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CHARACTERIZATION THEOREMS IN VON NEUMANN ALGEBRAS

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CHARACTERIZATION THEOREMS IN VON NEUMANN ALGEBRAS

by

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CONTENTS

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INTRODUCTION

It is well-known that C^*-algebras can be defined in two different ways. At first it is defined abstractly as a Banach algebra $\mathcal{U}$ such that for each element in $\mathcal{U}$ there exists an "adjoint" element in $\mathcal{U}$ which satisfies certain properties. Gelfand, Naimark and Segal proved that any such abstract C^*-algebra is isomorphic to a norm-closed *-subalgebra of the algebra of bounded linear operators on some Hilbert space [9]. Hence a C^*-algebra can be represented as a norm-closed *-subalgebra of the algebra of bounded linear operators on a Hilbert space.

J von Neumann made a study of operator algebras and in 1930 for the first time defined this class of operator algebras, later known as von Neumann algebras, in terms of a representation on a Hilbert space. After the studies of Gelfand, Naimark and Segal (cf also [2]), von Neumann algebras were defined as *-subalgebras of bounded operators on a Hilbert space which are weak operator closed.

A further concrete characterization of von Neumann algebras follows from von Neumann's well-known double commutant theorem [2] which characterizes von Neumann algebras in an algebraic way. Von Neumann tried to characterize these algebras in a more abstract or representation-independent manner. It was only in the mid fifties that two mathematicians named Kadison and Sakai, almost simultaneously published two abstract characterizations of von Neumann algebras [4], [8]. In our paper we give a description and proof of these von Neumann algebra characterization results.

In the first section of chapter one we state important results on projections and operators that are later needed to prove a few propositions and theorems. In the second section of the first chapter we state the important spectral theorem [7], [10] and a few results on Borel calculus. We prove a theorem of Baire in a unique way by using L-sets. We then use this theorem together with the spectral theorem to extend the Gelfand Naimark *-isomorphism to a *-homomorphism between all the bounded complex Borel functions on the spectrum $\sigma(T)$ of an operator T and the von Neumann algebra generated by T and I [10].

In chapter 2 we discuss Kadison's characterization. During his studies it became clear to him that a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(H)$ satisfies two conditions, firstly each increasing net of self-adjoint operators that is bounded from above has a least upper bound and secondly the states (positive linear functionals of norm 1) in $\mathcal{A}$ which are normal, separate $\mathcal{A}$. We prove that it is exactly these two conditions that characterize a von Neumann algebra. In the latter part of chapter two we construct Kadison's example which shows that both conditions are necessary [4]. We construct a commutative C^*-algebra with the upper bound property, but without any normal states. Since all vector states in a von Neumann algebra are normal, this shows that this algebra can't be isomorphic to a von Neumann algebra.

It is well-known that one of the basic features of von Neumann algebras is that for each von Neumann algebra $\mathcal{A} \subset \mathcal{B}(H)$ there exists a unique Banach space $\mathcal{A}_*$ such that $\mathcal{A}$ is the dual of $\mathcal{A}_*$ [2]. Thus $\mathcal{A}$ is a C^*-algebra with a predual. In chapter 3 we show that a C^*-algebra $\mathcal{U}$ is isomorphic to a von Neumann algebra if and only if there exists a Banach space $\mathcal{U}_*$ such that $(\mathcal{U}_*)^* = \mathcal{U}$. This theorem was first proved by Sakai in 1956. We give Tomiyama's proof for this characterization. Tomiyama made a study of conditional expectations (projections of norm one) by generalising conditional expectations from commutative measure theory to non-commutative measure spaces. By using this technique and results on the well-known universal representation [6], Tomiyama gave an elegant proof of Sakai's result in 1957 [13]. The exposition hereof is contained in chapter 3.

We conclude this thesis with an Appendix where we mention a few basic results on some useful locally convex topologies defined on $\mathcal{A}$. As far as the references are concerned, the main sources used in this work are [2], [4], [5], [6], and [10]. More detailed references are given throughout the chapters. The notations and conversions used are also defined at the beginning of each section.

The author could find no reference of a proof of Baire's theorem in chapter 1, hence the proof given is his own. Another original piece of work is the proof of Lemma 1.1.12 using Borel calculus. The classical proof (which depends on many other results) can be found in [6]. Apart from these there are a few interesting remarks (cf Remark 3.4).

CHAPTER 1 : PROJECTIONS AND OPERATORS

I The lattice of projections

Let H be a Hilbert space and denote the C^*-algebra of all bounded linear operators on H by $\mathcal{B}(H)$. We denote by $\mathcal{A}$ a *-subalgebra of $\mathcal{B}(H)$, which is a von Neumann algebra. A von Neumann algebra is defined as a *-subalgebra of $\mathcal{B}(H)$ which is closed in the weak operator topology on $\mathcal{B}(H)$. Hence clearly $\mathcal{B}(H)$ itself is a von Neumann algebra. If F is a subset of $\mathcal{B}(H)$ then the commutant F' of F is the set $F' = \{T \in \mathcal{B}(H) : TS = ST$ for all $S \in F\}$.
The well-known double commutant theorem of von Neumann ([2], Theorem 3.5.2) states that a von Neumann algebra is a unital *-subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ such that $\mathcal{A} = \mathcal{A}''$. One of the most important classes of operators in a von Neumann algebra $\mathcal{A}$ is the class of so-called projections. It is well-known that there exists a one-to-one correspondence between closed subspaces of H and projections in $\mathcal{B}(H)$ (P is a projection if $P^* = P$ and $P^2 = P$). The aim of this part is to study the properties of projections in a von Neumann algebra. The set of all projections in $\mathcal{A}$ is denoted by $\mathcal{P}(\mathcal{A})$. It is easy to see that the order relation $\leq$, defined by $E \leq F$ iff $EF = E$ iff $E(H) \subset F(H)$ gives a partial order on $\mathcal{P}(\mathcal{A})$. From these equivalences it follows that the partial ordering of projections corresponds to the partial ordering of closed subspaces by the inclusion relation $(\subseteq)$. If we now consider a family of projections, say $\{E_i\}_{i\in I}$ then $\bigcap_{i\in I} E_i(H)$ and $[\bigcup_{i\in I} E_i(H)]$ are closed subsets of H where $[\bigcup_{i\in I} E_i\ (H)]$ is the closed linear span of $\bigcup_{i\in I} E_i(H)$. Let E and F be the projections in $\mathcal{B}(H)$ corresponding to $\bigcap_{i\in I} E_i(H)$ and $[\bigcup_{i\in I} E_i(H)]$ respectively. It is now clear that $\sup_{i\in I} E_i = F$ and that $\inf_{i\in I} E_i = E$, for if $E_i \leq G$ for each i then $\bigcup_{i\in I} E_i(H) \subset G(H)$. Hence $[\bigcup_{i\in I} E_i(H)] \subset G(H)$ which implies that $F \leq G$. Thus since $E_i \leq F$ for each $i \in I$ it follows that $\sup_{i\in I} E_i = F$. The fact that

$\inf_{i \in I} E_i = E$ follows from a similar argument. Our next lemma now shows that if $\{E_i\}_{i \in I} \subset \mathcal{A}$, then $\sup_{i \in I} E_i$ and $\inf_{i \in I} E_i$ are elements of $\mathcal{P}(\mathcal{A})$.

Lemma 1.1 ([12], V, Proposition 1.1)

If $\mathcal{A}$ is a von Neumann algebra, then the set of all projections $\mathcal{P}(\mathcal{A})$ is a complete lattice.

Proof

To prove that $\mathcal{P}(\mathcal{A})$ is a complete lattice we must show that $\sup_{i \in I} E_i$ and $\inf_{i \in I} E_i$ are in $\mathcal{P}(\mathcal{A})$ where $\{E_i\}_{i \in I}$ is a family of projections in $\mathcal{A}$. Let E_0 be a projection of H onto the closed subspace $\bigcap_{i \in I} E_i(H)$ of H. Let U be unitary and $U \in \mathcal{A}'$, then

$$\begin{aligned} U(E_i(H)) &= E_i U(H) \quad (U \in \mathcal{A}') \\ &= E_i(H) \quad \text{(a unitary operator is onto)} \end{aligned}$$

Thus any unitary element in $\mathcal{A}'$ leaves each $E_i(H)$ invariant. Hence any unitary element in $\mathcal{A}'$ leaves $\bigcap E_i(H)$ invariant, which implies that $E_0 U E_0 = U E_0$. A similar argument applied to $U^* \in \mathcal{A}'$ also implies that $E_0 U^* E_0 = U^* E_0$. If we take adjoints it follows that $E_0 U E_0 = E_0 U$. Thus $E_0 U = U E_0$ for every unitary $U \in \mathcal{A}'$. Since every element in $\mathcal{A}'$ is a linear combination of four unitaries ([10], Proposition 2.24), it follows that $E_0 T = T E_0$ for every $T \in \mathcal{A}'$. Thus $E_0 \in \mathcal{A}'' = \mathcal{A}$.

Since the mapping $E \longrightarrow I - E$ reverses the ordering of projections we have $\sup_{i \in I} E_i = I - \inf_{i \in I}(I - E_i) \in \mathcal{P}(\mathcal{A})$. The equality follows since $\inf_{i \in I}(I - E_i) \leq I - E_i$. Thus $I - \inf_{i \in I}(I - E_i) \geq I - (I - E_i) = E_i$.

Now, if $E_i \leq G$ for all $i \in I$ then $\inf_{i \in I} (I - E_i) \geq I - G$ for all i. Thus $I - \inf_{i \in I}(I - E_i) \leq I - (I - G) = G$ for all i.

Hence $I - \inf_{i \in I}(I - E_i) = \sup_{i \in I} E_i$.

The following result follows directly from the last part of our previous proof. Since we are going to use it in the chapters that follow, we state it as a corollary.

Corollary 1.2

If $\{E_i\}$ is a family of projections in H, $\{E_i\}$ has a greatest lower bound $\inf_{i \in I} E_i$ and a smallest upper bound $\sup_{i \in I} E_i$. The mapping $E \longrightarrow I - E$ reverses the order of projections and $\sup_{i \in I} (I - E_i) = I - \inf_{i \in I} E_i$; $\inf_{i \in I} (I - E_i) = I - \sup_{i \in I} E_i$.

With each bounded linear operator T acting on a Hilbert space we associate a null space and a range space. The null space $\{x \in H : Tx = 0\}$ and the range space (which is the closure $[T(H)]$ of the range $T(H)$ where $T(H) = \{Tx : x \in H\}$ of T) have corresponding projections namely the null projection, $N(T)$ and the range projection, $R(T)$. When E is a projection, $R(E) = E$ and $N(E) = I - E$.

Lemma 1.3 ([5], Proposition 2.5.13)

If T is a bounded linear operator acting on a Hilbert space H, then $R(T) = I - N(T^*)$, $N(T) = I - R(T^*)$ and $R(T^*T) = R(T^*)$.

Proof

Since the set $\{x \in H : Tx = 0\}$

$$= \{x \in H : \langle Tx, y\rangle = 0 \text{ for each } y \text{ in } H\}$$
$$= \{x \in H : \langle x, T^* y\rangle = 0 \text{ for each } y \text{ in } H\}$$
$$= T^*(H)^{\perp}$$
$$= [T^*(H)]^{\perp}$$

it follows that $N(T) = I - R(T^*)$. If we replace T by T^* we get $N(T^*) = I - R(T)$. Now for each x in H, $\|Tx\|^2 = \langle Tx,Tx\rangle = \langle T^*Tx,x\rangle$. Thus $Tx = 0$ if and only if $T^*Tx = 0$. That is $N(T) = N(T^*T)$ from which it follows that $R(T^*T) = I - N(T) = R(T^*)$

Lemma 1.4 ([5], Proposition 2.5.14]

If E and F are projections acting on a Hilbert space H, then $R(E + F) = \sup \{E,F\}$. Hence if $EF = 0$ then $R(E + F) = E + F = \sup \{E,F\}$.

Proof

Since $\|Ex\|^2 + \|Fx\|^2$

$= \langle Ex,x\rangle + \langle Fx,x\rangle$ (E and F are projections)

$= \langle (E + F)x,x\rangle$

for each vector x it follows that $(E + F)x = 0$ iff $Ex = Fx = 0$.

Thus $N(E + F) = \inf \{N(E),N(F)\}$

$= \inf \{(I - R(E)), (I - R(F))\}$

$= \inf \{(I - E), (I - F)\}$

From Lemma 1.3 it follows that $R(E + F) = I - N(E + F)$.

Thus $R(E + F) = I - \inf \{(I - E), (I - F)\}$ and from Corollary 1.2 $\inf \{(I - E), (I - F)\} = I - \sup \{E,F\}$. Hence

$R(E + F) = I - (I - \sup \{E,F\})$

$= \sup \{E,F\}$.

In chapter two we are going to state two conditions which a C^*-algebra $\mathcal{U}$ has to satisfy to be isomorphic to a von Neumann algebra. One of these conditions is that each increasing net of self-adjoint operators in $\mathcal{U}$ that is bounded above has a least upper bound in $\mathcal{U}$. In the following proposition we prove that if $\mathcal{A}$ is a von Neumann algebra, then this condition is satisfied. Moreover we show that this least upper bound of the increasing net is also the strong operator limit. (Note that for $T,S \in \mathcal{A}$, $T \leq S$ iff $\langle Tx,x\rangle \leq \langle Sx,x\rangle$ for each $x \in H$.)

Proposition 1.5 ([5], Lemma 5.1.4)

If $\{T_i\}_{i\in I}$ is a monotone increasing net of self-adjoint operators in $\mathcal{A}$ and $T_i \leq kI$ for all $i \in I$ and k a constant, then $\{T_i\}$ is strong operator convergent to a self-adjoint operator T. Thus $T \in \mathcal{A}$ and T is the least upper bound of $\{T_i\}$.

Proof

Since the convergence of $\{T_i\}_{i\in I}$ and that of $\{T_i,\ i \geq i_0\}$ are equivalent we may assume that $\{T_i\}$ is bounded below (by T_{i_0}) as well as above. Thus $-\|T_{i_0}\|I \leq T_i \leq kI$, and so $\{T_i\}$ is a bounded set of operators. Since a closed ball S in $\mathcal{B}(H)$ is weak operator compact (Banach Aloaglu, [9]) there exists a subset $\{T_\ell\}$ of $\{T_i\}$ which is weak operator convergent to a T in $\mathcal{B}(H)$. Since $\mathcal{A}$ is weak operator closed, $T \in \mathcal{A}$. As $\{T_i\}$ is monotone increasing $\langle T_\ell x,x\rangle \geq \langle T_{i_1}x,x\rangle$ when $\ell \geq i_1$ and $x \in H$. Since $\langle Tx,x\rangle = \lim_\ell \langle T_\ell x,x\rangle \geq \langle T_{i_1}x,x\rangle$ for all $x \in H$ we have that $T \geq T_{i_1}$ for all i_1 (the order relation is to be interpreted in the operator sense). If $i \geq \ell$ then $0 \leq T - T_i \leq T - T_\ell$, and $0 \leq \langle (T - T_i)x,x\rangle = \|(T - T_i)^{1/2}x\|^2 \leq \langle (T - T_\ell)x,x\rangle$. Hence $\{(T - T_i)^{1/2}\}$ is strong operator convergent to zero. The strong operator continuity of multiplication on bounded sets of operators allows us to conclude that $\{T - T_i\}$ is strong operator convergent to 0. We have noted that T is an upper bound for $\{T_i\}$. If $S \geq T_i$ for all i, then $\langle Sx,x\rangle \geq \langle T_i x,x\rangle \xrightarrow{i} \langle Tx,x\rangle$. Hence $\langle Sx,x\rangle \geq \langle Tx,x\rangle$ for all $x \in H$ so $S \geq T$. Therefore T is the least upper bound of $\{T_i\}$.

Lemma 1.6 ([5], Lemma 5.1.5]

If T is a bounded operator on the Hilbert space H and $0 \leq T \leq I$ then $\{T^{1/n}\}$ is a monotone increasing sequence of operators whose strong operator limit is the projection $R(T)$.

Proof

Let $\mathcal{U}(T,I)$ be the commutative C^*-algebra generated by I and T. Then by the Gelfand Naimark theorem (which will be stated in Chapter 1, II) $\mathcal{U}(T,I)$ is isometric isomorphic to $\mathcal{C}(\sigma(T))$. ($\mathcal{C}(\sigma(T))$ is the continuous functions on the spectrum $\sigma(T)$ of T). This isomorphism is also an order isomorphism. Now let $f \in \mathcal{C}(\sigma(T))$ be the function corresponding to T under this isomorphism. Then $0 \leq f \leq 1$ and $(f^{1/n})$ is a monotone increasing sequence in $\mathcal{C}(\sigma(T))$ bounded above by 1. Hence $\{T^{1/n}\}$ is a monotone increasing sequence bounded above by I. Thus $\{T^{1/n}\}$ has a strong operator limit E and this limit is the least upper bound of $\{T^{1/n}\}$ (cf Lemma 1.5). Since multiplication is jointly continuous on bounded parts with respect to the strong operator topology, $\{T^{2/n}\}$ is strong operator convergt to E^2. $\{T^{1/n}\} = \{T^{2/2n}\}$ is a sub-sequence of $\{T^{2/n}\}$ so that $E = E^2$. Thus E is a projection.

If we apply the Stone-Weierstrass theorem ([9], p160) to the function algebra representing $\mathcal{U}(T)$, we see that $T^{1/n}$ is the norm limit of polynomials, without constant term in T.

Thus $T^{1/n}x = 0$ if $Tx = 0$ and $Ex = 0$ when $Tx = 0$. If $Ex = 0$ then $0 = \langle Ex,x\rangle \geq \langle T^{1/n}x,x\rangle = \|T^{1/2n}x\|^2$. Thus $T^{1/2n}x = 0$ and $Tx = 0$. We have now proved that E and T have the same null space. From Lemma 1.3 we know that

$$R(T) = I - N(T^*) = I - N(T) = I - N(E) = R(E) = E.$$

Remark

If $T \in \mathcal{A}$ ($\mathcal{A}$ a von Neumann algebra) and $0 \leq T \leq I$ it follows from Proposition 1.5 and Lemma 1.6 that $R(T) \in \mathcal{A}$. Now let $S \in \mathcal{A}$ be arbitrary. We show that $R(S) \in \mathcal{A}$. Since $R(S) = R(SS^*)$ (cf Lemma 1.3) it suffices to show that $R(S) \in \mathcal{A}$ for S positive, and since $R(S) = R(\alpha S)$ for each positive scalar α we may assume that $0 \leq S \leq I$. Hence from the argument above $R(S) \in \mathcal{A}$ for an arbitrary $S \in \mathcal{A}$.

The following lemma will be crucial in the proof of one of our characterization theorems. This lemma states that each projection is the union of an orthogonal family of cyclic projections. A projection E is said to be cyclic in a von Neumann algebra $\mathcal{A}$ if its range is $[\mathcal{A}'x]$ for some vector x. We call x the generating vector for E under $\mathcal{A}'$ ($\mathcal{A}'$ the commutant of $\mathcal{A}$).

Lemma 1.7 ([5], Proposition 5.5.9]

If E is a cyclic projection in a von Neumann algebra $\mathcal{A}$ with generating vector x and F is a projection in $\mathcal{A}$ such that $F \leq E$, then F is cyclic in $\mathcal{A}$ with generating vector Fx. Moreover each projection in $\mathcal{A}$ is the union of an orthogonal family of cyclic projections in $\mathcal{A}$.

Proof

Firstly we must note that $[T'Fx : T' \in \mathcal{A}'] = [FT'x : T' \in \mathcal{A}']$. Since F is continuous and $\{T'x : T' \in \mathcal{A}'\}$ is dense in E(H), $\{FT'x : T' \in \mathcal{A}'\}$ is dense in $FE(H) = F(H)$. Thus F is cyclic in $\mathcal{A}$ and Fx is a generating vector for F. Suppose E is an arbitrary projection in $\mathcal{A}$. If E is 0 then E is cyclic in $\mathcal{A}$ with generating vector 0. If $E \neq 0$ and x is some non-zero vector in its range then $[\mathcal{A}'x]$ is the range of a cyclic projection E_0. The following with regard to the ranges of E_0 and E are true.

If $T \in \mathcal{A}'$ and $x \in E(H)$ then

$$\begin{aligned} Tx = TEx &= ETx \quad (T \in \mathcal{A}') \\ &\in E(H). \end{aligned}$$

Thus $T(E(H)) \subset E(H)$ and the range of E is stable under $\mathcal{A}'$. Similarly E_0 is stable under $\mathcal{A}'$.

Now if $x \in E(H)$, choose any $T \in \mathcal{A}'$ then since $Tx \in E(H)$ for any T we have $\mathcal{A}'x \subset E(H)$, but since E(H) is a closed subspace of H, it follows that $[\mathcal{A}'x] \subseteq E(H)$. Hence $E_0 \leq E$. To prove that $E_0 \in \mathcal{A}'' = \mathcal{A}$ we must show that $E_0T = TE_0$ for any $T \in \mathcal{A}'$. Let $y \in H$, then from the range stability we know that $TE_0y \in E_0(H)$. Thus $E_0(TE_0y) = TE_0y$ for all $y \in H$ for which $E_0TE_0 = TE_0$

follows. Similarly we can show for $T^* \in \mathcal{A}'$ that $E_0 T^* E_0 = T^* E_0$. If we take adjoints we have $E_0 T E_0 = E_0 T$ $(E_0^* = E_0)$. Thus $TE_0 = E_0 T$ for any $T \in \mathcal{A}$.

From the above arguments E has a non-zero cyclic subprojection if $E \neq 0$. The set of orthogonal families of non-zero cyclic subprojections of E is non-empty and the union of each totally ordered subset is an upper bound for that subset under inclusion ordering. Zorn's lemma guarantees the existence of a maximal orthogonal family $\{E_a\}$ of non-zero cyclic subprojections of E. If $E - \sup_a E_a$ is not 0, it contains by the argument above a non-zero cyclic subprojection E_0. Adjoining E_0 to $\{E_a\}$ contradicts the maximality of $\{E_a\}$. Thus E is the union of the orthogonal family $\{E_a\}$ of non-zero cyclic projections.

II The spectral theory and Borel calculus

In this part we're going to take a look at some of the important theorems needed to prove some of the characterization theorems of von Neumann algebras. The well-known and important spectral theorem for self-adjoint operators is used throughout this writing. In [2], [5], [12], etc proofs are stated for the spectral theorem, we'll give the proof sketched by Stratila and Zsido ([10], paragraph 2.19). To enable us to do this we need some preliminaries on elementary C^*-algebra theory. An important class of elements in a C^*-algebra is the so-called positive elements. An element $T \in \mathcal{U}$ is called positive if T is self-adjoint and $\sigma(T) \subset [0,\infty)$, where $\sigma(T)$ is the spectrum of T. If $\mathcal{U} = \mathcal{B}(H)$ it can easily be seen ([7], Theorem 9.2-1 and Theorem 9.2-3) that this definition coincides with the classical definition for positive operators (i.e. $T \geq 0$ iff $\langle Tx,x\rangle \geq 0$ for every $x \in H$). In a general C^*-algebra $\mathcal{U}$, the following condition is equivalent to the above-mentioned:

$$T \geq 0 \text{ iff } T = S^*S \text{ for some } S \in \mathcal{U}.$$

The following important lemma that gives a $*$-isomorphism between

$\mathcal{C}(\{T,I\})$ and $\mathcal{C}(\sigma(T))$ where $\mathcal{C}(\sigma(T))$ is all the continuous complex functions in the spectrum and $\mathcal{C}(\{T,I\})$ is the C^*-algebra generated by T and the identity I. We state the lemma without proof.

Lemma 1.8 (Gelfand-Naimark; [10], Theorem 2.6)

Let $T \in \mathcal{B}(H)$ be a self-adjoint operator. Then there exists a unique mapping $\tau : \mathcal{C}(\sigma(T)) \longrightarrow \mathcal{B}(H)$ taking a $f \in \mathcal{C}(\sigma(T))$ and mapping it onto $f(T) \in \mathcal{B}(H)$ and

(i) if f is a polynomial, $f(\lambda) = a_0 + a_1\lambda + \ldots + a_n\lambda^n$, then $f(T) = a_0 I + a_1 T + \ldots + a_n T^n$.

(ii) τ is isometric.

It is further true that this mapping is a *-isomorphism of the C^*-algebra $\mathcal{C}(\sigma(T))$ onto the C^*-algebra $\mathcal{C}(\{T,I\})$ and this isomorphism is an order isomorphism (i.e. $f \leq g$ iff $\tau(f) \leq \tau(g)$).

In the latter part of this section we extend this *-isomorphism to a *-homomorphism between all the bounded complex Borel functions in the spectrum $\sigma(T)$ and the von Neumann algebra generated by T and I. To prove this extension we need a theorem of Baire which we'll discuss after the spectral theorem. Before we discuss and prove the very important spectral theorem, we need the following lemma.

Lemma 1.9 ([10], Lemma 2.18)

Let $T \in \mathcal{B}(H)$ be a self-adjoint operator and let $\{f_n\}$ and $\{g_n\}$ be two bounded increasing sequences of positive functions from $\mathcal{C}(\sigma(T))$, such that $\sup_n f_n(\lambda) \leq \sup_n g_n(\lambda)$, $\lambda \in \sigma(T)$. Then $\sup_n f_n(T) \leq \sup_n g_n(T)$.

Proof

Since $\{f_n\}$ (respectively $\{g_n\}$) is an increasing sequence which is bounded it follows from the fact that the Gelfand-isomorphism is an order isomorphism that $\{f_n(T)\}$ (respectively $\{g_n(T)\}$) is increasing and bounded above. The existence of the elements $\sup_n f_n(T)$ and $\sup g_n(T)$ follows then from Lemma 1.5.

Let n be a natural number and $\epsilon > 0$. For any $\lambda \in \sigma(T)$ we have

$$f_n(\lambda) - \epsilon < f_n(\lambda) \leq \sup_m f_m(\lambda) \leq \sup g_m(\lambda).$$

Consequently there exists a neighbourhood V_λ of λ and a natural number m_λ, such that $f_n(\mu) - \epsilon < g_{m\lambda}(\mu)$, $\mu \in V_\lambda$.

Clearly the set $\{V_\lambda : \lambda \in \sigma(T)\}$ is an open covering for $\sigma(T)$ and since $\sigma(T)$ is compact there exist $\lambda_1,\ldots,\lambda_n \in \sigma(T)$ such that $\{V_{\lambda i} : 1 \leq i \leq n\}$ covers $\sigma(T)$. Now for each λ_i there is a natural number $m_{\lambda i}$ such that $f_n(\mu) - \epsilon < g_{m_{\lambda i}}(\mu)$ $(\mu \in V_{\lambda i})$. Let $m_n = \max \{m_{\lambda i},\ldots,m_{\lambda n}\}$, then it follows that $f_n - \epsilon \leq g_{m_n}$ in $\mathcal{C}(\sigma(T))$. Since there exists an order isomorphism between the C^*-algebra $\mathcal{C}(\sigma(T))$ and the C^*-algebra $\mathcal{C}(\{T,I\})$ it follows that

$$f_n(T) - \epsilon \leq g_{m_n}(T) \leq \sup_m g_m(T).$$

Since ϵ was chosen arbitrarily greater than zero, we have

$$f_n(T) \leq \sup_m g_m(T)$$

and since n was arbitrary it follows that

$$\sup_n f_n(T) \leq \sup_m g_m(T).$$

Let $T \in \mathcal{B}(H)$ be a self-adjoint operator with

$$m(T) = \inf \{\lambda : \lambda \in \sigma(T)\} \text{ and } M(T) = \sup \{\lambda : \lambda \in \sigma(T)\}.$$

Since $\sigma(T)$ is compact : $m(T)$ and $M(T)$ will also be in $\sigma(T)$. For any $\lambda \in \mathbb{R}$ we shall consider the continuous functions

$$f_n^\lambda(t) = \begin{cases} 1 & \text{if } t \in (-\infty,\lambda-\frac{1}{n}] \\ n(\lambda - t) & \text{if } t \in [\lambda-\frac{1}{n},\lambda] \\ 0 & \text{if } t \in [\lambda,\infty). \end{cases}$$

We then have $f_n^\lambda(t) \ n\uparrow\infty \ \chi_{(-\infty,\lambda)}(t)$ for all $t \in \mathbb{R}$.

Since the Gelfand isomorphism is also an order isomorphism we have from Lemma 1.8 that $\{f_n^\lambda(T)\}$ is an increasing sequence which is bounded above, hence from Lemma 1.5 there exists a self-adjoint $E_\lambda \in \mathcal{B}(H)$ such that

$$f_n^\lambda(T) \uparrow E_\lambda.$$

Moreover $\{f_n^\lambda(T)\}$ converges in the strong operator limit to E_λ.

We now show that E_λ is a projection.

Since multiplication of operators is strong operator continuous on bounded parts it is clear that

$$f_n^\lambda(T)^2 \longrightarrow E_\lambda^2 \text{ in the strong operator limit.}$$

On the other hand it also follows that

$$f_n^\lambda(t)^2 \uparrow \chi_{(-\infty,\lambda)}(t) \text{ for } t \in \mathbb{R}.$$

Using the same argument as above and Lemma 1.8

$$f_n^\lambda(T)^2 \longrightarrow E_\lambda.$$

Thus $E_\lambda^2 = E_\lambda$ and E_λ is a projection in $\mathcal{B}(H)$.

For a self-adjoint operator T the spectral theory consists of the following properties. (E_λ) is known as the spectral family.

Theorem 1.10 ([10], Paragraph 2.19)

(i) $E_\lambda \in \mathcal{A}(\{T,I\})$, the von Neumann algebra generated by T and I.

(ii) If $\lambda_1 \leq \lambda_2$ then $E_{\lambda_1} \leq E_{\lambda_2}$.

(iii) If $\lambda_n \uparrow \lambda$ then $E_{\lambda n} \uparrow E_\lambda$.

(iv) If $\lambda \leq m(T)$ then $E_\lambda = 0$ and if $\lambda > M(T)$ then $E_\lambda = I$.

(v) $TE_\lambda \leq \lambda E_\lambda$ and $T(I - E_\lambda) \geq \lambda(I - E_\lambda)$.

(vi) $T = \int_{-\infty}^{\infty} \lambda dE_\lambda = \int_{m(T)}^{M(T)+0} \lambda dE_\lambda$.

Proof

(i) Since there exists an $*$-isomorphism between $\mathcal{C}(\sigma(T))$ and the C^*-algebra $\mathcal{C}(\{T,I\})$, we have from the facts that $f_n^\lambda(T) \in \mathcal{C}(\{T,I\})$ and $\mathcal{A}(\{T,I\})$ is the weak operator closure of $\mathcal{C}(\{T,I\})$ that $E_\lambda \in \mathcal{A}(\{T,I\})$ and in particular E_λ commutes with any operator commuting with T.

(ii) For any n if $\lambda_1 \leq \lambda_2$ we have

$$f_n^{\lambda_1} \leq f_n^{\lambda_2} \text{ in } \mathcal{C}(\sigma(T)).$$

Thus $f_n^{\lambda_1}(T) \leq f_n^{\lambda_2}(T)$ and it follows from Lemma 1.9 that $\sup f_n^{\lambda_1}(T) \leq \sup f_n^{\lambda_2}(T)$, hence $E_{\lambda_1} \leq E_{\lambda_2}$.

(iii) It is clear that $f_n^{\lambda_n} \uparrow \chi_{(-\infty,\lambda)}$ pointwise as $\lambda_n \uparrow \lambda$. Since $\lambda_n \leq \lambda$ for each n it follows from (ii) that $E_{\lambda_n} \leq E_\lambda$. Since $\sup_m f_m^{\lambda_n} \leq \sup_n f_n^{\lambda_n}$ pointwise it follows from Lemma 1.9 that $E_{\lambda_n} = \sup_m f_m^{\lambda_n}(T) \leq \sup_n f_n^{\lambda_n}(T) = E_\lambda$. Hence $E_\lambda \geq E_{\lambda_n} \geq f_n^{\lambda_n}(T) \uparrow E_\lambda$. Therefore $E_{\lambda_n} \uparrow E_\lambda$.

(iv) If $\lambda \leq m(T)$, then $\sigma(T) \subset [\lambda,\infty)$; therefore $f_n^\lambda(t) = 0$ for all $t \in \sigma(T)$. Thus $E_\lambda = 0$. If $\lambda > M(T)$, then $\sigma(T) \subset (-\infty,\lambda-\frac{1}{n}]$ for n sufficiently great, therefore $f_n(t) = 1$ for all $t \in \sigma(T)$. Thus $E_\lambda = 1$.

(v) To prove (v) we must first show that

$$tf_n^\lambda(t) \leq \lambda f_n^\lambda(t) \quad \text{for } t \in \mathbb{R} \text{ and}$$

$$t(1 - f_n^\lambda(t)) \geq (\lambda - \tfrac{1}{n})(1 - f_n^\lambda(t)) \; : \; t \in \mathbb{R}.$$

If $t \in (-\infty,\lambda - \frac{1}{n}]$ then $tf_n^\lambda(t) = t\cdot 1 = t < \lambda$ and since $f_n^\lambda(t) = 1$, $tf_n^\lambda(t) < \lambda f_n^\lambda(t)$.

If $t \in [\lambda - \frac{1}{n},\lambda]$ then

$$tf_n^\lambda(t) = tn(\lambda - t) \leq \lambda n(\lambda - t) = \lambda f_n^\lambda(t).$$

If $t \in [\lambda,\infty)$ then $tf_n^\lambda(t) = 0$ but $\lambda f_n^\lambda(t) = 0$ as well. Thus $tf_n^\lambda(t) \leq \lambda f_n^\lambda(t)$ for all $t \in \mathbb{R}$.

Similarly we can show that

$$t(1 - f_n^\lambda(t)) \geq (\lambda - \tfrac{1}{n})(1 - f_n^\lambda(t)) \quad \text{for all } t \in \mathbb{R}.$$

From $tf_n^\lambda(t) \leq \lambda f_n^\lambda(t)$ for all $t \in \mathbb{R}$
we have $Tf_n^\lambda(T) \leq \lambda f_n^\lambda(T)$. Thus $TE_\lambda \leq \lambda E_\lambda$.
Since $t(1 - f_n^\lambda(t) \geq (\lambda - \frac{1}{n})(1 - f_n^\lambda(t))$, $t \in \mathbb{R}$
we have $T(I - f_n^\lambda(T)) \geq (\lambda - \frac{1}{n})(I - f_n^\lambda(T))$.
If we take the strong operator limits on both sides we obtain $T(I - E_\lambda) \geq (I - E_\lambda)$.

(vi) If $\mu \leq \lambda$ we know from (ii) that $E_\mu \leq E_\lambda$ but from (v) we further know that $TE_\mu \leq \mu E_\mu$ and also $TE_\lambda \leq \lambda E_\lambda$.
Further $T(I - E_\mu) \geq \mu(I - E_\mu)$ and $T(I - E_\lambda) \geq \lambda(I - E_\lambda)$.
We'll now use the following fact, if S,T and R are bounded, self-adjoint linear operators on a complex Hilbert space H and $R \geq 0$ and commutes with S and T then if $S \leq T$ one has $SR \leq TR$ ([cf [7], Theorem 9.3-1 for a proof).
Now since $E_\lambda \geq 0$ and E_λ commutes with $T(I - E_\mu)$ and with $\mu(I - E_\mu)$, we have $T(I - E_\mu)E_\lambda \geq \mu(I - E_\mu)E_\lambda$.
Thus since $E_\mu E_\lambda = E_\mu$ $(E_\mu \leq E_\lambda)$ we have
$T(E_\lambda - E_\mu) \geq \mu(E_\lambda - E_\mu)$.
Since $(I - E_\mu)$ commutes with TE_λ and λE_λ, we have
$TE_\lambda(I - E_\mu) \leq \lambda E_\lambda (I - E_\mu)$ from which it follows that
$T(E_\lambda - E_\mu) \leq \lambda(E_\lambda - E_\mu)$.

Thus $\mu(E_\lambda - E_\mu) \leq T(E_\lambda - E_\mu) \leq \lambda(E_\lambda - E_\mu)$.

Let $\delta > 0$ and $\epsilon > 0$ be given and let

$$\Delta = \{m(T) = \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_n = M(T) + \delta\}$$

be a partition of the interval $[m(T), M(T) + \delta]$ whose norm is $\|\Delta\| = \sup \{\lambda_i - \lambda_{i-1},\ i = 1,2,\dots,n\} < \epsilon$.

We shall now consider the following sums

$$ss(\Delta) = \sum_{i=1}^{n} \lambda_{i-1} (E_{\lambda i} - E_{\lambda i-1})$$

$$S(\Delta) = \sum_{i=1}^{n} \lambda_i (E_{\lambda i} - E_{\lambda i-1}).$$

Since $\lambda_{i-1} \leq \lambda_i$ it follows from the inequality proved above

$$\lambda_{i-1}(E_{\lambda i} - E_{\lambda i-1}) \leq T(E_{\lambda i} - E_{\lambda i-1}) \leq \lambda_i(E_{\lambda i} - E_{\lambda i-1}).$$

But since $\sum_{i=1}^{n} T(E_{\lambda i} - E_{\lambda i-1}) = T \sum_{i=1}^{n} (E_{\lambda i} - E_{\lambda i-1})$

$$= T(E_{\lambda n} - E_{\lambda_0})$$

$$= T(1 - 0) = T$$

it follows that $ss(\Delta) = \sum_{i=1}^{n} \lambda_{i-1} (E_{\lambda i} - E_{\lambda i-1}) \leq T$.

We also have $S(\Delta) = \sum_{i=1}^{n} \lambda_i (E_{\lambda i} - E_{\lambda i-1}) \geq T$. Further

$$\begin{aligned} \|S(\Delta) - ss(\Delta)\| &= \|\sum_{i=1}^{n} \lambda_i (E_{\lambda i} - E_{\lambda i-1}) \\ &\quad - \sum_{i=1}^{n} \lambda_{i-1} (E_{\lambda i} - E_{\lambda i-1})\| \\ &= \|\sum_{i=1}^{n} (\lambda_i - \lambda_{i-1})(E_{\lambda i} - E_{\lambda i-1})\| \\ &\leq \sup_i \{(\lambda_i - \lambda_{i-1}),\ i = 1,\ldots,n\} \\ &< \epsilon. \end{aligned}$$

Now if one has $0 \leq S \leq T$ then $\|S\| \leq \|T\|$.

Thus $0 \leq T - ss(\Delta) \leq S(\Delta) - ss(\Delta)$ which implies that

$$\|T - ss(\Delta)\| \leq \|S(\Delta) - ss(\Delta)\| < \epsilon.$$

These enable us to approximate T with a Riemann sum.

Thus $T = \int_{-\infty}^{\infty} \lambda dE_\lambda = \int_{m(T)}^{M(T)+0} \lambda dE_\lambda$

where the integral is to be considered as a Lebesgue-Stieltjes integral which converges with respect to its norm.

Before we can extend the *-isomorphism which we had in Lemma 1.8 by means of a theorem on operational calculus with Borel functions, we are now going to take a good look at a theorem of Baire. Let $\mathcal{B}(\mathbb{R})$ be the class of all bounded real-valued Borel functions on $\mathbb{R}$ and $\mathcal{C}(\mathbb{R})$ the class of all bounded continuous functions.

Definition 1.11

A set $F(\mathbb{R})$ of bounded functions is called an L-set if

(i) $\mathcal{C}(\mathbb{R}) \subsetneq F(\mathbb{R})$

(ii) If (f_n) in $F(\mathbb{R})$ is a bounded sequence (i.e. $\sup_n \|f_n\| < \infty$) such that $f = \lim_n f_n$ pointwise, then $f \in F(\mathbb{R})$.

Since the collection of all functions contains the bounded continuous functions it is clear that the collection of all functions is an L-set. Let $Q = \bigcap_{F \in \text{L-sets}} F$. Obviously Q itself is now an L-set and Q is contained in every L-set.

Lemma 1.12

The following is true in Q.

(i) If $f,g \in Q$ then $f + g$, fg, $\sup\{f,g\}$, $\inf\{f,g\}$ and $|f|$ are all elements of Q.

(ii) If $(f_n) \subset Q$ and the $\sup_n \|f_n\| < \infty$ then $\sup_n f_n = \lim_n \sup\{f_1,f_2,\ldots,f_n\} \in Q$.

Proof

(i) If $f \in \mathcal{C}(\mathbb{R})$ then the set of all functions $g : \mathbb{R} \longrightarrow \mathbb{R}$ for which $f + g \in Q$ is clearly an L-set, hence contains Q (Q is contained in every other L-set). Thus if $f \in \mathcal{C}(\mathbb{R})$ and $g \in Q$ then $f + g \in Q$. Further if $g \in Q$, the set of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ for which $f + g \in Q$ is an L-set which contains Q. Thus if $f \in Q$ and $g \in Q$ then $f + g \in Q$. If $f,g \in Q$ we can in a similar manner show that $fg \in Q$. We conclude (i) by showing that $f \in Q$ implies $|f| \in Q$.

Let $\mathcal{K} = \{f : \mathbb{R} \longrightarrow \mathbb{R} : |f| \in Q\}$. It is clear that $\mathcal{C}(\mathbb{R}) \subset \mathcal{K}$ and if $(f_n) \subset \mathcal{K}$, $\sup_n \|f_n\| < \infty$ with $f = \lim_n f_n$ then $|f| = \lim_n |f_n|$ and $\sup \||f_n|\| < \infty$, hence $|f| \in Q$. Thus we've shown that $\mathcal{K}$ is an L-set containing Q (Q the smallest L-set), thus if $f \in Q$ then $f \in \mathcal{K}$, implying $|f| \in Q$.

Since $\sup\{f,g\} = \frac{(f + g + |f - g|)}{2}$

and $\inf\{f,g\} = \frac{(f + g - |f - g|)}{2}$ ([9], p 159)

the rest of (i) follows.

(ii) This part follows directly from (i) and the definition of an L-set.

Proposition 1.13

Let $\Omega = \{E \subset \mathbb{R} \mid \chi_E \in Q\}$, then the following properties hold:

(i) $\phi \in \Omega$, $\mathbb{R} \in \Omega$

(ii) If $A,B \in \Omega$, then $A \cup B$, A^c, $B/A \in \Omega$

(iii) If $\{A_i\}_{i=1}^{\infty} \subset \Omega$, then $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i \in \Omega$

(iv) Ω contains all open sets.

Proof

(i) Since $\mathcal{C}(\mathbb{R}) \subset Q$ and χ_ϕ , $\chi_{\mathbb{R}}$ are the zero and identity functions respectively, (i) is trivial.

(ii) Since $\chi_{A \cup B} = \sup\{\chi_A, \chi_B\}$ it follows from Lemma 1.12 (i) that $A \cup B \in \Omega$. From the same lemma and the following relations $\chi_{\mathbb{R}\setminus A} = \chi_{\mathbb{R}} - \chi_A$, $\chi_{B\setminus A} = \chi_B - \chi_A$ it follows that A^c and $B\setminus A \in \Omega$.

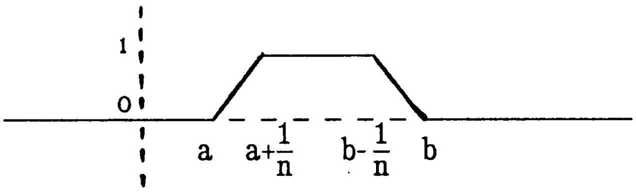
(iii) If $\{A_i\}_{i=1}^{\infty} \subset \Omega$, let $A = \bigcup_{i=1}^{\infty} A_i$.

Since $\chi_A = \lim_{n\to\infty} \sup\{\chi_{A_1}, \chi_{A_2}, \ldots, \chi_{A_n}\}$ it follows from the second part of Lemma 1.12 that $\chi_A \in Q$. Hence $A \in \Omega$.

Since $\bigcap_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} (\mathbb{R}\setminus A_i)$, (iii) follows.

(iv) Since any open set is a countable union of open intervals it is enough to prove the result for open intervals. Let (a,b) be any open interval. We are now going to show that $\chi_{(a,b)} \in Q$.

$$\text{Define } f_n(x) = \begin{cases} 0 & \text{on } (-\infty,a) \cup (b,\infty) \\ 1 & \text{on } [a+\frac{1}{n}, b-\frac{1}{n}] \\ n(x-a) & \text{on } [a, a+\frac{1}{n}] \\ n(b-x) & \text{on } [b-\frac{1}{n}, b] \end{cases}$$



Then clearly $(f_n) \subset Q$, $\sup \|f_n\| = 1 < \infty$ and $\lim_{n\to\infty} f_n = \chi_{(a,b)}$. Thus $\chi_{(a,b)} \in Q$.

Proposition 1.14 ([14], Theorem 16.7)

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. The following conditions are equivalent:

(i) $f \in Q$

(ii) For all $\alpha \in \mathbb{R}$, $\{x : f(x) \geq \alpha\} \in \Omega$.

Proof

Suppose $f \in Q$, then let

$M = \{f : \mathbb{R} \longrightarrow \mathbb{R} : \{x : f(x) \geq \alpha\} \in \Omega \text{ for all } \alpha\}$.

From Proposition 1.13 (iii) and (iv) it is clear that $C(\mathbb{R}) \subset M$.

Suppose $(g_n) \subset M$, $\sup \|g_n\| < \infty$ and $g = \lim_n g_n$.

Since $\{x \in \mathbb{R} : g(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} \{x : g_j(x) \geq \alpha - \frac{1}{n}\}$,

it is clear from Proposition 1.13 (iii) that $\{x \in \mathbb{R} : g(x) \geq \alpha\} \in \Omega$. Hence $g \in M$. Thus M is an L-set containing Q. We've now shown that (i) implies (ii).

Conversely, suppose $f \in M$, for $\alpha \in \mathbb{R}$ let $A(\alpha) := \{x : f(x) \geq \alpha\}$. Then $\chi_{A(\alpha)} \in Q$ and $\chi_{\mathbb{R}\setminus A(\alpha)} \in Q$ (Proposition 1.13).

Since $\alpha\chi_{A(\alpha)}(x) = \begin{cases} \alpha\cdot 1 & \text{if } x \in A(\alpha) \\ 0 & \text{if } x \notin A(\alpha) \end{cases}$

$= \begin{cases} \alpha & \text{if } f(x) \geq \alpha \\ 0 & \text{if } f(x) < \alpha \end{cases}$

it follows that $\sup_{\substack{\alpha\geq 0 \\ \alpha\in\mathbb{Q}}} \alpha\chi_{A(\alpha)}(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$

$= \sup\{f,0\}(x).$

Similarly $\sup\{-f,0\} = \sup_{\substack{\alpha\geq 0 \\ \alpha\in\mathbb{Q}}} \alpha\chi_{\mathbb{R}\backslash A(-\alpha)}$.

Since $f = \sup\{f,0\} - \sup\{-f,0\}$ and $\sup\{f,0\}$, $\sup\{-f,0\} \in Q$ we know from Lemma 1.12 that $f \in Q$.

Theorem 1.15 (Baire's Theorem)

If $\mathcal{B}(\mathbb{R})$ is the bounded Borel measurable functions on $\mathbb{R}$, then $\mathcal{B}(\mathbb{R}) = Q$ i.e. $\mathcal{B}(\mathbb{R})$ is the smallest class of bounded functions which contains the bounded continuous functions and which is closed with regard to pointwise convergence of bounded sequences of functions.

Proof

Since Ω as defined in Proposition 1.13 contains all open sets and is a σ-algebra it is clear that the σ-algebra $\psi = \{E \in \mathbb{R} : \chi_E \in \mathcal{B}(\mathbb{R})\}$ is contained in Ω. Hence if $f \in \mathcal{B}(\mathbb{R})$ then for all $\alpha \in \mathbb{R}$, $\{x : f(x) \geq \alpha\} \in \psi \subset \Omega$.
Thus $f \in Q$ (cf Proposition 1.14 (i)), so $\mathcal{B}(\mathbb{R}) \subseteq Q$. Conversely it is well-known from the properties of bounded Borel functions that $\mathcal{B}(\mathbb{R})$ is an L-set, thus $Q \subseteq \mathcal{B}(\mathbb{R})$ and we have proved that $\mathcal{B}(\mathbb{R}) = Q$.

This theorem can now easily be extended to complex valued bounded Borel functions which we denote by $\mathcal{B}(\mathbb{C})$ where $\mathcal{B}(\mathbb{C}) = \mathcal{B}(\mathbb{R}) + i\mathcal{B}(\mathbb{R})$.

Theorem 1.16

$\mathcal{B}(\mathbb{C})$ is the smallest class of complex valued functions containing the continuous functions and which is closed with regard to pointwise convergence of bounded sequences of functions.

Proof

Clearly $\mathcal{B}(\mathbb{C})$ is an L-set or such a class. The existence of the smallest class is easily verified, let $Q = \mathcal{D} + i\mathcal{E}$ be the smallest class. Let $\mathcal{R}$ now denote all those functions where $\mathcal{E}$ is zero, thus the imaginary part is zero. Since $Q \subseteq \mathcal{B}(\mathbb{C})$ it follows that $\mathcal{R} \cap Q \subseteq \mathcal{R} \cap \mathcal{B}(\mathbb{C})$ and thus $\mathcal{D} \subseteq \mathcal{B}(\mathbb{R})$. We know that Q contains all real valued continuous functions, thus $\mathcal{D}$ also contains them and since Q is closed with respect to pointwise convergence so is $\mathcal{D}$. Thus $\mathcal{D}$ is an L-set and then $\mathcal{B}(\mathbb{R}) \subset \mathcal{D}$ ($\mathcal{B}(\mathbb{R})$ is the smallest such class (cf Theorem 1.15)), hence $\mathcal{B}(\mathbb{R}) = \mathcal{D}$.

It is easily verified that $-iQ$ contains all continuous functions if Q contains them and also if Q is closed with regard to pointwise convergence, so is $-iQ$. Since $Q \subseteq \mathcal{B}(\mathbb{C})$, $-iQ \subseteq -i\mathcal{B}(\mathbb{C})$. Hence from $\mathcal{B}(\mathbb{R}) + i\mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{C})$ it is clear that $-iQ \subseteq \mathcal{B}(\mathbb{C})$. If we now use exactly the same argument as before, it follows that $\mathcal{E} = \mathcal{B}(\mathbb{R})$ and hence $Q = \mathcal{B}(\mathbb{C})$.

In our previous theorem the boundedness of functions can also be dropped. In fact we can prove the following result which is a corollary of Theorem 1.16. Let $\breve{\mathcal{B}}$ be the smallest class of all complex valued Borel functions.

Corollary 1.17

$\breve{\mathcal{B}}$ is the smallest class of complex valued functions which contains the continuous functions and which is closed with regard to pointwise convergence.

Proof

Clearly $\breve{\mathcal{B}}$ is such a class. If we take the intersection of all such classes it can easily be verified that this will be the smallest class, let it be Q. Thus $Q \subseteq \breve{\mathcal{B}}$. From the fact that Q contains bounded continuous functions as well as the fact that Q is closed with regard to pointwise convergence of bounded sequences, it follows that $\mathcal{B}(\mathbb{C}) \subseteq Q$ since $\mathcal{B}(\mathbb{C})$ is the smallest such class (cf Theorem

1.16). Any $f \in \mathcal{B}$ is the pointwise limit of a sequence (f_n) in $\mathcal{B}(\mathbb{C}) \subseteq Q$ (this follows from the fact that any positive measurable function is a pointwise limit of simple functions). Thus since Q is an L-set $f \in Q$. Thus $\mathcal{B} \subseteq Q$ and it follows that $\mathcal{B} = Q$.

We now state and prove the main theorem of this section.

Theorem 1.18 ([10], Theorem 2.20)

Let $T \in \mathcal{B}(H)$ be a self-adjoint operator. There exists a unique mapping $f \longmapsto f(T)$ from $\mathcal{B}(\sigma(T))$ to $\mathcal{B}(H)$ such that:

(i) if f is a polynomial and $f(\lambda) = a_0 + a_1\lambda + \ldots + a_n\lambda^n$ then $f(T) = a_0 I + a_1 T + \ldots + a_n T^n$

(ii) if $f, f_n \in \mathcal{B}(\sigma(T))$, $\sup \|f_n\| \leq \infty$ and $f_n \longrightarrow f$ pointwise then $f_n(T) \longrightarrow f(T)$ in the strong operator topology in $\mathcal{B}(H)$.

Moreover this mapping is a *-homomorphism of the C^*-algebra $\mathcal{B}(\sigma(T))$ into the von Neumann algebra $\mathcal{A}\{(T,I)\}$ and it is an extension of the mapping τ mentioned in Lemma 1.8.

Proof

If φ is a mapping satisfying (i) and (ii), φ coincides with τ when φ is restricted to $\mathcal{C}(\sigma(T))$. Then

$\varphi : \mathcal{B}(\sigma(T)) \longrightarrow \mathcal{B}(H)$ and $\varphi(p) = \tau(p)$ where p is a polynomial. If $f \in \mathcal{C}(\sigma(T))$ it follows from the Stone-Weierstrass theorem ([9], p 161) that there exists a sequence (p_n) of polynomials converging uniformly to f on $\sigma(T)$ and hence pointwise. It follows from (ii) that $\varphi(p_n) \longrightarrow \varphi(f)$ in the strong operator topology. On the other hand $\tau(p_n) \longrightarrow \tau(f)$ uniformly (τ is an isometry), and hence strongly.

Since $\tau(p_n) = \varphi(p_n)$ for each n, it follows that $\tau(f) = \varphi(f)$.

We now show that a mapping φ satisfying (i) and (ii) is unique (if it exists): Suppose $\psi : \mathcal{B}(\sigma(T)) \longrightarrow \mathcal{B}(H)$ is another such mapping. Then $\varphi|_{\mathcal{C}(\sigma(T))} = \psi|_{\mathcal{C}(\sigma(T))}$.

Since any bounded Borel function on $\sigma(T)$ is a pointwise limit of a bounded sequence of simple functions which is Borel measurable and the open sets of $\sigma(T)$ generate the σ-algebra of Borel measurable sets it suffices to show that $\varphi(\chi_{\mathcal{O}}) = \psi(\chi_{\mathcal{O}})$ by using properties (i) and (ii). By $\chi_{\mathcal{O}}$ we denote the characteristic function of an open subset $\mathcal{O}$ of $\sigma(T)$.

To prove this let $\mathcal{O}$ be any open subset of $\sigma(T)$. Then $\mathcal{O} = B \cap \sigma(T)$ where B is open in $\mathbb{R}$. Since B is a countable union of open intervals, say $B = \bigcup_{i=1}^{\infty} I_i$ one has that $\mathcal{O}$ is a countable union of sets of the form $I_i \cap \sigma(T)$.

Since $\chi_{\mathcal{O}} = \lim_n \sup (\chi_{I_1 \cap \sigma(T)}, \chi_{I_2 \cap \sigma(T)}, \dots, \chi_{I_n \cap \sigma(T)})$ it is sufficient from property (ii) to show that $\varphi(\chi_{I \cap \sigma(T)}) = \psi(\chi_{I \cap \sigma(T)})$ where I is an open interval in $\mathbb{R}$. If we let $I = (a,b)$ and define f_n as in the proof of Proposition 1.13 (iv), then $f_n \longrightarrow \chi_I$ pointwise. Let $g_n = f_n|_{\sigma(T)}$, then $g_n \in \mathcal{C}(\sigma(T))$ and $g_n \longrightarrow \chi_{I \cap \sigma(T)}$ pointwise. Since $\varphi(g_n) = \psi(g_n)$ the result follows from property (ii).

We now prove the existence of such a mapping as well as the other properties of the mapping described in the theorem:

Consider the spectral family of projections $(E_\lambda)_\lambda$, for any $\xi, \eta \in H$ we shall consider the function $E_{\xi,\eta}$ defined by the relation $E_{\xi,\eta}(\lambda) = \langle E_\lambda \xi, \eta \rangle$ with $\lambda \in \mathbb{R}$. We show that the function $E_{\xi,\eta}$ is of bounded variation. If we choose any partition of $[-\|T\|, \|T\|]$ say $\{\lambda_0, \dots, \lambda_n\}$ then $\sum_{i=1}^{n} |\langle E_{\lambda i} \xi, \eta \rangle - \langle E_{\lambda i-1} \xi, \eta \rangle|$

$$= \sum_{i=1}^{n} |\langle (E_{\lambda i} - E_{\lambda i-1}) \xi, \eta \rangle|$$

$$= \sum_{i=1}^{n} |\langle (E_{\lambda i} - E_{\lambda i-1})^2 \xi, \eta \rangle|$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} |<(E_{\lambda i} - E_{\lambda i-1})\xi,\ (E_{\lambda i} - E_{\lambda i-1})\eta>| \\
&\le \sum_{i=1}^{n} \|(E_{\lambda i} - E_{\lambda i-1})\xi\|\ \|(E_{\lambda i} - E_{\lambda i-1})\eta\| \\
&\le \left(\sum_{i=1}^{n} \|(E_{\lambda i} - E_{\lambda i-1})\xi\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} \|(E_{\lambda i} - E_{\lambda i-1})\eta\|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i=1}^{n} <(E_{\lambda i} - E_{\lambda i-1})\xi,\xi>\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} <(E_{\lambda i} - E_{\lambda i-1})\eta,\eta>\right)^{\frac{1}{2}} \\
&= (<(E_{\lambda n} - E_{\lambda o})\xi,\xi>)^{\frac{1}{2}}\ (<(E_{\lambda n} - E_{\lambda o})\eta,\eta>)^{\frac{1}{2}} \\
&= <\xi,\xi>^{\frac{1}{2}}\ <\eta,\eta>^{\frac{1}{2}} = \|\xi\|\ \|\eta\|
\end{aligned}
$$

If we take the supremum over all possible partitions we have $\sup\ (E_{\xi,\eta}) \le \|\xi\|\ \|\eta\|$, where $\sup\ (E_{\xi,\eta})$ is the total variation of $E_{\xi,\eta}$. Hence $E_{\xi,\eta}$ is of bounded variation. $E_{\xi,\eta}$ now defines a measure (called the Lebesgue-Stieltjes measure) on the σ-algebra of Borel sets on $\mathbb{R}$. For every $f \in \mathcal{B}(\sigma(T))$ we define

$$
\begin{aligned}
F_f(\xi,\eta) &= \int_{\sigma(T)} f(\lambda) dE_{\xi,\eta}(\lambda) \\
&= \int_{-\infty}^{\infty} f(\lambda) dE_{\xi,\eta}(\lambda) \ .
\end{aligned}
$$

(To get the second equality, we extend f to a Borel function on $\mathbb{R}$. Notice that if $\lambda \notin \sigma(T)$ we can find through the construction of the projection E_λ an open interval $(\lambda-\delta,\lambda+\delta)$ on which $E_\lambda = E_\mu$ for each $\mu \in (\lambda-\delta,\lambda+\delta)$. Thus the support of the measure $E_{\xi,\eta}$ is contained in $\sigma(T)$ ([7], Theorem 9.11-2).)

$F_f(\xi,\eta)$ is bounded and of sesquilinear form, because

$$
\begin{aligned}
|F_f(\xi,\eta)| &= \left|\int_{-\infty}^{\infty} f(\lambda) dE_{\xi,\eta}(\lambda)\right| \\
&\le \int_{-\infty}^{\infty} |f(\lambda)|\ |dE_{\xi,\eta}(\lambda)|
\end{aligned}
$$

$$\leq \|f\| \int_{-\infty}^{\infty} |dE_{\xi,\eta}(\lambda)|$$
$$= \|f\| \ \sup\ (E_{\xi,\eta})$$
$$\leq \|f\| \ \|\xi\| \ \|\eta\|$$

thus F_f is bounded and since for inner products $\langle x,ay\rangle = \bar{a}\langle x,y\rangle$ F_f is of sesquilinear form defined on $H \times H$. From the representation theorem of Riesz it follows that there exists a unique operator $f(T) \in \mathcal{B}(H)$ such that

$$\langle f(T)\xi,\eta\rangle = \int_{-\infty}^{\infty} f(\lambda)dE_{\xi,\eta}(\lambda), \quad \xi,\eta \in H.$$

We've now defined the mapping $f \longmapsto f(T)$ from $\mathcal{B}(\sigma(T)) \longrightarrow \mathcal{B}(H)$.
If $f,g \in \mathcal{B}(\sigma(T))$ and $\xi,\eta \in H$, the linearity follows since $\langle (f(T) + g(T))\xi,\eta\rangle$

$$= \langle f(T)\xi,\eta\rangle + \langle g(T)\xi,\eta\rangle$$
$$= \int_{-\infty}^{\infty} f(\lambda)dE_{\xi,\eta}(\lambda) + \int_{-\infty}^{\infty} g(\lambda)dE_{\xi,\eta}(\lambda)$$
$$= \int_{-\infty}^{\infty} (f(\lambda) + g(\lambda))dE_{\xi,\eta}(\lambda)$$
$$= \int_{-\infty}^{\infty} (f + g)(\lambda)dE_{\xi,\eta}(\lambda)$$
$$= \langle (f + g)(T)\xi,\eta\rangle$$

thus $f(T) + g(T) = (f + g)(T)$.
Our next step is to show that $\bar{f}(T) = (f(T))^*$.

$$\langle \bar{f}(T)\xi,\eta\rangle = \int_{-\infty}^{\infty} \bar{f}(\lambda)dE_{\xi,\eta}(\lambda)$$
$$= \overline{\int_{-\infty}^{\infty} f(\lambda)d\bar{E}_{\xi,\eta}(\lambda)}$$
$$= \overline{\int_{-\infty}^{\infty} f(\lambda)dE_{\xi,\eta}(\lambda)}$$
$$= \overline{\langle f(T)\xi,\eta\rangle}$$
$$= \langle \xi,f(T)\eta\rangle$$
$$= \langle (f(T))^*\xi,\eta\rangle$$

Thus $\bar{f}(T) = (f(T))^*$.

For $f \in B(\sigma(T))$ we have

$$E_{\xi,(f(T))^*\eta}(\lambda) = \langle E_\lambda \xi, (f(T))^* \eta \rangle \quad \text{with} \quad \xi,\eta \in H$$
$$= \langle f(T) E_\lambda \xi, \eta \rangle$$
$$= \int_{-\infty}^{\infty} f(\mu) dE_{E_\lambda \xi,\eta}(\mu)$$

If $\mu > \lambda$ it is true for the spectral family of projections $(E_\lambda)_\lambda$ that $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

Hence if $\mu_1, \mu_2 > \lambda$,

$$E_{E_\lambda \xi,\eta}(\mu_1) - E_{E_\lambda \xi,\eta}(\mu_2)$$
$$= \langle E_{\mu_1} E_\lambda \xi, \eta \rangle - \langle E_{\mu_2} E_\lambda \xi, \eta \rangle$$
$$= \langle E_\lambda \xi, \eta \rangle - \langle E_\lambda \xi, \eta \rangle = 0$$

Thus $E_{\xi,(f(T))^*\eta}(\lambda) = \int_{-\infty}^{\lambda} f(\mu) dE_{\xi,\eta}(\mu)$.

Suppose $g = \chi_{[a,b]}$ then $\int_{-\infty}^{\infty} \chi_{[a,b]} \, d \int_{-\infty}^{\lambda} f(\mu) dE_{\xi,\eta}(\mu)$

$$= \int_a^b d \int_{-\infty}^{\lambda} f(\mu) dE_{\xi,\eta}(\mu)$$
$$= \int_{-\infty}^b d \int_{-\infty}^{\lambda} f(\mu) dE_{\xi,\eta}(\mu) - \int_{-\infty}^a d \int_{-\infty}^{\lambda} f(\mu) dE_{\xi,\eta}(\mu)$$
$$= \int_a^b f(\mu) dE_{\xi,\eta}(\mu)$$
$$= \int_{-\infty}^{\infty} \chi_{[a,b]} f(\mu) dE_{\xi,\eta}(\mu)$$
$$= \int_{-\infty}^{\infty} (gf)(\mu) dE_{\xi,\eta}(\mu).$$

Thus $\langle f(T)g(T)\xi, \eta \rangle = \langle g(T)\xi, (f(T))^* \eta \rangle$

$$= \int_{-\infty}^{\infty} g(\mu) dE_{\xi,(f(T))^*\eta}(\mu)$$

$$= \int_{-\infty}^{\infty} g(\mu)d \int_{-\infty}^{\lambda} f(\mu)dE_{\xi,\eta}(\mu)$$

$$= \int_{-\infty}^{\infty} (gf)(\mu)dE_{\xi,\eta}(\mu).$$

By means of the Lebesgue's Monotone Convergence Theorem ([1], 5, Theorem 15), the above-mentioned relation can be extended to a positive $g \in \mathcal{B}(\sigma(T))$ and thus for any $g \in \mathcal{B}(\sigma(T))$. Hence $f(T)g(T) = (fg)(T)$ with $f,g \in \mathcal{B}(\sigma(T))$.

Consequently $f \longmapsto f(T)$ is a *-homomorphism of the C^*-algebra $\mathcal{B}(\sigma(T))$ into $\mathcal{B}(H)$.

If $f_0(\lambda) = 1$ and $\lambda \in \sigma(T)$ then from

$$\langle f_0(T)\xi,\eta\rangle = \int_{-\infty}^{\infty} f_0(\lambda)dE_{\xi,\eta}(\lambda)$$

$$= \int_{-\infty}^{\infty} dE_{\xi,\eta}(\lambda)$$

$= \langle\xi,\eta\rangle$ we have $f_0(T) = I$.

If $f_1(\lambda) = \lambda$ and $\lambda \in \sigma(T)$ then

$$\langle f_1(T)\xi,\eta\rangle = \int_{-\infty}^{\infty} f_1(\lambda)dE_{\xi,\eta}(\lambda)$$

$$= \int_{-\infty}^{\infty} \lambda dE_{\xi,\eta}(\lambda)$$

$= \langle T\xi,\eta\rangle$ therefore $f_1(T) = T$.

Since $f \longmapsto f(T)$ is a multiplicative mapping, we've proved that if

$f(\lambda) = a_0 + a_1\lambda^1 + a_2\lambda^2 + \ldots + a_n\lambda^n$

then $f(T) = a_0 I + a_1 T + a_2 T^2 + \ldots + a_n T^n$.

For any $f \in \mathcal{B}(\sigma(T))$ and any $\xi \in H$

$$\langle f(T)\xi, f(T)\xi\rangle = \langle (f(T))^* f(T)\xi,\xi\rangle$$

$$= \langle \bar{f}(T).\ f(T)\xi,\xi\rangle$$

$$= \langle |f|^2(T)\xi,\xi\rangle$$

thus $\|f(T)\xi\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 dE_{\xi,\xi}(\lambda).$

If $f_n \in \mathcal{B}(\sigma(T))$, $\sup \|f_n\| < \infty$ and $f_n \to f$ pointwise,

$$\|(f_n(T) - f(T))\xi\|^2 = \int_{-\infty}^{\infty} |f_n(\lambda) - f(\lambda)|^2 dE_{\xi,\xi}(\lambda)$$

by Lebesgue's Dominated Convergence Theorem ([1], Theorem 5.21) this integral tends to zero.

Thus $f_n(T) \to f(T)$ in the strong operator topology.

Finally, since the set $\{f \in \mathcal{B}(\sigma(T)): f(T) \in \mathcal{A}(\{T,I\})\}$ contains the polynomials and is closed with regard to pointwise convergence, our theorem of Baire (cf Theorem 1.15) applies; consequently the above-mentioned set is equal to $\mathcal{B}(\sigma(T))$. Hence the mapping is a *-homomorphism of the C^*-algebra $\mathcal{B}(\sigma(T))$ into the von Neumann algebra $\mathcal{A}(\{T,I\})$.

Our next lemma will be used in the proof of our dual-space characterization. A proof can be found in [6] (Theorem 6.8.8), but we provide an alternative proof using Borel calculus.

Lemma 1.19

If $\mathcal{K}$ is a weak operator closed left (or right) ideal in a von Neumann algebra $\mathcal{A}$ then $\mathcal{K} = \mathcal{A}E$ (or $\mathcal{K} = E\mathcal{A}$) for some projection E in $\mathcal{A}$. If $\mathcal{K}$ is a two-sided ideal E is a central projection in $\mathcal{A}$.

Proof

Let T be a positive operator in $\mathcal{K}$ and let $\{E_\lambda\}$ be the spectral resolution for T. Let $S_\lambda = \int_\lambda^{\|T\|} \frac{1}{\mu}\, dE_\mu$. Then it follows that $I - E_\lambda = TS_\lambda \in \mathcal{K}$. Since $I - E_\lambda = \chi_{[\lambda,\infty)}(T)$ (where $\chi_{[\lambda,\infty)}(T)$ is to be understood in the sense of the Borel calculus) it is clear via Theorem 1.12 that $I - E_{1/n}$ converges strongly to $R(T)$ ($= \chi_{(0,\infty)}(T)$, cf the proof of Theorem 2.4 (ii)). Hence $R(T) \in \mathcal{K}$.

Now if we let S be any operator in $\mathcal{K}$. Then S^*S is a positive element of $\mathcal{K}$ and from the argument above it follows that $R(S^*) = R(S^*S) \in \mathcal{K}$.

Since $\sup(E,F) = R(E+F)$ (cf Lemma 1.4) for $E,F \in \mathcal{P}(\mathcal{A})$, the union of a finite family of projections in $\mathcal{K}$ is in $\mathcal{K}$. Since $\mathcal{K}$ is weak operator closed it follows that $E := \sup_{T \in \mathcal{K}} R(T^*) \in \mathcal{K}$.

Since $ET^* = T^*$, $(ET^*)^* = (T^*)^*$

thus $TE = T$ for each T in $\mathcal{K}$ and $\mathcal{K} = \mathcal{A}E$.

If $\mathcal{K}$ is a right ideal, $\mathcal{K}^*$ is a left ideal. We've just shown that for some projection F in $\mathcal{A}$, $\mathcal{K}^* = \mathcal{A}F$, thus $\mathcal{K} = F\mathcal{A}$. If $\mathcal{K}$ is a two-sided ideal $\mathcal{K} = F\mathcal{A} = \mathcal{A}E$, thus $E = FE = F$ it now follows that $E\mathcal{A}E = \mathcal{A}E$ and the range of E is invariant under $\mathcal{A}$ ([5], p121). Since $\mathcal{A}$ is a self-adjoint family, E commutes with all the operators in $\mathcal{A}$ and E is a central projection.

We conclude this chapter with a few examples of von Neumann algebras to which we will refer in the chapters to follow.

Example 1.20

1. If $\mathcal{A} = \mathcal{B}(H)$ then clearly $\mathcal{A}$ is a von Neumann algebra.

2. Associate with each $x \in \ell^\infty$ the operator $T_x : \ell^2 \longrightarrow \ell^2$ defined by $T_x y = xy$. Then clearly $T_x \in \mathcal{B}(\ell^2)$ and $\|T_x\| = \|x\|$. Moreover its trivial to show that under this representation ℓ^∞ and $\mathcal{A} := \{T_x : x \in \ell^\infty\}$ are isomorphic as C^*-algebras. We show that $\mathcal{A}'' = \mathcal{A}$. This will imply that $\mathcal{A}$ is a von Neumann algebra. Let $e_n = (0,0,0,\ldots,0,1_n,0,0,\ldots)$. Then
   $\sup_n e_n = (1,1,1,\ldots)$.
   Then clearly T_{e_n} is a projection in $\mathcal{A}$ for each $n \in \mathbb{N}$ and the strong operator sum $\sum_{n=1}^{\infty} T_{e_n} = I$.
   Now if $T \in \mathcal{A}' = \{S \in \mathcal{B}(\ell^2) : ST_x = T_x S$ for all $x \in \ell^\infty\}$ and $T(e_n) = f_n = (f_n^k) \in \ell^2 \subset \ell^\infty$ then
   $TT_{e_n}(y) = T(ye_n) = T_y T(e_n) = f_n y = T_{f_n} y$ for all $y \in \ell^2$. ... (I)

Hence $TT_{e_n} = T_{f_n}$. Further from (1) it follows that $f_n = T(e_n) = T(e_n \cdot e_n) = T(T_{e_n} e_n) = f_n e_n$ for all $n \in \mathbb{N}$.
Define the sequence x by $x_k = f_k^k$ for $k = 1,2,3,\ldots$
Since $\|f_n\|_\infty = \|T_{f_n}\| = \|TT_{e_n}\| \leq \|T\|$ for all $n \in \mathbb{N}$, it is clear that $x \in \ell^\infty$.
Also $T_x T_{e_n} = T_{xe_n} = T_{f_n} T_{e_n} = TT_{e_n}$ for each $n \in \mathbb{N}$.
Since $\sum_{n=1}^{\infty} T_{e_n} = I$ it follows that

$$\begin{aligned} T_x = T_x I &= T_x \sum_{n=1}^{\infty} T_{e_n} \\ &= \sum_{n=1}^{\infty} T_x T_{e_n} \\ &= \sum_{n=1}^{\infty} TT_{e_n} \\ &= T \sum_{n=1}^{\infty} T_{e_n} \\ &= T. \end{aligned}$$

Thus $T \in \mathcal{A}$. This shows that $\mathcal{A}' \subset \mathcal{A}$. Since $\mathcal{A}$ is a commutative C^*-algebra it is clear that $\mathcal{A} \subset \mathcal{A}'$. Hence $\mathcal{A} = \mathcal{A}'$ which implies that $\mathcal{A}'' = (\mathcal{A}')' = (\mathcal{A})' = \mathcal{A}$. Hence $\mathcal{A}$ is a von Neumann algebra.

CHAPTER 2 : KADISON'S CHARACTERIZATION

I Kadison's characterization of von Neumann algebras

In Chapter 1 we defined von Neumann algebras among C^*-algebras as those C^*-algebras that are weak operator closed in their action on some Hilbert space H. In the same chapter we've also seen that von Neumann algebras satisfy the following order property, namely that each increasing net of operators in $\mathcal{A}$ that is bounded above has its supremum in $\mathcal{A}$. One significant structural difference between C^*-algebras and von Neumann algebras is that the weak operator closed algebras contain many projections, while in general C^*-algebras this is not the case. C^*-algebras with the above-mentioned order property contain many projections and possess similar properties than von Neumann algebras. It is thus natural to ask whether a C^*-algebra satisfying the order property mentioned above is isomorphic to a von Neumann algebra. The answer to this question is negative. In the mid fifties Kadison proved that a C^*-algebra satisfying this order property together with a separating condition is isomorphic to a von Neumann algebra.

In the latter part of this chapter we construct a C^*-algebra satisfying the order property, but not the separating condition. By proving that this C^*-algebra has no normal states we show that it can't be isomorphic to any von Neumann algebra. We now define a C^*-algebra satisfying these conditions.

Definition 2.1

A C^*-algebra $\mathcal{A}$ that satisfies the following two conditions is said to be a W^*-algebra:

(i) Any increasing net of self-adjoint operators with an upper bound, has a least upper bound.

(ii) The normal states of $\mathcal{A}$ separate $\mathcal{A}$ (i.e. if $T \in \mathcal{A}$ and $T \neq 0$ then there exists a normal state φ such that $\varphi(T) \neq 0$.)

Remember a state φ is a positive linear functional of norm one and φ is normal if it follows from $T_\alpha \uparrow T$ that $\varphi(T_\alpha) \uparrow \varphi(T)$. Since the vector states (φ is a vector state of $\mathcal{A}$ if $\varphi(T) = \langle Tx,x \rangle$ for some unit vector $x \in H$) separate $\mathcal{A}$ it follows from Proposition 1.5 that von Neumann algebras are definitely W^*-algebras.

Theorem 2.2 ([4], Lemma 4.1)

Let $\mathcal{U}$ be a C^*-algebra acting on a Hilbert space H. Suppose that each increasing net of operators in $\mathcal{U}$ that is bounded has its strong operator limit in $\mathcal{U}$, then:

(i) Each decreasing net of operators in $\mathcal{U}$ that is bounded from below has its strong operator limit in $\mathcal{U}$.

(ii) If S is an arbitrary projection in $\mathcal{U}$, then $R(S) \in \mathcal{U}$.

(iii) The union and intersection of each finite set of projections in $\mathcal{U}$ lie in $\mathcal{U}$.

(iv) The union and intersection of an arbitrary set of projections in $\mathcal{U}$ lie in $\mathcal{U}$.

(v) $E \in \mathcal{U}$, where E is a cyclic projection in $\bar{\mathcal{U}}^{\omega\text{-}o}$ with generating vector x provided that for each vector y in $(I - E)(H)$ there is a self-adjoint T_y in $\mathcal{U}$ such that $T_y x = x$ and $T_y y = 0$ ($\bar{\mathcal{U}}^{\omega\text{-}o}$ is the weak operator closure of $\mathcal{U}$).

(vi) $\bar{\mathcal{U}}^{\omega\text{-}o} = \mathcal{U}$ if each cyclic projection in $\bar{\mathcal{U}}^{\omega\text{-}o}$ lies in $\mathcal{U}$.

Proof

(i) If $\{T_n\}$ is a decreasing net in $\mathcal{U}$ that is bounded below with strong operator limit T, then $\{-T_n\}$ is an increasing net in $\mathcal{U}$ and $-T$ is its strong operator limit in $\mathcal{U}$ that bounds it from above (cf also Proposition 1.5).

(ii) This follows directly from the remark after Lemma 1.6 by noting that the only property of the von Neumann algebra $\mathcal{A}$ that was used in the remark, is the order property possessed by $\mathcal{U}$.

(iii) From Lemma 1.4 we know that $\sup\{E,F\} = R(E + F)$ and from (ii) we know that $R(E + F) \in \mathcal{U}$. Thus $\sup\{E,F\} \in \mathcal{U}$ when E and F are projections in $\mathcal{U}$. By induction we can easily show that the union of a finite family of projections is in $\mathcal{U}$.
Since we have $I - \sup_a (I - E_a) = \inf_a E_a$ from Corollary 1.2, the intersection of a finite family of projections in $\mathcal{U}$ is also in $\mathcal{U}$.

(iv) Let $\{E_a : a \in \mathcal{D}\}$ be any collection of projections in $\mathcal{U}$. Let $\mathcal{F}$ be the class of all finite subsets of $\mathcal{D}$. For $F \in \mathcal{F}$ let $E_F = \sup_{a\in F} E_a$. Clearly $E_F \in \mathcal{U}$ (cf (iii)). The family $\{E_F : F \in \mathcal{F}\}$ together with the order relation $E_{F_1} \leq E_{F_2}$ if and only if $F_1 \leq F_2$, is an increasing net with $\sup_{F\in\mathcal{F}} E_F = \sup_{F\in\mathcal{F}} (\sup_{a\in F} E_a) = \sup_{a\in\mathcal{D}} E_a$.
From Lemma 1.5 it is clear that $\{E_F, F \in \mathcal{F}\}$ is strong operator convergent to the projection $\sup_{a\in\mathcal{D}} E_a$. Hence by our assumption $\sup_{a\in\mathcal{D}} E_a \in \mathcal{U}$. Since $\inf_{a\in\mathcal{D}} E_a = I - \sup_{a\in\mathcal{D}} (I - E_a)$ (cf Corollary 1.2) it follows that $\inf_{a\in\mathcal{D}} E_a \in \mathcal{U}$.

(v) $R(T_y)(H) = \overline{T_y(H)}$

$$R(T_y)x = R(T_y)(T_y)x$$
$$= T_y x = x.$$

Since $T_y^* = T_y$ and $y \in N(T_y)$ one has

$$
\begin{aligned}
R(T_y)y &= (I - N(T_y^*))y \\
&= y - N(T_y^*)y \\
&= y - y \\
&= 0.
\end{aligned}
$$

Thus $Gx = x$ and $Gy = 0$ for each $y \in (I - E)H$ where $G = \inf_{y \in (I-E)H} R(T_y)$.

We know that the range projection of an operator lies in $\mathcal{U}$ from part (ii) of this theorem. From part (iv) we know that any intersection of an arbitrary set of projections lies in $\mathcal{U}$, thus $G \in \mathcal{U}$.

As E is cyclic under $\mathcal{U}'$ with generating vector x

$$
[\mathcal{U}'x] = E(H) \quad \text{with} \quad x \in G(H),
$$

$$
\begin{aligned}
\mathcal{U}'x &\subset \mathcal{U}'G(H) \\
&= G\mathcal{U}'(H) \\
&\subseteq G(H).
\end{aligned}
$$

Thus $E(H) = [\mathcal{U}'x] \subseteq G(H)$ and $E \leq G$.

To get the other inequality we choose any $y' \in H$ and let $y = (I - E)y'$. Hence $Gy = 0$. Consequently $G(I - E)y' = 0$ and $GIy' - G Ey' = 0$ for all $y' \in H$. It now follows that $G = GE$ and we have $G \leq E$. Thus $E = G \in \mathcal{U}$.

(vi) From Lemma 1.7 each projection is the union of an orthogonal family of cyclic projections and from part (iv) each union of an arbitrary set of projections in $\mathcal{U}$ lies in $\mathcal{U}$. It now follows that each projection in $\bar{\mathcal{U}}^{\omega\text{-}o}$ lies in $\mathcal{U}$. From the spectral theorem (cf Theorem 1.10) we know that each self-adjoint operator (which is the norm limit of projections) in $\bar{\mathcal{U}}^{\omega\text{-}o}$, lies in $\mathcal{U}$.
Since $\bar{\mathcal{U}}^{\omega\text{-}o}$ is a self-adjoint algebra containing $\mathcal{U}$, $\bar{\mathcal{U}}^{\omega\text{-}o} = \mathcal{U}$.

Remark

This theorem shows that a C^*-algebra is isomorphic to a von Neumann algebra iff (i) of Definiton 2.1 is satisfied and if each cyclic projection in $\bar{\mathcal{U}}^{\omega-o}$ lies in $\mathcal{U}$.

If $\mathcal{U}$ is a C^*-algebra we denote by $\mathcal{U}_h$ the self-adjoint operators in $\mathcal{U}$ (i.e. $T \in \mathcal{U}_h$ if $T = T^*$). By $\mathcal{U}_1$ (resp $(\mathcal{U}_h)_1$) we denote the unit ball in $\mathcal{U}$ (resp $\mathcal{U}_h$). For any $T \in \mathcal{U}_h$ it is easy to show that T can be written as the difference of two positive operators in $\mathcal{U}_h$ (i.e. $T = T^+ - T^-$).

Lemma 2.3

If $T \in \mathcal{U}_h$ there exist two positive operators T^+ and T^- in $\mathcal{U}_h$ such that $T = T^+ - T^-$ and $T^+T^- = 0$.

Proof

If $\mathcal{A}(T,I)$ is a commutative C^*-algebra generated by T and I, then $\mathcal{A}(T,I) \simeq \mathcal{C}(\sigma(T))$ (cf Lemma 1.8 (Gelfand-Naimark)). Let $f \in \mathcal{C}(\sigma(T))$ now be the function corresponding to T. Since

$$f = f^+ - f^- \text{ with } f^+ = \max\{f,0\} \in \mathcal{C}(\sigma(T))$$
$$\text{and } f^- = -\min\{0,f\} \in \mathcal{C}(\sigma(T))$$

there exist positive operators T^+ and T^- associated with f^+ and f^- such that $T = T^+ - T^-$ and $T^+T^- = 0$.

Lemma 2.4 ([4], Lemma 4.2)

Let $\mathcal{U}$ be a C^*-algebra acting on a Hilbert space H, such that each increasing net of operators in $\mathcal{U}$ that is bounded has its strong operator limit in $\mathcal{U}$. Suppose E is a cyclic projection in $\bar{\mathcal{U}}^{\omega-o}$ and x is a generating vector for $E(H)$ under $\mathcal{U}'$ with norm one. If y is a unit vector in $(I - E)(H)$ the following holds:

(i) There is a sequence $\{A_n\}$ in $(\mathcal{U}_h)_1$ such that $A_n x \to x$ and $A_n y \to 0$

$\|(A_n - A_{n-1})^+ x\| < 2^{1-n}$ and

$\|(A_n - A_{n-1})^+ y\| < 2^{1-n}$ where $A_0 = 0$.

(ii) If $T_n = [I + \sum_{k=1}^{n} (A_k - A_{k-1})^+]^{-1}$, then $\{T_n\}$ is a bounded monotone decreasing sequence of positive elements of $\mathcal{U}$ and $T^{\frac{1}{2}}(\sum_{k=1}^{n} (A_k - A_{k-1})^+)T^{\frac{1}{2}} \leq I$ for each n, where T is the strong operator limit of $\{T_n\}$ in $\mathcal{U}$.

(iii) For each j in $\{1,2,\ldots,n\}$, $\{T^{\frac{1}{2}}(\sum_{k=j}^{n} (A_k - A_{k-1})^+)T^{\frac{1}{2}}\}$ is monotone increasing with n, bounded above by I and if C_j is its strong operator limit then $0 \leq C_j \leq I$ and $\{C_j\}$ is decreasing; it also follows that

$$T^{\frac{1}{2}}(\sum_{k=1}^{n} (A_k - A_{k-1})^+)T^{\frac{1}{2}} + C_{n+1} = C_1$$

and $T^{\frac{1}{2}}A_n T^{\frac{1}{2}} + C_{n+1} = T^{\frac{1}{2}}(\sum_{k=1}^{n} - (A_k - A_{k+1})^-)T^{\frac{1}{2}} + C_1$.

(iv) $\{T^{\frac{1}{2}}A_n T^{\frac{1}{2}} + C_{n+1}\}$ is monotone decreasing and bounded and $T^{\frac{1}{2}}AT^{\frac{1}{2}} \in \mathcal{U}$ where A is a weak operator limit of $\{A_n\}$.

(v) $R(T) \in \mathcal{U}$, $R(T)x = x$ and $R(T)y = y$.

(vi) Each maximal abelian (self-adjoint) subalgebra of $\mathcal{U}$ is weak operator closed.

Proof

(i) From the Kaplansky density theorem ([2], Theorem 3.6.1) we know that $\overline{(\mathcal{U}_h)_1} = (\overline{\mathcal{U}_h})_1$ where the closure is taken in the strong operator topology. It follows that there exists a sequence $\{A_n\}$ in $(\mathcal{U}_h)_1$ such that $A_n x \longrightarrow x$ and $A_n y \longrightarrow 0$, since

$$E \in \overline{(\mathcal{U}_h)}_1 = \overline{(\mathcal{U}_h)_1}, \ Ex = x \ \text{ and } \ Ey = 0.$$

We consider convergent subsequences $\{A_n x\}$ and $\{A_n y\}$ such that $\|(A_n - A_{n-1})x\| < \epsilon = 2^{1-n}$ and $\|(A_n - A_{n-1})y\| < 2^{1-n}$.
For each self-adjoint T it follows from Lemma 2.3 that T^+ and T^- have orthogonal ranges, hence

$$\|Tz\|^2 = \|T^+z\|^2 + \|T^-z\|^2$$

thus $\|T^+z\| \leq \|Tz\|$.
It now follows for all $n \in \mathbb{N}$ that

$$\|(A_n - A_{n-1})^+x\| \leq 2^{1-n} \ \text{ and } \ \|(A_n - A_{n-1})^+y\| < 2^{1-n}.$$

(ii) If R is positive it is clear from the spectral mapping theorem ([12], Proposition 2.8) that $\sigma(I + R) \subseteq [1,\infty)$, hence $(I + R)^{-1}$ exists in $\mathcal{U}$. Thus T_n exists in $\mathcal{U}$ as it is defined in (ii).
If A is invertible and $0 \leq A \leq B$ we know (cf [5], Proposition 4.2.8) that B is invertible and $B^{-1} \leq A^{-1}$.
Since

$$I \leq I + \sum_{k=1}^{n} (A_k - A_{k-1})^+ \leq I + \sum_{k=1}^{n+1} (A_k - A_{k-1})^+$$

we now have $0 \leq T_{n+1} \leq T_n \leq I$ with

$$T_n = \left(I + \sum_{k=1}^{n} (A_k - A_{k-1})^+\right)^{-1}.$$

Since $\{T_n\}$ is decreasing and bounded below it follows from Theorem 2.2 (i) that $\{T_n\}$ converges strongly to a $T \in \mathcal{U}$.
With u a given unit vector in H and m large enough, $T_m^{\frac{1}{2}}u$ is close to $T^{\frac{1}{2}}u$ since the mapping $A \longrightarrow A^{\frac{1}{2}}$ is strong operator continuous on the unit ball of $\mathcal{B}(H)^+$ [cf Appendix, Lemma 2].

Now if $n \leq m$, $\langle T_m^{\frac{1}{2}}(\sum_{k=1}^{n}(A_k - A_{k-1})^+)\, T_m^{\frac{1}{2}}u,u\rangle$

$$= \langle(\sum_{k=1}^{n}(A_k - A_{k-1})^+)\, T_m^{\frac{1}{2}}u,\, T_m^{\frac{1}{2}}u\rangle$$

$$\leq \langle(\sum_{k=1}^{m}(A_k - A_{k-1})^+)T_m^{\frac{1}{2}}u,\, T_m^{\frac{1}{2}}u\rangle$$

$$= \langle T_m^{\frac{1}{2}}(\sum_{k=1}^{m}(A_k - A_{k-1})^+)T_m^{\frac{1}{2}}u,u\rangle$$

Since $\sum_{k=1}^{m}(A_k - A_{k-1})^+ = T_m^{-1} - I$ it follows that T_m and $\sum_{k=1}^{m}(A_k - A_{k-1})^+$ commute, hence $T_m^{\frac{1}{2}}$ and $\sum_{k=1}^{m}(A_k - A_{k-1})^+$ commute. Thus

$$\langle T_m^{\frac{1}{2}}(\sum_{k=1}^{m}(A_k - A_{k-1})^+)T_m^{\frac{1}{2}}u,u\rangle$$

$$\leq \langle T_m(\sum_{k=1}^{m}(A_k - A_{k-1})^+)u,u\rangle$$

$$= \langle T_m(T_m^{-1} - I)u,u\rangle$$

$$= \langle(I - T_m)u,u\rangle$$

$$\leq \langle Iu,u\rangle .$$

Hence $\lim_{m\to\infty} \langle T_m^{\frac{1}{2}}(\sum_{k=1}^{n}(A_k - A_{k-1})^+)T_m^{\frac{1}{2}}u,u\rangle$

$$= \langle T^{\frac{1}{2}}(\sum_{k=1}^{n}(A_k - A_{k-1})^+)T^{\frac{1}{2}}u,u\rangle$$

$$\leq \langle Iu,u\rangle$$

and $T^{\frac{1}{2}}(\sum_{k=1}^{n}(A_k - A_{k-1})^+)T^{\frac{1}{2}} \leq I$ for each $n \in \mathbb{N}$.

(iii) From (ii) it follows that

$$T^{\frac{1}{2}}(\sum_{k=j}^{n}(A_k - A_{k-1})^+)T^{\frac{1}{2}} \leq T^{\frac{1}{2}}(\sum_{k=1}^{n}(A_k - A_{k-1})^+)T^{\frac{1}{2}} \leq I$$

for each j in $\{1,2,\ldots,n\}$.

Thus $\{T^{\frac{1}{2}}(\sum_{k=j}^{n}(A_k - A_{k-1})^+T^{\frac{1}{2}}\}$ is an increasing sequence (over n) of operators in $\mathcal{U}$ bounded above by I. Let $C_j \in \mathcal{U}$ be the strong operator limit of this sequence.

Since a strong operator convergent sequence is weak operator convergent one can easily show that $0 \le \langle C_j x,x\rangle \le \langle Ix,x\rangle$ for each $x \in H$. Hence $0 \le C_j \le I$.

For each j and all n;

$$\sum_{k=j+1}^{n} (A_k - A_{k-1})^+ \le \sum_{k=j}^{n} (A_k - A_{k-1})^+$$

thus $\{C_j\}$ is a decreasing sequence.

Further for each n and m such that $n < m$

$$T^{\frac{1}{2}}\Big(\sum_{k=1}^{n} (A_k - A_{k-1})^+\Big)T^{\frac{1}{2}} + T^{\frac{1}{2}}\Big(\sum_{k=n+1}^{m} (A_k - A_{k-1})^+\Big)T^{\frac{1}{2}}$$

$$= T\Big(\sum_{k=1}^{n} (A_k - A_{k-1})^+\Big) + T\Big(\sum_{k=n+1}^{m} (A_k - A_{k-1})^+\Big)$$

$$= T\Big(\sum_{k=1}^{m} (A_k - A_{k-1})^+\Big)$$

$= T^{\frac{1}{2}}\Big(\sum_{k=1}^{m} (A_k - A_{k-1})^+\Big)T^{\frac{1}{2}}$ (Note that since $T_m^{\frac{1}{2}}$ and $\sum_{k=1}^{n} (A_k - A_{k-1})^+$ commute, so do $T^{\frac{1}{2}}$ and $\sum_{k=1}^{n} (A_k - A_{k-1})^+$.)

Thus $T^{\frac{1}{2}}\Big(\sum_{k=1}^{n} (A_k - A_{k-1})^+\Big)T^{\frac{1}{2}} + C_{n+1} = C_1$.

From Lemma 2.3 $A_n - A_{n-1} = (A_n - A_{n-1})^+ - (A_n - A_{n-1})^-$ and $A_0 = 0$ is given. Hence $\sum_{k=1}^{n} (A_k - A_{k-1}) = A_n$ and

$$\begin{aligned} T^{\frac{1}{2}}A_nT^{\frac{1}{2}} + C_{n+1} &= T^{\frac{1}{2}}\Big(\sum_{k=1}^{n} (A_k - A_{k-1})\Big)T^{\frac{1}{2}} + C_{n+1} \\ &= T^{\frac{1}{2}}\Big(\sum_{k=1}^{n} [(A_k - A_{k-1})^+ - (A_k - A_{k-1})^-]\Big)T^{\frac{1}{2}} + C_{n+1} \\ &= C_1 + T^{\frac{1}{2}}\Big(\sum_{k=1}^{n} [-(A_k - A_{k-1})^-]\Big)T^{\frac{1}{2}} \end{aligned}$$

(iv) Let $x \in H$ and $n \in \mathbb{N}$ be given, then

$$\langle (T^{\frac{1}{2}}A_nT^{\frac{1}{2}} + C_{n+1} - (T^{\frac{1}{2}}A_{n+1}T^{\frac{1}{2}} + C_{n+2}))x,x\rangle$$

$= \langle -T^{\frac{1}{2}}(-(A_{n+1} - A_n)^-)T^{\frac{1}{2}}x,x\rangle$ (from part (iii))

$$= \langle T^{\frac{1}{2}}(A_{n+1} - A_n)^-T^{\frac{1}{2}}x,x\rangle \ge 0$$

and $\{T^{\frac{1}{2}}A_nT^{\frac{1}{2}} + C_{n+1}\}$ is monotone decreasing.

Moreover $\|T^{\frac{1}{2}}A_nT^{\frac{1}{2}} + C_{n+1}\|$

$$\leq \|T\| \; \|A_n\| + \|C_{n+1}\|$$

$$< 1 + 1 = 2.$$

It is well-known that for a self-adjoint operator S $\|S\| = \sup_{\|x\|=1} |\langle Sx,x\rangle|$, hence $\{T^{\frac{1}{2}}A_nT^{\frac{1}{2}} + C_{n+1}\}$ is bounded below by $-2I$ and via Theorem 2.2 has a strong operator limit B in $\mathcal{U}$. At the same time $T^{\frac{1}{2}}AT^{\frac{1}{2}} + C$ is the weak operator limit of $\{T^{\frac{1}{2}}A_nT^{\frac{1}{2}} + C_{n+1}\}$ where C is the strong operator limit of $\{C_j\}$. Thus $T^{\frac{1}{2}}AT^{\frac{1}{2}} + C = B$ which implies $T^{\frac{1}{2}}AT^{\frac{1}{2}} = B - C$ and hence $T^{\frac{1}{2}}AT^{\frac{1}{2}} \in \mathcal{U}$.

(v) Since $T \in \mathcal{U}$, $R(T) \in \mathcal{U}$ (Theorem 2.2 (ii)). We show that $R(T)x = x$. To show that $R(T)y = y$ is exactly the same. Since $\|(A_k - A_{k-1})^+x\| \leq 2^{1-k}$ it is clear that $\sum_{k=1}^{\infty}(A_k - A_{k-1})^+x$ converges to some vector in H. If u is any vector in H, then

$$\langle T(x + \sum_{k=1}^{\infty}(A_k - A_{k-1})^+x), u\rangle$$

$$= \langle x + \sum_{k=1}^{\infty}(A_k - A_{k-1})^+x, Tu\rangle.$$

This can now closely be approximated by:

$$\langle x + \sum_{k=1}^{n}(A_k - A_{k-1})^+x, \; [I + \sum_{k=1}^{n}(A_k - A_{k-1})^+]^{-1}u\rangle$$

$= \langle x,u\rangle$ for large n with u in H.

Thus $T(x + \sum_{k=1}^{\infty}(A_k - A_{k-1})^+ x) = x$

and $x \in R(T)(H)$.

(vi) Suppose $\mathcal{A}$ is a maximal abelian self-adjoint subalgebra of $\mathcal{U}$. If $\{A_a\}$ is a bounded increasing net in $\mathcal{A}$, its strong operator limit S lies in $\mathcal{U}$ and commutes with $\mathcal{A}$. Hence that limit lies in $\mathcal{A}$, for if not, the algebra generated by $\{\mathcal{A},S\}$ will be an abelian self-adjoint sub- algebra of $\mathcal{U}$ containing $\mathcal{A}$. This is contrary to the fact that $\mathcal{A}$ is a maximal such algebra. We've now shown that $\mathcal{A}$ satisfies the same conditions as $\mathcal{U}$, hence it follows that everything we've proved for $\mathcal{U}$ is applicable to $\mathcal{A}$. Thus for this part of the proof we may now assume that $\mathcal{U}$ is abelian. Thus it follows from (iv) that $AT = T^{\frac{1}{2}}AT^{\frac{1}{2}} \in \mathcal{U}$. Since $Ax = x$ and $x \in R(T)$, $x \in R(AT)$. In addition $AT = TA$, hence AT is self-adjoint. Since $Ay = 0$, $TAy = 0$. Thus $R(TA) \in \mathcal{U}$, $R(TA)x = x$ and $R(TA)y = 0$. From Theorem 2.2 part (v) and (vi) it now follows that $\mathcal{U} = \overline{\mathcal{U}}^{\omega\text{-}o}$.
Hence $\mathcal{A} = \overline{\mathcal{A}}^{\omega\text{-}o}$ and $\mathcal{A}$ is consequently weak operator closed.

Theorem 2.5 ([4], Lemma 4.3)

With the notations and assumptions of Lemma 2.4 it follows that:

(i) MAN lies in $\mathcal{U}$ where M and N are spectral projections for T corresponding to bounded intervals with positive left endpoints.

(ii) M_mAF and FAM_m are in $\mathcal{U}$ where $F = R(T)$ and $\{M_m\}$ is a sequence of spectral projections for T corresponding to bounded intervals with positive endpoints such that $\sum_m M_m = F$.

(iii) $FAFAF \in \mathcal{U}$.

(iv) $FAFAFx = x$; $FAFAFy = 0$.

(v) $\mathcal{U} = \overline{\mathcal{U}}^{\omega\text{-}o}$.

Proof

(i) Let S be a bounded interval with positive left endpoint and let $g(t)$ be t^{-1} for t in S and 0 for t in $\mathbb{R}\backslash S$, so

$$g(t) = \begin{cases} t^{-1} & t \in S \\ 0 & t \in \mathbb{R}\backslash S \end{cases}$$

From Lemma 2.4 (vi) a maximal abelian subalgebra $\mathcal{A}$ of $\mathcal{U}$ containing T is weak operator closed in $\mathcal{B}(H)$ and therefore contains $g(T)$. ($g(T)$ makes sense via Theorem 1.18.) If M is the spectral projection for T corresponding to S it follows from Theorem 1.18 that $M = \chi_S(T)$ since we know that $\chi_S \in \mathcal{B}(\mathbb{R})$. From

$$g(t)(t) = \begin{cases} \frac{1}{t}\cdot t & \text{if } t \in S \\ 0\cdot t & \text{if } t \notin S \end{cases}$$

$$= \begin{cases} 1 & \text{if } t \in S \\ 0 & \text{if } t \notin S \end{cases}$$

it follows that $g(T)(T) = \chi_S(T) = M$.

Since $TAT \in \mathcal{U}$ (Lemma 2.4 (iv)) $MAT \in \mathcal{U}$ because $MAT = g(T)TAT$.

Similarly $MAN \in \mathcal{U}$ where N is another spectral projection for T corresponding to a bounded interval with positive left endpoint.

(ii) It is clear that there exists a sequence of projections $\{M_n\}$ as required: Since T is positive, T is self-adjoint and we can apply Theorem 1.18. Choose $M_1 = \chi_{(1,\infty)}(T)$, $M_n = \chi_{(\frac{1}{n},\frac{1}{n-1}]}(T)$ for $n \geq 2$.

Then clearly $\sum_{n=1}^{\infty} M_n = \chi_{(0,\infty)}(T) = R(T)$. We show the last equality: Since $\lambda\cdot\chi_{(0,\infty)} = \lambda$ if $\lambda \in (0,\infty)$ it follows that $T\cdot\chi_{(0,\infty)}(T) = T$ and this implies that $R(T)$ is lesser or equal to $\chi_{(0,\infty)}(T)$. On the other hand from $TR(T) = T$ it follows that $f(T)R(T) = f(T)$, first for f

a polynomial without constant term and then by tending to the limit for any $f \in \mathcal{B}(\sigma(T))$. Thus $f(0) = 0$ and we have $\chi_{(0,\infty)}(T)R(T) = \chi_{(0,\infty)}(T)$.

Therefore $\chi_{(0,\infty)}(T) \leq R(T)$.

Hence $\chi_{(0,\infty)}(T) = R(T)$.

Consider now $(M_mAM_n + M_n)(M_nAM_m + M_n)$

$$= M_mAM_nM_nAM_m + M_mAM_nM_n + M_nM_nAM_m + M_nM_n$$

$$= M_mAM_nAM_m + M_mAM_n + M_nAM_m + M_n$$

$$(M_nM_n = M_n^2 = M_n).$$

Since $F = \sum_{n=1}^{\infty} M_n \in \mathcal{U}$ we now show that $M_mAFAM_m \in \mathcal{U}$ and also $M_mAFAM_m + M_mAF + FAM_m + F \in \mathcal{U}$. This follows since multiplication is separately continuous in the strong operator topology

$$M_mAFAM_m = M_mA \sum_{n=1}^{\infty} M_nAM_m$$

$$= \sum_{n=1}^{\infty} M_mAM_nAM_m .$$

Let $S_k = \sum_{n=1}^{k} M_mAM_nAM_m$ then $\{S_k\}$ is an increasing sequence of positive operators which is bounded above and from the assumption on $\mathcal{U}$ its strong operator limit M_mAFAM_m is contained in $\mathcal{U}$. The fact that

$M_mAFAM_m + M_mAF + FAM_m + F \in \mathcal{U}$ follows similarly.

Since $F \in \mathcal{U}$ and from what we've just shown it follows that $M_mAF + FAM_m \in \mathcal{U}$.

From $M_m\,[M_mAF + FAM_m] = M_mAF + M_mAM_m$ the latter is also in $\mathcal{U}$.

M_mAM_m is in $\mathcal{U}$ and we can now conclude that $M_mAF \in \mathcal{U}$.

(iii) From (ii) we have $(M_mAF)^*(M_mAF)$

$$= F^*A^*M_m^*M_mAF$$

$$= FAM_mAF \in \mathcal{U}.$$

By using the same argument as in (ii) we now have

$$FAFAF = \sum_{m=1}^{\infty} FAM_m AF \in \mathcal{U}.$$

(iv) From Lemma 2.4 (v) we now have that $Fx = x$ and $Fy = y$ hence since $Ax = x$ and $Ay = 0$, $FAFAFx = x$ and $FAFAFy = 0$.

(v) From the conclusions of Theorem 2.2 and our proof thus far we see that for each cyclic projection E in $\bar{\mathcal{U}}^{\omega-o}$ with generating unit vector x and each unit vector y in $(I - E)(H)$ there is a self-adjoint operator $FAFAF$ in $\mathcal{U}$ such that $FAFAFx = x$ and $FAFAFy = 0$. This fulfills the conditions stated in parts (v) and (vi) of Theorem 2.2, thus $\mathcal{U} = \bar{\mathcal{U}}^{\omega-o}$.

We can now prove our characterization of a von Neumann algebra in terms of nets and normal states.

Theorem 2.6 ([4], Theorem 4.4)

A C^*-algebra $\mathcal{U}$ is $*$-isomorphic to a von Neumann algebra if and only if it is a W^*-algebra.

Proof

Suppose $\mathcal{U}$ is $*$-isomorphic to a von Neumann algebra $\mathcal{A}$. Thus there exists a $*$-isomorphism $\varphi : \mathcal{U} \longrightarrow \mathcal{A}$ where $\mathcal{A}$ is a weak operator closed self-adjoint subalgebra of some $\mathcal{B}(H)$.
If $a \leq b$ in $\mathcal{U}$, then $b - a \geq 0$ and there exists a c in $\mathcal{U}$ such that $b - a = c^*c$.

$$\begin{aligned}\varphi(b) - \varphi(a) = \varphi(b - a) &= \varphi(c^*c), \quad \varphi(b) \text{ and } \varphi(a) \in \mathcal{A}\\ &= \varphi(c^*)\varphi(c)\\ &= (\varphi(c))^*\varphi(c).\end{aligned}$$

But $(\varphi(c))^*\varphi(c) \geq 0$, thus $\varphi(b) - \varphi(a) \geq 0$
and $\varphi(b) \geq \varphi(a)$.

Thus increasing nets in $\mathcal{U}$ are mapped onto increasing nets in $\mathcal{A}$. If $\{b_a\}$ is bounded above and $b = \sup b_a$ then $\varphi(b) \geq \varphi(b_a)$ for all a and if $\varphi(b_a) \leq \varphi(c)$ for some $c \in \mathcal{U}$ then $\varphi(b) \leq \varphi(c)$.

Thus $\varphi(b)$ is a least upper bound of $\{\varphi(b_a)\}$. The *-isomorphism transforms increasing bounded nets onto such nets, also least upper bounds onto least upper bounds and normal states onto normal states. Thus $\mathcal{U}$ is a W^*-algebra in this case.

Conversely, suppose $\mathcal{U}$ is a W^*-algebra and $\varphi = \sum_{a \in A} \oplus \pi_{\eta(a)}$ where $\{\eta(a) : a \in A\}$ is the family of normal states of $\mathcal{U}$. Since $\{\eta(a)\}$ is separating for $\mathcal{U}$, φ is a *-isomorphism from $\mathcal{U}$ onto a subalgebra of some $\mathcal{B}(H)$ (cf the Gelfand-Naimark-Segal construction in Chapter 3 part I). Suppose $\{\varphi(A_b)\}$ with $b \in B$ is a bounded increasing net in $\varphi(\mathcal{U})$ with strong operator limit B. Then $\{A_b\}$ is a bounded increasing net in $\mathcal{U}$. Since $\mathcal{U}$ is a W^*-algebra, $\{A_b\}$ has a least upper bound A in $\mathcal{U}$ and $\{\eta(a)\ (A_b)\}$ tends to $\eta(a)(A)$ for each a in A (recall that each $\eta(a)$ is normal).

If we write x_a for $x_{\eta}(a)$ (cf Gelfand-Naimark-Segal) $\{\langle\varphi(A_b)x_a,x_a\rangle\}$ tends to $\langle\varphi(A)x_a,x_a\rangle$, but $\{\langle\varphi(A_b)x_a,x_a\rangle\}$ tends to $\langle Bx_a,x_a\rangle$ as well. Thus $\langle(\varphi(A) - B)x_a,x_a\rangle = 0$ for each a in A.

With T invertible in $\mathcal{U}$, $\{T^*A_bT\}$ has T^*AT as least upper bound. We show this: $\{T^*A_bT\}$ is an increasing net, for if $b_1 \leq b_2$

$$\begin{aligned} \langle T^*A_{b_1}Tx,x\rangle &= \langle A_{b_1}Tx,Tx\rangle \\ &\leq \langle A_{b_2}Tx,Tx\rangle \\ &= \langle T^*A_{b_2}Tx,x\rangle \end{aligned}$$

$$\text{thus } T^*A_{b_1}T \leq T^*A_{b_2}T.$$

We also know that $T^*A_bT \leq T^*AT$ for all b. We show that T^*AT is the least upper bound for T^*A_bT. If $S \in \mathcal{U}$ such that $T^*A_bT \leq S$ for all b then $(T^{-1})^*T^*A_bTT^{-1} \leq (T^{-1})^*ST^{-1}$.

Hence $A_b \leq (T^{-1})^*ST^{-1}$ for all b so $A \leq (T^{-1})^*ST^{-1}$ which implies $T^*AT \leq S$. Since φ is a *-isomorphism $\varphi(T^*A_bT) = \varphi(T)^*\varphi(A_b)\varphi(T)$ and has $\varphi(T^*)B\varphi(T)$ as strong operator limit. Thus $\langle(\varphi(A) - B)\varphi(T)x_a,\ \varphi(T)x_a\rangle = 0$ for each a in A and each invertible T in $\mathcal{U}$.

With S in $\mathcal{U}$ arbitrary; $S + nI$ is an invertible element of $\mathcal{U}$ for all large n, since one has that $\sigma(S + nI) \subseteq \sigma(S) + \{n\}$. Hence if we choose n large enough $0 \notin \sigma(S + nI)$ and consequently $S + nI$ is invertible.

Thus $<(\varphi(A) - B)\varphi(S + nI)x_a, \varphi(S + nI)x_a> = 0$ .... (I) for all large positive integers n.

But $<(\varphi(A) - B)\varphi(S + nI)x_a, \varphi(S + nI)x_a>$

$$= <(\varphi(A) - B)(\varphi(S) + n\varphi(I))x_a, (\varphi(S) + n\varphi(I))x_a>$$

$$= <(\varphi(A) - B)\varphi(S)x_a, \varphi(S)x_a> + <(\varphi(A) - B)\varphi(S)x_a, n\varphi(I)x_a>$$
$$+ <(\varphi(A) - B)n\varphi(I)x_a, \varphi(S)x_a)> + <\varphi(A) - b)n\varphi(I)x_a, n\varphi(I)x_a>$$

$$= <(\varphi(A) - B)\varphi(S)x_a, \varphi(S)x_a> + <(\varphi(A) - B)\varphi(S)x_a, nx_a>$$
$$+ <(\varphi(A) - B)nx_a, \varphi(S)x_a> + n^2 <(\varphi(A) - B)x_a,x_a>.$$

$$= <(\varphi(A) - B)\varphi(S)x_a, \varphi(S)x_a> + 2 \operatorname{Re} <(\varphi(A) - B)\varphi(S)x_a,nx_a>$$
$$+ n^2 <(\varphi(A) - B)x_a,x_a>, \text{ (since } z + \bar{z} = 2 \operatorname{Re} z).$$

And from (I) this equals zero when n is large.

But $<(\varphi(A) - B)x_a,x_a> = 0$ and $<(\varphi(A) - B)\varphi(S)x_a, \varphi(S)x_a>$ is independent of n, thus

$$<(\varphi(A) - B)\varphi(S)x_a, \varphi(S)x_a> = 0 \text{ for each } S \text{ in } \mathcal{U}.$$

Since x_a is a cyclic vector for the representation π_a and $\varphi = \underset{a\in A}{\oplus} \pi_a$ one has that $[\pi_a(S)x_a : S \in \mathcal{U}] = H$.

Now $<(\varphi(A) - B) x,y> = 0$ for any x and $y \in H = \underset{a\in A}{\oplus} H_a$.

Hence it follows that $\varphi(A) - B = 0$.

Thus $\varphi(A) = B$. Hence $\varphi(\mathcal{U})$ satisfies the conditions in Theorem 2.5 and $\varphi(\mathcal{U}) = \overline{\varphi(\mathcal{U})}^{w-o}$. Thus $\mathcal{U}$ is *-isomorphic to a von Neumann algebra.

We now prove the following strengthened version of Theorem 2.6 when $\mathcal{U}$ satisfies a certain "countability" assumption. This assumption is always fulfilled if H is separable.

Lemma 2.7 ([4], Lemma 4.5)

Let $\mathcal{U}$ be a C^*-algebra acting on a Hilbert space. Suppose that each bounded increasing sequence in $\mathcal{U}$ has its strong operator limit in $\mathcal{U}$ and that each orthogonal family of non-zero projections in $\mathcal{U}$ is countable. Then $\mathcal{U} = \bar{\mathcal{U}}^{\omega-0}$.

Proof

By studying the proofs of Lemmas 2.2, 2.4 and 2.5 we note that the only use of nets as opposed to sequences is to show that arbitrary unions of projections in $\mathcal{U}$ lie in $\mathcal{U}$. Thus we only have to show that the union of an increasing net of projections in $\mathcal{U}$ is in $\mathcal{U}$ under the present assumptions. We show that the union F of an arbitrary family $\{F_a : a \in A\}$ of projections in $\mathcal{U}$ lies in $\mathcal{U}$. Let $\{E_b : b \in B\}$ be a maximal orthogonal family of non-zero projections in $\mathcal{U}$ such that $E_b \leq F$ for each b. By assumption B is countable (possibly finite), so that we can denote the family $\{E_b\}$ by $\{E_1, E_2, \ldots\}$. Since $\{E_1, E_1+E_2, \ldots\}$ is an increasing sequence of projections in $\mathcal{U}$, its strong operator limit $\sum_n E_n$ is in $\mathcal{U}$. Let $E = \sum_n E_n$, we must now prove that $E = F$. Since $E \leq F$, $\sup\{E,F_a\} \leq F$ for each a in A. The range projection of $\frac{1}{2}(E + F_a)$ is $\frac{1}{2}\sup\{E,F_a\}$ (cf Lemma 1.4) and is the strong operator limit of the increasing sequence $\{[(E + F_a)/2]^{1/n}\}$ (cf Lemma 1.6). Thus $\sup\{E,F_a\}$ is in $\mathcal{U}$ as is $\sup\{E,F_a\} - E$. If $\sup\{E,F_a\} - E \neq 0$ it can be added to $\{E_1, E_2, \ldots\}$ to form a larger orthogonal family of non-zero projections in F. This contradicts the maximality of $\{E_1, E_2, \ldots\}$. Thus $\sup\{E,F_a\} - E = 0$, and hence $F_a \leq E$ for each a in A. Since $E_b \leq F$ for each b, we have $F = E \in \mathcal{U}$.

We can now characterize countable decomposable von Neumann algebras. Before we state the characterization we must first define countable decomposable. A projection E in a von Neumann algebra $\mathcal{A}$ is said to be countable decomposable relative to $\mathcal{A}$ when each orthogonal family of non-zero subprojections of E in $\mathcal{A}$, is countable. If I is countable decomposable relative to $\mathcal{A}$ we say that $\mathcal{A}$ is countable decomposable.

Theorem 2.8 ([4], Lemma 4.6)

A C^*-algebra $\mathcal{U}$ is $*$-isomorphic to a countable decomposable von Neumann algebra $\mathcal{A}$ if and only if each bounded increasing sequence in $\mathcal{U}$ has a least upper bound in $\mathcal{U}$, there is a separating family of normal states of $\mathcal{U}$ whose limits on such a sequence are their values at the least upper bound, and each orthogonal family of non-zero projections in $\mathcal{U}$ is countable.

Proof

Suppose $\mathcal{U}$ is $*$-isomorphic to a countable decomposable von Neumann algebra $\mathcal{A}$ acting on a Hilbert space H. Then as in the proof of Theorem 2.6 bounded increasing sequences in $\mathcal{U}$ map onto such sequences in $\mathcal{A}$ under the isomorphism and the least upper bound of the image sequence in $\mathcal{A}$ is the image of an element of $\mathcal{U}$ that is the least upper bound of the sequence in $\mathcal{U}$. Hence vector states of $\mathcal{A}$ composed with the isomorphism are normal states of $\mathcal{U}$ and the set of such forms a separating family for $\mathcal{U}$. This is true since if $\varphi(T) = \langle Tx,x\rangle$ and $\|x\| = 1$ (φ is called a vector state) then clearly φ is a normal state on $\mathcal{A}$ ($\|\varphi\| = 1$ and $\varphi(T) \geq 0$ if $T \geq 0$). Now if ψ is the $*$-isomorphism from $\mathcal{U}$ to $\mathcal{A}$, then clearly

$\gamma(a) := \varphi(\psi(a))$

$= \varphi\cdot\psi(a)$ for each a in $\mathcal{U}$ defines a normal state on $\mathcal{U}$.

We show that $\|\gamma\| = 1$.

$$\begin{aligned} |\gamma(a)| &= |\varphi(\psi(a))| \\ &\leq \|\varphi\| \, \|\psi\| \, \|a\| \\ &= \|a\| \quad \text{hence} \quad \|\gamma\| \leq 1. \end{aligned}$$

Also $\|\gamma\| \geq \frac{\|\varphi(\psi(I))\|}{\|I\|} = \frac{|\langle x,x\rangle|}{\|I\|}$

$$= \frac{\|x\|^2}{1} = 1.$$

If E_n and E_m are orthogonal in $\mathcal{U}$ the isomorphism φ will map them onto orthogonal elements in $\mathcal{A}$ since

$$\varphi(E_n)\varphi(E_m) = \varphi(E_nE_m) = \varphi(0) = 0.$$

Because $\varphi(E_n)^2 = \varphi(E_n^2) = \varphi(E_n)$, projections will also be mapped onto projections. Thus an orthogonal family of non-zero projections in $\mathcal{U}$ maps onto such a family in $\mathcal{A}$. Since $\mathcal{A}$ is countable decomposable, the family of projections is countable.

To prove the converse, we use the same argument as in Theorem 2.6 and apply it to a C^*-algebra in $\mathcal{U}$ satisfying the given conditions, but now we use sequences in stead of nets and the previous lemma. Thus $\mathcal{U}$ is *-isomorphic to a von Neumann algebra $\mathcal{A}$. From our assumption we know that orthogonal families of non-zero projections in $\mathcal{U}$ are countable, thus $\mathcal{A}$ is countable decomposable.

We've now shown that countable decomposable von Neumann algebras can also be characterized in terms of bounded increasing sequences and separating families of normal states. In the next part of this chapter we're going to take a look at an example that shows that a C^*-algebra satisfying the order property, but not the separating condition stated in Definition 2.1, is not a von Neumann algebra. We will construct a commutative C^*-algebra not isomorphic to any von Neumann algebra.

II The commutative case: A counter example

A lattice is a Banach space E endowed with a partial order which

(i) is compatible with the algebraic operations in the following way: if $x \leq y$ then $z + x \leq z + y$ $(x,y,z \in E)$ and if $\alpha \in \mathbb{R}^+$ then $\alpha x \leq \alpha y$

(ii) includes the sup (x,y) and the inf (x,y) if $x,y \in E$.

Definition 2.9

A boundedly complete lattice is a lattice in which each non-empty family of elements that has an upper bound, has a least upper bound.

Our first theorem concerns the algebra of all complex valued continuous functions on a compact Hausdorff space X, namely C(X). It is clear that under the following order relation C(X) is a lattice:

$$f \le g \text{ iff } f(x) \le g(x) \text{ for every } x \in X.$$

Theorem 2.10 ([4], Theorem 3.1)

If C(X) is a boundedly complete lattice, then each open set in X has an open closure.

Proof

Let $\mathcal{O}$ be an open subset of X and $\bar{\mathcal{O}}$ its closure. Also let $\mathcal{F}$ be the family of functions f in C(X) such that $0 \le f \le 1$ (since X is completely regular, $\mathcal{F}$ is non-empty) and $f(p') = 0$ if $p' \notin \mathcal{O}$.

Since C(X) is a boundedly complete lattice and 1 is an upper bound for $\mathcal{F}$, $\mathcal{F}$ has a least upper bound say f_0 with $f_0 \le 1$.

If $p \in \mathcal{O}$, there exists an $f \in \mathcal{F}$ such that $f(p) = 1$, so that $f_0(p) = 1$ for each p in $\mathcal{O}$ $(f \le f_0 \le 1)$ and since f_0 is continuous, $f_0(p) = 1$ also for each p in $\bar{\mathcal{O}}$. If $p' \notin \bar{\mathcal{O}}$, then there exists a $g \in C(X)$ such that $0 \le g \le 1$ and $g(p') = 0$ and $g(q) = 1$ with $q \in \bar{\mathcal{O}}$ (X is compact Hausdorff and thus normal). Hence g is an upper bound for $\mathcal{F}$ which implies that $f_0 \le g$.

Thus f_0 is 1 on $\bar{\mathcal{O}}$ and zero on $X\backslash\bar{\mathcal{O}}$. Since f_0 is continuous and $f^{-1}(\frac{1}{2},\frac{3}{2}) = \bar{\mathcal{O}}$, $\bar{\mathcal{O}}$ must be open.

We now define a space X with the property that each open set has an open closure as <u>extremely disconnected</u>. We also define a subset which is both open and closed as <u>clopen</u>. The converse of Theorem 2.10 is also true.

Theorem 2.11 ([4], Theorem 3.2)

If X is an extremely disconnected compact Hausdorff space, then $C(X)$ is a boundedly complete lattice.

Proof

Let $\{f_a : a \in A\}$ be a family of real valued functions in $C(X)$ which is bounded from above by a constant. We must now show that $\{f_a\}$ has a least upper bound, then $C(X)$ will be a boundedly complete lattice. We construct this least upper bound in five steps.

STEP 1

We firstly suppose that each f_a is the characteristic function of some clopen subset X_a of X. We will now show that $\overline{\bigcup_{a\in A} X_a}$ is a clopen set with characteristic function $\sup\{f_a : a \in A\}$ which is the least upper bound of $\{f_a\}$ in $C(X)$. We will also show that the interior of $\bigcap_{a\in A} X_a$ is also a clopen set with its characteristic function $\inf\{f_a : a \in A\}$, the greatest lower bound of $\{f_a\}$ in $C(X)$.

Since X_a is clopen, $\bigcup_{a\in A} X_a$ will be open. If $Y_0 = \overline{\bigcup_{a\in A} X_a}$, Y_0 will be closed. If g is an upper bound for $\{f_a\}$, then $1 \leq g(p)$ for every $p \in \bigcup_{a\in A} X_a$. If a net (p_λ) in $\bigcup_{a\in A} X_a$ converges to some p it follows from the continuity of g that $g(p_\lambda) \to g(p)$.

Thus $1 \leq g(p)$ for every $p \in Y_0$. Hence $\sup\{f_a : a \in A\}$ is the least upper bound of $\{f_a\}$.

Clearly $1 - f_a$ is the characteristic function on $X\backslash X_a$. Let $f_0 = \inf\{f_a : a \in A\}$, from the first part we now have that $1 - f_0$ which is the least upper bound of $\{1 - f_a\}$ is the characteristic function of $\overline{\bigcup_{a\in A}(X\backslash X_a)}$. Thus f_0 is the greatest lower bound of $\{f_a\}$ and f_0 is also the characteristic function of $X \setminus \overline{\bigcup_{a\in A}(X\backslash X_a)}$.

Since $X \setminus \overline{\bigcup_{a\in A}(X\backslash X_a)} = X \setminus \overline{(X\backslash \bigcap_{a\in A} X_a)}$ (De Morgan)

$$= \operatorname{int}(\bigcap_{a\in A} X_a)$$

f_0 is the characteristic function of the $\operatorname{int}(\bigcap_{a\in A} X_a)$.

STEP 2

We show next that $X_\lambda = X \setminus \overline{(\bigcup_{a\in A}\{x \in X : f_a(x) > \lambda\})}$ is a clopen subset of X and also that if Y is a clopen subset of X, with the property that $f_a(p) \leq \lambda$ for all $a \in A$ and $p \in Y$, then $Y \subseteq X_\lambda$.

Since X_λ is the complement in X of the closure of the union of open subsets of X, X_λ will be open. But since X is an extremely disconnected space, the closure of open subsets are also open, thus X_λ will also be closed. Thus X_λ is a clopen set.

If $p \in X_\lambda$, then for every a in A

$p \notin \{x \in X : f_a(x) > \lambda\}$, thus $f_a(p) \leq \lambda$.

We have assumed that Y is a subset with $f_a(p) \leq \lambda$ for all a in A, thus $Y \subseteq X\backslash f_a^{-1}(\lambda,\infty)$ so that $f_a^{-1}(\lambda,\infty) \subseteq X\backslash Y$ for all $a \in A$.

As Y is open, $X\backslash Y$ is closed.

Thus $\overline{(\bigcup_{a\in A} f_a^{-1}(\lambda,\infty))} \subseteq X\backslash Y$ and thus $Y \subseteq X_\lambda$.

STEP 3

Let e_λ be the characteristic function of X_λ, thus $e_\lambda(x) = 1$ if $x \in X_\lambda$ and $e_\lambda(x) = 0$ if $x \in X \backslash X_\lambda$. We know that $\{f_a\}$ is bounded from above, let k be the constant that bounds $\{f_a\}$ from above and such that $-k \leq f_{a'}$ for some $a' \in A$. We now prove that

(i) $e_\lambda = 0$ for $\lambda < -k$ and $e_\lambda = 1$ for $\lambda > k$;

(ii) $e_\lambda \leq e_{\lambda'}$ if $\lambda \leq \lambda'$ and

(iii) $e_\lambda = \inf\{e_{\lambda'} : \lambda' > \lambda\}$.

(Hence $\{e_\lambda\}$ is the spectral resolution for some f which we will later show is contained in $C(X)$.)

(i) If $\lambda < -k$ and $p \in X_\lambda$

then $p \notin \{x \in X : f_a'(x) > \lambda\}$.

But $-k \leq f_a'$, thus from $\lambda < -k$ it follows that

$\{x \in X : f_a'(x) > \lambda\} = X$.

Thus $X_\lambda = X \backslash X = \emptyset$ and $e_\lambda = 0$.

If $\lambda \geq k$ and $p \in X$

then $f_a(p) \leq \lambda$ for all $a \in A$ because $\{f_a\}$ is bounded from above by k.

Thus X is a clopen set where all f_a's take values not greater than λ. From step 2 we know that $X = X_\lambda$.

Hence $e_\lambda = 1$.

(ii) If $\lambda \leq \lambda'$

then $\{x \in X : f_a(x) > \lambda'\} \subseteq \{x \in X : f_a(x) > \lambda\}$.

Thus $X_\lambda' \supseteq X_\lambda$

and the result follows.

(iii) Since $e_\lambda \le e_{\lambda'}$ if $\lambda \le \lambda'$ (from (ii))

$e_\lambda \le \inf\{e_{\lambda'} : \lambda' > \lambda\}$.

If Y_λ is a set with characteristic function $\inf\{e_{\lambda'} : \lambda' > \lambda\}$ then $X_\lambda \subseteq Y_\lambda$.

Since $\inf\{e_{\lambda'} : \lambda' > \lambda\}$ is the characteristic function for Y_λ, $Y_\lambda \subseteq X_{\lambda'}$, for each $\lambda' > \lambda$.

Thus if $p \in Y_\lambda$, $f_a(p) \le \lambda'$ for each a in **A** and each $\lambda' > \lambda$.

Hence $f_a(p) \le \lambda$ for all a in **A** and Y_λ is a clopen set on which all f_a take values not exceeding λ (cf step 1).

From step 2 we have $Y_\lambda \subseteq X_\lambda$.

Thus $Y_\lambda = X_\lambda$

and $e_\lambda = \inf\{e_{\lambda'} : \lambda' > \lambda\}$.

STEP 4

We now show that $\int_{-k}^{k} \lambda de_\lambda$ converges in norm to a function f in C(X) and that X_λ is the largest clopen set on which all f_a take values not exceeding λ.

Let $\mathcal{P} = \{\lambda_0, \ldots, \lambda_n\}$ and $\mathcal{L} = \{\mu_0, \ldots, \mu_m\}$ be two partitions of $[-k,k]$ with $|\mathcal{P}|$ and $|\mathcal{L}|$ the lengths of the largest subintervals, $\lambda_j' \in [\lambda_{j-1}, \lambda_j)$ and $\mu_j' \in [\mu_{j-1}, \mu_j)$. If $\{\gamma_0, \ldots, \gamma_r\}$ is their common refinement and $\gamma_j' \in [\gamma_{j-1}, \gamma_j)$ then

$$|\sum_{j=1}^{n} \lambda_j' (e_{\lambda_j} - e_{\lambda_{j-1}}) - \sum_{k=1}^{r} \gamma_k' (e_{\gamma_k} - e_{\gamma_{k-1}})| \le |P| \sum_{k=1}^{r} (e_{\gamma_k} - e_{\gamma_{k-1}})$$

$$= |\mathcal{P}| (e_{\gamma_r} - e_{\gamma_0}) \le |\mathcal{P}|.$$

Hence $\|\sum_{j=1}^{n} \lambda_j' (e_{\lambda_j} - e_{\lambda_{j-1}}) - \sum_{k=1}^{r} \gamma_k' (e_{\gamma_k} - e_{\gamma_{k-1}})\| \le |\mathcal{P}|$.

Similarly $\|\sum_{j=1}^{m} \mu_j' (e_{\mu_j} - e_{\mu_{j-1}}) - \sum_{k=1}^{r} \gamma_k' (e_{\gamma_k} - e_{\gamma_{k-1}})\| \le |\mathcal{L}|$.

The triangle inequality gives the following:

$$\|\sum_{j=1}^{n} \lambda_j'(e_{\lambda_j} - e_{\lambda_{j-1}}) - \sum_{j=1}^{m} \mu_j'(e_{\mu_j} - e_{\mu_{j-1}})\| \leq |\mathcal{P}| + |\mathcal{L}|.$$

Thus the family of approximating Riemann sums to $\int_{-k}^{k} \lambda de_\lambda$, indexed by their corresponding partitions of $[-k,k]$ and the set of these partitions partially ordered by refinement, forms a Cauchy net in the norm topology on $C(X)$. Since $C(X)$ is complete in the norm topology it follows that the Cauchy net converges in the norm to a real valued function f in $C(X)$. Since each approximating Riemann sum has range in $[-k,k]$, f also has range in $[-k,k]$.

$$\text{Thus } f = \int_{-a}^{a} \lambda de_\lambda \text{ when } k \leq a.$$

Suppose now that $k \leq a$ and $\lambda \in [-a,a]$. If $\{\lambda_0,\dots,\lambda_n\}$ is a partition of $[-a,a]$ with λ as some λ_k such that

$$g = \sum_{j=1}^{n} \lambda_j'(e_{\lambda_j} - e_{\lambda_{j-1}})$$

is close to f in norm, then $\|fe_\lambda - ge_\lambda\|$ is small and

$$\begin{aligned} ge_\lambda &= \sum_{j=1}^{k} \lambda_j'(e_{\lambda_j} - e_{\lambda_{j-1}}) \quad (e_{\lambda_k} \leq e_{\lambda_j} \text{ for } j \geq k) \\ &\leq \sum_{j=1}^{k} \lambda_k(e_{\lambda_j} - e_{\lambda_{j-1}}) \\ &= \lambda_k \sum_{j=1}^{k} (e_{\lambda_j} - e_{\lambda_{j-1}}) \\ &= \lambda_k (e_{\lambda_k} - e_{\lambda_0}) \\ &= \lambda (e_\lambda - e_{-a}) \end{aligned}$$

But $e_\lambda = 0$ if $\lambda < -k$ and $-a < -k$ hence $e_{-a} = 0$.

Thus $ge_\lambda \leq \lambda e_\lambda$ and now $fe_\lambda \leq \lambda e_\lambda$.

$\|f(1 - e_\lambda) - g(1 - e_\lambda)\|$ is also small and subsequently

$$\begin{aligned} g(1 - e_\lambda) &= g - ge_\lambda \\ &= \sum_{j=k+1}^{n} \lambda_j'(e_{\lambda_j} - e_{\lambda_{j-1}}) \end{aligned}$$

$$\geq \sum_{j=k+1}^{n} \lambda_k(e_{\lambda_j} - e_{\lambda_{j-1}})$$
$$= \lambda(e_{\lambda_n} - e_{\lambda_k})$$
$$= \lambda(e_a - e_\lambda)$$

But $e_\lambda = 1$ if $\lambda > k$ and $a > k$ thus $e_a = 1$.
Hence $g(1 - e_\lambda) \geq \lambda(1 - e_\lambda)$ and thus $f(1 - e_\lambda) \geq \lambda(1 - e_\lambda)$.

Let $Y_\lambda = X \setminus \bar{Z}_\lambda$ with $Z_\lambda = f^{-1}((\lambda,\infty))$. Since $f^{-1}((\lambda,\infty))$ is the inverse image of an open set and f is continuous, $f^{-1}((\lambda,\infty))$ is open. Since X is extremely disconnected $\bar{Z}_\lambda$ will be a clopen set. Thus $X \setminus \bar{Z}_\lambda$ is a clopen subset of X on which f takes values not exceeding λ.

If Y is another clopen subset of X on which f takes values not exceeding λ then

$Y \subseteq X\setminus f^{-1}((\lambda,\infty))$ from STEP 2.
Thus $f^{-1}((\lambda,\infty)) \subseteq X\setminus Y$.
Since Y is open, $X\setminus Y$ will be closed.
Thus $\bar{Z}_\lambda \subseteq X\setminus Y$ and $Y \subseteq Y_\lambda$.

We've now proved that Y_λ is the largest clopen set in X on which f takes values not exceeding λ.

We've shown that $fe_\lambda \leq \lambda e_\lambda$, hence f takes values not exceeding λ on X_λ and thus $X_\lambda \subseteq Y_\lambda$.

As $e_\lambda = \inf \{e_{\lambda'} : \lambda' > \lambda\}$, X_λ is the largest clopen set in X contained in $\bigcap_{\lambda'>\lambda} X'_\lambda$. Now $\lambda' \leq f(p)$ if $p \in X\setminus X'_\lambda$ since $\lambda'(1 - e_\lambda) \leq f(1 - e_\lambda)$ so that $X\setminus X'_\lambda \subseteq \overline{f^{-1}((\lambda,\infty))}$ if $\lambda' > \lambda$.
Thus $X\setminus\bar{Z}_\lambda \subseteq X'_\lambda$ so $Y_\lambda \subseteq X'_\lambda$ when $\lambda' > \lambda$ and Y_λ is a clopen set contained in $\bigcap_{\lambda'>\lambda} X'_\lambda$ but X_λ is the largest clopen set in X contained in $\bigcap_{\lambda'>\lambda} X_\lambda$, thus $Y_\lambda \subseteq X_\lambda$.

Hence $Y_\lambda = X_\lambda$ and consequently X_λ is the largest clopen set on which f takes values not exceeding λ.

STEP 5

We now only have to show that f is the least upper bound of $\{f_a\}$. If $f(p) < f_a(p)$ for some $p \in X$ and $a \in \mathbf{A}$, choose λ,λ' such that $f(p) < \lambda < \lambda' < f_a(p)$. Let $Y = \overline{f^{-1}(-\infty,\lambda)} \cap \overline{f_a^{-1}(\lambda',\infty)}$.
Since f and f_a are continuous and X is extremely disconnected, Y is a clopen set containing p such that $f(q) \leq \lambda$ and $f_a(q) > \lambda$ for each q in Y. Thus $p \in Y \subsetneq X_\lambda$ from STEP 4.
But $p \in \{x \in X : f_a(x) > \lambda\}$ thus $p \notin X_\lambda$.
Thus a contradiction and $f_a \leq f$ for each a in $\mathbf{A}$, so f is an upper bound for $\{f_a\}$.

We must now finally show that f is a least upper bound. If g is another upper bound for $\{f_a\}$ and $g(p) < f(p)$ for some p in X, then again there exists a λ and a clopen set Y containing p such that $g(q) \leq \lambda < f(q)$ for each q in Y. Since $f_a \leq g$ for all a in $\mathbf{A}$, Y is a clopen set on which all f_a take values not exceeding λ. From STEP 2 we know that $p \in Y \subsetneq X_\lambda$ and from STEP 4 we now have $f(p) \leq \lambda$ since on X_λ f doesn't take values exceeding λ. This contradicts the fact that $\lambda < f(q)$ for each q (also p) in Y. Thus $f \leq g$ and f is the least upper bound of $\{f_a\}$ in $C(X)$.
It now follows that $C(X)$ is a boundedly complete lattice.

In the following lemma a few conditions on a compact Hausdorff space that are equivalent to the extremely disconnectedness of the space will be given. In the second part we'll see that a totally disconnected Hausdorff space X is extremely disconnected if the family of clopen subsets of X is a complete lattice. (Recall that X is totally disconnected if each pair of points can be separated by clopen sets.)

Lemma 2.12 ([4], Lemma 3.3)

Let X be a compact Hausdorff space.

(i) X is extremely disconnected if and only if each pair of disjoint open sets have disjoint closures.

(ii) X is extremely disconnected if and only if it satisfies the following two conditions:

(a) X is totally disconnected
(b) the family $\mathcal{C}$ of clopen subsets of X partially ordered by inclusion is a complete lattice.

Proof

(i) Suppose O_1 and O_2 are disjoint open subsets of X and X is extremely disconnected. Since O_2 is open, $X \backslash O_2$ is closed. Thus $\overline{O_1} \subseteq X \backslash O_2$ and hence $O_2 \subseteq X \backslash \overline{O_1}$.
We know that X is extremely disconnected, so that $\overline{O_1}$ is open and $X \backslash \overline{O_1}$ is closed. Hence $\overline{O_2} \subseteq X \backslash \overline{O_1}$ which means $\overline{O_1} \cap \overline{O_2} = \phi$.
Thus if X is extremely disconnected, each pair of disjoint open sets have disjoint closures. To prove the opposite of (i) we now suppose that disjoint open subsets of X have disjoint closures. Let O be an open subset of X. Then O and $X \backslash \bar{O}$ are disjoint open subsets of X. By our assumption $\bar{O}$ and the closure say F of $X \backslash \bar{O}$ is disjoint, but $X = F \cup \bar{O}$.
Hence $\bar{O}$ is the complement of F in X ($\bar{O} = X \backslash F$) and $\bar{O}$ is open. Thus each open subset has an open closure and X is thus extremely disconnected.

(ii) Assume that X is totally disconnected and $\mathcal{C}$ is a complete lattice. Since X is a compact Hausdorff space in which points can be separated by clopen subsets of X, a standard compactness argument shows that a point can be separated from a closed (compact) subset of X by clopen sets:

If $x \notin Y$ ($Y \subset X$ is closed) there exists for each $y \in Y$ disjoint clopen neighbourhoods V_y and U_y of x and y. Now $\{U_y : y \in Y\}$ is an open covering for Y and since Y is compact there exist $y_1, \ldots, y_n \in Y$ such that $\{U_{y_i} \mid y_i \in Y\}_{i=1}^{n}$ covers Y. Let $V = \bigcap_{i=1}^{n} V_{y_i}$ and $U = \bigcup_{i=1}^{n} U_{y_i}$, then clearly V and U are disjoint and clopen subsets containing x and Y respectively.

This in particular implies that each open set in X is the union of clopen sets. Let O_1 and O_2 be disjoint open subsets of X and let $\mathcal{C}_j$ be $\{X_0 \in \mathcal{C} \mid X_0 \subseteq O_j\}$ for j in $\{1,2\}$. $\mathcal{C}_1$ is a complete lattice by assumption and has a least upper bound, say X_1 in $\mathcal{C}$.

If $X_0 \in \mathcal{C}_2$ then $X \backslash X_0$ is a clopen subset containing O_1.

Hence $X \backslash X_0$ contains each element of $\mathcal{C}_1$.

Thus $X_1 \subseteq X \backslash X_0$ and $X_0 \subseteq X \backslash X_1$.

Since $O_2 = \cup\, \mathcal{C}_2$ $(\mathcal{C}_2 = \{X_0 \in \mathcal{C} : X_0 \subseteq O_2\})$, $O_2 \subseteq X \backslash X_1$.

But $X \backslash X_1$ is clopen so that $\overline{O_2} \subseteq X \backslash X_1$.

As X_1 is the least upper bound of $\mathcal{C}_1 = \{X_0 \in \mathcal{C} : X_0 \subseteq O_1\}$ it follows that $O_1 \subseteq X_1$ and X_1 is clopen thus $\overline{O_1} \subseteq X_1$.

Thus $\overline{O_1} \cap \overline{O_2} = \phi$. From (i) it now follows that X is extremely disconnected.

To prove the converse of (ii) we now assume that X is extremely disconnected. From Theorem 2.10 we know that $C(X)$ is a boundedly complete lattice. If we consider the characteristic functions associated with the clopen subsets of X it can easily be shown that $\mathcal{C}$ is also a complete lattice. Since X is extremely disconnected, X is also totally disconnected.

We will now construct a commutative C^*-algebra that is not isomorphic to a von Neumann algebra, although it satisfies the first condition of the definition of a W^*-algebra (cf Definition 2.1).

Definition 2.13

(i) In a topological space X, a subset is said to be meager when it is a subset of a countable union of subsets of X each of which is nowhere dense ($M \subset X$ is nowhere dense if its closure $\bar{M}$ has no interior points) in X.

(ii) An open subset of X is said to be regular when it coincides with the interior of its closure.

It is clear from our definition that a countable union of meager sets is meager and a subset of a meager set is meager. $(0,1)$ is an example of a regular set in $\mathbb{R}$ since the interior of $[0,1]$ is $(0,1)$ but $(-1,0) \cup (0,1)$ is not a regular set in $\mathbb{R}$ because the interior of $[-1,1]$ is $(-1,1)$ and not the above-mentioned set.

Lemma 2.14

Let X be a topological space and $\mathcal{F}$ be the family of all Borel sets in X. We call $S_1 \sim S_2$ in $\mathcal{F}$ if S_1 differs from S_2 by a meager set (i.e. $S_1 \backslash S_2 \cup S_2 \backslash S_1$ is meager). Then $\sim$ is an equivalence relation.

Proof

If $S_1, S_2, S_3 \in \mathcal{F}$ clearly $S_1 \sim S_1$ and if $S_1 \sim S_2$ then $S_2 \sim S_1$.
We show that if $S_1 \sim S_2$ and $S_2 \sim S_3$ then $S_1 \sim S_3$.

Since $S_1 \backslash S_3 \cup S_3 \backslash S_1$

$$= S_1 \backslash (S_2 \cup S_3) \cup (S_2 \cap S_3) \backslash S_1 \cup (S_1 \cap S_2) \backslash S_3 \cup S_3 \backslash (S_1 \cup S_2)$$

$$\subset S_1 \backslash S_2 \cup S_2 \backslash S_1 \cup S_2 \backslash S_3 \cup S_3 \backslash S_2$$

the result follows.

Lemma 2.15 ([4], Lemma 3.4)

Let X be a complete metric space.

(i) The interior of the closure of an open set and the interior of the complement of a regular open set in X are regular.

(ii) Each open subset of X differs from a regular open subset on a meager set.

(iii) Each Borel subset of X differs from a regular open subset on a meager (Borel) set.

(iv) There is a unique regular open subset of X that differs from a given Borel set on a meager (Borel) set.

(v) Let $\mathcal{F}_0$ be the family of regular open subsets of X partially ordered by inclusion. Then $\mathcal{F}_0$ is a complete lattice.

(vi) Let $\mathcal{F}$ be the family of Borel subsets of X and $\mathcal{M}$ be the σ-ideal of meager Borel subsets of X (a countable union of sets in $\mathcal{M}$ is in $\mathcal{M}$ and the intersection of a set in $\mathcal{M}$ with any set $\mathcal{F}$ is in $\mathcal{M}$). Let $\mathcal{F}/\mathcal{M}$ be the family of equivalence classes of sets in $\mathcal{F}$ under the relation $S \sim S'$ when S and S' differ by a meager set. With $\mathcal{L}$ and $\mathcal{L}'$ in $\mathcal{F}/\mathcal{M}$ define $\mathcal{L} \leq \mathcal{L}'$ when $S \subseteq S'$ for some S

in $\mathcal{L}$ and S' in $\mathcal{L}'$. The $\leq$ is a partial ordering of $\mathcal{F}/\mathcal{N}$ (the quotient of inclusion on $\mathcal{F}$ by the ideal $\mathcal{N}$). Each $\mathcal{L}$ in $\mathcal{F}/\mathcal{N}$ contains precisely one regular open set and the mapping that assigns to each $\mathcal{L}$ in $\mathcal{F}/\mathcal{N}$ the regular open set it contains, is an order isomorphism of $\mathcal{F}/\mathcal{N}$ onto $\mathcal{F}_0$. The partially ordered set $\mathcal{F}/\mathcal{N}$ is a complete lattice.

(vii) The algebra $\mathcal{B}(X)$ of bounded Borel functions on X is a commutative C^*-algebra and the family $\mathcal{N}_0$ of functions in $\mathcal{B}(X)$ that vanish on the complement of a meager Borel set is a closed ideal in $\mathcal{B}(X)$ and $\mathcal{B}(X)/\mathcal{N}_0$ is a commutative C^*-algebra.

(viii) Let Y be a compact Hausdorff space such that $\mathcal{B}(X)/\mathcal{N}_0 \simeq C(Y)$ (cf Lemma 1.8 Gelfand-Naimark). Then Y is totally disconnected and the family of clopen subsets Y partially ordered by inclusion is a complete lattice. Y is extremely disconnected and $C(Y)$ and $\mathcal{B}(X)/\mathcal{N}_0$ are boundedly complete lattices.

Proof

(i) Let Y be a closed subset of X and O be its interior. Since O is an open subset of X contained in $\bar{O}$, O is contained in the greatest open subset of X contained in $\bar{O}$ being the interior of $\bar{O}$ say O_0. Since $O_0 \subseteq \bar{O} \subseteq Y$ and O_0 is an open subset of X, O_0 is contained in the interior O of Y. Thus $O = O_0$, O is the interior of $\bar{O}$ and hence O is regular.

For the second part let $Y = \overline{\text{interior } (X\setminus O)}$ with O still a regular open set. From the proof above it is clear that the interior of $X\setminus O$ is regular. Thus the interior of the complement of a regular open set in X is regular.

(ii) Let O be an open subset of X and O_o the interior of $\bar{O}$. Then $(O_o \backslash O) \cup (O \backslash O_o)$

$$= O_o \backslash O \quad (O \subseteq O_o)$$

$$\subseteq \bar{O} \backslash O.$$

$\bar{O} \backslash O$ is a closed nowhere dense set (has no interior points). Hence O and O_o differ on a meager subset of X, thus $O \sim O_o$ and from (i) we know that O_o is regular.

(iii) Let $\mathcal{F}'$ be the family of Borel subsets of X that differs from a regular open set on a meager (Borel) set. If $S \in \mathcal{F}'$ and O_o is a regular open set such that $S \sim O_o$ (S differs from O_o on a meager set) then

$[(X \backslash S) \backslash (X \backslash O_o)] \cup [(X \backslash O_o) \backslash (X \backslash S)]$

$= (S \backslash O_o) \cup (O_o \backslash S)$ is meager.

Thus $X \backslash S \sim X \backslash O_o$ and from (i) the interior O_1 of $X \backslash O_o$ is regular and $O_1 \sim X \backslash O_o \sim X \backslash S$.

Thus $X \backslash S$ is a Borel set that differs from O_1 on a meager set. Thus $X \backslash S \in \mathcal{F}'$.

Suppose $\{S_j\}$ are in $\mathcal{F}'$. Let O_j be a regular open set such that $S_j \sim O_j$. Then $(S_j \backslash O_j) \cup (O_j \backslash S_j) = M_j$ is meager and by direct computation

$$\left[\left(\bigcup_{j=1}^{\infty} S_j\right) \backslash \left(\bigcup_{j=1}^{\infty} O_j\right)\right] \cup \left[\left(\bigcup_{j=1}^{\infty} O_j\right) \backslash \left(\bigcup_{j=1}^{\infty} S_j\right)\right] \subseteq \bigcup_{j=1}^{\infty} M_j .$$

As $\bigcup_{j=1}^{\infty} M_j$ is meager, we have

$$\bigcup_{j=1}^{\infty} S_j \sim \bigcup_{j=1}^{\infty} O_j .$$

From (i) we know that the interior O_o of $\overline{\left(\bigcup_{j=1}^{\infty} O_j\right)}$ is regular and $O_o \sim \overline{\bigcup_{j=1}^{\infty} O_j} \sim \bigcup_{j=1}^{\infty} S_j$.

Thus $\bigcup_{j=1}^{\infty} S_j \in \mathcal{F}'$ and $\mathcal{F}'$ is a σ-algebra containing the open sets and contained in $\mathcal{F}$ (the Borel subsets of X). Since $\mathcal{F}$ is the smallest σ-algebra of such nature, it follows that $\mathcal{F} = \mathcal{F}'$.
Thus each Borel subset of X differs from a regular open subset on a meager (Borel) set.

(iv) If $S \in \mathcal{F}$ (a Borel set) and $S \sim O_1$ and $S \sim O_2$ with O_1 and O_2 regular open sets, then $O_1 \sim O_2$. Since $\overline{O_2}$ is closed, it follows that if some $p \in O_1$ and $p \notin \overline{O_2}$ then some open set O containing p does not meet O_2 and

$$O \cap O_1 \subseteq O_1 \backslash O_2 .$$

But $O_1 \backslash O_2$ is meager and we know that meager sets in a complete metric space have empty interior. We have a contradiction, so $O_1 \subseteq \overline{O_2}$ and O_1 is also contained in the interior O_2 of $\overline{O_2}$. Thus $O_1 \subseteq O_2$. Symmetrically it follows that $O_2 \subseteq O_1$. Thus $O_1 = O_2$ and there is a unique regular open subset of X that differs from a given Borel set on a meager (Borel) set.

(v) Suppose $O_a \in \mathcal{F}_0$ for a in A. Let O_1 be the interior of $\overline{(\bigcup_{a \in A} O_a)}$. Then O_1 is regular from (i) since $\bigcup_{a \in A} O_a$ is open. Clearly by using the definition of an interior set, O_1 is an upper bound for $\{O_a : a \in A\}$. If $O \in \mathcal{F}_0$ is another upper bound for this set then $\overline{\bigcup O_a} \subseteq \overline{O}$ and the interior O of $\overline{O}$ contains O_1. Thus O_1 is the least upper bound of $\{O_a : a \in A\}$ and $\mathcal{F}_0$ is a complete lattice ($\mathcal{F}_0$ contains its least upper bound).

It now follows that the set of lower bounds of $\{0_a\}$ also has a least upper bound say 0_o and 0_o is the greatest lower bound of $\{0_a\}$.

(vi) Since $S \subseteq S$ for every $S \in \mathcal{L}$ we have

$\mathcal{L} \leq \mathcal{L}$ for every $\mathcal{L}$ in $\mathcal{F}/\mathcal{M}$.

If $\mathcal{L} \leq \mathcal{L}'$ and $\mathcal{L}' \leq \mathcal{L}''$ there exist a $S \subseteq T$ and $T \subseteq R$ for S in $\mathcal{L}$, T in $\mathcal{L}'$ and R in $\mathcal{L}''$, then $S \subseteq R$ and $\mathcal{L} \leq \mathcal{L}''$.

Suppose $\mathcal{L} \leq \mathcal{L}'$ and $\mathcal{L}' \leq \mathcal{L}$ with $\mathcal{L}$ and $\mathcal{L}'$ in $\mathcal{F}/\mathcal{M}$. Then there are an S_1 and S_2 in $\mathcal{L}$ and an S_1' and S_2' in $\mathcal{L}'$ such that $S_1 \subseteq S_1'$ and $S_2' \subseteq S_2$.

Let M be $(S_1 \backslash S_2) \cup (S_2 \backslash S_1)$ and

M' be $(S_1' \backslash S_2') \cup (S_2' \backslash S_1')$.

Since $S_1 \sim S_2$ and $S_1' \sim S_2'$, M, M' and $M \cup M'$ are meager and $S_1 \cup M = S_2 \cup M$, $S_1' \cup M' = S_2' \cup M'$.

Thus
$$\begin{aligned} S_1' \cup M' \cup M &= S_2' \cup M' \cup M \\ &\subseteq S_2 \cup M' \cup M \\ &= S_1 \cup M' \cup M \\ &\subseteq S_1' \cup M' \cup M. \end{aligned}$$

Hence $S_1' \cup M' \cup M = S_1 \cup M' \cup M$ from which it follows that S_1 and S_1' differ by a meager set.

Thus $S_1' \sim S_1$ and $\mathcal{L} = \mathcal{L}'$. This proves that $\leq$ is a partial ordering of $\mathcal{F}/\mathcal{M}$. From (iii) and (iv) we know that each S in $\mathcal{F}$ differs from a unique regular open set 0 by a meager set. Thus the equivalence class $\mathcal{L}$ of S containing 0 has no other regular open sets. This implies that the mapping that assigns $\mathcal{L}$ to 0 is a bijection. If $\mathcal{L}'$ is another equivalence class and $0'$ is the regular open set it contains, then if $0 \subseteq 0'$, $\mathcal{L} \leq \mathcal{L}'$ by the definition of $\leq$.

Conversely, to complete the proof of our order isomorphism we must still prove that if $\mathcal{L} \leq \mathcal{L}'$ then $O \subseteq O'$. If $\mathcal{L} \leq \mathcal{L}'$ then by definition of the relation $\leq$ there exist meager sets M' and M in $\mathcal{M}$ such that

$$O \cup M \subseteq O' \cup M'.$$

Thus $O \subseteq O' \cup M'$ so that

$$O \backslash \overline{O'} \subseteq O \backslash O' \subseteq M'$$

Since $O \backslash \overline{O'}$ is open and M' is meager

$$O \backslash \overline{O'} = \phi, \quad \text{that is} \quad O \subseteq \overline{O'}.$$

Hence O is contained in the interior O' of $\overline{O'}$. We've now proved that the mapping $\mathcal{L} \rightarrow O$ for $\mathcal{F}/\mathcal{M}$ onto $\mathcal{F}_0$ is an order isomorphism and from (v) we know, that $\mathcal{F}_0$ is a complete lattice. Thus $\mathcal{F}/\mathcal{M}$ is a complete lattice.

(vii) Let $\mathcal{B}(X)$ be the algebra of bounded Borel functions on X. If we define the norm $\|f\|_\infty = \sup |f(x)|$ on $\mathcal{B}(X)$ it is clear that $\mathcal{B}(X)$ becomes a Banach algebra. The operation of complex conjugation of functions is an adjoint operation in $\mathcal{B}(X)$. $\|f \cdot \bar{f}\| = \| \, |f|^2 \, \| = \| \, |f| \, \|^2 = \|f\|^2$.
Thus $\mathcal{B}(X)$ with the given norm and adjoint operation is a C^*-algebra. If f_1 and f_2 vanish outside the meager sets M_1 and M_2, then $f_1 + f_2$ vanishes outside $M_1 \cup M_2$ where $M_1 \cup M_2$ is a meager subset of X. Further ff_1 vanishes outside M_1 for each f in $\mathcal{B}(X)$.
Thus $\mathcal{M}_0$ as defined above is an ideal in $\mathcal{B}(X)$. Let $f \in \bar{\mathcal{M}}_0$. Then there exists a sequence $\{f_n\} \subset \mathcal{M}_0$ such that $\|f - f_n\| \xrightarrow[\infty]{n} 0$. Let M_n be the meager set such that $f_n(x) = 0$ for each $x \in X \backslash M_n$. Clearly $\bigcup_{n=1}^{\infty} M_n$ is meager.
We show that f vanishes on $X \backslash \bigcup_{n=1}^{\infty} M_n = \bigcap_{n=1}^{\infty} (X \backslash M_n)$.
Choose $\epsilon > 0$ arbitrarily. Then there exists an n_0 such that for all $n \geq n_0$, $\|f - f_n\| < \epsilon$.

Thus if we choose x in $\bigcap_{n=1}^{\infty} (X\backslash M_n)$ then

$$|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)|$$
$$< \epsilon \quad \text{for all} \quad n \geq n_0.$$

Thus $f(x) = 0$ which shows that $f \in \mathcal{M}_0$ and $\mathcal{M}_0$ is a closed (two-sided) ideal in $\mathcal{B}(X)$. With the supremum norm and complex conjugation as involution, $\mathcal{B}(X)$ is a commutative C^*-algebra.

Now let $f + M$ and $g + M$ be in $\mathcal{B}(X)/\mathcal{M}_0$.

Then $(f + M)\ (g + M) = fg + M$

$= gf + M$ ($\mathcal{B}(X)$ is commutative)

$= (g + M)\cdot(f + M)$.

Hence $\mathcal{B}(X)/\mathcal{M}_0$ is a commutative C^*-algebra.

(viii) Let $\mathcal{L}$ be in $\mathcal{F}/\mathcal{M}$ and e be the characteristic function of a set in $\mathcal{L}$ (clearly $e \in \mathcal{B}(X)$).

Define $\eta(\mathcal{L})$ to be the projection in $\mathcal{B}(X)/\mathcal{M}_0$ which is the image of e under the quotient mapping from $\mathcal{B}(X)$ to $\mathcal{B}(X)/\mathcal{M}_0$. If e' corresponds to another set in $\mathcal{L}$, then by definition of sets in $\mathcal{L}$, $e - e' \in \mathcal{M}_0$ so that e and e' have the same image in $\mathcal{B}(X)/\mathcal{M}_0$. Thus $\eta(\mathcal{L})$ is well-defined.

We show that η is order preserving:

If $\mathcal{L} \leq \mathcal{L}'$ there exist sets S in $\mathcal{L}$ and S' in $\mathcal{L}'$ such that $S \subseteq S'$. With e and e' the characteristic functions of S and S' respectively and since $S \subseteq S'$ we have $e \leq e'$ so that $\eta(\mathcal{L}) \leq \eta(\mathcal{L}')$. (Note that the quotient map preserves order.) Let E be a projection in $\mathcal{B}(X)/\mathcal{M}_0$ and f an element of $\mathcal{B}(X)$ mapping onto E. Then $f^2 - f$ maps onto $E^2 - E \in \mathcal{M}_0$.

Thus $f^2 - f$ vanishes outside some meager Borel set M.

Let $e(p) = \begin{cases} f(p) & p \in X\backslash M \\ 0 & p \in M \end{cases}$.

On $\mathcal{B}(X)$ we'll have $e^2(p) - e(p) = f^2(p) - f(p) = 0$.

Hence in $\mathcal{B}(X)$, e is an idempotent and consequently e is the characteristic function of a set S in $\mathcal{F}$.
If $\mathcal{L}$ in $\mathcal{F}/\mathcal{M}$ is the equivalence class of S then $\eta(\mathcal{L}) = E$. Hence we've proved that η is an order preserving mapping of $\mathcal{F}/\mathcal{M}$ onto the set $\mathcal{P}$ of projections in $\mathcal{B}(X)/\mathcal{M}_0$.

If E and E′ are in $\mathcal{P}$ and $E \leq E'$, there are sets $\mathcal{L}$ and $\mathcal{L}'$ in $\mathcal{F}/\mathcal{M}$ such that $\eta(\mathcal{L}) = E$ and $\eta(\mathcal{L}') = E'$.
By the definition of η there are sets S and S′ in $\mathcal{L}$ and $\mathcal{L}'$ whose characteristic functions e and e′ map onto E on E′ respectively.
Thus $2(e - e'e)$ maps onto $2E - 2E'E$
and
$$\begin{aligned} 2E - 2E'E &= E' - 2E'E + E - E' + E \\ &= (E')^2 - 2E'E + (E)^2 - E' + E \\ &= (E' - E)^2 - (E' - E) \\ &= 0. \end{aligned}$$
Hence $(e - e'e)$ is 0 on $X\backslash M'$ for some meager set M′.
It follows that $S\backslash S' \subseteq M'$ so that $S \subseteq S' \cup M'$ $(S \in \mathcal{L})$.
Since $S' \cup M' \in \mathcal{L}'$, $\mathcal{L} \leq \mathcal{L}'$.
If $\eta(\mathcal{L}) = \eta(\mathcal{L}') = E$ then $\mathcal{L} \leq \mathcal{L}'$ and $\mathcal{L}' \leq \mathcal{L}$.
Thus $\mathcal{L} = \mathcal{L}'$ from (vi).
Hence η is a one-to-one mapping and since η is order preserving it follows that η^{-1} is also order preserving.
We know from (vii) that $\mathcal{B}(X)/\mathcal{M}_0$ is a commutative C^*-algebra. It follows from Lemma 1.8 that $B(X)/\mathcal{M}_0 \simeq C(Y)$ for some compact Hausdorff space Y. If ϕ is the isomorphism of $\mathcal{B}(X)/\mathcal{M}_0$ onto C(Y) then $\phi\cdot\eta$ is an order isomorphism of $\mathcal{F}/\mathcal{M}$ with the set $\mathcal{P}'$ of idempotents in C(Y). We know from (vi) that $\mathcal{F}/\mathcal{M}$ is a complete lattice, hence $\mathcal{P}'$ is a complete lattice. Each continuous function in $\mathcal{B}(X)$ is approximable in norm by step functions as close as we wish. Since each $f \in \mathcal{B}(X)$ is a linear combination of four positive functions we only have to show this for a positive function. Let $c = \frac{\|f\|}{2}$ and define $g_1 = c\cdot\chi_{E_0}$ where $E_0 = \{x : f(x) > c\}$.
Then $\|f - g_1\| \leq c$ and $0 \leq g_1 \leq f$.

If we use the same argument for $0 \leq f - g_1$ we can find a $0 \leq g_2 \leq f - g_1$ such that $\|(f - g_1) - g_2\| \leq \frac{\|f - g_1\|}{2} \leq \frac{c}{2}$. By induction we obtain a simple function g_n such that $0 \leq g_n \leq f - (g_1 + \dots + g_{n-1})$ and $\|f - (g_1 + \dots + g_n\| \leq \frac{c}{2^{n-1}}$. If we choose $f_n = g_1 + \dots + g_n$ then $\|f - f_n\| \xrightarrow[\infty]{n} 0$ and f_n is a simple function. Thus linear combinations of idempotents lie dense in $\mathcal{B}(X)$, in $\mathcal{B}(X)/\mathcal{N}_0$ and in $C(Y)$. Hence Y is totally disconnected. From Lemma 2.12 we have that Y is extremely disconnected and from Theorem 2.11 it follows that $C(Y)$ is a boundedly complete lattice. From $\mathcal{B}(X)/\mathcal{N}_0 \simeq C(Y)$ we have that $\mathcal{B}(X)/\mathcal{N}_0$ is a boundedly complete lattice.

Theorem 2.16 ([4], Theorem 3.5)

With the notation of our previous lemma assume that X is $[0,1]$ and let q be a state of $C(Y)$.

(i) Suppose $q(\sup_n e_n) = \sum_{n=1}^{\infty} q(e_n)$ whenever $\{e_n\}$ is a countable family of orthogonal idempotents in $C(Y)$ (i.e. $e_n \cdot e_{n'} = 0$ unless $n = n'$).
Then $q(\sup_n f_n) \leq \sum_{n=1}^{\infty} q(f_n)$ for each countable set $\{f_n\}$ of idempotents f_n in $C(Y)$.

(ii) Enumerate the open intervals in $[0,1]$ with rational end-points and let $f_1, f_2, \dots$ be the idempotents in $C(Y)$ that are the images of their characteristic functions in $\mathcal{B}(X)$ under the composition of the quotient mapping of $\mathcal{B}(X)$ onto $\mathcal{B}(X)/\mathcal{N}_0$ and the isomorphism of $\mathcal{B}(X)/\mathcal{N}_0$ with $C(Y)$. For each j in $\{1,2,\ldots\}$, let e_j be an idempotent in $C(Y)$ such that $0 < e_j \leq f_j$. Then $\sup_j e_j = 1$.

(iii) With the notation of (ii) and a positive ϵ given, e_j can be chosen such that $q(e_j) \leq 2^{-j} \epsilon$. Moreover this will show that $C(Y)$ has no normal states. (Hence the C^*-algebra $C(Y)$ is not isomorphic to any abelian von Neumann algebra, although Y is extremely disconnected.)

Proof

(i) Let $f'_1 = f_1$ and

$f'_n = \sup \{f_1,f_2,\ldots,f_n\} - \sup \{f_1,\ldots,f_{n-1}\}$ for $n = \{2,3,\ldots\}$. If $m < n$, then $f_m \leq \sup \{f_1,\ldots,f_{n-1}\}$ so that $f'_m \leq \sup \{f_1,\ldots,f_{n-1}\}$ from which it follows that $f'_m \cdot f'_n = 0$. Moreover $f'_1 + f'_2 + f'_3 + \ldots + f'_n$

$$= f_1 + \sup \{f_1,f_2\} - f_1 + \sup \{f_1,f_2,f_3\} - \sup \{f_1,f_2\} + \ldots + \sup \{f_1,\ldots,f_n\} - \sup \{f_1,f_2,\ldots,f_{n-1}\}$$

$= \sup \{f_1,f_2,\ldots,f_n\}$ for each n in $\{1,2,\ldots\}$.

Since $f'_n \leq \sup \{f'_1,f'_2,\ldots,f'_n\}$ for all n and $f'_n \cdot f'_m = 0$ we have $f'_1 + f'_2 + \ldots + f'_n \leq \sup \{f'_1,f'_2,\ldots,f'_n\}$.

Conversely $f'_1 + f'_2 + \ldots + f'_n$ is an upper bound for $f'_1,\ldots,f'_n$, but $\sup \{f'_1,f'_2,\ldots,f'_n\}$ is the smallest upper bound for $f'_1,\ldots,f'_n$.

Thus $\sup \{f'_1,f'_2,\ldots,f'_n\} \leq f'_1 + f'_2 + \ldots + f'_n$.

Thus $\sup\{f'_1,f'_2,\ldots,f'_n\} = \sum_{i=1}^{n} f'_i = \sup \{f_1,\ldots,f_n\}$ for each $n = \{1,2,\ldots\}$ so that $\sup_n f'_n = \sup_n f_n$.

Now $q(\sup \{f_1,f_2,\ldots\}) = q(\sup \{f'_1,f'_2,\ldots\}) = \sum_{n=1}^{\infty} q(f'_n)$.

Now $\sup\{f_1, f_2, \ldots, f_n\} \le f_1 + f_2 + \ldots + f_n$ so that

$$\sum_{n=1}^{m} q(f_n') = q(\sup\{f_1, f_2, \ldots, f_m\}) \le \sum_{n=1}^{m} q(f_n) \le \sum_{n=1}^{\infty} q(f_n)$$

Thus $q(\sup\{f_1, f_2, \ldots\}) = \sum_{n=1}^{\infty} q(f_n') \le \sum_{n=1}^{\infty} q(f_n)$.

(ii) With the notation and results of the last part of Lemma 2.15 in mind, let $(\phi\cdot\eta)^{-1}(e_j)$ be $\mathcal{L}_j$. From (iii) in the previous lemma $\mathcal{L}_j$ contains a regular open set O_j. Let $O = \bigcup_{j=1}^{\infty} O_j$. If $p \in [0,1]\backslash\bar{O}$, there is some open interval (a,b) with rational endpoints that contains p and does not meet O. Let $\mathcal{L}$ be the equivalence class of (a,b) in $\mathcal{F}/\mathcal{N}$ and $f_j = (\phi\cdot\eta)(\mathcal{L})$.

Since $(\phi\cdot\eta)^{-1}$ is an order isomorphism it follows from $0 < e_j \le f_j$ that $\mathcal{L}_j \le \mathcal{L}$. Now (a,b) is regular and from Lemma 2.15 (iv) (a,b) is the only regular set in $\mathcal{L}$. Since $\mathcal{L}_j \le \mathcal{L}$ it follows from Lemma 2.15 (vi) that $O_j \subseteq (a,b)$. This contradicts the fact that (a,b) and $\bigcup_{j=1}^{\infty} O_j$ are disjoint.

Hence $\bar{O} = [0,1]$, from which it is clear that $\sup_j e_j = 1$.

(Note that $e_j = \phi(\chi_{O_j} + \mathcal{N}_0)$ thus $\sup_j e_j = \phi(1 + \mathcal{N}_0) = 1$.)

(iii) We construct this sequence of idempotents e_j in $C(X)$ by using the fact that $C(Y)$ contains no minimal non-zero idempotents. In fact if f were minimal it would have followed that $\mathcal{L} = (\phi\cdot\eta)^{-1}(f)$ is minimal in $\mathcal{F}/\mathcal{N}$ (recall that $(\phi\cdot\eta)^{-1}$ is an order isomorphism). This would have implied that the regular open set O in $\mathcal{L}$ is non-empty and minimal in $\mathcal{F}_0$. This contradicts the fact that O contains some open interval (a,b) (which is clearly regular) as a proper subset. Now let f be any non-zero idempotent in $C(Y)$. Then there exists an idempotent f' in $C(Y)$ such that $0 < f' < f$. Clearly $0 < f - f' < f$ also holds and one of $q(f')$ and $q(f - f')$ is not greater than $\frac{1}{2}q(f)$. Suppose this is true for $q(f')$, then we can choose an $f'' \in C(Y)$ such that $0 < f'' < f' < f$ and $q(f'') < \frac{1}{2}q(f') \leq \frac{1}{4}q(f)$. Continuing in this way, we can find an idempotent $\hat{f} \in C(Y)$ such that $0 < \hat{f} < f$ and $q(\hat{f}) \leq \epsilon$. Now if we apply this argument for each idempotent $f_j \in C(Y)$ we can find an idempotent e_j such that $0 < e_j < f_j$ and $q(e_j) \leq 2^{-j}\,\epsilon$. Suppose q is normal, then it follows from part (i) of this proof that:

$$\begin{aligned} 1 = q(1) = q(\sup_n e_n) &\leq \sum_{j=1}^{\infty} q(e_j) \\ &\leq \sum_{j=1}^{\infty} 2^{-j}\,\epsilon \\ &= \epsilon. \end{aligned}$$

Thus $\epsilon \geq 1$, but this is a contradiction since ϵ was chosen arbitrarily. Hence $C(Y)$ has no normal states. (Recall that normality of states defined after Definition 2.1 is equivalent to the assumption on q in (i), (cf [11], Section 5.2). If $C(Y)$ were isomorphic to a weak operator closed subalgebra of $\mathcal{B}(H)$ (for some Hilbert space H), then $\mathcal{A}$ (and hence $C(Y)$) would have had many normal states. In fact, any vector state on $\mathcal{A}$ (i.e. ρ is a vector state of $\mathcal{A}$ if there exists a unit vector $x \in H$ such that $\rho(T) = \langle Tx,x\rangle$), is normal.

CHAPTER 3 : SAKAI'S CHARACTERIZATION

In chapter two we showed that a necessary and sufficient condition for a C^*-algebra to be isomorphic to a von Neumann algebra is for it to be a W^*-algebra. In this chapter we present another "abstract" characterization for a C^*-algebra to be isomorphic to a von Neumann algebra.

It is well-known that if $\mathcal{A}$ is a von Neumann algebra and $\mathcal{A}_*$ is the Banach space of all σ-weak continuous (cf Appendix for a definition) linear functionals on $\mathcal{A}$, then $\mathcal{A} = (\mathcal{A}_*)^*$ (i.e. $\mathcal{A}$ is the Banach space dual of $\mathcal{A}_*$). So any von Neumann algebra $\mathcal{A}$ has a predual $\mathcal{A}_*$. In 1956 Sakai proved that it is exactly those C^*-algebras that are dual spaces of Banach spaces which are isomorphic to von Neumann algebras (cf [8]). In 1957 Tomiyama gave an elegant proof of this dual space characterization by using the Gelfand- Naimark-Segal construction together with results on so-called conditional expectations (cf [13]). In this chapter we present a version of Tomiyama's proof of Sakai's characterization. For this result we need a few results on the universal representation of a C^*-algebra $\mathcal{U}$.

I The universal representation

By a representation of $\mathcal{U}$ on a Hilbert space H we mean a *-homomorphism π from $\mathcal{U}$ into $\mathcal{B}(H)$. If π is one-to-one, we call it a faithful representation. By means of the Gelfand-Naimark-Segal construction we can associate with each state φ on $\mathcal{U}$ a representation π_φ of $\mathcal{U}$ on a Hilbert space H_φ and a unit cyclic vector x_φ for π_φ, so that

$$\varphi(T) = \langle\pi_\varphi(T)x_\varphi, x_\varphi\rangle \quad (T \in \mathcal{U}).$$

Let Φ be the state space of $\mathcal{U}$ and consider the family $\{(\pi_\varphi, H_\varphi) : \varphi \in \Phi\}$. If we let $H = \oplus_{\varphi\in\Phi} H_\varphi$ and define $\pi : \mathcal{U} \longrightarrow \mathcal{B}(H)$ by $\pi(T)\,(\oplus_{\varphi\in\Phi} \xi_\varphi) = \oplus_{\varphi\in\Phi} \pi_\varphi(T)\xi_\varphi$ then if follows that π is a faithful representation of $\mathcal{U}$ on H. By means of this representation $\mathcal{U}$ is isometric *-isomorphic to a norm closed $*$-subalgebra of $\mathcal{B}(H)$. We call this representation the universal representation for a C^*-algebra $\mathcal{U}$ (cf [5], Remark 4.5.7).

One of the important merits of the universal representation π, among other representations is that any bounded linear functional φ on $\pi(\mathcal{U})$ is weak operator continuous. This result will follow after the following lemma. If $\mathcal{B}$ is a C^*-algebra of operators on a Hilbert space H, we call a state φ on $\mathcal{B}$ a vector state if there exists a unit vector $x \in H$ such that $\varphi(T) = \langle Tx,x\rangle$ $(T \in \mathcal{B})$.

Lemma 3.1

Let (π,H_π) be the universal representation of a C^*-algebra $\mathcal{U}$. Each state of the C^*-algebra $\pi(\mathcal{U})$ is a vector state.

Proof

Let $\varphi \in \Phi$, then there exists a unique unit vector x_φ such that $\varphi(T) = \langle\pi_\varphi(T)x_\varphi, x_\varphi\rangle$. If we let $w_x(T) = \langle Tx,x\rangle$ $(T \in \mathcal{B}(H))$, then $\varphi = w_{x_\varphi}\cdot\pi_\varphi$. If we let $y = \underset{\rho\in\Phi}{\oplus} y_\rho$ where $y_\rho = 0$ for $\rho \neq \varphi$ and $y_\varphi = x_\varphi$, then $\varphi = w_y\cdot\pi$. Hence each state of $\mathcal{U}$ has the form $w_y\cdot\pi$ with y a unit vector in H_π. Now, let w be any state on $\pi(\mathcal{U})$ and define φ on $\mathcal{U}$ by $\varphi(T) = w(\pi(T))$. Since $\|\varphi\| = \varphi(I) = w((\pi(I)) = \|w\|$ (cf [5], Theorem 4.3.2) it is clear that φ is a state on $\mathcal{U}$. Hence, by the above arguments there exists a unit vector y in H_π such that $w(\pi(T)) = \langle\pi(T)y,y\rangle$.

Remark

This lemma actually shows more. In fact it can be seen from the proof that the mapping $\varphi \longrightarrow \varphi\cdot\pi^{-1}$ carries the state space of $\mathcal{U}$ onto the state space of $\pi(\mathcal{U})$ in a one-to-one manner.

Proposition 3.2 ([6], Proposition 10.1.1)

Let π be the universal representation of $\mathcal{U}$. Each bounded linear functional ρ on $\pi(\mathcal{U})$ is weak operator continuous and extends uniquely to a weak operator continuous linear functional $\bar{\rho}$ on $\overline{\pi(\mathcal{U})}^{w-o}$ with $\|\rho\| = \|\bar{\rho}\|$.
Moreover, the mapping $\gamma:\rho \longrightarrow \bar{\rho}$ is a Banach space isomorphism from the dual space $\pi(\mathcal{U})^*$ onto $(\overline{\pi(\mathcal{U})}^{w-o})_*$.

Proof
From Lemma 3.1 each state of $\pi(\mathcal{U})$ is a vector state w_x (for some $x \in H$) which is weak operator continuous. Hence w_x extends to a vector state $\bar{w}_x$ on $\overline{\pi(\mathcal{U})}^{w-o}$.
(Note $\bar{w}_x(T) = \langle Tx,x\rangle = \lim_{n\to\infty} \langle\pi(T_n)x,x\rangle$ where $\{\pi(T_n)\} \subset \pi(\mathcal{U})$ converges to $T \in \overline{\pi(\mathcal{U})}^{w-o}$ in the weak operator topology.) From [2] it follows that each bounded linear functional ρ on $\pi(\mathcal{U})$ can be written as a linear combination of at most four states on $\pi(\mathcal{U})$. Since these states are vector states, ρ extends to the corresponding linear combination $\bar{\rho}$ of vector states on $\overline{\pi(\mathcal{U})}^{w-o}$. Moreover $\bar{\rho}$ is weak operator continuous (all vector states are) which also implies that the extension is unique.
We show that $\|\rho\| = \|\bar{\rho}\|$. Clearly $\|\rho\| \le \|\bar{\rho}\|$. Consider the weak operator closed set $\{T \in \overline{\pi(\mathcal{U})}^{w-o} : |\bar{\rho}(T)| \le \|\rho\|\}$. It is clear that this set contains $\pi(\mathcal{U})_1$. Then since $\overline{(\pi(\mathcal{U})_1)}^{w-o} = (\overline{\pi(\mathcal{U})}^{w-o})_1$ (by the Kaplansky density theorem cf [2], Theorem 3.6.1), it follows that the set also contains $(\overline{\pi(\mathcal{U})}^{w-o})_1$. Thus $\|\bar{\rho}\| \le \|\rho\|$.
Define $\gamma : \pi(\mathcal{U})^* \longrightarrow (\overline{\pi(\mathcal{U})}^{w-o})_*$ by $\gamma(\rho) = \bar{\rho}$.
We show that γ is onto. If $w \in (\overline{\pi(\mathcal{U})}^{w-o})_*$, let $\rho = w|_{\pi(\mathcal{U})}$, then $\rho \in \pi(\mathcal{U})^*$ (σ-weak continuous linear functionals are norm

continuous) and since $\bar{\rho}$ and w coincide on $\pi(\mathcal{U})$ and both are σ-weak continuous on $\overline{\pi(\mathcal{U})}^{w-o}$ it follows that $\bar{\rho} = w$. (Note that $\bar{\rho}$ is a linear combination of vector states which are σ-weak continuous.) Now it is clear that γ is a Banach space isomorphism.

Proposition 3.3 ([6], Proposition 10.1.21)

Let π be the universal representation of the C^*-algebra $\mathcal{U}$ and for $\rho \in \pi(\mathcal{U})^*$ let $\bar{\rho} \in (\overline{\pi(\mathcal{U})}^{w-o})_*$ be the unique extension of ρ (cf Proposition 3.2). Then for each $T \in \overline{\pi(\mathcal{U})}^{w-o}$ the mapping $\hat{T} : \pi(\mathcal{U})^* \longmapsto \mathbb{C}$ defined by $\hat{T}(\rho) = \bar{\rho}(T)$ is a bounded linear functional on $\pi(\mathcal{U})^*$. Moreover the mapping $T \to \hat{T}$ is an isometric isomorphism from $\overline{\pi(\mathcal{U})}^{w-o}$ onto the bidual space $\pi(\mathcal{U})^{**}$. Its restriction to $\pi(\mathcal{U})$ is the canonical embedding of $\pi(\mathcal{U})$ into $\pi(\mathcal{U})^{**}$.

Proof

It is well-known that any von Neumann algebra has a predual. In fact it can be seen from [10] that for the algebra $\overline{\pi(\mathcal{U})}^{w-o}$ the mapping $\psi_T : (\overline{\pi(\mathcal{U})}^{w-o})_* \to \mathbb{C}$ defined by $\psi_T(w) = w(T)$ is a bounded linear functional on $(\overline{\pi(\mathcal{U})}^{w-o})_*$. Moreover, the mapping $\beta : T \to \psi_T$ defines an isometric isomorphism from $(\overline{\pi(\mathcal{U})}^{w-o})$ onto the dual space $(\overline{\pi(\mathcal{U})}^{w-o})_*^*$ of $(\overline{\pi(\mathcal{U})}^{w-o})_*$. In Proposition 3.2 we have shown that the mapping $\gamma : \rho \to \bar{\rho}$ from $\pi(\mathcal{U})^*$ onto $(\overline{\pi(\mathcal{U})}^{w-o})_*$ is an isometric isomophism. Now let $\alpha : (\overline{\pi(\mathcal{U})}^{w-o})_*^* \to \pi(\mathcal{U})^{**}$ be the adjoint operator of γ (i.e. $\alpha(\psi) = \varphi$ where $\varphi(\rho) = \psi(\gamma\rho)$
$= \psi(\bar{\rho})$).

Since γ is an isometric isomorphism, α is one. If we consider the composition $\alpha\cdot\beta : T \to \varphi$ it is an isometric isomorphism from $(\overline{\pi(\mathcal{U})}^{w-o}$ onto $\pi(\mathcal{U})^{**}$ where $\varphi_T(\rho) = \psi_T(\bar{\rho}) = \bar{\rho}(T)$.

Clearly $\alpha\cdot\beta|_{\pi(\mathcal{U})} : \pi(\mathcal{U}) \to \pi(\mathcal{U})^{**}$ is the canonical imbedding for if $T \in \pi(\mathcal{U})$ then $(\alpha\cdot\beta)(T) = \varphi_T$ where $\varphi_T(\rho) = \rho(T)$ $(\rho \in \pi(\mathcal{U})^*)$.

Remark 3.4

The result above may be misleading if it is not understood in the correct way. For instance let $\mathcal{U}$ be the C^*-algebra ℓ^∞. Then it was shown in Example 1.20 that there exists a representation Φ from ℓ^∞ into some $\mathcal{B}(H)$ such that $\overline{\Phi(\mathcal{U})}^{w-o} = \Phi(\mathcal{U})$. Hence ℓ^∞ is a von Neumann algebra. For the universal representation π of ℓ^∞, this is surely not the case that $\overline{\pi(\mathcal{U})}^{w-o} = \pi(\mathcal{U})$. For if $\overline{\pi(\mathcal{U})}^{w-o} = \pi(\mathcal{U})$ then it would follow from the above theorem that $\mathcal{U}$, $\pi(\mathcal{U})$ and $\pi(\mathcal{U})^{**}$ are as Banach spaces the same! Hence ℓ^∞ will be reflexive - a contradiction.

Remark 3.5

Note that from now on if we write $\mathcal{U}$ for a C^*-algebra it will always mean that $\mathcal{U}$ is represented as a closed $*$-subalgebra of some $\mathcal{B}(H)$ under the *universal representation.* (So we shall always write $\mathcal{U}$ in stead of $\pi(\mathcal{U})$.)

II Tomiyama's proof

We've already mentioned that Tomiyama gave an elegant proof for Sakai's characterization. Apart from the universal representation, Tomiyama used conditional expectations which he generalised from commutative measure spaces to non-commutative measure spaces.

Definition 3.6

A linear mapping φ from a C^*-algebra $\mathcal{U}$ into another C^*-algebra $\mathcal{B}$ is said to be positive if $\varphi(H) \geq 0$ when $H \in \mathcal{U}^+$. If $\mathcal{B}$ is also a subalgebra of $\mathcal{U}$, $\varphi(I) = I$ and $\varphi(BAC) = B\varphi(A)C$ when $B,C \in \mathcal{B}$ and $A \in \mathcal{U}$ then φ is said to be a conditional expectation from $\mathcal{U}$ onto $\mathcal{B}$.

Proposition 3.7

If φ is a conditional expectation from $\mathcal{U}$ onto $\mathcal{B}$ ($\mathcal{B}$ is a subalgebra of $\mathcal{U}$) then φ is a projection of norm one from $\mathcal{U}$ onto $\mathcal{B}$.

Proof

By means of the Gelfand-Naimark Theorem (cf Lemma 1.8) it can be shown that each self-adjoint element of $\mathcal{U}$ is the difference of two positive elements of $\mathcal{U}$. Hence φ maps self-adjoint elements onto self-adjoint elements and φ is hermitian (adjoint-preserving).

For each A in $\mathcal{U}$ and B in $\mathcal{B}$

$$0 \leq (A - B)^*(A - B)$$

thus

$$\begin{aligned} 0 &\leq \varphi((A - B)^*(A - B)) \qquad (\varphi \text{ is positive}) \\ &= \varphi(A^*A) - \varphi(A^*B) - \varphi(B^*A) + \varphi(B^*B) \\ &= \varphi(A^*A) - \varphi(A^*)B - B^*\varphi(A) + B^*B. \end{aligned}$$

If we replace B by $\varphi(A)$ we get:

$$\begin{aligned} 0 &\leq \varphi(A^*A) - \varphi(A^*)\varphi(A) - \varphi(A)^*\varphi(A) + \varphi(A)^*\varphi(A) \\ &= \varphi(A^*A) - \varphi(A^*)\varphi(A). \end{aligned}$$

Thus $\varphi(A^*)\varphi(A) \leq \varphi(A^*A)$, which holds for each A in $\mathcal{U}$. It can easily be shown that $\langle A^*Ax,x\rangle \leq \langle \|A\|^2Ix,x\rangle$ hence $A^*A \leq \|A\|^2I$ and it follows that

$$0 \leq \varphi(A^*)\varphi(A) \leq \varphi(A^*A) \leq \|A\|^2I$$

and $\|\varphi(A)\| \leq \|A\|$ and $\|\varphi\| \leq 1$.

Since $\varphi(I) = I$, $\|\varphi\| = 1$ and

$$\begin{aligned} \varphi(B) &= \varphi(B\cdot I\cdot I) \\ &= B\varphi(I)\cdot I \qquad (\varphi \text{ a conditional expectation}) \\ &= B \text{ for each } B \text{ in } \mathcal{B}. \end{aligned}$$

Thus $\varphi(\varphi(A)) = \varphi(A)$ and φ is idempotent.

Hence φ is a projection of norm one mapping $\mathcal{U}$ onto $\mathcal{B}$.

We give a few examples of conditional expectations.

(i) Let $\mathcal{U}$ be a C^*-algebra and ρ any state on $\mathcal{U}$. Define $\Phi(T) = \rho(T)I$. Then Φ is a conditional expectation on $B = \{S : S = \lambda I, \lambda \in \mathbb{C}\}$.

Clearly since ρ is positive, Φ will be positive and

$$\Phi(I) = \rho(I)I = \|\rho\|I = I.$$

If $S = \lambda I$ then
$$\begin{aligned}\Phi(ST) &= \Phi(\lambda IT)\\ &= \lambda\rho(T)I\\ &= \lambda\Phi(T)\\ &= \lambda I\Phi(T)\\ &= S\Phi(T).\end{aligned}$$

Hence Φ is a conditional expectation.

There are surely other non-trivial examples.

(ii) Let $\mathcal{A}$ be a finite von Neumann algebra (i.e. I is a finite projection). Then it is well-known that there exists a unique central valued trace $\Phi : \mathcal{A} \longrightarrow \mathcal{Z}$ where $\mathcal{Z} = \mathcal{A} \cap \mathcal{A}'$ which satisfies all the properties of a conditional expectation (cf [10]).

Theorem 3.8 ([4], Lemma 5.1)

Let $\mathcal{B}$ be a C^*-subalgebra of the C^*-algebra $\mathcal{U}$ and let φ_0 be an idempotent bounded linear mapping of $\mathcal{U}$ onto $\mathcal{B}$ such that $\|\varphi_0\| = 1$. Let $\bar{\mathcal{U}}^{w-o}$ be the weak operator closure of $\mathcal{U}$, then φ_0 is a positive linear mapping of $\mathcal{U}$ onto $\mathcal{B}$ such that $\varphi_0(I) = I$ and φ_0 extends uniquely to a weakly continuous idempotent linear mapping φ of $\bar{\mathcal{U}}^{w-o}$ onto $\bar{\mathcal{B}}^{w-o}$ such that $\|\varphi\| = 1$ and φ is a positive linear mapping.

Proof

Since φ_0 is onto, we can choose an $A \in \mathcal{U}$ such that $\varphi_0(A) = B$. As φ_0 is idempotent we have $\varphi_0(B) = \varphi_0(\varphi_0(A)) = \varphi_0(A) = B$. In particular $\varphi_0(I) = I$. If ρ is a state on $\mathcal{B}$ it follows from $\rho(I) = \|\rho\| = 1$ (cf Appendix, Lemma 1) that $(\rho\cdot\varphi_0)(I) = \rho(I) = 1$. Since $\|\rho\cdot\varphi_0\| \le \|\rho\|\ \|\varphi_0\| = 1$ one has from the same reference that $\rho\cdot\varphi_0$ is a state on $\mathcal{U}$.

Suppose $H \in \mathcal{U}^+$, then $\rho(\varphi_0(H)) \geq 0$ for each state ρ on $\mathcal{B}$. Since $\varphi_0(H) \in \mathcal{B}$ it follows from ([5], Theorem 4.3.4) that $\varphi_0(H) \in \mathcal{B}^+$. Thus φ_0 is a positive, linear mapping from $\mathcal{U}$ onto $\mathcal{B}$.

Let ω be a σ-weakly continuous linear functional on $\mathcal{B}$. Since a σ-weakly continuous functional is norm continuous it follows that $\omega \cdot \varphi_0$ is a bounded linear functional on $\mathcal{U}$. Hence by Proposition 3.2 $\omega \cdot \varphi_0$ is σ-weakly continuous on $\mathcal{U}$. Since it is clear from Proposition 3.2 that the σ-weakly and weak operator topologies coincide under the universal representation on $\mathcal{U}$ and $\mathcal{B}$, we have that $\varphi_0 : \mathcal{U} \longrightarrow \mathcal{B}$ is σ-weakly continuous:
If $T_\alpha \longrightarrow T$ σ-weakly in $\mathcal{U}$ we know that for each x in H $\omega_x \cdot \varphi_0(T_\alpha) \longrightarrow \omega_x \cdot \varphi_0(T)$ ($\omega_x(T) = \langle Tx,x\rangle$ is σ-weakly continuous). Hence $\varphi_0(T_\alpha) \longrightarrow \varphi_0(T)$ weakly, hence σ-weakly. Since $\bar{\mathcal{U}}^{\sigma-\omega} = \bar{\mathcal{U}}^{w-o}$ and $\bar{\mathcal{B}}^{\sigma-\omega} = \bar{\mathcal{B}}^{w-o}$, $\mathcal{U}$ and $\mathcal{B}$ is σ-weakly dense in $\bar{\mathcal{U}}^{w-o}$ and $\bar{\mathcal{B}}^{w-o}$ respectively. Thus φ_0 can uniquely be extended to a mapping φ from $\bar{\mathcal{U}}^{w-o}$ into $\bar{\mathcal{B}}^{w-o}$ which is σ-weakly continuous.
We now show that $\|\varphi\| = \|\varphi_0\| = 1$, since $\|\varphi_0\| = 1$ we only have to prove that $\|\varphi\| = 1$. Clearly since φ is an extension of φ_0, $1 = \|\varphi_0\| \leq \|\varphi\|$.
Since φ is σ-weakly operator continuous the set $S = \{T \in \bar{\mathcal{U}}^{\sigma-\omega} : \|\varphi(T)\| \leq 1\}$ is weak operator closed. Now for any $T \in \mathcal{U}_1$ it follows that $\|\varphi(T)\| = \|\varphi_0(T)\| \leq \|\varphi_0\| \, \|T\| \leq 1$. Thus $\mathcal{U}_1 \subset S$. Hence it follows from Kaplansky's density theorem ([2], Theorem 3.6.1) that $(\bar{\mathcal{U}}^{\sigma-\omega})_1 = (\bar{\mathcal{U}}_1)^{\sigma-\omega} \subset S$.
Thus $\|\varphi\| = \sup_{\substack{\|T\|\leq 1 \\ T\in\bar{\mathcal{U}}^{\sigma-\omega}}} \|\varphi(T)\| \leq 1$ and $\|\varphi\| = \|\varphi_0\| = 1$.

Since $\bar{\mathcal{B}}^{w-o} \subset \bar{\mathcal{U}}^{w-o}$, $\varphi\cdot\varphi$ is well-defined, σ-weakly continuous and coincides on $\mathcal{U}$ with $\varphi_0\cdot\varphi_0 = \varphi_0 = \varphi|_{\mathcal{U}}$. Since the σ-weak continuous mappings $\varphi\cdot\varphi$ and φ coincides on $\mathcal{U}$ and $\mathcal{U}$ is σ-weakly dense in $\bar{\mathcal{U}}^{w-o}$ it is clear that $\varphi\cdot\varphi = \varphi$ on $\bar{\mathcal{U}}^{w-o}$. Since $\bar{\mathcal{U}}^{w-o}$ is a von Neumann algebra one has that $\bar{\mathcal{U}}^{w-o} = (\bar{\mathcal{U}}^{w-o})_*^{\ *}$. Hence from Banach-Alauglo ([9]) the unit ball of $\bar{\mathcal{U}}^{w-o}$ is $\sigma(\bar{\mathcal{U}}^{w-o}, (\bar{\mathcal{U}}^{w-o})_*)$ compact, hence σ-weakly compact. Since $\mathcal{B}_1 \subset \mathcal{U}_1$ and $\|\varphi\| = 1$, φ maps the σ-weak compact set $(\bar{\mathcal{U}}^{w-o})_1$ onto a σ-weak compact (hence closed) subset of $\bar{\mathcal{B}}^{w-o}$ that contains $\mathcal{B}_1$. From Kaplansky it follows that $\overline{(\mathcal{B}_1)}^{w-o} = (\bar{\mathcal{B}}^{w-o})_1$. It is now easy to see that $\varphi(\bar{\mathcal{U}}^{w-o}) = \bar{\mathcal{B}}^{w-o}$.
Since $\|\varphi\| = 1$ and $\varphi(I) = I$ it can be shown in a similar way as for φ_0 that φ is positive.

Remark 3.9

It is important to notice that φ is the identity on $\bar{\mathcal{B}}^{w-o}$. It is clear at the beginning of the proof of Theorem 3.8 that $\varphi_0(B) = B$ for each $B \in \mathcal{B}$. If we take $B \in \bar{\mathcal{B}}^{w-o}$ then there exists a sequence B_n converging to B σ-weakly. Since φ is σ-weakly continuous from $\bar{\mathcal{U}}^{w-o}$ onto $\bar{\mathcal{B}}^{w-o}$ it is clear that $\varphi(B) = B$.

Recall that a linear functional ρ on a C^*-algebra $\mathcal{U}$ is positive if $\rho(A) \geq 0$ for any positive element A of $\mathcal{U}$.

Lemma 3.10 ([12], I, Lemma 9.5)

If ρ is a positive linear functional on a C^*-algebra $\mathcal{U}$, then $\rho(B^*A) = \overline{\rho(A^*B)}$ and $|\rho(B^*A)|^2 \leq \rho(A^*A)\rho(B^*B)$.

Proof

It follows by direct computation that

$$4B^*A = \sum_{n=0}^{3} i^n (A + i^nB)^*(A + i^nB)$$

and

$$4AB^* = \sum_{n=0}^{3} i^n \ (A + i^nB)(A + i^nB)^*.$$

Hence $\rho(B^*A) = \overline{\rho(A^*B)}$.

Since for any $\lambda \in \mathbb{C}$, $0 \leq \rho((\lambda A + B)^*(\lambda A + B))$

$$= |\lambda|^2\rho(A^*A) + 2\ \mathrm{Re}(\lambda\rho(A^*B)) + \rho\ (B^*B)$$

$$\leq \rho(A^*A)\ |\lambda|^2 + 2|\rho(A^*B)|\ |\lambda| + \rho(B^*B)$$

it follows that the discriminant of the parabola in $|\lambda|$ is negative. Hence $|\rho(A^*B)|^2 = |\rho(B^*A)|^2$

$$\leq \rho(A^*A)\rho(B^*B).$$

Definition 3.11

A state ρ of a C^*-algebra is said to be definite on a self-adjoint element A of $\mathcal{U}$ when $\rho(A^2) = \rho(A)^2$.

The following two properties of a definite state will be crucial in this chapter.

Proposition 3.12

Let $\mathcal{U}$ be a C^*-algebra and suppose a state ρ is definite on a self-adjoint $A \in \mathcal{U}_0$. Then $A - \rho(A)I$ is in the kernel of ρ and $\rho(AB) = \rho(BA)$ for each $B \in \mathcal{U}$.

Proof

The proof will follow by direct computation.

Consider $\rho((A - \rho(A)I)^2)$

$$= \rho(A)^2 - 2\rho(A)^2 + \rho(A)^2$$

$$= 0.$$

Thus $\rho((A - \rho(A)I)^2) = 0$ and since $\rho(A - \rho(A)I)$ is real it is clear that $\rho(A - \rho(A)I) = 0$.

Now if $B \in \mathcal{U}$, then it follows from Lemma 3.10 that $\rho(B(A - \rho(A)I)) = \rho((A-\rho(A)I)B) = 0$ from which it is clear that $\rho(BA) = \rho(B)\rho(A) = \rho(AB)$.

The following important fact will be used in the next proposition.

Lemma 3.13

For all T and S in $\mathcal{B}(H)$ and each projection E it follows that $\|ET(I - E) + (I - E) SE\| = \max \{\|ET(I - E)\|; \|(I - E)SE\|\}$.

Proof

Let x be a unit vector in H, then

$$\begin{aligned}
&\|ET(I - E)x + (I - E)SEx\|^2 \\
&= \|ET(I - E)x\|^2 + \|(I - E)SEx\|^2 \quad (E \text{ and } I - E \text{ are orthogonal}). \\
&\leq \|ET(I - E)\|^2 \|(I - E)x\|^2 + \|(I - E)SE\|^2 \|Ex\|^2 \\
&\qquad\qquad (I - E \text{ and } E \text{ are idempotent}) \\
&\leq \max \{\|ET(I - E)\|^2, \|(I - E)SE\|^2\} \\
&\qquad\qquad \text{since } \|(I - E)x\|^2 + \|Ex\|^2 = 1.
\end{aligned}$$

On the other hand $\|ET(I - E)\|$

$$\begin{aligned}
&= \sup \{\|ET(I - E)y\| : \|y\| \leq 1\} \\
&= \sup \{\|ET(I - E)z\| : z = (I - E)y : \|z\| \leq 1\} \\
&\leq \|ET(I - E) + (I - E)SE\|
\end{aligned}$$

(since $(I - E)SE(I - E)(H) = (I - E)S(E - E^2)(H) = \{0\}$).

Similarly $\|(I - E)SE\| \leq \|ET(I - E) + (I - E)SE\|$

thus $\max \{\|ET(I - E)\|; \|(I - E)SE\|\}$

$\leq \|ET(I - E) + (I - E)SE\|$ from which equality follows.

Proposition 3.14 ([4], Lemma 5.2)

With the notation and assumptions of Theorem 3.8, let E be a projection in $\bar{\mathcal{B}}^{w-o}$ and x be a unit vector either in $E(H)$ or in $(I - E)H$. Then:

(i) $w_x \cdot \varphi$ is a state of $\bar{\mathcal{U}}^{w-o}$ which is definite on E.

(ii) $E\varphi(EA)E = E\varphi(AE)E = E\varphi(A)E$, $E\varphi(EAE)E = E\varphi(A)E$ and $(I - E)\varphi(EA)(I - E) = (I - E)\varphi(AE)(I - E) = 0$ for each A in $\bar{\mathcal{U}}^{w-o}$.

(iii) $\varphi(EAE) = E\varphi(A)E$ for each A in $\bar{\mathcal{U}}^{w-o}$.

(iv) $\varphi(EA(I - E)) = (I - E)\varphi(EA(I - E))E + E\varphi(EA(I - E))(I - E)$ for each A in $\bar{\mathcal{U}}^{w-o}$.

(v) $(I - E)\varphi(EA(I - E))E = 0$.

(vi) $\varphi(EA) = E\varphi(A)$ and $\varphi(AE) = \varphi(A)E$ for each A in $\bar{\mathcal{U}}^{w-o}$.

Proof

(i) $w_x(T) = \langle Tx,x\rangle$ by definition and since $\varphi(I) = \varphi_0(I) = I$

$(w_x\cdot\varphi)I = w_x(\varphi(I)) = w_x(I) = \langle Ix,x\rangle = \|x\|^2 = 1$.

Let $T \geq 0$, from Lemma 3.8 φ is a positive linear mapping of $\bar{\mathcal{U}}^{w-o}$ onto $\bar{\mathcal{B}}^{w-o}$ so that

$$\begin{aligned} w_x\cdot\varphi(T) &= w_x(\varphi(T)) \\ &= \langle\varphi(T)x,x\rangle \\ &\geq 0 \text{ since } \varphi \text{ is a positive mapping.} \end{aligned}$$

Thus $w_x\cdot\varphi$ is a state of $\bar{\mathcal{U}}^{w-o}$ (cf Appendix, Lemma 1). As $E^2 = E$, the states ρ of $\bar{\mathcal{U}}^{w-o}$ that are definite on E are those such that $\rho(E) = \rho(E^2) = \rho(E)^2$ and that is precisely those that take the value 0 or 1 at E.

Since $E \in \bar{\mathcal{B}}^{w-o}$ and φ is idempotent with range $\bar{\mathcal{B}}^{w-o}$, there exists an $A \in \bar{\mathcal{U}}^{w-o}$ such that $\varphi(A) = E$.

Thus $\varphi(E) = \varphi(\varphi(A)) = \varphi^2(A) = \varphi(A) = E$ and now $(w_x\cdot\varphi)(E) = w_x(\varphi(E)) = w_x(E)$.

When $x \in (I - E)H$, $(w_x\cdot\varphi)(E) = w_x(E)$

$$= \langle Ex,x\rangle$$
$$= 0 \text{ because } x \in (I - E)H$$

and when $x \in E(H)$, $(w_x\cdot\varphi)E = \langle Ex,x\rangle$

but $\langle Ex,x\rangle = \|x\|^2 = 1$ if $x \in E(H)$.

Thus when $x \in E(H)$, $(w_x\cdot\varphi)E = 1$. Thus $w_x\cdot\varphi$ is definite on E when x is a unit vector in either $E(H)$ or $(I - E)H$.

(ii) Since $w_x \cdot \varphi$ is a state on the C^*-algebra $\bar{\mathcal{U}}^{w-o}$ which is definite on E, it follows from the proof of Proposition 3.12 that $w_x \cdot \varphi(EA) = w_x \cdot \varphi(E) \,.\, w_x \cdot \varphi(A)$

$$= w_x(E) \,.\, (w_x \cdot \varphi)(A).$$

Hence for all $x \in E(H)$ one has

$$<\varphi(EA)x,x> = <\varphi(A)x,x>.$$

Now if $x \in H$ arbitrary then $Ex \in E(H)$ and it follows that

$$\begin{aligned} &<E\varphi(EA)Ex,x> \\ &= <\varphi(EA)Ex,Ex> \\ &= <\varphi(A)Ex,Ex> \\ &= <E\varphi(A)Ex,x> \end{aligned}$$

From this it follows that $E\varphi(EA)E = E\varphi(A)E$.

Since $w_x \cdot \varphi(EA) = w_x \cdot \varphi(AE)$ (cf (i) and Proposition 3.12) it is clear that we also have $E\varphi(EA)E = E\varphi(AE)E = E\varphi(A)E$.

With x a unit vector in $(I - E)H$, from $w_x \cdot \varphi(EA) = 0$ we have $<\varphi(EA)x,x> = 0$ and this holds for all $x \in (I - E)H$.

Now $(I - E)\varphi(EA)(I - E) = 0$.

Similarly $(I - E)\varphi(AE)(I - E) = 0$.

Hence $E\varphi(AE)E = E\varphi(A)E$ and $(I - E)\varphi(AE)(I - E) = 0$ for all A in $\bar{\mathcal{U}}^{w-o}$. Thus

$$E\varphi(EAE)E = E\varphi(E(AE))E = E\varphi(AE)E = E\varphi(A)E.$$

(iii) We firstly show that $-\|A\|E \le EAE \le \|A\|E$

$$\begin{aligned} |<EAEx,x>| &= |<AEx,Ex>| \\ &\le \|AEx\| \, \|Ex\| \quad \text{(Cauchy-Schwarz)} \\ &\le \|A\| \, \|Ex\|^2 \\ &= \|A\| <Ex,Ex> \\ &= \|A\| <Ex,x> \\ &= <\|A\|Ex,x> \end{aligned}$$

Thus $<-\|A\|Ex,x> \le <EAEx,x> \le <\|A\|Ex,x>$ and it follows that

$$-\|A\|E \le EAE \le \|A\|E.$$

Since φ is a positive linear mapping we have that

$$-\|A\|E = -\|A\|\varphi(E) \le \varphi(EAE) \le \|A\|\varphi(E) = \|A\|E.$$

For $x \in (I - E)H$ it follows that

$$|<\varphi(EAE)x,x>| \le <\|A\|Ex,x> = 0.$$

Thus $\langle\varphi(EAE)x,x\rangle = 0$ for all $x \in (I - E)H$.
If $x \in H$ arbitrary, then $x = Ex + (I - E)x$
so $\varphi(EAE)x = \varphi(EAE)Ex + 0$ thus $\varphi(EAE) = \varphi(EAE)E$.
Now similarly one has $\varphi(EA^*E) = \varphi(EA^*E)E$ for all $A^* \in \bar{\mathcal{U}}^{w-o}$.

$$\begin{aligned} \text{Thus}\quad \varphi(EAE) = [\varphi(EA^*E)]^* &= [\varphi(EA^*E)E]^* \\ &= E[\varphi(EA^*E)]^* \\ &= E\varphi(EAE). \end{aligned}$$

Thus $\varphi(EAE) = E\varphi(EAE) = E\varphi(EAE)E$.
So that $\varphi(EAE) = E\varphi(EAE)E = E\varphi(A)E$ follows from (ii).

(iv) Since $(I - E)\varphi(EA(I - E))(I - E)$

$$= \varphi(EA(I - E)) - \varphi(EA(I - E))E - E\varphi(EA(I - E))(I - E)$$

we have

$$\varphi(EA(I - E)) = E\varphi(EA(I - E))E + (I - E)\varphi(EA(I - E))E + E\varphi(EA(I - E))(I - E) + (I - E)\varphi(EA(I - E))(I - E).$$

But $E\varphi(EA(I - E))E$

$$= E\varphi(EA)E - E\varphi(EAE)E = E\varphi(A)E - E\varphi(A)E \quad \text{(from (iii))}$$
$$= 0$$

and $(I - E)\varphi(EA(I - E))(I - E)$

$$= (I - E)\varphi(EA)(I - E) - (I - E)\varphi(EAE)(I - E) \quad (\varphi \text{ linear}).$$

But $(I - E)\varphi(EA)(I - E) = 0$ (from (ii)).
Thus $(I - E)\varphi(EA(I - E))(I - E)$

$$\begin{aligned} &= -(I - E)\varphi(EAE)(I - E) \\ &= -\varphi(EAE) + \varphi(EAE)E + E\varphi(EAE) - E\varphi(EAE)E \\ &= -E\varphi(A)E + E\varphi(A)E + E\varphi(A)E - E\varphi(A)E \\ &= 0. \end{aligned}$$

Thus $\varphi(EA(I - E))$

$$= (I - E)\varphi(EA(I - E))E + E\varphi(EA(I - E))(I - E).$$

(v) Suppose $(I - E)\varphi(EA(I - E))E \neq 0$, then for all large positive integers n, $\|E\varphi(EA(I - E))(I - E)\|$

$$\leq \|n(I - E)\varphi(EA(I - E))E\|.$$

Hence $n\|(I - E)\varphi(EA(I - E))E\|$

$$\begin{aligned} &= \max\{\|n(I - E)\varphi(EA(I - E))E\|; \|E\varphi(EA(I - E))(I - E)\|\} \\ &= \|E\varphi(EA(I - E))(I - E) + n(I - E)\varphi(EA(I - E))E\| \end{aligned}$$

[cf Lemma 3.13]

$$= \|E\varphi(EA(I - E))(I - E) + (I - E)\varphi(EA(I - E))E + (n - 1)(I - E)\varphi(EA(I - E))E\|$$

$$= \|\varphi(EA(I - E)) + (n - 1)(I - E)\varphi(EA(I - E))E\| \quad \text{(from (iv))}$$

$$= \|\varphi(EA(I - E)) + (n - 1)\varphi(I - E)\varphi^2(EA(I - E))\varphi(E)\| \quad \text{(cf Remark 3.9)}$$

$$= \|\varphi[EA(I - E) + (n - 1)(I - E)\varphi(EA(I - E))E]\| \quad (\varphi \text{ is linear})$$

$$\leq \|EA(I - E) + (n - 1)(I - E)\varphi(EA(I - E))E\| \quad (\|\varphi\| \leq 1)$$

$$= \max\{\|EA(I - E)\|; \ \|(n - 1)(I - E)\varphi(EA(I - E))E\|\}$$

$$= (n - 1)\ \|(I - E)\varphi(EA(I - E))E\| \quad \text{since } EA(I - E) = 0$$

(recall for n large enough $\|EA(I - E)\| \leq \|(n - 1)(I - E)\varphi(EA(I - E))E\|$).

This is a contradiction since n is a positive integer.

Thus $(I - E)\varphi(EA(I - E))E = 0$.

(vi) From (iv) and (v) we have

$$\varphi(EA(I - E)) = E\varphi(EA(I - E))(I - E).$$

Thus for each A in $\bar{\mathcal{U}}^{W-O}$ we have from (iii) that

$$\varphi(A) = \varphi(EAE) + \varphi(EA(I - E)) + \varphi((I - E)AE) + \varphi((I - E)A(I - E))$$

$$= E\varphi(EAE)E + E\varphi(EA(I - E))(I - E) + (I - E)\varphi((I - E)AE)E + (I - E)\varphi((I - E)A(I - E))(I - E)$$

so that

$$E\varphi(A) = E^2\varphi(EAE)E + E^2\varphi(EA(I - E))(I - E) + 0 + 0$$

$$= E\varphi(EAE)E + E\varphi(EA(I - E))(I - E)$$

$$= \varphi(EAE) + \varphi(EA(I - E))$$

$$= \varphi(EAE) + \varphi(EA) - \varphi(EAE)$$

$$= \varphi(EA).$$

Since $[\varphi(A)]^* = \varphi(A^*)$ and $E\varphi(A) = \varphi(EA)$ it is clear that $E\varphi(A^*) = \varphi(EA^*)$ and so $[E\varphi(A^*)]^* = [\varphi(EA^*)]^*$.

Thus $\varphi(A)E = \varphi(AE)$.

Theorem 3.15 ([4], Theorem 5.3)

With the notations and assumptions of Theorem 3.8, φ_0 from $\mathcal{U}$ onto $\mathcal{B}$ and φ from $\bar{\mathcal{U}}^{W-O}$ onto $\bar{\mathcal{B}}^{W-O}$ are conditional expectations.

Proof

From Theorem 3.8, φ is a positive linear mapping from $\bar{\mathcal{U}}^{w-o}$ onto $\bar{\mathcal{B}}^{w-o}$ and $\varphi(I) = \varphi_0(I) = 1$. Since φ_0 maps $\mathcal{U}$ onto $\mathcal{B}$ and is the restriction of φ to $\mathcal{U}$, it will follow that φ_0 is a conditional expectation from $\mathcal{U}$ onto $\mathcal{B}$ if we can prove that φ is a conditional expectation from $\bar{\mathcal{U}}^{w-o}$ onto $\bar{\mathcal{B}}^{w-o}$.

We now have to show that $\varphi(BA) = B\varphi(A)$ and $\varphi(AB) = \varphi(A)B$ for each A in $\bar{\mathcal{U}}^{w-o}$ and B in $\bar{\mathcal{B}}^{w-o}$. First let B be a self-adjoint element in the von Neumann algebra $\bar{\mathcal{B}}^{w-o}$. Let $\epsilon > 0$ be given; from the spectral theory there is a finite orthogonal family $\{E_1, E_2, \ldots, E_n\}$ of projections in $\bar{\mathcal{B}}^{w-o}$ and real scalars $a_1, a_2, \ldots, a_n$ such that

$$\|B - \sum_{j=1}^{n} a_j E_j\| < \epsilon/(2\|A\|) \quad \text{with } A \text{ in } \bar{\mathcal{U}}^{w-o}.$$

It now follows that

$$\begin{aligned}
&\|\varphi(BA) - B\varphi(A)\| \\
&\leq \|\varphi(BA) - \varphi((\sum_{j=1}^{n} a_j E_j)A)\| + \|\varphi((\sum_{j=1}^{n} a_j E_j)A) - B\varphi(A)\| \\
&\leq \|BA - (\sum_{j=1}^{n} a_j E_j)A\| + \|(\sum_{j=1}^{n} a_j E_j)\varphi(A) - B\varphi(A)\| \\
&\qquad\qquad (\text{from } \|\varphi\| = 1 \text{ and Proposition 3.14(vi)}) \\
&\leq \|(B - \sum_{j=1}^{n} a_j E_j)A\| + \|((\sum_{j=1}^{n} a_j E_j) - B)\varphi(A)\| \\
&< \frac{\epsilon}{2\|A\|}\|A\| + \frac{\epsilon}{2\|A\|}\|\varphi(A)\| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2\|A\|}\|A\| \qquad (\|\varphi(A)\| \leq \|A\|) \\
&= \epsilon.
\end{aligned}$$

Thus $\varphi(BA) = B\varphi(A)$ and similarly we can prove that $\varphi(AB) = \varphi(A)B$. Since any $B \in \bar{\mathcal{B}}^{w-o}$ is a linear combination of self-adjoint elements in $\bar{\mathcal{B}}^{w-o}$, the general case follows.

Theorem 3.16 ([4], Theorem 5.4)

Suppose the C^*-algebra $\mathcal{U}$ is the norm dual of a Banach space $\mathcal{U}_*$. η is the natural injection of $\mathcal{U}_*$ into $\mathcal{U}^*(= (\mathcal{U}_*)^{**})$ and $\mathcal{U}$ acting on H is the universal representation of $\mathcal{U}$.

(i) If ν is an element of $\mathcal{U}^{**}$, then $\nu\cdot\eta = A$ for a unique A in $\mathcal{U}$.

(ii) Let $A \longrightarrow \hat{A}$ be the natural isometric isomorphism between $\bar{\mathcal{U}}^{w-o}$ and $\mathcal{U}^{**}$ obtained in Proposition 3.3. Let $\varphi(A)$ be the unique element obtained in (i) such that $\hat{A}\cdot\eta = \varphi(A)$. Then φ is a conditional expectation from $\bar{\mathcal{U}}^{w-o}$ onto $\mathcal{U}$.

(iii) If $\mathcal{K} = \varphi^{-1}(0)$ then $\mathcal{K}$ is a weak operator closed two-sided ideal in $\bar{\mathcal{U}}^{w-o}$ and $\mathcal{K} = \bar{\mathcal{U}}^{w-o}P$ for some central projection P in $\bar{\mathcal{U}}^{w-o}$.

(iv) $\bar{\mathcal{U}}^{w-o}(I - P) = \mathcal{U}(I - P)$, hence $\mathcal{U}(I - P)$ is a von Neumann algebra.

(v) $\mathcal{U}$ is *-isomorphic to the von Neuman algebra $\mathcal{U}(I - P)$.

Proof

(i) Suppose $\xi \in (\mathcal{U}_*)_1$ then since η is an isometry

$$\begin{aligned}\|(\nu\cdot\eta)\xi\| &\le \|\nu\|\ \|\eta(\xi)\| \\ &= \|\nu\|\ \|\xi\| \\ &\le \|\nu\| \quad (\|\xi\| \le 1).\end{aligned}$$

Hence $\nu\cdot\eta$ is a bounded linear functional on $\mathcal{U}_*$. By assumption $\mathcal{U}$ is the norm dual of $\mathcal{U}_*$. Thus there exists a unique A in $\mathcal{U}$ such that $\nu\cdot\eta = A$.

(ii) Suppose that $A \in \mathcal{U}$. We show that $\varphi(A) = A$. Choose $\xi \in \mathcal{U}_*$, then

$$\begin{aligned}\varphi(A)(\xi) &= (\hat{A}\cdot\eta)(\xi)\\ &= \hat{A}(\eta(\xi))\\ &= \eta(\xi)(A)\\ &= A(\xi) \quad (\mathcal{U} = \mathcal{U}_*{}^*).\end{aligned}$$

Hence $\varphi(A) = A$ for each $A \in \mathcal{U}$. Then clearly φ is a linear mapping from $\bar{\mathcal{U}}^{w-o}$ onto $\mathcal{U}$. φ is idempotent for if $A \in \bar{\mathcal{U}}^{w-o}$, $\varphi(A) \in \mathcal{U}$ and it follows that $\varphi(\varphi(A)) = \varphi(A)$.
We now show that $\|\varphi\| = 1$:
If $B \in (\bar{\mathcal{U}}^{w-o})_1$, then $B \in (\mathcal{U}^{**})_1$ and

$$\begin{aligned}\|\varphi(B)(\xi)\| &= \|(\hat{B}\cdot\eta)(\xi)\|\\ &= \|\eta(\xi)B\|\\ &\leq \|\eta(\xi)\|\ \|B\|\\ &\leq \|\xi\|.\end{aligned}$$

Hence $\|\varphi(B)\| \leq 1$ for all $B \in (\mathcal{U}^{**})_1$. This implies that $\|\varphi\| \leq 1$. Since $\varphi(I) = 1$ it follows that $\|\varphi\| = 1$. Theorem 3.15 now implies that φ is a conditional expectation for $\bar{\mathcal{U}}^{w-o}$ onto $\mathcal{U}$.

(iii) We first show that $\mathcal{K}$ is weak operator closed. We know that $A \in \mathcal{K}$ if and only if $\varphi(A) = 0$, thus if and only if $(\hat{A}\cdot\eta)(\xi) = 0$ for all ξ in U_*, hence if and only if $\eta(\xi)(A) = 0$ for all $\xi \in \mathcal{U}_*$. Now $\eta(\xi) \in \mathcal{U}^*$ and $\mathcal{U}$ acting on H is the universal representation of $\mathcal{U}$, so that there are vectors $x(\xi)$ and $y(\xi)$ in H such that $\eta(\xi) = w_{x(\xi),y(\xi)}|\mathcal{U}$. Thus $A \in \mathcal{K}$ if and only if $w_{x(\xi),y(\xi)}(A) = 0$ for all ξ in $\mathcal{U}_*$. Since the null space of $w_{x(\xi),y(\xi)}$ is weak operator closed, it follows that $\mathcal{K}$ is weak operator closed (note that $w_{x(\xi),y(\xi)}$ is weak operator continuous). Since φ is a conditional expectation form $\bar{\mathcal{U}}^{w-o}$ onto $\mathcal{U}$, $\varphi(BAC) = B\varphi(A)C$ for each A in $\bar{\mathcal{U}}^{w-o}$ and B, C in $\mathcal{U}$. Thus if $A \in \mathcal{K}$, $0 = B\varphi(A)C = \varphi(BAC)$ and thus $BAC \in \mathcal{K}$.
By weak operator continuity of left (and then right) multiplication $BAC \in \mathcal{K}$ for $B, C \in \bar{\mathcal{U}}^{w-o}$ and $A \in \mathcal{K}$.

Hence $\mathcal{K}$ is a weak operator closed two-sided ideal in $\bar{\mathcal{U}}$. From Lemma 1.19 there is central projection P in $\bar{\mathcal{U}}$ such that $\mathcal{K} = \bar{\mathcal{U}}^{w-o}P$.

(iv) Since φ is idempotent, $\varphi(A - \varphi(A)) = \varphi(A) - \varphi(\varphi(A))$
$= \varphi(A) - \varphi(A) = 0.$

Thus $A - \varphi(A) \in \mathcal{K}$ for each A in $\bar{\mathcal{U}}^{w-o}$.
Now $A - \varphi(A) \in \bar{\mathcal{U}}^{w-o}P$, say $A - \varphi(A) = SP$ with $S \in \bar{\mathcal{U}}^{w-o}$ then $(A - \varphi(A))P = SPP = SP = A - \varphi(A)$ (P a projection).
Thus from $A - \varphi(A) = (A - \varphi(A))P$ it follows that

$$A - AP = \varphi(A) - \varphi(A)P$$
$$A(I - P) = \varphi(A)(I - P) \in \mathcal{U}(I - P).$$

Hence $\bar{\mathcal{U}}^{w-o}(I - P) = \mathcal{U}(I - P)$.

(v) If $A \in \mathcal{U}$ and $0 \neq A$ then $A \notin \mathcal{K}$ (since $A = \varphi(A) \neq 0$).
Thus $A \notin \bar{\mathcal{U}}^{w-o}P$ hence $A \neq AP$ and $A - AP \neq 0$.
Since P is a central projection it commutes with $\mathcal{U}$ and the mapping $A \longrightarrow A(I - P)$ of $\mathcal{U}$ onto $\mathcal{U}(I - P)$ is a *-homomorphism

$$([A(I - P)]^* = (I - P)A^* = A^*(I - P) \in \mathcal{U}(I - P))$$

and since it follows from $A \neq 0$ that $A - AP \neq 0$, the mapping is one-to-one and hence a *-isomorphism from $\mathcal{U}$ onto $\mathcal{U}(I - P)$. From (iv) we know that

$$\bar{\mathcal{U}}^{w-o}(I - P) = \mathcal{U}(I - P).$$

Thus $\mathcal{U}$ is *-isomorphic to the von Neumann algebra $\mathcal{U}(I - P)$ acting on $(I - P)(H)$.

From this result and the fact that $(\ell^1)^* = \ell^\infty$ ([7], Example 2.10-6) we now have that ℓ^∞ is *-isomorphic to a von Neumann algebra. Hence any C^*-algebra with a predual will be *-isomorphic to a von Neumann algebra. Hence with this more abstract characterization it is much easier to prove that certain C^*-algebras are isomorphic to von Neumann algebras.

APPENDIX

LOCALLY CONVEX TOPOLOGIES ON A VON NEUMANN ALGEBRA (cf [11])

Let $\mathcal{A}$ be a von Neumann algebra i.e. $\mathcal{A}$ is a *-subalgebra of $\mathcal{B}(H)$, containing an identity $I \in \mathcal{A}$ such that $\mathcal{A} = \mathcal{A}''$. As stated in Chapter 1, this is the equivalent of saying that $\mathcal{A}$ is a *-subalgebra of $\mathcal{B}(H)$ which is closed in the weak operator topology on $\mathcal{B}(H)$ (the double commutant theorem). The weak operator topology on $\mathcal{A}$ is the topology generated by the family of seminorms

$$T \in \mathcal{A} \longrightarrow |(Tx,y)| \quad x,y \in H.$$

If $\mathcal{A}_\sim$ is the linear hull of the set of all weak operator continuous functionals on $\mathcal{A}$, then this weak operator topology is nothing but the $\sigma(\mathcal{A},\mathcal{A}_\sim)$-topology. The strong operator topology on $\mathcal{A}$ is the locally convex topology determined by the family of seminorms

$$T \in \mathcal{A} \longrightarrow \|Tx\| \quad x \in H.$$

The σ-weak operator topology on $\mathcal{A}$ is the locally convex topology determined by the family of seminorms

$$T \in \mathcal{A} \longrightarrow \sum_{n=1}^{\infty} (Tx_n,y_n) \quad \text{where } \sum_{n=1}^{\infty} \|x_n\|^2 < +\infty \text{ and } \sum_{n=1}^{\infty} \|y_n\|^2 < +\infty.$$

Let $\mathcal{A}_*$ be the set of all σ-weak continuous linear functionals on $\mathcal{A}$. It can be shown that every $f \in A_*$ is of the form $f(T) = \sum_{n=1}^{\infty} (Tx_n,y_n)$ for some sequences (x_n), $(y_n) \subseteq H$ with $\sum_{n=1}^{\infty} \|x_n\|^2 < +\infty$ and $\sum_{n=1}^{\infty} \|y_n\|^2 < +\infty$ and that the σ-weak operator topology on $\mathcal{A}$ is exactly the $\sigma(\mathcal{A},\mathcal{A}_*)$ topology on $\mathcal{A}$. The locally convex topology determined by the family of seminorms

$$T \in \mathcal{A} \longrightarrow (\sum_{n=1}^{\infty} \|Tx_n\|^2)^{\frac{1}{2}}, \quad \sum_{n=1}^{\infty} \|x_n\|^2 < +\infty$$

where (x_n) is a sequence in H, is called the σ-strong operator topology on $\mathcal{A}$ The topology given by the norm $\|T\|$ is called the norm topology on $\mathcal{A}$. If "<" means the left-hand side is finer than the right-hand side, the relation between these various operator topologies defined on $\mathcal{A}$ is as follows:

$$\begin{array}{ccccc} \text{norm} & < & \sigma\text{-strong} & < & \sigma\text{-weak} \\ & & \wedge & & \wedge \\ & & \text{strong} & < & \text{weak} \end{array}$$

It can be shown that the σ-strong and strong (resp. σ-weak and weak) operator topologies coincide on bounded parts of $\mathcal{A}$. Consider $\mathcal{A}_*$ and $\mathcal{A}_\sim$ as defined above. Then, by using the general duality theory of Banach spaces it can be shown that $\mathcal{A}_*$ is a closed subspace of the conjugate space $\mathcal{A}^*$ of $\mathcal{A}$ and $\mathcal{A}_\sim$ is dense in $\mathcal{A}_*$ with respect to the norm topology. Furthermore, $\mathcal{A}$ is isometrically isomorphic to the conjugate space of the Banach space $\mathcal{A}_*$ under the natural correspondence $T \in \mathcal{A} \longrightarrow \hat{T} \in (\mathcal{A}_*)^*$ where $\hat{T}(\omega) = \omega(T)$ for every $\omega \in \mathcal{A}_*$. We call $\mathcal{A}_*$ the predual of $\mathcal{A}$. Since $\mathcal{A}$ is a convex subset of $\mathcal{B}(H)$ there is a well-known result in the duality theory of Banach spaces from which it follows that the closures of $\mathcal{A}$ in all these locally convex topologies are the same. Since $\mathcal{A}$ is weak operator closed, it is closed in all these locally convex topologies on $\mathcal{A}$. For the proof of these statements we refer to ([2], Sections 3.1 to 3.4) and [12].

One merit of all the locally convex topologies defined above, is that multiplication is separately continuous. This means that the mappings $T \in \mathcal{A} \longrightarrow TS \in \mathcal{A}$, $T \in \mathcal{A} \longrightarrow ST \in \mathcal{A}$ are continuous for every $S \in \mathcal{A}$. We show this for the weak operator topology on $\mathcal{A}$ (the proofs for the others are similar). If $T_\alpha \longrightarrow 0$ weakly, one has that $|(T_\alpha x,y)| \longrightarrow 0$ for every $x,y \in H$ ($\{T_\alpha\}$ a net in $\mathcal{A}$). Thus $|(T_\alpha x,S^*y)| \longrightarrow 0$ for every $x,S^*y \in H$. Hence $|(ST_\alpha x,y)| \longrightarrow 0$ for every $x,y \in H$. This proves that $ST_\alpha \longrightarrow 0$ weakly. The same procedure is used to show that $T \in \mathcal{A} \longrightarrow TS \in \mathcal{A}$ is weak operator continuous. Another merit of the weak and σ-weak topology on $\mathcal{A}$ is that the mapping $T \in \mathcal{A} \longrightarrow T^* \in \mathcal{A}$ is continuous. The proof of this proceeds as above. This is not true in the strong and σ-strong operator topologies. The following is also true: Multiplication is jointly continuous on bounded parts in the strong operator topology on $\mathcal{A}$. Moreover if $T_\lambda \longrightarrow T$, $S_\lambda \longrightarrow S$ and $\|S_\lambda\| \leq k$ for all λ then the relation

$$\|(S_\lambda T_\lambda - ST)x\| \leq k \|(T_\lambda - T)x\| + \|(S_\lambda - S)Tx\|$$

implies that $(T,S) \in \mathcal{A}x\mathcal{A}^b \longrightarrow TS \in \mathcal{A}$ is continuous where $\mathcal{A}^b$ is a uniformly bounded subset of $\mathcal{A}$.

Lemma 1 ([5], Theorem 4.3.2)

If D is a self-adjoint subspace of a C^*-algebra $\mathcal{U}$ and contains the unit I of $\mathcal{U}$, a linear functional ρ on D is positive if and only if ρ is bounded and $\|\rho\| = \rho(I)$.

Proof

Let ρ be positive, a be a scalar with $|a| = 1$ and A be in D such that $a\rho(A) \geq 0$. Let H be the real part of aA. Then $\|H\| \leq \|A\|$, $H \leq \|H\|I \leq \|A\|I$ and $\|A\|I - H \geq 0$.

Hence $\|A\|\rho(I) - \rho(H) = \rho(\|A\|I - H) \geq 0$,

$$\begin{aligned} \text{thus } |\rho(A)| = \rho(aA) &= \overline{\rho(aA)} \\ &= \rho(\bar{a}A^*) \\ &= \rho(\tfrac{1}{2}(aA + \bar{a}A^*)) \\ &= \rho(H) \\ &\leq \rho(I)\|A\|. \end{aligned}$$

Hence ρ is bounded and $\|\rho\| = \rho(I)$ follows easily.

Conversely if we suppose ρ is bounded with $\|\rho\| = \rho(I)$ we only have to consider the case where $\|\rho\| = 1$. With A in D^+, let $\rho(A) = a + ib$. With $s \geq 0$ and small, $\sigma(I - sA) = \{1 - st : t \in \sigma(A)\} \subseteq [0,1]$. Since $\sigma(A) \subseteq \mathbb{R}^+$, $\|I - sA\| = r(I - sA) \leq 1$ with $r(I - sA)$ the spectral radius of $I - sA$.

$1 - sa \leq |1 - s(a + ib)| = |\rho(I - sA)| \leq 1$, hence $a \geq 0$. With B_n in D defined as $A - aI + inbI$ for each integer n it follows that

$$\|B_n\|^2 = \|B_nB_n^*\| = \|(A - aI + inbI).\ (A - aI - inbI)\|$$
$$= \|(A - aI)^2 + n^2b^2I\|$$
$$\leq \|A - aI\|^2 + n^2b^2$$

since $\rho(B_n) = \rho(A - aI + inbI) = \rho(A) - a + inb$

$$= a + ib - a + inb$$
$$= ib(1 + n)$$

Thus $|\rho(B_n)|^2 = |ib(1 + n)|^2$

$$= |-b^2(1 + 2n + n^2)|$$
$$= b^2(1 + 2n + n^2).$$

Hence $(1 + 2n + n^2)b^2 = |\rho(B_n)|^2 \leq \|A - aI\|^2 + n^2b^2$

and $(1 + 2n)b^2 \leq \|A - aI\|^2$ for $n = 1,2,\ldots$

Thus $b = 0$ and $\rho(A) = a \geq 0$ and consequently ρ is positive.

Lemma 2 ([5], Proposition 5.3.2)

Each continuous real or complex valued function f is strong operator continuous on bounded sets of (self-adjoint or normal) operators on the Hilbert space H.

Proof

We may assume that the bounded set of normal operators under consideration is contained in the ball of radius r. If T_0 is a normal operator in this ball, $\epsilon > 0$ and $x_1,\ldots,x_n$ is a set of vectors in H, we want to find vectors $y_1,\ldots,y_m$ and a positive δ such that $\|(f(T) - f(T_0))x_k\| < \epsilon$ if $\|(T - T_0)y_j\| < \delta$ where T is normal and $\|T\| \leq r$, to prove continuity.

We only have to show this for a single vector x_0, because we can do it for $x_1,\ldots,x_n$ by increasing the y's successively. If we now replace ϵ by $\epsilon\|x_0\|^{-1}$ and $\|(f(T) - f(T_0))\frac{x_0}{\|x_0\|}\| < \frac{\epsilon}{\|x_0\|}$ then $\|(f(T) - f(T_0))x_0\| < \epsilon$, hence we may assume that $\|x_0\| = 1$.

From the Weierstrass approximation theorem ([9], Theorem 36A) there exists a polynomial p in z and $\bar{z}$, such that $\|f - p\|_{D(o,r)} < \frac{\epsilon}{3}$. ($D(0,r)$ is a closed disk in $\mathbb{C}$ with center 0 and radius r). Since multiplication on bounded sets of operators and the adjoint operation on the set of normal operators (cf [5], Remark 3.4.15) is strong operator continuous, there exist vectors $y_1,\ldots,y_m$ and a $\delta > 0$ such that $\|(p(T) - p(T_0))x_0\| < \frac{\epsilon}{3}$ if $\|(T - T_0)y_j\| < \delta$ and T is normal with $\|T\| < r$. We have

$$\begin{aligned}\|(f(T) - f(T_0))x_0\| &\leq \|(f(T) - p(T))x_0\| + \|(p(T) - p(T_0))x_0\| \\ &\quad + \|(p(T_0) - f(T_0))x_0\| \\ &\leq \|f(T) - p(T)\| \, \|x_0\| + \frac{\epsilon}{3} + \|p(T_0) - f(T_0)\| \, \|x_0\| \\ &= \|f(T) - p(T)\| + \frac{\epsilon}{3} + \|p(T_0) - f(T_0)\| \\ &\leq 2\|f - p\|_{D(o,r)} + \frac{\epsilon}{3} < \epsilon.\end{aligned}$$

The fact that both $\|f(T) - p(T)\|$ and $\|p(T_0) - f(T_0)\| \leq \|f - p\|_{D(o,r)}$ follows from the Gelfand-Naimark theorem (cf Lemma 1.8) where T is represented by z and T^* by $\bar{z}$ and $f(T)$ and $p(T)$ are respectively represented by $f|_{\sigma(T)}$ and $p|_{\sigma(T)}$. Since the Gelfand-Naimark representation is an isometry we have $\|f(T) - p(T)\| = \|f - p\|_{\sigma(T)}$.

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CHARACTERIZATION THEOREMS IN VON NEUMAN ALGEBRAS

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SUMMARY

The aim of this thesis is to study the characterization theorems in von Neumann algebras. This class of operator algebras was defined for the first time in 1930 by J von Neumann in terms of a representation on a Hilbert space.

After the studies of Gelfand, Naimark and Segal, von Neumann algebras were defined as *-subalgebras of bounded operators on a Hilbert space which are weak operator closed. Von Neumann himself was intrigued by the question how to characterize von Neumann algebras in a more abstract, hence representation-independent way. By studying the features of von Neumann algebras, Kadison and Sakai almost simultaneously solved this problem in the mid-fifties.

Chapter one contains important results on projections and operators that are needed to prove the characterization theorems later. The well-known spectral theory and a few important facts on Borel calculus are also stated here. By using a theorem of Baire we extend the Gelfand-Naimark *-isomorphism to a *-homomorphism between all the bounded complex Borel functions on the spectrum of an operator T and the von Neumann algebra generated by T and I.

In the second chapter Kadison's characterization is discussed. He proved that a C^*-algebra with the properties that an increasing net of self-adjoint operators bounded from above, has a least upper bound and secondly that the normal states separate the algebra, is a von Neumann algebra. We also show that both these conditions are necessary by giving a counter example in the case where only the first condition is satisfied. Kadison constructed this complicated example.

In the last chapter Sakai's characterization stating that a C^*-algebra $\mathcal{U}$ with a predual is a von Neumann algebra, is discussed. An interesting proof by Tomiyama based on the universal representation and conditional expectations (projections of norm one) is given. We conclude this thesis with results on locally convex topologies and a few lemmas in the Appendix.

KARAKTERISERINGSTELLINGS IN VON NEUMANN ALGEBRAS

deur

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OPSOMMING

Die doel van hierdie verhandeling is om verskillende karakteriseringstellings in von Neumann algebras te bestudeer. Hierdie klas van operator-algebras was vir die eerste keer in 1930 deur J. von Neumann in terme van 'n representasie op 'n Hilbert-ruimte gedefinieer.

Na Gelfand, Naimark en Segal se onderskeie bestuderings van representasies is von Neumann algebras gedefinieer as *-deelalgebras van begrensde operatore op Hilbert-ruimtes wat swak operator geslote is. Von Neumann self was reeds gefasineer met die vraag hoe om von Neumann algebras representasie-onafhanklik, dit wil sê op 'n meer abstrakte manier te karakteriseer. In die middel vyftigs het Kadison en Sakai bykans gelyktydig hierdie probleem opgelos na bestudering van die eienskappe eie aan von Neumann algebras.

Hoofstuk 1 bevat belangrike resultate in verband met projeksies en operatore wat benodig word om die karakteriseringstellings te bewys. Die bekende spektraalstelling en belangrike resultate van Borel-calculus word ook hier weergegee. Met behulp van 'n stelling van Baire brei ons die Gelfand-Naimark *-isomorfisme uit na 'n *-homomorfisme tussen alle begrensde komplekse Borel-funksies op die spektrum van 'n operator T en die von Neumann algebra voortgebring deur T en I.

In die tweede hoofstuk word Kadison se karakterisering bespreek. Hy het bewys dat 'n C^*-algebra met die eienskappe dat 'n toenemende net van selftoegevoegde operatore wat van bo begrens is, 'n kleinste bogrens besit en tweedens dat die normale state die algebra skei, 'n von Neumann algebra is. Ons toon verder aan dat beide hierdie eienskappe noodsaaklik is deur 'n teenvoorbeeld te gee wat nie 'n von Neumann algebra is nie, alhoewel aan die eerste voorwaarde voldoen word. Hierdie ingewikkelde voorbeeld is deur Kadison gekonstrueer.

In die laaste hoofstuk word Sakai se karakterisering bespreek. Sakai definieer 'n von Neumann algebra as 'n C^*-algebra met 'n preduaal. 'n Interessante bewys hiervoor wat gebaseer is op die universele representasie en projeksies van norm een deur Tomiyama word gegee. Die verhandeling word afgesluit met 'n bylae waarin resultate oor lokaal konvekse topologieë en 'n paar lemmas gegee word.