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Total variation diminishing nonstandard finite difference schemes for conservation laws

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ABSTRACT

Nonstandard finite difference schemes for conservation laws preserving the property of diminishing total variation of the solution are proposed. Computationally simple implicit schemes are derived by using nonlocal approximation of nonlinear terms. Renormalization of the denominator of the discrete derivative is used for deriving explicit schemes of first or higher order. Unlike the standard explicit methods, the solutions of these schemes have diminishing total variation for any time step-size.

1. Introduction

The general setting of this work is conservation laws in the form

$$u_t + f(u)_x = 0, \quad x \in \mathbb{R},\ t \in [0, T], \tag{1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \tag{2}$$

We assume that the data functions f and u_0 are such that Eqs. (1)–(2) have a unique entropy solution, e.g. f smooth and uniformly convex and $u_0 \in L^\infty(\mathbb{R})$, see [1, Section 3.4]. As typical for partial differential equations, Eqs. (1)–(2) cannot be completely solved by analytic techniques. Consequently, numerical simulations are of fundamental importance in gaining some useful insights on the solutions. More precisely, it is crucial to design numerical methods, which replicate essential physical properties of the solutions. This motivates the following concept of stability [2]:

Definition 1. Assume that the solution of (1)–(2) satisfies some property (P). A numerical method approximating (1)–(2) is called qualitatively stable with respect to (P) or P-stable if the numerical solutions satisfy property (P) for all values of the involved step sizes.

The nonstandard finite difference method introduced by Mickens in the late 1980s appear to be powerful in producing qualitatively stable schemes. A formal definition is as follows [2]:

Definition 2. A finite difference method for (1)–(2) is called nonstandard if at least one of the following is met

- In the discrete derivatives the traditional denominator Δt or Δx is replaced by a nonnegative function $\varphi(\Delta t)$ or $\varphi(\Delta x)$ such that

$$\varphi(z) = z + O(z^2) \quad \text{as } 0 < z \to 0. \tag{3}$$

- Nonlinear terms are approximated in a nonlocal way, i.e. by a suitable function of several points of the mesh.

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The following properties have received extensive attention in the design of qualitatively stable nonstandard finite difference schemes: fixed points and their stability [3,2], conservation of energy, monotonicity [4,5], positivity and boundedness [6,7], etc.

This paper is concerned with a physical property which to the best of the authors' knowledge has not yet been exploited in the context of the nonstandard finite difference method. That is, the total variation diminishing of the entropy solution of (1)–(2). More precisely we have that the total variation with respect to x does not increase with time [8](Chapter 16), [9](Chapter 2):

$$TV(u(\cdot,t_1)) \geq TV(u(\cdot,t_2)) \quad \text{for } 0 \leq t_1 < t_2. \tag{4}$$

Our aim is to design and analyze nonstandard finite difference methods for (1)–(2), which are qualitatively stable with respect to the total variation diminishing property (4). A large variety of methods is available for this equation including total variation diminishing (TVD) methods. The preservation of the diminishing total variation property is also discussed in [10] within the context of the more general concept of strong stability. It has been shown that schemes with such qualitative stability resolve discontinuities in the solution without spurious oscillations which are often displayed by numerical solutions [11,12]. One problem associated with the explicit TVD methods is a restriction on the time step-size which in some cases could be rather severe. This is particularly pronounced in higher order methods, e.g. methods of Runge–Kutta type [13,14]. On the other hand, the computational complexity of TVD implicit methods is significantly higher particularly when nonlinear functions are involved.

Our approach is to use the tools of the nonstandard finite difference method in constructing total variation diminishing schemes which have the advantages of being computationally simpler (in the case of implicit schemes) and have no step-size restriction (in the case of explicit schemes).

The rest of the paper is organized as follows. In Section 2 we give some preliminary settings and results including Harten's lemma. In Section 3 we formulate an implicit nonstandard finite difference scheme using nonlocal approximation of nonlinear terms. Section 4 deals with explicit nonstandard finite difference schemes where renormalization of the denominator is used. Numerical results by both the implicit and the explicit methods are presented in Section 5. Some final remarks are given in the conclusion.

2. Preliminaries

Following a space discretization, Eq. (1) is written as a system of ODEs of the form

$$v_t = L(v), \tag{5}$$

where $v = (v_j)$ and $v_j(t) \approx u(x_j, t)$. We consider the case when the operator L in (5) is obtained from spacial discretization using the Lax–Friedrichs numerical flux. More precisely, we have

$$(L(v))_j = \frac{1}{\Delta x}\left(\hat{f}_{j-\frac{1}{2}} - \hat{f}_{j+\frac{1}{2}}\right), \tag{6}$$

where we assume that the mesh in the space dimension is uniform with a step-size Δx and

$$\hat{f}_{j+\frac{1}{2}} = \frac{1}{2}\left[f(v_{j+1}) + f(v_j) - \alpha(v_{j+1} - v_j)\right], \tag{7}$$

$$\alpha = \max_u |f'(u)|, \tag{8}$$

the maximum being taken over the relevant range of u. Hence

$$(L(v))_j = \frac{1}{2\Delta x}\left[\alpha(v_{j+1} - 2v_j + v_{j-1}) - f(v_{j+1}) + f(v_{j-1})\right]. \tag{9}$$

Let a mesh $t_n = n\Delta t$, $n = 0, 1, \ldots$, in the time direction be given. As usual v^n denotes an approximation of v at $t = t_n$. The total variation of v^n is given by

$$TV(v^n) = \sum_j |v^n_{j+1} - v^n_j|.$$

The discrete analogue of (4) is as follows. A numerical scheme is called TVD if $TV(v^n)$ is decreasing with respect to n, that is,

$$TV(v^n) \geq TV(v^{n+1}), \quad n = 0, 1, 2, \ldots. \tag{10}$$

The TVD property of numerical methods is often proved by using Harten's lemma. We give below a version dealing with both the explicit and the implicit cases [15,10].

Lemma 1 (*Harten*). *Consider the explicit scheme*

$$v^{n+1}_j = v^n_j + C_{j+\frac{1}{2}}(v^n_{j+1} - v^n_j) - D_{j-\frac{1}{2}}(v^n_j - v^n_{j-1}), \tag{11}$$

and the implicit scheme

$$v_j^{n+1} = v_j^n + C_{j+\frac{1}{2}}(v_{j+1}^{n+1} - v_j^{n+1}) - D_{j-\frac{1}{2}}(v_j^{n+1} - v_{j-1}^{n+1}), \tag{12}$$

where $C_{j+\frac{1}{2}}$ and $D_{j-\frac{1}{2}}$ are functions of v^n and/or v^{n+1} at various (usually neighboring) grid points. If $C_{j+\frac{1}{2}} \geq 0$ and $D_{j-\frac{1}{2}} \geq 0$ then the scheme (12) is TVD. If in addition to these conditions we have $C_{j+\frac{1}{2}} + D_{j-\frac{1}{2}} \leq 1$ then the scheme (11) is TVD.

3. Implicit nonstandard schemes

In this section, we design nonstandard schemes by exploiting the nonlocal approximation of nonlinear terms as stated in the second bullet of Definition 2. We consider nonlocal approximation of the function L for deriving nonstandard TVD schemes for Eq. (5). Below we propose a scheme of Euler type.

$$v_j^{n+1} = v_j^n + \frac{\Delta t}{2\Delta x}\left(\alpha(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}) - (v_{j+1}^{n+1} - v_{j-1}^{n+1})\frac{f(v_{j+1}^n) - f(v_{j-1}^n)}{v_{j+1}^n - v_{j-1}^n}\right). \tag{13}$$

We should note that the linear terms in (9) are evaluated at $t = t_{n+1}$. The expression $f(v_{j+1}) - f(v_{j-1})$ is multiplied and divided by $v_{j+1} - v_{j-1}$, where the multiplier is evaluated at $t = t_{n+1}$ and the remaining part of the expression evaluated at $t = t_n$.

Theorem 1. *The scheme (13) is qualitatively stable with respect to the total variation diminishing property (4).*

Proof. The scheme (13) can be written as

$$v_j^{n+1} = v_j^n + \frac{\Delta t}{2\Delta x}\left(\left(\alpha - \frac{f(v_{j+1}^n) - f(v_{j-1}^n)}{v_{j+1}^n - v_{j-1}^n}\right)(v_{j+1}^{n+1} - v_j^{n+1}) - \left(\alpha + \frac{f(v_{j+1}^n) - f(v_{j-1}^n)}{v_{j+1}^n - v_{j-1}^n}\right)(v_j^{n+1} - v_{j-1}^{n+1})\right).$$

Therefore the scheme (13) can be represented in the form (12) with

$$C_{j+\frac{1}{2}} = \frac{\Delta t}{2\Delta x}\left(\alpha - \frac{f(v_{j+1}^n) - f(v_{j-1}^n)}{v_{j+1}^n - v_{j-1}^n}\right),$$

$$D_{j-\frac{1}{2}} = \frac{\Delta t}{2\Delta x}\left(\alpha + \frac{f(v_{j+1}^n) - f(v_{j-1}^n)}{v_{j+1}^n - v_{j-1}^n}\right).$$

Using (8), we obtain that $C_{j+\frac{1}{2}} \geq 0$ and $D_{j-\frac{1}{2}} \geq 0$ for all j. Hence it follows from Lemma 1 that the scheme (13) satisfies (10). ∎

Using standard techniques of numerical analysis one can easily obtain that for linear systems the scheme (13) is consistent and unconditionally stable. Moreover, the qualitative stability of the scheme (13) also does not impose any condition on Δx and/or Δt.

We should note that one step in the time dimension requires the solutions of a tridiagonal linear system. Hence the computation effort is similar to the one for explicit methods. Furthermore, the suggested scheme is not unique. One may use a different kind of nonlocal approximation to obtain a TVD scheme. For example, the scheme

$$v_j^{n+1} = v_j^n + \frac{\Delta t}{2\Delta x}\left(\alpha(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}) - (v_{j+1}^{n+1} - v_j^{n+1})\frac{f(v_{j+1}^n) - f(v_j^n)}{v_{j+1}^n - v_j^n} - (v_j^{n+1} - v_{j-1}^{n+1})\frac{f(v_j^n) - f(v_{j-1}^n)}{v_j^n - v_{j-1}^n}\right)$$

is also TVD.

4. Explicit nonstandard schemes

The schemes in this section are based on the renormalization of the denominator of the discrete derivatives, see Definition 2. This means that the denominator Δt in the discrete time derivative is replaced by a function $\varphi(\Delta t)$ satisfying (3). In order to obtain elementary stable schemes, that is, schemes which are qualitatively stable with respect to fixed points of the differential equation and their stability, the following renormalization was considered, [3,2]:

$$\varphi(\Delta t) = \frac{\phi(q\Delta t)}{q}, \tag{14}$$

where the function ϕ is such that

$$\phi(z) = z + O(z^2) \quad \text{as } z \to 0, \tag{15}$$

$$0 < \phi(z) < 1 \quad \text{for } z > 0, \tag{16}$$

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and $q = \max\{|\lambda|\}$, λ tracing the eigenvalues of the Jacobian $J(v)$ of the right-hand side of Eq. (5) at the fixed points of the equation. We will show that similar renormalization also ensures the TVD property of the scheme. We consider function φ as given by (14) where the value of q is suitably determined by the function L.

Let us consider first the Euler scheme

$$\frac{v^{n+1} - v^n}{\varphi(\Delta t)} = L(v^n). \tag{17}$$

The next theorem shows that the above scheme is TVD.

Theorem 2. *The scheme (17) where* $\varphi(z) = \frac{\phi(\frac{\alpha z}{\Delta z})}{\alpha/\Delta x}$, $z > 0$, *and* ϕ *satisfies conditions (15)–(16) is qualitatively stable with respect to the total variation diminishing property (4).*

Proof. The scheme (17) can be written in the form

$$\begin{aligned} v_j^{n+1} &= v_j^n + \frac{\varphi(\Delta t)}{2\Delta x}\left(\alpha(v_{j+1}^n - 2v_j^n + v_{j-1}^n) - f(v_{j+1}^n) + f(v_{j-1}^n)\right) \\ &= v_j^n + \frac{\varphi(\Delta t)}{2\Delta x}\left(\left(\alpha - \frac{f(v_{j+1}^n) - f(v_j^n)}{v_{j+1}^n - v_j^n}\right)(v_{j+1}^n - v_j^n) - \left(\alpha + \frac{f(v_j^n) - f(v_{j-1}^n)}{v_j^n - v_{j-1}^n}\right)(v_j^n - v_{j-1}^n)\right). \end{aligned}$$

Therefore (17) can be represented in the form (11) with

$$C_{j+\frac{1}{2}} = \frac{\varphi(\Delta t)}{2\Delta x}\left(\alpha - \frac{f(v_{j+1}^n) - f(v_j^n)}{v_{j+1}^n - v_j^n}\right)$$

$$D_{j-\frac{1}{2}} = \frac{\varphi(\Delta t)}{2\Delta x}\left(\alpha + \frac{f(v_j^n) - f(v_{j-1}^n)}{v_j^n - v_{j-1}^n}\right).$$

Using the definition of α, see (8), it is easy to see that $C_{j+\frac{1}{2}} \geq 0$ and $D_{j-\frac{1}{2}} \geq 0$. Furthermore, we have

$$\begin{aligned} C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} &= \frac{\varphi(\Delta t)}{2\Delta x}\left(\alpha - \frac{f(v_{j+1}^n) - f(v_j^n)}{v_{j+1}^n - v_j^n}\right) + \frac{\varphi(\Delta t)}{2\Delta x}\left(\alpha + \frac{f(v_{j+1}^n) - f(v_j^n)}{v_{j+1}^n - v_j^n}\right) \\ &= \frac{\phi\left(\frac{\alpha\Delta t}{\Delta x}\right)}{\frac{\alpha}{\Delta x}2\Delta x}(2\alpha) = \phi\left(\frac{\alpha\Delta t}{\Delta x}\right). \end{aligned}$$

Then it follows from (16) that $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1$. Hence we can apply Harten's lemma, see Lemma 1, and obtain that the scheme is TVD. ■

Renormalization can also be used in higher order methods, e.g. Runge–Kutta methods. For investigation of TVD properties a Runge–Kutta method is typically written in the so-called Shu–Osher form, [14], namely

$$\begin{aligned} y^{(0)} &= v^n \\ y^{(i)} &= \sum_{j=1}^{i-1}(\lambda_{ij}y^{(j)} + \Delta t\mu_{ij}L(y^{(j)})), \quad i = 1, \ldots, m \\ v^{n+1} &= y^{(m)}. \end{aligned}$$

By consistency $\sum_{j=0}^{i-1}\lambda_{ij} = 1, i = 1, \ldots, m$. Therefore in each intermediate step of the method $y^{(i)}$ is a convex combination of Euler forward operators:

$$y^{(i)} = \sum_{j=1}^{i-1}\lambda_{ij}\left(y^{(j)} + \Delta t\frac{\mu_{ij}}{\lambda_{ij}}L(y^{(j)})\right).$$

If these operators are TVD then the Runge–Kutta method is also TVD. Following the result of Theorem 2 we will obtain a TVD scheme if the Euler operator involving $y^{(j)}$ above is renormalized by

$$\varphi_{ij}(\Delta t) = \frac{\phi\left(\frac{\alpha\mu_{ij}\Delta t}{\lambda_{ij}\Delta x}\right)}{\frac{\alpha\mu_{ij}}{\lambda_{ij}\Delta x}}$$

where the function ϕ satisfies conditions (15)–(16). Note that this function might have to satisfy additional conditions for

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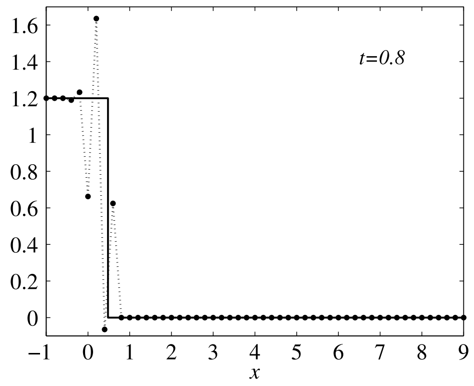

Fig. 1. Numerical solution by the standard Euler method with $\Delta x = \Delta t = 0.2$.

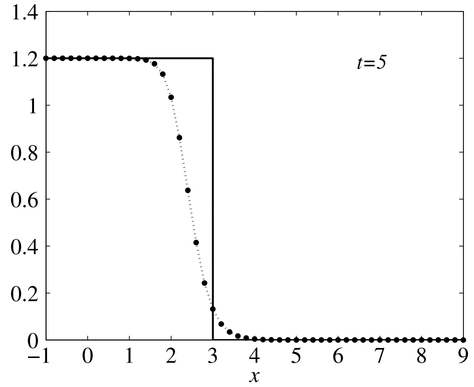

Fig. 2. Numerical solution by the scheme (22) with $\Delta x = \Delta t = 0.2$.

the scheme to be of particular order. We will illustrate by an example. Following the discussion above the two stage scheme given below is TVD:

$$y^{(1)} = v^n + \phi\left(\frac{\alpha\Delta t}{\Delta x}\right)\frac{\Delta x}{\alpha}L(v^n) \tag{18}$$

$$v^{n+1} = \frac{1}{2}v^n + \frac{1}{2}y^{(1)} + \frac{1}{2}\phi\left(\frac{\alpha\Delta t}{\Delta x}\right)\frac{\Delta x}{\alpha}L(y^{(1)}). \tag{19}$$

Using standard techniques one can also obtain that it is of order two provided that $\phi(z) = z + O(z^3)$.

5. Numerical results

We apply the schemes considered in Sections 3 and 4 to Burger's equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0. \tag{20}$$

It is well known that the entropy solution of this equation develops discontinuities (shocks) even for smooth initial condition. To simplify the matters we take

$$u(x,0) = \begin{cases} 1.2 & \text{if } x \le 0, \\ 0 & \text{if } x > 0. \end{cases} \tag{21}$$

After some obvious transformations the scheme (13) can be written in the form

$$v_j^{n+1} = v_j^n + \frac{\Delta t}{2\Delta x}\left(\alpha(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) - \frac{1}{2}(v_{j+1}^{n+1} - v_{j-1}^{n+1})(v_{j+1}^n + v_{j-1}^n)\right). \tag{22}$$

It was shown in [13] that non-TVD methods typically produce oscillations around the points of discontinuity. Fig. 1 shows such oscillations produced by the standard Euler method applied to problem (20)–(21). Figs. 2–4 show the numerical solution of the problem (20)–(21) by the implicit scheme (22) for various time steps. One can observe that while an increase in Δt affects the accuracy of the solution it nevertheless remains TVD and free of spurious oscillations.

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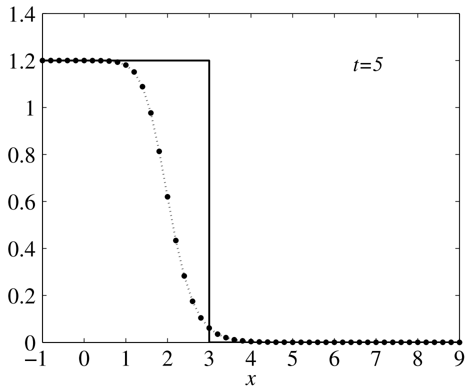

Fig. 3. Numerical solution by the scheme (22) with $\Delta x = 0.2$, $\Delta t = 0.5$.

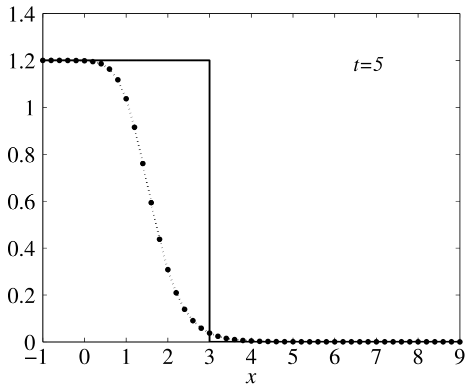

Fig. 4. Numerical solution by the scheme (22) with $\Delta x = 0.2$, $\Delta t = 1.0$.

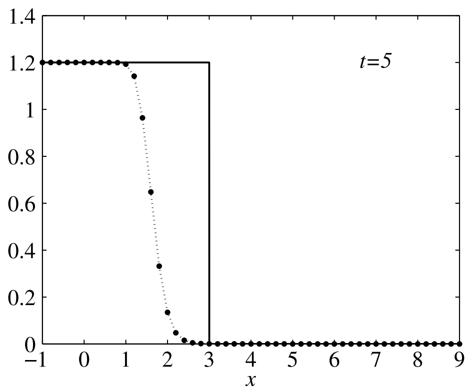

Fig. 5. Numerical solution by the scheme (23) with $\Delta x = \Delta t = 0.2$.

Similar results are obtained using the explicit schemes in Section 3. For the considered problem (20)–(21) Euler's method (17) can be written as

$$v_j^{n+1} = v_j^n + \frac{1}{2\alpha}\phi\left(\frac{\alpha\Delta t}{\Delta x}\right)(\alpha(v_{j+1}^n - 2v_j^n + v_{j-1}^n) - (v_{i+1}^n)^2 + (v_{j-1}^n)^2), \tag{23}$$

where we take $\phi(z) = 1 - e^{-z}$. The numerical solution computed with $\Delta x = \Delta t = 0.2$ is presented on Fig. 5. Fig. 6 represents the solution produced by the Runge–Kutta method (18)–(19) with renormalizing function $\phi(z) = \frac{1-e^{-t^2}}{t}$ so that the method is of order two. Let us note that since the exact solution is discontinuous, a higher order method does not necessarily give a better approximation. Naturally, the accuracy can be improved by decreasing the step sizes. However, the major point here is that irrespective of the step sizes the numerical solution is free of spurious oscillations and its total variation does not increase with time, for this particular equation it is in fact constant.

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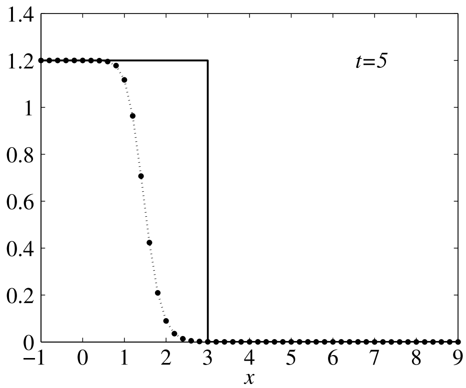

Fig. 6. Numerical solution by the scheme (18)–(19) with $\Delta x = \Delta t = 0.2$.

6. Conclusion

Schemes preserving the essential physical property of diminishing total variation are of great importance in practice. Such schemes are free of spurious oscillations around discontinuities. In this paper, we have discussed nonstandard finite difference schemes, which have this qualitative stability property. We used Mickens' rules of approximating nonlinear terms in a nonlocal way and of renormalizing denominators. The obtained schemes are computationally simple. Furthermore, they require no restriction on the time step-size as typical for qualitatively stable nonstandard schemes [2].

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