An analysis of grade 11 learners' errors in trigonometric function graphs

By

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Submitted in fulfilment of the requirements for the degree

MAGISTER EDUCATIONIS

Department of Science, Mathematics and Technology Education
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DECLARATION

I declare that this dissertation entitled "**An analysis of grade 11 learners' errors in trigonometric function graphs",** is hereby submitted for the degree, Med in Mathematics Education at the University of Pretoria. I acknowledge that this is my own work and has not been previously submitted by me or any other person for a degree at this institution or any other tertiary institution.

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The author, whose name appears on the title page of the dissertation, has obtained the required research ethics approval for the research described in this work. The author declares that she has adhered to the ethical standards required in terms of the code of ethics for research and policy guidelines for responsible research at the University of Pretoria.

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DEDICATION

I dedicate my dissertation to God Almighty, whose grace has seen me through this study. Also, to the memory of my late father, Alhaji Kareem Oladele Adewale. To my mum for her prayers and emotional support.

Special dedications to my family, my husband and sons. To my supportive husband, Dr Oluwakemi Adebayo, your unrelenting efforts, push, and advice have seen me through this journey. Ayoola and David, thanks for your understanding, encouragement and support you have given. I love you all.

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To God be all the Glory.

ABSTRACT

This research study explored the types of errors that Grade 11 learners make in trigonometric function graphs and the possible causes of these errors. The investigation was done in the quest for answers to these two research sub-questions: 1) Which types of errors do Grade 11 learners make in trigonometric function graphs? 2) What causes Grade 11 learners to make these errors? Brown and Skow's (2016), Newman's (1977), Oktaviani's (2017), Radatz's (1979), and Smith et al.'s (1993) research were used to guide the deductive data analysis process of this study. The investigation was an exploratory case study conducted at three secondary schools in Tshwane, Gauteng Province-South Africa. Qualitative data were generated within the interpretive paradigm based on the researcher's experience and insight into errors made in trigonometric function graphs. Thirty sampled learners' test scripts were analysed for error types, while fifteen of those learners were interviewed for possible causes of errors. Content analysis of the data generated from the test administered and the interview scheduled was done. There were 17 items in the administered test and were divided into four categories of concepts for the purpose of data analysis.

The findings from the test revealed that Grade 11 learners committed comprehension error in Concept 1 and Concept 3. It was further revealed that in Concept 4, encoding error was prevalent, lastly, misconceptions were notable errors in Concept 1 and 2. Also, this investigation identified the possible causes of these errors as: difficulties in obtaining spatial information; deficient mastery of pre-requisite skills, facts, and concept and the application of irrelevant rules or strategies.

Keywords: Error, Error analysis, trigonometry, functions, concepts, misconceptions

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TO WHOM IT MAY CONCERN

The dissertation "An analysis of grade 11 learners' errors in trigonometric function graphs" by Ronke Adebayo has been proofread and edited for language by me.

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List of Acronyms

AMESA	Association for Mathematics Education of South Africa
BPA	Bloodstain Pattern Analysis
CAPS	Curriculum Assessment Policy Statement
CGI	Cognitively Guided Instruction
DBE	Department of Basic Education
DH	Departmental Heads
DNA	Did not attempt
FET	Further Education and Training
GDE	Gauteng Department of Education
HOTS	Higher Order Thinking Skills
NBT	National Benchmark Tests
NSC	National Senior Certificate
PC	Partially Correct
PCK	Pedagogic content knowledge
SMTE	Science, Mathematics and Technology Education

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CHAPTER ONE: INTRODUCTION AND CONTEXTUALISATION

1.1 Introduction

Trigonometry is one of the content areas of the mathematics curriculum, dealing with the relationships between sides and angles in triangles (Orhun, 2010). Trigonometry is regarded as one of the important topics in the secondary school curriculum, which requires integration with algebraic, geometric, and graphical reasoning. Trigonometry integrates with other content areas in mathematics and various other disciplines. For example, in astronomy, the triangulation technique is used to measure the distance to nearby stars, while in geography, triangulation is used to measure distances between landmarks and satellite navigation systems. Makovický et al. (2013, p. 20) stated that the Bloodstain Pattern Analysis (BPA) method "permits exact analysis of the dynamic and characteristic properties of bloodstains after impact on surfaces such as floors, walls, and ceilings". These authors inferred that trigonometric models were utilised in BPA as a forensic method that belongs to the category of biological methods. Accordingly, without trigonometry, we cannot make numerous calculations in everyday life, such as finding "the height, angle of impact or area of convergence" of blood spatter at the scene of a crime involving bloodstains (Guerra, 2014, p. 4).

However, the learning of trigonometry in South Africa is confronted with poor conceptual understanding, which is evident from the Department of Basic Education's (DBE) National Report on Grade 12 performances of the November Examination - National Senior Certificate (NSC) Examination (DBE, 2020, 2021, 2022). Also, learners experience a significant level of difficulty when attempting trigonometric problems, due to lack of the development of requisite schemas and inadequate acquisition of basic mathematical concepts as they progress from one grade to another (Ngcobo et al., 2019). In addition, students' understanding of more challenging trigonometric function topics is hampered because of a lack of prerequisite knowledge of the concept of radians (Walsh et al., 2017). This leads me to ask, what is the possible cause of this poor conceptual understanding amongst learners? It may be the existence of learner errors and misconceptions in the content taught.

Gur (2009), in his study conducted in Turkey, averred that trigonometry is an area of mathematics that learners believe and experience as being difficult and abstract compared to the other branches of mathematics. Nanmumpuni and Retnawati (2021) confirmed the arduous nature of trigonometry through the result generated from the investigation of five Grade 10 learners in High Schools in a district in the Province of Yogyakarta, Indonesia. Likewise, Wijaya et al.'s (2020) identification of problems faced by high school learners in China in solving trigonometric problems made them explore using the Hawgent mathematics software application to improve learners' performance in trigonometry. From the foregoing, it is clear that we need more research regarding the challenges experienced by learners in trigonometry. Additionally, this study will benefit the academic discourse in the field of mathematics in South Africa, and the mathematics community at large, and in particular, empower and encourage learners to become future scientists in specializations such as forensic science and other professional fields.

1.2 Problem Statement

The 2019 NSC Diagnostic Report (DBE, 2020) indicated that learners performed worst in trigonometry relative to the other nine content areas in the FET band. Additionally, the 2021 and 2022 National Diagnostic Reports stressed that the poor performance of Grade 12 learners in trigonometry in the NSC Examination is still an issue of concern (DBE, 2021, 2022). Because trigonometry integrates with many other content areas, such as Euclidean geometry, analytical geometry, and functions, it is evident that this poor performance influences learners' general mathematics performance. Sasman (2011, p. 10) found that "trigonometry was the most poorly answered section in Paper 2". It was found that failure to understand trigonometric concepts has a ripple effect on the success rate of learners in mathematics. The researcher argued that some candidates lack trigonometry fundamentals. It was alarming that several students obtained a negative radius number, a common misconception or a sign of not thinking about their responses. Therefore, it suffices to say that Grade 12 learners' poor performance in trigonometry has been a recurring challenge in South Africa for many years. The 2014 NSC examination diagnostic report on Mathematics Paper 2 highlighted that "performance in the trigonometry section was a cause for concern as candidates performed poorly in questions that tested basic knowledge" (DBE, 2014, p. 121).

Although my study is based on Grade 11 learners, this study used Grade 12 learners' results as a reference frame because the research was informed by the results from the standardised and endorsed promotional examinations of the NSC examinations (2019 & 2020). It should also be noted that a substantial percentage of work covered in Grade 11 in trigonometry is examined at the Grade 12 level. In addition, the emphasis of this study is based on Grade 11 learners' errors and misconceptions because the Grade 11 academic session is just six months away from Grade 12. This research, in my view, aims to provide empirical information that can assist in addressing the stated problem by highlighting the Grade 11 learners' errors and misconceptions of trigonometry function graphs in South Africa.

1.3 Aim and Objectives of the Study

The study intends to achieve the following objectives:

- To determine the types of errors Grade 11 learners make in trigonometric function graphs by analysing the learners' test answers.
- To investigate possible causes of errors Grade 11 learners make in trigonometric function graphs.

The knowledge of the errors and their possible causes in this study might help teachers to teach in ways that will make learners avoid such errors/misconceptions in assessments given to them.

1.4 Research Questions

The following primary and secondary research questions guided the study:

1.4.1 Primary Research Question

How can learners' errors in trigonometric function graphs be described?

The following secondary research questions are asked to answer the primary research question:

1.4.2 Secondary Research Questions

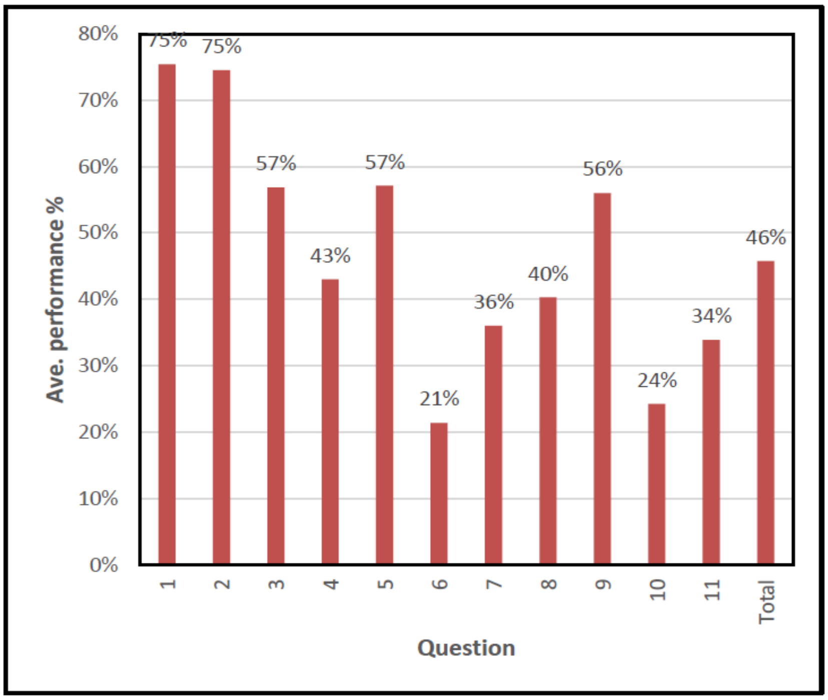
1 Which type of errors do Grade 11 learners make in trigonometric function graphs?

2 What causes Grade 11 learners to make these errors in trigonometric function graphs?

1.5 Motivation for the Study

As a secondary school teacher who has taught mathematics in the GET and the FET phases for seven years in Nigeria and seven years in South Africa, I observed the generally negative attitudes of learners in the mathematics classroom, particularly in trigonometric function graphs, in South Africa. This negative attitude could be attributed to the learners' weak cognitive ability, their prior knowledge, lack of support from home, and the fear of the subject, to mention a few. In my quest to be part of the solution, I studied the departmental and national reports to determine the content area(s) of concern. The NSC is a certificate one receives, commonly referred to as a "matric certificate". The certificate signifies the culmination of twelve years of formal education and is mainly used as an indicator to reveal how healthy the education system of South Africa is. In South Africa, the Report on the 2019 NSC Diagnostic Report in Mathematics showed that learners performed worst in the trigonometry questions in Paper 2 of the examination (DBE, 2020). As revealed in the 2020 Diagnostic report, there were noticeable errors and misconceptions in trigonometric function content based on responses of candidates in the mathematics NSC P2 examination. It was reported that learners' performance in trigonometric function graphs was below expectation. My search into these documents drew my attention to trigonometry as one of the topics in which learners perform poorly, consequently impeding their overall success in mathematics.

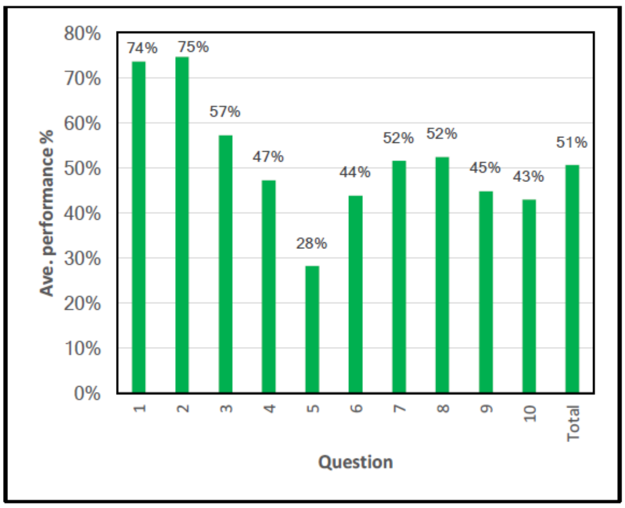
Figures 1, 2 and 3 show the learners' average percentage performance in the three trigonometry questions (Questions 5, 6, & 7 for 2019 and 2020, while 2021 also had Question 8 on trigonometry), compared to the other content areas, for Paper 2 in the 2019, 2020 and 2021 NSC mathematics examination of South Africa. For this study, I will focus on Grade 12 learners' average performance in trigonometric function graphs in 2019, 2020 and 2021, respectively.



Q	Topic(s)
1	Data Handling
2	Data Handling
3	Analytical Geometry
4	Analytical Geometry
5	Trigonometry
6	Trigonometry
7	Trigonometry
8	Trigonometry
9	Euclidean Geometry
10	Euclidean Geometry
11	Euclidean Geometry

Figure 1.1: Average percentage performance per question for Paper 2

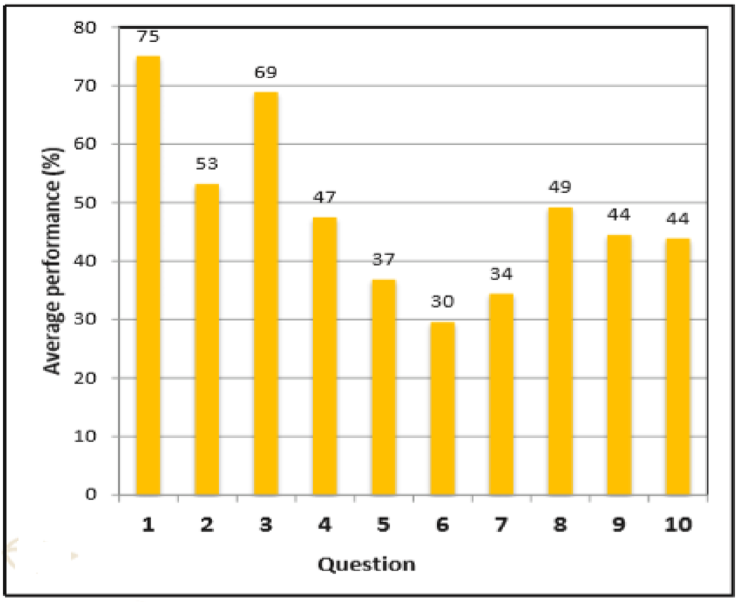
Sourced from DBE (2022, p. 177)



Q	Topic/s
1	Data Handling
2	Data Handling
3	Analytical Geometry
4	Analytical Geometry
5	Trigonometry
6	Trigonometry
7	Trigonometry
8	Euclidean Geometry
9	Euclidean Geometry
10	Euclidean Geometry

Figure 1.2: Average percentage performance per question for Paper 2

Sourced from DBE (2021, p. 195)



Q1	Data Handling
Q2	Data Handling
Q3	Analytical Geometry
Q4	Analytical Geometry
Q5	Trigonometry
Q6	Trigonometry
Q7	Trigonometry
Q8	Euclidean Geometry
Q9	Euclidean Geometry
Q10	Euclidean Geometry

Figure 1.3. Average percentage performance per question for Paper 2

Sourced from DBE (2020, p. 192)

From the 2020 NSC diagnostic report on the overview of learner performance, statistics reveal that of all tested concepts in Paper 2, learners grappled most with the understanding of trigonometric function graphs. The bar graph reflects an average performance of 28% in Question 5 of Paper 2 - See Figure 1.3. In Figure 1.2, the average percentage performance for Questions 5, 6, and 7 of Paper 2 were 37%, 30%, and 34% respectively. Lastly, Figure 1.1 which shows an extract from the 2022 NSC diagnostic report, revealed that Question 7, which tested candidates' understanding of the trigonometric function concept, had a 36% recorded average performance. The Department of Basic Education (DBE) (2011, p. 56) described a performance range of 30-39% as “elementary achievement or Level 2” out of 7 levels in its codes and percentages for recording and reporting. Given the previous statement, it can be deduced that trigonometry topics, relative to the other topics in Paper 2, were poorly attempted by the learners that wrote the Paper 2 examination in 2019. In the 2019 NSC report, a trigonometric function graph was tested in Question 6 while in the 2020 NSC report, a trigonometric function graph was tested in Question 5. From Figure 1, the trigonometric function graph's average performance was 30% in the year 2019, which dropped to 28% in 2020. It, therefore, means that there has been no improvement in learners' performances in trigonometric function graphs in this period of years.

Thus, if more research is focused on the concept of trigonometric function graphs, it would help teachers to improve the pedagogy of teaching trigonometric functions. Also, curriculum experts would be informed on trigonometric aspects to consider when the curriculum is reviewed. Lastly, the DBE may provide more resources and training for in-service teachers on how to improve trigonometric function graphs delivery to learners in the classroom, which will ultimately improve matric results and encourage further studies in the field of mathematics, science, technology and engineering.

1.5.1 The South African School Curriculum

The school years are divided into two bands in the South African Education system, called the General Education and Training (GET) band (Grades 0-9) and the Further Education and Training (FET) band (Grades 10-12). Mathematics in the FET band covers ten main content areas contributing to acquiring specific learner knowledge and skills. Progression in terms of concepts and skills across the FET grades for each content area is inter-connected. The South African curriculum is referred to as the Curriculum Assessment Policy Statement (CAPS). In the South African CAPS mathematics curriculum, Grade 11 learners are expected to "work on the relationships between variables in terms of numerical, graphical, verbal, and symbolic representations" (DBE, 2011, p. 13). Moreso outlines that Grade 11 learners are expected to be able to do the following:

- "Sketch basic graphs defined by y = $\sin\theta$, y = $\cos\theta$ and y = $\tan\theta$ for $\theta \epsilon$ [-360^0, 360^0] by doing a point-to-point plotting.
- Investigate the effect of the parameter k on the graphs of the functions defined by y = $\sin kx$, y = $\cos kx$, and y = $\tan kx$.
- Investigate the effect of the parameter p on the graphs of the functions defined by y = $\sin(x + p)$, y = $\cos(x + p)$, and y = $\tan(x + p)$.
- Draw sketch graphs defined by: y =$a \sin k(x + p)$, y = $a \cos k(x + p)$, and y = $a \tan k(x + p)$ at most two parameters at a time" (DBE, 2011, p.32).

Trigonometry is introduced in the FET mathematics curriculum in Grade 10 with a weighting of about 33% in the November examination paper, 35% in the Grade 11 November examination paper, and about 30% of 150 marks in the Grade 12 National Certificate Examination (DBE, 2011). From the aforementioned, it is important to stress the two primary purposes of weighting the trigonometric content area on acquiring the learning objectives. Firstly, the weighting guides the time needed to address the content area adequately. In addition, the weighting guides the content spread in the examination, especially the end-of-year summative assessment (DBE, 2011).

1.6 Methodological Considerations

This study used an interpretive qualitative research approach within a case study design to answer the above research questions (Cohen & Arieli, 2011). "Qualitative data analysis involves organizing, accounting for and explaining the data; in short, making sense of data in terms of the participants' definitions of the situation, noting patterns, themes, categories and regularities" (Cohen & Arieli, 2011, p. 537). A case study is an empirical inquiry that investigates a contemporary phenomenon within a real-life context by addressing the "how" or "why" questions concerning the phenomenon of interest (Yin, 2018). Qualitative research and a descriptive case study require multiple ways of collecting data. According to Joubish et al. Haider (2011), qualitative research has no single reality, but each reality is interpreted by the varying world views held by different individuals. Based on this, the data of my study were collected from administered trigonometric achievement tests and semi-structured individual interviews. The participants were the Grade 11 learners purposively selected from three public schools in Tshwane South, District 4. The test focused on the type of errors learners made in trigonometric function graphs and was used to identify the errors learners made. The interview focused on the reasons why learners made those errors. The interview was also used to validate learners' errors and clarify why these errors were made. A deductive data analysis strategy was used to analyse the data generated from this research process.

1.7 Clarification of Concepts

It is important for me to inform the reader of my working definitions for specific terms or concepts used in this study. These are given in Table 1.1.

Table 1.1. List of concept clarifications

TERM/CONCEPT	MEANING/CLARIFICATION
Error	An error could result from a misconception or be caused by several other factors, such as "carelessness, problems in reading or interpreting a question, and lack of 'number' knowledge" (Ryan & Williams, 2007). An error could, therefore, be "a mistake, slip, blunder, or inaccuracy and a deviation from accuracy" (Luneta & Makonye, 2010).
Error Analysis	Error analysis is a method often used to note the cause of student errors when they make frequent mistakes. It is a process of looking through a student's work and then finding patterns of errors (Radatz, 1979).
FET	FET refers to education and training provided from Grade 10 to Grade 12, the last three years of schooling in SA. This FET band also includes career-oriented education and training offered in technical, community, and private colleges.
Learners	Learners refer to children from Grade 1- Grade 12, while students refer to people studying at tertiary institutions.
Misconception	A misconception is "the product of a lack of understanding or, in most cases, the misuse of a 'rule' or mathematical generalisation, resulting in incorrect answers" (Ryan & Williams, 2007). Misconceptions are systematic errors that are symptomatic of a faulty line of thinking. Misconceptions could be referred to as repeating or recurring errors (Green et al., 2008; Nesher, 1987; Riccomini, 2005).
Mathematics Paper 1	This paper represents one of the two divisions of the South African FET mathematics curriculum. The contents in Paper 1 are "Algebra and Equations (Inequalities included), patterns and sequences; finance, growth and decay; Functions and Graphs; Differential Calculus; and probability. Differential Calculus is the only topic not done in Grades 10 and 11" (DBE, 2014, p. 121).
Mathematics Paper 2	This paper represents the second division of the South African FET mathematics curriculum. The mathematics content in Paper 2 includes Statistics; Analytical Geometry; Trigonometry; Euclidean Geometry; and Measurement.
Trigonometry	Trigonometry is a branch of mathematics that describes relations between sides of triangles and angles between the sides.

1.8 The Possible Contribution of This Study

The DBE and mathematics teachers could use this study's findings to understand the errors and misconceptions Grade 11 mathematics learners make in trigonometric function graphs better. Taking cognisance of these errors may assist teachers and education experts to re-think their teaching approaches. Furthermore, analysing learners' errors in trigonometric function graphs will enable learners to get suitable interventions from their teachers. An intervention, such as Cognitively Guided Instruction (CGI), where teachers build on learners' knowledge and skills, will help to enhance learners' understanding of trigonometry and will consequently improve learners' general performance in mathematics. I will raise awareness of the extent of common errors and misconceptions in trigonometric function graphs identified through my study. In conclusion, the awareness of learners' errors may provide relevant information to educational practitioners on the types and possible causes of learners' errors and misconceptions in sketching and analysing trigonometric graphs. This awareness and knowledge may then be used to inform teachers' lesson preparation and teaching, all hoping to enhance learners' understanding and improve their performances.

1.9 Structure of the Dissertation

Chapter 1 outlines the background of the research. This chapter is divided into sections which uniquely treat aspects that deal with the background of the study, providing a general picture of the study. The sections provide an overview of the investigation of research done on mathematics performance, past reports on the NSC Examination in South Africa as a way to put the problem into perspective. This chapter provided a setting for the study whereby the research problem, the purpose of the study, research questions and the significance of the study as it applies to the South African education system and the community of mathematics practice in general. In addition, the research aims and objectives and concept clarifications were elucidated in this chapter.

Chapter 2 is delineated to examine a range of literature and theories that underpin my study. This chapter begins with an introduction and extends to establish the connections between the objectives of my study and its theoretical basis. Both local and international literature were explored to determine how findings from other studies related to my study, which investigated errors and misconceptions that Grade 11 learners make in trigonometric function graphs. It further revealed possible and plausible reasons for errors that learners make in their responses to mathematical problems, particularly in trigonometric function graphs.

Chapter 3 details the research methodology adopted for this study. I use a qualitative approach and interpretive paradigm to analyse and interpret the data generated from this study and an exploratory case study as my research design. Data were collected by the administration of a test and the scheduling of interviews. Other aspects, such as the study's trustworthiness and ethical considerations, are also presented.

Chapter 4 presented the analysis and findings of the data collected in this research study using a test instrument and interview schedules. The researcher began by categorising the test items into four categories and obtained some useful statistics from this process. After that, the information revealed through this process facilitates the generation of categories for the types of errors through data analysis. The findings from the test analysis and interview dialogue are analysed and revealed.

Chapter 5 elucidates the research findings and discussions obtained through content analysis of test results and interviews conducted with the Grade 11 learners. The findings are presented and discussed based on the literature and the conceptual framework.

CHAPTER TWO: LITERATURE REVIEW AND CONCEPTUAL FRAMEWORK

2.1 Introduction

This chapter reviews previous studies on trigonometric function graphs and thereafter present the conceptual framework of the study. The chapter covers relevant material on the nature of mathematics, errors and misconceptions in mathematics, the importance of errors and misconceptions, error analysis and instruction, mathematics teaching and learning, mathematical conceptual understanding, trigonometry as subject, trigonometric function graphs in CAPS, errors and misconceptions in trigonometric functions, learners' errors in trigonometric function graphs, challenges in teaching and learning trigonometric function graphs, gaps in reviewed literature, and lastly, the conceptual framework that guides this study.

2.2 Errors and Misconceptions in Mathematics

Errors and misconceptions as constructs in the mathematics education field have been of growing interest for several years (Brodie, 2014; Chege, 2015; Gore, 2016; Luneta & Makonye, 2010; Mulungye et al., 2016; Shalem et al., 2014). Even though there is no common ground amongst researchers on whether errors and misconceptions are inseparable/separable terms, it is important to establish the difference between the two constructs. Generally stated, a mathematical error is an incorrect or wrong calculation in a given mathematical task which arises when learners use wrong procedures and/or inappropriate conceptions to present mathematical solutions. Spooner (2012) averred that mathematical errors might be caused by misconceptions, carelessness, problems in reading or misinterpretation of problems, and lack of knowledge about numbers or subject-specific theory. Misconceptions, as part of mathematical errors, is the result of lack of understanding, misapplication of "rules" or over-generalization (Spooner, 2012).

There are various general views and opinions about errors, so there are various views in categorising errors. For example, Luneta and Makonye (2010) described errors and misconceptions as two different constructs, where an error refers to a mistake, slip, blunder, or inaccuracy and a deviation from accuracy. Misconceptions, on the other hand, are systematic errors that are symptomatic of a faulty line of thinking that could be wrong answers or mistakes which are regular, planned, and repeated again and again. For instance, while learners are constructing concepts, they sometimes construct "incomplete, immature, alternative and transitional concepts" (Makonye, 2013, p. 47). Similar to Luneta and Makonye (2010), but using other terms, Olivier (1989) distinguished between two types of mistakes learners make, called "slips" and "bugs". The author described slips (careless errors) as random errors in declarative lack of procedural knowledge, which do not indicate systematic misconceptions or conceptual problems. Conversely, bugs are errors learners make that result from their lack of conceptual understanding; for example, learners make mathematical errors on the principles and ideas connected to the mathematical problem, relationship among numbers, characteristics, and properties of shapes.

Gagatsis and Kyriakides (2000) characterised errors as a natural and global phenomenon, which are made by learners, students or anyone regardless of age, country or ability. This nature of errors explains their pervasiveness and persistence, such that, regardless of the teaching method teachers use, errors will always arise in the mathematics learning process (Chauraya & Mashingaidze, 2017). More so, Allsopp et al. (2007), Brodie (2014), and Nesher's (1987) opinionated view is that errors are systematic, persistent and pervasive mistakes made by learners across a range of contexts. In fact, errors are reasoned and reasonable for learners and appear systematically in learners' work. Also, Oktaviani (2017) believed that most learners' errors result from misconceptions. To further elaborate, Ryan and Williams (2007) believed that an error is principally formed within the surface level of knowledge. Kshetree (2021) averred that learning errors are committed through learners' faulty thought processes and not due to carelessness. In the words of Kshetree et al. (2021), "If errors are committed, it is said that they arise because the children are thinking and not because they are careless" (Kshetree et al., 2021, p. 8).

It is necessary to explore a few more views about misconceptions, as misconceptions lead to many of the learners' errors made. Vermeulen and Meyer (2017) regarded misconceptions as an intelligent effort made by learners which is based on learners' inaccurate or partial prior experience. Scholars say this perspective on misunderstandings may contribute to constructivist theories considering mistakes as part of misconceptions or conceptual structures due to underlying misconceptions. In this way, misunderstandings may lead to mistakes, making them part of the same misunderstanding construct (Bush, 2011; Tendere & Mutambara, 2020). Skemp (1987) stated that misconceptions might occur when information is not incorporated into an appropriate schema resulting from instrumental understanding during the learning process. Ojose (2015) and Smith et al. (1993) defined a misconception as a misapplication of a rule, or an over- or under-generalisation of rules. In this study, a "misconception" was viewed as a type of error in addition to other types of errors.

Relatedly to some of the views mentioned and informative for this study, Brown and Skow (2016) and Lai (2012) classified errors into factual, procedural, or conceptual errors. Procedural errors are general mistakes, slips and blunders, and factual errors resulting from a lack of basic mathematical facts, a misinterpretation of signs or digits and lack of formulae knowledge. Conversely, conceptual errors are errors in understanding the concepts used to solve a problem.

Figure 2.1 depicts a tree diagram, which shows the divisions and sub-divisions of errors by researchers.

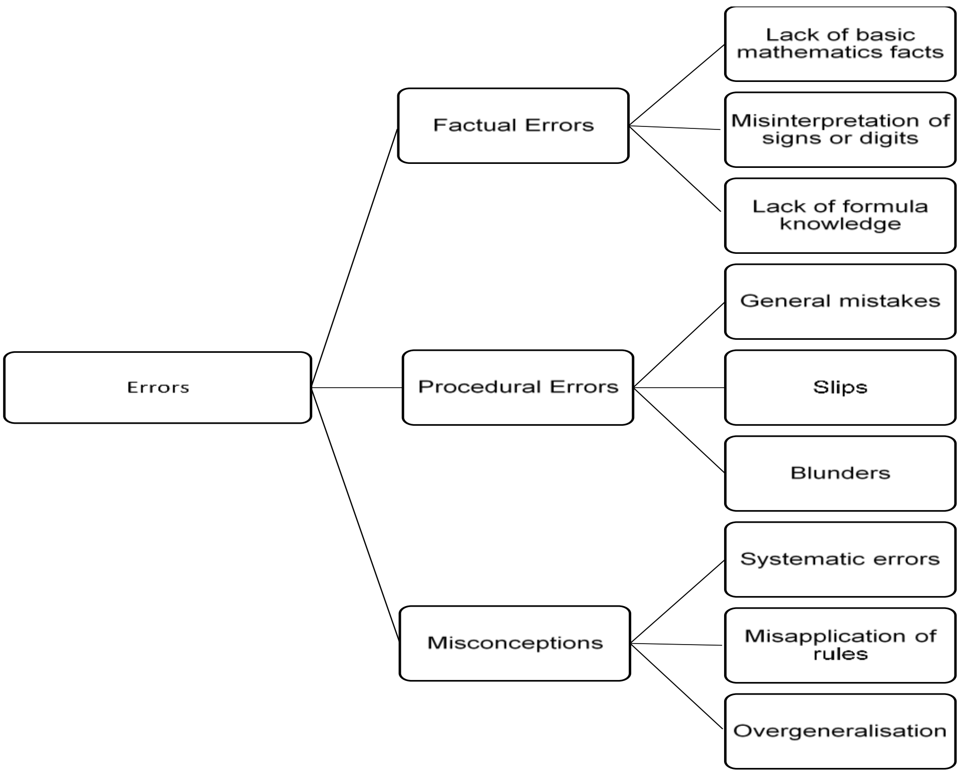


Figure 2.1. The divisions and sub-divisions of errors and misconceptions

Sourced from Brown and Skow (2016), Olivier (1989) and Smith et al. (1993)

The tree diagram above details the three divisions of errors and their sub-divisions, as it has been touched on earlier in researchers' work.

2.2.1 The Usefulness of Errors and Misconceptions in Teaching Mathematics

Hill et al. (2008) and McGuire (2013) argued that the ability of teachers to remediate learners' persistent errors and misconceptions underlies Shulman's (2015) definition of pedagogical content knowledge. Shulman (2015) re-examined his theory of teachers' pedagogic content knowledge (PCK). He suggested that "a teacher's knowledge of students' levels of comprehension contributes to an understanding of the mathematics learning process and an awareness of the mathematical concepts that students struggle to comprehend" (Shulman, 2015, p. 23).

Errors are reasonable; errors made by learners in their seatwork, homework, and other assessment forms are normal and necessary for learning mathematics (Hill et al., 2008; McGuire, 2013). Errors made by students provide teachers with insight into their students' current mathematical practices and, as a result, offer opportunities for future development in their mathematical practices (Brodie, 2014; Chege, 2015; Gore, 2016). Errors made by learners are therefore not a problem but allow all education stakeholders to reflect, learn and re-strategise.

Additionally, feedback from the errors learners make allows them to probe their current ways of thinking. Consequently, learners' continual reconstruction of their existing knowledge may produce "the observed intermediate states of understanding and eventual mastery of a domain" (Smith et al., 1993, p.117). In addition, they argued that the knowledge system framework makes it easier to decipher how novice conceptions can play major roles in developing expertise, despite their flaws and mistakes. I, therefore, argue that if the observed intermediate states of understanding are weak, research work must address the flawed concept so that pedagogical action is taken to enable the mastery of the content.

Furthermore, focusing on learners' errors helps teachers adjust how they engage with the content and the learners in the classroom situation, as well as revise their teaching strategy by using, for example, Cognitively Guided Instruction (CGI) as an effective instructional strategy. This teaching strategy may be suggested to teachers to address learners' errors and eventually enhance their understanding and performance. According to Carpenter and Franke (2004), CGI is particularly useful to guide teachers in adapting their instructional methodologies to include the use of their learners' understanding. Carpenter et al. (2000) advanced the employment of a CGI teacher development programme to promote learners' mathematical thinking. Also, Carpenter et al. (2000) added that teachers' beliefs, knowledge and practices influence their understanding of learners' thinking. So, since teaching is not just about curriculum coverage but about learning, it is recommended that teachers' notions about what learners know, how they learn, and the outcome of learners' achievement through the knowledge about their mathematical thought processes will improve learning outcomes. The construction of conceptual maps of the development of children's mathematical thinking in certain content domains through the awareness of errors and misconceptions may also help promote effective learning.

Also, Metcalfe (2017) argued that corrective feedback, including analysis of the reasoning leading up to the mistake, is an essential pedagogical effort needed to improve learners' overall performance in mathematics. Errors educate instructors and stimulate active, inquisitive, and creative learning. The researcher highlighted that if the aim of mistake analysis is optimum performance in high-stakes scenarios, it may be advantageous to accept and even encourage students to make and fix errors in low-stakes learning settings rather than avoiding them at all costs.

2.2.2 Error Analysis and Instruction

Error analysis relates to examining errors in learners' work to find explanations for the learners' reasoning for making the errors (Radatz, 1979). Error analysis is otherwise referred to as error pattern analysis, a multi-faceted process that teachers and researchers undertake. Additionally, Riccomini (2016) stressed that the overall purpose of error analysis is to improve student learning using a more effective instructional strategy by noting error patterns in learners' work. Riccomini (2016) reiterated that identified error patterns in learners' work comprise a database for determining what content to teach and the strategies that are needed to teach the content.

Moreover, error analysis can be done using erroneous examples (Jaeger et al., 2020). The authors argued that erroneous examples, when given to learners as tasks, are valuable for learning because the erroneous examples enhance learners' ability to identify errors and misconceptions. McLaren et al. (2015) reported that error analysis entails using erroneous examples as an instructional strategy to advance learning for learners in mathematics. This pedagogical process requires the teacher to present learners with erroneous examples with the intent for learners to identify some wrong steps in the examples presented. Subsequently, the teachers would be able to carry out error analysis based on the errors made by learners when providing suitable reasons for the right solutions in the erroneous examples.

A study conducted by Rushton (2014) in Northern Utah revealed that mathematical knowledge was significantly increased when error analysis was used as a teaching strategy. For example, in the treatment group, a mean of 9.56 was recorded, while the control group had a recorded mean of 8.23. The research findings of this study suggested that error analysis may promote richer learning experiences, which births a deeper understanding of the mathematical concept. Besides this, using erroneous examples in the mathematics classroom effectively deals with misconceptions because it provides opportunities for learners to reflect on their errors. Lastly, error analysis fosters learning and longer retention of content knowledge.

A research study conducted by Centillas and Larisma (2016) in Leyte, Philippines, showed that the first-year students of Palompon Institute of Technology had problems with learning previous and new concepts in trigonometry. The findings of this study elicit higher information-processing errors. The scholars posit that the findings of their study can help many educators develop instructional materials, worksheets, and modules by using students' misconceptions. On the other hand, the records of errors could serve as a useful database for administrators.

Furthermore, evaluating learners' work to determine an appropriate instructional focus to correct errors is one of the main objectives of remedial or corrective education for all learners (Fuchs et al., 1994; Salvia & Ysseldyke, 2004). These scholars argued that the identification and analysis of arithmetic errors could improve instructional planning, ultimately improving learners' performance. Furthermore, "analysing learners' errors may reveal the erroneous problem-solving process and thus provide information on the understanding of and the attitudes towards mathematical problems" (Sarwadi & Shahrill, 2014).

2.3 Mathematical conceptual understanding

Mathematics syllabi and curriculum documents in most countries place great emphasis on building learners' conceptual understanding (Bernard, Akbar, Ansaris & Filiestianto, 2019; Morsanyi, Prado, & Richland, 2018; Nabie, Akayuure, Ibrahim-Bariham & Sofo, 2018). Similarly, the National Council of Teachers of Mathematics (NCTM) (2000) recommended in its document called Principles and Standards for School Mathematics, that students learn mathematics with understanding by actively building new knowledge from experience and prior knowledge. The NCTM (2000) further advocates the uttermost importance for the mathematics community to build good and rich conceptual understanding for learners in mathematics.

In accordance with these studies on the need for enhancement in building mathematical conceptual understanding, South Africa's DBE in its document titled 'Mathematics Teaching and Learning Framework', justifies the significance of conceptual understanding as:

> "Conceptual understanding allows learners to see mathematics as a connected web of concepts. They should be able to explain the relationships between concepts and make links between concepts and related procedures. Conceptual knowledge enables learners to apply ideas and justify their thinking". (DBE, 2018, p. 9)

Further to the preceding excerpt on mathematical, conceptual understanding, Figure 2.2 below shows the framework model of a mathematics learner-centred classroom which will guide education professionals in consciously aiming at building conceptual understanding, among other components. The framework model explains the characteristics of the learner-centred classroom that helps in achieving the primary goal of teaching and learning objectives.

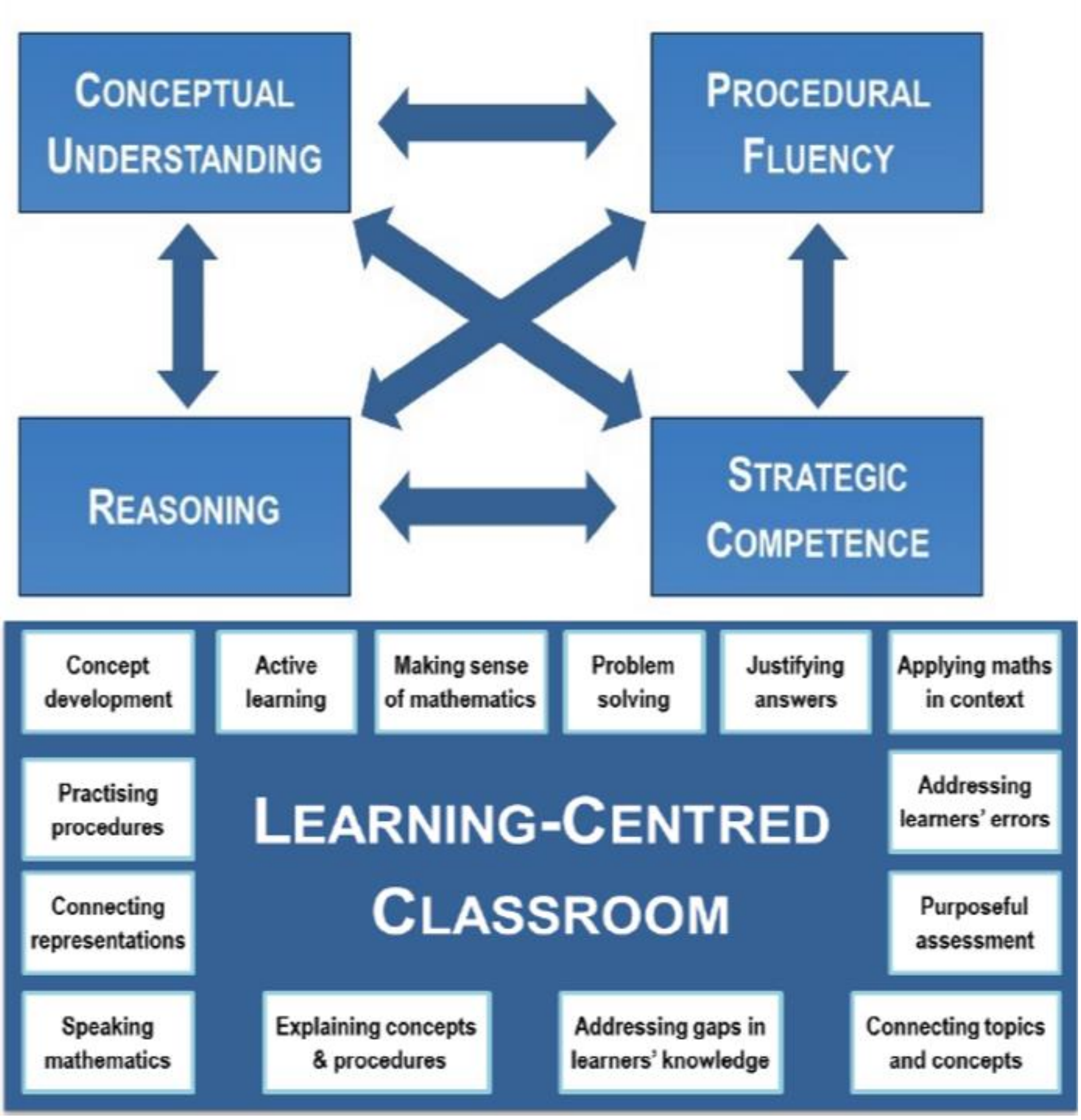


Figure 2.2. Framework model of mathematics teaching and learning (DBE, 2018, p.11)

The mathematics framework model aims at providing a multi-dimensional approach to transform the teaching and learning of mathematics in South Africa by guiding the mathematics education body. This framework model was adapted from the work of Kilpatrick (2001). In the document, four strands for mathematical proficiency was identified: conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning. Conceptual understanding is referred to as the knowledge of concepts, relations and patterns which reflects in learners' interrelated and functional knowledge of mathematical concepts and ideas. Learners with conceptual understanding are able to compare, relate, infer and engage in fundamental higher order thinking. Procedural fluency could be referred to as the ability to have the necessary skills to carry out mathematical actions in the correct sequence. Learners with procedural fluency are able to recognise symbols and use rules to do mathematical tasks. Strategic competence refers to the ability of learners to make sensible decisions on the strategies to use in solving mathematical problems. Adaptive reasoning provides learners with the capacity for logical thought, reflection, explanation and justification. It avails learners multiple and different opportunities to develop their mathematical reasoning skills. These components interact with each other to foster educational gains in the mathematics education community.

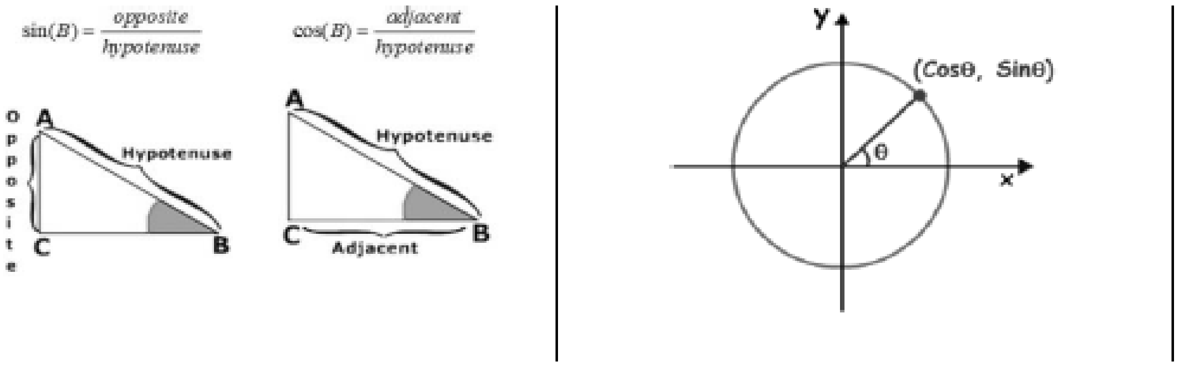
The box at the bottom of the framework model indicates the components of a learner-centred classroom. Taking this model into consideration, the driving forces for this research study are: addressing learners' errors, addressing gaps in learners' knowledge, connecting topics and concepts, purposeful assessment, connecting representations, making sense of mathematics and speaking mathematics. Dewi, Waluya, and Firmasari (2020) concurred that adaptive reasoning abilities, procedural fluency, logical thinking in choosing the right concepts and situations are the gateway to students' success. Possessing excellent adaptive reasoning and strategic competence imply that students will show success in learning which makes them mathematically proficient. I, therefore, argue that if learners lack the four basic skills: conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning as earlier mentioned in this model, there is a high likelihood of errors and misconceptions in the assessment of their learning.

2.4 Trigonometry

Trigonometry describes relations between sides of triangles and angles between the sides (Ogbonnaya & Mogari, 2014). The scholars argued that trigonometric topics are a prerequisite to other mathematics topics in many career courses in higher education across the globe and in many career practices. Examples of topics of application in higher education are Laplace transformation, matrices, complex numbers, differentiation, differential equations, integration, Fourier series, analytical geometry, and systems of equations (Bourne & Weaver, 2018; Ferrao, 2018). Furthermore, the researchers assert that trigonometry is also useful in many career practices such as medicine, surveying, astronomy, architecture, music production, engineering, and electronics. Based on the CAPS document (DBE, 2011), there are three contexts for the learning of trigonometry:

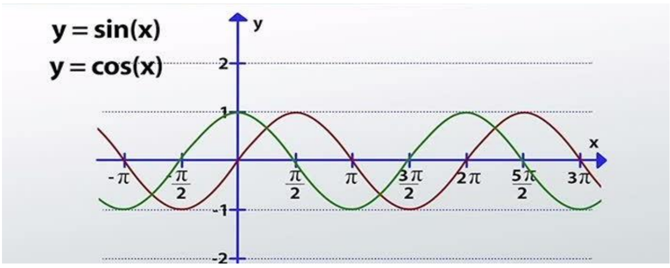
- Triangle trigonometry: learning trigonometry as ratios of a right-angle triangle.
- Unit circle trigonometry: learning trigonometry by describing coordinates of points based on rotational angles on the unit circle.
- Trigonometric function graphs: learning trigonometric functions as a domain of real numbers (Weber, 2005). Trigonometric functions are described as the “domain of angles in degrees” in CAPS. According to CAPS, “trigonometric functions as one of the most important topics in the secondary school curriculum, require the integration of algebraic, geometric, and graphical representations” (CAPS, 2011).

Abar (2013) and Demir and Heck (2013) also identified the three divisions of learning trigonometry as the ratio of a triangle, unit circle, and trigonometric functions. The scholars confirmed that learning trigonometry as a unit circle bridges the other two contexts of learning trigonometry. Therefore, the three contexts of trigonometric functions are inseparable. These contexts, as described by Demir and Heck (2013), are shown in the figures below:



(a) Triangle trigonometry

(b) Cartesian context



(c) Sine and Cosine function

Figure 2.3. The three distinct contexts of introducing sine and cosine functions

Sourced from Demir and Heck (2013, p.120)

As shown in Figure 2.3 (a), Demir and Heck (2013) opined that the triangle context applies in plane geometry, where sine and cosine are defined as ratios of sides in a right-angled triangle. Secondly, Figure 2.3 (b), the Cartesian context, relates to analytical geometry, which shows the connections between the coordinates of the intersection of a ray through the unit circle. Lastly, in Figure 2.3 (c), the scholars averred that sine and cosine are represented as functions through tables and graphs where properties of functions such as domain, range, period, and amplitudes are considered.

In my view, the latter three contexts identified by Demir and Heck (2013) are similar to the ones I have mentioned in my preceding paragraphs in that both sub-divisions focus on the same contents of trigonometry irrespective of the constituent contexts in which trigonometry is presented.

2.4.1 Trigonometric Function Graphs in CAPS (Grades 10 -12)

The South African curriculum is structured so that the function concept has many aspects, for example, linear, quadratic, hyperbolic, exponential, logarithmic, and trigonometric functions. My study was based on trigonometric function graphs (a graphical representation of trigonometric concepts). The trigonometric function graph visibly explores the properties of trigonometry, which combine the learned previous concepts like algebra, functions, and transformation geometry to give trigonometric function graphs. Trigonometric functions imply a correspondence between two sets, namely the domain and the range, where every element in the domain has one element in the range to which it is mapped. This concept is what translates into a graphical presentation with its properties embedded. Trigonometric function graphs integrate the prior knowledge of the properties of the trigonometric functions in their interpretations. Properties such as domain, range, period, the transformation of the graphs, and determining the equations are considered.

Trigonometric functions offer learners a wide range of problem-solving and visual representation opportunities. The trigonometric function concept communicates mathematical ideas through pictorial, diagrammatic, and symbolic representations (Murphy, 2013; Tuna, 2013). The researchers concurred that the visual representation of mathematical concepts is one of the different modes of representing mathematical instruction. A graphical representation of the concept in mathematics can provide a pictorial representation that can help learners better understand. Also, creating and interpreting visual representations during mathematical instruction is a crucial skill in learning mathematics as it stimulates learners' interest and motivates their learning (Mayer, 2014). As good as this may seem, visual representations of mathematical instruction could also be an obstacle that may negatively impact learning (Radatz, 1979). The scholars affirmed that the different forms of iconic instructions, diagrams and visualizations of mathematical activities place high demands on the learners' spatial competence and capacity for visual discrimination.

The function concept has a significant value in mathematics as its graphical presentation enhances learners' reasoning and communication ability. However, the difficulties learners face in grappling with sketching and interpreting the function concept challenge the curriculum objectives. Table 2.1 below gives the specification of contents to show the progression in the concept of the function in the FET phase of the South African mathematics curriculum document (DBE, 2011, p. 12)**.** The table displays the strong connections between the trigonometric function graphs and linear, quadratic, hyperbolic, and exponential functions. It shows the inseparable connections between all the different representations of functions.

Table 2.1. Overview of the topics covered under functions

Grade 10	Grade 11	Grade 12
"Work with relationships between variables in terms of numerical, graphical, verbal, and symbolic representations of functions and convert flexibly between representations (tables, graphs, words, and formulae), include linear and some quadratic polynomial functions, exponential functions and some rational functions and trigonometric functions" (DBE, 2011, p.12)	"Extend Grade 10 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between representations (tables, graphs, words and formulae), include linear and some quadratic polynomial functions, exponential functions and some rational functions and trigonometric functions" (DBE, 2011, p. 12).	"Introduce a more formal definition of a function and extend Grade 11 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between representations (tables, graphs, words and formulae), include linear quadratic and some cubic polynomial functions, exponential and logarithmic functions and some rational functions" (DBE, 2011, p. 13).
"Generate as many graphs as necessary initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generate the effect of the parameter which results in a vertical shift and that which result in vertical stretch and/or a reflection about the x-axis" (DBE, 2011, p. 12).	"Generate as many graphs as necessary initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generate the effect of the parameter which results in a horizontal shift and that which results in horizontal stretch and/or a reflection about the y-axis" (DBE, 2011, p. 12)	"The inverses of prescribed functions and be aware of the fact that, in the case of a many-to-one functions, the domain has to be restricted if the inverse is to be a function" (DBE, p.13)

It is evident from the above table, that the secondary school curriculum (FET band) in South Africa is planned so that a broad family of functions is covered. It is particularly prescribed that learners in this phase fully understand all functions and their graphs. I argue that conceptual understanding of other connecting functions needs to be emphasised to aid better application in trigonometric function graphs.

2.4.2 Errors and Misconceptions in Trigonometric Functions

Research has shown that learning complexity is associated with understanding trigonometric functions (DBE, 2020; Fi, 2003; Malambo, 2015; Ogbonnaya & Mogari, 2014). Moreover, the NSC Diagnostic Report of 2020 reported that learners confused amplitude with period and did not understand the difference between period and domain. According to the diagnostic report, “candidates were unable to read-off values from the graph correctly” (DBE, 2020, p. 200). It was also reported that in the 2020 Mathematics NSC examination, “many candidates solve the equation $f(x-10°)$= $g(x-10°)$ instead of reading off from the graph” (DBE, 2020, p. 200). Thus, I argue that more research needs to be done in Grade 11 to investigate learning errors in trigonometric function graphs so that timely remediation is offered to learners before they sit for their NSC examination.

Mensah (2017) discovered that the errors in learning trigonometry amongst Ghanaian senior high school students were associated with memorisation during learning and manipulations of trigonometric ratios, amongst others. In addition, the author alluded that students did not develop the concepts well enough in their previous grades, thereby making the same mistakes persistently. Constructing conceptual maps of the development of learners’ mathematical thinking in certain content domains in mathematics through the awareness of errors and misconceptions may also help promote effective learning.

2.4.3 Learners’ Errors in Trigonometric Function Graphs

The challenges learners experience and must deal with when learning trigonometric functions include sketching and interpreting sketched graphs. This requirement does not exclude the basic understanding of the function concept and other representations of trigonometric functions (Dubinsky & Wilson, 2013). For example, the algebraic arguments linked with trigonometric function graphs need considerable understanding. Weber (2005, p. 1) asserted that “the initial stages of learning trigonometric functions are fraught with difficulty” and that many secondary school learners are not used to the high level of reasoning expected in the FET band. Orhun (2004) also agreed that serious misconceptions arise when new trigonometry concepts are introduced abstractly. He advised that teachers continuously assess learners’ understanding of new concepts to avoid serious and lasting misconceptions.

Orhun (2010) further argued that students demonstrate difficulties in the multiplication of $(\sin x)(\sin x)$ in a calculus class at university. Furthermore, Chauke (2013) explained that the Grade 12 mathematics learners in the Gauteng province, one of the nine provinces in South Africa, had difficulties with trigonometric functions in the 2012 end-of-year examination. More so, the diagnostic reports in recent years (DBE, 2020, 2021) show that learners’ performance in trigonometric function graphs has been below expectations compared to other trigonometry concepts. Additionally, Chigonga (2016) averred that learners commit errors in solving trigonometric functions, and educators likewise have challenges teaching that topic. For instance, learners choose the wrong quadrants, and learners divide both sides of a function by a variable expression. For example:

Question: Solve $\sin\theta\tan\theta = \sin\theta$ over the interval $[0^0; 360^0]$.

Learner’s response: $\tan\theta - 1 = 0$

$$\tan\theta = 1$$

$$\theta = 45^0$$

Chigongoa added that learners did not check the validity of their solutions and that they also lacked knowledge about the periodicity of trigonometric functions.

Also, from the 2019 NSC Diagnostic Report on common errors and misconceptions on Question 6 of the November NSC mathematics examination (DBE, 2020), it was observed that many candidates could not make a distinction between range and domain in Question 6.1. It was also reported that candidates could not identify graph f correctly, while some candidates wrote down the interval incorrectly. Further, in Question 6.2, it was noted that candidates struggled to read off the critical values flawlessly and that this resulted in a faulty response to the question. Lastly, in Question

6.3, many candidates unintentionally calculated the distance between the two points on the graphs as PQ=0.

The 2020 NSC Diagnostic Report (DBE, 2021) also described the errors and misconceptions committed by candidates that wrote the examination in its report. Candidates also attempted question 5 (NSC, 2020, p.7) without adequately understanding trigonometric function graph concepts. It was outlined in the Diagnostic report that candidates could not tell the difference between period and amplitude, while some candidates erroneously wrote the period of $g\ as\ 1/2$, while other candidates misconceived the period as an interval of $(0°; 360°)$ in Question 5.1. Amplitude was presented as a negative value $(-1/2\ )$ by some of the candidates. More so, in Question 5.3, candidates could not read the information correctly from the graph. Also, in Question 5.4.1, candidates could not use the graphs to determine the values of $x,$ they rather solved the equation. This solving of an equation made them to arrive at $-1/2\ \cos(x-10°) = \sin[(x-10°)+30°],$ which could not be solved further by the candidates. On the other hand, some candidates could not demonstrate their understanding of the effect of p. Finally, Question 5.4.2 was passed by 14% of the candidates, with an indication that many candidates did not attempt this question.

The 2021 NSC Diagnostic Report (DBE, 2022) also revealed the errors and misconceptions of candidates in the national mathematics examination Question 7 (DBE, 2022, p.8). It was noted that candidates could not sketch the required graph in Question 7.1 correctly. The candidates indicated incorrect turning points and *x*-intercepts. In Question 7.2, there was evidence of a lack of understanding of the concept of the period as they were required to write down the period of $f(3x)$. The report gave insights into candidates' misconceptions leading to using the wrong operation in determining the period. For example, some candidates multiplied 360° by 3 instead of dividing 360° by 3. In other cases, candidates took the period as 180º and, after that, divided by 3 to obtain 60º. Furthermore, Question 7.3 was reported to be poorly attempted by candidates despite the familiar nature of the question. More rigour was put into Question 7.3 without candidates considering the marks allotted to the question. Some candidates solved this question! It was also revealed that there was an indication of insufficient understanding of the concept of transformation as depicted in candidates' marked scripts for the 2021 NSC examination. Some candidates found the range of the original function and failed to consider determining the range of the transformed function. From the review of the last three years NSC matric examinations (2019, 2020 and 2021), it can be deduced that trigonometric function graphs were inadequately attempted in the high-stake examination (DBE, 2020, 2021, 2022).

Scholars have reported the difficult nature of trigonometric function graphs. Bohlmann et al. (2017) analysed the mathematical errors made by high-performing candidates writing the National Benchmark Tests (NBTs) between May and November 2012 and 2013. According to these researchers, the NBTs provided a service to higher education institutions concerning selection and placement, assisted curriculum development and assessed the relationships between higher education entry-level requirements and school-level exit outcomes. The researchers argued that assumptions are made by candidates that any sine function, for example: $g(x) = 1 - \sin x$, has the same range as the mother function, $y = \sin x\,; y \in$ [-1; 1]. In the researchers' words, "unless the concept of a range is properly understood, the mistakes will continue" (Bohlmann et al., 2017, p. 7). In addition, the scholars said the trigonometric ratio is poorly expanded later in trigonometric functions; learners remember the ratios but cannot move beyond this context.

Table 2.2 below is an extract from a study on errors made by the students that wrote the NBTs assessment.

Table 2.2. Errors made by students in the 2012 & 2013 NBT assessment

Item ID	**Outline of the mistakes made**	**Percentage of candidates in the upper third making the mistake**
T31	Don't understand the effect of the coefficient in 28%; in 1 test front of x (shrink or stretch)	28%; in 1 test
TG41	Assume sin 2A= 2sinA	24%; in 2 tests
T10	"Range of shifted sine: assume that all sine curves have the same range (i.e.[-1; 1])" (Bohlman et al., 2017, p. 7)	20%-21%; in 2 tests
TG58	"Range of a shifted and stretched sine over a specific domain: ignore that the stretch changes the original	63%; in 1 test

Item ID	Outline of the mistakes made	Percentage of candidates in the upper third making the mistake
	domain, and hence, the range" (Bohlman et al., 2017, p.7).	
F46 TG46	"Left shift means minus"	21%-29%; in 8 tests 35%; in 1 test
F136	"Plus indicates a right shift"	33%-34%; in 4 tests
T116	"Minus indicates a left shift"	21%-25%; in 2 tests
T21 T20 TG18 TG49	Value of trigonometric expression: numerically correct but sign wrong- wrong quadrant used (i.e., only right triangle considered)	20%-26%; in 2 tests 23%-27%; in 2 tests 22% in 1 test 24% in 1 test

Sourced from Bohlmann et al. (2017, p. 7)

The above table revealed that entry-level university students also experienced learning difficulties in trigonometric functions, despite the expected level of competence that should be displayed in their work.

The following are the recommendations on errors students made in trigonometric functions, as pointed out by Bohlmann et al. (2017):

- Generic graphs need to be used for clarification of the properties.
- "If learners understand the graphical meaning of parameter change and the link between algebraic and graphical representations of transformed graphs, they may be less likely to memorise various transformation rules" (Bohlman, 2017, p. 7).

According to these researchers, there were noticeable errors and misconceptions in learners' work across all cadres, from the lower, middle, and higher groups of learner performances. In line with the preceding statement on the complex nature of trigonometric functions, Cetin (2015) affirmed that despite the high level of students' perceptions of trigonometric functions, they were unsuccessful in understanding the conceptual development of the content.

Ogbonnaya and Mogari (2014) suggested that to improve learners' achievement in trigonometry, we must be cognisant of the factors contributing to learners' learning. The researchers stressed that understanding the reasons for learners' poor achievement may be the first step to planning an effective intervention initiative to address the problems of learners' poor achievement.

2.4.4 Challenges in Teaching and Learning of Trigonometric Function Graphs

Learners have challenges with the understanding of trigonometric function graphs as was reflected in the general performance in the 2019 and 2020 NSC examinations (DBE, 2020, 2021). Studies show that trigonometric function graphs should be easy content to understand because they provide learners with visual learning opportunities. However, despite the learning opportunity trigonometric graphs offer through visual representation, it is reported that learners make mistakes in this concept, as reflected in the findings of some researchers in the field of mathematics (Cetin, 2015; DBE, 2020, 2021; Daher, 2020; Fi, 2003; Kamber & Takaci, 2017; Malambo, 2015; Moore, 2009; Nabie et al., 2018; Ogbonnaya & Mogari, 2014; Weber, 2005; Winslow, 2016). Nabie et al. (2018) researched pre-service teachers' responses to a Trigonometric Assessment Test (n = 94), and it was reported that 89,4% of the teachers could not sketch the sine function graph and none could explain why $sin^2\,x + cos^2\,x$= 1. In the words of these scholars, "[i]t is interesting to observe that over 72% of pre-service teachers could not apply their knowledge on the expansion of $\cos(A + B)$ to compute $\cos 120^0$"(p. 12).

This low performance is not in any way different from the work of past and current scholars that show the learning complexity in this mathematics content area. Fahrudin (2019) reported that the performance indicator of students in trigonometric problems at the National examination was 39,68%. The researcher implied that this low performance by learners indicates a low understanding of the concept taught.

Similarly, a study conducted by Malambo (2020) on pre-service mathematics teachers' understanding of the tangent function found that there is teaching and learning difficulty in trigonometry. The research findings showed that only one pre-service teacher (5%) accurately completed the task, 50% of these teachers (11) did not provide graphs, and 45% drew flawed graphs. The researcher stressed that the incompetence of the pre-service teachers in trigonometric concepts suggests mere memorisation of tangent graphs. Students at the tertiary level have little understanding of the concept of the trigonometric function (Kamber & Takaci, 2017; Weber, 2008).

Cetin (2015) further argued that pre-learning experience is a prerequisite for meta-learning experiences. The pre-knowledge of what an angle means is also a prerequisite for defining trigonometric functions. The findings of Cetin's (2015) study showed that students with a high level of perception of trigonometric functions were unsuccessful in understanding the conceptual development of trigonometric functions. In addition, students with a visual image of trigonometric functions could not use their mathematical content knowledge to develop their conceptual understanding.

The researchers stressed that if an assessment is to make sense, it needs to advance learning. To achieve this, teachers need to engage with clear examples of the problems their learners display in assessment tasks to communicate to the learners the underlying mathematics that will promote better understanding. In other words, "assessment for learning" could be used as a possible tool to address learning challenges to inform teaching. For this study, I did an "assessment of learning" of trigonometric function graphs by Grade 11 learners since data were collected after the curriculum coverage in the selected schools.

2.5 Gaps in the Reviewed Literature

The literature review above highlighted the views and works of researchers, curriculum developers, and other stakeholders in mathematics and education in general. Based on the reviewed literature, the views of scholars on errors and misconceptions in trigonometric functions in Grade 11 have not received enough attention. More so, of the research done on trigonometric function graphs, there has been scarce coverage of the tangent functions. Hence, my study covered errors and misconceptions on all aspects of trigonometric functions in Grade 11.

2.6 Conceptual Framework

This research sought to describe learners' errors and misconceptions in trigonometric function graphs based on the noticeable errors in the participants' tests. It involves establishing the nature, type, and causes of errors and misconceptions that highlighted in their responses to the items in the written trigonometric achievement test. The errors and misconceptions were classified according to the different types of errors and misconceptions as described by Brown and Skow (2016), Newman (1977), Oktaviani (2017), and Smith et al. (1993). The possible causes of errors were conceptualised using the scholarly work of Radatz (1979).

This study was conceptualised around the assumption that errors are the overarching concept that could be referred to as careless mistakes that can be easily rectified or systematic and persistent mistakes caused by misunderstandings generated from faulty knowledge acquisition. From the work of Brown and Skow (2016), Newman (1977), Oktaviani (2017), Radatz (1979), and Smith et al. (1993), a conceptual framework was designed to guide the data analysis process of this study. Figure 2.4 below illustrates the conceptual framework that allowed me to analyse the data.

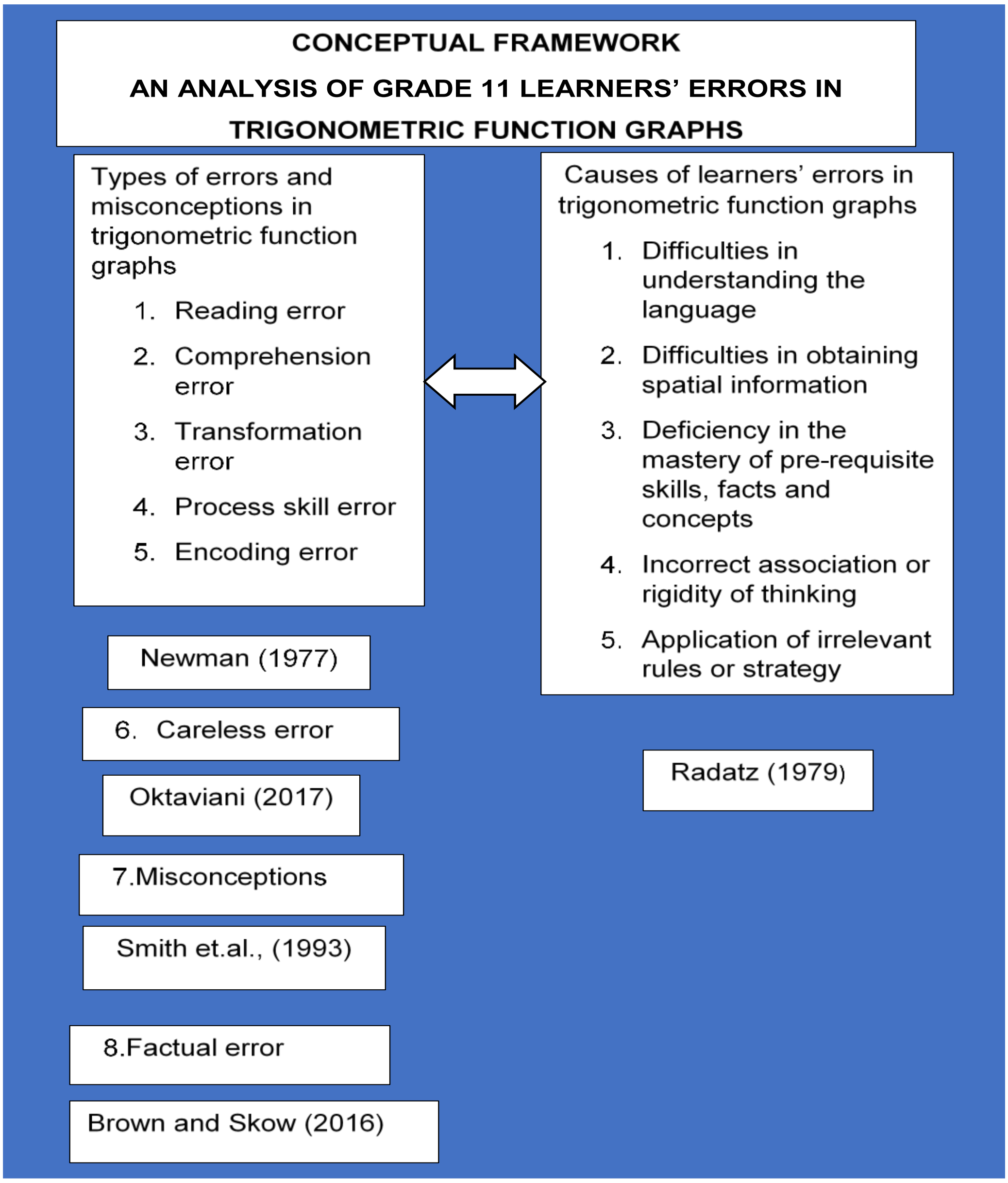


Figure 2.4. Conceptual framework: Types and causes of errors and misconceptions in trigonometric function graphs.

2.6.1 Types of Errors

Newman's error analysis model was utilised to describe learners' errors in trigonometric function graphs and to analyse the type of errors Grade 11 learners make in trigonometric function graphs. These error types have been confirmed and found useful by many researchers (Abdullah et al., 2015; Alhassora et al., 2017; Kristianto & Saputro, 2019; Sumule et al., 2018; Zamzam & Patricia, 2018) in categorising errors in learners' work which arise as a result of conceptions/preconceptions in learners' learning framework and are usually not easy to dispel except by pedagogical interventions.

According to Newman (1977), there are five types of errors:

1. Reading errors, which occur as a result of learners' inability to read the mathematical problems given correctly and identify the sentences and mathematical symbols that are involved.

2. Comprehension errors, which are the inability of learners to understand the given mathematical problems. These errors will reflect in the learner's inability to recognise "what is asked" and "what is given" from the test items (Hadi et al., 2018). The researchers reiterated that this kind of error could easily be identified in learners' work when they can write down important and needed information on the test item. If the written answer of the learner does not represent what was asked, it will not yield the correct answer. Consequent to this, a comprehension error will be made, which indicates the learner's difficulty in understanding the test item. Similarly, Abdullah et al. (2015, p.136) submitted that students commit comprehension errors when they can read but fail to understand "the wants and needs".

3. Transformation errors, which occur "when students understand the meaning of all the words in a problem but cannot compile a mathematical model used to solve a problem" (Wijaya et al., 2020). In support of Newman's error model, Prakitipong and Nakamura (2006); and Karimah et al. (2018) explained that this error occurs when learners can understand the questions but are not able to change the questions to a correct mathematical form or use the appropriate mathematical approach to solve the problem.

4. Process skills errors, where learners will display their ability to read and understand the question correctly and will be able to identify the mathematical operations involved but fail with the calculation procedures (Abdullah et al., 2015). Consistent with the work of the researchers, Hadi et al. (2018) concurred that these errors are noticed in learners' test items, in their inability to implement mathematical processes appropriately, in mathematical calculation and algebraic manipulations. Likewise, Singh et al. (2010) and Setiawan et al. (2018) agreed that learners' process skills errors manifest when they cannot follow all the solution processes involved in the problem through. Accordingly, Shinariko et al. (2020) admitted that processing skills errors occur in a learner's work when the learner does not continue the procedure or steps to complete the steps, so the learner does not find the right solution. In short, a procedural error results from an incorrect performance of steps in a mathematical process.

5. Errors of encoding, which are the inability of students to express the final answer. This error is displayed in learners' work when they cannot define where to conclude solutions. At this stage of the solution, the student fails to write the desired answer correctly.

In addition to the five Newman (1977) categories of errors, Oktaviani (2017) identified careless error as an error that occurs when learners write down wrong numbers and do not follow the right steps to get the expected answer. Matuku (2017) added that careless errors are mistakes learners make when mathematical problems are solved carelessly, even when they can get the required solution. Matuku opined that this type of error might sometimes be due to learners' inattentiveness during the lesson or carefree attitude in learners' responses to questions asked. The researcher said slips are not caused by inherent misunderstanding of concepts; but occur due to memory deficits, impulsivity, or visual-motor integration problems. Additionally, Agustyaningrum et al. (2018) hold that regardless of learners acquiring needful conceptual knowledge, they may still be unable to solve mathematical problems because of the following highlighted reasons:

- Wrong labelling
- Copying a wrong question

- Writing roughly
- Inability to follow the needed procedure
- Using wrong operations/symbol/notation

6. Misconceptions

Sarwadi and Shahrill (2014) opined that misconception results from gaps created in the learning of mathematical concepts. This classification is consistent with the classification of errors done by Luneta (2015) in his research study, entitled "Understanding students' misconceptions". The scholar categorised errors that were due to non-conceptual understanding as conceptual errors. In agreement with the scholars mentioned earlier, Brown and Skow (2016) agreed that "conceptual errors are errors due to misconceptions or a faulty understanding of the underlying principles and ideas connected to the mathematical problem, for example, the relationship among numbers, characteristics, and properties of shapes" (Brown & Skow, p. 183). The researchers also see conceptual errors as bugs. Bugs occur as a result of an incomplete grasp of specific mathematics concepts. Bugs are referred to as misconceptions by the researchers mentioned earlier and are one of the types of errors that guided this study. (Brown & Skow, 2016).

7. Factual errors are errors that occur when learners are unable to identify information contained in a given problem (Muthukrishnan et al., 2019; Oktaviani, 2017; Setiawan, 2020). Additionally, Brown and Skow (2016) opined that factual errors occur when learners lack factual information on basic mathematics facts, formulae knowledge and interpretation of signs or digits.

2.6.2 Causes of Errors

In addition to the description of learners' errors, as posited by Brown and Skow (2017), Newman (1977), Oktaviani (2017), and Smith et al. (1993), this study was further conceptualised around the work of Radatz (1979) on the possible causes of errors committed by learners. Radatz (1979, p. 164) averred that "various causes of errors that cut across mathematical content topics can be identified by examining the mechanisms used in obtaining, processing, retaining, and reproducing the information contained in mathematical tasks”. The scholar identified the causes of errors likely to be found in learners’ work as follows:

1. Errors due to language difficulties

Radatz (1979) averred that learning mathematical concepts, symbols and vocabulary in a foreign language pose problem to learners. He argued that the misunderstanding of the semantics of mathematical text is often the source of learning errors. Apart from Radatz’s (1979) identified error causes, Jha (2012) also argued that problems in reading language fluently and abstract understanding that promote effective reading and understanding of the meaning of problems is one of the hurdles that cause student errors. Khalo et al.’s (2022) research findings revealed a relationship between the language difficulties experienced by learners and the errors they make in financial mathematics.

2. Errors due to difficulties in obtaining spatial information

The iconic representations of mathematical information as visuals has been adopted by teachers as a useful instructional approach in mathematics classrooms in elementary and secondary schools (Radatz, 1979, p.165). The scholar submitted that the visual way of learning mathematics had placed heavy demands on pupils’ spatial abilities and capacity for visual discrimination. He said this visual instructional style is less content-specific and more representation-specific for all school mathematics content. Errors might have been caused using this instructional style. Studies showed that students are often faced with challenges when dealing with visual information, for example, learners cannot differentiate between distinguishing geometric solids and flat shapes (Demitriadou et al., 2020).

3. Errors due to the deficient mastery of pre-requisite skills, facts, and concepts

This error is caused by learners’ insufficient content-and- problem-specific knowledge required for the successful performance of the mathematical activity. Examples are “ignorance of the algorithm required to solve a mathematical problem, inadequate mastery of basic facts, incorrect procedures in applying mathematical techniques, and deficient knowledge of needful concepts and symbols” (Radatz, 1979, p.165). Learners’ lack of pre-requisite skills, facts and concepts, which results in an error, might be due to teachers' approaches to teaching trigonometry which do not allow learners to understand sine and cosine as functions (Altman & Kidron, 2016) Furthermore, Rohimah and Prabawanto (2020) contend that the challenges learners encounter in trigonometric functions are due to the incomprehension of trigonometric concepts.

4. Errors due to incorrect association or rigidity of thinking

According to Cui et al. (2006, p.1) “transfer is the ability to extend what has been learned in one context to new contexts”. Radatz (1979) realised that some errors are committed due to negative transfers by learners. He stressed that learners are often inflexible in decoding and encoding emerging information. This inflexibility implies that when learners encounter similar problems, they experience similar rigidity of thinking. In this source of errors, learners develop cognitive operations and continually use their formed operations even when the task's required mathematical processes change. Some parts of rigid thoughts persist in learners’ minds, preventing information processing.

5. Errors due to the application of irrelevant rules or strategies

This error is caused by the application of irrelevant rules, incorrect algorithms and the application of insufficient mathematical strategies in responses to mathematical tasks. Learners tend to use familiar rules they previously used in a certain content in a new situation. This error may result from a deficiency in the mastery of prerequisite knowledge needed to complete a task (Radatz, 1979, p.166). The survey findings by Khalo et al. (2022) affirmed that the participants (Grade 10 learners) did not apply the formula for simple interest incorrectly when providing the solutions for the content-based questionnaire in the financial mathematics topic.

2.7 Chapter Summary

Chapter 2 is delineated to examine a range of literature and theories that underpin my study. This chapter begins with an introduction and extends to establish the connections between my study’s objectives and its theoretical basis. Both local and international literature were explored on how it relates to my study, which investigated errors and misconceptions that Grade 11 learners make in trigonometric function graphs. It further revealed the types of errors and possible and plausible reasons for errors that learners make in their responses to mathematical problems, particularly in trigonometric function graphs.

CHAPTER THREE: RESEARCH DESIGN AND METHODOLOGY

3.1 Introduction

This chapter describes the research methodology used in this study. Research methods are referred to as the process used to collect and analyse data (Mcmillan & Schumacher, 2010). Thus, this chapter aims to give insight into how the study was carried out to achieve the stated objectives of the study. The research paradigm, paradigmatic assumptions, research approach and design, research site, population and sampling procedures, data collection process and instruments, data analysis and interpretation, study trustworthiness, ethical considerations, and a summary of this chapter are presented.

3.2 Research Paradigm and Assumptions

This study analysed the errors and misconceptions experienced by Grade 11 learners in trigonometric function graphs. To obtain an in-depth understanding of the phenomenon being studied, as a researcher, I needed to put this phenomenon into perspective. I asked myself: What is my view of the world? How do I view understanding? (Cohen et al., 2011). These questions necessitate thinking about the paradigmatic research assumptions regarding ontology, epistemology and methodology.

3.2.1 Research Paradigm

The term "paradigm" has been broadly defined by many scholars in copious related ways. For instance, Denzin and Lincoln (2011) and Bertram and Christiansen (2013) defined paradigms as human constructions, indicating the researcher's philosophical construction of meaning embedded in the data, which depicts individuals' own views. Kivunja and Kuyini (2017) also argued that a research paradigm informs us on how meaning will be constructed from the collected data based on individual experiences. These researchers further elucidated the importance of working within a paradigm, as it entails a set of beliefs and assumptions which influenced and guided me on what I should study, how it should be studied, and how the study's results should be interpreted. In my study, the underpinning paradigm is interpretivism, based on my experience and insight into the phenomenon being studied. For this study, I constructed meaning from a social constructivist perspective. Creswell (2014) contends that social constructivism is often combined with interpretivism. Hence, I found social constructivism a useful paradigm for my study because the participants' knowledge was co-created through social processes. Furthermore, the researcher's view on social constructivism is that individuals seek an understanding of the world in which they live by looking for the complexity of views in varied and multiple meanings generated from the world around them. I constructed meaning of the learners' errors by questioning them on their views and errors made in trigonometric function graphs, as guided by my conceptual framework. Relatedly, Creswell and Poth (2018) stressed the importance of learners' social environment as an agency in constructing meaningful learning. Therefore, considering constructivist epistemology enhanced the rigour of my study. The specific focus of my study was prioritised in my construction of meaning based on my reviewed literature.

3.2.2 Paradigmatic Assumptions

Ontology, epistemology and methodology are key premises that constitute the interpretive frameworks used in qualitative research (Denzin & Lincoln, 2011). Therefore, this section will clarify the two paradigmatic assumptions, namely, the ontological and epistemological assumptions used for this study. The ontological assumption addresses the nature of reality and its characteristics (Creswell, 2007). I used quotes from the interviews and themes in the words of my participants to provide evidence of their perspectives on the errors they made in trigonometric function graphs. The ontological assumption that guided this study is the nominalist assumption. According to Lincoln and Guba (2013, p. 39), ontology deals with the questions, "What is there that can be known?" or, "What is the nature of reality?" This approach supports observing human behaviour and using the participants' words as data. Maree (2012) averred that the nominalist approach is often used where a large amount of qualitative data is categorised. Moreover, the interpretive paradigm enabled me to have many possible interpretations of the learners' reasoning in solving trigonometric problems. It also helped me understand learners' "thought processes" leading to their errors.

Epistemology refers to how knowledge is constructed and what is deemed as acceptable knowledge (Wahyuni, 2012). The epistemological assumption is constructivism as I played a subjective role by focusing on learners' perceptions and interpretations of the phenomena under investigation (Leedy et al., 2019). The self-expressive tone of the interviewees enabled the researcher to know possible sources of challenges learners encountered in trigonometric function graphs, which led to errors and misconceptions in the concept. The methodological framework follows in the following two sections.

3.3 Research Approach and Design

This study used a qualitative research approach within a case study design. My choice of this approach was informed by the need to justify the reasoning for learners' errors through open-ended questions and probes. Kumar (2019) recommended a qualitative research approach as it reveals learners conceived and misconceived meaning of a phenomenon under study using open-ended questions and probes. I considered a case study inquiry the most suitable design to answer my research questions. A case study is an empirical inquiry that investigates a contemporary phenomenon within a real-life context by addressing the "how" or "why" questions concerning the phenomenon of interest (Yin, 2018). The scholar said, in general, the "what" questions may be either used in exploratory research, archival analysis or survey. An exploratory case study is a method of research which answers the questions "why" and "what"; and it is often used in research that does not produce a single set of outcomes (Seaton & Schwier, 2014). Therefore, this is an exploratory case study where the Grade 11 learners' errors were categorised according to the different categories listed in the conceptual framework. The questions that begged for answers were: How can learners' errors in trigonometric function graphs be described? Which types of errors do Grade 11 learners make in trigonometric function graphs? What causes Grade 11 learners to make these errors in trigonometric function graphs?

The case in this study is the Grade 11 learners from Tshwane District 4, Gauteng province. An in-depth study was conducted on 150 learners using answers to test items and interviews to collect the data. Although the researcher used some descriptive quantitative data during the data analysis, the study is qualitative in nature, analysing and describing learners' errors, and is therefore not a mixed method approach. Descriptive statistics such as mean, frequency tables, and line graphs were used to transform and summarise the data generated from the scores obtained from the achievement test. It also gave a visual overview of the data (Bertram &

Christiansen, 2014). A self-constructed trigonometric function graphs achievement test, which was moderated by my Departmental Head and my supervisors, was administered to learners. The conceptual framework guided me in analysing the learners' errors. Semi-structured interviews were done with learners whose performance ranged between low, average, and high in the administered test. The high-performing learners were also interviewed to understand what common misunderstandings learners have, irrespective of their mathematical abilities. These interviews allow for an in-depth discourse about learners' errors made in trigonometric function graphs as it reflects in their responses to the test instrument.

3.4 Research Population Sampling and Site

Alvin (2016) defined a population as all the members who meet the specific criteria needed for a research study. On the other hand, a sample is referred to as a group or a relatively smaller group of people selected from a population for research purposes. Similarly, Brooks et al. (2018) described sampling as selecting participants from the population to participate in a research study. Manterola and Otzen (2017) admitted that sample analysis permits researchers to generalise conclusions on the target population with a high level of certainty, such that a sample is considered representative of the target population. Burrell (2009) described a research site as the "stage" for the social processes being studied. Creswell (2003), in agreement, affirmed that a research site is a natural setting where the researcher goes to conduct research. Thus, all the Grade 11 mathematics learners in South Africa form this study's population. The sample I used for this study is one hundred and fifty Grade 11 mathematics learners in Tshwane South District.

This study adopted a non-probability sampling procedure for data collection. Non-probability sampling is a procedure in which the population unit is not randomly selected. Accordingly, Creswell and Poth (2018) concurred that non-probability sampling is intentionally done to sample a group of people that can best inform the researcher about the research problem under investigation. Convenience and purposive sampling were used as types of non-probability sampling in selecting the schools (research sites) and participants for the study. The selection was convenient because I selected schools in close proximity to my workplace saving time, money, and effort. A sample of 150 Grade 11 mathematics learners were purposively chosen.

Brooks et al. (2018) concurred that qualitative sampling is purposive in nature, focusing on selecting participants with a deep understanding of the phenomenon being studied. I was able to select the participants that gave insight into perceived errors and misconceptions Grade 11 learners make in trigonometric function graphs.

This study's research sites (schools) were in Tshwane District, Gauteng Province. Tshwane District is one of the 15 Districts managed by the Gauteng Department of Education, South Africa. Data were collected from three Section 21 public schools. The South African School Act 84 of 1996 defines Section 21 schools as “schools that are allocated finances by the department and are responsible for ordering stationery, textbooks, paying water and lights accounts and undertaking their own maintenance.” Schools can also decide what subjects they can offer and what sports and other extramural activities the learners can take.

I obtained consent from the Departmental Heads in the participating schools to use the days assigned for weekly tests to administer my test (during lesson periods). Forty marks were allotted to the test (converted to a percentage), and the test was administered to 150 purposively sampled Grade 11 mathematics learners in the selected schools. Based on their test results and assent to participate in the study, I randomly chose learners from the three purposively selected performance categories, namely two learners obtaining top-rated marks (60%-75%), three learners with medium marks (50%- 55%), and five learners with low marks (30%-40%) from each school on the trigonometric function graphs test. I had a relatively greater number of interviewees in the low marks category (30-45%) because I was able to get more information from this group of learners based on the errors committed, while the medium marks category (50-55%) also accounted for some information on errors made. Lastly, I also interviewed some top performers, as I needed to know what errors they committed. However, the top performers (85%+) were excluded from the interview section because their academic performance may not make them a dependable source to generate information for this research study. Miles and Huberman (1994) stressed the importance of a randomly purposive sampling strategy, they said that randomly purposive sampling adds credibility to the sample.

It is important to stress that only learners who gave consent to use their scripts were selected for this analysis. Five learners were purposively interviewed per school. The data findings informed the number of learners being interviewed. Two top-performing, one middle-range performer, and two low performers in the administered test were interviewed in each school. A total number of 15 learners were interviewed. Learners were purposively selected for the test and interview using the criteria in Table 3.1 below.

Table 3.1. Inclusion and exclusion criteria

Criteria	Schools	Learners	Test scripts	Interview
Inclusion	Three Section 21 public schools in Tshwane South	• Grade11 mathematics learners • Male and female learners	Two top-rated marks (60%-75%), three medium marks (50%-55%), and five low marks (30-40%) from each school.	Two top-performing, One middle-range performer, and two low performers in the administered test from each participating school.
Exclusion	Independent Schools	Mathematical literacy learners	Top performing learners (85%+)	Top performing learners (85%+)

3.5 Data Collection Instruments and Process

Qualitative research and an exploratory case study require multiple data collection methods (Creswell, 2014; Yin, 2018). In my study, administering a test and conducting interviews with learners based on the test outcomes were used to achieve this requirement. The data collection process and instruments used are further discussed below.

3.5.1 Data Collection Process

This process involved 150 Grade 11 mathematics learners writing a trigonometric achievement test based on graphs at their respective classes during weekly scheduled mathematics test periods in the sampled schools. I prepared copies of test papers two days before the test was written. These test scripts were given to the departmental heads in the participating schools to give to the mathematics teachers for final administration to learners in their respective mathematics classes (as some schools had more than one mathematics teacher). I requested that the teacher collect the test scripts from the learners just after the test was written. As it is a requirement that the subject heads should moderate written tests, I requested the assistance of the departmental heads in these schools to help moderate the marked scripts after I had finished marking. After the testing process was done, I conducted the interviews with individual learners. The number of learners interviewed was restricted to 15. I recorded the interview sessions with learners and transcribed them verbatim to text data afterwards. To ensure accuracy, the learners and I read all transcripts together. Table 3.2 below shows the data collection timeline of the different times data were collected in the three public schools. Pseudonyms were assigned to the schools and learners.

Table 3.2. Data collection timeline

Date	Data collection Instrument	Periods	School
24/7/2022	Test scripts	3 & 4 (45minutes/period)	A
25/7/2022	Test scripts		B
26/7/2022		3 sessions (30-minute periods)	

Date	Data collection Instrument	Periods	School
26/7/2022	Test scripts	6 & 7 (45-minute periods)	C
1/8/2022 until 15/8/2022	Interview schedule	After school hours (30 minutes/learner)	Schools (A-C) Based on test outcome

3.5.2 Test on Trigonometric Graphs

The testing instrument used for this study was guided by the South African Grade 11 CAPS document. The instrument was patterned around the November National examination questions so that standardised questions were tested on the learners. I developed all the items in the test and made necessary modifications after the feedback from my subject heads and supervisors. The literature review findings (Chapter 2) facilitated the preparation of the test items. The test consisted of four questions (see Appendix C). Each question comprised four to five open-ended questions, totalling f seventeen items. Furthermore, each item was allotted marks consistent with the nationally assigned standard with a presumed matched rigour invested by participants and complexity of the assessed concepts. Some tested questions with similar concepts were repeated in different forms (Questions 1.1, 2.5 & 3.4; 1.3 & 2.3; 2.1 & 4.1). For example:

> 1.1 If the graph of *f* is shifted 45° to the left, write down the equation of the new function.
>
> 2.5 Determine the equation of the new function h, if *h* is the image of the graph *g* shifted 45° to the right.
>
> 3.4 Determine the equation of the new function g, if g is the image of *f* shifted 45° to the left.

My reason for this allocation was to ascertain that errors and misconceptions in these concepts are not due to carelessness but rather a lack of understanding, which births other types of errors. The total mark for the test was 40. Table 3.3 shows the marks distribution across the test items.

Table 3.3. Distribution of marks (Q=Question number, M= Allotted mark)

Q	1.1	1.2	1.3	1.4	2.1	2.2	2.3	2.4	2.5	3.1	3.2	3.3	3.4	4.1	4.2	4.3	4.4
M	2	4	2	2	5	1	2	2	2	1	3	1	2	4	2	3	2

The trigonometric achievement test was used to assess learners' ability to display their understanding of: the effect of the parameter k, *p and q* on the graphs of the functions defined by *y* = sin (*x* + *p*), $\sin kx$, y = cos (*x* - *p*), and *y* = tan kx, concept of range, asymptotes, periodicity and amplitude. The trigonometric achievement test also assessed learners' ability to make deductions from graphs of $f(x)$ and $g(x)$ as well as drawing the graphs of $\cos(x - 30°)$and $g(x) = \sin 2x$ using point-by-point plotting, supported by available technology.

The trigonometric questions were moderated by educational experts viz-a-viz experienced teachers, subject specialists, academic scholars in the Department of Education and the Higher Education Institutions, and my supervisors. These educational personnel, referred to in the previous sentence, assisted in ascertaining the instrument's validity. Any inappropriate items were modified as suggested by the experts and my supervisors to suit the study's purpose. The moderation process helped to ensure that the test content was appropriate for learners' cognitive level as stipulated in the CAPS document outlined in my literature review and that the teachers were also satisfied with the test.

The test duration was one-and-a-half hours; however, learners were allowed extra time to avoid situations where errors generated from learners' scripts may have been caused by insufficient time. The mathematics teachers administered the test in three of the sampled schools when learners had double consecutive periods. Each period in these three schools consists of 45 minutes. I obtained the full support of the departmental heads in all the participating schools in ensuring that the subject teachers viewed test credibility as important by keeping test scripts safe from learners as is normally done in schools' formal assessments.

3.5.3 Semi-Structured Interviews

According to Yin (2013), an interview is one of the best ways to obtain data for a case study. The scholar defined an interview as a two-way dialogue in which the interviewer poses questions to the interviewee (respondent) to collect data and learn about the interviewee's behaviour, opinions, ideas, and views about the phenomenon of interest. Correspondingly, Warren and Kamer (2015) confirmed that an interview is considered a social interaction based on a conversation. Creswell (2014), Merriam and Tisdell (2016), and Punch and Oancea (2014) also identified the common forms used in conducting interviews as structured, semi-structured and unstructured.

To guide my individual interviews, I used face-to-face semi-structured interviews because of the exploratory nature of the study, which required the use of follow-ups, probing, and prompting (Bell, 2014). The interview schedule was subjected to my supervisors' expert judgement, who recommended follow-up questions that probed respondents' understanding and reasoning for all test items. My interview was scheduled for 30 minutes per learner and was conducted in the Grade 8 mathematics classroom in two of the schools as these classes were mostly available for use after school hours, and in the departmental head classroom in the third school. The interviews took two weeks to complete, as the interviews were done after school hours because learners were tired after the day's work, so I reduced the contact time by ensuring that learners did not stay beyond 30 minutes after the school day.

The purpose of the interviews was to obtain reasons for the Grade 11 mathematics learners' errors and misconceptions in trigonometric function graphs, which was guided by my conceptual framework. The questions, including the order, were not rigid. The respondents were asked questions based on the outcome of their written test to elicit the possible patterns and categories. Also, Cohen and Arieli (2011) recommended that it is important for the researcher to ensure that all the essential aspects of the study are reflected in the interview questions. I utilised an interview schedule with open-ended questions on errors learners made in the administered test (see Appendix B). Open-ended questions are more flexible and provided greater opportunity for the Grade 11 learners to express their reasons for committing the identified errors, which enabled me to obtain more information.

Additionally, Kallio et al. (2016) opined that a quality semi-structured interview would help enhance the objectivity and trustworthiness of a study and make the findings more plausible. The topic of trigonometric function graphs formed the basis for questioning; however, the sequencing of questions was participant-led. Follow-up questions were asked based on the responses of the interviewees. As mentioned in the data collection process, my interview was audio-recorded and transcribed verbatim.

3.6 Data Analysis

Qualitative data analysis entails the organisation of data, accounting for and explaining data and making sense of data in terms of the participants' view and experience of the situation by identifying patterns, themes, categories and regularities (Cohen & Arieli, 2011). Creswell and Poth (2018) and Hesse-Biber and Leavy (2011) described qualitative data analysis as an interpretive, helical and iterative process which involves the systematic organisation of data, a careful reading of data, coding, identifying of categories, developing of themes, representing and interpreting of the collected data.

For this study, a deductive approach was used to analyse the data from tests and interviews. The study's exploratory nature informed the approach used for the data analysis process. Casula et al. (2021, and Pearse (2019) declared that a deductive approach is most suitable for exploratory research. The authors claim that deductive analysis engages the need for "a priori theorising" and building upon prior bodies of knowledge. The test and interview data were analysed according to the pre-determined categories in my conceptual framework as informed by the literature. These prior bodies of knowledge from Brown and Skow (2016), Newman (1977), Oktaviani (2017), Radatz (1979), and Smith et al. (1993) informed the conceptual framework. The learners' responses to the test items were analysed quantitatively and qualitatively. The quantitative analysis of the test data was done using descriptive statistics, while the qualitative analysis was done using content analysis according to the pre-determined components of the conceptual framework. The interview data were qualitatively analysed as these data were the words of the interviewees.

3.6.1 Quantitative and Qualitative Analysis of Test Data

The data obtained from the research participants in the form of test scripts were analysed through content analysis. Krippendorff (2018, p. 10) defined content analysis as “a systematic reading of a body of texts, images and symbols”. Leedy et al. (2019) asserted that the examination of contents is to identify patterns, themes, or biases. Content analysis of the test scripts from the study was done after the test scripts had been scored to look for possible aspects of learners’ work where errors had occurred. The following steps were taken to start the analysis process:

- Step 1: Scoring and capturing of marks

Quantitative data analysis involves scoring of test scripts. I did this aspect of my research by using a memorandum to score learners’ written responses. Even though the study is qualitative, the learners’ scripts needed to be scored and recorded quantitatively to reveal the prevalence of errors in learners’ work. The memorandum used to assess the learners’ responses to the test items is in Appendix X. After marking the tests, I captured the marks in an Excel spread sheet which is in the form of a tabulated mark sheet which is reflected in Appendix Y. The marked scripts were moderated by an experienced colleague in the Department of Mathematics before the final capturing of marks was done.

- Step 2: Selection of samples for analysis

I purposively selected 30 learners’ marked scripts for my test analysis. My choice of 30 scripts was based on the recorded mean of the 30 learners (30.74%), which is a representative sample of the 150 learners that took part in the study. Also, the recorded mean of 30.74% was achieved by the 150 participants in the test. I randomly chose 30 scripts because this study has a limited scope and it was impossible for me to analyse 150 scripts for this research. I, therefore, based my random selection of scripts on all test responses per school until I was able to get to the point of saturation.

- Step 3: Classifying and organising the data

This stage of my data analysis started with identifying related items in the test instrument by doing a categorisation of learners’ responses according to the concepts in the CAPS document. I tallied the frequency of the performance range of learners on each concept to know which concept learners struggled with the most. This range of performance, as seen in the CAPS document, was used to organise learners per concept DBE (2011, p. 56). I created four categories of concepts; as mentioned earlier based on questions repeating in different forms, to ascertain if truly learning errors occur in learners' presumed learning. This categorisation process enhanced the qualitative exploration of Grade 11 learners' responses to the test items through the use of the error descriptors as postulated by Brown and Skow (2016), Newman (1977), Oktaviani (2017) and Smith et al. (1993). Basically, the Grade 11 learners' calculations, identifications, use of rules, establishing a correct understanding of the effect of parameters k and $p,$ sketching of graphs, determining the defining equations of functions from sufficient data and making deductions from graphs were explored for patterns in conformity with the types of errors identified in the conceptual framework.

Table 3.4 below shows all 17 test items that were categorised according to specific trigonometric functions concepts, as the test data analysis was first analysed based on the concept categories. This analysis put into focus the concepts that needed further investigation in the course of the interview process.

Table 3.4. Concept categories of test items

Question Number	Concept Number	Concept being investigated	Marks Allocated Per question	Total Marks
1.1 p 2.5 p 3.1 k 3.4 p 4.2 q	1	Investigate the effect of the parameter k, *p* *and q* on the graphs of the functions defined by *y* = sin (*x* + *p*), $\sin kx$, y = cos (*x* - *p*), and *y* = tankx.	2 2 1 2 2	9
1.3Range 1.4 reflection 2.2 Amplitude 2.3Range 3.2Asymptote 3.3Period	2	Identifying the range, asymptotes; periodicity and amplitude.	2 2 1 2 3 1	11

Question Number	Concept Number	Concept being investigated	Marks Allocated Per question	Total Marks
1.2 Difference 2.4Intersection 4.3Intersection 4.4Inequalities	3	Making deductions from graphs of $f(x)$ and $g(x)$.	4 2 3 2	11
2.1Draw 4.1Draw	4	Drawing the graphs of $\cos(x - 30°)$ and $g(x) = \sin 2x$ by means of point-by-point plotting, supported by available technology	5 4	9

I designated codes to learners' test scripts. For example, the first test script was assigned code from TS1 until TS30. TS1 means Test Script of the first learner, TS2 means Test Script of the second learner, until TS30, which means the Test Script of the thirtieth learner. All erroneous responses under each concept were analysed. This analysis was done for all concepts considering items erroneously attempted by learners, both partially correct and incorrect responses.

- Step 4: Summarising the data

The classified and organised data were, thereafter, summarised using the performance descriptors such as 30-40%, 50-55% and 60-75%. The frequency of each level of performance served as an indicator of which concept learners struggle with the most. The learners' errors were categorised using the pre-determined error types. This information on learners' performance per concept was also a basis for my research findings.

- Step 5: Interpretation of the findings

I used descriptive statistics such as frequency tables, means, percentages and line graphs to present my findings (Bertram & Christiansen, 2014). The use of frequency tables and line graphs helped me to transform and interpret the results obtained from the participants' test results and also helped to distinguish more explicitly between the test results and interviews. Also, the mean of the learners' marks indicated learners' academic ability in trigonometric function graphs. Ultimately, using these statistics helped me to determine the questions and concepts in which the learners performed poorly because this poor performance in the noted concept will guide curriculum planners and implementers on which part of trigonometric function graphs needs strengthened teaching. The next section discusses the processes used to analyse the interview data.

3.6.2 Qualitative Analysis of Interview Data

The interview data analysis employed a deductive analysis approach based on the causes of errors as postulated by Radatz (1979). The predetermined causes of errors were aimed at answering Research Sub-Question 2: "What causes Grade 11 learners to make these errors in trigonometric function graphs?" Content analysis of the interviewee's words was useful in achieving this purpose. Kumar (2018) believed that content analysis involves analysing the contents of interviews to identify the main themes arising from the interviewees' responses. For this phase of my data analysis, I was guided by the combined work of Creswell (2014), Kumar (2018) and Leedy and Ormrod (2019) in highlighting the steps that I took in this section. They were as follows:

- Step 1: Organising and preparing the data for analysis

I commenced analysing my interview data by transcribing the audio recordings of the responses from the interviewee. These data were analysed according to the pre-determined categories from my conceptual framework. The five causes of errors identified in my conceptual framework are: Errors due to language difficulties, errors due to difficulties in obtaining spatial information, errors due to the deficient mastery of pre-requisite skills, facts, and concepts; errors due to incorrect association or rigidity of thinking, and errors due to the application of irrelevant rules or strategies.

- Step 2: Reading, reflecting and identifying the main themes

This step involves the process of data examination. I became familiar with the data by thoroughly reading and reflecting on the data. Kumar (2018) suggested that the researcher needs to carefully go through descriptive responses given by the interviewee to figure out the meaning of the respondents' communication.

- Step 3: Categorising textual data

After reading, reflecting and identifying the main categories from the data, I began by using pseudonyms for all the respondents that took part in the interview so that the pledged ethical principle was not compromised. Just as was done in the test analysis, I asked the interviewee semi-structured questions according to the erroneous responses item-wise and concept-wise. I continued the categorisation process until I reached the 15th transcript of the interview. I could categorise each participant based on their experience and words about trigonometric function graphs by using the predetermined and precisely defined characteristics (Leedy & Ormrod, 2019). I identified related patterns to the causes of learners' errors in trigonometric function graphs.

- Step 4: Interpretation of findings

The last step of my interview data analysis proffered an opportunity for the analysis of the identified themes. These themes and responses were integrated into the text of my report. This report included the selection of excerpts from interview transcripts and connecting the excerpts back to the research objectives, theory and literature.

3.7 Trustworthiness of the Data

The procedures that are used to evaluate the trustworthiness of the data analysis need to be continually kept in mind, as this is the crucial test for the data analysis, the findings and conclusions (Nieuwenhuis, 2020). Also, the scholar declared that one of the ways in which qualitative researchers can enhance the trustworthiness of their studies is through the use of multiple data collection (Nieuwenhuis, 2020). In pursuit of ensuring the trustworthiness of the study, this study's data were collected through a trigonometric function graphs test in addition to interviews. Based on learners' individual performances, some of the tested items were further investigated during the interview. The interviews provided an opportunity to explore Grade 11 learners' errors and misconceptions as revealed by the test and to strengthen the meaning that might have been given to errors and misconceptions by the test. In summary, the test and interview instruments enabled me to "corroborate findings across data sets and thus reduce the impact of potential biases that existed in a single study" (Eisner, 2017).

In addition to the above procedures, the following were also done to enhance the trustworthiness of this study: continued responsibility, adherence to scheduled timelines, audit trials, availability of data, data analysis and the dependability of data. The continued responsibility I adhered to includes: Involving only the participants that met the inclusion and exclusion criteria during the data collection process, proper collection and documentation of informed consent, accurate record keeping, proper communication and increased monitoring by my supervisors. The early familiarity I established with my participants and schools from which data were collected enhanced my meeting of the scheduled timelines. Also, frequent debriefing sessions with my supervisors and Departmental Heads (DH) in respective schools helped in the aspect of conformity with time schedules.

I was guided by the steps Leedy and Ormrod (2019) recommended in the audit trial procedure. These scholars said an audit trail entails logging and detailing data collection and analysis activities as they occur. In pursuit of this trustworthiness procedure, I used the data of poor performances in past NSC examinations on trigonometric function graphs, as shown in Chapter 1, as sufficient reasons why this study was adapted for Grade 11. These detailed descriptions earlier given by me through the introduction and contextualisation chapter, enhanced my data collection and data analysis. I also made concerted efforts to base the findings of this study on the actual data collected as much as I could and my data collection and data analysis contained considerable detail such that other researchers might draw similar conclusions from similarly collected and analysed data. I ensured that the findings of this research were based on participants' responses and not on my personal judgement or biases.

3.8 Ethical Considerations

Ethics is defined as the "mutual trust, acceptance, cooperation, promises, and well accepted conventions and expectations between all parties involved in a research project" (Strydom, 2011, p. 113). As a researcher, I took the dignity of my participants and the gatekeepers into account during my study. My study considered various ethical issues to respect the rights of the people concerned in my study. I received consent from the Ethics Department of the University of Pretoria and the Gauteng Department of Education (GDE) before conducting the study. On obtaining permission to proceed with my study, I went to the selected schools and met with the principals. Upon arrival at these schools, letters of consent were given to the principals (Appendix C). I requested that the principals set up a meeting so I could meet the mathematics teachers to explain the study. My first meeting with the teachers allowed me to serve them the consent letters. Consequently, with the help of the teachers, I had meetings with the Grade 11 learners who wrote the test to ask for their assent. Learners were served the informed letter of assent and informed letter of consent to parents/guardians (see Appendix E) to sign before the commencement of data collection. My letter (Appendix F) to the participating learners addressed the following ethics procedures:

- Learners' voluntary participation was ensured by verbally communicating the nature of the research and the role of the participants. A letter of informed consent was given to learners and their parent(s)/ guardian to detail the research study's scope and reaffirm their consent. Also, the letter echoed the participants' freedom to withdraw from the research study when they deemed fit without coercion or negative consequences. The researcher used no incentives to bribe potential participants.

My application of consent to the Gauteng Department of Education as well as the University of Pretoria's Ethics Department, addressed the following ethical issues:

- The sensitive nature of the research work, research design and methodology, research participants' details, data collection process, and informed consent of participants (voluntary participation)
- Confidentiality and anonymity – I will/did not discuss the schools, teachers, or learners with anyone. Furthermore, pseudonyms/codes were used to hide the identity of the participants and schools to ensure confidentiality and anonymity in the sample selection phase. The selected respondents' geographical addresses, email addresses, and voiceprints were/will be protected. All identifying information about the sample will only be known by the researcher and supervisors. Data collected will be secured in encrypted files.

- Identifiers were done away with at the reporting stage of the research.
- Pseudonyms were also used for the interviewees, and the learners' responses were not linked to them in the course of data collection and afterwards.

3.9 Summary

This chapter discussed the paradigm and the ontological, epistemological and methodological assumptions underpinning the study. This study used a qualitative approach with an exploratory case study design. Discussions on data collection, such as developing and administering the data collection instruments, were done. Additionally, samples for the test process and the interview process of this study and the sampling techniques used in choosing the relevant schools and learners were explained. Also, the choice of using deductive data analysis was discussed. Lastly, the trustworthiness and ethical issues of this research were explained. In Chapter 4, the presentation and analysis of the test results and interview transcripts for trigonometric function graphs are provided.

CHAPTER FOUR: DATA ANALYSIS, PRESENTATION AND DISCUSSION OF FINDINGS

4.1 Introduction

The purpose of this chapter is to present the data collected from the learners' test scripts and interviews, and discuss the deductive analysis and findings thereof. This chapter was guided by the conceptual framework of two parts: 1) identifying the possible types and 2) possible causes of learners' errors. The analysis and findings of the data were done in accordance with the interpretivist research paradigm based on the researcher's experience and insight of the phenomenon. It is further aligned with constructivism as an epistemological assumption where I was subjectively involved and gained knowledge and meaning through personal experience. The first section of this chapter presents the data obtained from the test administered to the learners and the findings, while the second section presents the data that emanated from the interview and the findings.

4.2 Data Presentation and Analysis

For this study, the data presentation, analysis and findings were done separately for the written test and interviews. Burnard et al. (2008) assert that in writing up qualitative research findings, a researcher could simultaneously incorporate the discussion into the findings chapter. Therefore, this study presents its data alongside the discussions to avoid repetition during the data analysis.

The test data analysis answered the research sub-question 1: Which types of errors do Grade 11 learners make in trigonometric function graphs? Errors were categorised based on the conceptual framework. The interview analysis was used to answer research sub-question 2: What causes Grade 11 learners to make these errors in trigonometric function graphs? The causes of errors were also categorised based on Radatz's (1979) identified sources of errors.

4.3 Data presentation and analysis for written test

In the data-capturing process, the test scripts of learners that participated in this study were marked according to the memo, after which a senior colleague moderated the marking to increase the data's trustworthiness. The 17 test items were divided into four categories of concepts because some items in the test were repeated and tested a similar concept as in the CAPS document. The categories of concepts gave this study more focus on the types of errors committed in each concept. Table 4.1 below summarises the test items and learner results that emanated from the marked leaners' test scripts that participated in this study.

Table 4.1: Summary of test items and results

Concept Number	Concept being investigated	Question Number	Marks Allocated Per questi	Total Marks	Performance levels as per CAPS	Number of learners who achieved the respective levels	Percentage fail < 30% (DBE standard)	Percentage pass ≥ 30% (DBE standard)
1	Investigate the effect of the parameter k, *p and q* on the graphs of the functions defined by *y* = sin (*x* + *p*), $\sin kx$, y = cos (*x* - *p*), and *y* = tankx.	1.1 p 2.5 p 3.1 k 3.4 p 4.2 q	2 2 1 2 2	9	Level 1 (0-29%) 2 (30-39%) 3 (40-49%) 4 (50-59%) 5 (60-69%) 6 (70-79%) 7 (80-89%)	72 34 8 23 5 8 0	48%	52%
2	Identifying the range, asymptotes, periodicity and amplitude.	1.3 Range 1.4 Reflect 2.2 Amplitude 2.3 Range 3.2 Asymptote 3.3 Period	2 2 1 2 3 1	11	Level 1 (0-29%) 2 (30-39%) 3 (40-49%) 4 (50-59%) 5 (60-69%) 6 (70-79%) 7 (80-89%)	34 14 22 29 15 15 21	22.7%	77.3%
3	Making deductions from graphs of $f(x)$ and $g(x)$.	1.2 Difference 2.4 Intersection 4.3 Intersection 4.4 Inequalities	4 2 3 2	11	Level 1 (0-29%) 2 (30-39%) 3 (40-49%) 4 (50-59%) 5 (60-69%) 6 (70-79%) 7 (80-89%)	141 4 4 0 0 1 0	94%	6%
4	Drawing the graphs of $\cos(x - 30°)$and $g(x) = \sin 2x$ by means of point-b point plotting, supported b available technology	2.1 Draw 4.1 Draw	5 4	9	Level 1 (0-29%) 2 (30-39%) 3 (40-49%) 4 (50-59%) 5 (60-69%) 6 (70-79%) 7 (80-89%)	86 36 6 8 7 3 4	57.3%	42.7%

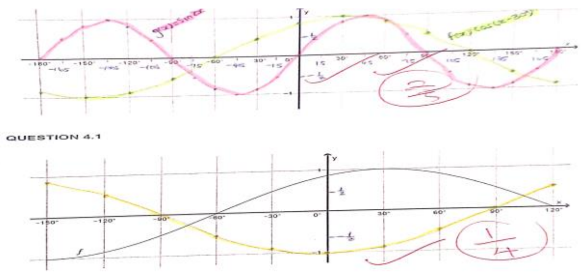
As mentioned earlier, 17 test items tested learners' understanding of different aspects of trigonometric function graphs. Column 1 displays the number of different concepts being assessed. Column 2 consists of the contents of the concept description as per the CAPS document. Column 3 displays the item numbers indicating the concepts being assessed. Column 4 shows the marks allocated for each item in the test, Column 5 shows the total marks per concept, and Column 6 displays the performance levels as prescribed in CAPS (DBE, 2011). Further to these, Column 7 indicates the number of learners who achieved the respective levels in the categorised concepts, Column 8 displays the percentage of learners who failed in that concept (<30% - DBE standard), and the last column indicates the percentage of learners who passed (≥ 30% - DBE standard). These test results informed me of the test items that needed to be focused on for the learners' interviews. The results show that the learners performed poorly with concepts 1, 3 and 4, suggesting that they have errors and misconceptions about these concepts. However, my data analysis was based on all the concepts since there was no exceptional performance (≥75%) in Concept 2.

Table 4.2 below shows the types of errors (based on the conceptual framework), test item number and content, error indicators and examples from learners' test scripts according to which the test data were analysed. The table indicates some erroneous solutions found in the scripts of learners that participated in this study. It is important to point out that in the table below, a reading error is not part of the errors indicated. This is because English language is the medium of instruction in the selected schools (the language of teaching and learning), and I could establish through the interview section with the 15 selected learners that the learners understood the language used in trigonometric function graphs and could also read fluently. More so, past research revealed that learners rarely committed reading errors in an administered test (Mensah, 2017; Sartika & Fatmanissa, 2020; Wardhani & Argaswari, 2022). The last column of the table contains various examples from learners' test scripts on the different errors committed by learners. Also, the examples of errors in the last column for referenced learner test script were coded. For instance, TS24 means "Test Script Number 24". Coding test scripts was the first action that began the analysis process.

Table 4.2. Rubric for data analysis

Types of errors	Item number & content	Error Indicators	Examples of errors (Script content)
Comprehension Error	1.1 Given the graphs of the functions $f(x) = 2\ \cos x + 1$ and $g(x) = 1 - \sin x$ for the interval $x \in [-90°; 360°]$. If the graph of *f* is shifted 45° to the left, write down the equation of the new function.	Script contents were completely different from the question. There is no connection between what was asked and what the learner wrote.	TS24: $f(x) = g(x)$ $2\cos x + 1 = 1 - \sin x$ $2co$ (end of the answer from learner's script) TS23: $f(x) = 3\cos x + 1$ TS22: $f(x) = \cos(2x - 90)$ • Evidence of item responses does not show an understanding of the effect of 'p'.
	1.2 Determine graphically, the value(s) of x for which: 1.2.1 $f(x) - g(x) = 0$ 1.2.2 $f(x) - g(x) = 2$ (See 1.1 above for functions)	Learner work shows inability to change the information to the correct form that would yield the desired result. For example, "determine graphically" required learners to use graphs in arriving at the solution, and not using substitution.	TS23: 1.2.1 $f(x) - g(x) = 0$ $2\cos(-90) - 1 - 〖\sin〗(-90) = 0$ 1.2.2 $f(x) - g(x) = 2$ $2\cos(270 + 1 - 1 - \sin 270 = 2$ • Substituted (-90) where y=0 and for item 1.2.2, replaced the value of (x) when $y = 2$ TS7: 1.2.1. $f(x) - g(x)$ =0 $2cosx + 1 - 1 - \sin x = 0$ $2cosx - \sin x = -1 + 1$ $(2cosx - sin\, x)/2$ = 0 $x - 1/2\ \sin x = \cos\text{^}(-1)\ (0)$ $x - \sin x = 90/0$ 1.2.2 $f(x) - g(x) = 2$ $2cosx + 1 - 1 - \sin x = 2 - 1 + 1$ $2cosx - \sin x = -1 + 1$

			• The learner interpreted the question by substitution. Got through by solving algebraically- algebraic substitution of $g(x)$ was not well done.
	1.4 The function g is reflected about the $x -$ axis to form a new function. Write down the equation of the new function in the form $y = \cdots$		TS10: $y = \ -2cosx + 1$ • The answer does not link with the provided information. TS22: $y = \sin(x - 30)$ • Learner does not know the difference between translation and reflection.
Transformation error	4.4 Hence, determine the values of x for the interval $-150° \leq x \leq 120°$ for which $\sin(x + 60°) + \cos x > 0$.	Learner work shows inability to change the information to the correct form that would yield the desired result. For example, learners were to find the points on the graph where the graph of $f(x) \ > \ k(x)$,	TS13: $= -120° < x < 120°$ • The answer does not show that learner knew what to do first in order to answer the question. TS15 $\sin(120 + 60^0) + \cos(60) > 0$ • The learner substituted one of the interval values in order to provide answer to the problem.
Process skill error	1.2 Determine graphically, the value(s) of x for which: 1.2.1 $f(x) - g(x) = 0$ (See 1.1 above for functions)	Learner knew what to do but failed in their calculation procedure by not estimating correctly the precise value of the grid boxes in order to get the right values.	TS20: $f(x) - g(x) = 0$ $f(x) = g(x)$ $x = -80°$ $x = 121°$ $x = 275°$ • The learner did not calculate the actual values of x by estimating correctly.
	2.4 Use your graph to estimate the x –coordinates of the points of intersection between *f* and g : $f(x) = \cos(x - 30°)$ and $g(x) = \sin 2x$.		TS8: $x = -75$ • The learner could not read-off the correct value of x at the point of intersection of the graphs.
Encoding error	3.4 Determine the equation of the new function g, if g is the image of *f* shifted 45° to the left.	The solution was incomplete. Step(s) needed to be done to earn full marks.	TS5: $\tan 2(x + 45)$

	$f(x) = \tan 2x$		• Learner did not finish up the solution. The learner did not remove the bracket. Expanding the solution would have shown that the learner understands that the horizontal shift does not change the period.
	1.4 The function g is reflected about the $x -$ axis to form a new function. Write down the equation of the new function in the form $y = \cdots$ $g(x) = 1 - \sin x$		TS8: $y = -(1 - \sin x)$ • The learner failed to expand the solution using algebraic process.
	2. Consider the functions $f(x)$=$cos\,(x - 30°)$ and $g\,(x) = \sin 2x$ On the grid paper provided in Attachment A, draw the graphs of f and g for $x \in$ [180°;180°]. Clearly show ALL intercepts with the axes, turning points and end points.		TS3: Learner did not label ALL intercepts with the axes as well as the coordinates of the turning points and end points of the graph. 
Careless error	3.4 Determine the equation of the new function g, if g is the image of f shifted 45° to the left. Given that $f(x) = \tan bx$	The learner omitted one of the basic information needed in the solution.	TS17: $\tan b\,(x + 45)$ • The learner omitted the value of b
Factual error	1.1 Given the graphs of the functions $f(x) = 2\ \cos x + 1$ and $g(x) = 1 - \sin x$ for the interval $x \in$ $[-90°; 360°]$. If the graph of f is shifted 45° to the left, write down the equation of the new function.	The concept of left shift and right shift are factual concept that distinguishes trigonometric functions in Grade 10 from Grade 11, if learners do not give the	TS1: $2\cos(x - 45^0) + 1$ • The error committed by this learner is a factual error. One of the basics learnt in Grade 11 trigonometric function graphs is the symbol attached to a left shift and right shift.

		right symbol, this counted as a factual error.	
Misconceptions	1.1 Given the graphs of the functions $f(x) = 2\ \cos x + 1$ and $g(x) = 1 - \sin x$ for the interval $x \in [-90°; 360°]$. If the graph of f is shifted 45° to the left, write down the equation of the new function. 1.4 The function g is reflected about the $x -$ axis to form a new function. Write down the equation of the new function in the form $y = \cdots$	In this study, all recorded factual errors were regarded as misconceptions. This is because misconceptions manifest in learners' incomplete grasp of a mathematical concept. In addition to this, if a learner failed to apply a previously learnt concept in a particular item, then the learner's error counted as a misconception.	TS1: $2\ \cos(x - 45^0\ ) + 1$ • The error recorded here is not just factual, but also a misconception. TS2: $\sin x + 1$ • Learner changed the x to the opposite.

4.3.1 Analysis and Findings of the Test Results

This section presents three out of the seven error types that were pre-determined in the conceptual framework. These errors are comprehension, encoding and misconception. It was discovered in the analysis process that other error types were rarely committed by the learners' whose scripts were analysed. The other possible errors are transformation, process skill, factual and careless errors. Therefore, the types of errors were analysed according to the four concepts identified in Table 4.1. Further, the error indicators in Table 4.2 guided the allocation of each learner's errors on a frequency table based on the criteria in the rubrics for data analysis in Table 4.2. In this section, I also present and discuss the data findings. Table 4.3 shows the aggregate percentage of errors made from all the previously conceptualised error types concerning all the incorrect solutions. The line graph in Section 4.3 gives a visual analysis of the rate at which these errors investigated occurred in the trigonometric achievement test.

Table 4.3. Percentage errors made from the different types of errors

Types of errors	Percentage error (relative to incorrect solutions)
Comprehension error	33%
Transformation	9%
Process skill	3%
Encoding	17%
Careless	6%
Factual	3%
Misconception	31%

Tables 4.4 – 4.7 show the types of errors per each of these four concepts. For each of the concepts, the largest total(s) indicated the type(s) of errors that occurred most frequently for the specific concept. Furthermore, I identified the test items with the most occurrences (≥ 5 in each item).

Table 4.4: Types of errors in Concept 1

Types of errors	Items number and number of learners that made the errors					Total
	1.1	**2.5**	**3.1**	**3.4**	**4.2**	
Comprehension	***8***	**3**	***7***	***7***	**4**	***29***
Transformation	**1**	**1**	**0**	**0**	**0**	2
Process	**1**	**0**	**0**	**0**	**1**	2
Encoding	**2**	**4**	**0**	**4**	**0**	10
Careless	**0**	**0**	**0**	**2**	**1**	3
Factual	**4**	**3**	**0**	**2**	**0**	9
Misconception	***5***	***16***	**0**	***14***	**0**	***35***

In Table 4.4, the frequency of error occurrences in Concept 1 is presented. Comprehension errors and misconceptions are the most often committed errors. The notable factual error in this concept was also accounted as a misconception because of the repeated occurrence of the same mistakes in items that tested similar concepts. Thus, analysis is presented on both types of error.

Table 4.5: Types of errors in Concept 2

Types of errors	Item number and number of learners that made the errors						Total
	1.3	**1.4**	**2.2**	**2.3**	**3.2**	**3.3**	
Comprehension	0	9	6	1	0	0	16
Transformation	0	0	0	0	0	0	0
Process	0	0	0	0	0	0	0
Encoding	0	0	0	0	6	0	6
Careless	14	0	0	0	0	0	14
Factual	0	0	0	0	0	0	0
Misconception	***5***	***16***	4	***12***	4	***16***	***57***

Table 4.5 displays the errors committed in Concept 2. As revealed, the errors that were made by learners in this concept are comprehension, carelessness and misconception errors. A misconception is the most prevalent type of error in this concept.

Table 4.6: Types of errors in Concept 3

Types of errors	Items number and number of learners that made the errors					Total
	1.2.1	**1.2.2**	**2.4**	**4.3**	**4.4**	
Comprehension	***16***	***14***	***8***	***16***	***1***	***55***
Transformation	0	0	0	0	15	15
Process	2	0	0	2	2	6
Encoding	0	1	0	0	0	1
Careless	0	0	0	0	0	0
Factual	0	0	0	0	0	0
Misconception	0	2	0	0	0	2

A comprehension error was revealed as the most pronounced type of error in Table 4.6, with 55 occurrences in the five items. Errors such as transformation, process skills, encoding, factual and misconception errors were rarely identified in learners' test scripts.

Table 4.7: Types of errors in Concept 4

Types of errors	Items number and number of learners that made the errors		Total
	2.1	4.1	
Comprehension	0	0	0
Transformation	4	5	9
Process	0	0	0
Encoding	***17***	***15***	***32***
Careless	0	0	0
Factual	0	0	0
Misconception	0	0	0

In Table 4.7, the encoding error was the most common type of error in Concept 4. Other noticeable type of error in this concept was the transformation error. However, a transformation error is not prevalent, so my presentation of data is based on the encoding error.

Table 4.8: Types of errors and item numbers presented and discussed

Paragraph number	Type of error	Concept number	Test item numbers
4.3.1.1	Comprehension	1	1.1; 3.1 & 3.4
		3	1.2 & 4.3
4.3.1.2	Encoding	4	2.1 & 4.1
4.3.1.3	Misconception	1	1.1; 2.5 & 3.4
		2	1.3; 1.4; 2.3 & 3.3

These types of errors and item numbers (See Table 4.8) are the ones that are then further presented and discussed.

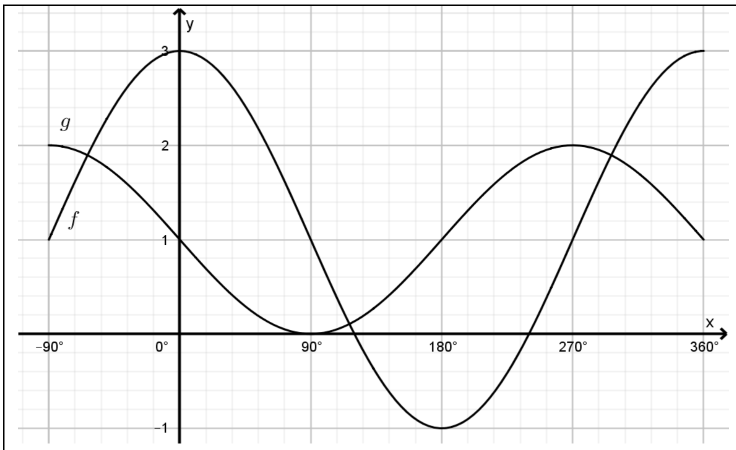
4.3.1.1 Comprehension Error.

Concept 1

Comprehension error is the inability of learners to understand the given mathematical problems. This type of error manifests in a learner's work if the learner fails to recognise "what is asked" and "what is given" from the test items (Hadi et al., 2018). For items 1.1, 3.1 and 3.4, comprehension errors were analysed because these items had the most frequent number of comprehension errors. As shown in Table 4.4, comprehension errors and misconceptions were most common in Concept 1. The reason for this is that misconceptions in learners' test scripts became higher due to additions from factual errors due to repeated occurrences of the same error in concepts that tested a similar understanding. For this analysis, a report will be given on both partially correct and flawed responses.

- Item 1.1

Given the graphs of the functions $f(x) = 2\ \cos x + 1$ and $g(x) = 1 - \sin x$ for the interval $x \in\ [-90°; 360°]$.



1.1 If the graph of *f* is shifted 45° to the left, write down the equation of the new function.

Learners' understanding of parameter "p" in the function: $f(x) = 2\cos x + 1$ was tested. Item 1.1 assessed learners' ability to display their understanding of the effect of the parameter "*p*" on the graph of function *f*. This question assessed learners' understanding of the horizontal shift of the trigonometric function graphs, that is, the effect of the parameter p on the functions of the form: $f(x) = cos(x - p)$. The excerpts below reveal the written work of learners with comprehension errors.

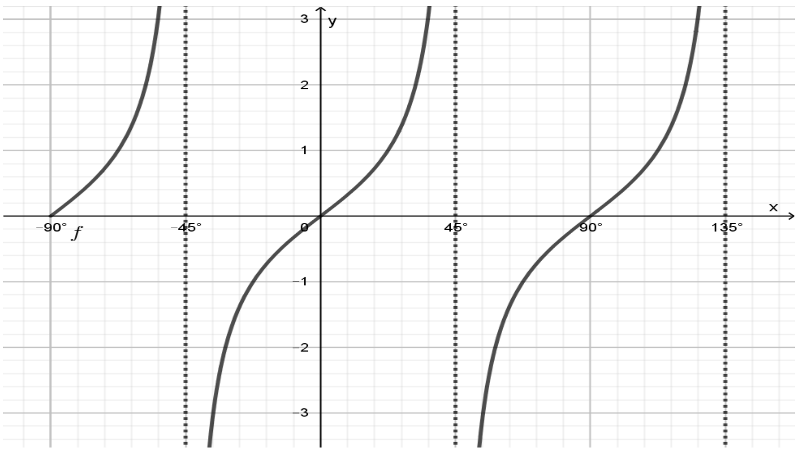
Test script number	**Excerpts from learners' scripts**
TS15	1.1 $1\cos(x) + 1$
TS23	1.1 $f(x) = 3\cos x + 1$

Only 13 out of 30 learners gave the correct response to the item. All learners attempted this question. Out of the 17 learners with erroneous responses, eight learners committed comprehension errors. It, therefore, means that in this item, 47% of the errors were comprehension errors. From the excerpts that display the contents of learners with TS15 and TS23, it is evident that these two learners could not recognise "what was asked" and "what was given" based on their responses to the test items.

They did not understand the question, as their answers indicated they could not use the information given to produce the right response. The answers "$\cos(x) + 1$" and "$f(x) = 3cosx + 1$" do not have any connection with $f(x) = 2\cos x + 1$, showing that the question was not well interpreted.

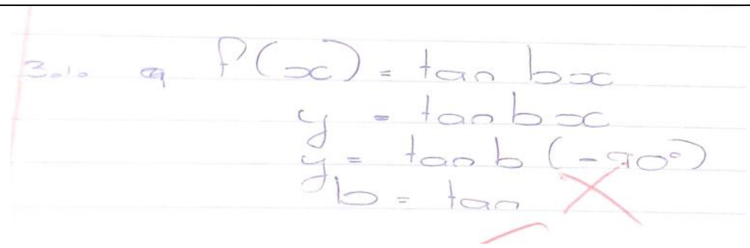
Item 3.1

Given the graph of $f(x) = tanbx$ for the interval $x \in [-90°; 135°)$.



3.1 Determine the value of b.

In Item 3.1, learners' understanding of the effect of change in period was investigated. For example, it is expected that learners can know that if: 0<b<1, then the graph is stretched, and the period increases. Also, if b>1, the graph shrinks, and the period decreases. In this item, learners were supposed to compare the asymptotes of the 'mother function' to the function of: $f(x) = tanbx$ to get the value of b.

Test script number	**Excerpts from learners' scripts**
TS19	 3.1 a) $f(x) = \tan bx$ $y = \tan bx$ $y = \tan b(-90°)$ $b = \tan$

TS24	$3.1)\ 0 = tan b$ $\therefore -90 = tan b$

Twenty-three out of 30 learners gave the correct response to the item. The rest of the learners gave erroneous responses. All seven erroneous responses resulted from learners' inability to understand the given problem. As shown in the excerpts above, the learner with TS19 substituted x with -90° that he used from the given interval (-90º; 0), an arbitrary point, into the function to get $b = tanx$. On the other hand, the learner with TS24 did this question by equating $tanb$ to zero, and not linking it at all with the period of the function.

- Item 3.4

Determine the equation of the new function g, if g is the image of *f* shifted 45° to the left.

In item 3.4, learners had to display their understanding of the effect of the parameter p on the function: $y = tan(x + p)$. Learners are expected to display their understanding of the effect of horizontal shift on the "mother function". For example, if p^0>0, the graph shifts pº to the left; if p^0<0, the graph shifts p^0 to the right. They further have to include the answer to Item 3.1 in their answer.

Test script number	Excerpts from learners' scripts
TS25	$g = tan\ 3(x)$

TS30	34) $f(x) = tan x$ X

Two learners did not answer this question, and only one out of the remaining 28 learners got the response correct. Four out of 27 learners gave partially correct answers, while 23 gave wrong responses. Seven out of 27 learners' responses had comprehension errors; this accounts for 26% of the group of learners with erroneous responses. The learner with TS25 did not show an understanding of the question. Even when 45º was given in the question, the learner's solution does not indicate what was given. If we look back at the learner with the TS25 solution in Item 3.1, the learner wrote the correct answer but failed to apply the determined value of $'b'$ in Item 3.4. This response shows that the learner does not clearly understand the requirements in Item 3.4. Similarly, the learner with TS30 also wrote $f(x) = tanx$, the "mother" $\tan$ function. If we view the learner's responses to Item 3.1, the learner wrote 3 as the value of "b". The learner's response to this item does not show that the learner was able to connect Item 3.1 to Item 3.4.

Concept 3

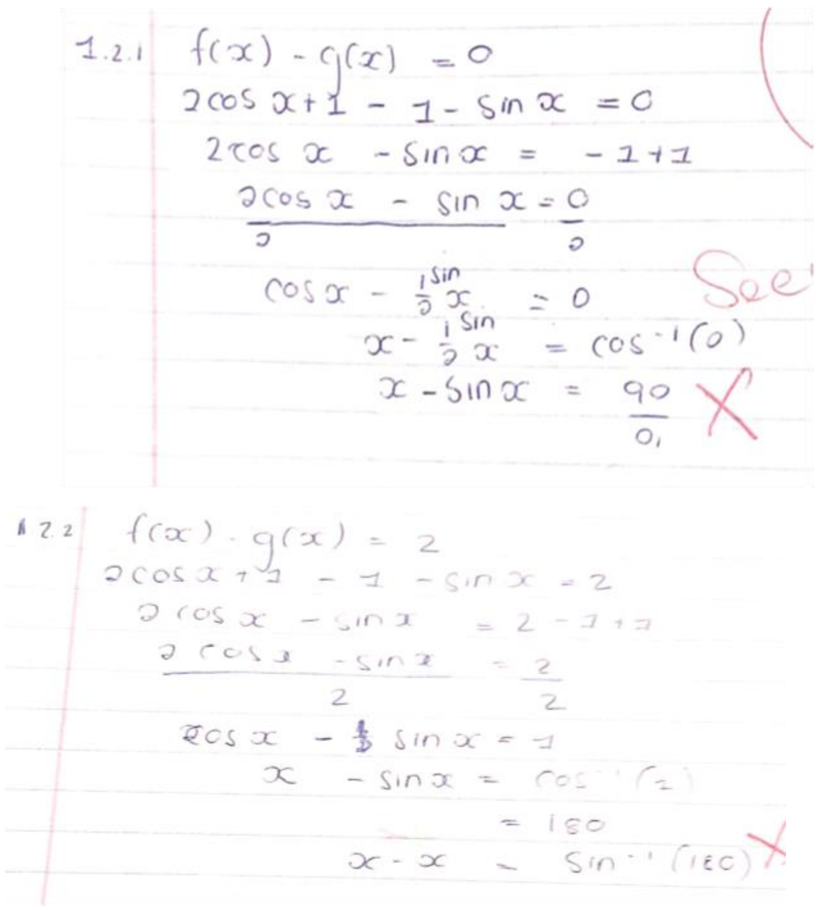
Abdullah et al. (2015) and Sartika and Fatmanissa (2020) confirmed that whenever the strategy learners employ in interpreting the question is less precise, comprehension error abounds. Arhin and Hokor's (2021) investigation of students' errors in solving trigonometric problems indicated that the students' comprehension level in the study conducted was low. Items 1.2.1; 1.2.2 & 4.3 were analysed.

Item 1.2

1.2	Determine graphically, the value(s) of *x* for which:	
	1.2.1	$f(x) - g(x) = 0$
	1.2.2	$f(x) - g(x) = 2$

This item tested if learners could use basic mathematics to interpret the “-” symbol, which indicates the difference in elementary mathematics. They had to determine graphically where the difference between the two graphs is zero, or transpose the function "$g(x)$" to the right so that the two functions were equal. By doing the latter, the learner can see at a glance that the x – values of the points of intersection of the two functions are required. The excerpts below reveal learners’ written responses to the test.

Test script number	**Excerpts from learners’ scripts**
TS6	1.2.1 $2\cos x + 1 - (1 - \sin x) = 0$ $2\cos x + \sin x = 0$ $2\cos x + \cos(x + 90) = 0$ $\cos^{-1} \times \cos x(2 + x + 90) = 0 \times \cos^{-1}$ $2 + x + 90^\circ = 90$ $2 + x = 90 - 90$ $2 + x = 0 \quad x = -2$ 1.2.2 $2\cos x + 1 - (1 - \sin x) = 2$ $2\cos x + 1 - 1 + \sin x = 2$ $2\cos x + \sin x = 2$ $2\cos x + \cos(x - 90) = 2$ $\cos^{-1} \times \cos x(2 + x - 90) = 2 \times \cos^{-1}$ $-2 + 2 + x - 90 = 0$ $x - 90 \quad x = 90$

TS7	 1.2.1 $f(x) - g(x) = 0$ $2\cos x + 1 - 1 - \sin x = 0$ $2\cos x - \sin x = -1+1$ $\frac{2\cos x - \sin x}{2} = \frac{0}{2}$ $\cos x - \frac{1\sin}{2} x = 0$ $x - \frac{1\sin}{2} x = \cos^{-1}(0)$ $x - \sin x = \frac{90}{0,}$ See 1.2.2 $f(x) \cdot g(x) = 2$ $2\cos x + 1 - 1 - \sin x = 2$ $2\cos x - \sin x = 2 - 1 + 1$ $\frac{2\cos x - \sin x}{2} = \frac{2}{2}$ $\cos x - \frac{1}{2}\sin x = 1$ $x - \sin x = \cos^{-1}(1)$ $= 180$ $x - x = \sin^{-1}(180)$

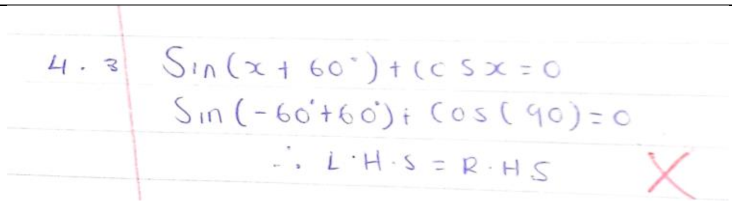
The error analysis in Table 4.5 shows a high number of learners committing comprehension errors in the concept that tested their ability to make deductions from the graphs of trigonometric functions. As revealed previously in Table 4.1, 94% of the learners scored below 30% in this concept. The items that follow show the content of the concept that was tested. Exactly 10 learners did not attempt Question 1.2.1. All 20 learners that attempted the question gave flawed responses, with two learners having partially correct responses and 18 learners having incorrect responses. From these 20 flawed responses, 16 test scripts had comprehension errors. This result manifested in the form in which the solutions were presented. Item 1.2.2 had one non-attempt more than item 1.2.1 (11), with only one learner having a correct solution. Of the remaining 18 learners with solution errors, only one partially correct solution was recorded, and the rest of the scripts had completely wrong solutions.

In the wrong solutions learners gave to Item 1.2.2, comprehension errors manifested in 14 learners' test scripts. Instead of solving x graphically, both learners tried to solve the equations algebraically, but also failed in their attempts. Eight out of 30 learners solved the equation. They probably did not think about the problem graphically, as the x-value(s) are where the difference between the **graphs** is either 0 or 2. Or they could think about Item 1.2.1 as where the two graphs are equal after rewriting the equation as $f(x) = g(x)$. It is interesting to mention briefly how these learners attempted to solve the problems. Both used substitution and the learner with TS6 tried to use a reduction formula in Item 1.2.1 to write his equation only in terms of $cos\ x$, but that was also done incorrectly. In the next step, he factorised, which was also incorrect. Already in the first step, the learner with TS7 substituted wrongly in Item 1.2.1, and his response was further flawed with errors in an attempt to solve x.

- Item 4.3

Use your graph to estimate the value(s) for *x* if: $\sin(x + 60°) + \cos x = 0$

Firstly, it is worthy of note to mention that in this item, the graph of $f(x) = \sin(x + 60°)$ was given and $k(x) = -\cos x$ was asked to be drawn in the first sub-question of this question. The concept tested in this item is similar to that of Item 1.2.1. The difference is only in the form of the posed question. The learners had to realise that the answer should be determined graphically and that the read-off values would only be estimates. Isolating the two functions should have been the first step to take. By doing this, learners would have been able to deduce that they were actually required to find the x –values at which the two graphs intersect. The following excerpts show the solutions proffered by learners with TS22 and TS29 from the test given.

Test script number	**Excerpts from learners' scripts**
TS22	 4.3 Sin(x + 60°) + cos x = 0 Sin(-60°+60°) + Cos(90) = 0 ∴ L·H·S = R·H·S

TS29	4.3) $\sin(x+60°)+\cos x=0$ $-180°$ -1.8 | $-150°$ -1.8 | -120 -1.3 | $-90°$ -0.5 | -60 0.5 | -30 1.3 | $0°$ 1.8 | $30°$ 1.8 | 60 | $90°$

This item was poorly attempted by all learners in that 12 learners did not attempt this question. For those that attempted the question, there was no correct response; three partially correct solutions and 15 wrong responses were observed. Comprehension error was the most prevalent error in learners' test scripts. Comprehension error accounted for 89% of the detected errors from the written test.

The errors displayed above are the learners' inability to understand the meaning of the term 'graphically'. It was noticed that if they failed to do so in Item 1.2, they continued to fail to do so in Item 4.3. The learner with TS22 attempted to make both terms on the left-hand side equal to zero to solve the equation. The learner realised that $sin0^0$ and $cos90^0$ are zero, so they just allocated different values to x to make the terms zero. It should be mentioned that the learner with TS29 used the graphs by setting a table and finding the difference between the two graphs at various x –values. This was done in an attempt to find an answer which is zero.

4.3.1.2 Encoding Error

Concept 4

An encoding error occurs in learners' work when learners display an inability to express final answers. This error is expressed in learners' work when they present incomplete solutions. At this stage of the solution, the student fails to write the desired answer correctly. In my quest to answer the research question, it was discovered that Items 2.1 and 4.1 had the greatest number of encoding errors. I observed that learners could draw the desired graphs but could not clearly show ALL intercepts with the axes, turning points and endpoints. This is one of the reasons for low performance in this concept. Therefore, Items 2.1 and 4.1 were analysed for encoding errors. The two items' analysis are presented together.

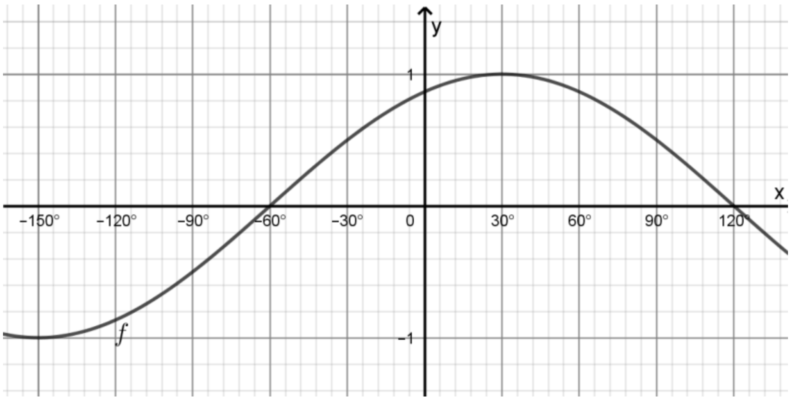
- Item 2.1 and 4.1

Consider the functions $f(x) = \cos(x - 30°)$ and $g(x) = \sin 2x$.

2.1 On the grid provided in Attachment A, draw the graphs of f and g for $x \in$ [-180°;180°].

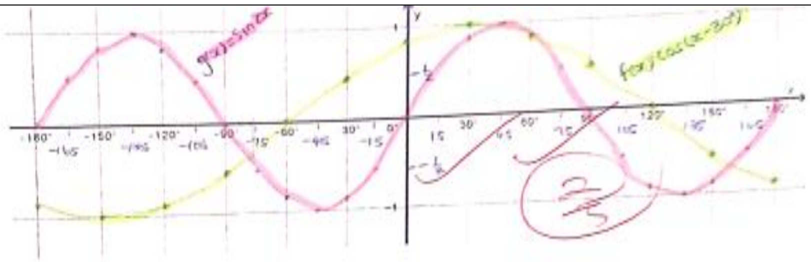
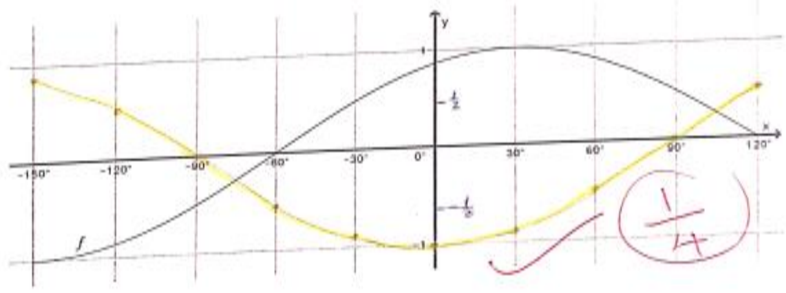
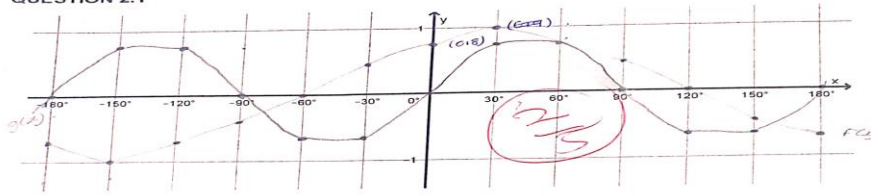
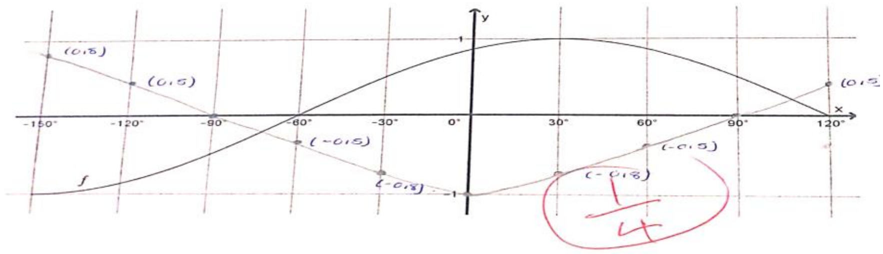
Clearly show ALL intercepts with the axes, turning points and endpoints.

4. In the diagram, the graph of $f(x) = \sin(x + 60°)$ is drawn on the interval $-150° \leq x \leq 120°$.



4.1 On the Attachment A provided, draw the graph of $k(x) = -\cos x$ for the interval -150°≤ x ≤120°. Show ALL the intercepts with the axes as well as the coordinates of the turning points and end points of the graph.

In the South African mathematics curriculum, curriculum planners expect that Grade 11 learners are able to draw trigonometric function graphs employing point-by-point plotting supported by available technology (DBE, 2011). Learners were expected to clearly label all the critical points as indicated in the question. The excerpts below display the graphs drawn by learners with TS3 and TS23.

Test script number	Excerpts from learners' scripts
TS3	 QUESTION 4.1 
TS23	QUESTION 2.1  QUESTION 4.1 

Four learners did not attempt the question in Item 2.1. The graph was well drawn, and five learners fully indicated the information needed. Seventeen of the 21 remaining learners gave a partially correct solution, and four drew the wrong graph. All 17 learners committed encoding errors, and this error type accounts for 81% of the errors in this item. Learners were marked-down because they failed to indicate all the critical points, such as: ALL intercepts with the axes, turning points and endpoints.

In Item 4.1, only two out of 30 learners did not attempt the question, while eight out of the remaining 28 obtained full marks. Of the 20 remaining learners, 15 gave a partially correct response, and five gave wrong solutions. Encoding errors were committed by 75% of the learners. As stated, most of the learners that committed the encoding error drew the correct kind of graph but did not put all the necessary information to earn full marks on the drawn graphs. Items 2.1 and 4.1 revealed learners' inability to write details even when the instruction given clearly states so. The learners in the above excerpts would have scored appreciable marks if only they had shown the critical points as demanded in the questions.

4.3.1.3 Misconceptions

Concept 1

Misconceptions are systematic errors that are symptomatic of a faulty line of thinking that could cause wrong answers or regular, planned, and repeated mistakes (Luneta & Makonye, 2010). The point of departure from scholarly work is that misconception is pervasive and, if not corrected, continues and becomes an entrenched obstacle in learners' academic path. To analyse this error, Items 1.1, 2.5 and 3.4 were analysed.

- Item 1.1

If the graph of f is shifted 45° to the left, write down the equation of the new function.

The concept tested in this question is learners' understanding of the effect of the parameter *p* on the graphs of the functions defined by: $f(x) = a\ \cos(x + p) + q$ where $f(x) = 2\ \cos x + 1$. This item's tested concept is similar to the concept tested in Item 3.4. Learners are expected to show their understanding of the effect of horizontal leftward shift on the cosine graph. Item 1.1 assessed learners' ability to display their understanding of the effect of the parameter "*p*" on the graph of function *f*. This question assessed learners' understanding of the horizontal shift of the trigonometric function graphs, that is, the effect of the parameter p on the functions of the form: $f(x) = cos\,(x - p)$.

Test script number	**Excerpts from learners' scripts**
TS16	$f(x) = 2\cos(x - 45° + 1$
TS28	1.1) $h(x) = 2\cos(x - 45°) + 1$

In Table 4.7, statistics revealed a high number of misconceptions committed by the learners under investigation. In this item, all learners attempted this question. Thirteen out of the 30 learners got full marks on this question. The remaining 17 learners gave wrong solutions. Misconceptions accounted for 29% of the wrong responses. Learners with TS16 and TS28 indicate a learning misconception because this error was repeatedly committed in other items that tested similar concepts. The sequence of errors reveals the pervasiveness of the learning "bug".

In Item 1.1, learners who committed factual errors due to not displaying their knowledge of horizontal shift by using the right symbol, inadvertently misconceived the concept taught. At the surface level of analysis, one would see this as a fact-lacking error but may not consider the multiplier effect of this error. The categorisation of items into four concepts helped to shed light on the nature of the error made. The learning error in Item 1.1 also repeated itself in Item 2.5 and Item 3.4 for the same set of learners. This showcased the pervasiveness of this error. Hence, for this analysis, I did not only see this error committed by learners as just factual, but also as a misconception.

- Item 2.5

Determine the equation of the new function h, if h is the image of the graph g shifted 45° to the right.

In this item, learners had to display the effect of the horizontal rightward shift on the graph of $g(x) = \sin 2x$. A clear understanding of this concept implies that learners will not only consider the rightward shift, but will also be able to exclude the parameter "k" which determines the period. In short, learning misconceptions manifested in learners' work when they consistently used the wrong symbol. The excerpts below tell about the learning misconceptions displayed by learners with TS16 and TS28.

Test script number	Excerpts from learners' scripts
TS16	2.5) $h(x) = \sin 2(x + 45°)$
TS16	3.4) $h(x) = \tan 2(x - 45°)$
TS28	2.5 $h(x) = \sin 2(x + 45°)$
TS28	3.4 $g(x) = \tan(x - 45°)$

In item 2.5, it was discovered that four learners did not do the question. Two out of the 26 remaining learners gave correct answers, while 24 learners' solutions were not without flaws. Sixteen learners' test scripts showed evidence of learning misconceptions. These misconceptions account for 67% of the flawed responses.

It could be seen that the above scripts tell that learners TS16 and TS28 did not have a complete grasp of this concept. Evidence of learning misconceptions in the above learners' script for Item 2.5 revealed that learners did not know when to use the right symbol to show their understanding of parameter "p". Misconceptions that emanated from learners' test responses were not limited to the wrong usage of symbols, but there was also a misconception on over-generalisation on the application of the effect of parameter "p" on the period of the function. In some instances, learners wrote: $h(x) = \sin(2x - 45°)$, which is a misinterpretation of the concept of horizontal shift as against an expected response of: $h(x) = \sin 2(x - 45°)$. On a final note, this becomes $h(x) = \sin(2x - 90°)$.

- Item 3.4

Determine the equation of the new function g, if g is the image of f shifted 45° to the left.

In Item 3.4, learners are expected to show their understanding of the effect of horizontal leftward shift on the tangent graph. For example, a 45° shift to the left will have a “+” symbol on the “mother” tangent graph. On the other hand, a 45° shift to the right, implies that the “mother” tangent graph will have a “-” symbol. Excerpts from learners’ test scripts below show more learning misconceptions.

Test script number	**Excerpts from learners’ scripts**
TS11	3.4) $g(x) = \tan(2x + 45)$ ✗
TS15	3.4) $g(x) = \tan(2x + 45)$ ✗

It came to light in the analysis process that two learners did not attempt the question. Of the remaining 28 learners, only one gave a flawless solution, while 23 learners gave an incorrect solution, and four partially correct solutions were evident in learners’ test scripts. Fourteen out of 27 flawed responses were misconceptions. The learning misconceptions recorded varied from misuse of symbols for the horizontal shift to overgeneralisation of rules. In these excerpts, there was a misconception of overgeneralisation of the rules on the effect of parameter "p" on the tangent function.

Concept 2

Items 1.3, 1.4, 2.3 and 3.3 were analysed to analyse this error because of the prevalence of misconceptions in the marked scripts. A misconception is one of the recurring errors in the trigonometric achievement test that was administered to learners and was the most prevalent error in Concept 1 as discussed earlier, as well as Concept 2. Misconceptions often occur in learners' work due to an incomplete grasp of mathematical concepts. Item 1.3 below gives a view of the question that was tested.

- Item 1.3

State the range of the function g.

The range of a function is referred to as the possible y-values for which the graph is drawn. Therefore, in Item 1.3, learners were to apply this in context. It means that learners were expected to focus on the graph of $g(x) = 1 - \sin x$ presented, identify the lowest q (y) point of the function and also look at the highest q (y) point of the graph to use the right interval notation or inequality to present the solution. In short, learners need to bring to light the effect of "q" units shift on the new graph $g(x) = 1 - \sin x$ in comparison to the standard (mother) graph. The excerpts below show some erroneous work of learners which accounts for learning misconception in Concept 2.

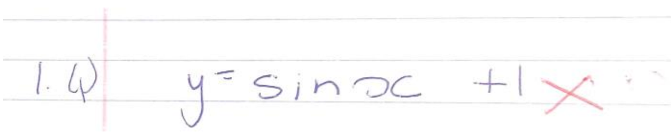
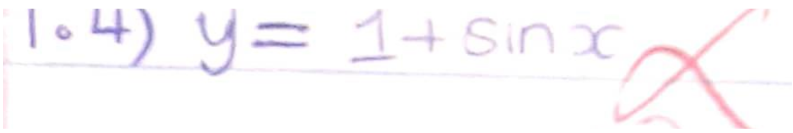
Test script number	**Excerpts from learners' scripts**
TS12	1.3 $R_g = [-90; 360]$ ✗
TS15	1.3 range = 3 ✗

All learners attempted Item 1.3. However, not all attempts gave correct solutions. Eleven out of 30 did this item without a flaw, while 19 learners had incorrect solutions, with two partially correct and 17 wrong solutions. Out of the incorrect solutions, five learners' errors resulted from misconception. For instance, the learner with TS12 misconstrued the concept of range by swapping the function's domain as the range. On the other hand, the learner with TS15 understood the range as a single value.

Item 1.4

The function g is reflected about the x-axis to form a new function. Write down the equation of the new function in the form $y = \cdots$

The concept of reflection was taught in previous grades as one of the components of transformation geometry. In this item, learners were expected to apply what they have learned on reflection about the x-axis in proffering a solution to Item 1.4. It is required that learners can tell, based on their responses, that a reflection about the x- axis changes the y coordinate and leaves the x coordinate the same. A view of the excerpts below tells what the learners' thoughts on Item 1.4 were.

Test script number	**Excerpts from learners' scripts**
TS11	
TS18	

Misconception as a type of error accounted for 62% of the errors recorded out of 26 wrong responses in Item 4. One out of 30 learners' scripts had partially correct solutions while 4 of the 30 had a fully correct solution. All learners attempted this question. The expected interpretation of this item is that learners had to apply basic principles of reflection as they have learnt in prior grades. For example: if we consider a point A (5;3) to be reflected about the x-axis, we expect it to produce a mirror image such as A' (5; -3).

It could be seen from learners' responses that they do not understand the given concept from previous grades. These learners wrote $y = 1 + \sin x$ as the result of reflecting $g(x)$ about the x-axis. The learners misconceived the co-ordinate meant to change due to the transformation. They rather changed $\sin x$ to the opposite instead of changing "y". This is a learning misconception as this is traceable to an incomplete grasp of previously learned concepts. Most of the learners that gave incorrect responses wrote the same content as the learners' scripts displayed for viewing in the preceding excerpts. Therefore, there is a misunderstanding of which coordinate is affected by the reflection in focus based on the participants' scripts. Learners were expected to write: $-y = 1 - \sin x$ (a reflection about the $x - axis$) and then $y = -(1 - \sin x)$. The final presentation of solution should have been $y = -1 + \sin x$.

- Item 2.3

State the range of f.

In Item 2.3, the concept of range was repeated similarly to how it was tested in Item 1.3. This repetition was done to ascertain if there is truly a learning misconception or not. Learners were expected to display their understanding of the concept of range on the graph of the function: $g(x) = \sin 2x$. It is expected that learners know that a change in the period of a function does not affect the range of the function when we compare the new graph to the standard graph. The excerpts below reveal the misconceptions observed in learners' scripts.

Test script number	**Excerpts from learners' scripts**
TS12	2.3 $R_f = [-180° ; 180°]$
TS15	2.3 range = 1

All 30 sampled learners attempted this question of which 16 gave a correct solution, 13 responded incorrectly to the item and one gave a partially correct solution. Misconceptions accounted for 86% of the incorrect responses.

- Item 3.3

Write down the period of *f*.

Item 3.3 tested learners' understanding of the concept of a period. It is expected that learners can make deductions from the drawn graph using the information provided about the asymptotes of the graph of $f(x) = \tan 2x$ to determine the period in relation to the standard tangent graph. Learners' test scripts with the manifestation of learning misconceptions are reflected below.

Test script number	**Excerpts from learners' scripts**
TS21	3.3) 135°
TS29	3.3) 45°

In this item, all learners attempted the question and 14 learners' responses were fully correct. There were 16 learners' scripts with incorrect responses, all due to misconceptions. These learners possibly assumed an arbitrary point on the given graph as the period of the graph. Learners with TS21 took the first asymptote on the right as the period, while learners with TS29 wrote the second asymptote as the graph's period. While I agree that there is a strong link between the asymptote of a tangent graph and the period, this should be applied with the knowledge of how these concepts help to make distinctions between different tangent graphs, and one can determine the period at a given point in time. So, learners' incomplete understanding of using this connection to arrive at a desirable solution led to the misconception recorded in this item. For instance, responses such as the period of $f(x) = 45^0$/135º are misconceptions. The learner with TS21 wrote the period of the function $f(x) = \tan 2x$ as 45º. Likewise, the learner with TS29 wrote the period as 135º. Of course, Item 3.1 could have been a pre-requisite to answering Item 3.3, but not absolutely, as the learner may use the given graph to interpret Item 3.3, because the asymptote of this graph has been given. Learners would only have to compare these asymptotes with the 'mother function' to deduce what the period of this graph is.

4.4 Summary of Learner Performances

Table 4.9 gives an overview of the general performance of the 30 sampled learners, which was used to guide the interview process on the items that needed clarification regarding the causes of errors committed by learners.

Table 4.9. General performance of the 30 sampled learners

Item Number	Description of learners' responses and the number of learners			
	Correct (C)	Partially Correct (PC)	Wrong (W)	Did not attempt (DNA)
1.1	13	0	17	0
1.2.1	0	2	18	10
1.2.2	1	1	17	11
1.3	11	2	17	0
1.4	4	1	25	0
2.1	5	17	4	4
2.2	19	0	10	1
2.3	16	1	13	0
2.4	3	5	15	7
2.5	2	4	20	4
3.1	23	0	7	0
3.2	19	5	4	1
3.3	14	0	16	0
3.4	1	4	23	2
4.1	8	15	5	2
4.2	11	0	15	4
4.3	0	3	15	12
4.4	1	0	18	11
Total	**151**	**60**	**259**	**69**

The table clearly shows the findings in terms of the number of learners who correctly responded to the question (C), the number whose responses were partially correct (PC), the number whose responses were wrong (W), and lastly the number of learners who did not attempt (DNA) the question in the achievement test. In Table 4.9 above, the total number of answer opportunities provided was 539 (when all the totals are added). This results in about 72% of responses that were either PW, W or DNA. When we omit the DNA, the PW and W answers are 59%. Because of these high percentages of incorrect answers, it is necessary to also attend to the percentages of each type of error occurrence.

It must be clarified that the error made by learners in some of the test items are not limited to one type of error. For example, in some test items, learners committed factual errors and misconceptions. Also, there were cases whereby a transformation error and a misconception were noticed in learners' test scripts during data analysis. Therefore, one could say that some errors are not mutually exclusive. To find the (approximated) percentage occurrence for each type of error, for example, the comprehension error, there was a total of 100 comprehension errors that occurred out of a total of 303 occurrences of errors in all four concepts, resulting in 33%. The occurrence of a transformation error was 9%, process skill errors 3%, encoding errors 17%, careless errors 6%, factual errors 3%, and misconceptions 31%. The line graph in Figure 4.1 shows the aggregate percentage of error occurrence committed by the 30 sampled learners in the written test.

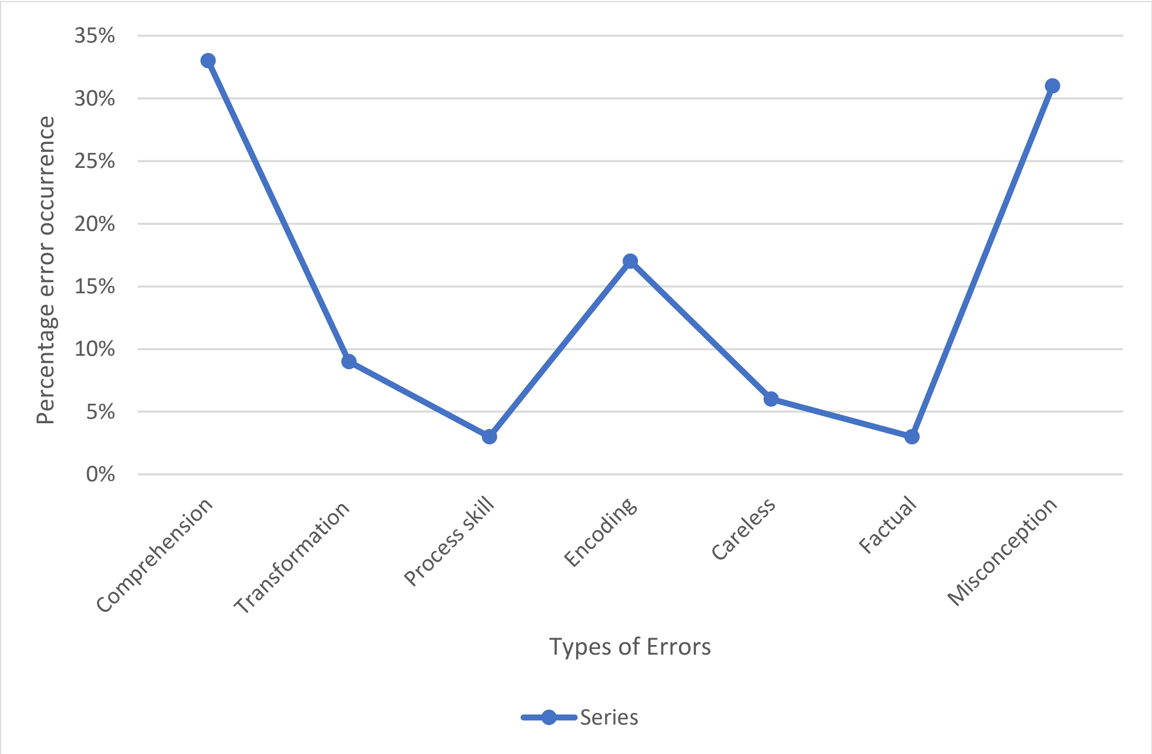


Figure 4.1: Line graph showing the number of error occurrences

This error analysis was followed by semi-structured interviews to shed some light on the possible causes of errors. The chosen questions to be used during the interviews were based on the percentage occurrence per error and the responses that had PW and W responses.

4.5 Data Presentation and Analysis of Interview

A semi-structured individual interview was conducted with 15 learners that were purposively selected from the group of 30 sampled learners for which the analysis of the test data were presented. This section will inform research Sub-Question 2 on the possible causes of errors as expressed in the interviewee's words. Categories were identified from the transcripts of the 15 learners interviewed based on the pre-determined causes of errors presented in the conceptual framework. However, a reading error was not an identified type of error in the achievement test. The reason for this is evident from the interactions with the interviewee. I was able to determine through the interview section with the 15 selected learners that the learners understood the language used in trigonometric function graphs and could also read English, which is the medium of instruction in the selected schools, fluently. Apart from this, past research revealed that reading and language errors were rarely committed by learners in administered tests (Mensah, 2017; Sari & Wutsqa, 2019; Sartika & Fatmanissa, 2020; Usman & Hussaini, 2017). Wardhani and Argaswari (2022) averred that using the language of teaching and learning in research reduces the chances of learners committing language errors. Therefore, in this section, the following causes, as found in the interview transcripts, are discussed:

- Difficulty in obtaining spatial information
- Application of irrelevant rules or strategies
- Poor mastery of pre-requisite skills, facts, and concepts

4.5.1 Difficulty in Obtaining Spatial Information

The visual way of learning some mathematics contents demands that students exhibit skilfulness in their spatial abilities and capacity for visual discrimination (Radatz, 1979). He said this visual instructional style is less content-specific and more representation-specific for all school mathematics content. Errors caused due to learners' inability to relate to spatial information were identified in learners' test responses and interview responses.

In this study, errors caused as a result of inadequate ability to obtain spatial information were found in eight out of 15 respondents; however, a report will only be presented on three interviewees. It is of utmost importance to state that most comprehension errors recorded in the test were as a result of learners not knowing what to do when a particular mathematics terminology is used, for example: 'graphically'. As a result, learners solved the problem. The transcripts below are the dialogue that ensued:

The dialogue with Amaka below is an excerpt that shows learners' inability to relate with spatial information.

Researcher: *"What is the meaning of value(s) in the introductory part of Question 1.2?"*

Amaka: *"It means it may be one or more."*

Researcher: *"Please show me the part of the graph that satisfies the functions. That is, your solution to Question 1.2.1."*

Amaka: $[\,f(x) - g(x) = 0$

$\quad + \quad +$

$\quad - \quad -]$

Researcher: *"Why is there a + and – at the bottom of Question 1.2.1?"*

Amaka: *"This is the way my teacher taught me. But I forgot how to do it!"*

Researcher: *"May you please show me how you arrived at your solution to Question 1.2.2?"*

Amaka: "$2cosx + 1 - 1 - sinx$

$\frac{2\cos x+1}{2} - 1 - cos(90° - x) = 0$

$x + 1 - 1 - (90° - x) = 0$

$x - 1 - (90° - x) = 0$

$x - 1 - 90° + x = 0$"

$2x -$

Researcher: "*What were you required to do in Question 1.2.2?*

Do you think the 'minus' symbol carries the same interpretation as it is used in other mathematical problems?"

Amaka: *"Yes, ma'am."*

Amaka, in this dialogue, does not understand most of the mathematical languages used in these items and then could not use the correct mathematical approach, which may seem as an error caused by the irrelevant application of a strategy so, in the end, it turned out to be an avoidance of working within cartesian space. The first error I picked up in Amaka's thought processes was the result of rote learning in Question 1.2.1. Let's say Amaka's strategy will give the desired result, but Amaka must have misconstrued the approach used by the teacher. The teacher must have used this approach for the function: $y > 0$. Amaka could not pull through this solution path. In Question 1.2.2, Amaka used the substitution method with the equation resulting in "0" and not "2". We can tell that Amaka avoided using the graphs when the examiner's intention was not clear to her.

The next extract is a short interview section with Zinhle. Zinhle preferred to use the table of value to provide solution to Question 1.2.

Researcher: *"Why did you use table of value in Question 1.2.1?"*

Zinhle: *"If I use the table of value, I can know which value is 0".*

Researcher: *"In Question 1.2.2, you also used the table of values. Why?"*

Zinhle: ***"****So that I can know which value is 2".*

Zinhle used a table of values to give the solutions to Question 1.2. This is the major reason for her erroneous response because the table is not a relevant strategy to obtain the answer to this question. This may imply that Zinhle did not know how best to go about interpreting the question.

Dimho's interview extract indicates learners' inability to deal with spatial information when the need arises. Let's take a look.

Researcher: *"Please talk me through your solution to Question 1.2.1."*

Dimpho: *"It is either you use the functions or use the values in the graph to solve the question."*

Researcher: *"Does the 'minus' symbol mean the same as you have known it to be in basic mathematics?"*

Dimpho: *"No, it is either it could be in the simplest form or distributed it into the equation."*

Researcher: "*Please apply what you just said in providing solutions to Question 1.2.1 and 1.2.2."*

1.2.1. "$f\ (x) - g(x) = 0$

$2cosx + 1 - (1 - sinx)$

$2cosx + 1 - 1 + sinx$

$2\,cos(-90) + sin(90)$"

Researcher: *"Please write 1.2.1 in another form."*

Dimpho: *[-*$f(x) + g(x)$*] = 0]*

"I am having difficulty in writing the exact values."

Dimpho: *[writes*

1.2.2 $f(x) - g(x) = 2$

=$2cosx + 1 - (1 - sinx)$

=$2cosx + 1 - 1 + sinx$

=$2cosx + sinx$

=$2(1) = 2$*]*

Dimpho claimed that the minus symbol did not carry the same meaning as when it is used in everyday mathematics. This may have been one of the reasons Dimpho had difficulty in responding correctly to this question. The use of the words "*simplest form or distributed into the equation*" explains the solution proffered. Dimpho substituted to answer the question. This approach is another avoidance by the learner to use the graphic space because the learner did not know what to do.

4.5.2 Application of Irrelevant Rules or Strategies

Radatz (1979), in his study, realised that some errors committed are attributable to negative transfers by learners. He stressed that learners are often inflexible in deciphering and encoding emerging information. The researcher noted the implication of this incompetence which becomes manifest when learners encounter similar problems. In this error source, learners develop cognitive operations and continually use their formed operations even when the task's required mathematical processes change. Some parts of rigid thoughts persist in learners' minds, preventing information processing.

For example, the dialogue that follows indicates inherent rigidity of thoughts in learners' thoughts in the course of interview.

Researcher: ***"****Please answer Question 1.4. But before you do that, let's have a look at this coordinate: A (3;4).*

Given that A is reflected about the x-axis, please tell me the new image of A."

Bola: *"A (3;4) → A'(-3; -4)"*

Researcher: *"Consider B (2; -1), reflect B about the y-axis."*

Bola: *"B (2; -1) → B' (-2;1)"*

Researcher: *"Please tell me your solution to Question 1.4."*

Bola: "y =1 + $sinx$"

It could be seen from this dialogue that Bola brought the misunderstood idea of reflection to the current grade. The learner could not state the images that emerged from the two reflections correctly (x*-axis* and y*-axis)*. This became a negative transfer in interpreting Question 1.4. The dialogue between the researcher and Daniel also provides evidence of this cause of error in the written test.

Researcher: *"Is there a remarkable difference between the reflection about the x-axis and a reflection about y-axis."*

Daniel: *"Yes, the y-axis reflects vertically which is the x-axis. It will reflect up and down*."

Researcher: *"Please tell me how you applied this in Question 1.4."*

Daniel: *"Normally, since they give me the $g(x)$ graph. I will just put the negative on the graph."*

Researcher: *"With this in mind, show me your solution to Question 1.4."*

Daniel: "$g(x) = 1 - sinx$

$\therefore y = 1 + sinx$"

Daniel: *"(Affirms) Yes!"*

It is evident from this conversation the learner does not understand the concept of reflection in his previous learning. Therefore, this made Daniel focus on changing the negative sign to the opposite. The learner did not apply this transformational change in the function correctly. The last transcript also reveals the existence of negative transfer in the thoughts of the learner that was engaged with the interview section.

Researcher: *"You learned the effect of 'q' in Grade 10. What is the effect of 'p' on trigonometric graph? Tell me if it affects the x –axis, the y –axis, the period or both x-axis and period."*

Grace: *"It affects the x-axis and the period."*

Researcher: ***"****Please answer Question 2.5."*

Grace: "$sin(2x - 45°)$"

Researcher: *"Please do Question 3.4 as well."*

Grace: "$h(x) = tan(2x + 45°)$"

The dialogue above indicates that Grace misunderstood the effect of parameter "p". According to this learner, the transformation of $\sin 2x,\ \ 45^0$ to the right, also impacts the function's period, which is the reason for erroneous responses from most of the learners that participated in the test. The misunderstanding might have been because learners generalised about the application of the effect of "p" as learned in Grade 10.

4.5.3 Poor Mastery of Pre-Requisite Skills, Facts, and Concepts.

The South African mathematics framework model (DBE, 2018) pointed out that learners with conceptual understanding can compare, relate, infer and engage in fundamental higher-order thinking, while learners with procedural fluency can recognise symbols and use rules to do mathematical tasks. I, therefore, argue that if learners lack these two basic skills, conceptual understanding and procedural fluency, there is a high likelihood of errors and misconceptions in the assessment of their learning. In the light of this, a dialogue between the researcher and Kenny is presented.

Researcher: *"Kenny, in Question 3.1 what does' b' indicate in the graph?"*

Kenny: *"It indicates the period of the graph."*

Researcher: "*Then, what is the value of b?"*

Kenny*: "b is 2."*

Researcher: *[Referred learner to Question 3.4] "What is the period of a tan graph?"*

Kenny: *"The period is when the graph starts to repeat itself, it will be zero."*

Researcher: "*What is the period of the mother functions i.e.$y = tan\, x$?"*

Kenny: *"0. I wanna say zero"*

Researcher: *"Why is it 0"?*

Kenny: *"Because it is repeating itself again at zero."*

In the above dialogue, Kenny could identify the period in the function but could not display an understanding of the concept. Kenny said the period of a tan graph is zero. This misunderstanding is a possible reason for the "DNA" that was reported in Kenny's test.

The following dialogue is more evidence of an error caused due to learner's deficient mastery of pre-requisite skills, facts and concepts. Themba was given another attempt to correct whatever mistake he made during the test. However, Themba still repeated the mistake he made in the test. This misconception needs to be addressed so it does not last too long. The transcript below reveals the misconception that Themba had with Item 1.1.

Researcher: *"Do you think you understand Question 1.1?"*

Themba: *"Yes, ma'am."*

Researcher: *"If I give you another opportunity to respond to this question – what are you required to do in Question 1.1."*

Themba: *"Yes- I need to write the new formula of the function."*

Researcher: *"Of the function? Then, if you are to write it, what would you write?"*

Researcher: *[Gives the learner a folio paper to write.]*

Themba*:* $[f(x) = 2\ cos(x - 45°) + 1]$

Researcher: *"When a graph is shifted to the left, which variable is changing?"*

Themba*: *"The x-axis."*

Researcher: *"The x-axis is changing. Thank you."*

Researcher: *"When there is a left shift on the x-axis, should we write $+45°$ or $-45°$."*

Themba: $" - 45"°$

Researcher: *"Alright, I am going to give you your paper so that you tell me what you feel you did wrong in Question 1.1."*

Themba: *"The reason, ma'am, why I made a mistake is that I took the $x(-45°)$ to the y side, and it was supposed to be on the left side – which is on the q-side and the q is always the x-axis shift in the graph."*

From the conversation above there is a lot of learning misconceptions Themba had with Item 1.1. For example, Themba said "*and the 'q' is always the x-axis shift in the graph.*" The researcher tried to see the level of misunderstanding that Themba had with knowing the basics of trigonometric function graphs, in doing this, the dialogue below ensued:

Researcher: *“We would look at another similar question to Question 1.1 – Question 2.5.”*

Themba: *[reads Question 2.5]*

Researcher: *“How would you write this? Which of these functions would you use to build the solution of the question”*

Themba*: *“I would use $g(x)$."*

Researcher: *[Re-iterates what the learner must do.]*

Themba: *[Writes $-g(x) = sin2x$]*

Researcher: *“$h(x)$ is shifted to the right.”*

Themba: "$g(x) = sin2(x + 45°)$"

From the above transcript, one could tell that there is a serious learning misconception displayed by Themba in understanding the effect of the parameter ‘*p’*. The dialogue further explains that the error observed is not a result of carelessness but a deficient mastery of the requisite concept. The next dialogue shows the transcript from the interview section with Rhema.

Researcher: *“Please, show me on the graph where you would find the range in Question 2.3 by using the graph I have provided.”*

Rhema: "R_g= *(-90°, 360°)”*

Researcher: *“Please tell me the difference between the domain and the range.”*

Rhema: *“The range is the point where the graph start repeating itself while the domain is the intercept between which it intercepts the graph.”*

Researcher: *“What do you understand by the term amplitude?”*

Rhema: *“I don’t really understand.”*

Researcher: *“What is the range of f?"*

Rhema: *"The range is 90*"°

Researcher: *"You gave so many solutions in Question 2.4."*

Rhema: *"I looked at x –coordinates. Same as Questions 4.3 and 4.4."*

From the concluded dialogue, the researcher wanted to be sure if the error made by the learner was a factual error or a misconception. The transcript revealed that the error made by the learner was a factual error based on the learner's response: " - R_g = *(-90*°*, 360*°*)". The facts are that the range is never stated in degrees, it is determined by the x- coordinates of the function. Also, the round bracket "()", was wrongly used by the learner. The error made was not just factual but also learning misconception based on this extract – "*The range is 90*°"*.* The first extract where the learner stated that the range is *(-90*°*, 360*°) must have been generated by the learner thinking that the range of the function is on the x-axis. To me, it looks like the learner became confused between the concepts of domain and range. The second extract, where the learner stated that the range was *90*°*,* must have been a learning misunderstanding from the past, which, in my view, the learner generated from what was learnt about the period of $\tan 2x$. The error did not arise from the learner's inability to draw the graph because a graph was provided, but this error was caused by her insufficient mastery of required skills, facts and concepts.

4.6 Discussion of Findings

From the analysis of test and interview data, the following discussions on the findings of this study are delineated.

4.6.1 Discussion of Comprehension Errors

Abdullah et al. (2015); Arhin and Hokor (2021); Santoso et al. (2017); Sartika and Fatmanissa (2020) findings on learners' errors are consistent with this study. The scholars discovered that one of the errors that learners commit the most is the comprehension error. This is evident in this current study where this type of error was highly made in Concept 1 and 3. From the excerpts presented in the presentation of the test data, it was revealed that the two learners did not know that a horizontal shift affects the x-value of the function. Also, in Concept 3, most of the learners that wrote the trigonometric achievement test did not understand the term 'graphically'. Because of this, teaching instructions should cognitively guide the learners to know and be able to interpret questions more appropriately to reduce this type of error.

4.6.2 Discussion of Encoding Errors

The report from the NSC (DBE, 2021) examination made known that in Question 7.1, many candidates did not sketch the graph of $\cos(x - 60^0)$ correctly. It was also reported that their graphs had incorrect turning points and x-intercepts, and the graphs drawn by candidates revealed the use of calculators to generate the points on the graph, but that these were not the critical points required for the sketch. Furthermore, Malambo (2020) and Nabie et al. (2018) admitted that there is learning complexity in the drawing and sketching of graphs. Nabie et al.'s (2018) research on pre-service teachers indicated that 89,4% of the teachers could not sketch the sine function graph. Similarly, Malambo (2020) reported in his findings that only one pre-service teacher accurately completed the task; 11 of these teachers did not provide graphs; and ten teachers drew flawed graphs. The reports from these scholars resonate with the work of scholars in the field of mathematics.

Dubinsky and Wilson (2013) stated that one of the challenges that learners must deal with in trigonometry is sketching graphs and interpreting sketched graphs. The research work of scholars says it all. If pre-service teachers who will eventually become in-service teachers struggle to sketch and draw trigonometric graphs, one begins to wonder what problem this may pose for the learners they would teach when they are fully employed. Although this study differs slightly from the aforementioned study, many learners could draw the graphs required but could not adhere to the instructions given. For this error not to re-occur in subsequent assessments, I would suggest that sketching of graphs is drawn with its adaptations from mother graphs. For instance, if we consider the functions: $y = \sin x$ and $y = \sin(x - 60^0)$ for the interval $y \in [0^0; 360^0]$ the coordinates of the critical points of the function $y = \sin x$ are: (0º; 0), (90º; 1), (180º; 0), (270º; -1) and (360º; 0). To draw the graph of the function $y = \sin(x - 60^0)$, the critical points using the existing coordinates are: (60º,0), (150º; 1), (240º; 0), (330º; -1). In my view, the transformation of the graph from $y = \sin x$ to $y = \sin(x - 60^0)$ needs to be taught without using a calculator so that learners can clearly understand the progression in concepts from Grade 10 to Grade 11. This approach will also assist learners in using the critical points to sketch the graphs; consequently, the reinforcement of indication of points will fall in place.

4.6.3 Discussion of Misconceptions

The error made by the learner with TS12 (page 89) corroborates the findings in Question 6 of the NSC examination (DBE, 2020, p.8***)***. It was reported that many candidates could not distinguish between range and domain in Question 6. Research showed that learners had misunderstood the concept of domain, range, period and amplitude. The NSC Diagnostic Report (DBE, 2020) disclosed that many candidates could not distinguish between range and domain in as per Question 6.1 of the 2019 NSC examination. Similarly, Bohlmann et al. (2017) also argued that assumptions are made that any sine function, for example: $g(x) = 1 - \sin x$, has the same range as the mother function, $y = \sin x$, $\mathcal{R}_f = [-1; 1]$. In the researchers' words, "unless the concept of range is properly understood, the mistakes will continue" (Bohlmann et al., 2017, p. 7). Additionally, Bohlmann et al.'s (2017) findings revealed that 63% of candidates in the upper third of learners, made mistakes in the concept of range. However, the excerpts above did not reveal the same error as indicated by the mentioned researchers. Nevertheless, it is important to state that some of the misconceptions generated from learners' incorrect responses revealed similar errors from literature.

Similarly, it was reported in the 2020 NSC Diagnostic documents (DBE, 2021) that candidates could not tell the difference between period and amplitude, while some candidates erroneously wrote the period of $g\ as$ ½. At the same time, other candidates misconceived the period as an interval of $(0°; 360°)$ in Question 5.1. Contrary to this report, the errors committed by learners were dissimilar to those above, coming in a different shade.

4.6.4 Discussion of Interview Findings

The prevailing causes of errors, as revealed from the dialogue with the learners, are: Errors due to difficulties in obtaining spatial information, errors due to the application of irrelevant rules or strategies, errors due to the deficient mastery of pre-requisite skills, facts, and concepts. The revealed causes of errors gave information on how we could re-strategise the teaching of this topic to better improve performance in trigonometric function graphs in future teaching. As reviewed in the literature, learners writing the NSC examination do not understand the difference between period, amplitude, range and domain of a given function, as corroborated in the dialogue sections with the learners (DBE, 2020). Also, it was reported that when learners were to make deductions from graphs, they substituted, this, as manifested by most of the learners in Items 1.2, 2.4, 4.3 and 4.4.

The findings from the interview dialogues with Amaka, Zinhle, and Dimpho revealed that these learners did not engage with the graphs in an attempt to answer the questions. This finding is consistent with the report from the NSC examination for 2019 and 2020. This may have been due to the lack of spatial intelligence on the part of the learners. Yaumi and Ibrahim (2013) defined spatial intelligence as the ability to perceive the spatial world accurately and transform spatial perception in various forms. Further to this, Riastuti et al. (2017) advanced the course of developing spatial intelligence in learners so that teachers can achieve the desired learning outcomes from their teaching, and learners' conceptual understanding of a concept is optimised. Radatz (1979) identified learners' inability to obtain spatial information and the application of irrelevant strategies or rules as possible dual causes of errors.

On the other hand, a research study conducted on 50 second-year students of Nceva Ecija University of Science and Technology (Ancheta, 2022) found that students experienced difficulty in pre-requisite subjects such as basic concepts and laws in algebra, trigonometry, trigonometric functions, analytical geometry, slope of line and graphing functions. The researcher pointed out that the lack of knowledge retention, misconceptions and skills deficits are the root causes of errors in mathematics. Also, Luneta (2015) reported that weak conceptual knowledge in mathematics makes learners commit errors in solving mathematical problems.

4.7 Chapter Summary

Chapter 4 presented the analysis and findings of the data collected in this research study using test instruments and interview schedules. The researcher began by categorising the test items into four categories and obtained useful statistics from this process. After that, information revealed through this process facilitated the generation of categories on the types of errors through data analysis. The findings from the test analysis and interview dialogue were revealed. The next chapter presents the conclusion of this study.

CHAPTER FIVE: CONCLUSIONS AND IMPLICATIONS

5.1 Introduction

Chapters 1-4 provided an overview of the study; a literature review and discussion of the conceptual framework used to guide the study; the methodology used to gather and analyse the data; and lastly, the presentation and discussion of findings. This chapter presents responses (answers) to the research questions, reflections on the study and on a personal level, the limitations of the study, recommendations for further studies, and conclusions of the study.

5.2 Answering the Research Questions

My motivation for this study was that candidates writing the NSC examination performed poorly in trigonometric functions, resulting in poor performance thereby impacting negatively on the overall success in mathematics. Therefore, this study aimed to investigate the types of errors Grade 11 learners make in trigonometric function graphs and the causes thereof. To do so, the following main research question was formulated: How can learners' errors in trigonometric function graphs be described? To address this main question, the following two sub-questions guided the enquiry:

1. Which types of errors do Grade 11 learners make in trigonometric function graphs?
2. What causes Grade 11 learners to make these errors in trigonometric function graphs?

In light of the interpretivist paradigm, particularly the social constructivism paradigm in which I worked, the first research sub-question is answered based on the meaning I made from learners' responses to the test and interview questions. I constructed meaning regarding the second research sub-question from a social constructivist perspective (Creswell, 2014) through my interviews with participants in a social context. The qualitative approach and case study design used were appropriate since they provided in-depth data, although it is a small case and generalisations cannot be made, it was informative. The findings can be used in similar cases. It should also be reiterated that deductive data analysis was conducted based on the pre-determined categories and causes of errors as informed by the literature.

5.2.1 Sub-Question 1

Which types of errors do Grade 11 learners make in trigonometric function graphs?

To answer sub-question 1, I carried out a content analysis of the trigonometric achievement test done by the Grade 11 learners. I used my conceptual framework and rubrics to analyse the different types of errors made according to the four pre-defined concepts the test covered.

Findings from the test data analysis revealed that Grade 11 learners make comprehension errors in Concepts 1 and 3 that tested learners' understanding of the effect of the parameters $k, p\ and\ q$ on the graphs of the functions defined by $y = sin\ (x + p), sinkx, y = cos\ (x - p), and\ y = tankx$. Also, learners could not display their understanding of the term 'graphically' in order to make deductions from trigonometric function graphs in Concept 3. Notably, in the analysis, learners did not understand the right approach to proffering a meaningful solution to the stated problems. Saifiyah and Retnawati (2019), and Usman and Hussaini (2017) alluded that students often misunderstand the question's demand, and this may be due to a lack of emphasis on the concept taught by the teacher, but also rote learning on the part of the learner, thus learning without a conceptual understanding of the concept.

The second error that learners made was the encoding error. This error type was mostly detected in Concept 4 with the drawing of graphs for $cos(x - 30°)\ and\ g(x) = sin2x$. It was also confirmed that learners often have problems drawing graphs (D Ancheta, 2022; Jaelani, 2017; Setiawan, 2021). In fact, studies showed that making connections between trigonometric function representations and the interpretation of their graphs pose challenges to students (Maknun et al., 2020; Mosese & Ogbonnaya, 2021). However, the findings of this study revealed that learners struggled with making deductions from graphs and could not label the critical points when the graphs had been drawn (encoding).

The last error that occurred in the trigonometric achievement test was a misconception error. Misconceptions occurred in Concepts 1 that tested learners understanding of the effect of parameters $k, p\ and\ q$ on trigonometric graphs. The test analysis presentation also revealed that learners struggle with identifying the range, asymptotes, periodicity, and amplitude in Concept 2. Many scholars in the field of mathematics education have discussed the cognitive way of learning, how it occurs, where it occurs, and factors that affect the process (Clark, 2018; Maton et al., 2015). Thus, this study would not be complete without referring to the knowledge gap created by an insufficient grasp of trigonometric functions, resulting in misconceptions. Therefore, I argue that the lack of conceptual understanding in the learning of trigonometric function graphs is the cause of misconceptions identified in the assessment of learning exhibited in the administered test.

5.2.2 Sub-Question 2

What causes Grade 11 learners to make these errors in trigonometric function graphs?

The findings from this, as pre-determined in the conceptual framework, revealed that trigonometric function errors are caused by: difficulties in obtaining spatial information; deficient mastery of pre-requisite skills, facts, and concepts and application of irrelevant rules or strategies. One of the challenges that learners face is the need to spatially relate algebraic representations with graphical context to produce a flawless result. Radatz (1979) affirmed that the different forms of iconic instructions, diagrams, and visualisations of mathematical activities place high demands on learners' spatial competence and capacity for visual discrimination. Trigonometric function graphs as a mathematical concept, like any other aspect of functions, need graphical competence and translation within all forms of representations to produce the desired result (Mosese & Ogbonnaya, 2021). Reports from the DBE (2020, 2021) have it that in the NSC examination that tested the deductions from graphs, candidates could not use the graphs effectively; they rather solved the equation, which caused erroneous solutions. Also, Rushton (2014) disclosed that higher tier candidates had difficulty with solving equations graphically when there was a need to make deductions from graphs. The common error made by these candidates was to solve the equations algebraically instead.

In my view, I see a trend in learners' avoidance of having to make deductions from graphs using the graphical method because of inadequate understanding of terms and symbols used. These learners chose convenience over the needful mathematical approach. It, therefore, suffices to say that learners' difficulty in obtaining spatial information, deficient mastery of pre-requisite skills, facts, and concepts and application of irrelevant rules or strategies inadvertently result in poor performance in trigonometric functions.

Donaldson (1963) identified the error caused by deficient mastery of pre-requisite skills, facts, and concepts as a structural error. The scholar said a structural error arose due to mistakes in the way learners perceive the nature of mathematical concepts, some failure to make connections between the relationships involved in the problem given, or an inability to grasp some principle or essential rule to find the solution.

Apart from the error cause identified above, Hirst (2003) also noted that procedural extrapolation error is an error that is caused by learners' attempts to extend a previously learnt procedure to a new situation similar to one learnt in the past. It is an overgeneralisation of a valid procedure in new situations that causes errors. In this study, Radatz (1979) opined that these source of errors are caused by the application of irrelevant rules or strategies.

5.2.3 Primary Research Question

How can learners' errors in trigonometric function graphs be described?

In this study, I investigated the types of errors committed by Grade 11 learners in trigonometric function graphs, together with the causes of these errors. These two areas of investigation provided insights to discuss the primary research question. The study's key findings revealed the prevailing errors in trigonometric function graphs as comprehension, encoding and misconceptions. These errors were investigated to be caused by difficulties in obtaining spatial information, deficient mastery of pre-requisite skills, facts, and concepts, and the application of irrelevant rules or strategies.

Table 5.1. gives the summary of research findings as it answers the study's research questions.

Table 5.1: Summary of Research Findings

	Research Sub-Question	Overall findings
1	Which types of errors do Grade 11 learners make in trigonometric function graphs?	Grade 11 learners make the following errors: comprehension error, encoding error, and having several misconceptions.
2	What causes Grade 11 learners to make these errors in trigonometric function graphs?	Grade 11 errors are caused due to the following: Difficulties in obtaining spatial information; deficient mastery of pre-requisite skills, facts, and concepts; and the application of irrelevant rules or strategies.

Table 5.1 summarises the types of errors that emerged from the data generated from the written test and the causes of errors identified from the interview schedule.

5.3 Limitations to the Study

It is crucial to note the non-generalisable intent of this study. This study's report is presented based on the findings from a case study of 30 purposively selected Grade 11 learners sampled from a group of 150 learners who wrote the test in three homogeneous public schools. Hence, the findings of this investigation may not be generalised to all Grade 11 learners in the year this research study was carried out. Also, the 15 information-rich participants' opinions and thoughts from the interview may not be generalisable to all Grade 11 learners.

Another aspect is the limited time that was spent on the content. It is also a concern that the curriculum planners and the implementers have not given trigonometric function graphs as a topic serious consideration. It is a topic that is paced in such a way that it clashes with the mid-year examination most of the time. Most schools actually teach it at a rapid pace to just tick the box. Also, I intended to use five schools with different characteristics, but two other schools did not grant me access because some other curricular activities needed to be done without fail. Therefore, I feel that some of the errors recorded might be minimised if trigonometric function graphs content is given preference like any other topic in the family of functions for it to gain its rightful level of success.

5.4 Implications of the Study

- During curriculum planning at various district levels, priority should be given to trigonometric function graphs.
- Educators may use error analysis of trigonometric function graphs as a teaching strategy.
- Educators should especially focus on making deductions from the graphs of trigonometric functions and also emphasise the need to follow all instructions for drawing graphs.
- This study may contribute as a reference for future research studies in a similar area of study.
- Teachers may develop tasks to help learners improve their understanding and ability in trigonometric functions.
- The awareness of trigonometric function graphs' errors through providing meaningful examples, may challenge teachers to create engaging learning environments that could contribute to learners developing deep understanding and mastery of the content.
- The information provided on learning errors in this study, may serve as a motivation to the DBE to provide more resources and training for in-service teachers on how to improve the delivery of trigonometric function graphs to learners, which will ultimately improve Grade 12 results and encourage further studies in the field of mathematics, science, technology and engineering.

5.5 Recommendations for the Study

In view of the findings of this study, it is then recommended that further research is done on:

- Investigating the relationship between allocated teaching time (as espoused in the South African mathematics school curriculum) and performance in trigonometric function graphs;

- Exploring errors made by Grade 11 learners in trigonometric function graphs using a mixed method approach; and
- Replicating this study with a large sample of learners.

5.6 Reflections

In this section, I reflect on the study and my personal experiences. These reflections may contribute to further studies in this field.

5.6.1 Reflections on the Study

The conceptual framework that was designed in this study was sufficient and served the study's objectives, namely, different types of errors and possible causes of errors. Combining different researchers' work in this framework was interesting and informative. This framework can also be used in studies that focus on any other mathematical topic. The description of the types of errors guided the analysis of data, and it also served as a springboard to facilitate the interview process. Furthermore, the descriptions of the causes of errors in the conceptual framework assisted me in identifying the possible causes of errors that emanated from the test.

The CAPS document and past examination questions were helpful in guiding the construction of the test items. Also, the categorisation of the test items into four concepts gave the study structure, especially when I had to analyse the data and report on the findings. Apart from this, the purposive sampling of learners' scripts provided useful insights into the errors made in trigonometric function graphs. The trigonometric achievement test and semi-structured interviews were suitable for the nature of the study and effective for collecting the relevant data. For example, the test gave the learners an opportunity to present their understanding on the concept learned, while the semi-structured interviews were useful in the exploration of causes of errors made. Some test items were repeated to see the consistency/inconsistency in errors made to get rich data from the test. Also, to get rich data from the interviews, the sampled learners were probed and prompted to explain what informed their test solutions.

However, the data used for this research study could have been richer if all 30 sampled learners had been interviewed. These interviews were not done because this study is of limited scope, and time was a major constraint. As a result of these factors, the interviewees were limited to 15 learners.

5.6.2 Personal Reflections

My journey into the teaching profession has placed a burden on me of the need to contribute my quota to improving mathematics education in South Africa. In my workspace as a mathematics teacher in a secondary school, I often ponder and ask myself a pertinent question: 'Why do learners make mistakes?' I sometimes also try in my teaching to reflect on learners' incorrect responses. Every time I just realised the importance of taking note of learners' errors and misunderstandings, I let that inform my planning and teaching. With this in mind, I decided to pursue my postgraduate studies in mathematics education, focusing on analysing learning errors in any topic of mathematics that is red-flagged for poor performance.

In the course of this study, I was able to gain insights into some of the challenges learners encounter in mathematics. In my bid to have a researchable topic, I obtained a diagnostic report, a document that comprehensively gives all stakeholders feedback on matric performance yearly. In short, the journey has improved my research skills and abilities, instilling in me the spirit of a life-long learner. However, this experience of doing my postgraduate studies has been hectic, time-consuming and financially demanding for me.

5.7 Conclusion

In this chapter, a conclusion to the research study is presented. This qualitative study, through an exploratory case study, was located within the social constructivist interpretive paradigm. It was conducted through the administering of a test and interviews with Grade 11 learners from three public schools in Gauteng province. The conceptual framework that guided the study was designed from the work of Brown and Skow (2016), Newman (1977), Oktaviani (2017), Radatz (9), and Smith et al. (1993). Deductive data analysis was used to analyse the data according to pre-determined categories in the framework. Grade 11 learners' errors in trigonometric function graphs are comprehension, encoding, and misconception errors. These errors are found to be caused by: difficulty experienced by Grade 11 learners in obtaining spatial information, poor mastery of pre-requisite skills, facts, and concepts, and application of irrelevant rules or strategies.

I hope that the findings of this study will inform learners on which errors to avoid and also be incorporated as a feedback tool for teachers to enrich the feedback given to their learners, and that all these findings may positively influence other teachers' practices.

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6. APPENDICES

6.1 Appendix A Requesting Permission: Letter to the Gauteng Department of Education

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

Mrs R.A. Adebayo
SMTE Department
Groenkloof campus
University of Pretoria
ronkeadebayo_fds@yahoo.com
0746546271

30 June 2021

Dear Sir/madam

LETTER OF CONSENT TO THE GAUTENG DEPARTMENT OF EDUCATION (GDE) TSHWANE SOUTH (D4)

I am Mrs Ronke Adebayo, a mathematics teacher at a secondary school in Tshwane District. I have enrolled for my Master's degree at the University of Pretoria at the Department of Science, Mathematics and Technology Education. Part of the requirements for awarding this degree is the successful completion of a significant research project in the field of education. The title of my approved research study is **"**An analysis of Grade 11 learners' errors in trigonometric graphs".

The purpose of the study is to investigate the errors that Grade 11 learners make in trigonometric function graphs to suggest ways of mitigating the problem learners have in trigonometry Five schools in the Tshwane district will be invited to participate in this research project.

To gather the information that is needed for this research, I will administer a trigonometric function graph test to all the Grade 11 learners of the schools that are participating in this study. I request your permission to administer a trigonometric function graph achievement test on these learners. The teachers will be asked to assist in administering the test. The test will take about 60 minutes. Based on the test results, I will purposefully choose six learners and invite them for an interview so they can provide more clarity on their thought processes. I will do an individual semi-structured interview with the learners that will take about 45minutes. Ethical issues that relate to the COVID-19 global pandemic will be strictly adhered to in the course of testing and interview, such as social distancing, wearing of face masks and sanitizing of hands.

Learners' participation is voluntary and their participation will not be advantageous to them, nor will withdrawal at any stage of the study result in any disadvantage or penalty. All the information obtained during the research study will be treated confidentially. At no time will either the Grade 11 learners' names or participating schools be mentioned by name or indeed be allowed to be identified in any manner or means whatsoever in the research report. At the end of the research study, you will be provided with a copy of the research report containing both the findings and recommendations of the study.

Thanking you in anticipation.

Botha

Mrs R. A. Adebayo
Student researcher
University of Pretoria
ronkeadebayo_fds@yahoo.com
074 654 6271

Dr J.J. Botha
Supervisor
University of Pretoria
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+27 12 420 562

6.2 Appendix B Gauteng Department of Education Research Approval Letter

GAUTENG PROVINCE
Department: Education
REPUBLIC OF SOUTH AFRICA

8/4/4/1/2

GDE RESEARCH APPROVAL LETTER

Date:	**14 October 2021**
Validity of Research Approval:	**08 February 2022– 30 September 2022** **2021/283A**
Name of Researcher:	**Adebayo R.A**
Address of Researcher:	**Unit 0052, 504 Jorissen Street**
	Sunnyside.
	Pretoria
Telephone Number:	**074 654 6271**
Email address:	**ronkeadebayo_fds@yahoo.com**
Research Topic:	**An analysis of Grade 11 learners' errors in trigonometric function graphs.**
Type of qualification	**M-Ed**
Number and type of schools:	**5 Secondary Schools**
District/s/HO	**Tshwane South**

Re: Approval in Respect of Request to Conduct Research

This letter serves to indicate that approval is hereby granted to the above-mentioned researcher to proceed with research in respect of the study indicated above. The onus rests with the researcher to negotiate appropriate and relevant time schedules with the school/s and/or offices involved to conduct the research. A separate copy of this letter must be presented to both the School (both Principal and SGB) and the District/Head Office Senior Manager confirming that permission has been granted for the research to be conducted.

15/10/2021

The following conditions apply to GDE research. The researcher may proceed with the above study subject to the conditions listed below being met. Approval may be withdrawn should any of the conditions listed below be flouted:

1. *Letter that would indicate that the said researcher/s has/have been granted permission from the Gauteng Department of Education to conduct the research study.*

1

Making education a societal priority

Office of the Director: Education Research and Knowledge Management
7th Floor, 17 Simmonds Street, Johannesburg, 2001
Tel: (011) 355 0488
Email: Faith.Tshabalala@gauteng.gov.za
Website: www.education.gpg.gov.za

3. ***Because of COVID 19 pandemic researchers can ONLY collect data online, telephonically or may make arrangements for Zoom with the school Principal. Requests for such arrangements should be submitted to the GDE Education Research and Knowledge Management directorate. The approval letter will then indicate the type of arrangements that have been made with the school.***
4. ***The Researchers are advised to make arrangements with the schools via Fax, email or telephonically with the Principal.***
5. *A copy of this letter must be forwarded to the school principal and the chairperson of the School Governing Body (SGB) that would indicate that the researcher/s have been granted permission from the Gauteng Department of Education to conduct the research study.*
6. *A letter / document that outline the purpose of the research and the anticipated outcomes of such research must be made available to the principals, SGBs and District/Head Office Senior Managers of the schools and districts/offices concerned, respectively.*
7. *The Researcher will make every effort obtain the goodwill and co-operation of all the GDE officials, principals, and chairpersons of the SGBs, teachers and learners involved. Persons who offer their co-operation will not receive additional remuneration from the Department while those that opt not to participate will not be penalised in any way.*
8. *Research may only be conducted after school hours so that the normal school programme is not interrupted. The Principal (if at a school) and/or Director (if at a district/head office) must be consulted about an appropriate time when the researcher/s may carry out their research at the sites that they manage.*
9. *Research may only commence from the second week of February and must be concluded before the beginning of the last quarter of the academic year. If incomplete, an amended Research Approval letter may be requested to conduct research in the following year.*
10. *Items 6 and 7 will not apply to any research effort being undertaken on behalf of the GDE. Such research will have been commissioned and be paid for by the Gauteng Department of Education.*
11. *It is the researcher's responsibility to obtain written parental consent of all learners that are expected to participate in the study.*
12. *The researcher is responsible for supplying and utilising his/her own research resources, such as stationery, photocopies, transport, faxes and telephones and should not depend on the goodwill of the institutions and/or the offices visited for supplying such resources.*
13. *The names of the GDE officials, schools, principals, parents, teachers and learners that participate in the study may not appear in the research report without the written consent of each of these individuals and/or organisations.*
14. *On completion of the study the researcher/s must supply the Director: Knowledge Management & Research with one Hard Cover bound and an electronic copy of the research.*
15. *The researcher may be expected to provide short presentations on the purpose, findings and recommendations of his/her research to both GDE officials and the schools concerned.*
16. *Should the researcher have been involved with research at a school and/or a district/head office level, the Director concerned must also be supplied with a brief summary of the purpose, findings and recommendations of the research study.*

The Gauteng Department of Education wishes you well in this important undertaking and looks forward to examining the findings of your research study.

Kind regards

....

Mrs Faith Tshabalala
Acting Director: Education Research and Knowledge Management

DATE: 15/10/2021

6.3 Appendix C Requesting Permission: Letter to Principal

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

Mrs R.A. Adebayo
SMTE Department
Groenkloof campus
University of Pretoria
ronkeadebayo_fds@yahoo.com
0746546271

2 May 2022

Dear Sir/Madam

REQUEST FOR PERMISSION TO CONDUCT RESEARCH AT YOUR SCHOOL

I am Mrs Ronke Adebayo, a mathematics teacher at a secondary school in Tshwane District. I have enrolled for my Master's degree at the University of Pretoria at the Department of Science, Mathematics and Technology Education. Part of the requirements for awarding this degree is the successful completion of a significant research project in the field of education. The title of my approved research study is **"An analysis of Grade 11 learners' errors in trigonometric graphs"**.

The purpose of the study is to investigate the errors that Grade 11 learners make in trigonometric function graphs to suggest ways of mitigating the problem learners have in trigonometry Five schools in the Tshwane district will be invited to participate in this research project.

To gather information, I request your permission to administer a trigonometric graph achievement test on the learners so that I can identify the errors and misconceptions that are associated with the learning of trigonometric graphs. The test will be written by all Grade 11 learners. The Grade 11 mathematics teacher(s) will be requested to assist in administering the test. Based on the test results, I will purposefully choose six learners and invite them for individual semi-structured interviews so they can provide more clarity on their thought processes. The test will take about 60 minutes and the interview about 45 minutes per learner. Ethical issues that relate to the COVID-19 global pandemic will be strictly adhered to by their teachers and the researcher, such as social distancing, wearing of face mask, and sanitizing of hands.

Learners' participation is voluntary and will not result in any advantage, nor withdrawal from participation at any stage of the study result in any disadvantage or penalty. All the information obtained during the research study will be treated confidentially. At no time will either the Grade 11 learners' names or participating schools be mentioned by name or indeed be allowed to be identified in any manner or means whatsoever in the research report. At the end of the research study, you will be provided with a copy of the research report containing both the findings and recommendations of the study.

I would like to request your permission to use your data, confidentially and anonymously for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and using the data for future teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Should you have any questions or concerns pertaining to this study, you may contact the researcher Mrs R.A. Adebayo or the supervisor Dr JJ Botha.

Thanking you in anticipation.

Mrs R.A. Adebayo
Student researcher
University of Pretoria
ronkeadebayo_fds@yahoo.com
074 654 6271

Dr J.J. Botha
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LETTER OF INFORMED CONSENT: PRINCIPAL

SCHOOL AS PARTICIPANT

VOLUNTARY PARTICIPATION IN THE RESEARCH PROJECT ENTITLED
An Analysis of Grade 11 Learners' Errors in Trigonometry Graphs

I, ____, the principal of

____, hereby voluntarily and willingly agree to allow the school under my jurisdiction to participate in the above-mentioned project introduced and explained to me by Mrs Ronke Adebayo, currently, a student enrolled for an MEd degree at the University of Pretoria.

I further declare that I understand, as explained to me by the researcher, the aim, scope, purpose, possible consequences, benefits, and methods of collecting information proposed by the researcher, as well as how the researcher will attempt to ensure the confidentiality and integrity of the information that she collects.

________________________________ ______________________
Full name Signature

____
Date

6.4 Appendix D Letter of Consent to the Mathematics Educator

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

Mrs. R.A. Adebayo
SMTE Department
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ronkeadebayo_fds@yahoo.com
0746546271

2 May 2022

Dear Sir/Madam

LETTER OF INFORMED CONSENT TO THE GRADE 11 MATHEMATICS TEACHER

I am Mrs Ronke Adebayo, a mathematics teacher at a secondary school in Tshwane District. I have enrolled for my Master's degree at the University of Pretoria at the Department of Science, Mathematics and Technology Education. Part of the requirements for awarding this degree is the successful completion of a significant research project in the field of education. The title of my approved research study is **"An analysis of Grade 11 learners' errors in trigonometric graphs"**.

I have obtained consent from the GDE, the data for the research will be collected through a trigonometric function graph test administered to all the Grade 11 learners to enhance the reliability and validity of the study. I will request your expertise in setting the test questions to enhance the validity of my instruments. I will also request your help in administering the test to reduce contact with your learners. Furthermore, 30 Grade 11 mathematics learners' scripts will be randomly selected from all marked scripts for data analysis by the researcher. Thereafter, I will do an individual semi-structured interview with the learners which will take about 45 minutes. These learners will be six, depending on the outcomes of the test.

All participation is voluntary and participating learners may withdraw from this study at any time. Pseudonyms will be used for all the parties (schools, teachers, and learners) involved to guarantee confidentiality and anonymity. Only my supervisors and I will have access to the audio recordings which will be password protected. The study will be conducted in English and there will be no incentives for the participating schools or teachers.

After the successful completion of my Master's degree, I will give feedback to the GDE in the form of a written report, and if the GDE is willing, I would like to do a PowerPoint presentation of my findings to the mathematics subject facilitators and teachers.

I would like to request your permission to use your data, confidentially and anonymously for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and using the data for future teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Should you have any questions or concerns pertaining to this study, you may contact the researcher Mrs R.A. Adebayo or the supervisor Dr JJ Botha.

Thanking you in anticipation.

Mrs R. A. Adebayo
Student researcher
University of Pretoria
ronkeadebayo_fds@yahoo.com
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Dr J.J. Botha
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TEACHER CONSENT FORM

VOLUNTARY PARTICIPATION IN THE RESEARCH PROJECT ENTITLED
An Analysis of Grade 11 Learners' Errors in Trigonometric Graphs

I hereby grant consent to Mrs R.A. Adebayo to conduct her research with my Grade 11 Mathematics learners for her Master's research project. I hereby also grant consent to Mrs R. A. Adebayo to, with my assistance, administer a test to the learners.

Grade 11 teacher's name: ____________________

Grade 11 teacher's signature: ___________________

Date: __________________________

Email address: _____________________________

Contact number: ___________________________

6.5 Appendix E Letter of Consent to the Parent(S)/Guardians

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

Mrs. R. A. Adebayo
SMTE Department
Groenkloof campus
University of Pretoria
ronkeadebayo_fds@yahoo.com
0746546271

2 May 2022

Dear Sir/Madam

LETTER OF INFORMED CONSENT TO THE PARENT(S)/GUARDIAN(S)

I am Mrs. Ronke Adebayo, a mathematics teacher at a secondary school in Tshwane District. I have enrolled for my Master's degree at the University of Pretoria at the Department of Science, Mathematics and Technology Education.

My research is aimed at investigating the errors Grade 11 learners make in trigonometric function graphs. The research will be reported on in my Master's dissertation at the University of Pretoria. To do the research, I request your permission to administer one trigonometric achievement test to your child with the help of his/her teacher. Your child may be interviewed, based on the outcome of the test. The interview will help to provide more clarity on his/her thought processes. Ethical issues relating to the COVID-19 global pandemic will be considered by your child's mathematics teacher and the researcher in the course of testing and interview. Your child’s health and safety will be prioritised above any other interest.

All learner’s confidentiality and anonymity will be protected at all times and only my supervisor and I will have access to the interview recordings and test results. By signing this letter, you will be permitting me to conduct this research with your child with the help of your child’s mathematics teacher.

Thanking you in anticipation

Mrs R. A. Adebayo
Student researcher
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6.6 Appendix F Letter of Assent to the Learners

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Faculty of Education

Mrs R.A. Adebayo
SMTE Department
Groenkloof campus
University of Pretoria
ronkeadebayo_fds@yahoo.com
0746546271

2 May 2022

Dear learner

LETTER OF INFORMED ASSENT TO THE LEARNERS

I am Mrs. Ronke Adebayo, a mathematics teacher at a secondary school in Tshwane District. I have enrolled for my Master's degree at the University of Pretoria at the Department of Science, Mathematics and Technology Education. My research project aims at investigating the errors Grade 11 learners make in trigonometric graphs.

I will be administering a trigonometric graph achievement test to you. if you are willing to participate. I may also do an individual semi-structured interview with you based on your test outcome. The interview will help to provide more clarity on your thought processes. The test will take about 60 minutes and the interview will take about 45 minutes. Ethical issues that relate to the COVID-19 global pandemic will be considered in the course of testing and interview. Your health safety will be prioritised above any other interest.

I would like to request your permission to use your data, confidentially and anonymously for further research purposes, as the data sets are the intellectual property of the University of Pretoria. Further research may include secondary data analysis and using the data for future teaching purposes. The confidentiality and privacy applicable to this study will be binding on future research studies.

Should you have any questions or concerns pertaining to this study, you may contact the researcher Mrs R.A. Adebayo or the supervisor Dr JJ Botha.

Thanking you in anticipation.

Mrs R. A. Adebayo
Student researcher
University of Pretoria
ronkeadebayo_fds@yahoo.com
074 654 6271

Dr J.J. Botha
Supervisor
University of Pretoria
hanlie.botha@up.ac.za
+27 12 420 5623

LEARNER ASSENT FORM

VOLUNTARY PARTICIPATION IN THE RESEARCH PROJECT ENTITLED:

An Analysis of Grade 11 Learners' Errors in Trigonometric Graphs

I hereby grant permission to Mrs R.A. Adebayo to participate in the test.

I also assent to be interviewed ☐ or I do not assent to be interviewed after the test

☐ (Please tick one of the boxes).

If you have any questions, you may contact me at any time.

Learner's name: ____________________

Learner's signature: ___________________

Date: __________________________

Grade (e.g. 11A): ___________________________

6.7 Appendix G Test Instrument

MATHEMATICS GRADE 11
TRIGONOMETRIC FUNCTION GRAPHS TEST

Time: 1½ Hours

Moderator: Mr AG Bennie

Examiner: Mrs RA Adebayo

Total: 40 Marks

Name of Learner:____

Gender:______________**Age:**____

Name of Subject Teacher:____

Name of School:____

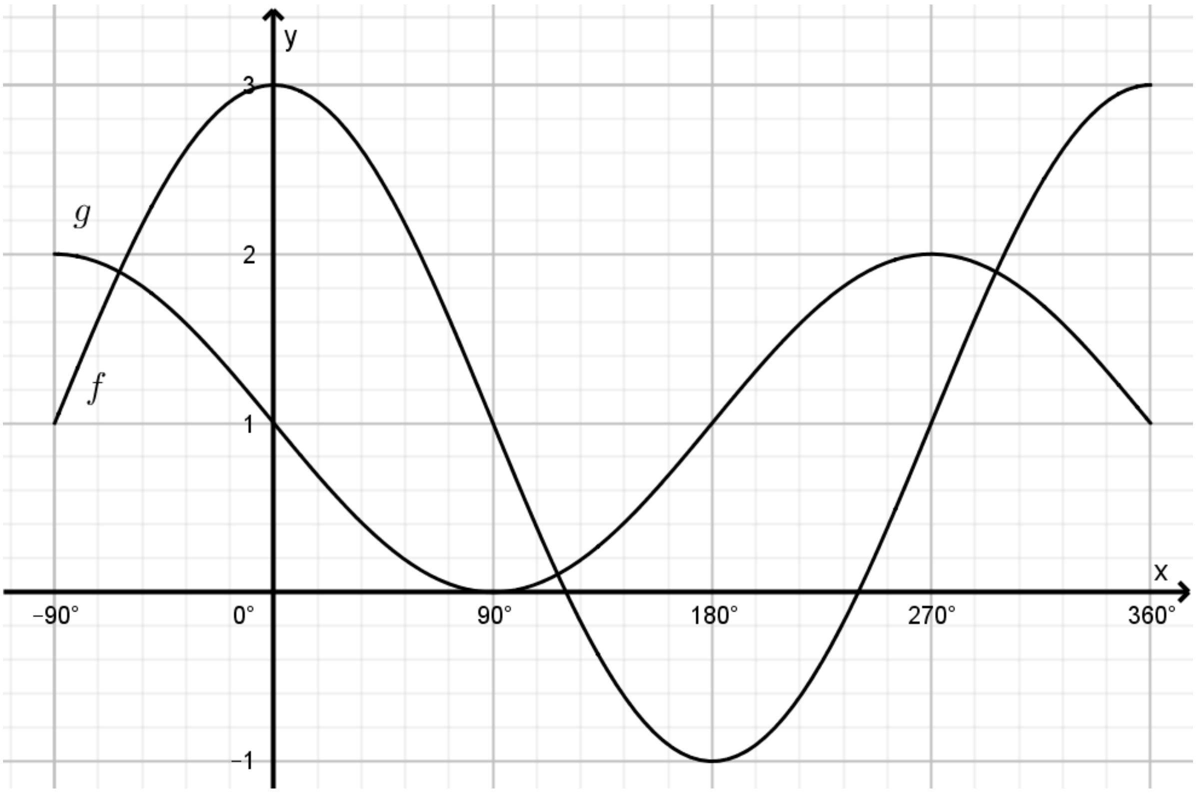
Date of Assessment:____

INSTRUCTIONS AND INFORMATION

1. This Question Paper consists of FOUR compulsory questions. Answer all of them.
2. Number the answers correctly according to the numbering system used in this question paper.
3. Clearly show ALL calculations, graphs, diagrams etcetera that you have used in determining your answers.
4. You may use an approved scientific calculator BUT a Programming and Graphical Calculator may NOT BE used.
5. If necessary, round-off answers to TWO decimal places UNLESS otherwise stated.
6. Write neatly and legibly.
7. Use Attachment A at the end of the question paper to draw the graphs required in Questions 2.1 and . 4.1.

QUESTION 1

Given the graphs of the functions $f(x) = 2\cos x + 1$ and $g(x) = 1 - \sin x$ for the interval $x \in [-90°; 360°]$.



1.1	If the graph of f is shifted 45° to the left, write down the equation of the new function.	(2)
1.2	Determine graphically, the value(s) of x for which:	
	1.2.1 $f(x) - g(x) = 0$	(2)
	1.2.2 $f(x) - g(x) = 2$	(2)
1.3	State the range of the function g.	(2)
1.4	The function g is reflected about the x-axis to form a new function. Write down the equation of the new function in the form $y = \cdots$	(2)

[10]

QUESTION 2

Consider the functions $f(x) = \cos(x - 30°)$ and $g(x) = \sin 2x$.

On the grid provided in Attachment A, draw the graphs of f and g for $x \in [-180°; 180°]$. (5)

Clearly show ALL intercepts with the axes, turning points and end points.

Write down the amplitude of g. (1)

State the range of f. (2)

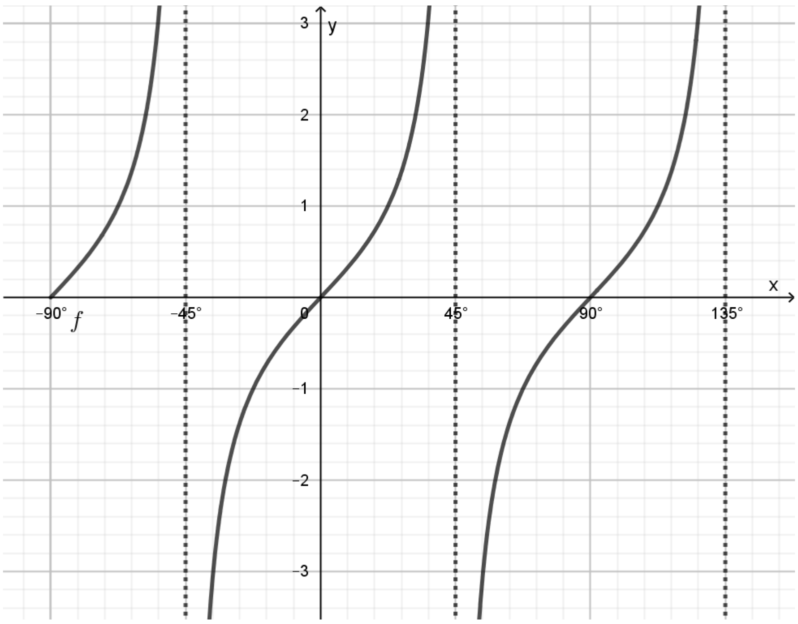
Use your graph to estimate the x –coordinates of the points of intersection between f and g. (2)

Determine the equation of the new function h, if h is the image of the graph g shifted 45° to the right. (2)

[12]

QUESTION 3

Given the graph of $f(x) = \tan bx$ for the interval $x \in [-90°; 135°)$.



3.1 Determine the value of *b*. (1)

3.2 State the asymptotes of f. (3)

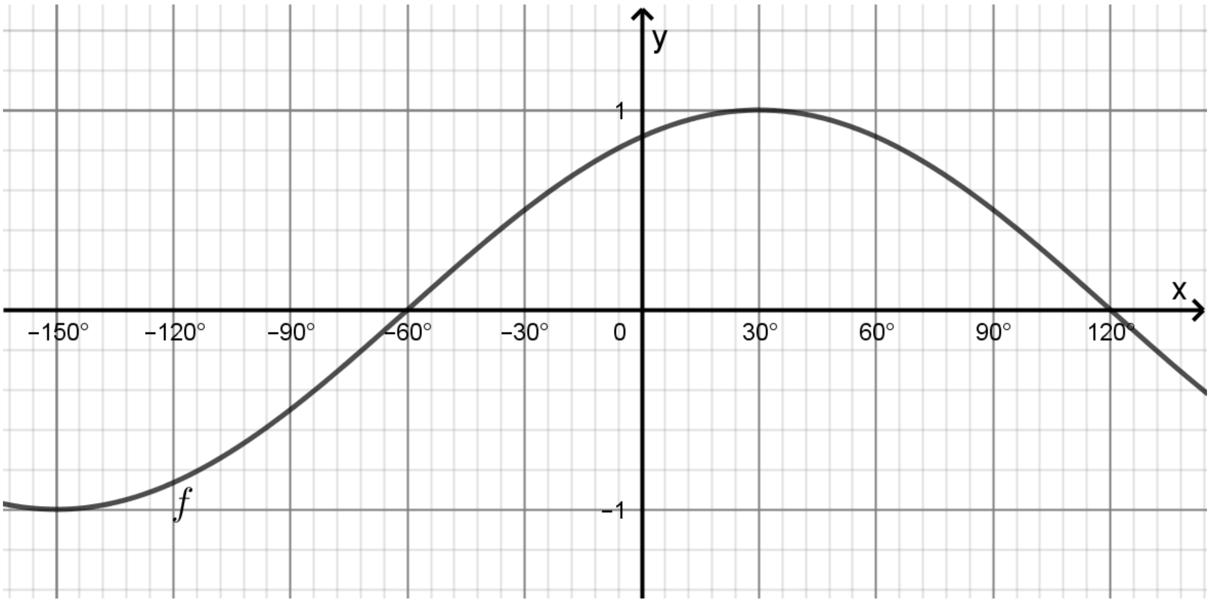
3.3 Write down the period of *f*. (2)

3.4 Determine the equation of the new function g, if g is the image of *f* shifted 45° to the left. (2)

[7]

QUESTION 4

In the diagram, the graph of $f(x) = \sin(x + 60°)$ is drawn on the interval $-150° \leq x \leq 120°$.



4.1 On the Attachment A provided, draw the graph of $k(x) = -\cos x$ for the interval $-150° \leq x \leq 120°$. Show ALL the intercepts with the axes as well as the coordinates of the turning points and end points of the graph. (4)

4.2 Determine the minimum value of $h(x) = \sin(x + 60°) - 4$. (2)

4.3 Use your graph to estimate the value(s) for *x* if: (3)

$$\sin(x + 60°) + \cos x = 0$$

4.4 Hence, determine the values of x for the interval $-150° \leq x \leq 120°$ for which $\sin(x + 60°) + \cos x > 0$. (2)

[11]

TOTAL: 40

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA

Attachment to the test

MATHEMATICS GRADE 11
TRIGONOMETRIC FUNCTION GRAPHS TEST

Name of Learner:____

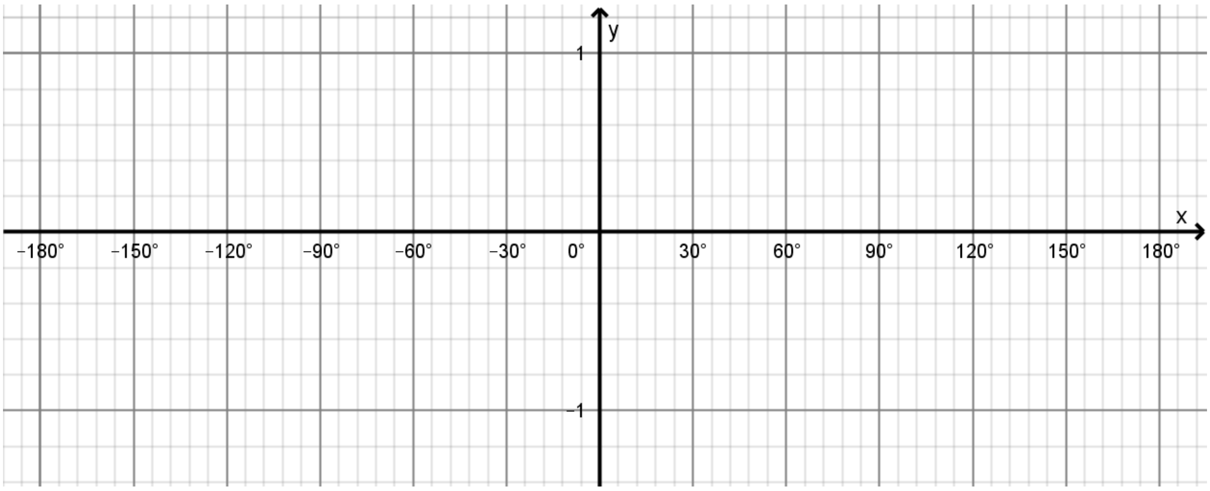
Gender:____________**Age:**_____________________________________

Name of Subject Teacher:_______________________________________

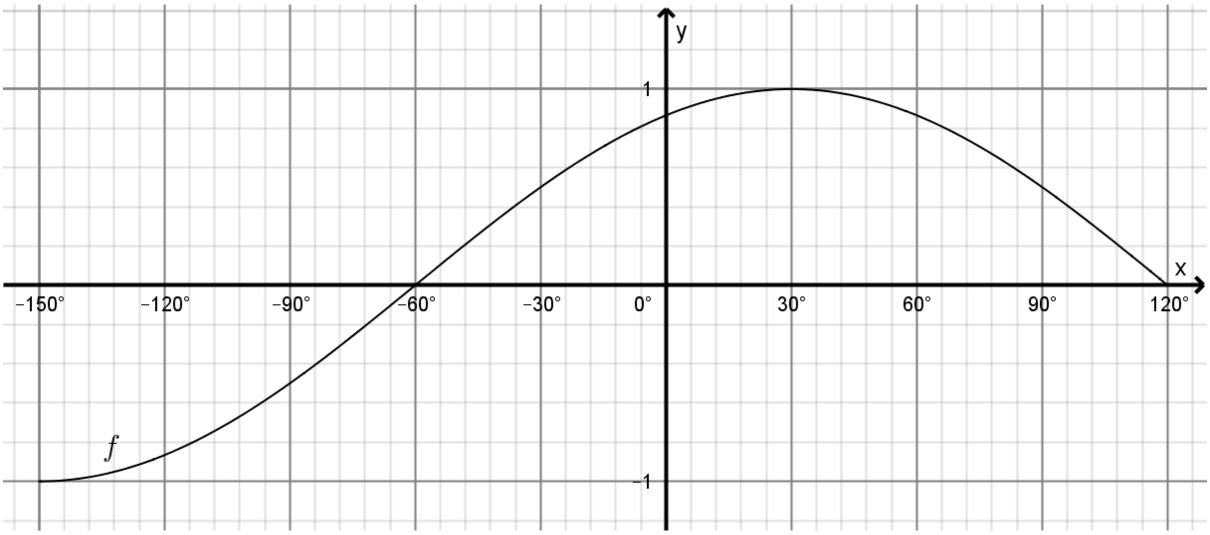
Name of School:____

Date of Assessment:____

QUESTION 2.1



QUESTION 4.1



UNIVERSITEIT VAN PRETORIA
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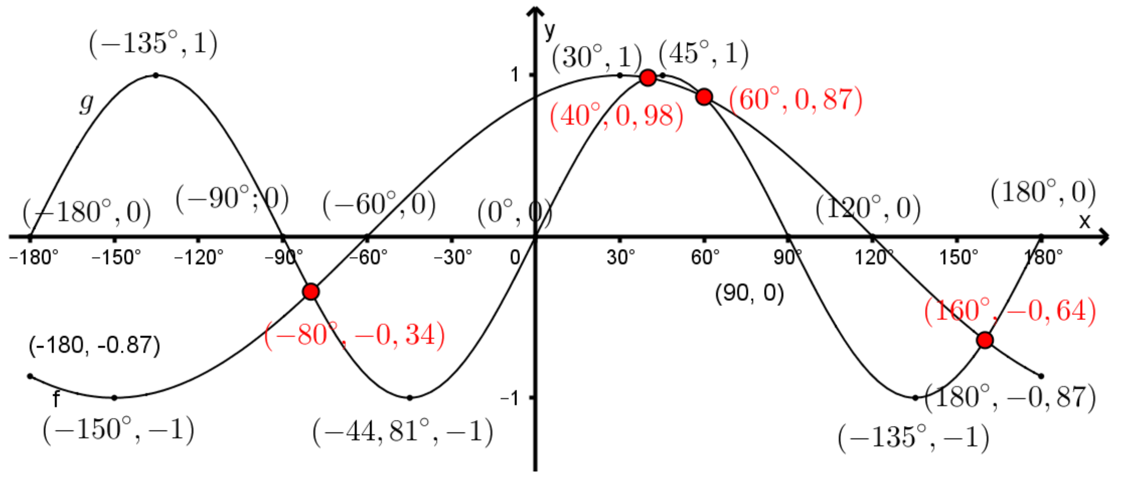
MARKING GUIDELINES

MATHEMATICS GRADE 11
TRIGONOMETRIC FUNCTION GRAPHS TEST

MARKING GUIDELINES

1.1	$y = f(x + 45°)$ $= 2\cos(x + 45°) + 1$ ✓✓	✓✓ Answer only (2)
1.2.1	$f(x) - g(x) = 0$ $\therefore f(x) = g(x)$ $\therefore x \in \{-63°, 117°, 297°\}$ **Allow $\pm 3°$ for the three estimated values.**	✓ one correct value ✓ all values correct (2)
1.2.2	$x \in \{0°; 180°; 360°\}$	✓ one correct value ✓ all values correct (2)
1.3	$\mathcal{R}_g = [0; 2]$ ✓✓	✓ critical values ✓ notation (2)
1.4	$y = -g(x)$ $= -(1 - \sin x)$ $= -1 + \sin x$ ✓✓	✓✓ answer only (2)
		[10]

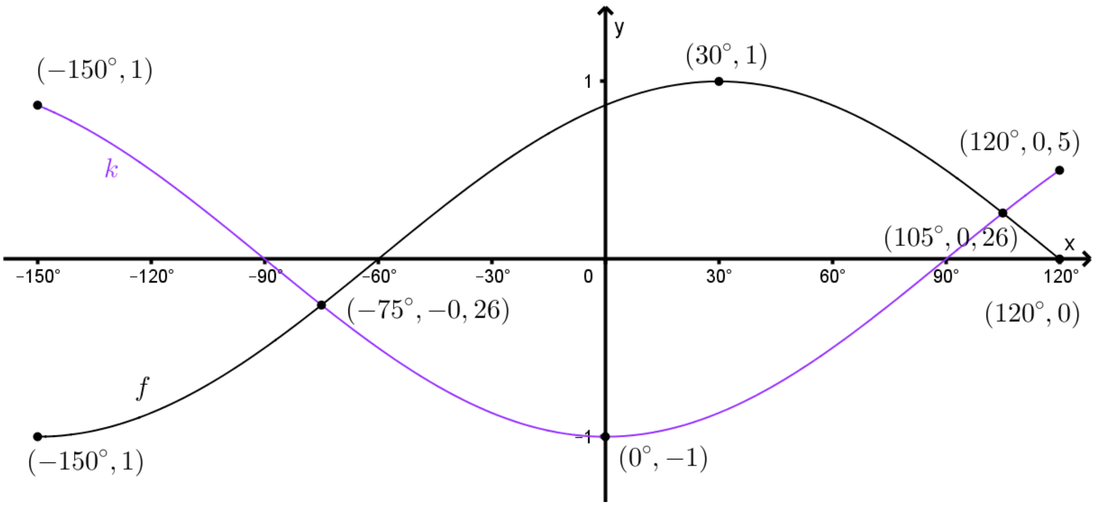
2.1	✓ Shape of f ✓ Shape of g ✓ Turning points identified	✓ *f*'s x-intercepts ✓*g*'s intercepts (5)



2.2	1✓	✓ Answer only (1)
2.3	$\mathcal{R}_f = [-1✓; 1]✓$	✓ critical values ✓ notation (2)
2.4	$x \in \{-80° \pm 3°; 40° \pm 3°; 60°; 160° \pm 3°\}$	**CA MARKING!** ✓ half (rounded down) of the points identified ✓ all the points identified **ALLOW ±3° IN TERMS OF ACCURACY** (2)
2.5	$h(x) = g(x - 45°)$ $= \sin(2(x - 45°))$ ✓ $= \sin(2x - 90°)$✓	✓ replace x with x – 45 ✓ simplification (2)
		[12]

3.1	$b = 2$✓	✓ Answer only (1)
3.2	$x = \pm 45°$ ✓✓ and $x = 135°$✓	✓✓✓ each equation (3) **NOTE:** no other answers are acceptable
3.3	$90°$✓	✓Answer only (1)
3.4	$g(x) = f(x + 45°)$ $= \tan\big(2(x + 45°)\big)$ ✓ $= \tan(2x + 90°)$ ✓	✓ replace x with x + 45 ✓ final answer (2)
		[8]

4.1	✓ correct intercept with the axes ✓ Shape of k	✓ All turning points identified ✓ All end points identified (4)



4.2	$\min h = -1 - 4 = -5$✓✓	✓✓ Answer only (2)
4.3	$\sin(x + 60°) + \cos x = 0$ $\sin(x + 60°) = -\cos x$ $f(x) = k(x)$ $\therefore x \in \{-75° \pm 3°$✓$; 105° \pm 3°$✓$\}$	**CA MARKING!** ✓ for half (rounded down) of the points identified ✓for all points identified (2) **ALLOW ±3° IN TERMS OF ACCURACY**
4.4	$\sin(x + 60°) + \cos x > 0$ $\sin(x + 60°) > -\cos x$ $f(x) > k(x)$ $\therefore x \in (-75°; 105°)$✓✓	✓ Concept of “between” the critical values ✓ Critical values (2)

6.8 Appendix H Learner Interview Schedule

SEMI-STRUCTURED INTERVIEW WITH LEARNERS

Time: 30minutes

Interviewer: Mrs RA Adebayo

Name of Interviewee:____

Gender:_____________**Age:**____

Name of Subject Teacher:____

Name of School:____

Date of Interview:____

QUESTIONS ON THE CAUSES OF ERRORS

Question 1.1

- What is the meaning of value(s) in the introductory part of the question?
- What are you required to do?
- Please write the new function for me on the question paper.
- When a graph is shifted to the left, which variable is changing?
- Why do we write +45° when we say the graph is shifted to the left?
- Talk me through your solution in your answer script.

Question 1.2

- What does graphically in the question imply?
- Do you think the (–) symbol means the same as difference in other mathematics problems?
- The symbol '- 'was used in question 1.2.1 and 1.2.2, do you think it means the same thing as you have used previously in Mathematics? Example: 8 – 2 = 6
- Talk me through your thinking and solution in your answer script.

Question 1.3

- Can you show me range on your graph?
- What is the difference between range and domain?
- In Question 1.3, which part of the graph helps you to figure out your range?

Question 1.4

- Please read Question 1.4 aloud.
- Please take a look at coordinate $P\ (-7;6)$, please reflect this coordinate about the x-axis.
- What would be the image of P if it is reflected about the y-axis?

Question 2.1

- In Question 2.1, please show me how you were able to determine the points on the graphs drawn.
- How do you determine the corresponding y-values to all the x-values based on the restrictions given?
- The instruction in this question required you to clearly show ALL intercepts with the axes, turning points and end points. Why did you not do this?

Question 2.2

- How best do you describe 'amplitude' in Question 2.2?
- Show me the two points on the graph of $g(x)$ that helps you to determine the amplitude.
- Now tell me what the amplitude of Question 2.2 is.

Question 2.4

- In Question 2.4 the word 'intersect' was used, what do you understand by this word?
- Please show me the part of the graph that satisfies the functions.

Question 2.5

- If a graph is said to be shifted to the left, should we write +45° or -45°?
- Can you show me your new function?

Question 3.1

- In Question 3.1 what does' b' indicate in the graph?
- Please state the asymptote of: $y = tanx$.
- Consider the function y = $\frac{tanx}{2}$, State the asymptote of the function.
- Please state the period of the pre-listed functions respectively.

Question 3.2

- Now in Question 3.2, state the asymptotes of 'f'.

Question 3.3

- Please state the period of the pre-listed functions in Question 3.1 respectively.

Question 4.2

- In Question 4.2, what kind of transformation took place? Which of this is correct? Reflection, rotation or translation?
- Shifted “-4” means what? Upward shift or downward shift?
- With this in mind, please show me the equation of your new function.

Question 4.3

- In Question 4.3 what would you do first in order to give an answer to this question?
- What does ‘hence’ mean in this question?
- In Question 4.3 you were told to use your graph to estimate the value(s) for x if: $\sin(x + 60°) + \cos x = 0$. Please point these points to me on this graph.
- Do you think there is anything common to Question 2.4 and 4.3?

Question 4.4

- Do you see any relationship between Question 4.3 and 4.4?
- Please spot the difference between the two questions.
- What solution would you give for Question 4.4?

6.9 Appendix I Table of Learners' Scores

		Q1	Q2	Q3	Q4	TOTAL	% Scores
		10	12	7	11	40	100%
Lwazi		1	5	0	1	7	18
Thato 2		0	5	0	1	6	15
Nulayiso		0	1	4	3	8	20
Kabelo		4	0	3	2	9	23
Koletso		0	4	3	1	8	20
Skosana		4	1	1	0	6	15
Oyama		4	0	3	0	7	18
Motshweni		2	0	3	0	5	13
Thandiwe		2	3	3	0	9	23
Thapelo		2	3	3	0	8	20
Kamogelo		0	3	2	0	5	13
Unathi		4	3	0	0	7	18
Iwm		6	4	0	0	10	25
Thato		2	3	3	0	8	20
Kelly		2	1	3	0	6	15
Daniels		2	0	3	0	5	13
Tsholofelo		0	3	2	0	5	13
Legwabe		2	2	2	4	10	25
Lesedi		0	0	2	3	5	13
Lindiwe		2	3	4	0	9	23
Baloyi		4	6	5	6	21	53
Avdrey		0	1	4	3	8	20
Modiba		0	3	4	1	8	20
Simbarashe		0	3	0	4	8	20
Ngobeni		0	2	3	2	8	20
Rebecca		0	4	3	1	8	20
Olivia Moke		0	4	4	2	6	15
Mabena		0	1	3	1	5	13
Maluleke		0	3	0	1	4	10
Mhuyi		0	1	5	0	6	15
Kutloana		0	2	3	0	5	13
Joyce		3	2	1	0	6	15
Reolebohile		6	0	1	0	7	18
Bonisile Siyengo		1	2	4	1	8	20
Junior		0	4	2	1	7	18

Themane		0	2	4	2	8	20
Goergina		2	2	4	1	9	23
Anonymous		3	0	4	3	10	25
Mandisa		8	7	5	7	30	75
TM		3	2	2	0	7	18
OM		7	6	4	1	18	45
AB		3	4	6	3	16	40
MO		4	9	5	3	21	53
China		2	7	5	1	17	43
Queen		8	7	5	1	18	45
MR		2	4	3	1	12	30
Kgalalelo		4	2	3	0	9	23
KNM		3	3	4	0	9	23
Sakalunda		4	4	4	2	15	38
Luke		4	3	3	0	10	25
Dior		6	2	4	1	15	38
No Name		2	2	3	3	10	25
Robertson		5	2	5	4	19	48
Lulendo		2	5	1	3	10	25
LM		4	3	4	1	12	30
Hein Mnguni		3	0	4	3	10	25
KM		4	1	4	0	9	23
Palesa		0	2	4	1	7	18
Dlamini		0	4	3	1	8	20
EE		0	2	2	1	5	13
Sithole		2	2	1	1	6	15
Lubuya		0	2	3	1	6	15
Anonymous		2	4	0	0	6	15
OSK		2	4	3	1	10	25
Ques 1		2	1	4	1	8	20
MK		2	3	4	0	9	23
MN		2	0	4	2	8	20
JJ		2	4	2	0	8	20
Leshabane		2	4	2	3	13	33
Angel		4	4	5	4	20	50
Redney		4	6	4	3	17	43
Neo		0	4	5	1	10	25
Masongo		0	5	5	1	11	28
Wisdom		2	3	3	4	14	35
Osiame		4	3	5	2	16	40
Dimpho		5	3	5	2	16	40
Keamogetswe		6	5	5	3	20	50
Asonala		6	5	4	0	16	40
Tadiwa		1	4	4	3	12	30
Kamvelihe		2	3	5	0	12	30

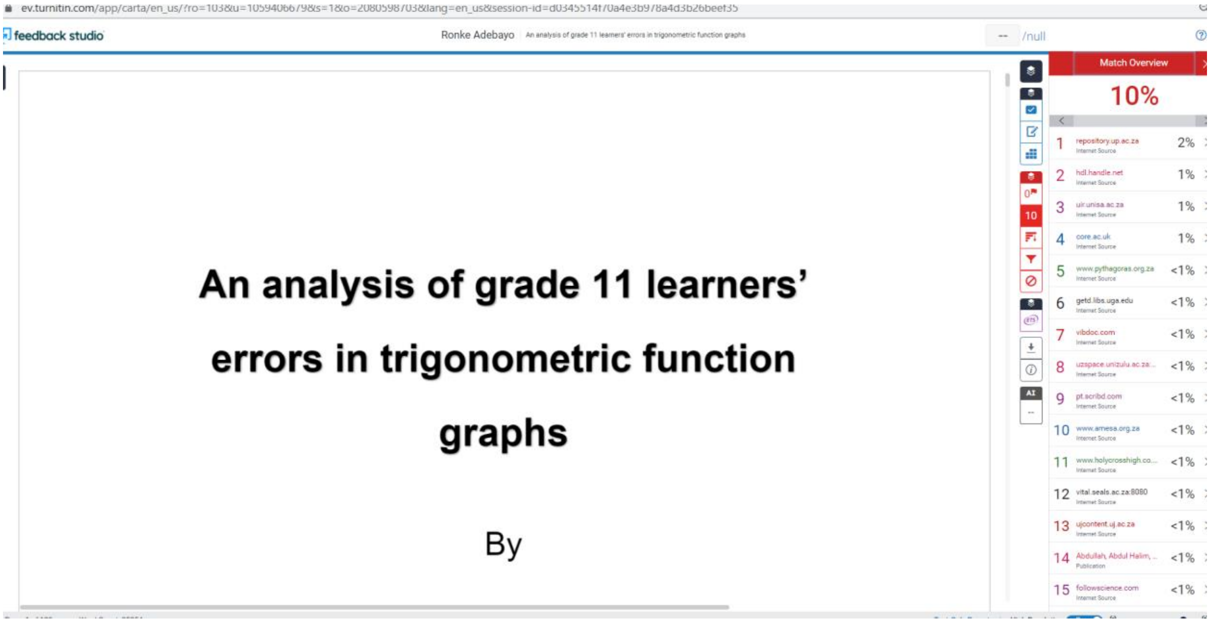
Josh		4	5	4	1	14	35
Lickson		5	7	5	3	22	55
Ramasodi		6	3	4	2	15	38
Lesedi 2		2	3	4	3	12	30
Mbongeni		4	3	5	0	15	38
Bogwana		3	1	5	2	12	30
Anani		6	4	5	1	17	43
Khanyisile		6	0	4	2	13	33
Mpho		2	4	3	1	10	25
Ndangi		3	2	3	2	11	28
Leandra		6	2	4	3	17	43
Hannah		6	2	1	0	10	25
Karabo		0	2	5	3	10	25
Maria		6	9	4	6	25	63
Muzingwa		0	3	5	3	11	28
Yusuff		2	9	5	5	22	55
Unarine		4	6	5	2	17	43
Mbali		2	3	5	1	11	28
Tshiamo		2	3	3	2	10	25
Ryam		4	9	5	3	23	58
Murungwa		2	3	2	3	11	28
Mogale		3	5	1	5	16	40
Rambuda		2	6	1	3	12	30
Segwana		4	5	4	4	17	43
Mercy		3	6	5	4	18	45
Fortune		4	3	5	1	12	43
Aziza		2	3	5	0	10	25
Sadoc		4	4	3	0	9	23
Searegu		2	8	5	2	17	43
Tshegofatso		2	2	4	1	9	23
Nyathi		0	5	3	5	13	33
Mhosa		6	7	6	5	24	60
Boitumelo		4	4	4	2	14	35
Palesa		4	7	5	1	18	45
Batlile		4	6	3	1	13	33
Siyavuya		4	3	3	3	12	30
X X X X X X		2	6	2	3	15	38
Leago		3	6	3	1	12	30
Sibidu		4	4	3	1	12	30
thomdo		7	7	6	1	20	50
X X X X X X		2	3	2	3	9	23
Mmahle		4	2	5	0	11	28
Luvuyo		4	6	0	0	8	20
Daniel		2	5	6	0	13	33

Sande		2	5	4	4	15	38
Thapyy		4	5	6	0	15	38
Zinhle		4	5	4	3	16	40
Nathan		1	4	4	2	11	28
Yui		2	3	4	1	10	25
Onthy		2	3	5	0	10	25
Maru		4	1	5	0	10	25
Amaka		2	5	5	7	19	48
Rhema		0	3	5	4	15	38
Sphosy		4	8	3	4	19	48
Owet		4	8	5	6	23	58
Hamil		2	5	3	3	13	33
Olly		2	6	2	3	13	33
Molly		6	4	4	0	13	33
Gail		5	1	4	2	9	23
X X		2	3	3	3	11	28
Harm		4	3	5	3	15	38
Ambrie		2	7	2	3	14	35
Oras		4	1	5	3	13	33
Phetty		4	8	4	5	21	53
Katlego		2	6	4	3	14	35
Ntam		4	9	6	6	25	63
Ova		2	7	0	5	14	35
Jeo		4	8	2	3	17	43
Amani		2	8	5	3	18	45
Lese		2	5	5	4	16	40

6.10 Appendix J: Turnitin Report Summary

Botha

8/5/2023



Analysis of grade 11 learners' errors in trigonometric function graphs

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