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FACULTY OF NATURAL AND AGRICULTURAL SCIENCES

DEPARTMENT OF STATISTICS

Quantile-based generalized logistic distribution

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ABSTRACT

This dissertation proposes the development of a new quantile-based generalized logistic distribution GLD_{QB}, by using the quantile function of the generalized logistic distribution (GLO) as the basic building block. This four-parameter distribution is highly flexible with respect to distributional shape in that it explains extensive levels of skewness and kurtosis through the inclusion of two shape parameters. The parameter space as well as the distributional shape properties are discussed at length. The distribution is characterized through its L-moments and an estimation algorithm is presented for estimating the distribution's parameters with method of L-moments estimation. This new distribution is then used to fit and approximate the probability of a data set.

Keywords: Generalized logistic distribution, L-moments, quantile function.

DECLARATION

I **BRENDA V. OMACHAR** hereby declare that this dissertation which I submit for the degree Msc in Mathematical Statistics at the University of Pretoria, contains no material that has been previously published or written by another person except where due reference has been made in the text. This dissertation has not been previously submitted for any other qualification at any other tertiary institution.

Signed:

Dated: 03/11/2014

ACKNOWLEDGEMENTS

I would like to thank the Almighty God for the ability to bring this dissertation to completion.

I thank my mother Beatrice for her unconditional love and support in the pursuit of my studies and my brother Alfred who has always been a constant challenge. I also thank you Edgar for your incessant words of encouragement.

I thank my supervisor Dr Paul J. van Staden for his tireless effort, assistance and time that he invested in this work.

I would like to thank the Department of Statistics at the University of Pretoria and the Head of the Department, Prof Andriette Bekker for all assistance provided. I am also grateful for all the financial assistance obtained from the Department, and University of Pretoria.

1 INTRODUCTION

1.1 AIMS AND OBJECTIVES.

The aim of this dissertation is the development of a new quantile-based generalized logistic distribution GLD_{QB}. The methodology used is proved in Proposition 2.7.1 in Chapter 2, where the four-parameter quantile-based distribution is constructed by taking the sum of the quantile function of an asymmetric distribution on half infinite support and the quantile function of the reflected asymmetric distribution. The afore-mentioned methodology is utilized in Definition 4.2.1 in Chapter 4. The distribution's parameters are estimated with method of L-moments estimation.

1.2 COMPOSITION OF DISSERTATION.

Chapter 2 gives an overview of quantile functions highlighting the relationships between the probability-based functions and the quantile-based functions. Different quantile-based functions are defined as well as the various construction rules defined by Gilchrist (2000) and examples of quantile-based distributions are discussed. Descriptive functions of the location, spread and shape of quantile-based distributions through their conventional moments as well as L-moments developed by Hosking (1990) will be illustrated. Various measures of location, spread (MacGillivray & Balanda, 1988) and shape (Bowley, 1902; MacGillivray, 1986; MacGillivray & Balanda, 1988) will be examined.This chapter concludes with the proposition that provides the methodology used in the construction of the proposed quantile-based distribution. The distributional form is outlined whilst making use of the construction rules discussed by Gilchrist (2000).

Chapter 3 lays emphasis on the building block that is used in the construction of the Freimer-Mudholkar-Kollia-Lin (FMKL) Type of the GLD, introduced by Freimer *et al.* (1988). In addition, the building block of the new quantile-based distribution is discussed. In particular, the functions, various measures and distributional properties are considered.

In Chapter 4, Proposition 2.7.1 from Chapter 2 is used to construct the new quantile-based distribution. The distribution is specified in terms of its quantile function in Definition 4.2.1 as well as its quantile-based functions in Eqs. (4.5) and (4.6). Moreover, the distribution is characterized in terms of the L-moments due to their simplicity as compared to the conventional moments. Various quantile-based measures and shape properties are analyzed in this chapter. The tail behavior of the distribution is discussed in this chapter.

Chapter 5 highlights an estimation algorithm for estimating the distribution's parameters with method of L-moments estimation. The new distribution is then used to fit and approximating the probability of a data set.

1.3 CONTRIBUTIONS OF DISSERTATION.

The new contributions of this dissertation are listed below.

Chapter 4:

- A new distribution is constructed and defined using Proposition 2.7.1 of Chapter 2 in Section 4.2.
- Section 4.3 takes and in-depth look at the distributional properties and shape characteristics of the new quantile-based distribution.
- The parameter space and support as well as the classes is defined for this distribution in Section 4.4 and 4.5.
- The expressions of the moments of this distribution are defined and derived in Section 4.6.
- In Section 4.7, the new distribution is characterized through its L-moments.
- The various expressions for the quantile-based measure of location, spread and shape are defined.

- Section 4.9 summarizes the values obtained for the density and the slope of the density curve.

Chapter 5:

- An estimation algorithm for estimating the distribution’s parameters with method of L-moments estimation is presented in Section 5.1.

2 QUANTILE-BASED MODELLING

2.1 INTRODUCTION.

The main aim of the process of statistical modelling, is to find and define various functions and measures that can be used to model the probability laws or distribution of a random variable. Suppose that $X_1, X_2, \ldots \ldots, X_n$ is a set of real-valued random variables. If the data analysis intention is to model the cumulative distribution function, $F(x)$, and the probability density function, $f(x)$, then it is said to be probability-based or probability-centered. In this case, the set of data is assumed to be independent and $F(x)$ is considered continuous.

However, in addition to the probability-based functions, Parzen (1979) discussed alternative functions and measures that can be used to model the distribution of a random variable. In his work, he established a set of functions that depend on the quantiles of a random variable, as well as a discussion on the modelling process through the use of these functions.

In this chapter, an overview of quantiles is discussed in Section 2.2, highlighting the relationships between the probability-based functions and the quantile-based functions. Moreover, different quantile-based functions are defined. Section 2.3 defines the quantile-based distributions as well as the rules that are used to construct these distributional models (Gilchrist, 2000). The section is concluded with examples of these type of distributions that have been modelled.

Section 2.4 describes the conventional moments, both the central and non-central moments. These can be defined in terms of the quantile functions, making them applicable in describing the location, spread and shape of quantile-based distributions.

The L-moments (Hosking, 1990) are defined and illustrated in Section 2.5. As with conventional moments, they can be used to describe and measure the location, spread and shape of a distribution. However, with L-moments the measurement is done through use of order statistics. The L-moment ratios will also be defined. Specifically the L-skewness and L-kurtosis, which are of great importance in explaining the shape of a distribution, will be represented.

In section 2.6, the quantile-based measures of location, spread and shape are highlighted. Various measures of location, spread (MacGillivray & Balanda, 1988) and shape (Bowley, 1902; MacGillivray, 1986; MacGillivray & Balanda, 1988) will be explained.

This chapter concludes with Section 2.7 in which the proposition that provides the methodology used in the construction of the proposed quantile-based distribution is represented. The distributional form is outlined in Section 2.7, making use of the construction rules discussed by Gilchrist (2000).

2.2 QUANTILES.

For a general distribution $F(x)$ which is continuous from the right, a quantile function $Q(p)$ is defined as

$$Q(p) = F^{-1}(p) = \inf\{x: F(x) \geq p\}, \quad 0 < p < 1$$

An important relationship is that, if $-\infty < x < \infty$ and $0 < p < 1$, then

$$F(x) \geq p \; if \; and \; only \; if \; Q(p) \leq x.$$

Provided that X is a continuous random variable, then

$$Q(p) = F^{-1}(p) = \inf\{x: F(x) = p\},$$

implying that the quantile function is equivalent to the inverse of the distribution function.

It is then observed that $F(Q(p)) = p$, a composite function.

It follows from the results above that various quantile-based functions are expressed in terms of the quantile function.

- **Quantile-density function.**

  This function, also referred to as the sparsity function (Tukey, 1965), is acquired by taking the derivative of the quantile function in terms of *p*. It is defined as

$$q(p) = Q'(p) = \frac{dQ(p)}{dp}, \quad 0 < p < 1\,. \tag{2.1}$$

- **Density quantile function.**

  It was earlier stated that one of the consequences of a continuous random variable was that $F(Q(p)) = p$. The density quantile function is obtained by taking derivatives of both sides of the equation. Therefore,

$$\frac{dF\big(Q(p)\big)}{dp} = \frac{dp}{dp} = 1$$

$$\Rightarrow f\big(Q(p)\big)q(p) = 1$$

$$\Rightarrow f\big(Q(p)\big) = \frac{1}{q(p)} \tag{2.2}$$

  The density quantile function is thus the inverse of the quantile density function and can be denoted by $f_p(p)$, since it is the density function expressed in terms of p instead of x.

- **Score function**

  Hájek and Šidák (1967, p19) defined this function as

$$J(p) = -\frac{f'\big(Q(p)\big)}{f\big(Q(p)\big)} = f'\big(Q(p)\big)q(p) \tag{2.3}$$

  Seeing as $\frac{df(Q(p))}{dp} = f'\big(Q(p)\big)q(p)$, it then follows that $J(p) = (fQ)'(p)$.

  In effect, the score function is the derivative of the density quantile function.

2.3 QUANTILE-BASED DISTRIBUTIONS.

There are some real-valued random variables that cannot be defined in terms of probability-based functions and measures. This is because there are no closed form expressions for either the distribution function, $F(x)$, or and the probability density function, $f(x)$.

As a result, they are specified in terms of their quantile function, $Q(p)$, quantile density function ,$q(p)$, and their density quantile function $f_p(p)$.

2.3.1 QUANTILE MODELLING RULES.

The advantage of quantile-based modelling as compared to probability-based modelling is that the models can be constructed through the addition, multiplication and various other transformations of the quantile-based functions. Gilchrist (2000) emphasized certain construction rules that must be adhered to in the modelling process. These rules are implied for both discrete and continuous variables.

The golden rule is that the quantile function of the model, $Q(p)$, should be non-decreasing for $0 < p < 1$. In the discrete case, it can have step increases. In this thesis the focus is on continuous variables.

Two construction rules highlighted by Gilchrist (2000) are discussed below.

- **Reflection rule.**

  If a random variable X has a quantile function $Q(p)$, the reflected quantile function is $-Q(1-p)$, about the line $x = 0$. Therefore $-Q(1-p)$ is the quantile function of $-X$. If $X \in (0, \infty)$ then $-X \in (-\infty, 0)$ and vice versa. It can be concluded that the distributions of $-X$ and X are reflections of one another

- **Addition rule.**

  Let X and Y be two random variables defined by their non-decreasing quantile functions $Q_X(p)$ and $Q_Y(p)$ respectively. Then $Q_Z(p) = Q_X(p) + Q_Y(p)$, is also a non-decreasing and hence the quantile function of the random variable Z.

  Similarly, since the derivative of the sum of two functions will be the sum of the derivatives of the functions, then the quantile density function of Z will be $q_Z(p) = q_X(p) + q_Y(p)$. This rule can be extended to cases with more than two variables.

  Now consider $X \in (0, \infty)$ with quantile function $Q_X(p)$ and $Y \in (-\infty, 0)$ with quantile function $Q_Y(p) = -Q_X(1-p)$. That is, let X and Y have distributions that are reflective of each other. Then $Q_Z(p) = Q_X(p) + Q_Y(p) = Q_X(p) - Q_X(1-p) = -Q_Y(1-p) + Q_Y(p) = -Q_Z(1-p)$ and the distribution of Z is symmetric.

2.3.2 EXAMPLES OF QUANTILE-BASED DISTRIBUTIONS.

Well-known examples of quantile-based distributions include Tukey's lambda distribution (Tukey, 1960), various types of lambda distributions (Ramberg et al., 1979; Freimer et al., 1988; van Staden, 2013) and the Davies distribution (Hankin and Lee, 2006). Of particular interest is the generalized lambda distribution of which the Freimer-Mudholkar-Kollia-Lin type (GLD_{FMKL}) (Freimer et al., 1988) will be of great importance in the modelling procedure.

2.4 MOMENTS.

The moments and the moment ratios are used to describe a distribution in terms of its location, spread and shape.

The uncorrected moments of a given distribution are defined as

$$\boldsymbol{\mu}'_r = \int_0^1 (Q(p))^r \, dp, \tag{2.4}$$

where

$$\boldsymbol{\mu_1'} = \int_0^1 Q(p)\,dp \tag{2.5}$$

is the mean of the distribution, in effect $\mu_1' = \mu$.

In the same manner, the corrected or central moments are defined in terms of the quantile function as

$$\boldsymbol{\mu_r} = \int_0^1 (Q(p) - \mu)^r\,dp, \tag{2.6}$$

where $\mu_2 = \sigma^2$, the variance of the distribution.

It can be noted that the moments and the moment ratios of a quantile-based distribution are not always easily obtained, since the integrals in Eqs. (2.5) and (2.6) contain polynomial functions of (p) . Often closed-form expressions can only be obtained after extensive simplification.

The skewness and kurtosis moment ratios are defined as $\alpha_3 = \frac{\mu_3}{\sigma^3}$ and $\alpha_4 = \frac{\mu_4}{\sigma^4}$. These two functions are used to describe the shape of the distribution.

2.5 ***L*-MOMENTS.**

L-moments as defined by Hosking (1990), are expectations of linear combinations of order statistics. They summarize the properties of a probability distribution in terms of location, scale and shape.

Suppose that X is a real-valued random variable with a cumulative distribution function $F(x)$ and quantile function $Q(p)$. Let $X_{1:n} \leq X_{2:n} \leq \cdots X_{n:n}$ be the order statistics of a random sample of size *n*.

The L-moments can be defined in terms of the order statistics as

$$L_r = r^{-1} \sum_{k=0}^{r-1} \left((-1)^k \binom{r-1}{k} E(X_{r-k:r}) \right). \tag{2.7}$$

However, Hosking (1990) compiled results in terms of the quantile function by making use of the definition of expectation of an order statistic, David (1981), given as

$$E(X_{j:r}) = \frac{r!}{(j-1)!(r-j)!}\int_0^1 x\big(F(x)\big)^{j-1}\big(1-F(x)\big)^{r-j}dF(x)\ . \tag{2.8}$$

Lemma 2.5.1. *Hosking (1990). The r^{th} order L-moment can be obtained in terms of its quantile function as*

$$L_r = \int_0^1 Q(p)P_{r-1}^*(p)dp, \tag{2.9}$$

where

$$P_r^*(p) = \sum_{k=0}^{r}\left((-1)^{r-k}\binom{r}{k}\binom{r+k}{k}p^k\right) \tag{2.10}$$

is the r^{th} order shifted Legendre polynomial.

Note that $P_r^*(p) = (-1)^r P_r^*(1-p)$ *and* $P_0^*(p) = 1$, *while* $\int_0^1 P_r^*(p)dp = 0$ *for r>0.* ■

All the L-moments of a real-valued random variable exist if and only if the random variable has a finite mean. The distribution is then uniquely defined by the L-moments (Hosking, 1990).

The first order L-moment, L_1, is referred to as the L-location. It is equivalent to the mean. L_2 is the second order L-moment. It is referred to as the L-scale, as it is a measure of scale.

As shown in Gilchrist (2000), the L-moments of an order greater than two are usually susceptible to large variability. For this reason they are transformed into L-moment ratios so that they are independent of the measurement units of the random variable. Therefore

$$\tau_r = \frac{L_r}{L_2}\ for\ r > 2\ , \tag{2.11}$$

where τ_3 is the L-skewness and τ_4 is the L-kurtosis. These two quantities are measures of skewness and kurtosis respectively and are bounded by the constraints

$$-1 < \tau_3 < 1 \text{ and } \frac{1}{4}(5{\tau_3}^2 - 1) \leq \tau_4 < 1, \tag{2.12}$$

proven by Hosking (1990) and Jones (2004).

2.6 QUANTILE-BASED MEASURES OF LOCATION, SPREAD AND SHAPE.

These are functions that are based on the quantiles of a distribution. Unlike the conventional moments or the L-moments, they always exist for all parameter values of a distribution. Various measures of location, spread and shape will be considered.

Location.

Given the property of robustness that is upheld by the median, it is considered an appropriate measure of location. The median is defined as

$$me = Q\left(\frac{1}{2}\right). \tag{2.13}$$

Spread.

MacGillivray & Balanda (1988) showed that the spread function can be defined as

$$S(u) = Q(u) - Q(1-u) \;\; for \;\; \frac{1}{2} < u < 1. \tag{2.14}$$

Since $u > 1-u$ it follows that $Q(u) > Q(1-u)$. As a result, $S(u) > 0$, proving that it is a valid measure of spread.

Special cases of the spread function include the inter-quartile range (IQR) and the inter-decile range, for which the values of u are $\frac{3}{4}$ and $\frac{9}{10}$ respectively.

Shape.

- $\boldsymbol{\gamma - Functional}$

  The first functional is an asymmetry functional that was defined in MacGillivray (1986) as

$$\gamma(u) = \frac{Q(u)+Q(1-u)-2Q\left(\frac{1}{2}\right)}{Q(u)-Q(1-u)} = \frac{Q(u)+Q(1-u)-2me}{S(u)} \quad , \quad \frac{1}{2} < u < 1 \qquad . \tag{2.15}$$

As can be seen, the functional is a function of the difference between the quantile function evaluated at u and $1-u$, and twice the median in the numerator. It is however scaled by the spread function in the denominator. As the numerator difference increases, the functional value increases. As a result, King & MacGillivray (2007, Theorem 6) showed that the γ-functional is bounded by -1 and 1. A special case of $\gamma(u)$ is Bowley's quartile-based measure of skewness (Bowley, 1902). This is obtained by setting $u=\frac{3}{4}$ in Eq. (2.15).

- **Ratio-of-spread functions.**

As introduced by MacGillivray & Balanda (1988), the ratio-of-spread functions is a measure of kurtosis. It is used to describe the position of the probability mass in the tails of the distribution. This is measured for any pairs of values u and v. This function is denoted as

$$R(u,v)=\frac{S(u)}{S(v)} \quad , \quad \frac{1}{2}<u<1\,. \tag{2.16}$$

Since $S(u)>0$ for $\frac{1}{2}<u<1$, it follows that Eqs. (2.15) and (2.16) are also positive.

2.7 PROPOSITION: QUANTILE-BASED DISTRIBUTIONS.

This proposition provides the procedure for the formation of four-parameter quantile-based families of distributions.

PROPOSITION 2.7.1.

Suppose that X is a real-valued random variable that is asymmetric with a bounded or half-infinite support. Let $Q_{X:0}(p)$ denote the quantile function of the standard distribution of X, with the shape parameter, λ, set as λ_3.

Consider $Y=-X$, such that the standard distribution of Y is simply the reflection of the standard distribution of X, about the line $x=0$.

Notably, the quantile function of Y is $Q_{Y:0}(p)=-Q_{X:0}(1-p)$, where the shape parameter λ, is set as λ_4.

A random variable Z that is characterized by the quantile function with the form

$$Q_Z(p) = \alpha + \frac{1}{\beta}\left(-Q_{X:0}(1-p) + Q_{X:0}(p)\right), \tag{2.17}$$

where α is a location parameter, $\beta > 0$ is a scale parameter, λ_3 and λ_4 are shape parameters, is a quantile-based distribution. ■

<u>Example 2.7.1.</u>

Assume that X has a standard generalized Pareto distribution (GPD) with quantile function given as $Q_{X:0}(p) = \frac{1-(1-p)^{\lambda}}{\lambda}$, and support given as $[0,\infty)$ for $\lambda \leq 0$ and $\left[0,\frac{\beta}{\lambda}\right)$ for $\lambda > 0$.

As a result of Proposition 2.7.1, $Y = -X$ has a standard reflected GPD with quantile function given by $Q_{Y:0}(p) = -Q_{X:0}(1-p) = \frac{p^{\lambda}-1}{\lambda}$. Let $\lambda = \lambda_3$ denote the shape parameter of the GPD, and let $\lambda = \lambda_4$ denote the shape parameter of the reflected GPD.

Then the quantile function of Z can be constructed by taking the sum of $Q_{Y:0}(p)$ and $Q_{X:0}(p)$ as stated in the proposition, and further modifying the result by adding a location parameter and a scale parameter as in Eq. (2.17).

Therefore,

$$Q_Z(p) = \alpha + \frac{1}{\beta}\left(\frac{p^{\lambda_3}-1}{\lambda_3} - \frac{(1-p)^{\lambda_4}-1}{\lambda_4}\right). \tag{2.18}$$

The result in Eq. (2.18) is the Freimer-Mudholkar-Kollia-Lin (FMKL) Type of the GLD that was introduced by Freimer *et al.* (1988) and denoted as GLD_{FMKL}, with $\alpha = \lambda_1$ and $\beta = \lambda_2$.

Although not shown by Freimer *et al.* (1988), it is noted that the quantile function of the standard generalized Pareto distribution is used as the building block in constructing the quantile function of the GLD_{FMKL}. Note that if $\lambda_3 = \lambda_4$, then the GLD_{FMKL} simplifies to the symmetric Tukey lambda distribution with single shape parameter λ. The GLD_{FMKL} has no closed-form expression for its cumulative distribution function $F(x)$ or its probability density function $f(x)$. It is therefore defined by its quantile function. See King (1999) and van Staden (2013) for detailed discussions on the GLD_{FMKL}.

3 DESCRIPTION OF THE BUILDING BLOCKS

3.1 INTRODUCTION.

This chapter lays emphasis on the building blocks that will be used in the modelling process of the proposed model. The distributional properties and functions will be defined and studied.

Section 3.2 examines the family of distributions from which the proposed model will obtain its form. This family is the generalized lambda distribution, from which a specific type will be considered for the construction process.

The Freimer-Mudholkar-Kollia-Lin (FMKL) Type of the GLD was introduced by Freimer *et al.* (1988). In particular, the construction form for this specific distribution is noted in Section 3.3. The building block for this model, which is the generalized Pareto distribution (Pickands, 1975), is documented in Section 3.4.

An analysis of the various functions that characterize the distribution, both the probability-based functions as well as the quantile-based functions are expressed. In addition, the conventional moments and the L-moments expressions are given as well.

Similarly, the building block for the proposed model is also given an in-depth look in the final section of this chapter. The generalized logistic distribution (Hosking, 1986, 1990) is detailed in terms of the probability-based functions as well as the quantile-based functions.

Likewise, the conventional moments and the L-moments expressions are given.

3.2 GENERALIZED LAMBDA DISTRIBUTION.

The generalized lambda distribution (GLD) was originally proposed by Ramberg & Schmeiser (1972, 1974) as a generalization of Tukey's one parameter lambda distribution (Tukey, 1960).

The generalization was initiated from a need to generate univariate random variates for Monte-Carlo studies.

This is a very broad family of continuous univariate probability distributions. The GLD is extremely versatile in its shape, and as a result, has been widely used to model a broad range of data.

The GLD is a four parameter distribution, defined through its quantile function. The quantile function of the Ramber-Schmeiser Type of the GLD denoted GLD_{RS}, is

$$Q(p) = \lambda_1 + \frac{p^{\lambda_3} - (1-p)^{\lambda_4}}{\lambda_2} \quad for\ 0 < p < 1\,, \tag{3.1}$$

where λ_1, λ_2, λ_3 and λ_4 are the location, scale and shape parameters respectively.

Karian & Dudewicz (2000, 2010) presented an in-depth study on this family of distributions, including its various functions, probabilistic properties and parameter estimation. Also see the doctoral theses of King (1999) and van Staden (2013) for additional results for the GLD_{RS}.
Some of the areas where the GLD_{RS} has been applied are in actuarial science (Balasooriya & Low, 2008), biochemistry (Ramos-Fernández *et al.*, 2008), computer science (Gautama & van Gemund, 2006), economics (Pacáková & Sipková, 2007), epidemiology (Ferguson *et al.*, 2002; Ghani *et al.*, 2003), forestry (Ivkovic & Rozenberg, 2004), inventory modelling (Lau *et al.*, 2002; Achary & Geetha; 2007), queuing theory (Robinson & Chen, 2003) and signal processing (Karvanen *et al.*, 2002)

3.2.1 FREIMER-MUDHOLKAR-KOLLIA-LIN TYPE (GLD_{FMKL}).

The Freimer-Mudholkar-Kollia-Lin (FMKL) Type of the GLD was introduced by Freimer *et al.* (1988) and denoted as GLD_{FMKL}. As with the GLD_{RS}, this distribution has no closed-form expression for its cumulative distribution function, $F(x)$, and its probability density function, $f(x)$. It is therefore defined by the quantile function

$$Q(p) = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{p^{\lambda_3} - 1}{\lambda_3} - \frac{(1-p)^{\lambda_4} - 1}{\lambda_4}\right)\,. \tag{3.2}$$

The quantile density function and density quantile function of the GLD_{FMKL} are respectively

$$q(p) = \frac{(p^{\lambda_3 - 1} + (1-p)^{\lambda_4 - 1})}{\lambda_2} \quad , \tag{3.3}$$

and

$$f_p(p) = \frac{\lambda_2}{(p^{\lambda_3 - 1} + (1-p)^{\lambda_4 - 1})} \quad . \tag{3.4}$$

Building process.

Quantile modelling as described by Gilchrist (2000) can be done using a set of construction rules that were clearly outlined in Section 2.3.1.

The construction procedure for the GLD_{FMKL} was outlined in Example 2.7.1, utilizing Proposition 2.7.1 that will be used in the modelling process of the new proposed quantile-based logistic distribution in this dissertation.

Although not shown by Freimer *et al.* (1988), it is noted that the quantile function of the standard generalized Pareto distribution (GPD) is used as the building block of quantile function of the GLD_{FMKL}, as clearly shown in the Example 2.7.1.

The standard GPD is defined by the quantile function

$$Q(p) = \frac{1-(1-p)^{\lambda}}{\lambda} \quad for\ 0 < p < 1\,, \tag{3.5}$$

with its reflected counterpart as

$$-Q(1-p) = \frac{p^{\lambda}-1}{\lambda} \quad for\ 0 < p < 1\,. \tag{3.6}$$

The parameter λ is the shape parameter of the GPD. This distribution is studied in Section 3.4, with the distributional properties highlighted.

3.3 GENERALIZED PARETO DISTRIBUTION.

The generalized Pareto distribution (GPD) is a three parameter distribution that was first introduced by Pickands (1975). It has a location, scale and shape parameter. This distribution is used in reliability studies, modelling environment extreme events such as flooding, and in cases where the exponential distribution may be used but in which some robustness is required against heavy or lighter tailed alternatives (Hosking & Wallis, 1987).

3.3.1 Definition and Special cases.

The GPD is characterized by a location parameter, α, scale parameter, $\beta > 0$, and shape parameter, λ. The GPD can be defined in terms of its cumulative distribution function, $F(x)$, probability density function, $f(x)$, quantile function $Q(p)$, quantile density function $q(p)$ and its density quantile function $f_p(p)$.

Definition 3.4.1 *Let X be a real-valued random variable, where* $X = \frac{\alpha(1-\exp(-\lambda y))}{\lambda}$ *and Y is a random variable with a standard exponential distribution. X is said to have a generalized Pareto distribution, denoted by* $X \sim GPD(\alpha, \beta, \lambda)$*, if it is defined by the following functions:*

Cumulative distribution function,

$$F(x) = \begin{cases} 1 - \left(1 - \lambda\left(\frac{x-\alpha}{\beta}\right)\right)^{\frac{1}{\lambda}}, & \lambda \neq 0 \\ 1 - exp\left(-\frac{x-\alpha}{\beta}\right), & \lambda = 0 \end{cases} \tag{3.7}$$

Probability density function,

$$f(x) = \begin{cases} \frac{1}{\beta}\left(1 - \lambda\left(\frac{x-\alpha}{\beta}\right)\right)^{\frac{1}{\lambda}-1}, & \lambda \neq 0 \\ \frac{1}{\beta}\exp\left[-\left(\frac{x-\alpha}{\beta}\right)\right], & \lambda = 0 \end{cases} \tag{3.8}$$

and Quantile function,

$$Q(p) = \begin{cases} \alpha + \frac{\beta}{\lambda}\left(1-(1-p)^{\lambda}\right), & \lambda \neq 0 \\ \alpha - \beta log[1-p] \,, & \lambda = 0 \,, \end{cases} \tag{3.9}$$

with support $[\alpha, \infty)$ for $\lambda \leq 0$ *and* $\left[\alpha, \alpha + \frac{\beta}{\lambda}\right]$ for $\lambda > 0$.

<u>Theorem 3.4.1.</u> *If* $X \sim GPD(\alpha, \beta, \lambda)$, *then its quantile density function is*

$$q(p) = \beta(1-p)^{\lambda-1} \;\; for \; 0 < p < 1 \tag{3.10}$$

and density quantile function is

$$f_p(p) = \frac{(1-p)^{1-\lambda}}{\beta} \; for \; 0 < p < 1 \,. \tag{3.11}$$

<u>Proof:</u> The expressions in Eqs. (3.10) and (3.11) are directly obtained using $q(p) = \frac{dQ(p)}{dp}$ and $f_p(p) = \frac{1}{q(p)}$ ■

Special cases of the GPD occur for the following values of λ:

- The exponential distribution is obtained as a limiting case of the GPD for $\lambda = 0$.
- $\lambda = 1$ results in the GLD reducing to the Uniform distribution with support $[\alpha, \alpha + \beta]$.
- For $\lambda < 0$, this distribution becomes the Pareto distribution.

3.3.2 MOMENTS.

The moments for the GPD are obtained by noting that $E\left[\left(1-\frac{\lambda x}{\alpha}\right)^r\right]=\frac{1}{1+\lambda r}$, if $1+\lambda r>0$. Therefore, the r^{th} moment exists if $>-\frac{1}{r}$. The expressions for the mean, variance, skewness moment ratio and kurtosis moment ratio are presented below.

$$E(x)=\alpha+\frac{\beta}{1+\lambda} \tag{3.12}$$

$$Var(x)=\frac{\beta^2}{(1+2\lambda)(1+\lambda)^2} \tag{3.13}$$

$$Skewness(x)=\frac{2(1-\lambda)(1+2\lambda)^{\frac{1}{2}}}{1+3\lambda} \tag{3.14}$$

$$Kurtosis(x)=\frac{3(1+2\lambda)(3-\lambda+2\lambda^2)}{(1+3\lambda)(1+4\lambda)}-3 \tag{3.15}$$

3.3.3 *L*-MOMENTS.

The L-moments for the GPD are defined for $\lambda>-1$. The first four L-moments are given in Hosking (1986) as follows

$$L_1=\alpha+\frac{\beta}{1+\lambda}, \tag{3.16}$$

$$L_2=\frac{\beta}{(1+\lambda)(2+\lambda)}, \tag{3.17}$$

$$L_3=\frac{\beta(1-\lambda)}{(1+\lambda)(2+\lambda)(3+\lambda)}, \tag{3.18}$$

$$L_4=\frac{\beta(1-\lambda)(2-\lambda)}{(1+\lambda)(2+\lambda)(3+\lambda)(4+\lambda)}, \tag{3.19}$$

with

$$L_r = \frac{\beta\Gamma(1+\lambda)\Gamma(r-1-\lambda)}{\Gamma(1-\lambda)\Gamma(r+1+\lambda)} \quad (3.20)$$

the general expression for the r^{th} order *L*-moment. Hosking (1986) also gave the general formula for the r^{th} order *L*-moment ratio,

$$\tau_r = \frac{\Gamma(3+\lambda)\Gamma(r-1-\lambda)}{\Gamma(1-\lambda)\Gamma(r+1+\lambda)}.$$

In particular, the *L*-skewness and *L*-kurtosis ratios are expressed as

$$\tau_3 = \frac{(1-\lambda)}{(3+\lambda)} \quad \text{and} \quad \tau_4 = \frac{(1-\lambda)(2-\lambda)}{(3+\lambda)(4+\lambda)}, \quad (3.21)$$

respectively.

3.4 GENERALIZED LOGISTIC DISTRIBUTION.

The generalized logistic distribution (GLO) was first studied and documented in Hosking (1986). The GLO is the generalized distribution that is obtained from the two parameter Logistic distribution by including a shape parameter that accommodates the presence of skewness. The GLO differs from other generalizations of the logistic distribution, such as those documented in Azzalini (1985), Johnson *et al*. (1995), Dubey (1969) and Gumbel (1944). As indicated by Hosking & Wallis (1997), it is a reparametrized form of the log-logistic distribution of Ahmad *et al.* (1988). The functional form of the GLO's distribution is similar to that of the GPD and the generalized extreme-value distribution, Hosking (1986).

3.4.1 Definition and Special cases.

The GLO is characterized by the location parameter,α, shape parameter, $\beta > 0$, and shape parameter, λ. Following the results in Hosking (1986), the GLO can be defined in terms of its cumulative distribution function, $F(x)$, probability density function, $f(x)$, quantile function, $Q(p)$, quantile density function, $q(p)$, and its density quantile function, $f_p(p)$.

Definition 3.5.1 *Let X be a real-valued random variable. X is said to have a generalized logistic distribution, denoted by $X \sim GLO(\alpha, \beta, \lambda)$, if it is defined by the following functions:*

Cumulative distribution function,

$$F(x) = \begin{cases} \dfrac{1}{1+\left(1-\lambda\left(\frac{x-\alpha}{\beta}\right)\right)^{\frac{1}{\lambda}}}, & \lambda \neq 0 \\ \dfrac{1}{1+exp\left[-\left(\frac{x-\alpha}{\beta}\right)\right]}, & \lambda = 0 \end{cases} \tag{3.22}$$

Probability density function,

$$f(x) = \begin{cases} \dfrac{1}{\beta}\dfrac{\left(1-\lambda\left(\frac{x-\alpha}{\beta}\right)\right)^{\frac{1}{\lambda}-1}}{1+\left(1-\lambda\left(\frac{x-\alpha}{\beta}\right)\right)^{\frac{1}{\lambda}}}, & \lambda \neq 0 \\ \dfrac{1}{\beta}\dfrac{\exp\left[-\left(\frac{x-\alpha}{\beta}\right)\right]}{\left(1+exp\left[-\left(\frac{x-\alpha}{\beta}\right)\right]\right)^2}, & \lambda = 0 \end{cases} \tag{3.23}$$

and Quantile function,

$$Q(p) = \begin{cases} \alpha + \frac{\beta}{\lambda}\left(1-\left(\frac{1-p}{p}\right)^{\lambda}\right), & \lambda \neq 0 \\ \alpha + \beta log\left[\frac{p}{1-p}\right], & \lambda = 0, \end{cases} \tag{3.24}$$

with support $\left[\alpha + \frac{\beta}{\lambda}, \infty\right)$ $for\ \lambda < 0$,$(-\infty, \infty)$ $for\ \lambda = 0$ *and* $\left(-\infty, \alpha + \frac{\beta}{\lambda}\right]$ $for\ \lambda > 0$.

Theorem 3.5.1 *If $X \sim GLO(\alpha, \beta, \lambda)$, then its quantile density function is*

$$q(p) = \frac{\beta(1-p)^{\lambda-1}}{p^{\lambda+1}} \quad for\ 0 < p < 1 \tag{3.25}$$

and density quantile function is

$$f_p(p) = \frac{p^{\lambda+1}}{\beta(1-p)^{\lambda-1}} \; for \; 0 < p < 1 \,. \tag{3.26}$$

Proof: The expressions in Eqs. (3.25) and (3.26) are directly obtained using $q(p) = \frac{dQ(p)}{dp}$ and $f_p(p) = \frac{1}{q(p)}$ ■

A special case of the distribution occurs when $\lambda = 0$, where the GLO becomes the logistic distribution.

3.4.2 MOMENTS.

The results of the conventional moments for the GLO have not featured anywhere else other than in Hosking (1986). The expressions for the mean, variance, skewness moment ratio and kurtosis moment ratio are presented below.

$$E(x) = \alpha + \frac{\beta}{\lambda}(1 - \mathrm{g}_1) \tag{3.27}$$

$$Var(x) = \frac{\alpha^2(\mathrm{g}_2 - {\mathrm{g}_1}^2)}{\lambda^2} \tag{3.28}$$

$$Skewness(x) = \frac{(-sign\, k)(\mathrm{g}_3 - 3\mathrm{g}_2\mathrm{g}_1 + 2{\mathrm{g}_1}^3)}{(\mathrm{g}_2 - {\mathrm{g}_1}^2)^{\frac{3}{2}}} \tag{3.29}$$

$$Kurtosis(x) = \frac{(\mathrm{g}_4 - 4\mathrm{g}_3\mathrm{g}_1 + 6\mathrm{g}_2{\mathrm{g}_1}^2 - 3{\mathrm{g}_1}^4)}{(\mathrm{g}_2 - {\mathrm{g}_1}^2)^2} \tag{3.30}$$

where $\mathrm{g}_r = \Gamma(1 + rk)\Gamma(1 - rk)$.

3.4.3 L-MOMENTS.

As indicated in Section 2.5, all the L-moments of a real-valued random variable exist if and only if the random variable has a finite mean. The distribution is then uniquely defined by the L-moments (Hosking, 1990). The L-moments and L-moment ratios for the GLO were first introduced by Hosking (1986).

Theorem 3.5.3. *If* $X \sim \mathrm{GLO}(\alpha, \beta, \lambda)$ *with* $-1 < \lambda < 1$ *then*

$$L_1 = \alpha + \frac{\beta}{\lambda}\big(1 - \Gamma(\lambda+1)\Gamma(-\lambda+1)\big), \tag{3.31}$$

$$L_2 = \beta\Gamma(\lambda+1)\Gamma(-\lambda+1), \tag{3.32}$$

$$\tau_3 = -\lambda, \tag{3.33}$$

and

$$\tau_4 = \frac{(1+5\lambda^2)}{6} \tag{3.34}$$

Proof: See Hosking (1986) for the results in Eqs. (3.31) - (3.34). There is no general expression that has been obtained for $L_r$or τ_r. Individual expressions can be obtained from Eq. (2.9) and Eq. (2.11) respectively. ■

4 QUANTILE-BASED GENERALIZED LOGISTIC DISTRIBUTION

4.1 INTRODUCTION.

This chapter discusses the quantile-based generalized logistic distribution (GLO_{QB}) and the modelling process utilized to construct this distribution. As indicated in Chapters 2 and 3, the construction rules that have been outlined in Gilchrist (2000) will be used in this modelling process. In Section 4.2, the outline of the model form as well as the building block will be considered. It was stated in Chapter 3 that the GLO will be used as the building block of the proposed distribution, the GLO_{QB}. The specific construction rules and the final quantile-based functions are defined in this section. These include the quantile function, the quantile density function and the density quantile functions.

Section 4.3 takes an in-depth look at the probability density function (pdf) curves of the proposed quantile-based model. The different shapes of the pdfs attained for different shape parameter values are examined. Various properties of the GLO_{QB} are noted from the probability density curves such as reflection of the distribution when the position of the shape parameters are interchanged in the quantile function structure.

From Section 4.3 stems the various domains of the support of the GLO_{QB} that are clearly outlined in Section 4.4. Section 4.5 presents a classification scheme for the GLO_{QB} in terms of possible distributional shapes attainable by the density curve of the GLO_{QB}, contingent on specific pairings of the shape parameters λ_3 and λ_4.

The conventional moments of the GLO_{QB} are obtained in Section 4.6. Due to the complicated nature of the moments, only the first uncorrected moment (the mean) as well as the second order corrected moment (the variance) are obtained. Higher order moments, howbeit complicated, can be obtained after extensive simplification. Because of the complexity of the moments' expressions, it is more expedient to characterize the GLO_{QB} with alternative measures. Specifically, from the results of Hosking (1986, 1990), the *L*-moments of the GLO_{QB} will be explored in Section 4.7. These will be used to obtain functions that explicate the location, scale, skewness and kurtosis. A formula that can be used to obtain *L*-moments for

$r > 1$ will be generated, and be extended to derive the first four L-moments as well as the L-skewness and L-kurtosis.

Section 4.8 focuses on quantile-based measures of location, spread and shape. Various functions such as the spread function (MacGillivray & Balanda, 1988), skewness functionals (MacGillivray, 1986; King, 1999) and the ratio-of-spread functions (MacGillivray & Balanda, 1988) will be attained for the GLO_{QB}. Section 4.9 takes a look at the behavior of the GLO_{QB} when the location and spread parameters change whilst holding all the other parameters constant. The final exploration of the GLO_{QB} will be with regards to the tail behavior in Section 4.10. The values that the density curve approaches in various classes as well as the slope of the density curve are examined and noted.

4.2 GENESIS.

The quantile function of the generalized logistic distribution (GLO) is used as the basic building block in the proposed derivation of the quantile-based generalized logistic distribution, as mentioned in Chapter 3.

The GLO was first introduced by Hosking (1986). This is a distribution with a location, scale and a single shape parameter.

Following Hosking (1986), the quantile function of the standard GLO is given by

$$Q(p) = \frac{1}{\lambda}\left(1 - \left(\frac{1-p}{p}\right)^{\lambda}\right), \lambda \neq 0. \tag{4.1}$$

As shown by Gilchrist (2000), the distribution with quantile function $-Q(1-p)$ is the reflection of the distribution with quantile function $Q(p)$. Therefore, the quantile function of the reflected standard GLO is

$$Q(p) = \frac{1}{\lambda}\left(\left(\frac{p}{1-p}\right)^{\lambda} - 1\right), \lambda \neq 0. \tag{4.2}$$

The limiting case, when $\lambda = 0$, is the logistic distribution.

The probability density functions of the standard GLO are shown in Figure 4.2.1. The density curve of the GLO is J-shaped for $\lambda \leq -1$, unimodal for $-1 < \lambda < 1$ and reversed J-shaped for $\lambda \geq 1$. The GLO is positively (negatively) skewed for $\lambda < (>)0$. Changing the sign of λ causes a reflection of the GLO's density curve. That is, the distributional shape of the GLO with shape parameter λ is the same as the distributional shape of the reflected GLO with shape parameter $-\lambda$.

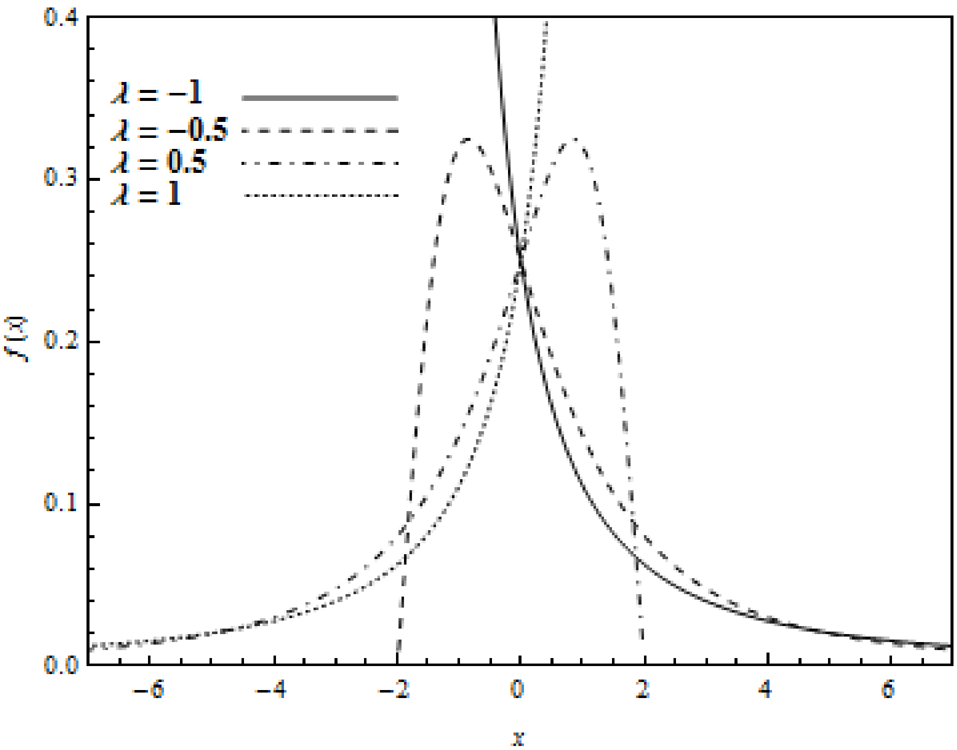

Figure 4.2.1: The probability density curves for the GLO, with $\alpha = 0, \beta = 1$, and $\lambda = -1, -0.5, 0.5$ and 1.

The quantile function of the Freimer-Mudholkar-Kollia-Lin Type of GLD (GLD_{FMKL}) in Eq. (2.18) was constructed using Proposition 2.7.1 in Chapter 2. As shown, the model form was as follows

$$Q(p) = \lambda_1 + \frac{1}{\lambda_2}\left(\mathrm{Q}_{Y:0}(\mathrm{p}) + \mathrm{Q}_{X:0}(\mathrm{p})\right)$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left(-\mathrm{Q}_{X:0}(1-\mathrm{p}) + \mathrm{Q}_{X:0}(\mathrm{p})\right) \tag{4.3}$$

where $-\mathrm{Q}_{X:0}(1-\mathrm{p})$ is the quantile function of the reflected standard GPD with shape parameter λ_3 and $\mathrm{Q}_{X:0}(\mathrm{p})$ is the quantile function of the standard GPD with shape parameter λ_4.

The form in Eq. (4.3) will be adapted in the construction of the proposed quantile-based distribution, the $\mathrm{GLO_{QB}}$. The quantile function of the $\mathrm{GLO_{QB}}$ will thus be attained by taking the sum of Eq. (4.1), setting $= \lambda_4$, and Eq. (4.2) with $\lambda = \lambda_3$. The same model form that was conveyed in Eq. (4.3) will be maintained.

<u>Definition 4.2.1</u>. *Let X be a real-valued random variable. X is said to have a quantile-based generalized logistic distribution, denoted as* $\sim\mathrm{GLO_{QB}}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *, if it has the quantile function*

$$Q(p) = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_3}\left(\left(\frac{p}{1-p}\right)^{\lambda_3} - 1\right) - \frac{1}{\lambda_4}\left(\left(\frac{1-p}{p}\right)^{\lambda_4} - 1\right)\right), 0 < p < 1, \tag{4.4}$$

where λ_1 is a location parameter, λ_2 is a scale parameter and λ_3 and λ_4 are shape parameters.■

Similar to the $\mathrm{GLD_{FMKL}}$, the $\mathrm{GLO_{QB}}$ will also be defined in terms of its quantile-based functions as it has no closed-form expressions for its cumulative distribution function, $F(x)$, and its probability density function, $f(x)$.

<u>Theorem 4.2.2</u> *If* $X\sim\mathrm{GLO_{QB}}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *, then its quantile density function is*

$$q(p) = \frac{1}{\lambda_2 p(1-p)}\left(\left(\frac{p}{1-p}\right)^{\lambda_3} + \left(\frac{1-p}{p}\right)^{\lambda_4}\right) \quad for\ 0 < p < 1 \tag{4.5}$$

and density quantile function is

$$f_p(p) = \lambda_2 p(1-p)\left(\left(\frac{p}{1-p}\right)^{\lambda_3} + \left(\frac{1-p}{p}\right)^{\lambda_4}\right)^{-1} \; for\; 0 < p < 1\,. \qquad (4.6)$$

<u>Proof:</u> The expressions in Eq. (4.5) and (4.6) are directly obtained using $q(p) = \frac{dQ(p)}{dp}$ and $f_p(p) = \frac{1}{q(p)}$ ∎

Akin to the $\mathrm{GLD_{FMKL}}$, the $\mathrm{GLO_{QB}}$ is symmetric when $\lambda_3 = \lambda_4$. In particular, when $\lambda_3 = \lambda_4 = 0$, $\mathrm{GLO_{QB}}$ reduces to the symmetric logistic distribution.

The plots of the probability density curves for various values of the shape parameters λ_3 and λ_4 will be disclosed and studied in Section 4.3.

4.3 PROBABILITY DENSITY CURVES.

The probability density curves for the $\mathrm{GLO_{QB}}$ are illustrated in Figures 4.3.1-4.3.4 for selected values of λ_3 and λ_4. Without loss of generality, $\lambda_1 = 0$ and $\lambda_2 = 1$.

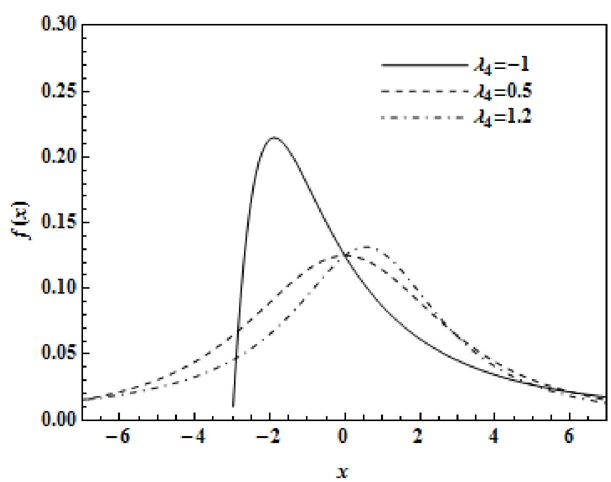

Figure 4.3.1: The probability density curves for the $\mathrm{GLO_{QB}}$, with $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 0.5$ and $\lambda_4 = -1, 0.5$ and 1.

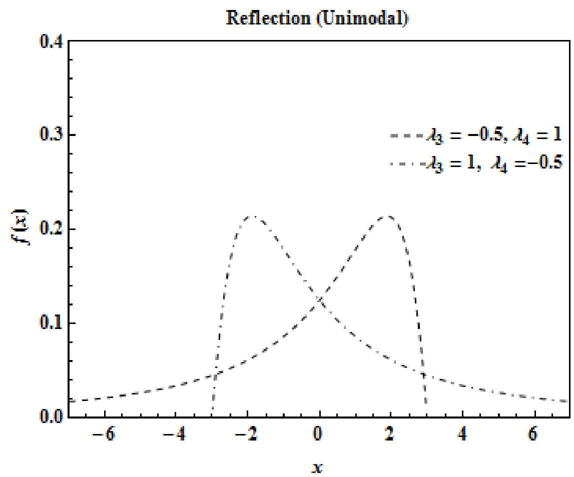

Figure 4.3.2: The probability density curves for the GLO_{QB}, with $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -0.5, 1$ and $\lambda_4 = -0.5, 1$.

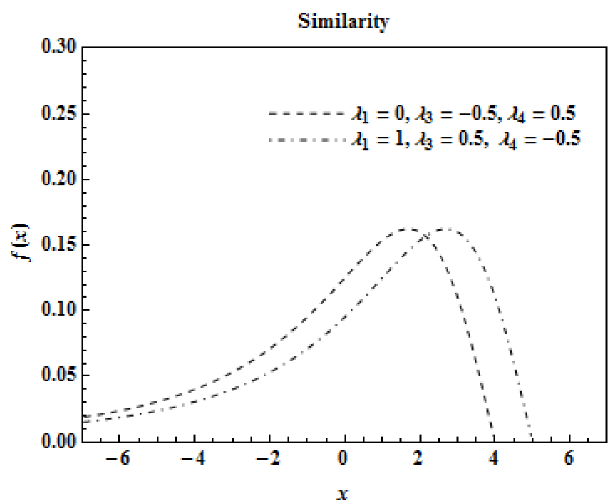

Figure 4.3.3: The probability density curves for the GLO_{QB}, with $\lambda_1 = 0,1$, $\lambda_2 = 1$, $\lambda_3 = -0.5, 0.5$ and $\lambda_4 = -0.5, 0.5$

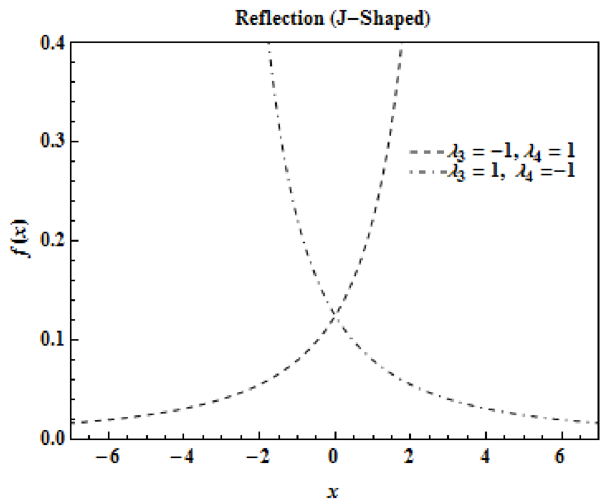

Figure 4.3.4: The probability density curves for the GLO_{QB}, with $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 1$ and $\lambda_4 = 1$

As can be seen from Figures 4.3.1-4.3.4, the density curves take on various shapes that are dependent on the combinations of the shape parameters λ_3 and λ_4. In particular, the following shape characteristics for different shape parameter values are obtained:

- The GLO_{QB} is symmetric for $\lambda_3 = \lambda_4$, positively skewed for $\lambda_3 > \lambda_4$ and negatively skewed for $\lambda_3 < \lambda_4$. This can be clearly seen when examining Figure 4.3.1, where $\lambda_3 = 0.5$. The solid line curve indicates positive skewness when $\lambda_4 = -1$. Similarly, when $\lambda_4 = 1.2$, the dot-dashed line indicates a negative skewness in the pdf of the GLO_{QB}. The dashed line reveals a symmetric distribution for $\lambda_3 = \lambda_4 = 0.5$.

  - The pdf of the GLO_{QB} with parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is the reflection of the pdf of GLO_{QB} with parameters $(\lambda_1, \lambda_2, \lambda_4, \lambda_3)$, about the line $x = \lambda_1$. This proof, as depicted in Karian & Dudewicz (2000, 2010), takes into account that $x = Q(p)$. It can be seen that the quantile function in Eq. (4.4) is a function of $(p, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Therefore,

$$x = Q(p, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_3}\left(\left(\frac{p}{1-p}\right)^{\lambda_3} - 1\right) - \frac{1}{\lambda_4}\left(\left(\frac{1-p}{p}\right)^{\lambda_4} - 1\right)\right)$$

$$= \lambda_1 + B.$$

Considering the reflection $x = Q(1-p)$, the reflected quantile function can also be constructed as above.

$$x = Q(1-p, \lambda_1, \lambda_2, \lambda_4, \lambda_3) = \lambda_1 - \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4}\left(\left(\frac{p}{1-p}\right)^{\lambda_4} - 1\right) - \frac{1}{\lambda_3}\left(\left(\frac{1-p}{p}\right)^{\lambda_3} - 1\right)\right)$$

$$= \lambda_1 - B.$$

Therefore, the two distributions are reflected about the line $x = \lambda_1$, when the shape parameters are interchanged.

From Figure 4.3.2, the pdf of the GLO_{QB} with $\lambda_3 = -0.5$ and $\lambda_4 = 1$, indicated by the dotted line, is the reflection of the pdf of the GLO_{QB} with $\lambda_3 = 1$ and $\lambda_4 = -0.5$, shown with a dot-dashed line, about the line $x = 0$.

- The pdf of the $\mathrm{GLO_{QB}}$ with parameters $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is the same as the pdf of $\mathrm{GLO_{QB}}$ with parameters $(\lambda_1, \lambda_2, -\lambda_4, -\lambda_3)$. The results follow using the same approach as in the reflective case above. From Figure 4.3.3, the pdf of the GLO_{QB} with $\lambda_1 = 0$, $\lambda_3 = -0.5$ and $\lambda_4 = 0.5$, indicated by the dashed line, has the same distributional shape as the pdf of the GLO_{QB} with $\lambda_1 = 1$, $\lambda_3 = 0.5$ and $\lambda_4 = -0.5$, shown with a dot-dashed line. Note that in Figure 4.3.3, in order to distinguish between the two density curves, the values of the corresponding location parameters are not equal, but set to zero and one respectively.

- It should also be noted that the pdf of the GLO_{QB} has a J-shape when $\lambda_3 \geq 1$ and $\lambda_4 \leq -1$, and also when $\lambda_3 \leq -1$ and $\lambda_4 \geq 1$. This can be seen in Figure 4.3.4 where, for $\lambda_3 = 1$ and $\lambda_4 = -1$ (dashed line), the curve has a J-shape to the right, while the curve takes on a J-shape to the left (also referred to as a reversed J-shape) when $\lambda_3 = -1$ and $\lambda_4 = 1$ (dot-dashed line). The probability density curve remains unimodal for all other combinations of λ_3 and λ_4.

From the probability density curves represented and discussed above, it is clear that the GLO_{QB} is highly flexible with respect to distributional shapes as well as the support that is attainable. These will be examined in Sections 4.4 and 4.5.

4.4 PARAMETER SPACE AND SUPPORT.

The parameter space of the $\mathrm{GLO_{QB}}$ can be divided into four distinct regions that are based on the shape parameters and the support in each region. According to the values listed in Table 4.4.1, the divisions are based on the different pairings of values that the two shape parameters λ_3 and λ_4 can take on. This (λ_3, λ_4)-space of the $\mathrm{GLO_{QB}}$ is depicted in Figure 4.5.1. Note that the dotted line, where $\lambda_3 = \lambda_4$, indicates symmetric distributions. When $\lambda_3 = \lambda_4 = 0$, the $\mathrm{GLO_{QB}}$ reduces to the logistic distribution which is indicated at the origin of the (λ_3, λ_4)-space.

Table 4.4.1: Parameter space and support of the $\mathrm{GLO_{QB}}$, in terms of Regions I, II, III and IV

Region	Shape parameter values	Support
Region 1	$\lambda_3 \geq 0 \; and \; \lambda_4 \geq 0$	$(-\infty, \infty)$
Region 2	$\lambda_3 < 0 \; and \; \lambda_4 > 0$	$\left(-\infty, \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right)\right)$
Region 3	$\lambda_3 \leq 0 \; and \; \lambda_4 \leq 0$	$(-\infty, \infty)$
Region 4	$\lambda_3 > 0 \; and \; \lambda_4 < 0$	$\left(\lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right), \infty\right)$

4.5 CLASSES.

An alternative classification scheme can be used for the $\mathrm{GLO_{QB}}$, in which the (λ_3, λ_4)-space is divided into four classes based on the distributional shape obtained by the density curve of the $\mathrm{GLO_{QB}}$. In this section, these four classes are presented and graphical examples of density curves from each class are given.

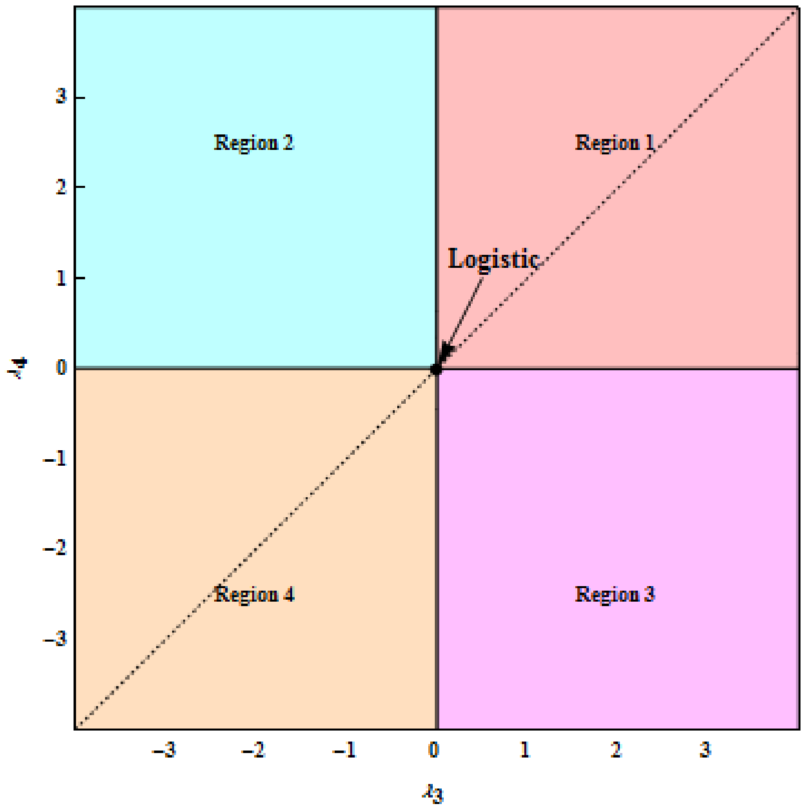

Figure 4.5.1: The parameter space of the GLO_{QB} in terms of Regions 1(a), 1(b), 2, 3(a), 3(b) and 4. The dashed line $\lambda_3 = \lambda_4$ indicates symmetric distributions. The Logistic distribution is attained when $\lambda_3 = \lambda_4 = 0$.

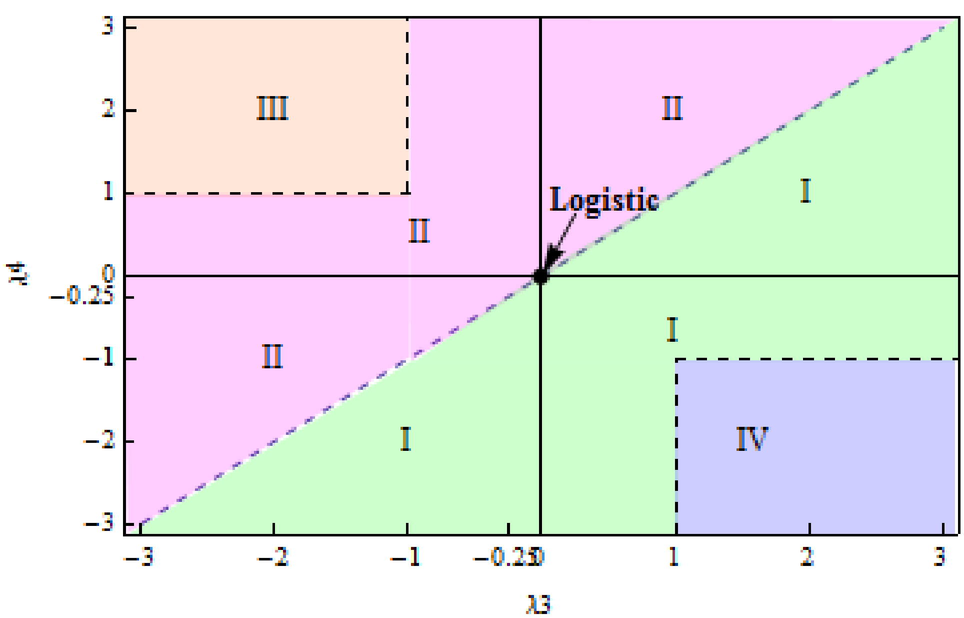

Figure 4.5.2: The parameter space of the GLO_{QB} in terms of Classes I, II, III and IV.

4.5.1 <u>CLASS I.</u>

Consider the distributional shapes of the $\mathrm{GLO_{QB}}$ from Figure 4.5.1.1 below. The values of λ_3 are fixed whilst the values of λ_4 are changed. In this class, $\lambda_3 > \lambda_4 > 0$ in Region 1, $\lambda_4 < \lambda_3 < 0$ in Region 3, and $-1 < \lambda_4 < 0$ and $\lambda_3 > 1$or $\lambda_4 < 0$ and $1 > \lambda_3 > 0$ in Region 2.

As noted from the probability density curves, they are unimodal and positively skewed. The closer the value of $\lambda_3$to λ_4, the less positively skewed the distribution is. In addition, this region contains members of the $\mathrm{GLO_{QB}}$ with infinite support for $\lambda_3 > \lambda_4 \geq 0$ in Region 1 and $\lambda_4 < \lambda_3 \leq 0$ in Region 3, and half infinite support for $-1 < \lambda_4 < 0$ and $\lambda_3 > 1$and for $\lambda_4 < 0$ and $1 > \lambda_3 > 0$ from Region 4.

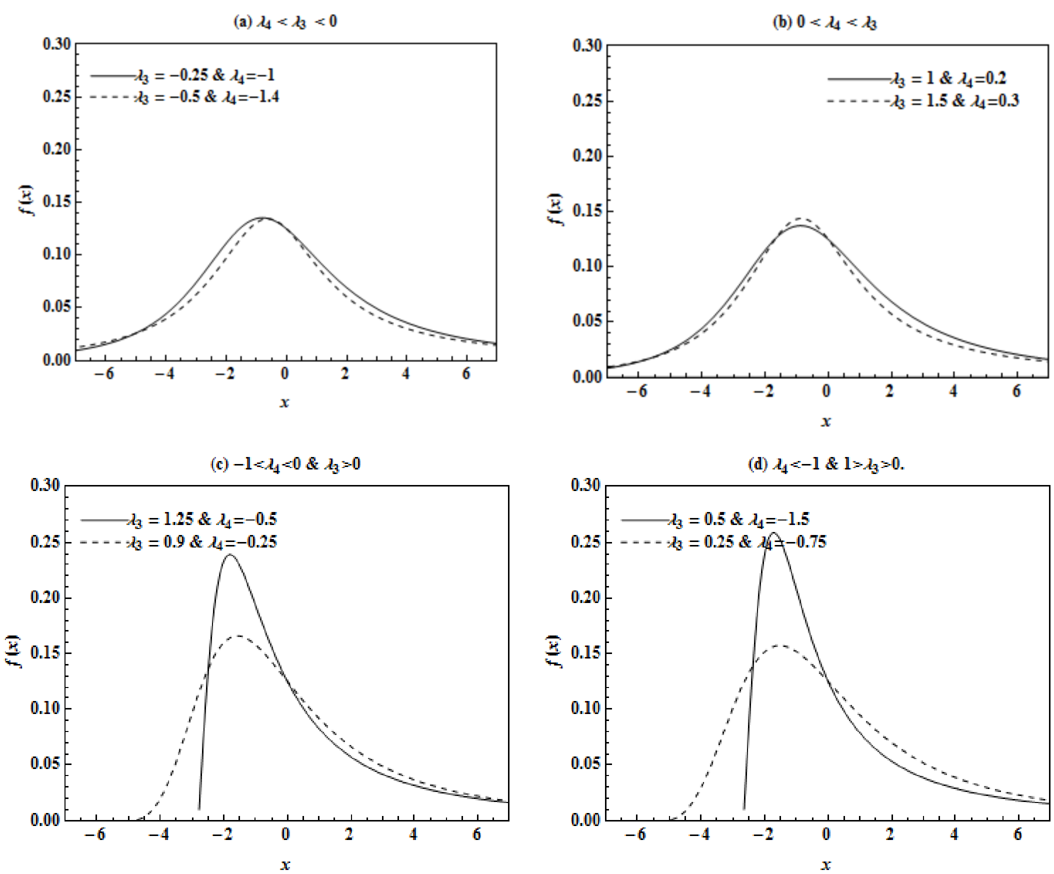

Figure 4.5.1.1. Probability density functions of members of the $\mathrm{GLO_{QB}}$ from Class I, all with $\mathrm{L_1} = 0$ and $\mathrm{L_2} = 1$

4.5.2 CLASS II.

Subsequently, Class II is characterized by distributions that depict negative skewness. This is an effect of $\lambda_4 > \lambda_3 \geq 0$ in Region 1, $\lambda_3 < \lambda_4 < 0$ in Region 3 and $-1 < \lambda_3 < 0$ and $\lambda_4 > 0$ as well as $\lambda_3 < -1$ and $0 < \lambda_4 < 1$ in Region 2. The distributional shape is still unimodal, while the $\mathrm{GLO_{QB}}$ maintains an infinite support for $\lambda_4 > \lambda_3 \geq 0$ and $\lambda_3 < \lambda_4 \leq 0$ in Regions 1 and 3 respectively, and half infinite support for Region 2 of Class II, in effect, for $-1 < \lambda_3 < 0$ and $\lambda_4 > 0$ and for $\lambda_3 < -1$ and $1 > \lambda_4 > 0$. This is evident from the curves in Figure 4.5.1.2.

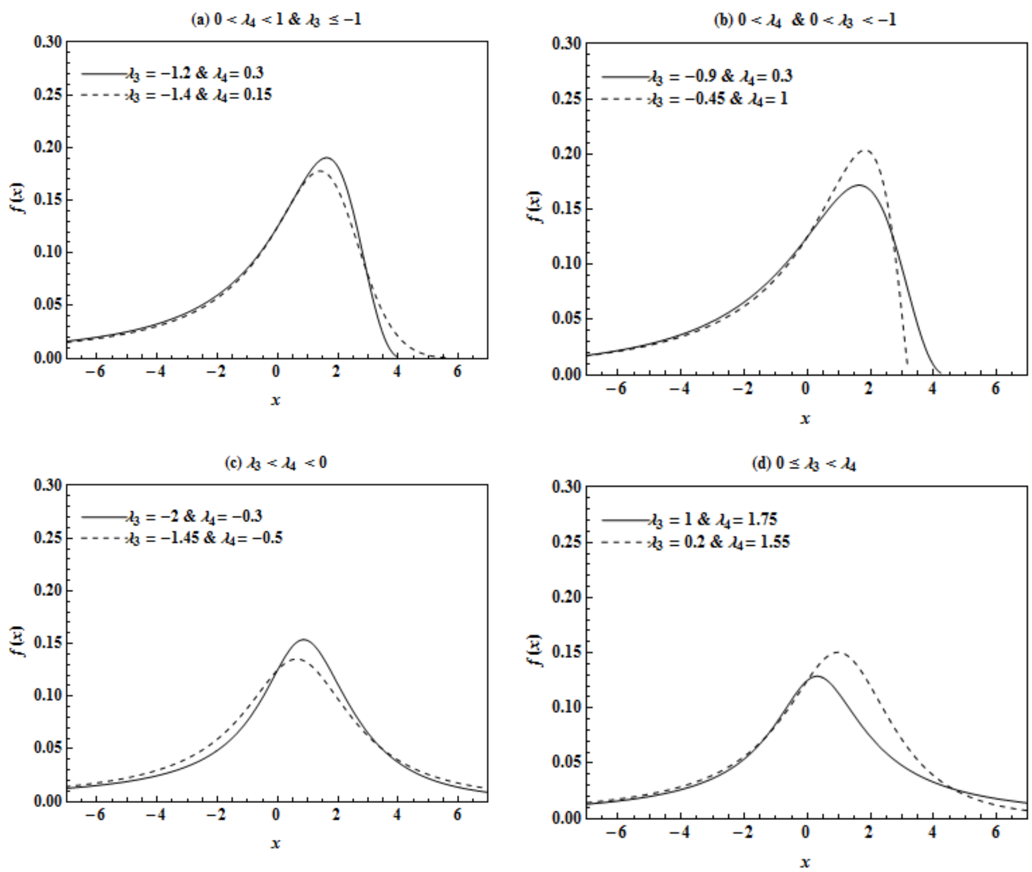

Figure 4.5.1.2. Probability density functions of members of the $\mathrm{GLO_{QB}}$ from Class II, all with $\mathrm{L_1} = 0$ and $\mathrm{L_2} = 1$

4.5.3 CLASS III.

The distributional shape in this class is J-shaped, as illustrated in Figure 4.5.1.3. It arises when $\lambda_4 \geq 1$ and $\lambda_3 \leq -1$. The tail behavior for Class III will be discussed in Section 4.9. Since Class III falls inside Region 2, the $\mathrm{GLO_{QB}}$ has half-infinite support in this class.

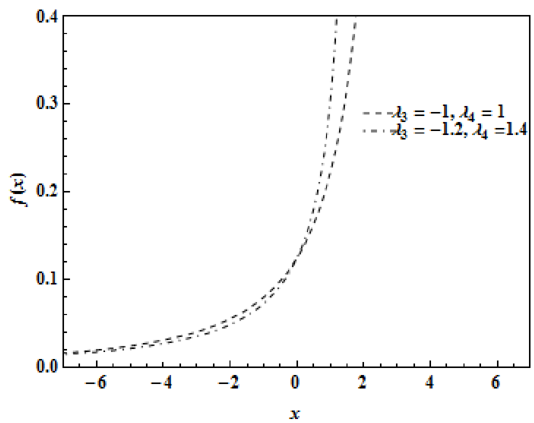

Figure 4.5.1.3. Probability density functions of members of the $\mathrm{GLO_{QB}}$ from Class III, all with $\mathrm{L}_1 = 0$ and $\mathrm{L}_2 = 1$.

4.5.4 CLASS IV.

The distributional shape in Class IV is reversed J-shaped, occurring when $\lambda_4 \leq -1$ and $\lambda_3 \geq 1$. Distributions from this class, which is part of Region 4, have half-infinite support. Examples of density curves from Class IV are shown in Figure 4.5.1.4.

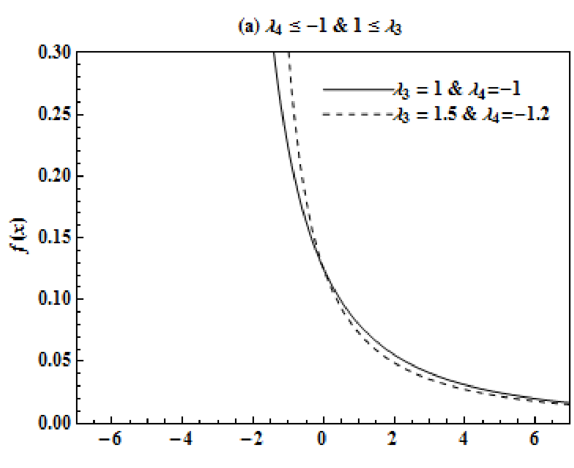

Figure 4.5.1.4. Probability density functions of members of the $\mathrm{GLO_{QB}}$ from Class IV, all with $\mathrm{L}_1 = 0$ and $\mathrm{L}_2 = 1$.

4.6 MOMENTS OF THE GLO$_{QB}$.

Although the GLO$_{QB}$ is a valid distribution for all values of λ_3 and λ_4, the r^{th}order moment of the GLO$_{QB}$ only exists if $-\frac{1}{r} < \lambda_3 < \frac{1}{r}$ and $-\frac{1}{r} < \lambda_4 < \frac{1}{r}$. Thus, only if $-\frac{1}{4} < \lambda_3 < \frac{1}{4}$ and $-\frac{1}{4} < \lambda_4 < \frac{1}{4}$, the mean, variance, skewness moment ratio and kurtosis moment ratio for the GLO$_{QB}$ exist. Furthermore, the moments of a quantile-based distribution are not easily obtained and the resulting expressions are not simple. In the case of the GLO$_{QB}$, the measure of location, μ, and the measure of spread, σ^2, will be obtained after extensive simplification. The first central moment is the mean of the GLO$_{QB}$. Substituting Eq. (4.4) into Eq. (2.4) and setting *r*=1, then

$$\mu_1' = \int_0^1 \left(\lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_3}\left(\left(\frac{p}{1-p}\right)^{\lambda_3} - 1\right) - \frac{1}{\lambda_4}\left(\left(\frac{1-p}{p}\right)^{\lambda_4} - 1\right)\right)\right) dp$$

$$=\lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \int_0^1 \left(\frac{1}{\lambda_3\lambda_2}\left(\frac{p}{1-p}\right)^{\lambda_3} - \frac{1}{\lambda_4\lambda_2}\left(\frac{1-p}{p}\right)^{\lambda_4}\right) dp$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \int_0^1 \frac{1}{\lambda_2\lambda_3}\left(\mathrm{p}^{\lambda_3}(1-\mathrm{p})^{-\lambda_3}\right) dp - \int_0^1 \frac{1}{\lambda_2\lambda_4}\left(\mathrm{p}^{-\lambda_4}(1-\mathrm{p})^{\lambda_4}\right) dp$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \frac{\Gamma(\lambda_3+1)\Gamma(-\lambda_3+1)}{\lambda_2\lambda_3\Gamma(2)} - \frac{\Gamma(\lambda_4+1)\Gamma(-\lambda_4+1)}{\lambda_2\lambda_4\Gamma(2)}$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \frac{\lambda_3\pi\mathrm{Csc}(\pi\lambda_3)}{\lambda_2\lambda_3 1!} - \frac{\lambda_4\pi\mathrm{Csc}(\pi\lambda_4)}{\lambda_2\lambda_4 1!}$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \frac{\pi\mathrm{Csc}(\pi\lambda_3)}{\lambda_2} - \frac{\pi\mathrm{Csc}(\pi\lambda_4)}{\lambda_2}$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left(\pi(\mathrm{Csc}(\pi\lambda_3) - \mathrm{Csc}(\pi\lambda_4)) + \frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right), \qquad (4.7)$$

where the final results above are obtained after extensive simplification.

The second noncentral moment can be used to obtain the variance of the GLO$_{QB}$. This is because $\sigma^2 = \mu_2' - (\mu_1')^2$. Therefore by substituting Eq. (4.4) into Eq. (2.4) and setting *r*=2, then

$$\mu_2' = \int_0^1 \left(\lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_3}\left(\left(\frac{p}{1-p}\right)^{\lambda_3} - 1\right) - \frac{1}{\lambda_4}\left(\left(\frac{1-p}{p}\right)^{\lambda_4} - 1\right)\right)\right)^2 dp$$

$$= \int_0^1 \lambda_1^2 \, dp + \frac{2\lambda_1}{\lambda_2}\int_0^1 \left(\frac{1}{\lambda_3}\left(\left(\frac{p}{1-p}\right)^{\lambda_3} - 1\right) - \frac{1}{\lambda_4}\left(\left(\frac{1-p}{p}\right)^{\lambda_4} - 1\right)\right) dp$$

$$+ \frac{1}{\lambda_2^2}\int_0^1 \frac{1}{\lambda_3^2}\left(\left(\frac{p}{1-p}\right)^{\lambda_3} - 1\right)^2 - \frac{2}{\lambda_3\lambda_4}\left(\left(\frac{p}{1-p}\right)^{\lambda_3} - 1\right)\left(\left(\frac{1-p}{p}\right)^{\lambda_4} - 1\right) + \frac{1}{\lambda_4^2}\left(\left(\frac{1-p}{p}\right)^{\lambda_4} - 1\right) dp$$

$$= \lambda_1^2 + \frac{2\lambda_1}{\lambda_2}\left(\frac{1}{\lambda_3}\left(\Gamma(\lambda_3 + 1)\Gamma(-\lambda_3 + 1) - 1\right) - \frac{1}{\lambda_4}\left(\Gamma(\lambda_4 + 1)\Gamma(-\lambda_4 + 1) - 1\right)\right)$$

$$+ \frac{1}{\lambda_2^2\lambda_3^2}\left(\frac{\Gamma(2\lambda_3+1)\Gamma(-2\lambda_3+1)}{\Gamma(2)} - \frac{\Gamma(2\lambda_3+1)\Gamma(-2\lambda_3+1)}{\Gamma(2)} + 1\right)$$

$$+ \frac{2}{\lambda_2^2\lambda_3\lambda_4}\left(\frac{\Gamma(\lambda_3-\lambda_4+1)\Gamma(1-\lambda_3+\lambda_4)}{\Gamma(2)} - \frac{\Gamma(\lambda_3+1)\Gamma(-\lambda_3+1)}{\Gamma(2)} - \frac{\Gamma(\lambda_4+1)\Gamma(-\lambda_4+1)}{\Gamma(2)} - 1\right)$$

$$+ \frac{1}{\lambda_2^2\lambda_4^2}\left(\frac{\Gamma(2\lambda_4+1)\Gamma(-2\lambda_4+1)}{\Gamma(2)} - \frac{\Gamma(2\lambda_4+1)\Gamma(-2\lambda_4+1)}{\Gamma(2)} + 1\right)$$

$$= \lambda_1^2 + \frac{2\lambda_1}{\lambda_2}\left(\pi(\mathrm{Csc}(\pi\lambda_3) - \mathrm{Csc}(\pi\lambda_4)) + \frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right)$$

$$+ \frac{1}{\lambda_2^2\lambda_3^2}\left(2\pi\lambda_3\mathrm{Csc}(2\pi\lambda_3) - \pi\lambda_3\mathrm{Csc}(\pi\lambda_3) + 1\right)$$

$$- \frac{2}{\lambda_2^2\lambda_3\lambda_4}\left(\pi(\lambda_3 - \lambda_4)\mathrm{Csc}\left(\pi(\lambda_3 - \lambda_4)\right) - \pi\lambda_3\mathrm{Csc}(\pi\lambda_3) - \pi\lambda_4\mathrm{Csc}(\pi\lambda_4) + 1\right)$$

$$+ \frac{1}{\lambda_2^2\lambda_4^2}\left(2\pi\lambda_4\mathrm{Csc}(2\pi\lambda_4) - \pi\lambda_4\mathrm{Csc}(\pi\lambda_4) + 1\right)$$

$$= \lambda_1^2 + \pi\mathrm{Csc}(\pi\lambda_3)\left(\frac{2\lambda_1}{\lambda_2} - \frac{\lambda_3}{\lambda_2^2\lambda_3^2} + \frac{2\lambda_3}{\lambda_2^2\lambda_3\lambda_4}\right) - \pi\mathrm{Csc}(\pi\lambda_4)\left(\frac{2\lambda_1}{\lambda_2} - \frac{\lambda_4}{\lambda_2^2\lambda_4^2} + \frac{2\lambda_4}{\lambda_2^2\lambda_3\lambda_4}\right)$$

$$+ \frac{\lambda_3}{\lambda_2^2\lambda_3^2}2\pi\mathrm{Csc}(2\pi\lambda_3) + \frac{\lambda_4}{\lambda_2^2\lambda_4^2}2\pi\mathrm{Csc}(2\pi\lambda_4) - \frac{2}{\lambda_2^2\lambda_3\lambda_4}\pi(\lambda_3 - \lambda_4)\mathrm{Csc}\left(\pi(\lambda_3 - \lambda_4)\right)$$

$$+ \left(\frac{2\lambda_1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \frac{1}{\lambda_2^2\lambda_3^2} - \frac{2}{\lambda_2^2\lambda_3\lambda_4} + \frac{1}{\lambda_2^2\lambda_4^2}\right) \qquad (4.8)$$

The variance of the GLO_{QB} is then obtained as follows:

$$\sigma^2 = \mu_2' - (\mu_1')^2$$

$$= -\frac{\pi^2 \text{Csc}[\pi\lambda_3]^2}{\lambda_2^2} + \frac{2\pi^2 \text{Csc}[\pi\lambda_3]\text{Csc}[\pi\lambda_4]}{\lambda_2^2} - \frac{\pi^2 \text{Csc}[\pi\lambda_4]^2}{\lambda_2^2} + \frac{2\pi \text{Csc}[2\pi\lambda_3]}{\lambda_2^2\lambda_3} + \frac{2\pi \text{Csc}[\pi(\lambda_3-\lambda_4)]}{\lambda_2^2\lambda_3} - \frac{2\pi \text{Csc}[\pi(\lambda_3-\lambda_4)]}{\lambda_2^2\lambda_4} + \frac{2\pi \text{Csc}[2\pi\lambda_4]}{\lambda_2^2\lambda_4} \qquad (4.9)$$

The moments of the GLO in Eqs. (3.27) - (3.30) are more complicated in form and not as easily obtained as the L-moments listed in Eqs. (3.31) - (3.34). As a result, it is expected that the L-moments that will be obtained in the next section will be much simpler in form than the conventional moments. This will lead to the use of the method of L-moments estimation in obtaining the parameters of the GLO_{QB} in Chapter 5.

4.7 L-MOMENTS OF THE GLO_{QB}.

As explained in Chapter 2, L-moments as defined by Hosking (1990), are expectations of linear combinations of order statistics. As seen in Eq. (2.9) in Chapter 2, the L-moments are defined in terms of the quantile function, rendering this method applicable for the GLO_{QB}.

In the case of the GLO_{QB}, the first four L-moments as well as the L-skewness and the L-kurtosis will be derived in Theorem 4.7.

Theorem 4.7.

If $X \sim \text{GLO}_{\text{QB}}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *with* $-1 < \lambda_3 < 1$ *and* $-1 < \lambda_4 < 1$ *then*

$$L_1 = \lambda_1 + \frac{1}{\lambda_2}\left(\pi(\text{Csc}(\pi\lambda_3) - \text{Csc}(\pi\lambda_4)) + \frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right), \qquad (4.10)$$

$$L_2 = \frac{1}{\lambda_2}(\{\lambda_3 \pi \text{Csc}(\pi\lambda_3) + \lambda_4 \pi \text{Csc}(\pi\lambda_4)\}), \qquad (4.11)$$

$$\tau_3 = \frac{L_3}{L_2} = \frac{Csc(\pi\lambda_3){\lambda_3}^2 - Csc(\pi\lambda_4){\lambda_4}^2}{Csc(\pi\lambda_3)\lambda_3 + Csc(\pi\lambda_4)\lambda_4} \qquad (4.12)$$

and

$$\tau_4 = \frac{L_4}{L_2} = \frac{\lambda_3 Csc(\pi\lambda_3)(1+5{\lambda_3}^2) + \lambda_4 Csc(\pi\lambda_4)(1+5{\lambda_4}^2)}{(Csc(\pi\lambda_3)\lambda_3 + Csc(\pi\lambda_4)\lambda_4)} \qquad (4.13)$$

where $Csc[\cdot]$ *is the Cosec function.*

Proof:

Using Eq. (2.9) and $P_0^*(p) = 1$ from Lemma 2.5.1,

$$\mathrm{L}_1 = \int_0^1 Q(p)\, P_{r-1}^*(p)dp$$

$$= \int_0^1 \left(\lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right)\right) dp + \int_0^1 \frac{1}{\lambda_2}\left(\frac{1}{\lambda_3}\left(\frac{\mathrm{p}}{1-\mathrm{p}}\right)^{\lambda_3} - \frac{1}{\lambda_4}\left(\frac{1-\mathrm{p}}{\mathrm{p}}\right)^{\lambda_4}\right) dp$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \int_0^1 \frac{1}{\lambda_2\lambda_3}\left(\mathrm{p}^{\lambda_3}(1-\mathrm{p})^{-\lambda_3}\right) dp - \int_0^1 \frac{1}{\lambda_2\lambda_4}\left(\mathrm{p}^{-\lambda_4}(1-\mathrm{p})^{\lambda_4}\right) dp$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \frac{\Gamma(\lambda_3+1)\Gamma(-\lambda_3+1)}{\lambda_2\lambda_3\Gamma(2)} - \frac{\Gamma(\lambda_4+1)\Gamma(-\lambda_4+1)}{\lambda_2\lambda_4\Gamma(2)}$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \frac{\lambda_3\pi\mathrm{Csc}(\pi\lambda_3)}{\lambda_2\lambda_3 1!} - \frac{\lambda_4\pi\mathrm{Csc}(\pi\lambda_4)}{\lambda_2\lambda_4 1!}$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \frac{\pi\mathrm{Csc}(\pi\lambda_3)}{\lambda_2} - \frac{\pi\mathrm{Csc}(\pi\lambda_4)}{\lambda_2}$$

$$= \lambda_1 + \frac{1}{\lambda_2}\left(\pi(\mathrm{Csc}(\pi\lambda_3) - \mathrm{Csc}(\pi\lambda_4)) + \frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right)$$

where the final results above are obtained (after simplification) using

$$\int_0^1 p^a(1-p)^{-a}dp = \Gamma(1+\mathrm{a})\Gamma(1-\mathrm{a}) = \pi\mathrm{a}\mathrm{Csc}(\pi\mathrm{a})\,. \tag{4.14}$$

Note that the integral only converges if $-1 < \lambda_3 < 1$ and $-1 < \lambda_4 < 1$.

For L-moments of an order greater than one, the quantile function of the $\mathrm{GLO_{QB}}$ in Eq. (4.4) can be rewritten as

$$Q(p) = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \frac{1}{\lambda_2}\left(\frac{\left(\frac{\mathrm{p}}{1-\mathrm{p}}\right)^{\lambda_3}}{\lambda_3} - \frac{\left(\frac{1-\mathrm{p}}{\mathrm{p}}\right)^{\lambda_4}}{\lambda_4}\right). \tag{4.15}$$

Using Eq. (4.15), $P_r^*(p) = (-1)^r\, P_r^*(1-p)$ *and* $\int_0^1 P_r^*(p)dp = 0$ for *r>0* from Lemma 2.5.1,

$$\mathrm{L}_r = \int_0^1 Q(p)\, P_{r-1}^*(p)dp$$

$$=\int_0^1 \left(\lambda_1 + \frac{1}{\lambda_2}\left(\frac{1}{\lambda_4} - \frac{1}{\lambda_3}\right) + \frac{1}{\lambda_2}\left(\frac{\left(\frac{\mathrm{p}}{1-\mathrm{p}}\right)^{\lambda_3}}{\lambda_3} - \frac{\left(\frac{1-\mathrm{p}}{\mathrm{p}}\right)^{\lambda_4}}{\lambda_4}\right)\right) P_{r-1}^*(p)dp$$

$$=\int_0^1\left(\lambda_1+\frac{1}{\lambda_2}\left(\frac{1}{\lambda_4}-\frac{1}{\lambda_3}\right)\right)P_{r-1}^*(p)dp+\int_0^1\left(\frac{1}{\lambda_2}\left(\frac{\left(\frac{p}{1-p}\right)^{\lambda_3}}{\lambda_3}-\frac{\left(\frac{1-p}{p}\right)^{\lambda_4}}{\lambda_4}\right)\right)P_{r-1}^*(p)dp$$

$$=\int_0^1\frac{1}{\lambda_2\lambda_3}p^{\lambda_3}(1-p)^{-\lambda_3}P_{r-1}^*(p)dp-\int_0^1\frac{1}{\lambda_2\lambda_4}p^{-\lambda_4}(1-p)^{\lambda_4}P_{r-1}^*(p)dp$$

$$=\int_0^1\frac{1}{\lambda_2\lambda_3}p^{\lambda_3}(1-p)^{-\lambda_3}P_{r-1}^*(p)dp-(-1)^{r-1}\int_0^1\frac{1}{\lambda_2\lambda_4}p^{-\lambda_4}(1-p)^{\lambda_4}P_{r-1}^*(1-p)dp$$

$$=\sum_{k=0}^{r-1}(-1)^{r-k-1}\binom{r-1}{k}\binom{r+k-1}{k}\left\{\int_0^1\frac{p^{\lambda_3}(1-p)^{-\lambda_3}p^k}{\lambda_2\lambda_3}dp-\quad(-1)^{r-1}\int_0^1\frac{p^{-\lambda_4}(1-p)^{\lambda_4}p^k}{\lambda_2\lambda_4}dp\right\}$$

$$=\sum_{k=0}^{r-1}(-1)^{r-k-1}\binom{r-1}{k}\binom{r+k-1}{k}\left\{\frac{\Gamma(\lambda_3+k+1)\Gamma(-\lambda_3+1)}{\lambda_2\lambda_3\Gamma(k+2)}-(-1)^{r-1}\frac{\Gamma(\lambda_4+k+1)\Gamma(-\lambda_4+1)}{\lambda_2\lambda_4\Gamma(k+2)}\right\}$$

$$=\frac{1}{\lambda_2}\sum_{k=0}^{r-1}(-1)^{r-k-1}\binom{r-1}{k}\binom{r+k-1}{k}\left\{\frac{\prod_{j=0}^{k}(\lambda_3+j)\pi\mathrm{Csc}(\pi\lambda_3)}{\lambda_3(k+1)!}-(-1)^{r-1}\frac{\prod_{j=0}^{k}(\lambda_4+j)\pi\mathrm{Csc}(\pi\lambda_4)}{\lambda_4(k+1)!}\right\}\tag{4.16}$$

The final results above are obtained (after simplification) using

$$\int_0^1 p^{a+k}(1-p)^{-a}dp=\frac{\Gamma(1+a+k)\Gamma(1-a)}{\Gamma(k+2)}=\frac{\prod_{j=0}^{k}(a+j)\pi\mathrm{Csc}(\pi a)}{(k+1)!},\text{for } k>0.\tag{4.17}$$

In order to obtain L_2, the L-scale, use $r=2$ and the values of $k=0$ and $k=1$ in Eq. (4.16) above. Therefore,

$$L_2=\frac{1}{\lambda_2}\sum_{k=0}^{2-1}(-1)^{2-k-1}\binom{2-1}{k}\binom{2+k-1}{k}\left\{\frac{\prod_{j=0}^{k}(\lambda_3+j)\pi\mathrm{Csc}(\pi\lambda_3)}{\lambda_3(k+1)!}-(-1)^{r-1}\frac{\prod_{j=0}^{k}(\lambda_4+j)\pi\mathrm{Csc}(\pi\lambda_4)}{\lambda_4(k+1)!}\right\}$$

$$=\frac{1}{\lambda_2}\left((-1)^{2-1}\binom{2-1}{0}\binom{2+0-1}{0}\left\{\frac{(\lambda_3+0)\pi\mathrm{Csc}(\pi\lambda_3)}{\lambda_3(1)!}-(-1)^{2-1}\frac{(\lambda_4+0)\pi\mathrm{Csc}(\pi\lambda_4)}{\lambda_4(1)!}\right\}\right)$$
$$+\frac{1}{\lambda_2}\left((-1)^{2-2}\binom{2-1}{1}\binom{2+1-1}{1}\left\{\frac{\lambda_3(\lambda_3+1)\pi\mathrm{Csc}(\pi\lambda_3)}{\lambda_3(1+1)!}-(-1)^{2-1}\frac{\lambda_4(\lambda_4+1)\pi\mathrm{Csc}(\pi\lambda_4)}{\lambda_4(1+1)!}\right\}\right)$$

$$=\frac{1}{\lambda_2}\left((-1)\{\pi\mathrm{Csc}(\pi\lambda_3)+\pi\mathrm{Csc}(\pi\lambda_4)\}\right)+\frac{1}{\lambda_2}\left((2)\left\{\frac{(\lambda_3+1)\pi\mathrm{Csc}(\pi\lambda_3)}{(2)!}+\frac{(\lambda_4+1)\pi\mathrm{Csc}(\pi\lambda_4)}{(2)!}\right\}\right)$$

$$=\frac{1}{\lambda_2}\left(\{\lambda_3\pi\mathrm{Csc}(\pi\lambda_3)+\lambda_4\pi\mathrm{Csc}(\pi\lambda_4)\}\right).$$

In the same way, expressions for L_3 and L_4 are obtained by substitution of the respective values of *r* and *k* into Eq. (4.16).

Consequently,

$$L_3 = \frac{1}{\lambda_2}\left(\pi\mathrm{Csc}(\pi\lambda_3)(\lambda_3{}^2) - \pi\mathrm{Csc}(\pi\lambda_4)(\lambda_4{}^2)\right), \tag{4.18}$$

and

$$L_4 = \frac{1}{6\lambda_2}\left(\lambda_3\pi\mathrm{Csc}(\pi\lambda_3)(1+5\lambda_3{}^2) + \lambda_4\pi\mathrm{Csc}(\pi\lambda_4)(1+5\lambda_4{}^2)\right). \tag{4.19}$$

L-moment ratios of the GLO$_{\mathrm{QB}}$ are obtained by using Eqs. (4.18) and (4.19) and substituting them into Eq. (2.11). Subsequently,

$$\tau_3 = \frac{L_3}{L_2} = \frac{Csc(\pi\lambda_3)\lambda_3{}^2 - Csc(\pi\lambda_4)\lambda_4{}^2}{Csc(\pi\lambda_3)\lambda_3 + Csc(\pi\lambda_4)\lambda_4} \tag{4.20}$$

and

$$\tau_4 = \frac{L_4}{L_2} = \frac{\lambda_3 Csc(\pi\lambda_3)(1+5\lambda_3{}^2) + \lambda_4 Csc(\pi\lambda_4)(1+5\lambda_4{}^2)}{(Csc(\pi\lambda_3)\lambda_3 + Csc(\pi\lambda_4)\lambda_4)}. \tag{4.21}$$

■

The (τ_3, τ_4) space covered by the GLO$_{\mathrm{QB}}$ is shown by the shaded area in Figure 4.7 below. The lower boundary at $\tau_4 = \frac{1}{6}(5\tau_3{}^2 + 1)$, is indicated by the solid curve and given by the GLO. The dashed curve at $\tau_4 = \frac{1}{4}(5\tau_3{}^2 - 1)$, is the lower boundary for all probability distributions. The logistic distribution, with $\lambda_3 = \lambda_4 = 0$, has (τ_3, τ_4)=$\left(0, \frac{1}{6}\right)$. From Figure 4.7 it follows that the minimum value for τ_4 obtainable by the GLO$_{\mathrm{QB}}$ is $\tau_4 = \frac{1}{6}$, in effect, the *L*-kurtosis value for the logistic distribution. Note that the logistic distribution is a leptokurtic distribution with heavier tails than the normal distribution. Hence the GLO$_{\mathrm{QB}}$ is a leptokurtic family of distributions. The GLO$_{\mathrm{QB}}$ is symmetric if $\lambda_3 = \lambda_4$(i.e., $\tau_3 = 0$), positively skewed if $\lambda_3 > \lambda_4$(i.e., $\tau_3 > 0$) and negatively skewed when $\lambda_3 < \lambda_4$ (i.e., $\tau_3 < 0$).

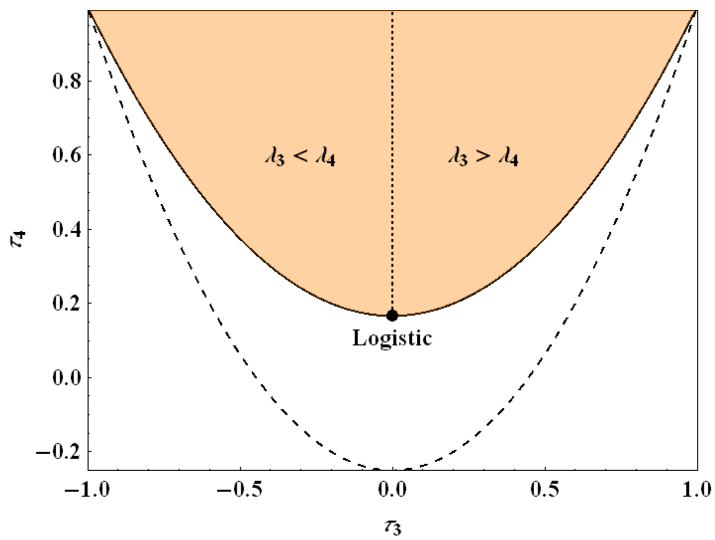

Figure 4.7. L-Moment ratio diagram for the $\mathrm{GLO_{QB}}$.

4.8 QUANTILE-BASED MEASURES OF LOCATION, SPREAD AND SHAPE.

Unlike the conventional moments or the L-moments, quantile-based measures of location, spread and shape exist for all parameter values of a distribution.

Theorem 4.8. *The median, spread function (MacGillivray and Balanda, 1988), γ-functional (MacGillivray, 1986) and ratio-of-spread functions (MacGillivray and Balanda, 1988) of $X \sim \mathrm{GLO_{QB}}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are*

$$me = Q\left(\frac{1}{2}\right) = \lambda_1, \tag{4.22}$$

$$S(u) = \frac{\left(\frac{u}{1-u}\right)^{\lambda_3} - \left(\frac{1-u}{u}\right)^{\lambda_3}}{\lambda_2\lambda_3} + \frac{\left(\frac{u}{1-u}\right)^{\lambda_4} - \left(\frac{1-u}{u}\right)^{\lambda_4}}{\lambda_2\lambda_4}, \quad for\,\frac{1}{2} < u < 1 \tag{4.23}$$

$$\gamma(u) = \frac{\left(\frac{u}{1-u}\right)^{\lambda_3} + \left(\frac{1-u}{u}\right)^{\lambda_3} - 2}{\lambda_2\lambda_3} - \frac{\left(\frac{1-u}{u}\right)^{\lambda_4} + \left(\frac{u}{1-u}\right)^{\lambda_4} - 2}{\lambda_2\lambda_4} \quad for\,\frac{1}{2} < u < 1 \tag{4.24}$$

$$R(u,v) = \frac{\left(\left(\frac{1-u}{u}\right)^{\lambda_4} + \left(\frac{u}{1-u}\right)^{\lambda_4} - 2\right)\lambda_3 - \left(\left(\frac{1-u}{u}\right)^{\lambda_3} + \left(\frac{u}{1-u}\right)^{\lambda_3} - 2\right)\lambda_4}{\left(\left(\frac{1-v}{v}\right)^{\lambda_4} + \left(\frac{v}{1-v}\right)^{\lambda_4} - 2\right)\lambda_3 - \left(\left(\frac{1-v}{v}\right)^{\lambda_3} + \left(\frac{v}{1-v}\right)^{\lambda_3} - 2\right)\lambda_4}, for\,\frac{1}{2} < v < u < 1 \tag{4.25}$$

Proof: The expressions in Eqs. (4.22) - (4.25) are attained by substituting Eq. (4.4) into Eq. (2.13) - (2.16) respectively and simplifying.

4.9 LOCATION AND SPREAD.

In this section the effect of changing the values of the location and spread parameters, whilst holding the two shape parameters constant, is considered. For the location, an increase in the value of λ_1 leads to an increase in the location position of the density curve. As can be seen in Figure 4.9(a), an increase in the value of λ_1 shifts the position of the density function to the right. Similarly a decrease in the location parameter leads to a shift towards the left side of the *x*-axis. As for the scale, an increase in the value of L_2 results in a decrease in the spread of the GLO_{QB}, as illustrated in Figure 4.9(b). This inverse relationship is evident from the expressions of the variance in Eq. (4.9) and of L_2, the L-scale, in Eq. (4.11).

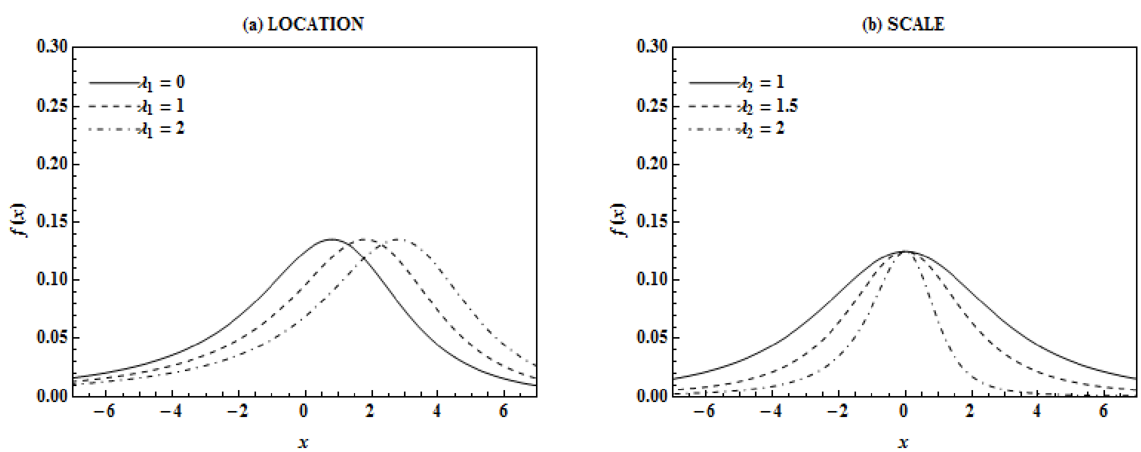

Figure 4.9. Probability density functions of members of the GLO_{QB} with varying location and spread. In (a),$(\lambda_2, \lambda_3, \lambda_4) = (1,0.25,0.5)$, where an increase in the value of $\lambda_1$leads to an increase in the value of the position of the location of the distribution. In (b), ,$(\lambda_1, \lambda_3, \lambda_4) = (0,0.5,0.75)$, with an increase in the value of λ_2 resulting in a decrease in the spread of the GLO_{QB}.

4.10 TAIL BEHAVIOUR.

The tail behavior of the density curve of a distribution is typically evaluated through the probability density function, $f(x)$. However, in the case of the quantile-based distributions, the tail behavior is evaluated through the density quantile function, $f_p(p)$, since no closed-form expression of the probability density function exists. The investigation involves determining the value that the density curve approaches at the endpoints, that is at the left and right tail. This is explored through computing $\lim_{p\to 0} f_p(p)$ and $\lim_{p\to 1} f_p(p)$. The expression for $f_p(p)$ for the GLO_{QB} was given in Eq. (4.6). In addition, the slope of the density curve at these two tails is also evaluated to determine the behavior of this distribution. This is done through attaining $\lim_{p\to 0} \xi(p)$ and $\lim_{p\to 1} \xi(p)$, where

$$\xi(p) = -\frac{dq(p)}{dp}\frac{1}{(q(p))^3}, \tag{4.26}$$

which was derived by King (1999) . This expression is the derivative of the probability density function expressed in terms of *p*. In the case of the GLO_{QB},

$$\xi(p) = \frac{p(1-p)\lambda_2^2\left((\lambda_3+2p-1)\left(\frac{p}{1-p}\right)^{\lambda_3}-\left(\frac{1-p}{p}\right)^{\lambda_4}(\lambda_4-2p+1)\right)}{\left(\left(\frac{1-p}{p}\right)^{\lambda_4}+\left(\frac{p}{1-p}\right)^{\lambda_3}\right)^3} \tag{4.27}$$

The values obtained for the density and the slope of the density curve are summarized below in Table 4.10.

There are various scenarios with regards to the value of the density curve when the limits at the end-points are obtained. When $\lambda_3 > \lambda_4 > 0$, the right tail and the left tail approach zero for Class I. Class II represents density curves for which $0 > \lambda_4 > \lambda_3$. The right tail approaches zero for all the values of λ_4 and λ_3, whereas the left tail approaches $(-\infty, -0.5{\lambda_2}^2, -0.125{\lambda_2}^2, 0)$ for various combinations of these shape parameters in this class.

Class III denotes density curves of the GLO_{QB} with $\lambda_4 \geq 1$ and $\lambda_3 \leq -1$. The density curves in this region are reversed J-shaped with the tail direction being to the left. The right tail approaches $(0.5\lambda_2, \lambda_2, \infty)$ for different pairings of λ_4 and λ_3 in this class, whilst the left tail of the density curves approaches $(0.5\lambda_2, 2\lambda_2, 3\lambda_2, \infty)$ for the combinations of λ_4 and λ_3 in this class. In Class IV, the density curves of the GLO_{QB} are J-shaped for $\lambda_3 \geq 1$ and $\lambda_4 \leq -1$, with the tail direction being to the right.

The investigation of the slope of the density curves of the GLO_{QB} reveals several results. For the unimodal density curves in Class I and Class II, the slope of the density is zero for the right tail, $\left(0, 0.125{\lambda_2}^2, 0.5{\lambda_2}^2, \infty\right)$ in Class I for the left tail and zero for the left tail of Class I. The slope of the density curves in Class III is zero for both tails. As for Class IV, the slope of the density curve of the right tail is $(0.5\lambda_2, \lambda_2, \infty)$ for $\lambda_3 \geq 1$ and $\lambda_4 \leq -1$ and $\left(-0.5{\lambda_2}^2, -2{\lambda_2}^2, -3{\lambda_2}^2, \infty\right)$ for the left tails.

Table 4.10 The values approached by the density curve and the slope of the density curve of the GLO_{QB} at the end-points of the tails.

Class	Shape parameter values	Density(right)	Density(left)	Slope(right)	Slope(left)
Class I	$\lambda_3 > \lambda_4 \geq 0$	0	0	0	0
	$0 > \lambda_3 > \lambda_4$	0	0	0	0
	$\lambda_3 = 0.5, \lambda_4 = -0.5$	0	0	0	$0.125{\lambda_2}^2$
	$0 < \lambda_3 < 1, -0.5 < \lambda_4 < 0$	0	0	0	0
	$0.5 < \lambda_3 < 1, \lambda_4 = -0.5$	0	0	0	$0.5{\lambda_2}^2$
	$0 < \lambda_3 < 0.5, \lambda_4 < 0$	0	0	0	0
	$\lambda_3 = 0.5, \lambda_4 < -0.5$	0	0	0	$0.5{\lambda_2}^2$
	$\lambda_3 > 1, -0.5 < \lambda_4 < 0$	0	0	0	0
	$\lambda_3 > 1, \lambda_4 = -0.5$	0	0	0	$0.5{\lambda_2}^2$
	$\lambda_3 > 1, -1 < \lambda_4 < -0.5$	0	0	0	∞
Class II	$0 \leq \lambda_3 < \lambda_4$	0	0	0	0
	$\lambda_3 < \lambda_4 < 0$	0	0	0	0
	$\lambda_3 = -0.5, \lambda_4 = 0.5$	0	$-0.125{\lambda_2}^2$	0	0
	$-0.5 < \lambda_3 < 0, 0 < \lambda_4 < 1)$	0	0	0	0
	$\lambda_3 = -0.5, 0.5 < \lambda_4 < 1$	0	$-0.5{\lambda_2}^2$	0	0
	$\lambda_3 < 0, 0 < \lambda_4 < 0.5$	0	0	0	0
	$\lambda_3 < -0.5, \lambda_4 = 0.5$	0	$-0.5{\lambda_2}^2$	0	0
	$-0.5 < \lambda_3 < 0, \lambda_4 > 1$	0	0	0	0
	$\lambda_3 = -0.5, \lambda_4 > 1$	0	$-0.5{\lambda_2}^2$	0	0
	$-1 < \lambda_3 < -0.5, \lambda_4 > 1$	0	$-\infty$	0	0
Class III	$\lambda_3 = -1, \lambda_4 = 1$	$-0.5\lambda_2$	0	$-0.5{\lambda_2}^2$	0
	$-2 < \lambda_3 < -1, \lambda_4 = 1$	$-\lambda_2$	0	$-\infty$	0
	$\lambda_3 = -2, \lambda_4 = 1$	$-\lambda_2$	0	$-3{\lambda_2}^2$	0
	$\lambda_3 = -1, \lambda_4 = 2$	$-\lambda_2$	0	$-3{\lambda_2}^2$	0
	$\lambda_3 < -2, \lambda_4 = 1$	$-\lambda_2$	0	$-2{\lambda_2}^2$	0
	$\lambda_3 < -1, \lambda_4 > 1$	$-\infty$	0	$-\infty$	0
	$\lambda_3 = -1, 1 < \lambda_4 < 2$	$-\lambda_2$	0	$-\infty$	0
	$\lambda_3 = -1, \lambda_4 > 2$	$-\lambda_2$	0	$-2{\lambda_2}^2$	0
Class IV	$\lambda_3 = 1, \lambda_4 = -1$	0	$0.5\lambda_2$	0	$0.5{\lambda_2}^2$
	$\lambda_3 = 1, -2 < \lambda_4 < -1$	0	λ_2	0	∞
	$1 < \lambda_3 < 2, \lambda_4 = -1$	0	λ_2	0	∞
	$\lambda_3 = 1, \lambda_4 = -2$	0	λ_2	0	$3{\lambda_2}^2$
	$\lambda_3 = 2, \lambda_4 = -1$	0	λ_2	0	$3{\lambda_2}^2$
	$\lambda_3 = 1, \lambda_4 < -2$	0	λ_2	0	$2{\lambda_2}^2$
	$\lambda_3 > 2, \lambda_4 = -1$	0	λ_2	0	$2{\lambda_2}^2$
	$\lambda_3 > 1, \lambda_4 < -1$	0	∞	0	∞

5 FITTING OF THE GLO_{QB} TO DATA

5.1 <u>METHOD OF L-MOMENTS ESTIMATION.</u>

This section entails the estimation of the parameters that will be required in order to fit the GLO_{QB} to an observed data set. The parameters that are to be estimated are λ_1, λ_2, λ_3 and λ_4, which are the location, spread and shape parameters respectively.

For these parameters to be estimated, it is required that a measure of location, spread and two measures of shape are obtained, from which the estimation method will proceed. The measures of shape that have been obtained for the GLO_{QB} are the L-skewness ratio (Eq. (4.12)) and the L –kurtosis ratio (Eq. (4.13)), using the L-moments in Chapter 4.

It was not sensible to use the skewness and kurtosis moment ratios for the GLO_{QB} based on the conventional moments, since they are computationally difficult as stated in Section 4.6. As a result, the method of moments will be an impractical estimation method to obtain the parameters for the GLO_{QB}.

Hosking (1990) introduced and used method of L-moments estimation, whereby the unknown parameters are estimated from linear combinations of an ordered data set.

As a result of their unbiasedness and accuracy in small samples, they can be extended to the estimation of the parameters of the underlying distribution.

Estimates based on L-moments are generally superior to standard moment based estimates. The method yields more accurate estimates since they are robust with respect to outliers. Being linear functions of data, they are less sensitive to sampling variability as compare to the conventional moments.

Moreover, they hold the property of unbiasedness and L-moments are less prone to bias estimation and approximation by asymptotic normal distribution is more accurate in finite samples.

Let $X_1, X_2, X_3, \dots \dots, X_n$ be a sample of size n, and $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be an ordered sample.

Hosking (1990) defined the r^{th}Sample ℓ-moment as

$$\ell_r = \binom{n}{r}^{-1} \underset{1 < i_1 < i_2 < \cdots < i_r < n}{\sum \sum \ldots\ldots\ldots\ldots\ldots\sum} r^{-1} \quad \sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} x_{i_{r-k:n}} \quad for\, r = 1, 2, 3, \ldots, n. \tag{5.1}$$

In particular,

$$\ell_1 = \frac{1}{n} \quad \sum_{i=1}^{n} x_i \,, \tag{5.2}$$

$$\ell_2 = \frac{1}{2} \binom{n}{2}^{-1} \sum_{i>j}^{n} \sum (x_i - x_j) \,, \tag{5.3}$$

$$\ell_3 = \frac{1}{3} \binom{n}{3}^{-1} \sum_{i>j>k}^{n} \sum \sum (x_i - 2x_j + x_k) \,, \tag{5.4}$$

and

$$\ell_4 = \frac{1}{4} \binom{n}{4}^{-1} \sum_{i>j>k>l}^{n} \sum \sum \sum (x_i - 3x_j + 3x_k - x_l) \tag{5.5}$$

are the first four sample ℓ-moments.

Notably, ℓ_1 is the sample mean, ℓ_2 is a scalar multiple of the Gini's mean difference statistic, $G = \binom{n}{2}^{-1} \sum_{i>j}^{n} \sum (x_i - x_j)$, (Gini, 1912).

ℓ_3 is used to test for normality in a distribution (Locke & Spurrier, 1976). ℓ_4 could be used to test a null hypothesis of normality against symmetric alternatives.

Similarly, the sample L-skewness and L-kurtosis can be defined as

$$t_3 = \frac{\ell_3}{\ell_2} \tag{5.6}$$

and

$$t_4 = \frac{\ell_4}{\ell_2} \tag{5.7}$$

respectively.

The four parameters of the GLO_{QB} can be obtained using the following estimation algorithm:

STEP 1:

The first four sample L-moments are calculated using Eq. (5.1). In essence, use Eqs. (5.2) - (5.5) to obtain the sample L-moments. The sample L-moment ratios, that is the sample L-skewness and L-kurtosis ratios, are then obtained using Eq. (5.6) and Eq. (5.7).

Before any further procedures are carried out in the estimation procedures, it is verified whether the values of t_3 and t_4 lie within the (τ_3, τ_4) space of the GLO_{QB} in Figure 4.7. If the values are found to lie within this space, then proceed to the second step. If this is not the case, then the GLO_{QB} cannot be fitted to the data.

STEP 2:

Since both the L-skewness and L-kurtosis ratios of the GLO_{QB} are functions of both the shape parameters, a numerical optimization method must be used to simultaneously obtain solutions for $\hat{\lambda}_3$ and $\hat{\lambda}_4$. The function that is used is the *FindRoot* function in *Mathematica* 8.0 (Wolfram, 2010). In order to find the solutions, Eq. (4.12) – (4.13) will be utilized concurrently.

STEP 3:

Solve for $\hat{\lambda}_2$ using Eq. (4.11) and then for $\hat{\lambda}_1$ using Eq. (4.10)

5.2 FITTING OF THE GLO$_{QB}$ TO DATA.

Consider the concentrations of the polychlorinated biphenyl (PCB) in the yolklipids of pelican eggs, used by Thas (2010). The data set of $n = 65$ observations, was used by him as an example data set with respect to goodness-of-fit testing. In Figure 5.2(a), a histogram of the data set is illustrated. The first four sample L-moments are calculated using Eqs. (5.2) – (5.7). The values of the sample L-location, L-scale, L-skewness ratio and L-kurtosis ratio for the data set are $\ell_1 = 210$, $\ell_2 = 39.793$, $t_3 = 0.104$ and $t_4 = 0.213$ respectively. The parameter estimates of the fitted GLO$_{QB}$ using the method of L-moments are $\hat{\lambda}_1 = 203.598$, $\hat{\lambda}_2 = 0.055$, $\hat{\lambda}_3 = 0.301$ and $\hat{\lambda}_4 = 0.12$. Figure 5.2(a) shows the probability density curve of the fitted GLO$_{QB}$ whilst Figure 5.2(b) shows the corresponding Q-Q plot, which indicates that the GLO$_{QB}$ provides an excellent fit to the data set.

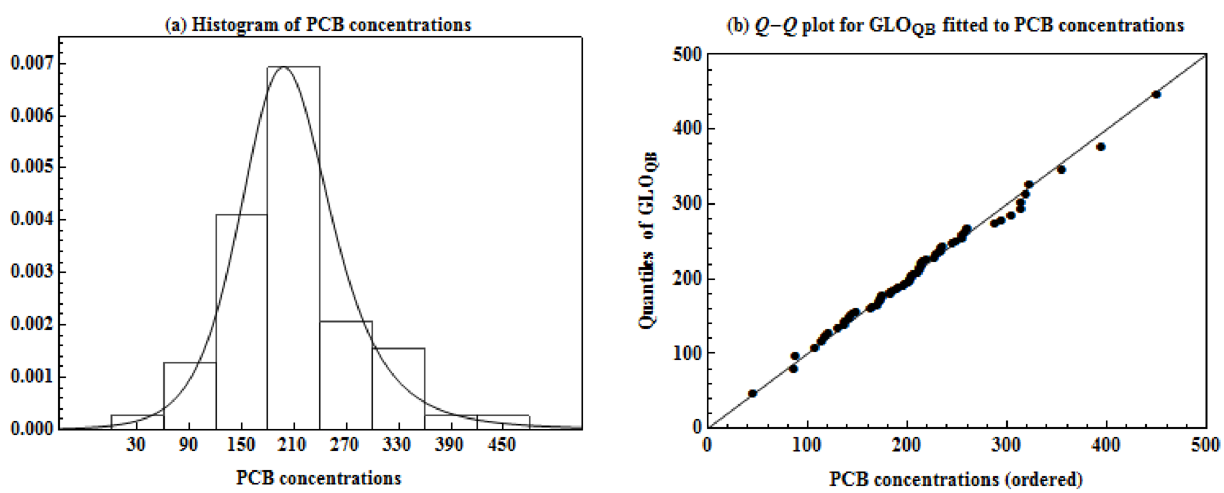

Figure 5.2. A histogram of the PCB concentrations of the polychlorinated biphenyl (PCB) in the yolklipids of the pelican eggs together with the probability density curve of the fitted GLO$_{QB}$ and the corresponding Q-Q Plot.

5.3 CONCLUSION.

By making use of the quantile function of the GLO with shape parameter λ as the building block and the Proposition 2.7.1 in Chapter 2, a new four-parameter quantile-based distribution is constructed. It is highly flexible with respect to distributional shape in that it explains extensive levels of skewness and kurtosis through the inclusion of two shape parameters, in addition to the location and scale parameters. This distribution is defined in terms of its quantile function in Definition 4.2.1 and denoted as GLO_{QB}.

The distributional properties and shape characteristics are discussed in detail. The parameter space and support as well as the classes is defined for this distribution in Section 4.4 and 4.5. The expressions of the moments of this distribution are defined and derived in Section 4.6. In Section 4.7, the new distribution is characterized through its L-moments, showing that they are simpler in characterizing the distribution. The various expressions for the quantile-based measure of location, spread and shape are defined. Section 4.9 summarizes the values obtained for the density and the slope of the density curve.

Chapter 5 presents an estimation algorithm for estimating the distribution's parameters with method of L-moments estimation, due to the ease in the relationship between the parameters and the L-moments. Using this method, the GLO_{QB} can be fitted to data sets and be used to approximate probability distributions.

5.4 FUTURE RESEARCH.

The aim of this dissertation was in the construction of a new quantile-based generalized logistic distribution. This was achieved by making use of the asymmetric GLO with a single shape parameter as the building block of the distribution. Of particular interest, it would be worth considering other distributions that can be constructed using the other Types of the GLO i.e. Type I, II, III and IV as the building blocks and evaluate the fit of the distributions to various data sets. The development of other estimation methods for the parameters is also worth exploring.

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