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Weighted Fejér, Hermite–Hadamard, and Trapezium-Type Inequalities for $(\mathfrak{h}_1, \mathfrak{h}_2)$–Godunova–Levin Preinvex Function with Applications and Two Open Problems

Abdullah Ali H. Ahmadini [1], Waqar Afzal [2,3], Mujahid Abbas [3,4,5,*] and Elkhateeb S. Aly [1]

1 Department of Mathematics, College of Science, Jazan University, P.O. Box 114, Jazan 45142, Saudi Arabia; aahmadini@jazanu.edu.sa (A.A.H.A.); elkhateeb@jazanu.edu.sa (E.S.A.)
2 Department of Mathematics, University of Gujrat, Gujrat 50700, Pakistan; waqar_afzal_22@sms.edu.pk
3 Department of Mathematics, Government College University, Katchery Road, Lahore 54000, Pakistan
4 Department of Medical Research, China Medical University, Taichung 406040, Taiwan
5 Department of Mathematics and Applied Mathematics, University of Pretoria, Lynnwood Road, Pretoria 0002, South Africa
* Correspondence: abbas.mujahid@gcu.edu.pk

Abstract: This note introduces a new class of preinvexity called $(\mathfrak{h}_1, \mathfrak{h}_2)$-Godunova-Levin preinvex functions that generalize earlier findings. Based on these notions, we developed Hermite-Hadamard, weighted Fejér, and trapezium type inequalities. Furthermore, we constructed some non-trivial examples in order to verify all the developed results. In addition, we discussed some applications related to the trapezoidal formula, probability density functions, special functions and special means. Lastly, we discussed the importance of order relations and left two open problems for future research. As an additional benefit, we believe that the present work can provide a strong catalyst for enhancing similar existing literature.

Keywords: Hermite-Hadamard; weighted Fejér-type inequality; trapezoid inequality; interval-valued; Godunova-Levin Preinvex

MSC: 05A30; 26D10; 26D15

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1. Introduction

Convexity is important in optimization, and many optimization problems can be formulated and solved using convex function and convex set properties. Here are some of the most important applications of convexity in optimization: These days, convex optimization is essential to many deep learning algorithms, including logistic regression and neural network training (see ref. [1]). Utility theory relies heavily on convexity since convex utility functions are frequently used to represent preferences (see ref. [2]). Furthermore, cooperative games fall under the category of convex games. This idea is used to analyse and model cooperation amongst logical decision makers (see ref. [3]). For more information on some other recent developments in other disciplines, see refs. [4–6].

A relaxation of some of the strict conditions imposed by convex sets and functions is known as generalized convexity, which is an extension of classical convexity. To accommodate a wider class of functions and sets, some modifications or generalizations of the convexity concept are introduced in generalized convexity. The following are some features and uses of generalised convexity: log-convex, p-convex, h-convex, preinvexity, Godunova–Levin, exponentially convex, harmonic convex, and many others (see refs. [7–13]). As a result of these different classes, various authors developed the following double inequality for convex function in different perspectives, and it is the most crucial factor in optimization [14].

$$\mathfrak{V}\left(\frac{\mathtt{a}+\mathtt{b}}{2}\right) \leq \frac{1}{\mathtt{b}-\mathtt{a}}\int_{\mathtt{a}}^{\mathtt{b}} \mathfrak{V}(\beta)\, d\beta \leq \frac{\mathfrak{V}(\mathtt{a})+\mathfrak{V}(\mathtt{b})}{2}. \tag{1}$$

It is clear from the extensive study of generalizations and variants of the Hermite–Hadamard inequality, as well as its extensive application in multiple fields, that this result carries a great deal of mathematical significance [15]. In the realm of mathematical optimization, the notion of invex functions is an intriguing advancement. Due to their properties, these functions are useful in many mathematical models and are appropriate for use in optimization problems. First, Hanson [16] introduced and studied invex functions, a more generalised class of convex mappings, and talked about some of their fascinating characteristics. In [17], the authors minimized the objective function by using preinvex mappings in multiobjective optimization. According to Suneja et al. [18], b-preinvex functions are generalizations of classical preinvex and b-vex functions. Drawing inspiration from this work, Noor first proposed the concept of h-preinvex mapping [19]. Later, in 2018, Awan proposed the concept of $(\mathtt{h}_1, \mathtt{h}_2)$-convex functions and created a new, more generalized version of the well-known double inequality [20].

Let $\mathfrak{V} : [\mathtt{a}, \mathtt{b}] \to \mathrm{R}$. Consider $\mathfrak{V}$ be h-G.L-preinvex mappings, and $\mathtt{h}\left(\frac{1}{2}\right) \neq 0$; then, the following double inequality holds [21]:

$$\frac{\mathtt{h}\left(\frac{1}{2}\right)}{2}\mathfrak{V}\left(\frac{2\mathtt{a}+\varsigma(\mathtt{b},\mathtt{a})}{2}\right) \leq \frac{1}{\varsigma(\mathtt{b},\mathtt{a})}\int_{\mathtt{a}}^{\mathtt{a}+\varsigma(\mathtt{b},\mathtt{a})} \mathfrak{V}(\varrho)\, d\varrho \leq \int_0^1 \frac{d\mathfrak{y}}{\mathtt{h}(\mathfrak{y})}.$$

A set-valued analysis is a useful tool for dealing with uncertainties and errors in data and computations. An interval-valued integral inequality is used to study functions with outputs determined by intervals rather than sets of arbitrary shapes. Initially, in 1993 Wolfgang [22], developed the continuity of generalized convex mapping through set-valued mappings. In [23], the authors linked famous double inequalities in the settings of interval-valued functions (I.V.F.S) using $(\mathtt{h}_1, \mathtt{h}_2)$-convex mappings, and demonstrated that our result becomes more general when our interval is degenerated. By utilizing the idea of preinvex functions and the fractional integral via interval-valued functions, the authors in [24] developed the well-known double inequality along with several new variations. Liu used the idea of strongly preinvex type functions on fuzzy invex sets and used differentiability to look into some of its properties [25]. Kalsoom et al. [26] developed some new generalized forms of double inequalities on coordinates with two types of preinvex mappings by using quantum integrals. By using conformable fractional integrals with various integral identities, Khurshid and his co-workers proposed a Hermite–Hadamard–Fejér inequality for symmetric preinvex mappings [27]. Barani developed the mean value inequality in invexity analysis by utilizing the concept of Cartan–Hadamard manifolds and explained its significant characteristics [28]. Nasir et al. [29] developed some Ostrowski-type results using a fractional approach by using preinvex functions via second derivatives. In [30], the authors developed new variants of double inequalities based on preinvex functions on a real plane. As a result of using preinvex mappings in fractal space, Yu et al. [31] found error bounds on parameterized integral inequalities. In [32], the authors created a number of inequalities connected to these developed results by using fractional integral utilizing interval mapping via generalized $(\mathtt{h}_1, \mathtt{h}_2)$-preinvex functions. Zhou et al. [33] employed the pre-invex exponential type definition in the context of interval-valued mapping through fractional integrals, yielding several findings associated with these. Khan et al. [34] generalized a number of results pertaining to these developed results using the concept of U-d preinvex mappings via fractional operator in the fuzzy environment. Afzal et al. [35] employed the midpoint and center radius interval order relations through set-valued mappings, and derived several conclusions via harmonical $(\mathtt{h}_1, \mathtt{h}_2)$–Godunova–Levin. In [36], the concept of (s, m, φ)-type functions was used to develop a number of novel double inequalities with some intriguing characteristics. Regarding a few additional recent developments concerning developed outcomes, see [37].

Novelty and Significance

This study is novel and significant because we introduce a more generalized class for the first time, referred to as $(\mathtt{h}_1, \mathtt{h}_2)$–Godunova–Levin preinvex functions, which generalize the results of authors mentioned in abstract as well as unify various other results by using different sets of non-negative arbitrary functions and bifunctions ς. Futhermore, we have found error bounds of the trapezoidal formula using standard as well as Kulisch- and Miranke-type order relations. In addition, we found error bounds for trapezoidal-type formulas in a set-valued setting for the first time in the literature, and the result generalizes when the interval is degenerated. The application of preinvexity to special means and random variables also lends an interesting note to researchers through this novel class of preinvexity.

The rich literature on developed results, and specifically [21,38], motivated us to create new extensions, improved forms of Hermite–Hadamard, trapezium, and weighted forms of Fejér-type results, as well as applications spanning a wide range of areas using this new class of preinvex functions. This article follows the following structure. In Section 2, first of all, we recall some known definitions and interval calculus results that are necessary to progress. Next, we developed some improved forms of Hermite–Hadamard and Fejér-type results that generalize various previous findings in Section 3. In Section 4, we apply some novel results to numerical quadrature rules and applications to special means involving set-valued mappings. In Section 5, we include a discussion on the developed results and conclusions with some future recommendations.

2. Preliminaries

We now define some existing definitions and results that may lend support to the main findings presented in the article. Let $\Theta \subset \mathsf{R}$, and $\varsigma(\cdot,\cdot) : \Theta \times \Theta \longrightarrow \mathsf{R}$ is a bifunction.

Definition 1 (see [39])**.** *A set Θ is considered to be invex with reference to the bifunction $\varsigma(\cdot,\cdot)$, iff*

$$\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b}, \mathtt{a}) \in \Theta,$$

for all $\mathtt{a}, \mathtt{b} \in \Theta$, and $\mathfrak{y} \in [0, 1]$.

Example 1. *Let $\Theta = [-4, -3] \cup [-2, 3]$ be considered to be invex with reference to bifunction $\varsigma(\cdot,\cdot)$ and defined as:*

$$\varsigma(\mathtt{a}, \mathtt{b}) = \begin{cases} \mathtt{a} - \mathtt{b} & \text{if } 3 \geq \mathtt{a} \geq -2, 3 \geq \mathtt{b} \geq -1; \\ \mathtt{a} - \mathtt{b} & \text{if } -4 \leq \mathtt{a} \leq -3, -4 \leq \mathtt{b} \leq -3; \\ -4 - \mathtt{b} & \text{if } -2 \leq \mathtt{a} \leq 3, -4 \leq \mathtt{b} \leq -2; \\ -2 - \mathtt{b} & \text{if } -4 \leq \mathtt{a} \leq -3, -2 \leq \mathtt{b} \leq 3. \end{cases}$$

Example 2. *Let $\Theta = \left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$,*

$$\varsigma(\mathtt{a}, \mathtt{b}) = \begin{cases} \cos(\mathtt{a} - \mathtt{b}), & 0 < \mathtt{b} \leq \frac{\pi}{2}, 0 < \mathtt{a} \leq \frac{\pi}{2}; \\ -\cos(\mathtt{a} - \mathtt{b}), & \frac{-\pi}{2} \leq \mathtt{b} < 0, \frac{-\pi}{2} \leq \mathtt{a} < 0; \\ \cos \mathtt{b}, & 0 < \mathtt{b} \leq \frac{\pi}{2}, \frac{-\pi}{2} \leq \mathtt{a} < 0; \\ -\cos \mathtt{b}, & \frac{-\pi}{2} \leq \mathtt{b} < 0, 0 < \mathtt{a} \leq \frac{\pi}{2}. \end{cases}$$

Then, in both examples, Θ is considered to be invex with reference to bifunction $\varsigma(\cdot,\cdot)$. In contrast, Θ is not convex.

Definition 2 (see [39])**.** *Consider Θ be invex with reference to bifunction $\varsigma(\cdot,\cdot)$. A mappping $\mathfrak{V} : \Theta \to \mathsf{R}$ is known as preinvex with reference to ς if*

$$\mathfrak{V}(\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b}, \mathtt{a})) \leq \mathfrak{y}\mathfrak{V}(\mathtt{b}) + (1 - \mathfrak{y})\, \mathfrak{V}(\mathtt{a}),$$

for all $\mathtt{a}, \mathtt{b} \in \Theta$, *and* $\mathfrak{y} \in [0,1]$.

Example 3. *The function* $\mathfrak{V}(\rho) = \frac{1-|2\rho-1|}{2}$, $(\rho \in \mathsf{R})$, *in light of the following*

$$\eta(\mathtt{a},\mathtt{b}) = \begin{cases} 2(\mathtt{a}-\mathtt{b}), & \mathtt{b} \geq \frac{1}{2}, \mathtt{a} \geq \frac{1}{2}, \mathtt{a} \geq \mathtt{b}, \\ 0, & \mathtt{b} > \frac{1}{2}, \mathtt{a} \geq \frac{1}{2}, \mathtt{a} < \mathtt{b}, \\ 0, & \mathtt{b} < \frac{1}{2}, \mathtt{a} \leq \frac{1}{2}, \mathtt{a} > \mathtt{b}, \\ \mathtt{a}-\mathtt{b}, & \mathtt{b} \leq \frac{1}{2}, \mathtt{a} \leq \frac{1}{2}, \mathtt{a} \leq \mathtt{b}, \\ 1-\mathtt{a}-\mathtt{b}, & \mathtt{b} < \frac{1}{2}, \mathtt{a} > \frac{1}{2}, \mathtt{a}+\mathtt{b} \geq 1, \\ 0, & \mathtt{b} < \frac{1}{2}, \mathtt{a} > \frac{1}{2}, \mathtt{a}+\mathtt{b} \leq 1, \\ 0, & \mathtt{b} > \frac{1}{2}, \mathtt{a} < \frac{1}{2}, \mathtt{a}+\mathtt{b} \geq 1, \\ 1-\mathtt{a}-\mathtt{b}, & \mathtt{b} > \frac{1}{2}, \mathtt{a} < \frac{1}{2}, \mathtt{a}+\mathtt{b} \leq 1, \end{cases}$$

is considered to be a preinvex mapping on R, *while not being convex in general.*

Definition 3 (see [19])**.** *Consider* Θ *to be a invex with reference to bifunction* $\varsigma(\cdot,\cdot)$. *A mapping* $\mathfrak{V} : \Theta \to \mathsf{R}$ *is known as* G.L *preinvex with respect to* ς *if*

$$\mathfrak{V}(\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b},\mathtt{a})) \leq \frac{\mathfrak{V}(\mathtt{b})}{\mathfrak{y}} + \frac{\mathfrak{V}(\mathtt{a})}{(1-\mathfrak{y})},$$

for all $\mathtt{a}, \mathtt{b} \in \Theta$, *and* $\mathfrak{y} \in (0,1)$.

Definition 4 (see [19])**.** *Consider* Θ *be invex with reference to bifunction* $\varsigma(\cdot,\cdot)$, *and* $h : (0,1) \to (0,\infty)$, *where* $\mathsf{h} \neq 0$. *A mapping* $\mathfrak{V} : \Theta \to \mathsf{R}$ *is known as* h*-preinvex with reference to* ς *if*

$$\mathfrak{V}(\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b},\mathtt{a})) \leq \mathsf{h}(\mathfrak{y})\mathfrak{V}(\mathtt{b}) + \mathsf{h}(1-\mathfrak{y})\, \mathfrak{V}(\mathtt{a}),$$

for all $\mathtt{a}, \mathtt{b} \in \Theta$, *and* $\mathfrak{y} \in [0,1]$.

Definition 5 (see [19])**.** *Consider* Θ *to be invex with reference to bifunction* $\varsigma(\cdot,\cdot)$, *and* $h : (0,1) \to (0,\infty)$, *where* $\mathsf{h} \neq 0$. *A mapping* $\mathfrak{V} : \Theta \to \mathsf{R}$ *is known as* h-G.L *preinvex with reference to* ς *if*

$$\mathfrak{V}(\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b},\mathtt{a})) \leq \frac{\mathfrak{V}(\mathtt{b})}{\mathsf{h}(\mathfrak{y})} + \frac{\mathfrak{V}(\mathtt{a})}{\mathsf{h}(1-\mathfrak{y})},$$

for all $\mathtt{a}, \mathtt{b} \in \Theta$, *and* $\mathfrak{y} \in (0,1)$.

Remark 1. *A note should be made to the effect that not all Godunova–Levin functions belong to the class of h-Godunova–Levin or h-convex functions, and this inconvenience can be avoided by omitting the assumption that* $\mathfrak{V}$ *is positive (see ref. [40]).*

Definition 6 (see [32])**.** *Let* $\Theta \subset \mathsf{R}^n$ *be an invex set with reference to* $\varsigma(\cdot,\cdot)$. *For all* $\mathtt{a}, \mathtt{b} \in \Theta$ *and* $\mathfrak{y} \in [0,1]$, *we have*

$$\varsigma(\mathtt{b}, \mathtt{b} + \mathfrak{y}\, \varsigma(\mathtt{a},\mathtt{b})) = -\mathfrak{y}\, \varsigma(\mathtt{a},\mathtt{b}) \tag{2}$$

and

$$\varsigma(\mathtt{a}, \mathtt{b} + \mathfrak{y}\, \varsigma(\mathtt{a},\mathtt{b})) = (1-\mathfrak{y})\, \varsigma(\mathtt{a},\mathtt{b}). \tag{3}$$

for all $\mathtt{a}, \mathtt{b} \in \Theta$ *and* $\mathfrak{y}_1, \mathfrak{y}_2 \in [0,1]$. *This is often called Condition C, that is*

$$\varsigma(\mathtt{b} + \mathfrak{y}_2\, \varsigma(\mathtt{a},\mathtt{b}), \mathtt{b} + \mathfrak{y}_1\varsigma(\mathtt{a},\mathtt{b})) = (\mathfrak{y}_2 - \mathfrak{y}_1)\varsigma(\mathtt{a},\mathtt{b}).$$

It was proven in [41] that if a differentiable mapping is invex on some set, it also meets the assumptions of preinvex-type mappings under Condition C and the corresponding bifunction. Note that the mapping "ς" in Example 1 is not defined on a convex set and that "ς" still holds with regard to Condition C.

There are vector-valued mappings that meet Condition C. For instance, consider $\Theta = \mathsf{R} - \{0\}$ and

$$\varsigma(\mathtt{a},\mathtt{b}) = \begin{cases} \mathtt{a}-\mathtt{b} & \text{if } \mathtt{a} \geq 0, \mathtt{b} \geq 0 \\ \mathtt{a}-\mathtt{b} & \text{if } \mathtt{a} \leq 0, \mathtt{b} \leq 0 \\ -\mathtt{b} & \text{otherwise} \end{cases}$$

It is clear from the above that Θ is an invex set and that ς satisfies Condition C.

2.1. *Interval Operations*

Consider $\mathsf{M_C}$ be the pack of all compact subsets of R in one-dimensional Euclidean space, that is,

$$\mathsf{M_C} = \{[\rho_x, \eta_x] : \rho_x, \eta_x \in \mathsf{R} \textit{ and } \rho_x \leq \eta_x\},$$

The Hausdorff metric on M_C is defined as

$$\mathcal{H}(\rho,\eta) = \sup\{d(\rho,\eta), d(\eta,\rho)\}, \tag{4}$$

where $d(\rho,\eta) = \sup_{\mathtt{a}\in\rho} d(\mathtt{a},\eta)$, and $d(\mathtt{a},\eta) = \min_{\mathtt{b}\in\eta} d(\mathtt{a},\mathtt{b}) = \min_{\mathtt{b}\in\eta} |\mathtt{a}-\mathtt{b}|$.

Remark 2. *A parallel representation of the Hausdorff metric, as stated in (4), is:*

$$\mathcal{H}([\underline{\mathtt{a}},\overline{\mathtt{a}}],[\underline{\mathtt{b}},\overline{\mathtt{b}}]) = \sup\{|\underline{\mathtt{a}}-\underline{\mathtt{b}}|, |\overline{\mathtt{a}}-\overline{\mathtt{b}}|\},$$

which is referred to as the Moore metric in interval space.

As is commonly known, $(\mathsf{M}_C, \mathcal{H})$ is a complete metric space. The following notations are fixed throughout this paper:

- $\mathsf{R_I}^+$: positive intervals of R;
- $\mathsf{R_I}^-$: negative intervals of R;
- $\mathsf{R_I}$: all intervals of R.
- $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$: interval degenerated;
- $\leq$: standard order relation;
- $\subseteq_{\mathcal{K}_C}$: Kulisch and Miranker relations.

Now, we define the scalar multiplication and Minkowski sum on M_C using

$$\rho + \eta = \{\mathtt{a}+\mathtt{b} \mid \mathtt{a}\in\rho, \mathtt{b}\in\eta\} \text{ and } \gamma\rho = \{\gamma\mathtt{a} \mid \mathtt{a}\in\rho\}.$$

In addition, if $\rho = [\underline{\mathtt{a}},\overline{\mathtt{a}}]$ and $\eta = [\underline{\mathtt{b}},\overline{\mathtt{b}}]$ are two closed and bounded intervals, we define the difference as follows:

$$\rho - \eta = [\underline{\mathtt{a}}-\overline{\mathtt{b}}, \overline{\mathtt{a}}-\underline{\mathtt{b}}],$$

with the product

$$\rho\cdot\eta = [\min\{\underline{\mathtt{a}}\underline{\mathtt{b}}, \underline{\mathtt{a}}\overline{\mathtt{b}}, \overline{\mathtt{a}}\underline{\mathtt{b}}, \overline{\mathtt{a}}\overline{\mathtt{b}}\}, \sup\{\underline{\mathtt{a}}\underline{\mathtt{b}}, \underline{\mathtt{a}}\overline{\mathtt{b}}, \overline{\mathtt{a}}\underline{\mathtt{b}}, \overline{\mathtt{a}}\overline{\mathtt{b}}\}],$$

and the division

$$\frac{\rho}{\eta} = \left[\min\left\{\frac{\underline{\mathtt{a}}}{\underline{\mathtt{b}}}, \frac{\underline{\mathtt{a}}}{\overline{\mathtt{b}}}, \frac{\overline{\mathtt{a}}}{\underline{\mathtt{b}}}, \frac{\overline{\mathtt{a}}}{\overline{\mathtt{b}}}\right\}, \sup\left\{\frac{\underline{\mathtt{a}}}{\underline{\mathtt{b}}}, \frac{\underline{\mathtt{a}}}{\overline{\mathtt{b}}}, \frac{\overline{\mathtt{a}}}{\underline{\mathtt{b}}}, \frac{\overline{\mathtt{a}}}{\overline{\mathtt{b}}}\right\}\right],$$

whenever $0 \notin \eta$. The order relation "$\subseteq_{\mathcal{K}_C}$" is defined as follows by Kulisch and Miranker in [38]:

$$[\underline{\mathtt{a}},\overline{\mathtt{a}}] \subseteq_{\mathcal{K}_C} [\underline{\mathtt{b}},\overline{\mathtt{b}}] \Leftrightarrow \underline{\mathtt{b}} \leq \underline{\mathtt{a}} \text{ and } \overline{\mathtt{a}} \leq \overline{\mathtt{b}}.$$

Remark 3. *We note that if* $[\mathtt{a},\mathtt{b}],[\mathtt{c},\mathtt{d}]$*, and* $[\mathtt{e},\mathtt{f}]$ *are intervals with positive endpoints, then*

$$[\mathtt{a},\mathtt{b}] \geq [\mathtt{e},\mathtt{f}] \Leftrightarrow \frac{[\mathtt{a},\mathtt{b}]}{[\mathtt{c},\mathtt{d}]} \geq \frac{[\mathtt{e},\mathtt{f}]}{[\mathtt{c},\mathtt{d}]},$$
$$[\mathtt{c},\mathtt{d}] \leq [\mathtt{e},\mathtt{f}] \Leftrightarrow \frac{[\mathtt{a},\mathtt{b}]}{[\mathtt{c},\mathtt{d}]} \geq \frac{[\mathtt{a},\mathtt{b}]}{[\mathtt{e},\mathtt{f}]}.$$

Next, we will describe how interval-valued functions are defined, followed by how these kinds of functions are integrated.

2.2. Integral of I.V.F.S

If $\mathcal{M} = [\mathtt{a},\mathtt{b}]$ is a interval and $\mathfrak{Y} : \mathcal{M} \to \mathsf{M}_C$ is an interval-valued mapping, then one has

$$\mathfrak{Y}(\eta_o) = [\underline{\mathfrak{s}}(\eta_o), \overline{\mathfrak{s}}(\eta_o)],$$

where $\underline{\mathfrak{s}}(\eta_o) \leq \overline{\mathfrak{s}}(\eta_o), \forall \eta_o \in \mathcal{M}$. The endpoints of function $\mathfrak{Y}$ are denoted by the functions $\underline{\mathfrak{s}}(\eta_o)$ and $\overline{\mathfrak{s}}(\eta_o)$, respectively. For I.V.F, it is clear that $\mathfrak{Y} : \mathcal{M} \to \mathsf{M}_C$ is continuous at $\eta_o \in \mathcal{M}$ if

$$\lim_{\eta \to \eta_o} \mathfrak{Y}(\eta) = \mathfrak{Y}(\eta_o),$$

where the limit is considered from the metric space $(\mathsf{M}_C, \mathcal{H})$. Consequently, $\mathfrak{Y}$ is continuous at $\eta_o \in \mathcal{M}$ iff its terminal functions $\underline{\mathfrak{s}}(\eta_o)$ and $\overline{\mathfrak{s}}(\eta_o)$ are continuous at a given point.

Theorem 1 (see [35])**.** *Let* $\mathfrak{Y} : [\mathtt{a},\mathtt{b}] \to \mathsf{R}_\mathsf{I}$ *be considered to be* I.V.F*, defined as* $\mathfrak{Y}(\eta_o) = [\underline{\mathfrak{Y}}(\eta_o), \overline{\mathfrak{Y}}(\eta_o)]$. $\mathfrak{Y} \in \mathsf{IR}_{([\mathsf{a},\mathsf{b}])}$ *iff* $\underline{\mathfrak{Y}}(\eta_o), \overline{\mathfrak{Y}}(\eta_o) \in \mathsf{R}_{([\mathsf{a},\mathsf{b}])}$ *and*

$$(\mathsf{IR}) \int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{Y}(\eta_o)\, \mathfrak{d}\eta_o = \left[(\mathsf{R}) \int_{\mathsf{a}}^{\mathsf{b}} \underline{\mathfrak{Y}}(\eta_o)\, \mathfrak{d}\eta_o, (\mathsf{R}) \int_{\mathsf{a}}^{\mathsf{b}} \overline{\mathfrak{Y}}(\eta_o)\, \mathfrak{d}\eta_o \right],$$

where $\mathsf{R}_{([\mathsf{a},\mathsf{b}])}$ *is considered to be the pack of all interval-valued integrable functions. If* $\mathfrak{Y}(\eta_o) \subseteq \mathfrak{V}(\eta_o)$ *for all* $\eta_o \in [\mathtt{a},\mathtt{b}]$*, then the following holds:*

$$(\mathsf{IR}) \int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{Y}(\eta_o)\, \mathfrak{d}\eta_o \subseteq (\mathsf{IR}) \int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{V}(\eta_o)\, \mathfrak{d}\eta_o.$$

Some Novel Definitions via Kulisch and Miranker Inclusion Relations

Here, we introduce some new types of preinvex mappings based on inclusion relations for I.V.F.S, called $(\mathsf{h}_1, \mathsf{h}_2)$–Godunova–Levin preinvex functions, which generalize several existing definitions and unify many previously published studies. In what follows, let $\mathsf{H}(\mathtt{a},\mathtt{b}) = \mathsf{h}_1(\mathtt{a})\mathsf{h}_2(\mathtt{b})$.

Definition 7. *Let* $\mathfrak{V} : [\mathtt{a},\mathtt{b}]$ *be* I.V.F *given by* $\mathfrak{V} = [\underline{\mathfrak{V}}, \overline{\mathfrak{V}}]$*. Let* $\mathsf{h}_1, \mathsf{h}_2 : (0,1) \to (0,\infty)$ *where* $\mathsf{h}_1, \mathsf{h}_2 \neq 0$*. Then,* $\mathfrak{V}$ *defined on invex set* Θ *is known as* $(\mathsf{h}_1, \mathsf{h}_2)$-G.L*-preinvex with reference to* ς *if*

$$\mathfrak{V}\big(\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b},\mathtt{a})\big) \supseteq_{\mathcal{K}_C} \frac{\mathfrak{V}(\mathtt{b})}{\mathsf{H}(\mathfrak{y}, 1-\mathfrak{y})} + \frac{\mathfrak{V}(\mathtt{a})}{\mathsf{H}(1-\mathfrak{y}, \mathfrak{y})},$$

for all $\mathtt{a},\mathtt{b} \in \Theta$ *and* $\mathfrak{y} \in (0,1)$.

Remark 4.

- *Setting* $\mathsf{h}_1(\mathfrak{y}) = \frac{1}{\mathfrak{y}}, \mathsf{h}_2(\mathfrak{y}) = 1$*, Definition 7 becomes a preinvex function [32].*
- *Setting* $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$ *and* $\mathsf{h}_1(\mathfrak{y}) = \mathfrak{y}, \mathsf{h}_2(\mathfrak{y}) = 1$*, Definition 7 reduces to a* G.L*-preinvex function [21].*
- *Setting* $\varsigma(\mathtt{b},\mathtt{a}) = \mathtt{b} - \mathtt{a}$, $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$ *with* $\mathsf{h}_1(\mathfrak{y}) = \frac{1}{\mathsf{h}_1(\mathfrak{y})}, \mathsf{h}_2(\mathfrak{y}) = \frac{1}{\mathsf{h}_2(\mathfrak{y})}$*, Definition 7 becomes a* $(\mathsf{h}_1, \mathsf{h}_2)$*-convex function [20].*

- *Setting* $\varsigma(\mathsf{b},\mathsf{a}) = \mathsf{b}-\mathsf{a}$ *with* $\mathsf{h}_1(\eta) = \mathsf{h}(\eta), \mathsf{h}_2(\eta) = 1$, *Definition 7 becomes a* h-G.L *function [42].*
- *Setting* $\varsigma(\mathsf{b},\mathsf{a}) = \mathsf{b}-\mathsf{a}$ *with* $\mathsf{h}_1(\eta) = \frac{1}{\mathsf{h}(\eta)}, \mathsf{h}_2(\eta) = 1$, *Definition 7 becomes a* h-*convex function [43].*

3. Hermite–Hadamard-Type Inclusions for $(\mathsf{h}_1,\mathsf{h}_2)$–Godunova–Levin Preinvex Mappings

The objective of this section is to developed several novel Hermite–Hadamard inclusions for the $(\mathsf{h}_1,\mathsf{h}_2)$–Godunova–Levin–preinvex functions.

Theorem 2. *Consider* $\mathfrak{V}:[\mathsf{a},\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})]\to \mathsf{R}_\mathsf{I}$ *to be* $(\mathsf{h}_1,\mathsf{h}_2)$*–Godunova–Levin preinvex interval-valued mapping and hold the assumptions of Condition C. Then, one has*

$$\frac{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\mathfrak{V}\left(\frac{2\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}{2}\right) \supseteq_{\mathcal{K}_\mathcal{C}} \frac{1}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\mathfrak{V}(\varrho)\,d\varrho$$
$$\supseteq_{\mathcal{K}_\mathcal{C}} [\mathfrak{V}(\mathsf{a})+\mathfrak{V}(\mathsf{b})]\int_0^1 \frac{d\eta}{\mathsf{H}(\eta,1-\eta)}.$$

Proof. From the definition of $(\mathsf{h}_1,\mathsf{h}_2)$-G.L-preinvex functions, one has

$$\mathfrak{V}\left(\frac{2\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}{2}\right) \supseteq_{\mathcal{K}_\mathcal{C}} \frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]}[\mathfrak{V}(\mathsf{a})+\mathfrak{V}(\mathsf{b})].$$

Choosing $\mathsf{a}=\mathsf{a}+\eta\varsigma(\mathsf{b},\mathsf{a})$ and $\mathsf{b}=\mathsf{a}+(1-\eta)\varsigma(\mathsf{b},\mathsf{a})$, we have

$$\mathfrak{V}\left(\mathsf{a}+\eta\varsigma(\mathsf{b},\mathsf{a})+\frac{1}{2}\varsigma(\mathsf{a}+(1-\eta)\varsigma(\mathsf{b},\mathsf{a}),\mathsf{a}+\eta\varsigma(\mathsf{b},\mathsf{a}))\right)$$
$$\supseteq_{\mathcal{K}_\mathcal{C}} \frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]}[\mathfrak{V}(\mathsf{a}+\eta\varsigma(\mathsf{b},\mathsf{a}))+\mathfrak{V}(\mathsf{a}+(1-\eta)\varsigma(\mathsf{b},\mathsf{a}))].$$

This implies

$$\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]\mathfrak{V}\left(\frac{2\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}{2}\right) \supseteq_{\mathcal{K}_\mathcal{C}} [\mathfrak{V}(\mathsf{a}+\eta\varsigma(\mathsf{b},\mathsf{a}))+\mathfrak{V}(\mathsf{a}+(1-\eta)\varsigma(\mathsf{b},\mathsf{a}))]. \tag{5}$$

Integrating the aforementioned inclusion (5), we obtain

$$\begin{aligned}\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]\mathfrak{V}\left(\frac{2\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}{2}\right) &\supseteq_{\mathcal{K}_\mathcal{C}} \left[\int_0^1 \mathfrak{V}(\mathsf{a}+\eta\varsigma(\mathsf{b},\mathsf{a}))d\eta+\int_0^1 \mathfrak{V}(\mathsf{a}+(1-\eta)\varsigma(\mathsf{b},\mathsf{a}))d\eta\right]\\ &= \int_0^1 (\underline{\mathfrak{V}}(\mathsf{a}+\eta\varsigma(\mathsf{b},\mathsf{a}))+\underline{\mathfrak{V}}(\mathsf{a}+(1-\eta)\varsigma(\mathsf{b},\mathsf{a})))d\eta,\\ &\quad\int_0^1 (\overline{\mathfrak{V}}(\mathsf{a}+\eta\varsigma(\mathsf{b},\mathsf{a}))+\overline{\mathfrak{V}}(\mathsf{a}+(1-\eta)\varsigma(\mathsf{b},\mathsf{a})))d\eta\\ &= \frac{2}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\underline{\mathfrak{V}}(\varrho)\,d\varrho, \frac{2}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\overline{\mathfrak{V}}(\varrho)\,d\varrho\\ &= \frac{2}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\mathfrak{V}(\varrho)\,d\varrho.\end{aligned}$$

Taking into account previous developments, it is clear that

$$\frac{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\mathfrak{V}\left(\frac{2\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}{2}\right) \supseteq_{\mathcal{K}_\mathcal{C}} \frac{1}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\mathfrak{V}(\varrho)\,d\varrho. \tag{6}$$

From Definition 7, one has

$$\mathfrak{V}(a+\eta\varsigma(b,a)) \supseteq_{\mathcal{K}_C} \frac{\mathfrak{V}(b)}{H(\eta,1-\eta)} + \frac{\mathfrak{V}(a)}{H(1-\eta,\eta)}.$$

Integrating the aforementioned inclusion, we have

$$\int_0^1 \mathfrak{V}(a+\eta\varsigma(b,a))\, d\eta \supseteq_{\mathcal{K}_C} \mathfrak{V}(b)\int_0^1 \frac{d\eta}{H(\eta,1-\eta)} + \mathfrak{V}(a)\int_0^1 \frac{d\eta}{H(1-\eta,\eta)}.$$

This implies

$$\frac{1}{\varsigma(b,a)}\int_a^{a+\varsigma(b,a)} \mathfrak{V}(\varrho)\, d\varrho \supseteq_{\mathcal{K}_C} [\mathfrak{V}(a)+\mathfrak{V}(b)]\int_0^1 \frac{d\eta}{H(\eta,1-\eta)}. \tag{7}$$

By combining Equations (6) and (7), we obtain required result, that is,

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\mathfrak{V}\left(\frac{2a+\varsigma(b,a)}{2}\right) \supseteq_{\mathcal{K}_C} \frac{1}{\varsigma(b,a)}\int_a^{a+\varsigma(b,a)} \mathfrak{V}(\varrho)\, d\varrho$$
$$\supseteq_{\mathcal{K}_C} [\mathfrak{V}(a)+\mathfrak{V}(b)]\int_0^1 \frac{d\eta}{H(\eta,1-\eta)}.$$

This completes the proof. □

Remark 5.

- *Setting* $h_1(\eta) = \frac{1}{h(\eta)}, h_2(\eta) = 1$ *and* $\varsigma(b,a) = b-a$, *Theorem 2 incorporates result for h-convex functions ([43], Theorem 4.1).*
- *Setting* $h_1(\eta) = h(\eta), h_2(\eta) = 1$ *and* $\varsigma(b,a) = b-a$, *Theorem 2 incorporates results for h-G.L functions ([42], Theorem 2).*
- *Setting* $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$ *and* $h_1(\eta) = \eta, h_2(\eta) = 1$, *Definition 7 reduces to a G.L-preinvex function ([21], Theorem 1).*

Example 4. *Let* $\mathfrak{V}(\varrho) = [\varrho^2, 4-e^{\varrho}]$, $\varsigma(b,a) = b-a$, $a = 0$, *and* $b = 2$. *Then, for* $h_1(\eta) = \frac{1}{\eta}, h_2(\eta) = 1$, *we have*

$$\frac{\left[H\left(\frac{1}{2},\frac{1}{2}\right)\right]}{2}\mathfrak{V}\left(\frac{2a+\varsigma(b,a)}{2}\right) \approx [1, 1.2817],$$
$$\frac{1}{\varsigma(b,a)}\int_a^{a+\varsigma(b,a)} \mathfrak{V}(\varrho)\, d\varrho \approx [1.3333, 0.8054],$$
$$[\mathfrak{V}(a)+\mathfrak{V}(b)]\int_0^1 \frac{d\eta}{H(\eta,1-\eta)} \approx [2, 0.7091].$$

Consequently, Theorem 2 is correct.

$$[1, 1.2817] \supseteq_{\mathcal{K}_C} [1.3333, 0.8054] \supseteq_{\mathcal{K}_C} [2, 0.7091].$$

Theorem 3. *Based on the same hypotheses in Theorem 2, the successive inclusion relation can be defined as follows:*

$$\frac{1}{\varsigma(b,a)}\int_a^{a+\varsigma(b,a)} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \supseteq_{\mathcal{K}_C} M(a,b)\int_0^1 \frac{d\eta}{H^2(\eta,1-\eta)}$$
$$+ N(a,b)\int_0^1 \frac{d\eta}{H(\eta,\eta)H(1-\eta,1-\eta)}. \tag{8}$$

where

$$\mathsf{M}(\mathsf{a},\mathsf{b}) = \mathfrak{V}(\mathsf{a})\mathfrak{Y}(\mathsf{a}) + \mathfrak{V}(\mathsf{b})\mathfrak{Y}(\mathsf{b})$$

and

$$\mathsf{N}(\mathsf{a},\mathsf{b}) = \mathfrak{V}(\mathsf{a})\mathfrak{Y}(\mathsf{b}) + \mathfrak{V}(\mathsf{b})\mathfrak{Y}(\mathsf{a}).$$

Proof. By the definition of $(\mathsf{h}_1,\mathsf{h}_2)$-G.L-preinvex functions, we have

$$\mathfrak{V}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a})) \supseteq_{\mathcal{K}_{\mathcal{C}}} \frac{\mathfrak{V}(\mathsf{b})}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})} + \frac{\mathfrak{V}(\mathsf{a})}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}$$

and

$$\mathfrak{Y}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a})) \supseteq_{\mathcal{K}_{\mathcal{C}}} \frac{\mathfrak{Y}(\mathsf{b})}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})} + \frac{\mathfrak{Y}(\mathsf{a})}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}.$$

By multiplying the two previously mentioned outcomes, we obtain

$$\begin{aligned}
&\mathfrak{V}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\mathfrak{Y}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a})) \\
&\supseteq_{\mathcal{K}_{\mathcal{C}}} \left[\frac{\mathfrak{V}(\mathsf{b})}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})} + \frac{\mathfrak{V}(\mathsf{a})}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}\right]\left[\frac{\mathfrak{Y}(\mathsf{b})}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})} + \frac{\mathfrak{Y}(\mathsf{a})}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}\right] \\
&= \frac{[\mathfrak{V}(\mathsf{b})\mathfrak{Y}(\mathsf{b})]}{\mathsf{H}^2(\mathfrak{y},1-\mathfrak{y})} + \frac{[\mathfrak{V}(\mathsf{a})\mathfrak{Y}(\mathsf{a})]}{\mathsf{H}^2(1-\mathfrak{y},\mathfrak{y})} + \frac{[\mathfrak{V}(\mathsf{b})\mathfrak{Y}(\mathsf{a})] + [\mathfrak{V}(\mathsf{a})\mathfrak{Y}(\mathsf{b})]}{\mathsf{H}(\mathfrak{y},\mathfrak{y})\mathsf{H}(1-\mathfrak{y},1-\mathfrak{y})}.
\end{aligned} \tag{9}$$

Integrating the aforementioned inclusion (9) over $(0,1)$, we have

$$\begin{aligned}
&\int_0^1 \mathfrak{V}\big(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a})\big)\mathfrak{Y}\big(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a})\big)\, d\mathfrak{y} \\
&\supseteq_{\mathcal{K}_{\mathcal{C}}} [\mathfrak{V}(\mathsf{b})\mathfrak{Y}(\mathsf{b})]\int_0^1 \frac{d\mathfrak{y}}{\mathsf{H}^2(\mathfrak{y},1-\mathfrak{y})} + [\mathfrak{V}(\mathsf{a})\mathfrak{Y}(\mathsf{a})]\int_0^1 \frac{d\mathfrak{y}}{\mathsf{H}^2(1-\mathfrak{y},\mathfrak{y})} \\
&\quad + [\mathfrak{V}(\mathsf{b})\mathfrak{Y}(\mathsf{a}) + \mathfrak{V}(\mathsf{a})\mathfrak{Y}(\mathsf{b})]\int_0^1 \frac{d\mathfrak{y}}{\mathsf{H}(1-\mathfrak{y},1-\mathfrak{y})\mathsf{H}(\mathfrak{y},\mathfrak{y})}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\frac{1}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \supseteq_{\mathcal{K}_{\mathcal{C}}} [\mathfrak{V}(\mathsf{a})\mathfrak{Y}(\mathsf{a}) + \mathfrak{V}(\mathsf{b})\mathfrak{Y}(\mathsf{b})]\int_0^1 \frac{d\mathfrak{y}}{\mathsf{H}^2(1-\mathfrak{y},\mathfrak{y})} \\
&+ [\mathfrak{V}(\mathsf{a})\mathfrak{Y}(\mathsf{b}) + \mathfrak{V}(\mathsf{b})\mathfrak{Y}(\mathsf{a})]\int_0^1 \frac{d\mathfrak{y}}{\mathsf{H}(1-\mathfrak{y},1-\mathfrak{y})\mathsf{H}(\mathfrak{y},\mathfrak{y})} \\
&= \mathsf{M}(\mathsf{a},\mathsf{b})\int_0^1 \frac{d\mathfrak{y}}{\mathsf{H}^2(1-\mathfrak{y},\mathfrak{y})} + \mathsf{N}(\mathsf{a},\mathsf{b})\int_0^1 \frac{d\mathfrak{y}}{\mathsf{H}(1-\mathfrak{y},1-\mathfrak{y})\mathsf{H}(\mathfrak{y},\mathfrak{y})}.
\end{aligned}$$

As Theorem 3 indicates, we will eventually arrive at the intended outcome. □

Remark 6.

- *Setting* $\mathsf{h}_1(\mathfrak{y}) = \mathsf{h}(\mathfrak{y}), \mathsf{h}_2(\mathfrak{y}) = 1$ *and* $\varsigma(\mathsf{b},\mathsf{a}) = \mathsf{b}-\mathsf{a}$, *Theorem 3 incorporates results for* h-G.L *functions ([42], Theorem 4).*
- *Setting* $\mathsf{h}_1(\mathfrak{y}) = \frac{1}{\mathsf{h}(\mathfrak{y})}, \mathsf{h}_2(\mathfrak{y}) = 1$ *and* $\varsigma(\mathsf{b},\mathsf{a}) = \mathsf{b}-\mathsf{a}$, *Theorem 3 incorporates results for h-convex functions ([43], Theorem 4.5).*
- *Setting* $\mathsf{h}_1(\mathfrak{y}) = \frac{1}{\mathfrak{y}}, \mathsf{h}_2(\mathfrak{y}) = 1$ *and* $\varsigma(\mathsf{b},\mathsf{a}) = \mathsf{b}-\mathsf{a}$, *Theorem 3 incorporates results for classical convex functions, that is,*

$$\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \supseteq_{\mathcal{K}_{\mathcal{C}}} \frac{\mathsf{M}(\mathsf{a},\mathsf{b})}{3} + \frac{\mathsf{N}(\mathsf{a},\mathsf{b})}{6}.$$

Example 5. *Let* $\mathfrak{V}(\varrho) = [\varrho^2, 4 - e^{\varrho}], \mathfrak{Y}(\varrho) = [\varrho, 3 - \varrho^2]$, $\varsigma(\mathsf{b}, \mathsf{a}) = \mathsf{b} - \mathsf{a}, \mathsf{a} = 0$ *and* $\mathsf{b} = 1$. *Then, for* $\mathsf{h}_1(\mathfrak{y}) = \frac{1}{\mathfrak{y}}, \mathsf{h}_2(\mathfrak{y}) = 1$, *we have*

$$\frac{1}{\varsigma(\mathsf{b}, \mathsf{a})} \int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \approx [0.25, 6.2301],$$

and

$$\mathsf{M}(\mathsf{a}, \mathsf{b}) \int_0^1 \frac{d\mathfrak{y}}{\mathsf{H}^2(\mathfrak{y}, 1 - \mathfrak{y})} + \mathsf{N}(\mathsf{a}, \mathsf{b}) \int_0^1 \frac{d\mathfrak{y}}{\mathsf{H}(1 - \mathfrak{y}, 1 - \mathfrak{y})\mathsf{H}(\mathfrak{y}, \mathfrak{y})} d\mathfrak{y} \approx [0.3333, 5.4953].$$

Thus, we have

$$[0.25, 6.2301] \supseteq_{\mathcal{K}_{\mathcal{C}}} [0.3333, 5.4953].$$

Consequently, Theorem 3 is true.

Theorem 4. *Based on the same hypotheses as in Theorem 2, the successive inclusion relation can be defined as:*

$$\begin{aligned}
&\frac{\left[\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2}{2} \mathfrak{V}\left(\frac{2\mathsf{a} + \varsigma(\mathsf{b}, \mathsf{a})}{2}\right)\mathfrak{Y}\left(\frac{2\mathsf{a} + \varsigma(\mathsf{b}, \mathsf{a})}{2}\right) \\
&\supseteq_{\mathcal{K}_{\mathcal{C}}} \frac{1}{\varsigma(\mathsf{b}, \mathsf{a})} \int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho \\
&+ \mathsf{M}(\mathsf{a}, \mathsf{b}) \int_0^1 \frac{d\mathfrak{y}}{\mathsf{H}(1 - \mathfrak{y}, 1 - \mathfrak{y})\mathsf{H}(\mathfrak{y}, \mathfrak{y})} + \mathsf{N}(\mathsf{a}, \mathsf{b}) \int_0^1 \frac{d\mathfrak{y}}{\mathsf{H}^2(\mathfrak{y}, 1 - \mathfrak{y})}.
\end{aligned}$$

Proof. Since $\mathfrak{V}$ is a $(\mathsf{h}_1, \mathsf{h}_2)$-G.L-preinvex function, we have

$$\left[\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]\mathfrak{V}\left(\frac{2\mathsf{a} + \varsigma(\mathsf{b}, \mathsf{a})}{2}\right) \supseteq_{\mathcal{K}_{\mathcal{C}}} [\mathfrak{V}(\mathsf{a} + \mathfrak{y}\varsigma(\mathsf{b}, \mathsf{a})) + \mathfrak{V}(\mathsf{a} + (1 - \mathfrak{y})\varsigma(\mathsf{b}, \mathsf{a}))].$$

Utilizing Condition C defined above, we have

$$\begin{aligned}
\mathfrak{V}\left(\frac{2\mathsf{a} + \varsigma(\mathsf{b}, \mathsf{a})}{2}\right) &= \mathfrak{V}\left(\mathsf{a} + \mathfrak{y}\varsigma(\mathsf{b}, \mathsf{a}) + \frac{1}{2}\varsigma(\mathsf{a} + (1 - \mathfrak{y})\varsigma(\mathsf{b}, \mathsf{a}), \mathsf{a} + \mathfrak{y}\varsigma(\mathsf{b}, \mathsf{a}))\right) \\
&\supseteq_{\mathcal{K}_{\mathcal{C}}} \frac{1}{\left[\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]}[\mathfrak{V}(\mathsf{a} + \mathfrak{y}\varsigma(\mathsf{b}, \mathsf{a})) + \mathfrak{V}(\mathsf{a} + (1 - \mathfrak{y})\varsigma(\mathsf{b}, \mathsf{a}))]. \qquad (10)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathfrak{Y}\left(\frac{2\mathsf{a} + \varsigma(\mathsf{b}, \mathsf{a})}{2}\right) &= \mathfrak{Y}\left(\mathsf{a} + \mathfrak{y}\varsigma(\mathsf{b}, \mathsf{a}) + \frac{1}{2}\varsigma(\mathsf{a} + (1 - \mathfrak{y})\varsigma(\mathsf{b}, \mathsf{a}), \mathsf{a} + \mathfrak{y}\varsigma(\mathsf{b}, \mathsf{a}))\right) \\
&\supseteq_{\mathcal{K}_{\mathcal{C}}} \frac{1}{\left[\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]}[\mathfrak{Y}(\mathsf{a} + \mathfrak{y}\varsigma(\mathsf{b}, \mathsf{a})) + \mathfrak{Y}(\mathsf{a} + (1 - \mathfrak{y})\varsigma(\mathsf{b}, \mathsf{a}))]. \qquad (11)
\end{aligned}$$

Multiplying Equations (10) and (11), we have

$$
\begin{aligned}
&\mathfrak{V}\left(\frac{2\mathtt{a}+\varsigma(\mathtt{b},\mathtt{a})}{2}\right)\mathfrak{Y}\left(\frac{2\mathtt{a}+\varsigma(\mathtt{b},\mathtt{a})}{2}\right)\\
&\supseteq_{\mathcal{K}_{\mathcal{C}}} \frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]}[\mathfrak{V}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))+\mathfrak{V}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))]\times\\
&\qquad\frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]}[\mathfrak{Y}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))+\mathfrak{Y}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))]\\
&=\frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\Big[\underline{\mathfrak{V}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\underline{\mathfrak{Y}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\\
&\qquad+\underline{\mathfrak{V}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\underline{\mathfrak{Y}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\\
&\qquad+\underline{\mathfrak{V}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\underline{\mathfrak{Y}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\\
&\qquad+\underline{\mathfrak{V}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\underline{\mathfrak{Y}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a})),\\
&\qquad\overline{\mathfrak{V}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\overline{\mathfrak{Y}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\\
&\qquad+\overline{\mathfrak{V}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\overline{\mathfrak{Y}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\\
&\qquad+\overline{\mathfrak{V}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\overline{\mathfrak{Y}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\\
&\qquad+\overline{\mathfrak{V}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\overline{\mathfrak{Y}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\Big]\\
&=\frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\Big[\underline{\mathfrak{V}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\underline{\mathfrak{Y}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a})),\overline{\mathfrak{V}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\overline{\mathfrak{Y}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\Big]\\
&\quad+\frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\Big[\underline{\mathfrak{V}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\underline{\mathfrak{Y}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a})),\\
&\qquad\overline{\mathfrak{V}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\overline{\mathfrak{Y}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\Big]\\
&\quad+\frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\Big[\underline{\mathfrak{V}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\underline{\mathfrak{Y}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\\
&\qquad,\overline{\mathfrak{V}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\overline{\mathfrak{Y}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\Big]\\
&\quad+\frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\Big[\underline{\mathfrak{V}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\underline{\mathfrak{Y}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\\
&\qquad,\overline{\mathfrak{V}}(\mathtt{a}+(1-\mathfrak{y})\varsigma(\mathtt{b},\mathtt{a}))\overline{\mathfrak{Y}}(\mathtt{a}+\mathfrak{y}\varsigma(\mathtt{b},\mathtt{a}))\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\Big[\mathfrak{V}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\mathfrak{Y}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a})) \\
&\qquad + \mathfrak{V}(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\mathfrak{Y}(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\Big] \\
&\quad + \frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\Big[\mathfrak{V}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\mathfrak{Y}(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a})) \\
&\qquad + \mathfrak{V}(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\mathfrak{Y}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\Big] \\
&\supseteq_{\mathcal{K}_\mathcal{C}} \frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\Big[\mathfrak{V}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\mathfrak{Y}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a})) \\
&\qquad + \mathfrak{V}(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\mathfrak{Y}(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\Big] \\
&\quad + \frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\Bigg[\left(\frac{\mathfrak{V}(\mathsf{a})}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}+\frac{\mathfrak{V}(\mathsf{b})}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}\right)\left(\frac{\mathfrak{Y}(\mathsf{b})}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{\mathfrak{Y}(\mathsf{a})}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}\right) \\
&\qquad + \left(\frac{\mathfrak{V}(\mathsf{b})}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{\mathfrak{V}(\mathsf{a})}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}\right)\left(\frac{\mathfrak{Y}(\mathsf{a})}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}+\frac{\mathfrak{Y}(\mathsf{b})}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}\right)\Bigg] \\
&= \frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\Big[\mathfrak{V}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\mathfrak{Y}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a})) \\
&\qquad + \mathfrak{V}(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\mathfrak{Y}(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\Big] \\
&\quad + \frac{1}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\left[\mathsf{M}(\mathsf{a},\mathsf{b})\left[\frac{2}{\mathsf{H}(1-\mathfrak{y},1-\mathfrak{y})\mathsf{H}(\mathfrak{y},\mathfrak{y})}\right]+\mathsf{N}(\mathsf{a},\mathsf{b})\left[\frac{1}{\mathsf{H}^2(1-\mathfrak{y},\mathfrak{y})}+\frac{1}{\mathsf{H}^2(\mathfrak{y},1-\mathfrak{y})}\right]\right].
\end{aligned}
$$

With an integration over $(0,1)$, we obtain

$$
\begin{aligned}
&\mathfrak{V}\left(\frac{2\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}{2}\right)\mathfrak{Y}\left(\frac{2\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}{2}\right) \\
&\supseteq_{\mathcal{K}_\mathcal{C}} \frac{2}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}\Bigg\{\frac{1}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\,d\varrho \\
&+ \mathsf{M}(\mathsf{a},\mathsf{b})\int_0^1\frac{d\mathfrak{y}}{\mathsf{H}(1-\mathfrak{y},1-\mathfrak{y})\mathsf{H}(\mathfrak{y},\mathfrak{y})}+\mathsf{N}(\mathsf{a},\mathsf{b})\int_0^1\frac{d\mathfrak{y}}{\mathsf{H}^2(\mathfrak{y},1-\mathfrak{y})}\Bigg\}.
\end{aligned}
$$

This readily gives

$$
\begin{aligned}
&\frac{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]^2}{2}\mathfrak{V}\left(\frac{2\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}{2}\right)\mathfrak{Y}\left(\frac{2\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}{2}\right) \\
&\supseteq_{\mathcal{K}_\mathcal{C}} \frac{1}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\,d\varrho \\
&+ \mathsf{M}(\mathsf{a},\mathsf{b})\int_0^1\frac{d\mathfrak{y}}{\mathsf{H}(1-\mathfrak{y},1-\mathfrak{y})\mathsf{H}(\mathfrak{y},\mathfrak{y})}+\mathsf{N}(\mathsf{a},\mathsf{b})\int_0^1\frac{d\mathfrak{y}}{\mathsf{H}^2(\mathfrak{y},1-\mathfrak{y})}.
\end{aligned}
$$

The proof is now completed. □

Remark 7.

- *Setting* $h_1(\eta) = \frac{1}{h(\eta)}, h_2(\eta) = 1$, *and* $\varsigma(b, a) = b - a$, *Theorem 4 incorporates results for h-convex functions.*
- *Setting* $h_1(\eta) = h(\eta), h_2(\eta) = 1$, *and* $\varsigma(b, a) = b - a$, *Theorem 4 incorporates results for h-*G.L *functions.*

Example 6. *Furthermore, by Example 5, we have*

$$\frac{\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2}{2} \mathfrak{V}\left(a + \frac{1}{2}\varsigma(b, a)\right)\mathfrak{Y}\left(a + \frac{1}{2}\varsigma(b, a)\right) \approx [0.25, 12.9190],$$

and

$$\frac{1}{\varsigma(b, a)} \int_a^{a+\varsigma(b,a)} \mathfrak{V}(\varrho)\mathfrak{Y}(\varrho)\, d\varrho$$
$$+ M(a, b) \int_0^1 \frac{d\eta}{H(1-\eta, 1-\eta)H(\eta, \eta)} + N(a, b) \int_0^1 \frac{d\eta}{H^2(\eta, 1-\eta)} \approx [0.4166, 11.4390].$$

Thus, we have

$$[0.25, 12.9190] \supseteq_{\mathcal{K}_C} [0.4166, 11.4390].$$

Consequently, Theorem 4 is true.

Weighted Fejér-Type Inclusions For (h_1, h_2)-G.L-*Preinvex Functions*

Theorem 5. *Let* $\mathfrak{V} : [a, a + \varsigma(b, a)] \to R_I$ *be* I.V.F, *which is defined as* $\mathfrak{V}(\varrho) = [\underline{\mathfrak{V}}(\varrho), \overline{\mathfrak{V}}(\varrho)]$ *for all* $\varrho \in [a, b]$. *If* $\mathfrak{V} : [a, a + \varsigma(b, a)] \to R_I$ *is a* (h_1, h_2)-G.L-*preinvex function and* $\chi : [a, a + \varsigma(b, a)] \to R_I, \chi > 0$ *is symmetric with reference to* $a + \frac{1}{2}\varsigma(b, a)$, *then the following inclusion holds:*

$$\frac{1}{\varsigma(b, a)} \int_a^{a+\varsigma(b,a)} \mathfrak{V}(\varrho)\chi(\varrho)\, d\varrho \supseteq_{\mathcal{K}_C} [\mathfrak{V}(a) + \mathfrak{V}(b)] \int_0^1 \frac{\chi(a + \eta\varsigma(b, a))\, d\eta}{H(\eta, 1-\eta)}.$$

Proof. Since $\mathfrak{V}$ is a (h_1, h_2)-G.L-preinvex function and χ is symmetric with reference to $a + \frac{1}{2}\varsigma(b, a)$, we have

$$\mathfrak{V}(a + \eta\varsigma(b, a))\chi(a + \eta\varsigma(b, a))$$
$$\supseteq_{\mathcal{K}_C} \left[\frac{\mathfrak{V}(b)}{H(\eta, 1-\eta)} + \frac{\mathfrak{V}(a)}{H(1-\eta, \eta)}\right]\chi(a + \eta\varsigma(b, a)),$$

and

$$\mathfrak{V}(a + (1-\eta)\varsigma(b, a))\chi(a + (1-\eta)\varsigma(b, a))$$
$$\supseteq_{\mathcal{K}_C} \left[\frac{\mathfrak{V}(a)}{H(\eta, 1-\eta)} + \frac{\mathfrak{V}(b)}{H(1-\eta, \eta)}\right]\chi(a + (1-\eta)\varsigma(b, a)).$$

Adding the above two inclusions and integrating them, we have

$$\int_0^1 \mathfrak{V}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\chi(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\,\mathsf{d}\mathfrak{y} \tag{12}$$

$$+\int_0^1 \mathfrak{V}(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\chi(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\,\mathsf{d}\mathfrak{y}$$

$$\supseteq_{\mathcal{K}_C} \int_0^1 \left[\mathfrak{V}(\mathsf{a})\left(\frac{(\chi(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{\chi(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}\right)\right.$$

$$\left.+\mathfrak{V}(\mathsf{b})\left(\frac{(\chi(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{\chi(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}\right)\right]\mathsf{d}\mathfrak{y}$$

$$=2\mathfrak{V}(\mathsf{a})\int_0^1 \frac{\chi(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}\,\mathsf{d}\mathfrak{y}+2\mathfrak{V}(\mathsf{b})\int_0^1 \frac{\chi(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}\,\mathsf{d}\mathfrak{y}$$

$$=2[\mathfrak{V}(\mathsf{a})+\mathfrak{V}(\mathsf{b})]\int_0^1 \frac{\chi(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}\,\mathsf{d}\mathfrak{y}, \tag{13}$$

since

$$\int_0^1 \mathfrak{V}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\chi(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\,\mathsf{d}\mathfrak{y} \tag{14}$$

$$+\int_0^1 \mathfrak{V}(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\chi(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\,\mathsf{d}\mathfrak{y}$$

$$=\frac{2}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})} \mathfrak{V}(\varrho)\chi(\varrho)\,d\varrho, \tag{15}$$

By considering Equations (12) and (14), we obtain the required outcome. □

Remark 8.

- *If* $\mathsf{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}}, \mathsf{h}_2(\mathfrak{y})=1$, *then Theorem 5 incorporates results for preinvex functions, that is,*

$$\frac{1}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})} \mathfrak{V}(\varrho)\chi(\varrho)\,d\varrho \supseteq_{\mathcal{K}_C} [\mathfrak{V}(\mathsf{a})+\mathfrak{V}(\mathsf{b})]\int_0^1 \mathfrak{y}\chi(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\,\mathsf{d}\mathfrak{y}.$$

- *If* $\mathsf{h}_1(\mathfrak{y})=\frac{1}{\mathsf{h}(\mathfrak{y})}, \mathsf{h}_2(\mathfrak{y})=1$ *and* $\varsigma(\mathsf{b},\mathsf{a})=\mathsf{b}-\mathsf{a}$, *then Theorem 5 incorporates results for* h-G.L *functions, and this is also new as well, that is,*

$$\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{V}(\varrho)\chi(\varrho)\,d\varrho \supseteq_{\mathcal{K}_C} [\mathfrak{V}(\mathsf{a})+\mathfrak{V}(\mathsf{b})]\int_0^1 \frac{\chi((1-\mathfrak{y})\mathsf{a}+\mathfrak{y}\mathsf{b})}{\mathsf{h}(\mathfrak{y})}\,\mathsf{d}\mathfrak{y}.$$

- *If* $\mathsf{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}}, \mathsf{h}_2(\mathfrak{y})=1$ *and* $\varsigma(\mathsf{b},\mathsf{a})=\mathsf{b}-\mathsf{a}$, *then Theorem 5 incorporates results for convex functions, that is,*

$$\frac{1}{\mathsf{b}-\mathsf{a}}\int_{\mathsf{a}}^{\mathsf{b}} \mathfrak{V}(\varrho)\chi(\varrho)\,d\varrho \supseteq_{\mathcal{K}_C} [\mathfrak{V}(\mathsf{a})+\mathfrak{V}(\mathsf{b})]\int_0^1 \mathfrak{y}\chi((1-\mathfrak{y})\mathsf{a}+\mathfrak{y}\mathsf{b})\,\mathsf{d}\mathfrak{y}.$$

Example 7. *Let* $\mathfrak{V}(\varrho) = \left[\frac{1}{\varrho}, \varrho\right], \varsigma(\mathtt{b}, \mathtt{a}) = \mathtt{b} - \mathtt{a}, \mathtt{a} = 1,$ *and* $\mathtt{b} = 4$*. Then, for* $\mathsf{h}_1(\mathfrak{y}) = \frac{1}{\mathfrak{y}}, \mathsf{h}_2(\mathfrak{y}) = 1$ *and symmetric functions* $\chi(\varrho) = \varrho - 1$ *for* $\varrho \in \left[1, \frac{5}{2}\right]$ *and* $\chi(\varrho) = -\varrho + 4$ *for* $\varrho \in \left[\frac{5}{2}, 4\right]$*, we have*

$$\begin{aligned}
&\frac{1}{\varsigma(\mathtt{b}, \mathtt{a})} \int_{\mathtt{a}}^{\mathtt{a}+\varsigma(\mathtt{b},\mathtt{a})} \mathfrak{V}(\varrho)\chi(\varrho)\, d\varrho \\
&= \frac{1}{3} \int_1^4 \mathfrak{V}(\varrho)\chi(\varrho)\, d\varrho \\
&= \frac{1}{3} \int_1^{\frac{5}{2}} \left[\frac{1}{\varrho}(\varrho - 1), (\varrho - 1)\varrho\right] d\varrho \\
&\quad + \frac{1}{3} \int_{\frac{5}{2}}^4 \left[\frac{1}{\varrho}(-\varrho + 4), (-\varrho + 4)\varrho\right] d\varrho \\
&\approx [0.4048, 1.875],
\end{aligned}$$

and

$$\begin{aligned}
&[\mathfrak{V}(\mathtt{a}) + \mathfrak{V}(\mathtt{b})] \int_0^1 \frac{\chi(\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b}, \mathtt{a}))}{\mathsf{H}(\mathfrak{y}, 1 - \mathfrak{y})}\, \mathrm{d}\mathfrak{y} \\
&\approx [0.4687, 1.8671].
\end{aligned}$$

Thus, we have

$$[0.4048, 1.875] \supseteq_{\mathcal{K}_{\mathcal{C}}} [0.4687, 1.8671].$$

Consequently, Theorem 5 is correct.

Theorem 6. *Based on the same hypotheses as in Theorem 5, the successive inclusion relation can be defined as follows:*

$$\mathfrak{V}\left(\frac{2\mathtt{a} + \varsigma(\mathtt{b}, \mathtt{a})}{2}\right) \supseteq_{\mathcal{K}_{\mathcal{C}}} \frac{2}{\left[\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right] \int_{\mathtt{a}}^{\mathtt{a}+\varsigma(\mathtt{b},\mathtt{a})} \chi(\varrho)\, d\varrho} \int_{\mathtt{a}}^{\mathtt{a}+\varsigma(\mathtt{b},\mathtt{a})} \mathfrak{V}(\varrho)\chi(\varrho)\, d\varrho.$$

Proof. Since $\mathfrak{V}$ is a $(\mathsf{h}_1, \mathsf{h}_2)$-G.L-preinvex function, one has

$$\mathfrak{V}\left(\frac{2\mathtt{a} + \varsigma(\mathtt{b}, \mathtt{a})}{2}\right) \supseteq_{\mathcal{K}_{\mathcal{C}}} \frac{1}{\left[\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]} [\mathfrak{V}(\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b}, \mathtt{a}))\mathrm{d}\mathfrak{y} + \mathfrak{V}(\mathtt{a} + (1 - \mathfrak{y})\varsigma(\mathtt{b}, \mathtt{a}))\mathrm{d}\mathfrak{y}].$$

Multiplying the above inclusion with $\chi(\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b}, \mathtt{a})) = \chi(\mathtt{a} + (1 - \mathfrak{y})\varsigma(\mathtt{b}, \mathtt{a}))$ and integrating it, we have

$$\begin{aligned}
&\mathfrak{V}\left(\frac{2\mathtt{a} + \varsigma(\mathtt{b}, \mathtt{a})}{2}\right) \int_0^1 \chi(\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b}, \mathtt{a}))\, \mathrm{d}\mathfrak{y} \\
&\supseteq_{\mathcal{K}_{\mathcal{C}}} \frac{1}{\left[\mathsf{H}\left(\frac{1}{2}, \frac{1}{2}\right)\right]} \left[\int_0^1 \mathfrak{V}(\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b}, \mathtt{a}))\chi(\mathtt{a} + \mathfrak{y}\varsigma(\mathtt{b}, \mathtt{a}))\, \mathrm{d}\mathfrak{y}\right. \\
&\quad \left. + \int_0^1 \mathfrak{V}(\mathtt{a} + (1 - \mathfrak{y})\varsigma(\mathtt{b}, \mathtt{a}))\chi(\mathtt{a} + (1 - \mathfrak{y})\varsigma(\mathtt{b}, \mathtt{a}))\, \mathrm{d}\mathfrak{y}\right],
\end{aligned} \tag{16}$$

since

$$\begin{aligned}&\int_0^1 \mathfrak{V}(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\chi(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))\,d\mathfrak{y}\\ &=\int_0^1 \mathfrak{V}(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\chi(\mathsf{a}+(1-\mathfrak{y})\varsigma(\mathsf{b},\mathsf{a}))\,d\mathfrak{y}\\ &=\frac{1}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\mathfrak{V}(\varrho)\chi(\varrho)\,d\varrho \end{aligned} \tag{17}$$

and

$$\int_0^1 \chi(\mathsf{a}+\mathfrak{y}\varsigma(\mathsf{b},\mathsf{a}))d\mathfrak{y}=\frac{1}{\varsigma(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\chi(\varrho)\;\;d\varrho. \tag{18}$$

Using (17) and (18) in (16), we have

$$\mathfrak{V}\left(\frac{2\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}{2}\right)\supseteq_{\mathcal{K}_{\mathcal{C}}}\frac{2}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\chi(\varrho)\,d\varrho}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\mathfrak{V}(\varrho)\chi(\varrho)\,d\varrho.$$

The proof is now complete. □

Remark 9.

- *If* $\mathsf{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}},\mathsf{h}_2(\mathfrak{y})=1$, *then Theorem 6 incorporates results for preinvex functions, that is,*

$$\mathfrak{V}\left(\frac{2\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}{2}\right)\supseteq_{\mathcal{K}_{\mathcal{C}}}\frac{1}{\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\chi(\varrho)\,d\varrho}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\mathfrak{V}(\varrho)\chi(\varrho)\,d\varrho.$$

- *If* $\varsigma(\mathsf{b},\mathsf{a})=\mathsf{b}-\mathsf{a}$, *then Theorem 6 incorporates results for* $(\mathsf{h}_1,\mathsf{h}_2)$-G.L *function, that is,*

$$\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2}\right)\supseteq_{\mathcal{K}_{\mathcal{C}}}\frac{2}{\left[\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)\right]\int_{\mathsf{a}}^{\mathsf{b}}\chi(\varrho)\,d\varrho}\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{V}(\varrho)\chi(\varrho)\,d\varrho.$$

- *If* $\mathsf{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}},\mathsf{h}_2(\mathfrak{y})=1$ *and* $\varsigma(\mathsf{b},\mathsf{a})=\mathsf{b}-\mathsf{a}$, *then Theorem 6 incorporates results for convex functions, that is,*

$$\mathfrak{V}\left(\frac{\mathsf{a}+\mathsf{b}}{2}\right)\supseteq_{\mathcal{K}_{\mathcal{C}}}\frac{1}{\int_{\mathsf{a}}^{\mathsf{b}}\chi(\varrho)\,d\varrho}\int_{\mathsf{a}}^{\mathsf{b}}\mathfrak{V}(\varrho)\chi(\varrho)\,d\varrho.$$

Example 8. *Furthermore, by Example 7, we have*

$$\mathfrak{V}\left(\mathsf{a}+\frac{1}{2}\varsigma(\mathsf{b},\mathsf{a})\right)=[0.4,2.5],$$

and

$$\begin{aligned}&\frac{\mathsf{H}\left(\frac{1}{2},\frac{1}{2}\right)}{2\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\chi(\varrho)\,d\varrho}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\mathfrak{V}(\varrho)\chi(\varrho)\,d\varrho\\ &\approx[0.5397,2.471].\end{aligned}$$

Thus, one has

$$[0.4,2.5]\supseteq_{\mathcal{K}_{\mathcal{C}}}[0.5397,2.471].$$

Consequently, Theorem 6 is correct.

4. Applications of Some Novel Results to Numerical Integration Rule

This section aims to relate our results to some numerical integration trapezoidal-type formulas via $(\mathsf{h}_1,\mathsf{h}_2)$–Godunova—Levin preinvex-type mappings to find the error bounds on set-valued mappings. The following lemma will assist us in developing fresh findings in the configuration of interval-valued functions.

Lemma 1 (see [44]). *Let* $\mathfrak{V} : \Theta = [\mathsf{a}, \mathsf{a} + \varsigma(\mathsf{b}, \mathsf{a})] \to (0, \infty)$ *be a differentiable mapping, where* $\mathsf{a} < \mathsf{a} + \varsigma(\mathsf{b}, \mathsf{a})$. *If* $\mathfrak{V}' \in L_1[\mathsf{a}, \mathsf{a} + \varsigma(\mathsf{b}, \mathsf{a})]$. *Then, we have*

$$\frac{1}{\zeta(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\mathfrak{V}(\beta)d\beta - \frac{\mathfrak{V}(\mathsf{a})+\mathfrak{V}(\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a}))}{2} = \frac{\zeta(\mathsf{b},\mathsf{a})}{2}\left[\int_0^1 (1-2\mathfrak{y})\mathfrak{V}'(\mathsf{a}+\mathfrak{y}\zeta(\mathsf{b},\mathsf{a}))d\mathfrak{y}\right].$$

Theorem 7. *Let* $\mathfrak{V} : \Theta = [\mathsf{a}, \mathsf{a} + \varsigma(\mathsf{b}, \mathsf{a})] \to (0, \infty)$ *be a differentiable* I.V.F *on* a°, $\mathsf{a}, \zeta(\mathsf{b}, \mathsf{a}) \in \mathsf{a}^\circ$, *with* $\mathsf{a} < \mathsf{a} + \varsigma(\mathsf{b}, \mathsf{a})$. *If* $|\mathfrak{V}'|$ *is a* $(\mathsf{h}_1, \mathsf{h}_2)$*–Godunova–Levin preinvex on* $[\mathsf{a}, \mathsf{a} + \varsigma(\mathsf{b}, \mathsf{a})]$, *then we obtain the following inclusion:*

$$\begin{aligned}
&\mathcal{H}\left(\left[\frac{\underline{\mathfrak{V}}(\mathsf{a})+\underline{\mathfrak{V}}(\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a}))}{2},\frac{\overline{\mathfrak{V}}(\mathsf{a})+\overline{\mathfrak{V}}(\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a}))}{2}\right],\right.\\
&\left.\left[\frac{1}{\zeta(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\underline{\mathfrak{V}}(\beta)d\beta,\frac{1}{\zeta(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\overline{\mathfrak{V}}(\beta)d\beta\right]\right)\\
&\supseteq \frac{\zeta(\mathsf{b},\mathsf{a})}{2}\left[|\underline{\mathfrak{V}}'(\mathsf{a})|+|\underline{\mathfrak{V}}'(\mathsf{b})|,\left|\overline{\mathfrak{V}}'(\mathsf{a})\right|+\left|\overline{\mathfrak{V}}'(\mathsf{b})\right|\right]\\
&\times\int_0^1|1-2\mathfrak{y}|\left[\frac{1}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}. \qquad (19)
\end{aligned}$$

Proof. Taking into account Lemma 1 to prove Theorem 7, we have the following:

$$\begin{aligned}
&\mathcal{H}([\frac{\underline{\mathfrak{V}}(\mathsf{a})+\underline{\mathfrak{V}}(\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a}))}{2},\frac{\overline{\mathfrak{V}}(\mathsf{a})+\overline{\mathfrak{V}}(\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a}))}{2}],.\\
&[\frac{1}{\zeta(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\underline{\mathfrak{V}}(\beta)d\beta,\frac{1}{\zeta(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\overline{\mathfrak{V}}(\beta)d\beta])\\
&=\sup\{|\frac{\underline{\mathfrak{V}}(\mathsf{a})+\underline{\mathfrak{V}}(\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a}))}{2}-\frac{1}{\zeta(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\underline{\mathfrak{V}}(\beta)d\beta,.\\
&.\frac{\overline{\mathfrak{V}}(\mathsf{a})+\overline{\mathfrak{V}}(\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a}))}{2}-\frac{1}{\zeta(\mathsf{b},\mathsf{a})}\int_{\mathsf{a}}^{\mathsf{a}+\varsigma(\mathsf{b},\mathsf{a})}\overline{\mathfrak{V}}(\beta)d\beta|\Big\}\\
&\supseteq|\frac{\zeta(\mathsf{b},\mathsf{a})}{2}\int_0^1|1-2\mathfrak{y}|[\underline{\mathfrak{V}}'(\mathsf{a}+\mathfrak{y}\zeta(\mathsf{b},\mathsf{a})),\overline{\mathfrak{V}}'(\mathsf{a}+\mathfrak{y}\zeta(\mathsf{b},\mathsf{a}))]d\mathfrak{y}|\\
&\supseteq\frac{\zeta(\mathsf{b},\mathsf{a})}{2}\int_0^1|1-2\mathfrak{y}|[\underline{\mathfrak{V}}'(\mathsf{a}+\mathfrak{y}\zeta(\mathsf{b},\mathsf{a})),\overline{\mathfrak{V}}'(\mathsf{a}+\mathfrak{y}\zeta(\mathsf{b},\mathsf{a}))]d\mathfrak{y}\\
&\supseteq\frac{\zeta(\mathsf{b},\mathsf{a})}{2}\int_0^1|1-2\mathfrak{y}|[\frac{\underline{\mathfrak{V}}'(\mathsf{a})}{\mathsf{h}_1(\mathfrak{y})\mathsf{h}_2(1-\mathfrak{y})}+\frac{\underline{\mathfrak{V}}'(\mathsf{b})}{\mathsf{h}_1(1-\mathfrak{y})\mathsf{h}_2(\mathfrak{y})},\frac{\overline{\mathfrak{V}}'(\mathsf{a})}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{\overline{\mathfrak{V}}'(\mathsf{b})}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}]d\mathfrak{y}\\
&\supseteq\frac{\zeta(\mathsf{b},\mathsf{a})}{2}[|\underline{\mathfrak{V}}'(\mathsf{a})|+|\underline{\mathfrak{V}}'(\mathsf{b})|,|\overline{\mathfrak{V}}'(\mathsf{a})|+|\overline{\mathfrak{V}}'(\mathsf{b})|]\\
&\times\int_0^1|1-2\mathfrak{y}|[\frac{1}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}]d\mathfrak{y}\\
&=\frac{\zeta(\mathsf{b},\mathsf{a})}{2}[|\mathfrak{V}'(\mathsf{a})|+|\mathfrak{V}'(\mathsf{b})|]\times\int_0^1|1-2\mathfrak{y}|[\frac{1}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}]d\mathfrak{y}
\end{aligned}$$

□

Corollary 1. *Note that* $\int_0^1 \frac{d\mathfrak{y}}{H(\mathfrak{y},1-\mathfrak{y})} = \int_0^1 \frac{d\mathfrak{y}}{H(1-\mathfrak{y},\mathfrak{y})}$. *Using this fact in inclusion (19), we obtain this fresh result:*

$$\sup\{|\frac{\underline{\mathfrak{V}}(a)+\underline{\mathfrak{V}}(a+\varsigma(b,a))}{2} - \frac{1}{\zeta(b,a)}\int_a^{a+\varsigma(b,a)} \underline{\mathfrak{V}}(\beta)d\beta, . \frac{\overline{\mathfrak{V}}(a)+\overline{\mathfrak{V}}(a+\varsigma(b,a))}{2} - \frac{1}{\zeta(b,a)}\int_a^{a+\varsigma(b,a)} \overline{\mathfrak{V}}(\beta)d\beta|\right\} \supseteq \frac{\zeta(b,a)}{2}[|\underline{\mathfrak{V}}'(a)|+|\underline{\mathfrak{V}}'(b)|, |\overline{\mathfrak{V}}'(a)|+|\overline{\mathfrak{V}}'(b)|] \times \int_0^1 [\frac{|1-2\mathfrak{y}|}{H(\mathfrak{y},1-\mathfrak{y})}]d\mathfrak{y}. \quad (20)$$

Corollary 2. *Considering* $\zeta(b,a) = b - a$, $\underline{\mathfrak{V}} = \overline{\mathfrak{V}}$ *in inclusion (19), we obtain this fresh result:*

$$\left|\frac{\mathfrak{V}(a)+\mathfrak{V}(b)}{2} - \frac{1}{b-a}\int_a^b \mathfrak{V}(\beta)d\beta\right| \leq \frac{b-a}{2}\left[\left|\mathfrak{V}'(a)\right| + \left|\mathfrak{V}'(b)\right|\right]\int_0^1 \frac{|1-2\mathfrak{y}|}{H(\mathfrak{y},1-\mathfrak{y})}d\mathfrak{y}.$$

Example 9. *Consider* $[a,b] = [1,2]$, $h_1(\mathfrak{y}) = \frac{1}{\mathfrak{y}}$, $h_2(\mathfrak{y}) = 1$, *and* $\forall\, \mathfrak{y} \in (0,1)$. *If* $\mathfrak{V} : [a,b] \to R^+$ *is defined by*

$$\mathfrak{V}(\mathfrak{y}) = \mathfrak{y}^2 + 2,$$

then

$$\left|\frac{\mathfrak{V}(a)+\mathfrak{V}(b)}{2} - \frac{1}{b-a}\int_a^b \mathfrak{V}(\beta)d\beta\right| = \frac{1}{6}$$

$$\frac{b-a}{2}\left[\left|\mathfrak{V}'(a)\right| + \left|\mathfrak{V}'(b)\right|\right]\int_0^1 \frac{|1-2\mathfrak{y}|}{H(\mathfrak{y},1-\mathfrak{y})}d\mathfrak{y} = \frac{1}{4}.$$

Consequently,

$$\frac{1}{6} \leq \frac{1}{4}$$

This verifies Corollary 2.

4.1. Applications to Numerical Trapezoidal Formula on Set-Valued Mappings

It is well known that generalized convexity is applicable to a variety of research areas. We demonstrate here how to estimate errors accumulated by using the (h_1,h_2)–Godunova–Levin preinvex functions for numerical integration with the trapezoidal formula.

Consider $\mathfrak{d}$ to be a partition of interval $[a,b]$ (i.e., $\mathfrak{d} : a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$) of the quadrature formula, known as the trapezoidal rule. The corresponding approximation error is represented as $S(\mathfrak{V},\mathfrak{d})$.

$$\int_a^b \mathfrak{V}(\beta)d\beta \cong T(\mathfrak{V},\mathfrak{d}) + S(\mathfrak{V},\mathfrak{d}),$$

where

$$T(\mathfrak{V},\mathfrak{d}) = \sum_{i=0}^{n-1} \frac{\mathfrak{V}(a_i)+\mathfrak{V}(a_{i+1})}{2}(a_{i+1}-a_i)$$

and

$$V(\mathfrak{y}) = \begin{cases} 2\int_{\mathfrak{y}}^{\frac{1}{2}} \mathfrak{V}(\mathfrak{s}a + (1-\mathfrak{s})b)d\mathfrak{s} & \mathfrak{y} \in \left[0\frac{1}{2}\right]; \\ -2\int_{\frac{1}{2}}^{\mathfrak{y}} \mathfrak{V}(\mathfrak{s}a + (1-\mathfrak{s})b)d\mathfrak{s} & \mathfrak{y} \in \left[\frac{1}{2},1\right]. \end{cases}$$

Proposition 1. *Let* $\mathfrak{V} : \Theta = [a, a+\varsigma(b,a)] \to (0,\infty)$ *be a differentiable* I.V.F *on* a° *with* $a < a+\varsigma(b,a)$. *If* $|\mathfrak{V}'|$ *is a* (h_1,h_2)*–Godunova–Levin preinvex on* $[a, a+\varsigma(b,a)]$, *then for every disjoint* $\mathfrak{d}$ *of* $[a,b]$, *we obtain the following inclusion:*

$$\left|\underline{S_n}(\mathfrak{V},\mathfrak{d}),\overline{S_n}(\mathfrak{V},\mathrm{d})\right| \supseteq \frac{1}{2}\sum_{i=0}^{n-1}(\mathrm{a}_{i+1}-\mathrm{a}_i)^2\big(|\underline{\mathfrak{V}}'(\mathrm{a}_i)|+|\underline{\mathfrak{V}}'(\mathrm{a}_{i+1})|,$$
$$\left|\overline{\mathfrak{V}}'(\mathrm{a}_i)\right|+\left|\overline{\mathfrak{V}}'(\mathrm{a}_{i+1})\right|\Big)\times\int_0^1\frac{|1-2\mathfrak{y}|}{\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}d\mathfrak{y}$$
$$\supseteq \sum_{i=0}^{n-1}(\mathrm{a}_{i+1}-\mathrm{a}_i)^2\big[\sup\{|\mathfrak{V}'(\mathrm{a})|,|\mathfrak{V}'(\mathrm{b})|\},\sup\{|\mathfrak{V}'(\mathrm{a})|,|\mathfrak{V}'(\mathrm{b})|\}\big]\int_0^1\frac{|1-2\mathfrak{y}|}{\mathrm{H}(\mathfrak{y}.1-\mathfrak{y})}d\mathfrak{y}.$$

Proof. Take into account inclusion (19) on the interval $[\mathrm{a}_i,\mathrm{a}_{i+1}]$ for each $\mathrm{i}=0$ to $\mathrm{n}-1$ of the partition $\mathfrak{d}$. This provides the following:

$$\sup\{|\frac{\underline{\mathfrak{V}}(\mathrm{a}_i)+\underline{\mathfrak{V}}(\mathrm{a}_{i+1})}{2}-\frac{1}{(\mathrm{a}_{i+1}-\mathrm{a}_i)}\int_{\mathrm{a}_i}^{\mathrm{a}_{i+1}}\underline{\mathfrak{V}}(\beta)d\beta,.$$
$$\frac{\overline{\mathfrak{V}}(\mathrm{a}_i)+\overline{\mathfrak{V}}(\mathrm{a}_{i+1})}{2}-\frac{1}{(\mathrm{a}_{i+1}-\mathrm{a}_i)}\int_{\mathrm{a}_i}^{\mathrm{a}_{i+1}}\overline{\mathfrak{V}}(\beta)d\beta|\Big\}$$
$$\supseteq\frac{(\mathrm{a}_{i+1}-\mathrm{a}_i)^2}{2}[|\underline{\mathfrak{V}}'(\mathrm{a}_i)|+|\underline{\mathfrak{V}}'(\mathrm{a}_{i+1})|,|\overline{\mathfrak{V}}'(\mathrm{a}_i)|+|\overline{\mathfrak{V}}'(\mathrm{a}_{i+1})|]\times\int_0^1[\frac{|1-2\mathfrak{y}|}{\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}]d\mathfrak{y}. \quad (21)$$

Since $|\mathfrak{V}'|$ is $(\mathrm{h}_1,\mathrm{h}_2)$–Godunova–Levin preinvex and sums up the results from $\mathrm{i}=0$ to $\mathrm{i}=\mathrm{n}-1$, we obtain

$$\mathcal{H}\Big([\underline{\mathrm{T}}(\mathfrak{V},\mathrm{d}),\overline{\mathrm{T}}(\mathfrak{V},\mathrm{d})],\Big[\int_{\mathrm{a}}^{\mathrm{b}}\underline{\mathfrak{V}}(\beta)\mathrm{d}\beta,\int_{\mathrm{a}}^{\mathrm{b}}\overline{\mathfrak{V}}(\beta)\mathrm{d}\beta\Big]\Big)$$
$$=\sup\Big\{\Big|\underline{\mathrm{T}}(\mathfrak{V},\mathrm{d})-\int_{\mathrm{a}}^{\mathrm{b}}\underline{\mathfrak{V}}(\beta)\mathrm{d}\beta,\overline{\mathrm{T}}(\mathfrak{V},\mathrm{d})-\int_{\mathrm{a}}^{\mathrm{b}}\overline{\mathfrak{V}}(\beta)\mathrm{d}\beta\Big|\Big\}$$
$$\supseteq\frac{1}{2}\sum_{i=0}^{n-1}(\mathrm{a}_{i+1}-\mathrm{a}_i)^2\Big(|\underline{\mathfrak{V}}'(\mathrm{a}_i)|+|\underline{\mathfrak{V}}'(\mathrm{a}_{i+1})|,\left|\overline{\mathfrak{V}}'(\mathrm{a}_i)\right|+\left|\overline{\mathfrak{V}}'(\mathrm{a}_{i+1})\right|\Big)$$
$$\supseteq\Big[\sup\{|\underline{\mathfrak{V}}'(\mathrm{a}_i)|,|\underline{\mathfrak{V}}'(\mathrm{a}_{i+1})|\},\sup\Big\{\left|\overline{\mathfrak{V}}'(\mathrm{a}_i)\right|,\left|\overline{\mathfrak{V}}'(\mathrm{a}_{i+1})\right|\Big\}\Big]\sum_{i=0}^{n-1}(\mathrm{a}_{i+1}-\mathrm{a}_i)^2\int_0^1\frac{|1-2\mathfrak{y}|}{\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}d\mathfrak{y}$$
$$\supseteq\Big[\sup\{|\underline{\mathfrak{V}}'(\mathrm{a})|,|\underline{\mathfrak{V}}'(\mathrm{b})|\},\sup\Big\{\left|\overline{\mathfrak{V}}'(\mathrm{a})\right|,\left|\overline{\mathfrak{V}}'(\mathrm{b})\right|\Big\}\Big]\sum_{i=0}^{n-1}(\mathrm{a}_{i+1}-\mathrm{a}_i)^2\int_0^1\frac{|1-2\mathfrak{y}|}{\mathrm{H}(\mathfrak{y},1-\mathfrak{y})}d\mathfrak{y}$$

□

Remark 10. *With the assistance of* $(\mathrm{h}_1,\mathrm{h}_2)$*–Godunova—Levin preinvex mappings, the above inclusion provides an error bound of numerical quadrature trapezoidal-type formulas. Furthermore, our results generalize the results reported in [45] when our interval is degenerated and when setting* $\mathrm{h}_1(\mathfrak{y})=1,\mathrm{h}_2(1-\mathfrak{y})=1$.

4.2. Some Further Applications to Trapezoidal Formula and the Probability Density Function

By using interval-valued functions, we find the error bounds of the trapezoidal formula in the previous section. This section eliminates the Kulisch–Miranker order relation and develops the trapezium-type inequality under interval degeneration and finds the error bound for the trapezoidal formula and its applications to probability density functions using Definition 2.4, defined by Afzal et al. [46].

Theorem 8. *Consider a differentiable function* $\mathfrak{Y}:\mathrm{I}\subseteq\mathrm{R}\to\mathrm{R}$ *on* $\mathrm{I}^\circ,\mathrm{a},\mathrm{b}\in\mathrm{I}^\circ$ *and* $\mathfrak{V}:[\mathrm{a},\mathrm{b}]\to\mathrm{R}^+$ *as a function symmetric to* $\frac{\mathrm{a}+\mathrm{b}}{2}$. *If* $|\mathfrak{Y}'|$ *is a a* $(\mathrm{h}_1,\mathrm{h}_2)$*–Godunova–Levin mapping on* $[\mathrm{a},\mathrm{b}]$*, then*

$$\left|\frac{\mathfrak{Y}(\mathtt{a})+\mathfrak{Y}(\mathtt{b})}{2}\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{V}(\beta)d\beta-\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta\right|$$
$$\leq(\mathtt{b}-\mathtt{a})\left(\left|\mathfrak{Y}'(\mathtt{a})\right|+\left|\mathfrak{Y}'(\mathtt{b})\right|\right)\int_{\mathtt{a}}^{\frac{\mathtt{b}+\mathtt{a}}{2}}\int_0^{\frac{\beta-\mathtt{a}}{\mathtt{b}-\mathtt{a}}}\mathfrak{V}(\beta)\left[\frac{1}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y}d\beta.$$

Proof. By virtue of $\mathtt{V}(\mathfrak{y})$ and the $(\mathtt{h}_1,\mathtt{h}_2)$–Godunova–Levin convexity of $|\mathfrak{Y}'|$, this implies

$$\left|\frac{\mathfrak{Y}(\mathtt{a})+\mathfrak{Y}(\mathtt{b})}{2}\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{V}(\beta)d\beta-\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta\right|=\frac{(\mathtt{b}-\mathtt{a})^2}{2}\left|\int_0^1\mathtt{V}(\mathfrak{y})\mathfrak{Y}'(\mathfrak{y}\mathtt{a}+(1-\mathfrak{y})\mathtt{b})d\mathfrak{y}\right|$$
$$\leq\frac{(\mathtt{b}-\mathtt{a})^2}{2}\left\{\int_0^{\frac{1}{2}}|\mathtt{V}(\mathfrak{y})|\left|\mathfrak{Y}'(\mathfrak{y}\mathtt{a}+(1-\mathfrak{y})\mathtt{b})\right|d\mathfrak{y}+\int_{\frac{1}{2}}^1|\mathtt{V}(\mathfrak{y})|\left|\mathfrak{Y}'(\mathfrak{y}\mathtt{a}+(1-\mathfrak{y})\mathtt{b})\right|d\mathfrak{y}\right\}$$
$$=\frac{(\mathtt{b}-\mathtt{a})^2}{2}\left\{\int_0^{\frac{1}{2}}\mathtt{V}(\mathfrak{y})\left|\mathfrak{Y}'(\mathfrak{y}\mathtt{a}+(1-\mathfrak{y})\mathtt{b})\right|d\mathfrak{y}-\int_{\frac{1}{2}}^1\mathtt{V}(\mathfrak{y})\left|\mathfrak{Y}'(\mathfrak{y}\mathtt{a}+(1-\mathfrak{y})\mathtt{b})\right|d\mathfrak{y}\right\}$$
$$\leq\frac{(\mathtt{b}-\mathtt{a})^2}{2}\left\{2\int_0^{\frac{1}{2}}\int_{\mathfrak{y}}^{\frac{1}{2}}\mathfrak{V}(\mathfrak{s}\mathtt{a}+(1-\mathfrak{s})\mathtt{b})\left(\frac{|\mathfrak{Y}'(\mathtt{a})|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\mathfrak{Y}'(\mathtt{b})|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right)d\mathfrak{s}d\mathfrak{y}\right.$$
$$\left.+2\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^{\mathfrak{y}}\mathfrak{V}(\mathfrak{s}\mathtt{a}+(1-\mathfrak{s})\mathtt{b})\left(\frac{|\mathfrak{Y}'(\mathtt{a})|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\mathfrak{Y}'(\mathtt{b})|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right)d\mathfrak{s}d\mathfrak{y}\right\}.$$

Now, if we alter the integration's order, then

$$\left|\frac{\mathfrak{Y}(\mathtt{a})+\mathfrak{Y}(\mathtt{b})}{2}\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{V}(\beta)d\beta-\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta\right|$$
$$\leq(\mathtt{b}-\mathtt{a})^2\left\{\int_0^{\frac{1}{2}}\int_0^{\mathtt{s}}\mathfrak{V}(\mathfrak{s}\mathtt{a}+(1-\mathfrak{s})\mathtt{b})\left(\frac{|\mathfrak{Y}'(\mathtt{a})|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\mathfrak{Y}'(\mathtt{b})|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right)d\mathfrak{y}d\mathfrak{s}\right.$$
$$\left.+\int_{\frac{1}{2}}^1\int_{\mathtt{s}}^1\mathfrak{V}(\mathfrak{s}\mathtt{a}+(1-\mathfrak{s})\mathtt{b})\left(\frac{|\mathfrak{Y}'(\mathtt{a})|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\mathfrak{Y}'(\mathtt{b})|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right)d\mathfrak{y}d\mathfrak{s}\right\}.$$

Making use of the variable change $\beta=\mathfrak{s}\mathtt{a}+(1-\mathfrak{s})\mathtt{b}$, we obtain

$$\left|\frac{\mathfrak{Y}(\mathtt{a})+\mathfrak{Y}(\mathtt{b})}{2}\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{V}(\beta)d\beta-\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta\right|$$
$$\leq(\mathtt{b}-\mathtt{a})\left\{\int_{\frac{\mathtt{b}+\mathtt{a}}{2}}^{\mathtt{b}}\int_0^{\frac{\mathtt{b}-\beta}{\mathtt{b}-\mathtt{a}}}\mathfrak{V}(\beta)\left(\frac{|\mathfrak{Y}'(\mathtt{a})|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\mathfrak{Y}'(\mathtt{b})|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right)d\mathfrak{y}d\beta\right.$$
$$\left.+\int_{\mathtt{a}}^{\frac{\mathtt{b}+\mathtt{a}}{2}}\int_{\frac{\mathtt{b}-\beta}{\mathtt{b}-\mathtt{a}}}^1\mathfrak{V}(\beta)\left(\frac{|\mathfrak{Y}'(\mathtt{a})|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\mathfrak{Y}'(\mathtt{b})|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right)d\mathfrak{y}d\beta\right\}. \tag{22}$$

Given that the $\mathfrak{V}$ function is symmetric to $\frac{\mathtt{b}+\mathtt{a}}{2}$,

$$\int_{\frac{\mathtt{b}+\mathtt{a}}{2}}^{\mathtt{b}}\int_0^{\frac{\mathtt{b}-\beta}{\mathtt{b}-\mathtt{a}}}\mathfrak{V}(\beta)\left(\frac{|\mathfrak{Y}'(\mathtt{a})|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\mathfrak{Y}'(\mathtt{b})|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right)d\mathfrak{y}d\beta$$
$$=\int_{\mathtt{a}}^{\frac{\mathtt{b}+\mathtt{a}}{2}}\int_0^{\frac{\beta-\mathtt{a}}{\mathtt{b}-\mathtt{a}}}\mathfrak{V}(\beta)\left(\frac{|\mathfrak{Y}'(\mathtt{a})|}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{|\mathfrak{Y}'(\mathtt{b})|}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right)d\mathfrak{y}d\beta. \tag{23}$$

Replacing Equation (23) in (22) implies that

$$\left|\frac{\mathfrak{Y}(\mathtt{a})+\mathfrak{Y}(\mathtt{b})}{2}\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{V}(\beta)d\beta-\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta\right|$$
$$\leq(\mathtt{b}-\mathtt{a})\left(|\mathfrak{Y}'(\mathtt{a})|+|\mathfrak{Y}'(\mathtt{b})|\right)\int_{\mathtt{a}}^{\frac{\mathtt{b}+\mathtt{a}}{2}}\int_{0}^{\frac{\beta-\mathtt{a}}{\mathtt{b}-\mathtt{a}}}\mathfrak{V}(\beta)\left[\frac{1}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}\right]\mathrm{d}\mathfrak{y}d\beta. \quad (24)$$

□

Corollary 3. *Considering the hypothesis of Theorem 8, if* $|\mathfrak{Y}'|$ *is* $\mathfrak{s}$*-convex on* $[\mathtt{a},\mathtt{b}]$ *with* $\mathsf{h}_1(\mathfrak{y})=\mathsf{h}_2(1-\mathfrak{y})=1$*, then*

$$\left|\frac{\mathfrak{Y}(\mathtt{a})+\mathfrak{Y}(\mathtt{b})}{2}\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{V}(\beta)d\beta-\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta\right|$$
$$\leq\frac{(\mathtt{b}-\mathtt{a})}{1+\mathfrak{s}}\left(|\mathfrak{Y}'(\mathtt{a})|+|\mathfrak{Y}'(\mathtt{b})|\right)\int_{\mathtt{a}}^{\frac{\mathtt{b}+\mathtt{a}}{2}}\mathfrak{V}(\beta)\left[\left(\frac{\beta-\mathtt{a}}{\mathtt{b}-\mathtt{a}}\right)^{1+\mathfrak{s}}-\left(\frac{\mathtt{b}-\beta}{\mathtt{b}-\mathtt{a}}\right)^{1+\mathfrak{s}}+1\right]d\beta.$$

Corollary 4. *Considering the hypothesis of Theorem 8, if* $|\mathfrak{Y}'|$ *is convex on* $[\mathtt{a},\mathtt{b}]$ *with* $\mathsf{h}_1(\mathfrak{y})=\mathsf{h}_2(1-\mathfrak{y})=1$*, then*

$$\left|\frac{\mathfrak{Y}(\mathtt{a})+\mathfrak{Y}(\mathtt{b})}{2}\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{V}(\beta)d\beta-\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta\right|\leq\left(|\mathfrak{Y}'(\mathtt{a})|+|\mathfrak{Y}'(\mathtt{b})|\right)\int_{\frac{\mathtt{b}+\mathtt{a}}{2}}^{\mathtt{b}}\mathfrak{V}(\beta)(\mathtt{b}-\beta)d\beta.$$

Additionally, if we take $\mathfrak{V}=1$, we repeat the outcome found in [47].

$$\left|\frac{\mathfrak{Y}(\mathtt{a})+\mathfrak{Y}(\mathtt{b})}{2}-\frac{1}{\mathtt{b}-\mathtt{a}}\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{Y}(\beta)dx\right|\leq\frac{(\mathtt{b}-\mathtt{a})(|\mathfrak{Y}'(\mathtt{a})|+|\mathfrak{Y}'(\mathtt{b})|)}{8}.$$

4.3. Trapezoidal Formula

Consider $\mathfrak{d}$ to be a partition of interval $[\mathtt{a},\mathtt{b}]$ (i.e., $\mathfrak{d}:\mathtt{a}=\mathtt{a}_0<\mathtt{a}_1<\cdots<\mathtt{a}_{\mathtt{n}-1}<\mathtt{a}_{\mathtt{n}}=\mathtt{b}$) of a trapezoidal formula.

$$\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta=\mathcal{T}(\mathfrak{Y},\mathfrak{V},\mathfrak{d})+S(\mathfrak{Y},\mathfrak{V},\mathfrak{d}),$$

where

$$\mathcal{T}(\mathfrak{Y},\mathfrak{V},\mathfrak{d})=\sum_{i=0}^{\mathtt{n}-1}\frac{\mathfrak{Y}(\mathtt{a}_i)+\mathfrak{Y}(\mathtt{a}_{i+1})}{2}\int_{\mathtt{a}_i}^{\mathtt{a}_{i+1}}\mathfrak{V}(\beta)d\beta,$$

The corresponding approximation error is represented as $S(\mathfrak{Y},\mathfrak{V},\mathfrak{d})$. Consider the hypothesis of Theorem 8 to hold and take a subinterval $[\mathtt{a}_i,\mathtt{a}_{i+1}]$ for each $\mathtt{i}=0$ to $\mathtt{i}=\mathtt{n}-1$ of the partition $\mathfrak{d}$. This provides the following:

$$\left|\frac{\mathfrak{Y}(\mathtt{a}_i)+\mathfrak{Y}(\mathtt{a}_{i+1})}{2}\int_{\mathtt{a}_i}^{\mathtt{a}_{i+1}}\mathfrak{V}(\beta)d\beta-\int_{\mathtt{a}_i}^{\mathtt{a}_{i+1}}\mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta\right| \quad (25)$$

Considering the inequality (25), for each $i=0$ to $i=n-1$, employing a triangular inequality, we arrive at

$$
\begin{aligned}
&| \mathcal{T}(\mathfrak{Y},\mathfrak{V},\mathfrak{d}) - \int_{\mathtt{a}}^{\mathtt{b}} \mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta | \\
&= \left| \sum_{i=0}^{\mathtt{n}-1} \left[\frac{\mathfrak{Y}(\mathtt{a}_i)+\mathfrak{Y}(\mathtt{a}_{i+1})}{2} \int_{\mathtt{a}_i}^{\mathtt{a}_{i+1}} \mathfrak{V}(\beta)d\beta - \int_{\mathtt{a}_i}^{\mathtt{a}_{i+1}} \mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta \right] \right| \\
&\le \sum_{i=0}^{\mathtt{n}-1} \left| \frac{\mathfrak{Y}(\mathtt{a}_i)+\mathfrak{Y}(\mathtt{a}_{i+1})}{2} \int_{\mathtt{a}_i}^{\mathtt{a}_{i+1}} \mathfrak{V}(\beta)d\beta - \int_{\mathtt{a}_i}^{\mathtt{a}_{i+1}} \mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta \right| \\
&\le \sum_{i=0}^{\mathtt{n}-1} (\mathtt{a}_{i+1}-\mathtt{a}_i)\left[|\mathfrak{Y}'(\mathtt{a}_i)| + |\mathfrak{Y}'(\mathtt{a}_{i+1})|\right] \int_{\frac{\mathtt{a}_i+\mathtt{a}_{i+1}}{2}}^{\mathtt{a}_{i+1}} \int_0^{\frac{\mathtt{a}_{i+1}-\beta}{\mathtt{a}_{i+1}-\mathtt{a}_i}} \mathfrak{V}(\beta) \\
&\times \left[\frac{1}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})} + \frac{1}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})} \right] d\mathfrak{y}\, d\beta.
\end{aligned}
$$

This gives us the error bound:

$$
\begin{aligned}
|S(\mathfrak{Y},\mathfrak{V},\mathfrak{d})| \le& \sum_{i=0}^{\mathtt{n}-1} (\mathtt{a}_{i+1}-\mathtt{a}_i)\left[|\mathfrak{Y}'(\mathtt{a}_i)| + |\mathfrak{Y}'(\mathtt{a}_{i+1})|\right] \\
&\times \int_{\frac{\mathtt{a}_i+\mathtt{a}_{i+1}}{2}}^{\mathtt{a}_{i+1}} \int_0^{\frac{\mathtt{a}_{i+1}-\beta}{\mathtt{a}_{i+1}-\mathtt{a}_i}} \mathfrak{V}(\beta) \left[\frac{1}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})} + \frac{1}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})} \right] d\mathfrak{y}\, d\beta.
\end{aligned}
$$

Corollary 5. *Setting* $\mathsf{h}_1(\mathfrak{y}) = \frac{1}{\mathfrak{y}^{\mathtt{k}}}, \mathsf{h}_2(\mathfrak{y}) = 1$ *in (25), we obtain the following:*

$$
\begin{aligned}
|S(\mathfrak{Y},\mathfrak{V},\mathfrak{d})| \le& \frac{1}{\mathtt{k}+1} \sum_{i=0}^{\mathtt{n}-1} (\mathtt{a}_{i+1}-\mathtt{a}_i)\left[|\mathfrak{Y}'(\mathtt{a}_i)| + |\mathfrak{Y}'(\mathtt{a}_{i+1})|\right] \\
&\times \int_{\frac{\mathtt{a}_i+\mathtt{a}_{i+1}}{2}}^{\mathtt{a}_{i+1}} \left[\left(\frac{\mathtt{a}_{i+1}-\beta}{\mathtt{a}_{i+1}-\mathtt{a}_i} \right)^{\mathtt{k}+1} - \left(\frac{\beta-\mathtt{a}_i}{\mathtt{a}_{i+1}-\mathtt{a}_i} \right)^{\mathtt{k}+1} + 1 \right] \mathfrak{V}(\beta)d\beta. \qquad (26)
\end{aligned}
$$

Remark 11. *Additionally, if we set* $k = 1$ *and* $\mathfrak{V} = 1$ *in (26), we subsequently restate the inequality found in Prop. 4.1 in [47]:*

$$
|S(\mathfrak{Y},\mathfrak{d})| \le \frac{1}{8} \sum_{i=0}^{\mathtt{n}-1} \left[|\mathfrak{Y}'(\mathtt{a}_i)| + |\mathfrak{Y}'(\mathtt{a}_{i+1})|\right] (\mathtt{a}_{i+1}-\mathtt{a}_i)^2.
$$

4.4. Associating Probability Density Function with Trapezoidal-Type Inequality

Consider a probability density function $\mathfrak{V} : [\mathtt{a},\mathtt{b}] \to \mathsf{R}^+$ where $0 < \mathtt{a} < \mathtt{b}$. Then,

$$
\int_{\mathtt{a}}^{\mathtt{b}} \mathfrak{V}(\beta)d\beta = 1,
$$

which is symmetric with respect to $\frac{\mathtt{b}+\mathtt{a}}{2}$. Consider μ to be a moment where $\mu \in \mathsf{R}$. Then, one has

$$
\mathcal{E}_\mu(X) = \int_{\mathtt{a}}^{\mathtt{b}} \beta^\mu \mathfrak{V}(\beta)d\beta,
$$

which is finite. From Equation (8) and the fact that for any $\mathtt{a} \le \beta \le \frac{\mathtt{b}+\mathtt{a}}{2}$ we have $0 \le \frac{\beta-\mathtt{a}}{\mathtt{b}-\mathtt{a}} \le \frac{1}{2}$, the following result holds.

$$\left|\frac{\mathfrak{Y}(\mathtt{a})+\mathfrak{Y}(\mathtt{b})}{2}\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{V}(\beta)d\beta-\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta\right|\leq(\mathtt{b}-\mathtt{a})\left(|\mathfrak{Y}'(\mathtt{a})|+|\mathfrak{Y}'(\mathtt{b})|\right)$$
$$\times\int_{\mathtt{a}}^{\frac{\mathtt{b}+\mathtt{a}}{2}}\int_0^{\frac{1}{2}}\mathfrak{V}(\beta)\left[\frac{1}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}\right]\mathsf{d}\mathfrak{y}d\beta=\frac{(\mathtt{b}-\mathtt{a})}{2}\left(|\mathfrak{Y}'(\mathtt{a})|+|\mathfrak{Y}'(\mathtt{b})|\right)$$
$$\times\int_0^{\frac{1}{2}}\left[\frac{1}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathsf{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y},$$

From the fact that $\mathfrak{V}$ is symmetric and $\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{V}(\beta)d\beta=1$, one has $\int_{\mathtt{a}}^{\frac{\mathtt{b}+\mathtt{a}}{2}}\mathfrak{V}(\beta)d\beta=\frac{1}{2}$.

Example 10. *If we take into account*

$$\begin{cases}\mathfrak{Y}(\beta)=\frac{1}{\mu}\beta^{\mu}, & \beta>0,\mu\in(-\infty,0)\cup(0,1]\cup[2,+\infty);\\ \mathsf{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}^{\mathtt{k}}},\mathsf{h}_2(\mathfrak{y})=\frac{1}{8} & k\in(-\infty,-1)\cup(-1,1];\\ \mathfrak{V}(\beta)=1.\end{cases}$$

then $|\mathfrak{Y}'|$ *is* $(\mathsf{h}_1,\mathsf{h}_2)$*–Godunova–Levin; therefore, from Theorem 8, one has*

$$\left|\frac{\mathtt{a}^{\mu}+\mathtt{b}^{\mu}}{2\mu}-\mathcal{E}_{\mu}(X)\right|\leq\frac{\mu(\mathtt{b}-\mathtt{a})}{2}\left(\mathtt{a}^{\mu-1}+\mathtt{b}^{\mu-1}\right)\int_0^{\frac{1}{2}}\left[\frac{\mathfrak{y}^{\mathtt{k}}}{8}+\frac{(1-\mathfrak{y})^{\mathtt{k}}}{8}\right]d\mathfrak{y}$$
$$=\frac{\mu(\mathtt{b}-\mathtt{a})}{8(\mathtt{k}+1)}\left(\mathtt{a}^{\mu-1}+\mathtt{b}^{\mu-1}\right).$$

Hence, the required bound is

$$\left|\frac{\mathtt{a}^{\mu}+\mathtt{b}^{\mu}}{2\mu}-\mathcal{E}_{\mu}(X)\right|\leq\frac{\mu(\mathtt{b}-\mathtt{a})}{8(\mathtt{k}+1)}\left(\mathtt{a}^{\mu-1}+\mathtt{b}^{\mu-1}\right),$$

Remark 12. *If* $\mu=1,\mathsf{h}_2(\mathfrak{y})=1,k=1,\mathcal{E}(X)$ *is the random variable* X'*s expectation, we can obtain the following known bound from the inequality above.*

$$\left|\frac{\mathtt{b}+\mathtt{a}}{2}-\mathcal{E}(X)\right|\leq\frac{(\mathtt{b}-\mathtt{a})}{8(1+1)}\left(\mathtt{a}^{1-1}+\mathtt{b}^{1-1}\right),$$
$$\left|\frac{\mathtt{b}+\mathtt{a}}{2}-\mathcal{E}(X)\right|\leq\frac{\mathtt{b}-\mathtt{a}}{8}.$$

As a consequence of this remark, we concluded that the error bound of a trapezoid inequality in combination with a probability density function depends on the selection of non-negative functions h_1 and h_2.

4.5. Applications Associated with Special Functions

Here, we present bounds for trapezoidal-type inequality involving moment-generating and symmetric weighted functions as applications, representing their solution using special functions. In order to begin, we should recall the two special functions known as the gamma function and beta function:

$$\mathtt{B}(\mathtt{z},\mathtt{w})=\int_0^1\mathtt{t}^{\mathtt{z}-1}(1-\mathtt{t})^{\mathtt{w}-1}\mathtt{dt}=\int_0^1\mathtt{t}^{\mathtt{w}-1}(1-\mathtt{t})^{\mathtt{z}-1}\mathtt{dt}.$$

We have the following relationship between the gamma and the beta functions:

$$\mathtt{B}(\mathtt{z},\mathtt{w})=\frac{\Gamma(\mathtt{z})\Gamma(\mathtt{w})}{\Gamma(\mathtt{z}+\mathtt{w})},$$

when $\mathtt{z}$ and $\mathtt{w}$ have positive real parts. Based on Theorem 8, we arrive at the following conclusion:

$$\begin{aligned}
&\left|\frac{\mathfrak{Y}(\mathtt{a})+\mathfrak{Y}(\mathtt{b})}{2}\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{V}(\beta)d\beta-\int_{\mathtt{a}}^{\mathtt{b}}\mathfrak{Y}(\beta)\mathfrak{V}(\beta)d\beta\right| \\
&\leq \frac{(\mathtt{b}-\mathtt{a})}{2}\left(|\mathfrak{Y}'(\mathtt{a})|+|\mathfrak{Y}'(\mathtt{b})|\right)\int_0^{\frac{1}{2}}\left[\frac{1}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\frac{1}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right]d\mathfrak{y} \\
&\leq \frac{(\mathtt{b}-\mathtt{a})}{2}\left(|\mathfrak{Y}'(\mathtt{a})|+|\mathfrak{Y}'(\mathtt{b})|\right)\left[\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right].
\end{aligned}$$

Accordingly, $\mathtt{H}(\mathfrak{y},1-\mathfrak{y})=\mathtt{h}_1(\mathfrak{y})\mathtt{h}_2(1-\mathfrak{y})$, and taking into account the assumptions of Example 10, we have the following result:

$$\left|\frac{\mathtt{a}^{\mu}+\mathtt{b}^{\mu}}{2\mu}-\mathcal{E}_{\mu}(X)\right|\leq\frac{\mu(\mathtt{b}-\mathtt{a})}{2}\left(\mathtt{a}^{\mu-1}+\mathtt{b}^{\mu-1}\right)\left[\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}(\mathfrak{y},1-\mathfrak{y})}+\int_0^1\frac{d\mathfrak{y}}{\mathtt{H}(1-\mathfrak{y},\mathfrak{y})}\right]. \quad (27)$$

Remark 13. *If* $\mathtt{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}^{\mathtt{s}_1}},\mathtt{h}_2(\mathfrak{y})=\frac{1}{\mathfrak{y}^{\mathtt{s}_2}}$*, then under the assumptions of Theorem 8 and Example 10, the result presented in (27) becomes bounds for Breckner-type* $(\mathtt{s}_1,\mathtt{s}_2)$*-preinvex functions:*

$$\left|\frac{\mathtt{a}^{\mu}+\mathtt{b}^{\mu}}{2\mu}-\mathcal{E}_{\mu}(X)\right|\leq\frac{\mu(\mathtt{b}-\mathtt{a})}{2}\left(\mathtt{a}^{\mu-1}+\mathtt{b}^{\mu-1}\right)\mathtt{B}(\mathtt{s}_1+1,\mathtt{s}_2+1). \quad (28)$$

Remark 14. *If* $\mathtt{h}_1(\mathfrak{y})=\frac{1}{\mathfrak{y}^{-\mathtt{s}_1}},\mathtt{h}_2(\mathfrak{y})=\frac{1}{\mathfrak{y}^{-\mathtt{s}_2}}$*, then under the assumptions of Theorem 8 and Example 10, the result presented in (27) becomes bounds for Godunova–Levin-type* $(\mathtt{s}_1,\mathtt{s}_2)$*-preinvex functions:*

$$\left|\frac{\mathtt{a}^{\mu}+\mathtt{b}^{\mu}}{2\mu}-\mathcal{E}_{\mu}(X)\right|\leq\frac{\mu(\mathtt{b}-\mathtt{a})}{2}\left(\mathtt{a}^{\mu-1}+\mathtt{b}^{\mu-1}\right)\mathtt{B}(1-\mathtt{s}_1,1-\mathtt{s}_2). \quad (29)$$

where $\mathtt{B}(\mathtt{z},\mathtt{w})$ *denotes the beta as a special function.*

Remark 15. *As we connect the results of a generalized class of Godunova–Levin mappings with probability density functions and find bounds with the aid of moment-generating functions, it could be interesting for academics to take inspiration from these results and discuss other properties like variance, standard deviation, and various other applications of moments. Here, we refer to [48–52].*

4.6. Applications to Special Means

The mean plays an essential role in mathematics of all kinds, especially when it comes to ensuring accuracy. We now demonstrate the relationship between our developed results and special means under specific assumptions. Consider two positive numbers $\mathtt{a},\mathtt{b}$, where $\mathtt{a}\neq\mathtt{b}$, and define the means as follows:

- The arithmetic mean:

$$\mathtt{A}=\mathtt{A}(\mathtt{a},\mathtt{b})=\frac{\mathtt{a}+\mathtt{b}}{2};\mathtt{a},\mathtt{b}\in\mathrm{R},\text{ with }\mathtt{a},\mathtt{b}>0.$$

- The following defines the logarithmic mean in its generalized form:

$$\mathtt{L}_{\mathtt{s}}(\mathtt{a},\mathtt{b})=\left[\frac{\mathtt{b}^{\mathtt{s}+1}-\rho_{\mathtt{a}}^{\mathtt{s}+1}}{(\mathtt{s}+1)(\mathtt{b}-\mathtt{a})}\right]^{\frac{1}{\mathtt{s}}},\mathtt{s}\neq-1,0.$$

Using Theorem 7 and the aforementioned applications of special means, the following propositions are obtained.

Proposition 2. *Let* $0 < \mathtt{a} < \mathtt{b}$, *where* $s \geq 2$; *then, one has*

$$\begin{aligned}
&\mathcal{H}\left(\left[\underline{\mathtt{A}}(\mathtt{a}^{\mathtt{s}},\mathtt{b}^{\mathtt{s}}),\overline{\mathtt{A}}(\mathtt{a}^{\mathtt{s}},\mathtt{b}^{\mathtt{s}})\right],\left[\underline{\mathtt{L}}_{\mathtt{s}}^{\mathtt{s}}(\mathtt{a},\mathtt{b}),\overline{\mathtt{L}}_{\mathtt{s}}^{\mathtt{s}}(\mathtt{a},\mathtt{b})\right]\right)\\
&=\sup\left\{\left|\underline{\mathtt{A}}(\mathtt{a}^{\mathtt{s}},\mathtt{b}^{\mathtt{s}})-\underline{\mathtt{L}}_{\mathtt{s}}^{\mathtt{s}}(\mathtt{a},\mathtt{b}),\overline{\mathtt{A}}(\mathtt{a}^{\mathtt{s}},\mathtt{b}^{\mathtt{s}})-\overline{\mathtt{L}}_{\mathtt{s}}^{\mathtt{s}}(\mathtt{a},\mathtt{b})\right|\right\}\\
&\supseteq \frac{s(\mathtt{b}-\mathtt{a})}{2}\left[\underline{\mathtt{A}}\left(\left|\mathtt{a}^{\mathtt{s}-1}\right|,\left|\mathtt{b}^{\mathtt{s}-1}\right|\right),\overline{\mathtt{A}}\left(\left|\mathtt{a}^{\mathtt{s}-1}\right|,\left|\mathtt{b}^{\mathtt{s}-1}\right|\right)\right]\int_0^1\frac{|1-2\mathfrak{y}|}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}d\mathfrak{y}.
\end{aligned}$$

Corollary 6. *Proposition 2 is simplified to the following result when our interval is degenerated and this is its unique instance.*

$$|\mathtt{A}(\mathtt{a}^{\mathtt{s}},\mathtt{b}^{\mathtt{s}})-\mathtt{L}_{\mathtt{s}}^{\mathtt{s}}(\mathtt{a},\mathtt{b})|\leq\frac{s(\mathtt{b}-\mathtt{a})}{2}\mathtt{A}\left(\left|\mathtt{a}^{\mathtt{s}-1}\right|,\left|\mathtt{b}^{\mathtt{s}-1}\right|\right)\int_0^1\frac{|1-2\mathfrak{y}|}{\mathsf{H}(\mathfrak{y},1-\mathfrak{y})}d\mathfrak{y}$$

Proof. We derive this inequality from Corollary (2) and apply it to the $(\mathtt{h}_1,\mathtt{h}_2)$–Godunova–Levin preinvex function $\mathfrak{V}:\mathsf{R}\to\mathsf{R},\mathfrak{V}(\beta)=\beta^{\mathtt{s}}, s\geq 2$. □

Note: Our next objective is to demonstrate that the results developed in Theorem 8 are consistent with the arithmetic and generalized logarithm means.

Consider

$$\begin{cases}\mathfrak{V}(\beta)=\beta^{\mathtt{s}}, & \beta>0 \text{ and } \mathtt{s}\in(-\infty,-1)\cup(-1,0)\cup[1,\infty);\\ \mathtt{h}_1(\mathfrak{y})=\mathfrak{y}^{\mathtt{k}},\mathtt{h}_2(\mathfrak{y})=1, & \mathtt{k}\leq 1 \text{ and } \mathtt{k}\neq -1,-2\\ \mathfrak{Y}(\beta)=1.\end{cases}$$

Theorem 8 implies the following inequalities:

$$\begin{aligned}
&\left|\frac{\mathtt{a}^{\mathtt{s}}+\mathtt{b}^{\mathtt{s}}}{2}(\mathtt{b}-\mathtt{a})-\frac{1}{\mathtt{s}+1}\left[\mathtt{b}^{\mathtt{s}+1}-\mathtt{a}^{\mathtt{s}+1}\right]\right|\leq \mathtt{s}(\mathtt{b}-\mathtt{a})\left(|\mathtt{a}|^{\mathtt{s}-1}+|\mathtt{b}|^{\mathtt{s}-1}\right)\\
&\times\int_{\mathtt{a}}^{\frac{\mathtt{a}+\mathtt{b}}{2}}\int_0^{\frac{\beta-\mathtt{a}}{\mathtt{b}-\mathtt{a}}}\left[\mathfrak{y}^{\mathtt{k}}+(1-\mathfrak{y})^{\mathtt{k}}\right]d\mathfrak{y}d\beta=\frac{\mathtt{s}(\mathtt{b}-\mathtt{a})}{\mathtt{k}+1}\left(|\mathtt{a}|^{\mathtt{s}-1}+|\mathtt{b}|^{\mathtt{s}-1}\right)\\
&\times\int_{\mathtt{a}}^{\frac{\mathtt{a}+\mathtt{b}}{2}}\left[\left(\frac{\beta-\mathtt{a}}{\mathtt{b}-\mathtt{a}}\right)^{\mathtt{k}+1}-\left(\frac{\mathtt{b}-\beta}{\mathtt{b}-\mathtt{a}}\right)^{\mathtt{k}+1}+1\right]d\beta\\
&=\frac{\mathtt{s}}{(\mathtt{k}+1)(2+\mathtt{k})(\mathtt{b}-\mathtt{a})^{\mathtt{k}}}\left(|\mathtt{a}|^{\mathtt{s}-1}+|\mathtt{b}|^{\mathtt{s}-1}\right)\\
&\times\left[(\beta-\mathtt{a})^{\mathtt{k}+2}-(\mathtt{b}-\beta)^{\mathtt{k}+2}+(\mathtt{b}-\mathtt{a})^{\mathtt{k}+1}(\mathtt{k}+2)\beta\right]_{\mathtt{a}}^{\frac{\mathtt{a}+\mathtt{b}}{2}}\\
&\leq\frac{\mathtt{s}}{2(\mathtt{k}+1)(2+\mathtt{k})(\mathtt{b}-\mathtt{a})^{\mathtt{k}}}\left(|\mathtt{a}|^{\mathtt{s}-1}+|\mathtt{b}|^{\mathtt{s}-1}\right)(\mathtt{b}-\mathtt{a})^{2+\mathtt{k}}\left[2^{-\mathtt{k}}+k\right]\\
&=\frac{\mathtt{s}(\mathtt{b}-\mathtt{a})^2}{2(\mathtt{k}+1)(\mathtt{k}+2)}\left(|\mathtt{a}|^{\mathtt{s}-1}+|\mathtt{b}|^{\mathtt{s}-1}\right)\left[2^{-\mathtt{k}}+k\right].
\end{aligned}$$

Hence, one has

$$\left|\frac{\mathtt{a}^{\mathtt{s}}+\mathtt{b}^{\mathtt{s}}}{2}-\frac{\mathtt{b}^{\mathtt{s}+1}-\mathtt{a}^{\mathtt{s}+1}}{(\mathtt{s}+1)(\mathtt{b}-\mathtt{a})}\right|\leq\frac{\mathtt{s}(\mathtt{b}-\mathtt{a})}{2(\mathtt{k}+1)(\mathtt{k}+2)}\left(|\mathtt{a}|^{\mathtt{s}-1}+|\mathtt{b}|^{\mathtt{s}-1}\right)\left[2^{-\mathtt{k}}+k\right],$$

this implies that

$$|\mathtt{A}(\mathtt{a}^{\mathtt{s}},\mathtt{b}^{\mathtt{s}})-\mathtt{L}_{\mathtt{s}}^{\mathtt{s}}(\mathtt{a},\mathtt{b})|\leq\frac{\mathtt{s}(\mathtt{b}-\mathtt{a})}{(\mathtt{k}+1)(\mathtt{k}+2)}\mathtt{L}\left(|\mathtt{a}|^{\mathtt{s}-1},|\mathtt{b}|^{\mathtt{s}-1}\right)\left[2^{-\mathtt{k}}+k\right]. \tag{30}$$

Remark 16. *If we let* $\mathtt{k}=1$ *in (30), then we obtain Prop. 3.1 in [47].*

5. Discussion and Conclusions

This work introduces a generalized class of Godunova–Levin preinvex-type mappings via Kulisch- and Miranker-type order relations and develops a number of new generalizations of Hermite–Hadamard and Fejér-type inequalities. Preinvexity of this type unifies several definitions that have been developed recently with the inclusion of distinct non-negative arbitrary functions and associated bifunctions. Furthermore, we have calculated error bounds of the trapezoid-type numerical integration formula using interval-valued mappings based on Moore interval distance metrics. The last step in our analysis was to develop trapezoidal-type inequality and to show some further applications to trapezoidal formulas as well as probability density functions. As a result of the results obtained in this paper, similar inequalities can be derived for the fractional integrals with nonsingular kernels defined as:

$$\Im_{w}^{q}\mathfrak{V}(t) = \frac{1}{q}\int_{w}^{t} \mathcal{E}_{q,1}\left(-\frac{1-q}{q}(t-e)^{q}\right)\mathfrak{V}(e)de, t > w$$

and

$$\Im_{g}^{q}\mathfrak{V}(t) = \frac{1}{q}\int_{t}^{g} \mathcal{E}_{q,1}\left(-\frac{1-q}{q}(e-t)^{q}\right)\mathfrak{V}(e)de, t < g$$

where

$$\mathcal{E}_{q,\eta}(r) = \sum_{k=0}^{\infty}\frac{r^{k}}{\Gamma(qk+\eta)}.$$

Furthermore, since we know that inequalities are established using different types of integral operators as well as order relations, each with its own characteristics and limitations, it is apparent from reference [53] that Theorem 11 does not satisfy the assumptions with this order relation in interval-valued settings. Recently, Saeed et al. [35] used this order relation defined by Bhunia and developed several results. The beauty of this order relation is that it is full order, which means we can collate intervals with fewer constraints than inclusion order, so interested researchers may apply this type of order relation to Theorem 11. The second problem is that we used the Moore metric in interval space to arrive at these conclusions. Researchers could instead use the Hukuhara interval metric defined in [54] and do a comparative study to see which metric provide more precise results in terms of closer endpoint distances between intervals.

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