - L_\delta(f)(y_2) = (id - L_\delta(f))(y_2)$$

Case 2. $L(f)(y_1) < f(y_1)$. Then we have

$$I_{\frac{\delta}{2}}(f)(x) \le L_\delta(f)(y_1) < f(y_1) \quad \text{for all} \quad x \in \left[y_1 - \frac{\delta}{2}, y_1 + \frac{\delta}{2}\right] \bigcap \Omega. \tag{19}$$

In particular,

$$I_{\frac{\delta}{2}}(f)\left(y_1 + \frac{\delta}{2}\right) = \inf\{f(x) : x \in [y_1, y_1 + \delta] \cap \Omega\} \le L_\delta(f)(y_1) < f(y_1). \tag{20}$$

Considering the monotonicity of f on the interval $[x_1, x_2]$ the above inequality implies that $y_1 + \delta > y_2$. It further follows from (20) that for every $\varepsilon > 0$ there exists $y_\varepsilon \in [y_2, y_1 + \delta] \cap \Omega$ such that

$$f(y_\varepsilon) \le L_\delta(f)(y_1) + \varepsilon. \tag{21}$$

Hence we have

$$I_{\frac{\delta}{2}}(f)(x) \le L_\delta(f)(y_1)\ , \quad x \in \left[y_2 - \frac{\delta}{2}, y_1 + \frac{\delta}{2}\right] \bigcap \Omega \quad \text{(see (19))},$$
$$I_{\frac{\delta}{2}}(f)(x) \le f(y_\varepsilon) \le L_\delta(f)(y_1) + \varepsilon\ , \quad x \in \left[y_1 + \frac{\delta}{2}, y_2 + \frac{\delta}{2}\right] \bigcap \Omega \quad \text{(see (21))}.$$

Therefore

$$\begin{aligned} L_\delta(f)(y_2) &= \sup\left\{ I_{\frac{\delta}{2}}(f)(x) : x \in \left[y_2 - \frac{\delta}{2}, y_2 + \frac{\delta}{2}\right] \bigcap \Omega \right\} \\ &\le L_\delta(f)(y_1) + \varepsilon. \end{aligned} \tag{22}$$

Since ε in the inequality (22) is arbitrary, using also (18) we obtain $L_\delta(f)(y_2) = L_\delta(f)(y_1)$. Then by the monotonicity of f on $[x_1, x_2]$ we have

$$(id - L_\delta(f))(y_1) = f(y_1) - L_\delta(f)(y_1) \le f(y_2) - L_\delta(f)(y_2) = (id - L_\delta(f))(y_2).$$

■

5 Total variation preservation

The operators L_δ, U_δ and their compositions are smoothers. Therefore, one can expect that they reduce the Total Variation of the functions. This is indeed true, but in fact these operators satisfy a much stronger property, namely, total variation preservation. Denote by $BV(\Omega)$ the set of all real functions with bounded variation defined on Ω and denote by $TV(f)$ the total variation of a function $f \in BV(\Omega)$. Consider an operator $A : BV(\Omega) \to BV(\Omega)$. Since the total variation is a semi-norm on $BV(\Omega)$ we have

$$TV(f) \leq TV(A(f)) + TV((id - A)(f)) \ , \quad f \in BV(\Omega). \tag{23}$$

Definition 12 *The operator A is called* total variation preserving *if*

$$TV(f) = TV(A(f)) + TV((id - A)(f)) \ , \quad f \in BV(\Omega). \tag{24}$$

The above definition implies that for a total variation preserving operator the decomposition $f = A(f) + (id - A)(f)$ does not create additional total variation.

Theorem 13 *If the operators $A : BV(\Omega) \to BV(\Omega)$ and $B : BV(\Omega) \to BV(\Omega)$ are both total variation preserving then so is their composition $A \circ B$.*

Proof. Using the total variation preserving property of A and B and (23) we have

$$\begin{aligned} TV(f) &= TV(B(f)) + TV((id - B)(f)) \\ &= TV(A(B(f))) + TV((id - A)(B(f))) + TV((id - B)(f)) \\ &\geq TV((A \circ B)(f)) + TV(((id - A) \circ B + id - B)(f)) \\ &= TV((A \circ B)(f)) + TV((id - A \circ B)(f)) \end{aligned}$$

From (23) we also obtain $TV(f) \leq TV((A \circ B)(f)) + TV((id - A \circ B)(f))$. Therefore $TV(f) = TV((A \circ B)(f)) + TV((id - A \circ B)(f))$. ■

It is easy to see that $BV(\Omega) \subseteq \mathcal{A}(\Omega)$. Hence the operators L_δ, U_δ are defined on $BV(\Omega)$. We will show that L_δ, U_δ and their compositions are total variation preserving. The proof uses the following technical lemmas:

Lemma 14 *Let $a, b \in \Omega$, $a \leq b$.*

(a) If there exists $\varepsilon > 0$ such that $f(x) - L_\delta(f)(x) \geq \varepsilon$, $x \in [a,b]$, then $b - a < \delta$ and $L_\delta(f)(x)$ is a constant on $[a,b]$.

(b) If there exists $\varepsilon > 0$ such that $U_\delta(f)(x) - f(x) \geq \varepsilon$, $x \in [a,b]$, then $b - a < \delta$ and $U_\delta(f)(x)$ is a constant on $[a,b]$.

Proof. We will prove (a). Assume that $b - a \geq \delta$. Then

$$B_{\frac{\delta}{2}}\left(\frac{a+b}{2}\right) = \left[\frac{a+b-\delta}{2}, \frac{a+b+\delta}{2}\right] \subseteq [a,b]$$

and using Lemma 11 we obtain a contradiction as follows:

$$\begin{aligned} I_{\frac{\delta}{2}}(f)\left(\frac{a+b}{2}\right) &= I_{\frac{\delta}{2}}(L_\delta(f))\left(\frac{a+b}{2}\right) = \inf_{y\in\left[\frac{a+b-\delta}{2},\frac{a+b+\delta}{2}\right]} L_\delta(f)(y) \\ &\leq \inf_{y\in\left[\frac{a+b-\delta}{2},\frac{a+b+\delta}{2}\right]} f(y) - \varepsilon \ = \ I_{\frac{\delta}{2}}(f)\left(\frac{a+b}{2}\right) - \varepsilon \end{aligned}$$

Therefore $b - a < \delta$. Let $p = \sup_{y\in\left[b-\frac{\delta}{2},a+\frac{\delta}{2}\right]} I_{\frac{\delta}{2}}(f)(y)$. Since $\left[b-\frac{\delta}{2}, a+\frac{\delta}{2}\right] \subseteq B_{\frac{\delta}{2}}(x)$, $x \in [a,b]$, we have

$$p \leq \sup_{y\in B_{\frac{\delta}{2}}(x)} I_{\frac{\delta}{2}}(f)(y) = L_\delta(f)(x) \leq f(x) - \varepsilon\ , \quad x \in [a,b].$$

Therefore

$$p \leq \inf_{z\in[a,b]} L_\delta(f)(z) \leq \inf_{z\in[a,b]} f(z) - \varepsilon\ , \quad x \in [a,b]. \tag{25}$$

We will show next that

$$I_{\frac{\delta}{2}}(f)(y) \leq p \text{ for all } y \in \left[a - \frac{\delta}{2}, b + \frac{\delta}{2}\right] \cap \Omega. \tag{26}$$

If $y \in \left[b-\frac{\delta}{2}, a+\frac{\delta}{2}\right]$ the inequality (26) follows directly from the definition of p. Let $y > a + \frac{\delta}{2}$. Then $[b, a+\delta] \cap \Omega \subset B_{\frac{\delta}{2}}(y)$ which implies

$$I_{\frac{\delta}{2}}(f)(y) \leq \inf_{z\in[b,a+\delta]\cap\Omega} f(z). \tag{27}$$

Furthermore, using (25), we have

$$\begin{aligned} p &\geq I_{\frac{\delta}{2}}\left(a+\frac{\delta}{2}\right) = \min\left\{\inf_{z\in[a,b]} f(z), \inf_{z\in[b,a+\delta]\cap\Omega} f(z)\right\} \\ &\geq \min\left\{p+\varepsilon, \inf_{z\in[b,a+\delta]\cap\Omega} f(z)\right\}. \end{aligned}$$

Hence

$$\inf_{z\in[b,a+\delta]\cap\Omega} f(z) \leq p. \tag{28}$$

The inequality (26) follows from (27) and (28). The case $y < a+\frac{\delta}{2}$ is considered in a similar manner.

Since $B_{\frac{\delta}{2}}(x) \subset \left[a-\frac{\delta}{2}, b+\frac{\delta}{2}\right]$, $x\in[a,b]$, using (26) we obtain

$$L_\delta f(x) = \sup_{y\in B_{\frac{\delta}{2}}(x)} I_{\frac{\delta}{2}}(f)(y) \leq p\ , \quad x\in[a,b]. \tag{29}$$

The inequalities (25) and (29) imply that $L_\delta(f)(x) = p$ for $x\in[a,b]$. ■

Lemma 15 *Let $a,b\in\Omega$, $a\leq b$.*

(a) If $L_\delta(f)(a) \neq L_\delta(f)(b)$ then for every $\varepsilon>0$ there exists $c\in[a,b]$ such that

(i) $f(c) \leq \min\{f(a), f(b)\}$

(ii) $L_\delta(f)(c) \leq \min\{L_\delta(f)(a), L_\delta(f)(b)\} + \varepsilon$

(iii) $(id-L_\delta)(f)(c) \leq \min\{(id-L_\delta)(f)(a), (id-L_\delta)(f)(b)\}$

(b) If $U_\delta(f)(a) \neq U_\delta(f)(b)$ then there exists $c\in[a,b]$ such that

(i) $f(c) \geq \max\{f(a), f(b)\}$

(ii) $U_\delta(f)(c) \geq \max\{U_\delta(f)(a), U_\delta(f)(b)\}$

(iii) $(id-U_\delta)(f)(c) \geq \max\{(id-U_\delta)(f)(a), (id-U_\delta)(f)(b)\}$

Proof. We will prove (a) when $L_\delta(f)(a) < L_\delta(f)(b)$. The rest is done in a similar way.

Let $\mathcal{D} = \{y \geq a : \inf_{z\in[a,y]}(f(z) - L_\delta(f)(z)) > 0\}$. It follows from Lemma 14 that for every $y \in \mathcal{D}$ the function $L_\delta(f)$ is a constant on $[a, y]$ and that $y - a \leq \delta$. Therefore, b and $a + \delta$ are upper bounds of $\mathcal{D}$ and we have

$$d = \sup \mathcal{D} \leq \min\{b, a + \delta\}.$$

Moreover, for every $\eta > 0$ we have

$$\inf_{z\in[a,d+\eta]}(f(z) - L_\delta(f)(z)) = 0. \tag{30}$$

<u>Case 1.</u> There exists $c \in [a, d)$ such that

$$f(c) - L_\delta(f)(c) \leq \min\{f(a) - L_\delta(f)(a), f(b) - L_\delta(f)(b)\}.$$

Then (iii) is automatically satisfied. Furthermore, (ii) holds since $L_\delta(f)(c) = L_\delta(f)(a) < L_\delta(f)(b)$. The inequality (i) is a consequence of (ii) and (iii).
<u>Case 2.</u> For every $z \in [a, d)$ we have

$$f(z) - L_\delta(f)(z) \geq \min\{f(a) - L_\delta(f)(a), f(b) - L_\delta(f)(b)\}. \tag{31}$$

According to Lemma 14 $L_\delta(f)$ is constant on the interval $[a, d)$. Let $L_\delta(f)(x) = p$, $x \in [a, d)$. Assume that there exists $\xi > 0$ such that $\inf_{z\in[d,d+\xi]} f(z) > p$. Lemma 14 implies that $d + \xi < a + \delta$. Then, using also (31), $\Delta = \inf_{z\in[a,d+\xi]} f(z) > p$ and we have

$$\begin{aligned} p = L_\delta(f)(a) \geq I_{\frac{\delta}{2}}\left(a + \frac{\delta}{2}\right) &= \min\{\inf_{z\in[a,d+\xi])} f(z), \inf_{z\in[d+\xi,a+\delta]} f(z)\} \\ &\geq \min\{p - \Delta, \inf_{z\in[d+\xi,a+\delta]} f(z).\} \end{aligned}$$

Therefore

$$\inf_{z\in[d+\xi,a+\delta]} f(z) \leq p. \tag{32}$$

Using similar techniques as in the proof of Lemma 14 the inequality (32) implies that $L_\delta(f)(x) \leq p < p + \Delta \leq f(x)$, $x \in [a, d + \xi]$, which contradicts the definition of d. Hence

$$\inf_{z\in[d,d+\xi]} f(z) \leq p\ , \quad \xi > 0. \tag{33}$$

As a consequence of the above equation we have

$$L_\delta(f)(d) \leq p. \tag{34}$$

The function f, being a function of bounded variation, may have only discontinuities of first kind, that is, the left and right limit exist at every point. Then the inequality (33) means that

$$f(d) \le p \quad \text{or} \quad f(d^+) \le p. \tag{35}$$

The inequality (30) can be treated in a similar manner. Under the case assumption (31) the inequality (30) is equivalent to

$$\inf_{z \in [d, d+\eta]} (f(z) - L_\delta(f)(z)) = 0 \ , \quad \eta > 0,$$

which implies that

$$f(d) = L_\delta(f)(d) \quad \text{or} \quad f(d^+) = L_\delta(f)(d^+). \tag{36}$$

Case 2.1 $f(d) > p$. Then we also have $L_\delta(f)(d) \le p < f(d)$ so that (35) and (36) imply that $f(d^+) \le p$ and $f(d^+) = L_\delta(f)(d^+)$. Therefore for every $\epsilon > 0$ there exists $\mu(\epsilon) > 0$ such that

$$f(z) \le p + \epsilon \ , \quad f(z) - L_\delta(f)(z) < \epsilon \ , \quad z \in (d, d+\mu) \ .$$

Let $\epsilon = \min\left\{\varepsilon, \frac{1}{2}(f(a) - L_\delta(f)(a)), \frac{1}{2}(f(b) - L_\delta(f)(b))\right\}$. Then any $c \in (d, d+\mu(\epsilon))$ satisfies the conditions (i)–(iii).

Case 2.2 $f(d) = p$.

Case 2.2.1 $f(d) = p = L_\delta(f)(d)$. Then we take $c = d$.

Case 2.2.2 $f(d) = p > L_\delta(f)(d)$. Then it follows from (35) that $f(d^+) = L_\delta(f)(d^+)$. Assume that $L_\delta(f)(d^+) = f(d^+) > p$. Then there exists $\eta > 0$ such that $f(z) \ge L_\delta(f)(z) \ge p$, $z \in [a, d+\eta] \setminus \{d\}$ and $f(d) = p > L_\delta(f)(d)$. It is easy to see that this is impossible. Indeed, let $L_\delta(f)(d) < m < p$. Then there exists $y_1 \in B_{\frac{\delta}{2}}(a)$ and $y_2 \in B_{\frac{\delta}{2}}(d+\eta)$ such that $I_{\frac{\delta}{2}}(f)(y_1) > m$ and $I_{\frac{\delta}{2}}(f)(y_2) > m$. Using also that m is a lower bound of f on $[a, d+\eta]$ we obtain that $f(z) > m$, $z \in [\alpha, \beta]$ where $\alpha = \min\{y_1, y_2\} - \frac{\delta}{2}$, $\beta = \max\{y_1, y_2\} + \frac{\delta}{2}$. Since $d \in [\alpha, \beta]$ and $\beta - \alpha \ge \delta$ there exists $z \in B_{\frac{\delta}{2}}(d)$ such that $B_{\frac{\delta}{2}}(z) \subseteq [\alpha, \beta]$. Then $L_\delta(f)(d) \ge I_{\frac{\delta}{2}}(f)(z) > m$ which is a contradiction. Therefore $L_\delta(f)(d^+) = f(d^+) \le p$. Then the proof proceeds as in the Case 2.1. ■

Theorem 16 *For an arbitrary $\delta > 0$ the operators L_δ, U_δ and their compositions are all total variation preserving operators on $BV(\Omega)$.*

Proof. Let $\delta > 0$. We will only prove that L_δ is total variation preserving, since the total variation preserving property of U_δ is proved in a similar way and the statement for the compositions follows directly from Theorem 13. Let $\theta > 0$ and let $\{x_1, x_2, ...x_n\}$ be an arbitrary grid of points on Ω arranged in increasing order. We will show that there exist a finer grid $\{y_1, y_2, ...y_m\}$, $n \le m < 2n$, such that for every $i = 1, ..., m-1$ we have either

$$\begin{aligned} f(y_i) &\ge f(y_{i+1}) \\ L_\delta(f)(y_i) + \frac{\theta}{2n} &\ge L_\delta(f)(y_{i+1}) \\ (id - L_\delta)(f)(y_i) &\ge (id - L_\delta)(f)(y_{i+1}) \end{aligned} \tag{37}$$

or

$$\begin{aligned} f(y_i) &\le f(y_{i+1}) \\ L_\delta(f)(y_i) &\le L_\delta(f)(y_{i+1}) + \frac{\theta}{2n} \\ (id - L_\delta)(f)(y_i) &\le (id - L_\delta)(f)(y_{i+1}) \end{aligned} \tag{38}$$

This result is obtained from Lemma 15 with $\varepsilon = \frac{\theta}{n}$. If $L_\delta(f)(x_i) = L_\delta(f)(x_{i+1})$ trivially either (37) or (38) is satisfied for the points x_i and x_{i+1}. If $L_\delta(f)(x_i) \ne L_\delta(f)(x_{i+1})$ then according to Lemma 15(a) there exists $c_i \in [x_i, x_{i+1}]$ such that the inequalities (37) are satisfied for the points x_i and c_i and the inequalities (38) are satisfied for the points c_i and x_{i+1}. Thus by including in the grid $\{x_1, x_2, ...x_n\}$ a point c_i between x_i and x_{i+1} for all i such that $L_\delta(f)(x_i) \ne L_\delta(f)(x_{i+1})$ we obtain a finer grid $\{y_1, y_2, ...y_m\}$ satisfying either (37) or (38) for every two consecutive points. Using this property, for every $i = 1, ..., m-1$ we have

$$\begin{aligned} &|f(y_i) - f(y_{i+1})| \\ &\quad = |[L_\delta(f)(y_i) - L_\delta(f)(y_{i+1})] + [(id - L_\delta)(f)(y_i) - (id - L_\delta)(f)(y_{i+1})]| \\ &\quad \ge |L_\delta(f)(y_i) - L_\delta(f)(y_{i+1})| - \frac{\theta}{n} + |(id - L_\delta)(f)(y_i) - (id - L_\delta)(f)(y_{i+1})| \end{aligned}$$

Therefore

$$
\begin{aligned}
TV(f) &\geq \sum_{i=1}^{m-1} |f(y_i) - f(y_{i+1})| \\
&\geq \sum_{i=1}^{m-1} |L_\delta(f)(y_i) - L_\delta(f)(y_{i+1})| + \sum_{i=1}^{m-1} |(id - L_\delta)(f)(y_i) - (id - L_\delta)(f)(y_{i+1})| - \theta \\
&\geq \sum_{i=1}^{n-1} |L_\delta(f)(x_i) - L_\delta(f)(x_{i+1})| + \sum_{i=1}^{n-1} |(id - L_\delta)(f)(x_i) - (id - L_\delta)(f)(x_{i+1})| - \theta
\end{aligned}
$$

Since the grid $\{x_1, x_2, ...x_n\}$ and the number θ are arbitrary, the above inequality implies

$$TV(f) \geq TV(L_\delta(f)) + TV((id - L_\delta)(f)).$$

In view of (23) this completes the proof. ■

6 Conclusion

In this paper we extended the LULU operators from sequences to real functions defined on a real interval using the lower and upper δ-envelopes of functions. The obtained structure, although more general than the well known LULU structure of the discrete operators, retains some of its essential properties.

Of significant importance is the link obtained between properties of functions and sequences that are samplings of these. Particularly, we can easily observe that if a function f has a good approximation Af that is δ-monotone, then a sampling of Af at a uniform sampling interval of h with $h < \frac{\delta}{n+1}$ then the sampling is n-monotone, and a Discrete Pulse Transform will have no (high)-resolution components less than n. Thus we may call Af a "pulse limited" function, in the same sense as a sequence is called "band limited" in the theory of Wavelet Analysis when there are no high frequencies present.

Since the total variation of a function is the supremum of the total variations of all its samplings, we can derive that the total variation of a sequence of samplings does not exceed that of the function. If the functions is δ-monotone they are equal, provided the sampling interval h is smaller than $\frac{\delta}{n+1}$.

This is important in image processing, where Total Variation is used as an appropriate norm [8]. It may be illuminating to consider that the energy reaching the ear is appropriate as a natural norm, where the power spectrum yields important information for economical decomposition and storage of auditory signals.

The eye does not even see with the total illumination as norm, but rather the measure of contrast. It is well known that we perceive an image in the same way under different illumination intensities. The total Variation fits naturally as the sum of the absolute differences of intensity between neighbouring pixels. It turns out to be the natural norm in Discrete Pulse Transforms, as they have a naturally associated "Parseval's Identity" which can be considered analogous to the Parseval's Identity in Wavelet and Fourier Transforms, which is based on the energy distribution amongst resolution levels. We thus have a Pulse Spectrum associated with such a $LULU$-decomposition, which is useful for thresholding decisions for economical transportation and storage of the essentials of an image [8].

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