Commutativity in the lattice of topologizing filters of a commutative semiartinian ring

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Abstract.

The set Fil R_R of all right topologizing filters on a fixed but arbitrary ring R admits a monoid operation ':' that is in general noncommutative, even in cases where the ring R is commutative. Earlier results show (see [8]) that commutativity of the monoid operation ':', when imposed as a condition on Fil R_R, manifests as a type of finiteness condition on R. In a quite separate and much earlier study, Shores [6] has shown that if R is a commutative semiartinian ring, then R will be artinian precisely if the first two terms in the Loewy series for R_R, namely soc (R_R) and $\mathrm{soc}^2(R_R)$, are finitely generated. Shores goes further to produce examples which show that the finiteness of just soc (R_R) exercises no constraint whatsoever on the length of R_R. The main result of this paper asserts that a commutative semiartinian ring R will be artinian precisely if soc (R_R) is finitely generated and the monoid operation ':' on Fil R_R is commutative. A family of commutative semiartinian rings of Loewy length 3 is constructed and this used to delineate earlier theory. In particular, and within this family, rings R are exhibited such that (1) soc (R_R) and $\mathrm{soc}^2(R_R)/\mathrm{soc}(R_R)$ have infinite length, yet (2) the monoid operation ':' on Fil R_R is commutative

Keywords: topologizing filter, semiartinian, Loewy module, Loewy length, socle

1. Introduction

A *right topologizing filter* on a ring R (with identity) is a nonempty family $\mathfrak{F}$ of right ideals of R that satisfies the following three conditions:

F1. $A \in \mathfrak{F}$ implies $B \in \mathfrak{F}$ whenever B is a right ideal of R containing A;

F2. $A, B \in \mathfrak{F}$ implies $A \cap B \in \mathfrak{F}$;

F3. $A \in \mathfrak{F}$ and $r \in R$ implies $r^{-1}A \stackrel{\text{def}}{=} \{x \in R : rx \in A\} \in \mathfrak{F}$.

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The family of all right ideals of a ring R that are open with respect to a right linear topology on R is, by definition, a neighbourhood base for 0; it is also a right topologizing filter on R. Moreover, every right topologizing filter on R arises in this way.

The set of all right topologizing filters on some fixed ring R, which we shall denote by $\operatorname{Fil} R_R$, is a complete lattice with respect to the relation of inclusion. It also admits a monoid operation ':' (to be defined in Section 2.2) that distributes over finite meets. This property renders the order dual $[\operatorname{Fil} R_R]^{\text{du}}$ of $\operatorname{Fil} R_R$ a lattice ordered monoid.

The importance of $\operatorname{Fil} R_R$ lies in the fact that it encodes at least as much information about the ring R as does the ideal lattice $\operatorname{Id} R$, for there is an embedding (that is in general not onto) of $\operatorname{Id} R$ into $[\operatorname{Fil} R_R]^{\text{du}}$ that takes each $I \in \operatorname{Id} R$ onto the set of all right ideals of R containing I. This embedding is, moreover, structure preserving for it preserves the lattice operations and the monoid operation ':' in the sense that ':', when restricted to $\operatorname{Id} R$, coincides with ideal multiplication (see Theorem 3).

With the exception of Sections 3 and 6, the rings considered will always be commutative. The reader will observe that for commutative R, Condition F3 in the definition of a (right) topologizing filter is implied by F1, so that a topologizing filter on R is just a filter, in the purely lattice theoretic sense, on the ideal lattice $\operatorname{Id} R$ of R. There is, however, much structural complexity in $\operatorname{Fil} R$ that derives from the monoid operation ':' and its interaction with the lattice operations. Indeed, ':' need not be commutative, even in cases where the ring R is commutative. In this respect, $\operatorname{Fil} R$ differs from the smaller structure $\operatorname{Id} R$, for ideal multiplication always commutes in a commutative ring.

The purpose of Section 2, titled Preliminaries, is self-evident. Section 3 establishes connections between $\operatorname{Fil} R_R$ and $\operatorname{Fil} T_T$ in the case where R and T are arbitrary rings that are linked by a ring homomorphism $\varphi : R \to T$. A correspondence theorem (Theorem 7) shows that if I is any proper ideal of arbitrary ring R, then $\operatorname{Fil} (R/I)_{R/I}$ is isomorphic to an interval in $\operatorname{Fil} R_R$, a fact that we shall exploit in Section 7.

The main theorem (Theorem 17) of Section 4, shows that if S is a multiplicative subset of commutative ring R and RS^{-1} denotes the ring of fractions of R with respect to S, then the map $\hat{\varphi}_S$ from $[\operatorname{Fil} R]^{\text{du}}$ to $[\operatorname{Fil} RS^{-1}]^{\text{du}}$ defined by $\mathfrak{F} \mapsto \{IS^{-1} : I \in \mathfrak{F}\}$, is an onto homomorphism of lattice ordered monoids. This homomorphism induces a canonical congruence $\equiv_{\hat{\varphi}_S}$ on $[\operatorname{Fil} R]^{\text{du}}$ whose properties we shall explore in Section 5. In particular, we see (16) that

$$[\operatorname{Fil} RS^{-1}]^{\text{du}} \cong [\operatorname{Fil} R]^{\text{du}} / \equiv_{\hat{\varphi}_S} .$$

If $\operatorname{spec_m} R$ denotes the set of maximal ideals of commutative ring R and $S = R \backslash P$ with $P \in \operatorname{spec_m} R$, then $R_P \overset{\text{def}}{=} RS^{-1}$ is just the localization of R at P. We define $\operatorname{Rad}(\operatorname{Fil} R)$ to be the intersection of all congruences on $[\operatorname{Fil} R]^{\text{du}}$ of the form $\equiv_{\hat{\varphi}_{R \backslash P}}$ as P ranges through $\operatorname{spec_m} R$. Thus $\operatorname{Rad}(\operatorname{Fil} R)$ is the kernel of the canonical homomorphism

$$[\operatorname{Fil} R]^{\mathrm{du}} \longrightarrow \prod_{P \in \operatorname{spec}_{\mathrm{m}} R} [\operatorname{Fil} R_P]^{\mathrm{du}}$$

which takes $\mathfrak{F} \in \operatorname{Fil} R$ to $\{\hat{\varphi}_{R \setminus P}(\mathfrak{F})\}_{P \in \operatorname{spec}_{\mathrm{m}} R}$.

In general, $\operatorname{Rad}(\operatorname{Fil} R)$ is nontrivial (meaning, not equal to the identity congruence). Indeed, for a commutative Von Neumann Regular ring R, we show that $\operatorname{Rad}(\operatorname{Fil} R)$ is trivial if and only if R is noetherian and thus a finite product of fields (Proposition 20). If, however, R is a commutative ring such that $\operatorname{Fil} R$ is commutative (meaning, the monoid operation ':' on $\operatorname{Fil} R$ is commutative), then $\operatorname{Rad}(\operatorname{Fil} R)$ is trivial (Theorem 32). This has the consequence that for such rings R, the canonical homomorphism from $[\operatorname{Fil} R]^{\mathrm{du}}$ to $\prod_{P \in \operatorname{spec}_{\mathrm{m}} R} [\operatorname{Fil} R_P]^{\mathrm{du}}$ is a subdirect embedding (see Corollary 33).

Recall that a lattice ordered monoid L with monoid operation '$\cdot$' is said to be *left residuated* if for every $a, b \in L$, there exists a largest $x \in L$ such that $x \cdot b \leqslant a$. In this situation we call x the *left residual of a by b* and denote it ab^{-1}. Similarly, we say that L is *right residuated* if for every $a, b \in L$, there exists a largest $x \in L$, called the *right residual of a by b* and denoted $b^{-1}a$, such that $b \cdot x \leqslant a$. It is known that if R is an arbitrary ring, then the lattice ordered monoid $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is left, but in general not right, residuated (see Theorem 23). If $\operatorname{Fil} R_R$ is commutative, then clearly the notions left residuated and right residuated coincide making $[\operatorname{Fil} R_R]^{\mathrm{du}}$ a two-sided residuated lattice ordered monoid.

The study of rings R for which $\operatorname{Fil} R_R$ is commutative was initiated in [11], but later widened in [1] to include rings enjoying the more general two-sided residuation property. Further contributions to this project are made in Section 6. A main theorem (Theorem 28) shows that rings R for which $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated possess only finitely many minimal prime ideals.

In Section 7, theory is put to use to show that the only Prüfer domains R for which $\operatorname{Fil} R$ is commutative are noetherian and thus Dedekind domains. This result generalises [11, Corollary 32, page 102].

2. Preliminaries

The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. If A, B are sets, $f : A \to B$ a function, and $B' \subseteq B$, we define

$$f^{-1}[B'] \stackrel{\mathrm{def}}{=} \{a \in A : f(a) \in B'\}.$$

Throughout this paper R will denote an associative ring with identity and Mod-R the category of unital right R-modules. If $M, N \in$ Mod-R, $\operatorname{Hom}_R(M, N)$ shall denote the additive abelian group of all R-module homomorphisms $f : M \to N$. We write $N \leqslant M$ if N is a submodule of M. If X, Y are nonempty subsets of M, we define

$$Y^{-1}X \stackrel{\mathrm{def}}{=} \{r \in R : Yr \subseteq X\}.$$

If $Y = \{y\}$ [resp. $X = \{x\}$] is a singleton we write $y^{-1}X$ [resp. $Y^{-1}x$] in place of $\{y\}^{-1}X$ [resp. $Y^{-1}\{x\}$].

2.1. **Lattice ordered monoids.** A *lattice ordered monoid* is a structure $\langle L, \vee, \wedge, \cdot, e_L \rangle$ where:

L1. $\langle L, \vee, \wedge \rangle$ is a lattice;
L2. $\langle L, \cdot, e_L \rangle$ is a monoid with identity element e_L;
L3. $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$ and $(b \vee c) \cdot a = (b \cdot a) \vee (c \cdot a)$ for all $a, b, c \in L$.

In the interests of brevity, we shall refer to L as a lattice ordered monoid in cases where the monoid and lattice operations are understood and no ambiguity arises from their suppression in the notation.

Note that L3 entails the monoid operation '$\cdot$' is order preserving, that is, $a \leqslant b$ implies $a \cdot c \leqslant b \cdot c$ and $c \cdot a \leqslant c \cdot b$ for all $a, b, c \in L$.

The following result is ring theoretic folklore.

Proposition 1. *Let R be any ring (with identity). Then $\langle \operatorname{Id} R, +, \cap, \cdot, R \rangle$ is a complete, lattice ordered monoid, where the join is the operation $+$ of ideal addition, the meet is intersection $\cap$, and '$\cdot$' is the monoid operation of ideal multiplication.*

2.2. **Topologizing filters.** This section and the next, provide the torsion theoretic background that is necessary for what follows. A slightly more detailed exposition may be found in the early pages of [1]. For further background, we refer the reader to the texts [6], [7] and [10].

The set $\operatorname{Fil} R_R$ of right topologizing filters on ring R is closed under arbitrary intersections and thus has the structure of a complete lattice with respect to inclusion. The lattice join in $\operatorname{Fil} R_R$ has an internal description which we provide below. If $X \subseteq \operatorname{Fil} R_R$, then

$$\bigwedge X = \bigcap X, \text{ and}$$

$$\bigvee X = \left\{ K \leqslant R_R : K \supseteq \bigcap X' \text{ for some finite subset } X' \text{ of } \bigcup X \right\}. \tag{1}$$

The smallest element of $\operatorname{Fil} R_R$ is the singleton $\{R\}$ whilst the largest element is the family comprising all right ideals of R.

A key component of the structure of $\operatorname{Fil} R_R$ derives from a binary operation ':' defined by $\forall \mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R_R$,

$$\mathfrak{F} : \mathfrak{G} \stackrel{\text{def}}{=} \{K \leqslant R_R : \exists H \in \mathfrak{F} \text{ such that } H \supseteq K \text{ and } h^{-1}K \in \mathfrak{G} \ \forall h \in H\}.$$

It is easily seen that the smallest topologizing filter $\{R\}$ is an identity with respect to ':' and $\mathfrak{F} : \mathfrak{G} \supseteq \mathfrak{F} \vee \mathfrak{G}$ for all $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R_R$.

Theorem 2. [7, Proposition 4.1, page 43] *If R is any ring then the order dual of $\langle \operatorname{Fil} R_R, \vee, \cap, :, \{R\} \rangle$, henceforth denoted $[\operatorname{Fil} R_R]^{\mathrm{du}}$, is a complete, lattice ordered monoid.*

If I is an ideal of ring R, then the family

$$\eta(I) \stackrel{\text{def}}{=} \{K \leqslant R_R : K \supseteq I\}$$

is easily shown to constitute a right topologizing filter on R. If $\mathfrak{F} \in \operatorname{Fil} R_R$, then $\mathfrak{F} = \eta(I)$ for some $I \in \operatorname{Id} R$ if and only if $\mathfrak{F}$ is closed under arbitrary (not just finite) intersections [7, Proposition 1.14 and Corollary 1.15, page 9].

Theorem 3. [7, Proposition 2.7, page 17 and Proposition 3.4, page 31] *If R is any ring then the map from* $\operatorname{Id} R$ *to* $[\operatorname{Fil} R_R]^{\text{du}}$ *defined by $I \mapsto \eta(I)$ is a one-to-one homomorphism in respect of the binary join, meet, and multiplication operations, that also preserves arbitrary joins. Thus* $\operatorname{Id} R$ *is embedded in* $[\operatorname{Fil} R_R]^{\text{du}}$ *as a lattice ordered monoid.*

The above theorem allows us to interpret η as a mapping. If, in such a situation, ambiguity arises in relation to the choice of underlying ring R, we shall write η^R in place of η.

We call $\mathfrak{F} \in \operatorname{Fil} R_R$ a *right Gabriel topology* on R if $\mathfrak{F}$ is idempotent in the sense that $\mathfrak{F} : \mathfrak{F} = \mathfrak{F}$.

2.3. **Hereditary pretorsion classes.** A nonempty class $\mathcal{T}$ of right R-modules is called a *hereditary pretorsion class* if it is closed under (arbitary) direct sums, homomorphic images and submodules. The closure of $\mathcal{T}$ under, in particular, direct sums and homomorphic images, means that every right R-module M has a (unique) largest submodule $\mathcal{T}(M)$ called the $\mathcal{T}$*-torsion submodule* of M that belongs to $\mathcal{T}$. If $\mathcal{T}(M) = M$, or equivalently $M \in \mathcal{T}$, we say that M is $\mathcal{T}$*-torsion* and if $\mathcal{T}(M) = 0$, we say that M is $\mathcal{T}$*-torsion-free*.

A submodule U of $M \in$ Mod-R is called a *hereditary pretorsion submodule* of M if $U = \mathcal{T}(M)$ for some hereditary pretorsion class $\mathcal{T}$ of Mod-R.

If $\mathfrak{F} \in \operatorname{Fil} R_R$, define

$$\mathcal{T}_{\mathfrak{F}} \stackrel{\text{def}}{=} \{M \in \text{Mod-}R : x^{-1}0 \in \mathfrak{F} \ \forall x \in M\}. \tag{2}$$

It is easily checked that $\mathcal{T}_{\mathfrak{F}}$ is a hereditary pretorsion class in Mod-R that we shall call the *hereditary pretorsion class associated with* $\mathfrak{F}$. Thus, for each $M \in$ Mod-R,

$$\mathcal{T}_{\mathfrak{F}}(M) = \{x \in M : x^{-1}0 \in \mathfrak{F}\} = \{x \in M : xK = 0 \text{ for some } K \in \mathfrak{F}\}. \tag{3}$$

The map $\mathfrak{F} \mapsto \mathcal{T}_{\mathfrak{F}}$ constitutes a bijection from $\operatorname{Fil} R_R$ to the collection of all hereditary pretorsion classes in Mod-R [10, Proposition VI.4.2, page 145].

It follows from (2) that for each $K \leqslant R_R$,

$$\begin{aligned} R/K \in \mathcal{T}_{\mathfrak{F}} &\Leftrightarrow r^{-1}K \in \mathfrak{F} \ \forall r \in R \\ &\Leftrightarrow K \in \mathfrak{F} \ \text{[by Condition F3]}. \end{aligned} \tag{4}$$

Inasmuch as $H \in \mathfrak{F}$ if and only if $R/H \in \mathcal{T}_{\mathfrak{F}}$ by (4), and $h^{-1}K \in \mathfrak{G} \ \forall h \in H$ if and only if $H/K \in \mathcal{T}_{\mathfrak{G}}$ by (2), it follows that $K \in \mathfrak{F} : \mathfrak{G}$ if and only if there exists a short exact sequence

$$0 \longrightarrow H/K \longrightarrow R/K \longrightarrow R/H \longrightarrow 0$$

with $K \subseteq H \leqslant R_R$ such that $H/K \in \mathcal{T}_{\mathfrak{G}}$ and $R/H \in \mathcal{T}_{\mathfrak{F}}$. This can be generalised to: $M \in \mathcal{T}_{\mathfrak{F}:\mathfrak{G}}$ if and only if there exists a short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow L \longrightarrow 0$$

such that $N \in \mathcal{T}_{\mathfrak{G}}$ and $L \in \mathcal{T}_{\mathfrak{F}}$.

3. Change of rings

In this section we show how a ring homomorphism between two rings induces structure preserving maps between the rings' respective sets of topologizing filters. We derive a correspondence theorem (Theorem 7) in the process.

Proposition 4. *Let R and T be arbitrary rings and $\varphi : R \to T$ a ring homomorphism. Then the map $\varphi^* : \operatorname{Fil} T_T \to \operatorname{Fil} R_R$ defined by $\forall \mathfrak{F} \in \operatorname{Fil} T_T$,*

$$\varphi^*(\mathfrak{F}) \overset{\text{def}}{=} \{K \leqslant R_R : K \supseteq \varphi^{-1}[L] \text{ for some } L \in \mathfrak{F}\} \tag{5}$$

is a complete lattice homomorphism, that is to say, φ^* preserves arbitrary meets and joins. Moreover, $\varphi^*(\mathfrak{F} : \mathfrak{G}) \subseteq \varphi^*(\mathfrak{F}) : \varphi^*(\mathfrak{G})$ for all $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} T_T$.

Proof. That $\varphi^*(\mathfrak{F})$ is a right topologizing filter on R is easily established using the fact that for all $A, B \leqslant T_T$ and $r \in R$, $\varphi^{-1}[A \cap B] = \varphi^{-1}[A] \cap \varphi^{-1}[B]$ and $r^{-1}\varphi^{-1}[A] = \varphi^{-1}[\varphi(r)^{-1}A]$.

We show next that φ^* preserves arbitrary meets. To this end, let $\{\mathfrak{F}_\delta : \delta \in \Delta\}$ be a nonempty subset of $\operatorname{Fil} T_T$. Since φ^* is order preserving, the containment $\varphi^*(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta) \subseteq \bigcap_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$ is clear. To establish the reverse containment, take $K \in \bigcap_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$ so that $K \in \varphi^*(\mathfrak{F}_\delta)$ for all $\delta \in \Delta$. By (5), there exists, for each $\delta \in \Delta$, a right ideal $L_\delta \in \mathfrak{F}_\delta$ such that $K \supseteq \varphi^{-1}[L_\delta]$. It follows that $K \supseteq \sum_{\delta\in\Delta} \varphi^{-1}[L_\delta] = \varphi^{-1}[\sum_{\delta\in\Delta} L_\delta]$. Since $\sum_{\delta\in\Delta} L_\delta \supseteq L_\delta$ for each $\delta \in \Delta$, it follows that $\sum_{\delta\in\Delta} L_\delta \in \bigcap_{\delta\in\Delta} \mathfrak{F}_\delta$, so $K \in \varphi^*(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta)$. Thus $\varphi^*(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta) \supseteq \bigcap_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$, whence equality.

We now show that φ^* preserves arbitrary joins. The containment $\varphi^*(\bigvee_{\delta\in\Delta} \mathfrak{F}_\delta) \supseteq \bigvee_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$ is clear. Take $K \in \varphi^*(\bigvee_{\delta\in\Delta} \mathfrak{F}_\delta)$. Then $K \supseteq \varphi^{-1}[L]$ for some $L \in \bigvee_{\delta\in\Delta} \mathfrak{F}_\delta$. By (1), there exists a finite subset Δ' of Δ and a right ideal $A_\delta \in \mathfrak{F}_\delta$ for each $\delta \in \Delta'$ such that $L \supseteq \bigcap_{\delta\in\Delta'} A_\delta$. Then $K \supseteq \varphi^{-1}[L] \supseteq \varphi^{-1}[\bigcap_{\delta\in\Delta'} A_\delta] = \bigcap_{\delta\in\Delta'} \varphi^{-1}[A_\delta]$. Clearly $\varphi^{-1}[A_\delta] \in \varphi^*(\mathfrak{F}_\delta)$ for all $\delta \in \Delta'$, so $\bigcap_{\delta\in\Delta'} \varphi^{-1}[A_\delta] \in \bigvee_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$ by (1), whence $K \in \bigvee_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$. The containment $\varphi^*(\bigvee_{\delta\in\Delta} \mathfrak{F}_\delta) \subseteq \bigvee_{\delta\in\Delta} \varphi^*(\mathfrak{F}_\delta)$ is thus established, whence equality.

To complete the proof, it remains to show that $\varphi^*(\mathfrak{F} : \mathfrak{G}) \subseteq \varphi^*(\mathfrak{F}) : \varphi^*(\mathfrak{G})$ for all $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} T_T$. Take $K \in \varphi^*(\mathfrak{F} : \mathfrak{G})$. Then $K \supseteq \varphi^{-1}[L]$ for some $L \in \mathfrak{F} : \mathfrak{G}$, so there exists $H \in \mathfrak{F}$ such that $H \supseteq L$ and

$$t^{-1}L \in \mathfrak{G} \;\; \forall t \in H. \tag{6}$$

Since $H \in \mathfrak{F}$,

$$\varphi^{-1}[H] \in \varphi^*(\mathfrak{F}). \tag{7}$$

Observe that

$$\begin{aligned} r \in \varphi^{-1}[H] &\Rightarrow \varphi(r) \in H \\ &\Rightarrow \varphi(r)^{-1}L \in \mathfrak{G} \ \text{[by (6)]} \\ &\Rightarrow \varphi^{-1}[\varphi(r)^{-1}L] \in \varphi^*(\mathfrak{G}) \ \text{[by (5)]} \\ &\Rightarrow r^{-1}\varphi^{-1}[L] \in \varphi^*(\mathfrak{G}) \ [\text{because } \varphi^{-1}[\varphi(r)^{-1}L] = r^{-1}\varphi^{-1}[L]]. \end{aligned} \tag{8}$$

Statements (7) and (8) imply that $\varphi^{-1}[L] \in \varphi^*(\mathfrak{F}) : \varphi^*(\mathfrak{G})$, whence $K \in \varphi^*(\mathfrak{F}) : \varphi^*(\mathfrak{G})$. □

We point out, with reference to the previous result, that φ^* is, in general, not a monoid homomorphism with respect to ':'.

Now let I be a proper ideal of ring R and $\pi : R \to R/I$ the canonical ring epimorphism. Observe that in this situation, for each $\mathfrak{F} \in \mathrm{Fil}\,(R/I)_{R/I}$,

$$\begin{aligned} \pi^*(\mathfrak{F}) &= \{K \leqslant R_R : K \supseteq \pi^{-1}[L] \text{ for some } L \in \mathfrak{F}\} \\ &= \{K \leqslant R_R : K \supseteq I \text{ and } K/I \in \mathfrak{F}\}. \end{aligned} \tag{9}$$

We remind the reader that Mod-(R/I) may be interpreted as a subcategory of Mod-R: if $M \in$ Mod-(R/I) and $x \in M$, then

$$xr \overset{\text{def}}{=} x(r + I) \ \forall r \in R. \tag{10}$$

Now take $\mathfrak{F} \in \mathrm{Fil}\,(R/I)_{R/I}$ and let $M \in$ Mod-(R/I). Then

$$\begin{aligned} x \in \mathcal{T}_{\mathfrak{F}}(M) &\Leftrightarrow \exists K \leqslant R_R \text{ such that } K \supseteq I, K/I \in \mathfrak{F} \text{ and } x(K/I) = 0 \ \text{[by (3)]} \\ &\Leftrightarrow \exists K \in \pi^*(\mathfrak{F}) \text{ such that } x(K/I) = 0 \ \text{[by (9)]} \\ &\Leftrightarrow \exists K \in \pi^*(\mathfrak{F}) \text{ such that } xK = 0 \ [\text{because, by (10), } xK = x(K/I)] \\ &\Leftrightarrow x \in \mathcal{T}_{\pi^*(\mathfrak{F})}(M) \ \text{[by (3)]}. \end{aligned}$$

We have thus shown that, for every $M \in$ Mod-R/I,

$$\mathcal{T}_{\mathfrak{F}}(M) = \mathcal{T}_{\pi^*(\mathfrak{F})}(M). \tag{11}$$

Proposition 5. *Let I be a proper ideal of arbitrary ring R and $\pi : R \to R/I$ the canonical ring epimorphism. Then, for all $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\,(R/I)_{R/I}$,*

$$\pi^*(\mathfrak{F} : \mathfrak{G}) = [\pi^*(\mathfrak{F}) : \pi^*(\mathfrak{G})] \cap \eta(I).$$

Proof. By Proposition 4, $\pi^*(\mathfrak{F} : \mathfrak{G}) \subseteq \pi^*(\mathfrak{F}) : \pi^*(\mathfrak{G})$. It follows from (9) that for each $\mathfrak{F} \in \mathrm{Fil}\,(R/I)_{R/I}$, $\pi^*(\mathfrak{F}) \subseteq \{K \leqslant R_R : K \supseteq I\} = \eta(I)$. Thus $\pi^*(\mathfrak{F} : \mathfrak{G}) \subseteq [\pi^*(\mathfrak{F}) : \pi^*(\mathfrak{G})] \cap \eta(I)$.

To establish the reverse containment, take $K \in [\pi^*(\mathfrak{F}) : \pi^*(\mathfrak{G})] \cap \eta(I)$. Then there exists $H \leqslant R_R$ such that $H \supseteq K$, R/H is $\mathcal{T}_{\pi^*(\mathfrak{F})}$-torsion and H/K is $\mathcal{T}_{\pi^*(\mathfrak{G})}$-torsion. Inasmuch as $K \in \eta(I)$, $H \supseteq K \supseteq I$. This means that the short exact sequence

$$0 \longrightarrow H/K \longrightarrow R/K \longrightarrow R/H \longrightarrow 0$$

in Mod-R, induces the following short exact sequence in Mod-(R/I)

$$0 \longrightarrow (H/I)/(K/I) \longrightarrow (R/I)/(K/I) \longrightarrow (R/I)/(H/I) \longrightarrow 0.$$

Since $H \in \pi^*(\mathfrak{F})$ (because R/H is $\mathcal{T}_{\pi^*(\mathfrak{F})}$-torsion), it follows from (9) that $H/I \in \mathfrak{F}$. Since $(H/I)/(K/I) \cong H/K$ is $\mathcal{T}_{\pi^*(\mathfrak{G})}$-torsion, it follows from (11) that $(H/I)/(K/I)$ is $\mathcal{T}_{\mathfrak{G}}$-torsion. We conclude that $K/I \in \mathfrak{F} : \mathfrak{G}$, so $K \in \pi^*(\mathfrak{F} : \mathfrak{G})$. We have thus shown that $[\pi^*(\mathfrak{F}) : \pi^*(\mathfrak{G})] \cap \eta(I) \subseteq \pi^*(\mathfrak{F} : \mathfrak{G})$, whence equality. □

Let I be an ideal of arbitrary ring R. In general, the interval $[0, \eta(I)] \overset{\text{def}}{=} \{\mathfrak{F} \in \operatorname{Fil} R_R : \mathfrak{F} \subseteq \eta(I)\}$ of $\operatorname{Fil} R_R$ is not closed under the monoid operation ':', for $\eta(I) : \eta(I) = \eta(I \cdot I) = \eta(I^2)$ by Theorem 3, and $\eta(I^2)$ does not belong to $[0, \eta(I)]$ unless $I^2 = I$.

We define operation $:_I$ on $[0, \eta(I)]$ by $\forall \mathfrak{F}, \mathfrak{G} \in [0, \eta(I)]$,

$$\mathfrak{F} :_I \mathfrak{G} \overset{\text{def}}{=} (\mathfrak{F} : \mathfrak{G}) \cap \eta(I).$$

Remark 6. *Note that if, in the above definition, the ideal I is idempotent, that is to say $I^2 = I$, then $[0, \eta(I)]$ will be closed under the operation ':' which coincides with $:_I$.*

In light of the previous definition and Proposition 5, we see that $\forall \mathfrak{F}, \mathfrak{G} \in \operatorname{Fil}(R/I)_{(R/I)}$,

$$\pi^*(\mathfrak{F} : \mathfrak{G}) = \pi^*(\mathfrak{F}) :_I \pi^*(\mathfrak{G}), \tag{12}$$

which is to say, $\pi^* : \operatorname{Fil}(R/I)_{R/I} \to \langle [0, \eta(I)]; :_I \rangle$ is a monoid homomorphism.

Let I be a proper ideal of arbitrary ring R and $\pi : R \to R/I$ the canonical ring epimorphism. Define map $\pi_* : [0, \eta(I)] \to \operatorname{Fil}(R/I)_{R/I}$ by $\forall \mathfrak{F} \in [0, \eta(I)]$,

$$\pi_*(\mathfrak{F}) \overset{\text{def}}{=} \{K/I : K \in \mathfrak{F}\}. \tag{13}$$

It is easily checked that $\pi_*(\mathfrak{F})$ is indeed a member of $\operatorname{Fil}(R/I)_{R/I}$.

Theorem 7. *(Correspondence Theorem) Let I be a proper ideal of arbitrary ring R and $\pi : R \to R/I$ the canonical ring epimorphism. Then π^* and π_* are mutually inverse complete lattice and monoid isomorphisms between* $\operatorname{Fil}(R/I)_{R/I}$ *and* $\langle [0, \eta(I)]; :_I \rangle$.

Proof. We proved in Proposition 4 that π^* is a complete lattice homomorphism; it is, furthermore, a monoid homomorphism by (12). To complete the proof, it therefore suffices to show that π^* and π_* are mutually inverse maps. To this end, take $\mathfrak{F} \in \operatorname{Fil}(R/I)_{R/I}$ and $K \leqslant R_R$ with $K \supseteq I$. Then

$$K/I \in (\pi_* \circ \pi^*)(\mathfrak{F}) = \pi_*(\pi^*(\mathfrak{F})) \Leftrightarrow K \in \pi^*(\mathfrak{F}) \Leftrightarrow K/I \in \mathfrak{F}.$$

Thus $(\pi_* \circ \pi^*)(\mathfrak{F}) = \mathfrak{F}$.

Now take $\mathfrak{G} \in [0, \eta(I)]$. Then

$$\begin{aligned} K \in (\pi^* \circ \pi_*)(\mathfrak{G}) = \pi^*(\pi_*(\mathfrak{G})) &\Leftrightarrow K \supseteq I \text{ and } K/I \in \pi_*(\mathfrak{G}) \ \text{[by (9)]} \\ &\Leftrightarrow K \in \mathfrak{G} \ \text{[by (13)]}. \end{aligned}$$

Thus $(\pi^* \circ \pi_*)(\mathfrak{G}) = \mathfrak{G}$. We conclude that π^* and π_* are mutually inverse maps. □

Inasmuch as $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is a complete lattice ordered monoid for all rings R by Theorem 2, the following corollary to Theorem 7 is immediate.

Corollary 8. *Let I be a proper ideal of arbitrary ring R. Then $[\mathrm{Fil}\,(R/I)_{R/I}]^{\mathrm{du}}$ and $\langle [0, \eta(I)]; :_I \rangle^{\mathrm{du}}$ are isomorphic complete lattice ordered monoids.*

Let $I \in \mathrm{Id}\, R$ with $I \subset R$. Consider the map from $\mathrm{Fil}\, R_R$ to $[0, \eta(I)]$ given by $\mathfrak{F} \mapsto \mathfrak{F} \cap \eta(I)$, $\mathfrak{F} \in \mathrm{Fil}\, R_R$. That this map is onto and preserves arbitrary meets is obvious. Moreover, if $\mathfrak{F}, \mathfrak{G} \in \mathrm{Fil}\, R_R$, then

$$\begin{aligned}
[\mathfrak{F} \cap \eta(I)] :_I [\mathfrak{G} \cap \eta(I)] &= ([\mathfrak{F} \cap \eta(I)] : [\mathfrak{G} \cap \eta(I)]) \cap \eta(I) \\
&= ([\mathfrak{F} : \mathfrak{G}] \cap [\mathfrak{F} : \eta(I)] \cap [\eta(I) : \mathfrak{G}] \cap [\eta(I) : \eta(I)]) \cap \eta(I) \\
&= [\mathfrak{F} : \mathfrak{G}] \cap \eta(I) \ \ \text{[because } \mathfrak{F} : \eta(I),\ \eta(I) : \mathfrak{G} \text{ and } \eta(I) : \eta(I) \text{ all contain } \eta(I)\text{]}.
\end{aligned}$$

We have thus proved:

Proposition 9. *Let I be a proper ideal of arbitrary ring R. The map from $\mathrm{Fil}\, R_R$ to $\langle [0, \eta(I)]; :_I \rangle$ given by $\mathfrak{F} \mapsto \mathfrak{F} \cap \eta(I)$, $\mathfrak{F} \in \mathrm{Fil}\, R_R$, is onto, preserves arbitrary meets, and is a monoid homomorphism.*

4. Topologizing filters in rings of fractions

Throughout this section and the next, S shall denote a multiplicative subset of commutative ring R and RS^{-1} the ring of fractions of R with respect to S. We denote by

$$\begin{aligned} \varphi_S : R &\longrightarrow RS^{-1} \\ r &\longmapsto \tfrac{r}{1} \end{aligned}$$

the canonical ring homomorphism.

For each $M \in \text{Mod-}R$, $MS^{-1} \in \text{Mod-}RS^{-1}$ shall denote the module of fractions of M with respect to S and

$$\begin{aligned} \varphi_S^M : M &\longrightarrow MS^{-1} \\ x &\longmapsto \tfrac{x}{1} \end{aligned}$$

the canonical R-module homomorphism.

If $N \in \text{Mod-}R$ and $f \in \mathrm{Hom}_R(M, N)$, then

$$\begin{aligned} fS^{-1} : MS^{-1} &\longrightarrow NS^{-1} \\ \tfrac{x}{s} &\longmapsto \tfrac{f(x)}{s} \end{aligned}$$

denotes the canonical RS^{-1}-module homomorphism.

The associations $M \mapsto MS^{-1}$ and $f \mapsto fS^{-1}$ are easily shown to be functorial, thus allowing us to interpret $(\_)S^{-1}$ as a covariant functor from Mod-R to Mod-RS^{-1} which is known to be exact (see for example [9, Theorem 3.2, page 134]).

Proofs of the statements in Proposition 10 below, all of which are standard, may be found in [9, Chapter 3].

Proposition 10. *Let S be a multiplicative subset of commutative ring R and $M \in$ Mod-R. Then:*

(a) *For each RS^{-1}-submodule L of MS^{-1}, $((\varphi_S^M)^{-1}[L])S^{-1} = L$. Hence the map $N \mapsto NS^{-1}$ is an onto map from the set of R-submodules of M to the set of RS^{-1}-submodules of MS^{-1}. In particular, the map $I \mapsto IS^{-1}$ from* $\operatorname{Id} R$ *to* $\operatorname{Id} RS^{-1}$ *is onto.*
(b) *For each finite family $\{N_i : 1 \leqslant i \leqslant n\}$ of submodules of M, $\left(\bigcap_{i=1}^n N_i\right) S^{-1} = \bigcap_{i=1}^n N_i S^{-1}$.*
(c) *For every (possibly infinite) family $\{L_\delta : \delta \in \Delta\}$ of submodules of M, $\left(\sum_{\delta\in\Delta} L_\delta\right) S^{-1} = \sum_{\delta\in\Delta} L_\delta S^{-1}$.*
(d) *For each finite family $\{I_i : 1 \leqslant i \leqslant n\}$ of ideals of R, $(I_1 I_2 \dots I_n) S^{-1} = (I_1 S^{-1})(I_2 S^{-1}) \dots (I_n S^{-1})$.*
(e) *The map $I \mapsto IS^{-1}$ from* $\operatorname{Id} R$ *to* $\operatorname{Id} RS^{-1}$ *restricts to a bijection from the set of prime ideals of R disjoint from S, to the set of prime ideals of RS^{-1}.*

Remark 11. *Parts (a)-(d) of Proposition 10 tell us that the map $I \mapsto IS^{-1}$ from* $\operatorname{Id} R$ *to* $\operatorname{Id} RS^{-1}$ *is an onto homomorphism of lattice ordered monoids.*

For each $\mathfrak{F} \in \operatorname{Fil} R$, define

$$\hat{\varphi}_S(\mathfrak{F}) \overset{\text{def}}{=} \{IS^{-1} : I \in \mathfrak{F}\}.$$

It is easily seen that $\hat{\varphi}_S(\mathfrak{F})$ is a member of $\operatorname{Fil} RS^{-1}$. Indeed, since the ring RS^{-1} is commutative, to show that $\hat{\varphi}_S(\mathfrak{F})$ is a topologizing filter on RS^{-1}, it suffices to show that closure properties F1 and F2 hold and this is easily done with the aid of Proposition 10((a)-(c)). We may thus interpret $\hat{\varphi}_S$ as a map from $\operatorname{Fil} R$ to $\operatorname{Fil} RS^{-1}$.

To obtain a map in the reverse direction, we first remind the reader (see Proposition 4) that the canonical ring homomorphism $\varphi_S : R \to RS^{-1}$ induces a map $\varphi_S^* : \operatorname{Fil} RS^{-1} \to \operatorname{Fil} R$ where, for each $\mathfrak{F} \in \operatorname{Fil} RS^{-1}$,

$$\varphi_S^*(\mathfrak{F}) = \{I \leqslant R_R : I \supseteq \varphi_S^{-1}[L] \text{ for some } L \in \mathfrak{F}\}.$$

Proposition 12. *Let S be a multiplicative subset of commutative ring R. Then:*
(a) *The map $\hat{\varphi}_S : \operatorname{Fil} R \to \operatorname{Fil} RS^{-1}$ is onto.*
(b) *$\hat{\varphi}_S$ preserves arbitrary (possibly infinite) meets.*
(c) *$\hat{\varphi}_S$ preserves finite joins.*

Proof. (a) To show that $\hat{\varphi}_S$ is onto, it suffices to show that $(\hat{\varphi}_S \circ \varphi_S^*)(\mathfrak{F}) = \mathfrak{F}$ for all $\mathfrak{F} \in \operatorname{Fil} RS^{-1}$. Take $\mathfrak{F} \in \operatorname{Fil} RS^{-1}$. Then

$$\begin{aligned} K \in (\hat{\varphi}_S \circ \varphi_S^*)(\mathfrak{F}) &= \hat{\varphi}_S(\varphi_S^*(\mathfrak{F})) \\ &\Rightarrow K = AS^{-1} \text{ for some } A \in \varphi_S^*(\mathfrak{F}) \\ &\Rightarrow K \supseteq (\varphi_S^{-1}[L])S^{-1} \text{ for some } L \in \mathfrak{F} \\ &\Rightarrow K \supseteq L \text{ for some } L \in \mathfrak{F} \ \ [\text{because } (\varphi_S^{-1}[L])S^{-1} = L \text{ by Proposition 10(a)}] \\ &\Rightarrow K \in \mathfrak{F}. \end{aligned}$$

Thus $(\hat{\varphi}_S \circ \varphi_S^*)(\mathfrak{F}) \subseteq \mathfrak{F}$.

Now take $K \in \mathfrak{F}$ so that $\varphi_S^{-1}[K] \in \varphi_S^*(\mathfrak{F})$. Then $(\varphi_S^{-1}[K])S^{-1} \in \hat{\varphi}_S(\varphi_S^*(\mathfrak{F})) = (\hat{\varphi}_S \circ \varphi_S^*)(\mathfrak{F})$ and since $(\varphi_S^{-1}[K])S^{-1} = K$ by Proposition 10(a), it follows that $K \in (\hat{\varphi}_S \circ \varphi_S^*)(\mathfrak{F})$. Thus $\mathfrak{F} \subseteq (\hat{\varphi}_S \circ \varphi_S^*)(\mathfrak{F})$ which establishes the reverse containment.

(b) Let $\{\mathfrak{F}_\delta : \delta \in \Delta\} \subseteq \operatorname{Fil} R$. Since $\hat{\varphi}_S$ is order preserving, we must have $\hat{\varphi}_S(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta) \subseteq \bigcap_{\delta\in\Delta} \hat{\varphi}_S(\mathfrak{F}_\delta)$.

To establish the reverse containment, suppose $K \in \bigcap_{\delta\in\Delta} \hat{\varphi}_S(\mathfrak{F}_\delta)$. Then $K \in \hat{\varphi}_S(\mathfrak{F}_\delta)$ for each $\delta \in \Delta$, so there exists $B_\delta \in \mathfrak{F}_\delta$ for each $\delta \in \Delta$ such that $K = B_\delta S^{-1}$. Putting $B = \sum_{\delta\in\Delta} B_\delta$, we have $B \in \bigcap_{\delta\in\Delta} \mathfrak{F}_\delta$ and

$$\begin{aligned} BS^{-1} &= \left(\sum_{\delta\in\Delta} B_\delta\right) S^{-1} \\ &= \sum_{\delta\in\Delta} (B_\delta S^{-1}) \ \text{[by Proposition 10(c)]} \\ &= \sum_{\delta\in\Delta} K = K. \end{aligned}$$

Thus $K \in \hat{\varphi}_S(\bigcap_{\delta\in\Delta} \mathfrak{F}_\delta)$. We conclude that $\hat{\varphi}_S$ preserves arbitrary meets.

(c) Let $\{\mathfrak{F}_i : 1 \leqslant i \leqslant n\}$ be a finite subfamily of $\operatorname{Fil} R$. Since $\hat{\varphi}_S$ is order preserving, $\hat{\varphi}_S(\bigvee_{i=1}^n \mathfrak{F}_i) \supseteq \bigvee_{i=1}^n \hat{\varphi}_S(\mathfrak{F}_i)$.

For the reverse containment, take $K \in \hat{\varphi}_S(\bigvee_{i=1}^n \mathfrak{F}_i)$. Then $K = AS^{-1}$ for some $A \in \bigvee_{i=1}^n \mathfrak{F}_i$. It follows from (1) that there exists $L_i \in \mathfrak{F}_i$ for each $i \in \{1, 2, \ldots, n\}$ such that $A \supseteq \bigcap_{i=1}^n L_i$. Then

$$\begin{aligned} K &= AS^{-1} \\ &\supseteq \left(\bigcap_{i=1}^n L_i\right) S^{-1} \ \text{[because } A \supseteq \textstyle\bigcap_{i=1}^n L_i\text{]} \\ &= \bigcap_{i=1}^n L_i S^{-1} \ \text{[by Proposition 10(b)]} \\ &\in \bigvee_{i=1}^n \hat{\varphi}_S(\mathfrak{F}_i) \ \text{[by (1)]}. \end{aligned}$$

This implies that $K \in \bigvee_{i=1}^n \hat{\varphi}_S(\mathfrak{F}_i)$. Thus $\hat{\varphi}_S(\bigvee_{i=1}^n \mathfrak{F}_i) \subseteq \bigvee_{i=1}^n \hat{\varphi}_S(\mathfrak{F}_i)$, as required. □

The family

$$\mathfrak{F}_S \stackrel{\text{def}}{=} \{I \leqslant R : I \cap S \neq \emptyset\}$$

is a Gabriel topology on R [10, Proposition VI.6.1, page 148] and if $M \in \text{Mod-}R$ and $\varphi_S^M : M \to MS^{-1}$ is the canonical R-module homomorphism, then

$$\begin{aligned} \operatorname{Ker} \varphi_S^M &= \{x \in M : xs = 0 \text{ for some } s \in S\} \\ &= \{x \in M : x^{-1}0 \cap S \neq \emptyset\} \\ &= \{x \in M : x^{-1}0 \in \mathfrak{F}_S\} \\ &= \mathcal{T}_{\mathfrak{F}_S}(M) \ \text{[by (3)]}. \end{aligned} \tag{14}$$

Let R be an arbitrary (not necessarily commutative) ring and $\mathfrak{F} \in \operatorname{Fil} R_R$. We shall call a subset X of $\mathfrak{F}$ a *cofinal* set for $\mathfrak{F}$ if, given any $A \in \mathfrak{F}$, there exists $B \in X$ such that $A \supseteq B$.

We require the following result from [11, Lemma 3 and Remark 2, page 90].

Lemma 13. *Let R be an arbitrary ring and $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R_R$. If $\{I_\gamma : \gamma \in \Gamma\}$ is a cofinal set of finitely generated right ideals for $\mathfrak{F}$ and $\{J_\theta : \theta \in \Theta\}$ is a cofinal set of (two-sided) ideals for $\mathfrak{G}$, then $\{I_\gamma J_\theta : \gamma \in \Gamma, \theta \in \Theta\}$ is a cofinal set for $\mathfrak{F} : \mathfrak{G}$.*

Proposition 14. *Let S be a multiplicative subset of commutative ring R. Then, for all $\mathfrak{G} \in \operatorname{Fil} R$:*
(a) $\mathfrak{F}_S : \mathfrak{G} \subseteq \mathfrak{G} : \mathfrak{F}_S$.
(b) $\mathfrak{F}_S : \mathfrak{G} : \mathfrak{F}_S = \mathfrak{G} : \mathfrak{F}_S$.

Proof. (a) Note that $A \in \mathfrak{F}_S$ if and only if $t \in A$ for some $t \in S$, or equivalently, $A \supseteq tR$ for some $t \in S$. Thus $\{tR : t \in S\}$ is a cofinal set of principal (and thus finitely generated) ideals for $\mathfrak{F}_S$. Take $\mathfrak{G} \in \operatorname{Fil} R$. Inasmuch as every member of $\mathfrak{G}$ is a (two-sided) ideal of R, it follows from Lemma 13 that $\{tK : t \in S, K \in \mathfrak{G}\}$ is a cofinal set for $\mathfrak{F}_S : \mathfrak{G}$. Take $t \in S$ and $K \in \mathfrak{G}$ and consider the short exact sequence

$$0 \longrightarrow K/tK \longrightarrow R/tK \longrightarrow R/K \longrightarrow 0.$$

Observe that the right R-module K/tK is annihilated by t and is thus $\mathcal{T}_{\mathfrak{F}_S}$-torsion by (2). Since R/K is $\mathcal{T}_{\mathfrak{G}}$-torsion by (4), it follows that R/tK is $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion, whence $tK \in \mathfrak{G} : \mathfrak{F}_S$. Since the family $\{tK : t \in S, K \in \mathfrak{G}\}$ is cofinal in $\mathfrak{F}_S : \mathfrak{G}$, we conclude that $\mathfrak{F}_S : \mathfrak{G} \subseteq \mathfrak{G} : \mathfrak{F}_S$.

(b) Take $\mathfrak{G} \in \operatorname{Fil} R$. Since $\mathfrak{F} : \mathfrak{G} \supseteq \mathfrak{F} \vee \mathfrak{G}$ for all $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R$, it is easily seen that $\mathfrak{F}_S : \mathfrak{G} : \mathfrak{F}_S \supseteq \mathfrak{G} : \mathfrak{F}_S$. The reverse containment follows inasmuch as

$$\begin{aligned} \mathfrak{F}_S : \mathfrak{G} : \mathfrak{F}_S &\subseteq (\mathfrak{G} : \mathfrak{F}_S) : \mathfrak{F}_S \ \text{[because } \mathfrak{F}_S : \mathfrak{G} \subseteq \mathfrak{G} : \mathfrak{F}_S \text{ by (a)]} \\ &= \mathfrak{G} : (\mathfrak{F}_S : \mathfrak{F}_S) \\ &= \mathfrak{G} : \mathfrak{F}_S \ \text{[because } \mathfrak{F}_S \text{ is a Gabriel topology, so } \mathfrak{F}_S : \mathfrak{F}_S = \mathfrak{F}_S]. \end{aligned}$$

□

Lemma 15. *Let S be a multiplicative subset of commutative ring R and $\mathfrak{G} \in \operatorname{Fil} R$. The following statements are equivalent for a right R-module M:*
(a) MS^{-1} is a $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion right RS^{-1}-module;
(b) M is a $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion right R-module.

Proof. (a)⇒(b) Take $x \in M$. Since $\frac{x}{1} \in MS^{-1}$ and MS^{-1} is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion, $\left(\frac{x}{1}\right) I = 0$ for some $I \in \hat{\varphi}_S(\mathfrak{G})$ by (3). Put $I = AS^{-1}$ with $A \in \mathfrak{G}$. Then $\left(\frac{x}{1}\right) I = (xA)S^{-1} = 0$. This implies that the canonical R-homomorphism $\varphi_S^{xA} : xA \to (xA)S^{-1}$ is the zero map. Hence, by (14), xA is $\mathcal{T}_{\mathfrak{F}_S}$-torsion. Consider the short exact sequence

$$0 \longrightarrow xA \longrightarrow xR \longrightarrow xR/xA \longrightarrow 0.$$

By (4), R/A is $\mathcal{T}_{\mathfrak{G}}$-torsion because $A \in \mathfrak{G}$. It follows that xR/xA, being an epimorphic image of R/A, is also $\mathcal{T}_{\mathfrak{G}}$-torsion. We infer from the above short exact sequence that xR is $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion, so $x \in \mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}(M)$. We conclude that M is $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion.

(b)⇒(a) Take $\frac{x}{s} \in MS^{-1}$ with $x \in M$, $s \in S$. Put $x^{-1}0 = A$. It is easily seen that $\left(\frac{x}{s}\right)(AS^{-1}) = 0$. Since M is $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion, $A \in \mathfrak{G} : \mathfrak{F}_S$ by (2), so there exists $H \in \mathfrak{G}$ such that $H \supseteq A$ and H/A is $\mathcal{T}_{\mathfrak{F}_S}$-torsion. Consider the short exact sequence

$$0 \longrightarrow H/A \longrightarrow R/A \longrightarrow R/H \longrightarrow 0$$

in Mod-R. Inasmuch as the functor $(\_)S^{-1}$ is exact, the above sequence induces the following short exact sequence in Mod-RS^{-1}:

$$0 \longrightarrow (H/A)S^{-1} \longrightarrow (R/A)S^{-1} \longrightarrow (R/H)S^{-1} \longrightarrow 0.$$

Since H/A is $\mathcal{T}_{\mathfrak{F}_S}$-torsion, it follows from (14) that the canonical R-homomorphism $\varphi_S^{H/A} : H/A \to (H/A)S^{-1}$ is the zero map, and this is only possible if $(H/A)S^{-1} = 0$. We conclude from exactness of the above sequence that $(R/A)S^{-1}$ and $(R/H)S^{-1}$ are isomorphic right RS^{-1}-modules. It again follows from the exactness of $(\_)S^{-1}$ that $(R/H)S^{-1} \cong RS^{-1}/HS^{-1}$ and $(R/A)S^{-1} \cong RS^{-1}/AS^{-1}$ as right RS^{-1}-modules. Since $(R/A)S^{-1} \cong (R/H)S^{-1}$, we must have $AS^{-1} = HS^{-1}$. Inasmuch as $H \in \mathfrak{G}$, $AS^{-1} = HS^{-1} \in \hat{\varphi}_S(\mathfrak{G})$. Since $\left(\frac{x}{s}\right)(AS^{-1}) = 0$, $\frac{x}{s} \in \mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}(MS^{-1})$ by (3). We conclude that MS^{-1} is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion.

□

Proposition 16. *Let S be a multiplicative subset of commutative ring R. Then:*
(a) $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{F}_S) = \hat{\varphi}_S(\mathfrak{F}_S : \mathfrak{F}) = \hat{\varphi}_S(\mathfrak{F})$ for all $\mathfrak{F} \in \operatorname{Fil} R$.
(b) $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{G}) = \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$ for all $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R$.

Proof. We first show that $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{G}) \subseteq \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$ for all $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R$. Take $K \in \hat{\varphi}_S(\mathfrak{F} : \mathfrak{G})$. Then $K = AS^{-1}$ for some $A \in \mathfrak{F} : \mathfrak{G}$. There exists therefore some $H \in \mathfrak{F}$ containing A such that H/A is $\mathcal{T}_{\mathfrak{G}}$-torsion. Since $H \in \mathfrak{F}$, $HS^{-1} \in \hat{\varphi}_S(\mathfrak{F})$ and so RS^{-1}/HS^{-1} is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{F})}$-torsion by (4). Note also that $HS^{-1}/AS^{-1} \cong (H/A)S^{-1}$ is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion by Lemma 15((b)⇒(a)), noting that $H/A \in \mathcal{T}_{\mathfrak{G}} \subseteq \mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$. Now consider the short exact sequence

$$0 \longrightarrow HS^{-1}/AS^{-1} \longrightarrow RS^{-1}/AS^{-1} \longrightarrow RS^{-1}/HS^{-1} \longrightarrow 0.$$

Since HS^{-1}/AS^{-1} is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion and RS^{-1}/HS^{-1} is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{F})}$-torsion, it follows that $K = AS^{-1} \in \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$. Thus $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{G}) \subseteq \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$.

(a) Take $\mathfrak{F} \in \operatorname{Fil} R$. Certainly, $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{F}_S) \supseteq \hat{\varphi}_S(\mathfrak{F})$ since $\mathfrak{F} : \mathfrak{F}_S \supseteq \mathfrak{F}$. To establish the reverse containment, we note that by the above argument, $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{F}_S) \subseteq \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{F}_S)$. But $\hat{\varphi}_S(\mathfrak{F}_S) = \{RS^{-1}\}$, for if $I \in \hat{\varphi}_S(\mathfrak{F}_S)$, then $I = AS^{-1}$

for some $A \in \mathfrak{F}_S$ and this entails $A \cap S \neq \emptyset$, whence $I = AS^{-1} = RS^{-1}$. Since $\{RS^{-1}\}$ is the identity of $\operatorname{Fil} RS^{-1}$ with respect to the monoid operation, we obtain $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{F}_S) \subseteq \hat{\varphi}_S(\mathfrak{F}) : \{RS^{-1}\} = \hat{\varphi}_S(\mathfrak{F})$. Thus $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{F}_S) \subseteq \hat{\varphi}_S(\mathfrak{F})$, as required.

We omit the proof that $\hat{\varphi}_S(\mathfrak{F}_S : \mathfrak{F}) = \hat{\varphi}_S(\mathfrak{F})$ which is similar to the above.

(b) Take $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R$. It follows from the argument preceding the proof of (a) that $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{G}) \subseteq \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$. To establish the reverse containment, take $K = AS^{-1} \in \hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G})$. There exists therefore some ideal $H \in \hat{\varphi}_S(\mathfrak{F})$ containing K such that H/K is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion. Since $H \in \hat{\varphi}_S(\mathfrak{F})$, $H = BS^{-1}$ for some $B \in \mathfrak{F}$. Inasmuch as $(A+B)S^{-1} = AS^{-1} + BS^{-1} = BS^{-1}$ [Proposition 10(c)] and $A+B \in \mathfrak{F}$ because $A+B \supseteq B \in \mathfrak{F}$, no generality is lost if we replace B with $A+B$ and assume that $A \subseteq B$. Since $(\_)S^{-1}$ is exact, the short exact sequence

$$0 \longrightarrow B/A \longrightarrow R/A \longrightarrow R/B \longrightarrow 0 \tag{15}$$

in Mod-R induces the short exact sequence

$$0 \longrightarrow (B/A)S^{-1} \longrightarrow (R/A)S^{-1} \longrightarrow (R/B)S^{-1} \longrightarrow 0$$

in Mod-RS^{-1}. Since $H/K = BS^{-1}/AS^{-1} \cong (B/A)S^{-1}$ is $\mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}$-torsion, we infer from Lemma 15((a)⇒(b)) that B/A is $\mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}$-torsion. It follows from short exact sequence (15) that R/A is $\mathcal{T}_{\mathfrak{F}:\mathfrak{G}:\mathfrak{F}_S}$-torsion, hence $A \in \mathfrak{F} : \mathfrak{G} : \mathfrak{F}_S$ and $K = AS^{-1} \in \hat{\varphi}_S(\mathfrak{F} : \mathfrak{G} : \mathfrak{F}_S)$. This shows that $\hat{\varphi}_S(\mathfrak{F}) : \hat{\varphi}_S(\mathfrak{G}) \subseteq \hat{\varphi}_S(\mathfrak{F} : \mathfrak{G} : \mathfrak{F}_S)$. The required containment follows noting that $\hat{\varphi}_S(\mathfrak{F} : \mathfrak{G} : \mathfrak{F}_S) = \hat{\varphi}_S(\mathfrak{F} : \mathfrak{G})$ by (a).

□

The following theorem is an immediate consequence of Propositions 12 and 16(b).

Theorem 17. *Let S be a multiplicative subset of commutative ring R. Then the map $\hat{\varphi}_S : [\operatorname{Fil} R]^{\mathrm{du}} \to [\operatorname{Fil} RS^{-1}]^{\mathrm{du}}$ is an onto homomorphism of lattice ordered monoids.*

We thus obtain the following commutative diagram (see Figure 1) of lattice ordered monoids:

$$\begin{array}{ccc}
\mathfrak{F} & \longmapsto & \hat{\varphi}_S(\mathfrak{F}) \\
[\operatorname{Fil} R]^{\mathrm{du}} & \twoheadrightarrow & [\operatorname{Fil} RS^{-1}]^{\mathrm{du}} \\
\uparrow \eta^R & & \uparrow \eta^{RS^{-1}} \\
\operatorname{Id} R & \twoheadrightarrow & \operatorname{Id} RS^{-1} \\
I & \longmapsto & IS^{-1}
\end{array}$$

Figure 1

Observe that the vertical maps in Figure 1 are lattice ordered monoid embeddings by Theorem 3, whilst the horizontal maps are onto lattice ordered monoid homomorphisms (see Remark 11 and Theorem 17).

5. Congruences on Fil R

The kernel of the onto homomorphism $\hat{\varphi}_S : [\operatorname{Fil} R]^{\mathrm{du}} \to [\operatorname{Fil} RS^{-1}]^{\mathrm{du}}$ of lattice ordered monoids established in Theorem 17, is the congruence relation $\equiv_{\hat{\varphi}_S}$ on $[\operatorname{Fil} R]^{\mathrm{du}}$ defined by $\forall \mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R$,

$$\mathfrak{F} \equiv_{\hat{\varphi}_S} \mathfrak{G} \;\Leftrightarrow\; \hat{\varphi}_S(\mathfrak{F}) = \hat{\varphi}_S(\mathfrak{G}).$$

Theory tells us that

$$[\operatorname{Fil} RS^{-1}]^{\mathrm{du}} \cong [\operatorname{Fil} R]^{\mathrm{du}} / \equiv_{\hat{\varphi}_S} . \tag{16}$$

Proposition 18. *Let S be a multiplicative subset of commutative ring R. The following statements are equivalent for $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R$:*
(a) $\mathfrak{F} \equiv_{\hat{\varphi}_S} \mathfrak{G}$, i.e., $\hat{\varphi}_S(\mathfrak{F}) = \hat{\varphi}_S(\mathfrak{G})$;
(b) $\mathfrak{F} : \mathfrak{F}_S = \mathfrak{G} : \mathfrak{F}_S$.

Proof. (a)⇒(b)

$$\begin{aligned} K \in \mathfrak{F} : \mathfrak{F}_S &\Leftrightarrow R/K \text{ is } \mathcal{T}_{\mathfrak{F}:\mathfrak{F}_S}\text{-torsion [by (4)]} \\ &\Leftrightarrow (R/K)S^{-1} \text{ is } \mathcal{T}_{\hat{\varphi}_S(\mathfrak{F})}\text{-torsion [by Lemma 15((b)}\Rightarrow\text{(a))]} \\ &\Leftrightarrow (R/K)S^{-1} \text{ is } \mathcal{T}_{\hat{\varphi}_S(\mathfrak{G})}\text{-torsion [by (a)]} \\ &\Leftrightarrow R/K \text{ is } \mathcal{T}_{\mathfrak{G}:\mathfrak{F}_S}\text{-torsion [by Lemma 15((a)}\Rightarrow\text{(b))]} \\ &\Leftrightarrow K \in \mathfrak{G} : \mathfrak{F}_S \text{ [by (4)]}. \end{aligned}$$

Thus $\mathfrak{F} : \mathfrak{F}_S = \mathfrak{G} : \mathfrak{F}_S$.

(b)⇒(a) is an immediate consequence of Proposition 16(a). □

If R is a commutative ring we shall henceforth denote by $\operatorname{spec} R$ [resp. $\operatorname{spec}_{\mathrm{m}} R$] the set of all prime [resp. maximal] ideals of R.

If $P \in \operatorname{spec} R$, then $S = R \backslash P$ is a multiplicative subset of R. For such a choice of S we shall write R_P in place of RS^{-1}, and write[1] φ_P, $\hat{\varphi}_P$, I_P (where $I \in \operatorname{Id} R$) and $\mathfrak{F}_P$ in place of $\varphi_{R\backslash P}$, $\hat{\varphi}_{R\backslash P}$, $I(R\backslash P)^{-1}$ and $\mathfrak{F}_{R\backslash P}$, respectively.

For each commutative ring R, we define $\operatorname{Rad}(\operatorname{Fil} R)$ to be the intersection of congruences:

$$\operatorname{Rad}(\operatorname{Fil} R) \stackrel{\text{def}}{=} \bigcap_{P \in \operatorname{spec}_{\mathrm{m}} R} \equiv_{\hat{\varphi}_P} .$$

The family of lattice ordered monoid homomorphisms (see Remark 11) indexed by $P \in \operatorname{spec}_{\mathrm{m}} R$:

$$\begin{aligned} \operatorname{Id} R &\longrightarrow \operatorname{Id} R_P \\ I &\longmapsto I_P, \end{aligned}$$

induces a canonical homomorphism of lattice ordered monoids:

$$\begin{aligned} \operatorname{Id} R &\longrightarrow \textstyle\prod_{P \in \operatorname{spec}_{\mathrm{m}} R} \operatorname{Id} R_P \\ I &\longmapsto \{I_P\}_{P \in \operatorname{spec}_{\mathrm{m}} R}. \end{aligned}$$

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[1] An abuse in aid of a less cumbersome notation.

Theory tells us that the above homomorphism is monic (see for example [9, Proposition 3.17, page 163]).

In a similar vein, the family of lattice ordered monoid homomorphisms $\{\hat{\varphi}_P : P \in \operatorname{spec}_{\mathrm{m}} R\}$, induces a canonical homomorphism

$$\begin{aligned} [\operatorname{Fil} R]^{\mathrm{du}} &\longrightarrow \textstyle\prod_{P\in\operatorname{spec}_{\mathrm{m}} R} [\operatorname{Fil} R_P]^{\mathrm{du}} \\ \mathfrak{F} &\longmapsto \{\hat{\varphi}_P(\mathfrak{F})\}_{P\in\operatorname{spec}_{\mathrm{m}} R} \end{aligned}$$

which is easily seen to have kernel

$$\bigcap_{P\in\operatorname{spec}_{\mathrm{m}} R} \equiv_{\hat{\varphi}_P} = \operatorname{Rad}(\operatorname{Fil} R).$$

We thus obtain the following commutative diagram (see Figure 2) of lattice ordered monoids:

$$\begin{array}{ccc} \mathfrak{F} & \longmapsto & \{\hat{\varphi}_P(\mathfrak{F})\}_{P\in\operatorname{spec}_{\mathrm{m}} R} \\ [\operatorname{Fil} R]^{\mathrm{du}} & \longrightarrow & \prod_{P\in\operatorname{spec}_{\mathrm{m}} R} [\operatorname{Fil} R_P]^{\mathrm{du}} \\ \eta^R \Big\uparrow & & \Big\uparrow \prod_{P\in\operatorname{spec}_{\mathrm{m}} R} \eta^{R_P} \\ \operatorname{Id} R & \hookrightarrow & \prod_{P\in\operatorname{spec}_{\mathrm{m}} R} \operatorname{Id} R_P \\ I & \longmapsto & \{I_P\}_{P\in\operatorname{spec}_{\mathrm{m}} R} \end{array}$$

Figure 2

We note that whereas the canonical homomorphism from $\operatorname{Id} R$ to $\prod_{P\in\operatorname{spec}_{\mathrm{m}} R} \operatorname{Id} R_P$ is monic (as noted above), the kernel $\operatorname{Rad}(\operatorname{Fil} R)$ of the canonical homomorphism from $[\operatorname{Fil} R]^{\mathrm{du}}$ to $\prod_{P\in\operatorname{spec}_{\mathrm{m}} R}[\operatorname{Fil} R_P]^{\mathrm{du}}$ need not be trivial.

We shall see in the next section that if R is a commutative ring for which $\operatorname{Fil} R$ is commutative, then $\operatorname{Rad}(\operatorname{Fil} R)$ is trivial. However, in general, $\operatorname{Rad}(\operatorname{Fil} R)$ is not trivial as Proposition 20 below shows.

For each commutative ring R, put

$$\begin{aligned} \mathfrak{P} &= \bigvee_{P\in\operatorname{spec}_{\mathrm{m}} R} \eta(P) \\ &= \Big\{K \leqslant R : K \supseteq \bigcap \mathcal{P} \text{ for some finite subset } \mathcal{P} \text{ of } \operatorname{spec}_{\mathrm{m}} R\Big\} \text{ (see (1))}. \end{aligned} \tag{17}$$

Lemma 19. *Let R be a commutative Von Neumann Regular (VNR) ring. If $\mathfrak{P}$ is defined as in (17), then $\eta(0)$ and $\mathfrak{P}$ are congruent with respect to* $\operatorname{Rad}(\operatorname{Fil} R)$.

Proof. Since R is VNR, $P_P = 0$, whence $0 \in \hat{\varphi}_P(\eta(P))$ and $\hat{\varphi}_P(\eta(0)) = \hat{\varphi}_P(\mathfrak{P})$ for all $P \in \operatorname{spec_m} R$. This entails $\eta(0)$ and $\mathfrak{P}$ are congruent with respect to $\operatorname{Rad}(\operatorname{Fil} R)$. □

Proposition 20. *The following statements are equivalent for a commutative VNR ring R:*
(a) R is noetherian and thus a finite product of fields;
(b) $\operatorname{Rad}(\operatorname{Fil} R)$ *is trivial.*

Proof. (a)⇒(b) Suppose R satisfies (a) so that R is artinian. This implies that each $\mathfrak{F} \in \operatorname{Fil} R$ has a (unique) smallest member I, say, whence $\mathfrak{F} = \eta(I)$. With reference to Figure 2, we see that the canonical embeddings η^R and $\prod_{P \in \operatorname{spec_m} R} \eta^{R_P}$ are onto maps and thus isomorphisms. This implies that the canonical homomorphism from $[\operatorname{Fil} R]^{\mathrm{du}}$ to $\prod_{P \in \operatorname{spec_m} R} [\operatorname{Fil} R_P]^{\mathrm{du}}$, which has kernel $\operatorname{Rad}(\operatorname{Fil} R)$, is an embedding. Thus (b) holds.

(b)⇒(a) If $\mathfrak{P}$ is defined as in (17), then it follows from (b) and the previous lemma that $\eta(0) = \mathfrak{P}$. This implies that $\bigcap \mathcal{P} = 0$ for some finite subset $\mathcal{P}$ of $\operatorname{spec_m} R$, whence $R \cong \prod\{R/P : P \in \mathcal{P}\}$ is a finite product of fields. □

Remark 21. *With reference to the implication (a)⇒(b) of the previous proposition, we point out that if the requirement that R is VNR is dispensed with, then Statement (b) holds under conditions much weaker than (a). Indeed, within the class of all commutative rings, it is known (see* [11, Corollary 8, page 91]*) that* $\operatorname{Fil} R$ *is commutative whenever R is noetherian and, as we shall prove in the next section (Theorem 32),* $\operatorname{Rad}(\operatorname{Fil} R)$ *is trivial whenever* $\operatorname{Fil} R$ *is commutative.*

6. Residuation and commutativity in $\operatorname{Fil} R_R$

Throughout this section and unless stated otherwise, R shall denote an arbitrary (not necessarily commutative) ring.

Recall that a complete lattice ordered monoid L is said to be *left residuated* [resp. *right residuated*] if for every $a, b \in L$, there exists a (unique) largest $x \in L$ such that $x \cdot b \leqslant a$ [resp. $b \cdot x \leqslant a$]. In this situation we call x the *left residual of a by b* [resp. *right residual of a by b*] and denote it ab^{-1} [resp. $b^{-1}a$]. We say that L is *two-sided residuated* if it is both left and right residuated.

The lattice ordered monoid $\operatorname{Id} R$ (see Proposition 1) is two-sided residuated: if $I, J \in \operatorname{Id} R$, then $IJ^{-1} = \{r \in R : rJ \subseteq I\}$ is the left residual of I by J and $J^{-1}I = \{r \in R : Jr \subseteq I\}$ the right residual of I by J.

Proposition 22. [12, Proposition 5, page 429] *The following statements are equivalent for a complete lattice ordered monoid L:*
(a) L is right residuated;
(b) $a \cdot (\bigvee X) = \bigvee_{x \in X}(a \cdot x)$ for all $a \in L$ and $X \subseteq L$.

Theorem 23. [7, Proposition 4.1, page 43] *If R is an arbitrary ring, then* $[\operatorname{Fil} R_R]^{\mathrm{du}}$ *is left residuated.*

In general, even for commutative rings R, $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ need not be *right* residuated. Indeed, as shown in [1, Theorem 34, page 1007], if R is a valuation domain, then $[\mathrm{Fil}\, R]^{\mathrm{du}}$ is two-sided residuated if and only if R is noetherian and thus rank one discrete. Since $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is always left residuated (Theorem 23), it is clear that $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ will be two-sided residuated if $\mathrm{Fil}\, R_R$ is commutative. We point out that the converse is not true in general: if R is any right fully bounded noetherian ring, then $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated by [1, Theorem 21, page 1001], but $\mathrm{Fil}\, R_R$ will not be commutative if ideal multiplication does not commute in R. The two notions do coincide, however, for commutative rings R as shown in [1, Theorem 33, page 1005] (see Theorem 31).

Theorem 24. *Let R be an arbitrary ring and T a nonzero factor ring of R.*
(a) If $\mathrm{Fil}\, R_R$ is commutative, then so is $\mathrm{Fil}\, T_T$.
(b) If $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated, then so is $[\mathrm{Fil}\, T_T]^{\mathrm{du}}$.

Proof. Suppose $T \cong R/I$ with I a proper ideal of R.

It follows from Theorem 7 and Proposition 9 that the composition of maps

$$\mathfrak{F} \mapsto \mathfrak{F} \cap \eta(I) \mapsto \pi_*(\mathfrak{F} \cap \eta(I))$$

from $\mathrm{Fil}\, R_R$ to $\mathrm{Fil}\,(R/I)_{R/I}$ is onto, preserves arbitrary meets, and is a monoid homomorphism. It follows that any property of $\mathrm{Fil}\, R_R$ that is characterizable in terms of an identity involving only meets and the monoid operation, is passed from $\mathrm{Fil}\, R_R$ to $\mathrm{Fil}\,(R/I)_{R/I}$.

Since commutativity of $\mathrm{Fil}\, R_R$ is an identity involving only the monoid operation, Statement (a) follows.

Statement (b) also follows if we note that meets in $\mathrm{Fil}\, R_R$ correspond with joins in $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$, and that by Proposition 22, right residuation in $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is characterizable in terms of the identity: $\forall \mathfrak{F} \in \mathrm{Fil}\, R_R$, $\forall X \subseteq \mathrm{Fil}\, R_R$,

$$\mathfrak{F} : \left(\bigvee X\right) = \bigvee_{\mathfrak{G} \in X} (\mathfrak{F} : \mathfrak{G}).$$

□

We require the following result [1, Corollary 15, page 1000].

Proposition 25. *Let R be an arbitrary ring for which $[\mathrm{Fil}\, R_R]^{\mathrm{du}}$ is two-sided residuated. Then for all ideals I of R, $(R/I)_R$ satisfies the DCC on hereditary pretorsion submodules.*

Recall that an ideal I of arbitrary ring R is called a *left annihilator ideal* [resp. *right annihilator ideal*] if for some $A \in \mathrm{Id}\, R$,

$$I = 0A^{-1} = \{r \in R : rA = 0\} \;\; [\text{resp. } I = A^{-1}0 = \{r \in R : Ar = 0\}].$$

Observe that every left annihilator ideal of R is a hereditary pretorsion submodule of R_R, for $0A^{-1} = \mathcal{T}_{\eta(A)}(R_R)$ for every $A \in \mathrm{Id}\, R$.

If R is an arbitrary ring, the maps $A \mapsto A^{-1}0$ and $A \mapsto 0A^{-1}$ represent a Galois connection between the sets of left annihilator ideals of R, and right annihilator ideals of R. Thus R will satisfy the DCC on left annihilator ideals, precisely if it satisfies the ACC on right annihilator ideals. It is shown in [11, Theorem 19, page 98] that for an arbitrary ring R, if $\operatorname{Fil} R_R$ is commutative, then R satisfies the ACC on right annihilator ideals. The next result, which is an immediate consequence of Proposition 25, the fact that every left annihilator ideal of R is a hereditary pretorsion submodule of R_R, and the equivalence of the DCC on left annihilator ideals and ACC on right annihilator ideals, shows that this theorem remains valid if the requirement that $\operatorname{Fil} R_R$ is commutative is weakened to $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated.

Theorem 26. *Let R be an arbitrary ring for which $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated. Then R satisfies the DCC on left annihilator ideals, and the ACC on right annihilator ideals.*

Remark 27. *If R is a semiprime ring, then the notions left annihilator ideal and right annihilator ideal coincide, allowing us to omit the prefices left and right.*

It is known that the following statements are equivalent for a semiprime ring R:
(a) R satisfies the ACC on annihilator ideals;
(b) R satisfies the DCC on annihilator ideals;
(c) R is a finite subdirect product of prime rings.

It is known that a commutative noetherian ring has finitely many minimal prime ideals [8, Corollary 3.14(a), page 41]. Rings R for which $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated enjoy the same property as the next result shows.

Theorem 28. *If R is an arbitrary ring for which $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated, then R contains only finitely many minimal prime ideals.*

Proof. Suppose that $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated. Let $\operatorname{rad} R$ denote the prime radical of R. It follows from Theorem 24(b) that the two-sided residuation property is passed to factor rings. Hence no generality is lost if we replace R by $R/\operatorname{rad} R$ and assume that R is semiprime.

It follows from Theorem 26 and the previous remark that R is a finite subdirect product of prime rings. Hence there are prime ideals $P_1, P_2, \ldots, P_n$ of R such that $\bigcap_{i=1}^n P_i = 0$. Let Q be a minimal prime ideal of R. Since $Q \supseteq \bigcap_{i=1}^n P_i$, the primeness of Q entails $Q \supseteq P_i$ for some $i \in \{1, 2, \ldots, n\}$. The minimality of Q implies $Q = P_i$. Thus every minimal prime ideal of R is a member of $\{P_i : 1 \leqslant i \leqslant n\}$. □

If R is a ring for which $\operatorname{Fil} R_R$ is commutative, then $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated. The next theorem shows that the converse holds if the ring R is commutative.

Theorem 29. [1, Theorem 33, page 1005] *The following statements are equivalent for a commutative ring R:*
(a) $\operatorname{Fil} R$ is commutative;
(b) $[\operatorname{Fil} R]^{\mathrm{du}}$ is two-sided residuated.

Our next objective is to prove that $\operatorname{Rad}(\operatorname{Fil} R)$ is trivial whenever R and $\operatorname{Fil} R$ are commutative.

Lemma 30. *Let R be an arbitrary ring, $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R_R$ and $\mathfrak{H}_1, \mathfrak{H}_2 \in \operatorname{Fil} R_R$ with $\mathfrak{H}_1 \subseteq \mathfrak{H}_2$. Suppose $\mathfrak{H}_2$ is idempotent (that is to say, $\mathfrak{H}_2 : \mathfrak{H}_2 = \mathfrak{H}_2$, i.e., $\mathfrak{H}_2$ is a right Gabriel topology on R). If $\mathfrak{F} : \mathfrak{H}_1 \subseteq \mathfrak{G} : \mathfrak{H}_1$, then $\mathfrak{F} : \mathfrak{H}_2 \subseteq \mathfrak{G} : \mathfrak{H}_2$.*

Proof.
$$\begin{aligned} \mathfrak{F} &\subseteq \mathfrak{F} : \mathfrak{H}_1 \\ &\subseteq \mathfrak{G} : \mathfrak{H}_1 \ \text{[by hypothesis]} \\ &\subseteq \mathfrak{G} : \mathfrak{H}_2 \ \text{[because } \mathfrak{H}_1 \subseteq \mathfrak{H}_2\text{]}, \end{aligned}$$
hence $\mathfrak{F} : \mathfrak{H}_2 \subseteq (\mathfrak{G} : \mathfrak{H}_2) : \mathfrak{H}_2 = \mathfrak{G} : (\mathfrak{H}_2 : \mathfrak{H}_2) = \mathfrak{G} : \mathfrak{H}_2$ [because $\mathfrak{H}_2$ is idempotent]. □

Lemma 31. *If R is any commutative ring, then*
$$\bigcap_{P \in \operatorname{spec_m} R} \mathfrak{F}_P = \{R\}.$$

Proof. Suppose I is any proper ideal of R. Then $I \subseteq P$ for some $P \in \operatorname{spec_m} R$, so that $I \notin \mathfrak{F}_P$, whence $I \notin \bigcap_{P \in \operatorname{spec_m} R} \mathfrak{F}_P$. □

Theorem 32. *Let R be a commutative ring for which $[\operatorname{Fil} R]^{\mathrm{du}}$ is two-sided residuated, or equivalently by Theorem 29, $\operatorname{Fil} R$ is commutative. Then $\operatorname{Rad}(\operatorname{Fil} R)$ is trivial.*

Proof. Take $\mathfrak{F}, \mathfrak{G} \in \operatorname{Fil} R$. Then:

$\mathfrak{F} \equiv_{\hat{\varphi}_P} \mathfrak{G}$ for all $P \in \operatorname{spec_m} R$

$\Rightarrow \mathfrak{F} : \mathfrak{F}_P = \mathfrak{G} : \mathfrak{F}_P$ for all $P \in \operatorname{spec_m} R$ [by Proposition 18]

$$\Rightarrow \bigcap_{P \in \operatorname{spec_m} R} (\mathfrak{F} : \mathfrak{F}_P) = \bigcap_{P \in \operatorname{spec_m} R} (\mathfrak{G} : \mathfrak{F}_P)$$

$$\Rightarrow \mathfrak{F} : \left(\bigcap_{P \in \operatorname{spec_m} R} \mathfrak{F}_P \right) = \mathfrak{G} : \left(\bigcap_{P \in \operatorname{spec_m} R} \mathfrak{F}_P \right)$$ [by Proposition 22((a)⇒(b)), noting that $[\operatorname{Fil} R]^{\mathrm{du}}$ is right residuated by hypothesis]

$\Rightarrow \mathfrak{F} = \mathfrak{G}$ [by the previous lemma noting that $\{R\}$ is the identity of $\operatorname{Fil} R$ with respect to the monoid operation ':'].

We conclude that $\operatorname{Rad}(\operatorname{Fil} R) = \bigcap_{P \in \operatorname{spec_m} R} \equiv_{\hat{\varphi}_P}$ is trivial. □

The next corollary follows from Theorem 32 and the fact that $\operatorname{Rad}(\operatorname{Fil} R)$ is the kernel of the canonical homomorphism from $[\operatorname{Fil} R]^{\mathrm{du}}$ to $\prod_{P \in \operatorname{spec_m} R} [\operatorname{Fil} R_P]^{\mathrm{du}}$ (see Figure 2).

Corollary 33. *Let R be a commutative ring for which $\operatorname{Fil} R$ is commutative. Then the canonical homomorphism of lattice ordered monoids*
$$\begin{aligned} [\operatorname{Fil} R]^{\mathrm{du}} &\longrightarrow \textstyle\prod_{P \in \operatorname{spec_m} R} [\operatorname{Fil} R_P]^{\mathrm{du}} \\ \mathfrak{F} &\longmapsto \{\hat{\varphi}_P(\mathfrak{F})\}_{P \in \operatorname{spec_m} R}, \end{aligned}$$

is monic and thus constitutes a subdirect embedding.

7. An application to Prüfer domains

Recall that a *Prüfer domain* is a commutative domain R for which R_P is a valuation domain for all maximal ideals P of R. (We refer the reader to [3] for background information on valuation domains.)

Following [4] (see also [5, page 434]) we say that a Prüfer domain R is *almost Dedekind* if R_P is Dedekind (and thus a rank one discrete valuation domain) for all maximal ideals P of R.

Our main goal in this section (Theorem 37) is to prove that a Prüfer domain R for which $\operatorname{Fil} R$ is commutative, is necessarily noetherian and thus a Dedekind domain. This result extends [11, Corollary 32, page 102] which says that a valuation domain R for which $\operatorname{Fil} R$ is commutative, is noetherian and thus rank one discrete.

The following is an initial step towards this goal.

Proposition 34. *If R is a Prüfer domain for which $\operatorname{Fil} R$ is commutative, then R is almost Dedekind.*

Proof. Take $P \in \operatorname{spec_m} R$. By (16), $[\operatorname{Fil} R_P]^{\mathrm{du}} \cong [\operatorname{Fil} R]^{\mathrm{du}} / \equiv_{\hat{\varphi}_P}$. Inasmuch as $\operatorname{Fil} R$ is commutative, we must have that $\operatorname{Fil} R_P$ is also commutative. Since R is a Prüfer domain, R_P is a valuation domain. It is known, however, that if T is any valuation domain for which $\operatorname{Fil} T$ is commutative, then T is noetherian and thus rank one discrete [11, Corollary 32, page 102]. We conclude that R_P is rank one discrete. □

If R is a commutative ring we shall denote by $\dim R$ the *Krull dimension* of R.

The following property of almost Dedekind domains is noted in [4, Theorem 1, page 813]. We provide a short proof of this readily accessible fact.

Lemma 35. *If R is an almost Dedekind domain, then every nonzero prime ideal of R is maximal, that is to say, $\dim R \leqslant 1$.*

Proof. If Q is any nonzero prime ideal of almost Dedekind domain R, and P is a maximal ideal of R such that $Q \subset P$, then by Proposition 10(e), Q_P and P_P are nonzero prime ideals of R_P with $Q_P \subset P_P$. This contradicts the fact that R_P is rank one. □

The following result is standard in the theory of commutative rings (see for example [2, Lemma 1.2.21, page 18]). We offer a proof of this result, since one may be readily extracted from the lattice embeddings exhibited in Figure 2.

Proposition 36. *If R is a commutative ring such that:*
(a) R_P is noetherian for all $P \in \operatorname{spec_m} R$; and
(b) every proper nonzero ideal of R is contained in only finitely many maximal ideals of R,
then R is noetherian.

Proof. Suppose R satisfies (a) and (b). To show that R is noetherian it clearly suffices to show that R/I is noetherian for all proper nonzero ideals I of R. Let I

be such an ideal (if no such I exists, then R is a field and there is nothing to prove). By (b), $\mathcal{P} \overset{\text{def}}{=} \{P \in \text{spec}_{\text{m}} R : P \supseteq I\}$ is finite. Since $I \nsubseteq P$ for all $P \in (\text{spec}_{\text{m}} R)\backslash\mathcal{P}$, we have $I_P = R_P$ for all $P \in (\text{spec}_{\text{m}} R)\backslash\mathcal{P}$.

The canonical embedding (see Figure 2)

$$\begin{aligned} \text{Id}\, R &\longrightarrow \textstyle\prod_{P\in\text{spec}_{\text{m}} R} \text{Id}\, R_P \\ K &\longmapsto \{K_P\}_{P\in\text{spec}_{\text{m}} R}, \end{aligned}$$

maps the interval $[I, R]$ of $\text{Id}\, R$ into $\prod_{P\in\text{spec}_{\text{m}} R}[I_P, R_P] \subseteq \prod_{P\in\text{spec}_{\text{m}} R} \text{Id}\, R_P$. Inasmuch as $[I_P, R_P]$ is a singleton for all $P \in (\text{spec}_{\text{m}} R)\backslash\mathcal{P}$, we see that $\prod_{P\in\text{spec}_{\text{m}} R}[I_P, R_P]$ and $\prod_{P\in\mathcal{P}}[I_P, R_P]$ are isomorphic lattices. By (a), the interval $[I_P, R_P]$ satisfies the ACC for each $P \in \mathcal{P}$. From this we may infer that $\prod_{P\in\mathcal{P}}[I_P, R_P]$ and hence $[I, R]$ satisfies the ACC. This implies that the ring R/I is noetherian, as required. □

Theorem 37. *The following statements are equivalent for a Prüfer domain R:*
(a) R is noetherian and thus a Dedekind domain;
(b) $\text{Fil}\, R$ *is commutative.*

Proof. (a)⇒(b) is a consequence of the fact that $\text{Fil}\, R$ is commutative in any commutative noetherian ring R by [11, Corollary 8, page 91].

(b)⇒(a) We show that the hypotheses of Proposition 36 are satisfied. Condition (a) of Proposition 36 evidently holds since, by Proposition 34, R_P is Dedekind and thus noetherian for all $P \in \text{spec}_{\text{m}} R$.

Let I be a proper nonzero ideal of R. It follows from Proposition 34 and Lemma 35 that $\dim R \leqslant 1$, whence $\dim R/I = 0$. Since the commutativity of $\text{Fil}\, R$ is passed from R to any nonzero factor ring of R by Theorem 24, we may infer from Theorem 28 that the ring R/I has only finitely many minimal prime ideals, but such minimal primes are maximal because $\dim R/I = 0$. It follows that $\{P \in \text{spec}_{\text{m}} R : P \supseteq I\}$ is finite. Condition (b) of Proposition 36 is thus established. We conclude that R is noetherian. □

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