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# CAUSAL RELATIONS IN SUPPORT OF IMPLICIT EVOLUTION EQUATIONS

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This is a brief exposition of dynamic systems approaches that form the basis for linear implicit evolution equations with some indication of interesting applications. Examples in infinite-dimensional dissipative systems and stochastic processes illustrate the fundamental notions underlying the use of double families of evolution equations intertwined by the empathy relation. Kisyński's equivalent formulation of the Hille–Yosida theorem highlights the essential differences between semigroup theory and the theory of empathy. The notion of K-bounded semigroups, a more direct approach to implicit equations, and related to empathy in a different way, is included in the survey.

*Keywords: implicit equations; empathy theory; semigroups.*

*Dedicated to Professor Jan Kisyński on the occasion of his 85th birthday.*

## Introduction

The classic autonomous evolution equation in a Banach space $Y$ is

$$\frac{d}{dt}u(t) = Au(t) \tag{1}$$

with $A$ (usually) an unbounded linear operator in $Y$. The causal relation that underpins such an equation is the semigroup $E(t+s) = E(t)E(s)$. Indeed, the solution curve is of the form $u(t) = E(t)y$, but it is not the only trajectory defined by the semigroup. The method is effective if $A$ is the generator of the semigroup and the initial state $y$ is in the domain of $A$.

In fluid mechanics and in dynamic boundary condition problems, evolution equations of the form

$$\frac{d}{dt}[Bu(t)] = Au(t) \tag{2}$$

arise. Here two Banach spaces $X$ and $Y$ are involved with $A$ and $B$ unbounded linear operators defined on $\mathfrak{D} \subset X$ with values in $Y$. These are called *implicit evolution equations.*

In early studies of implicit equations, modelled according to equations of hydrodynamics, the operators $A$ and $B$ were strongly elliptic partial differential equations not necessarily of the same order. This led to an assumption that the operators involved were closed or could be extended separately to closed operators. This, in turn, made possible to use the initial condition $\lim\limits_{t\to 0+} u(t) = u_0$ so that $\lim\limits_{t\to 0+}[Bu(t)] = Bu_0 = y$. For dynamic boundary condition problems this setting can be untenable. Indeed, an example of the operator $B$ not closeable was given in 1982. The natural initial condition should be

$$\lim_{t\to 0+}[Bu(t)] = y. \tag{3}$$

For the classic evolution equation (in a single space) treatment by means of semigroups made it necessary that the operator $A$ be closed or closeable. Thus we are led to the question: what is the analog of the notion of semigroup when we treat implicit equations? It turned out to be important to consider two families of "evolution operators". The "solution operators" $S(t) : Y \to X$ which will represent the solution in the form $u(t) = S(t)y$ and another family $E(t) : Y \to Y$ that will describe the curve $v(t) = Bu(t)$ in $Y$. If we free ourselves temporarily from the operator $B$, this leads to the *empathy relation* $S(t+s) = S(t)E(s)$.

If causal relations such as semigroup and empathy are used to investigate the solvability of evolution equations, a major question not always kept in mind, is the nature of the initial states that evolve into solutions. Our approach to the problem is via the Laplace transform which leads in an almost natural way to an answer.

In what follows we give a survey of the development of the empathy relation. Some proofs are given to share some secrets of the trade. Long and technical proofs have been omitted. The flow of text is as follows: Section 1 gives the mathematical setting for empathy theory and the very basic results. The bearing of empathy theory on implicit evolution equations is discussed in Section 2 where we introduce the notion of *generator* of an empathy. Section 3 gives a very short indication of how the theory is applied. The important case of holomorphic empathies is briefly discussed in Section 4. As in the case of holomorphic semigroups, the "admissible" class of initial states is quite large. In Section 5 we discuss integrated empathy which is the analog of integrated semigroup. An adaptation of Jan Kisyński's algebraic approach to the Hille–Yosida construction to empathy theory is discussed in Section 6. Section 7 is devoted to the role of empathy-considerations in Markov processes. It leads to implicit Fokker–Planck equations. The topic of K-bounded semigroups, which represents an alternative approach to implicit evolution equations, is explained in Section 8. We conclude with Section 9 where additional background is supplied and references are given.

## 1. Empathy

Let $X$ and $Y$ be Banach spaces (real or complex). We consider two families of bounded linear operators, $\{S(t) : Y \to X\}$ and $\{E(t) : Y \to Y\}$ defined for $t > 0$, with the following properties:

$$S(t+s) = S(t)E(s) \quad \text{for all } t, s > 0. \tag{4}$$

For every $\lambda > 0$ and $y \in Y$ the Laplace transforms

$$P(\lambda)y = \int_0^\infty \exp\{-\lambda t\} S(t)y\, dt, \tag{5}$$

$$R(\lambda)y = \int_0^\infty \exp\{-\lambda t\} E(t)y\, dt \tag{6}$$

exist as Lebesgue (Bochner) integrals.

As stated before, the requirement (4) is known as the *empathy relation.* The conditions (5) and (6) are akin to similar requirements used in semigroup theory when the $C_0$-property is not imposed.

Direct calculations lead to

**Theorem 1.** *For arbitrary positive $\lambda$, $\mu$ and $t$ the following is true:*

$$P(\lambda)E(t) = S(t)R(\lambda), \tag{7}$$
$$P(\lambda)R(\mu) = P(\mu)R(\lambda),$$
$$P(\lambda) - P(\mu) = (\mu - \lambda)P(\lambda)R(\mu). \tag{8}$$

Additional to the assumptions (5) and (6) we make one more with far-reaching consequences:

THE INVERTIBILITY ASSUMPTION. *There exists $\xi > 0$ for which the linear operator $P(\xi) : Y \to X$ is invertible.*

**Theorem 2.** *The family $\{E(t)\}$ is a semigroup, strongly continuous in $(0, \infty)$. Moreover, the family $\{S(t)\}$ is strongly continuous in $(0, \infty)$.*

*Proof.* From (4) we see that $S(t+r+s)y = S(t)E(r+s)y = S(t+r)E(s)y = S(t)E(r)E(s)y$. Taking the Laplace transform at $\xi$ with respect to $t$ gives $P(\xi)[E(r+s)y - E(r)E(s)y] = 0$ and the semigroup property follows. Assumption (6) means that the function $t \to E(t)y$ is measurable and therefore continuous in $(0, \infty)$ [1, Theorem 10.2.3]. From (4) it is seen that if $h \in (0, t/2)$, $S(t-h)y = S(t/2)E(t/2-h)y \to S(t/2)E(t/2)y = S(t)y$ when $h \to 0$. Continuity from the right is easier.

□

We note that continuity at $t = 0$ makes no sense and $\{E(t)\}$ does not have to be of class $C_0$. As another important consequence we have the identities

$$R(\lambda)E(t) = E(t)R(\lambda), \tag{9}$$
$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu). \tag{10}$$

They are derived in the same way as the identities in Theorem 1. Thus we have arrived at two *pseudo-resolvent equations* namely (8) and (10). These will turn out to be crucial.

**Theorem 3.** *For every $\lambda > 0$ the operator $P(\lambda)$ is invertible.*

*Proof.* We consider the kernels of $P(\lambda)$ and $R(\lambda)$. First we note from (10) that $\ker R(\lambda) = \ker R(\mu) =: N_E$. Then, from (8), if $y \in N_E \cap \ker P(\mu)$ then $P(\lambda)y = (\mu - \lambda)P(\lambda)R(\mu)y = 0$. Hence $P(\lambda) \cap N_E = P(\mu) \cap N_E$ for arbitrary $\lambda$ and $\mu$. Suppose that $P(\lambda)y = 0$. From (8) it follows that $P(\xi)y = (\xi - \lambda)P(\xi)R(\lambda)y$ and from the invertibility assumption therefore, $y = (\xi - \lambda)R(\lambda)y$. Hence $R(\xi)y = (\xi - \lambda)R(\xi)R(\lambda)y$. It follows from (10) that $R(\lambda)y = 0$. Thus $y \in \ker P(\lambda) \cap N_E = \ker P(\xi) \cap N_E = \{0\}$.

□

As can be seen from the proof above, the pseudo-resolvent equations (8) and (10) lead to invariances. There are more. Let us define the domains $\mathfrak{D}_E := R(\lambda)[Y]$ and $\mathfrak{D} := P(\lambda)[Y]$. From (8) and (10) we readily see that the definitions do not depend on the choice of $\lambda$. Indeed we have, as a consequence of (7) and (9),

**Theorem 4.** *For all $t > 0$, $E(t)[\mathfrak{D}_E] \subset \mathfrak{D}_E$ and $S(t)[\mathfrak{D}_E] \subset \mathfrak{D}$.*

Some calculations are needed to obtain the following representations:
Let $y = R(\lambda)y_\lambda \in \mathfrak{D}_E$. Then

$$E(t)y = e^{\lambda t}[y - \int_0^t e^{-\lambda s} E(s)y_\lambda \, ds], \tag{11}$$

$$S(t)y = e^{\lambda t}[P(\lambda)y - \int_0^t e^{-\lambda s} S(s)y_\lambda \, ds]. \tag{12}$$

From these representations it is possible to prove

**Theorem 5.** *The following statements hold:*
(a) *For every* $y \in \mathfrak{D}_E$, $\lim_{t\to 0+} E(t)y = y$.
(b) *There exists a linear operator* ${}^{\iota}B_0 : \mathfrak{D}_E \to \mathfrak{D}$ *defined by* ${}^{\iota}By = \lim_{t\to 0+} S(t)y$.
(c) *The operators* $R(\lambda)$ *are invertible for all* $\lambda > 0$.
(d) *The operator* ${}^{\iota}B_0$ *is invertible,* ${}^{\iota}B = P(\lambda)R^{-1}(\lambda)$ *and* ${}^{\iota}B[\mathfrak{D}_E] = \mathfrak{D}$.

The notation ${}^{\iota}B_0$ indicates that this operator is the inverse of an operator yet to be introduced. To end this section we state a theorem, the proof of which is based on the dominated convergence theorem, to show that Theorem 5 is in accordance with asymptotic behaviour of the Laplace transform.

**Theorem 6.** *For* $y \in \mathfrak{D}_E$, $\lim\limits_{\lambda\to\infty} \lambda R(\lambda)y = y$, *and* $\lim\limits_{\lambda\to\infty} \lambda P(\lambda)y = {}^{\iota}By$.

## 2. The Generator of an Empathy

We shall refer to the pair $\langle S(t), E(t)\rangle$ as an *empathy* and proceed to define the notion of its *generator*. To begin with we define the operator $B : \mathfrak{D} \to \mathfrak{D}_E$ as the inverse of the operator ${}^{\iota}B$. By Theorem 5 this is possible. In fact, we have the representation $B = [{}^{\iota}B]^{-1} = R(\lambda)P^{-1}(\lambda)$ which is free of the choice of $\lambda$.

Next, we define the operators $A_\lambda := [\lambda R(\lambda) - I_Y]P^{-1}(\lambda)$. It takes some effort to prove that $A_\lambda = A_\mu =: A$ and that $P(\lambda) = (\lambda B - A)^{-1}$. We call the operator pair $\langle A, B\rangle$ the *generator* of the empathy. The following result explains the word.

**Theorem 7.** *Let* $u(t) = S(t)y$. *For* $y \in \mathfrak{D}_E$, $u(t)$ *is a solution of the Cauchy problem*

$$\left.\begin{aligned} &\frac{d}{dt}[Bu(t)] = Au(t), \\ &\lim_{t\to 0+}[Bu(t)] = y. \end{aligned}\right\} \tag{13}$$

*Proof.* From Theorem 4 we see that $u(t) \in \mathfrak{D}$ so that $Au(t)$ and $Bu(t)$ are well-defined. Let us obtain an expression for $Bu(t)$. If $y = R(\lambda)y_\lambda$ then, from the representation of $B$, (7) and (9) we obtain $Bu(t) = BS(t)R(\lambda)y_\lambda = R(\lambda)P^{-1}(\lambda)S(t)R(\lambda)y_\lambda = R(\lambda)E(t)y_\lambda = E(t)R(\lambda)y_\lambda = E(t)y$.

From Theorem 5 we immediately see that the initial condition is satisfied. To evaluate the derivative of $Bu(t)$ we use the representation (11) and note that, by Theorem 2, $Bu(t) = E(t)y$ is indeed differentiable. The derivative, with the aid of (9) and some manipulations, turns out to be $Au(t)$. □

Thus the notion of empathy, which involves two families of "evolution operators", turns out to be the replacement of semigroups when implicit evolution equations are concerned. We need to take the analogy further. In semigroup theory the generator needs to be a closed (or closeable) operator (see *e.g.* [1, Chapter XI]). What should it be in empathy theory? For this we note a fact that was not used before, namely that the operators $P(\lambda)$ are bounded [1, Theorem 3.8.2]. This means that the operators $P^{-1}(\lambda) = \lambda B - A$ are closed. What does that say about $A$ and $B$? For this we introduce the notion of *jointly closed*. Let $A, B : \mathfrak{D} \subset X \to Y$ be two linear operators. Then $A$ and $B$ are jointly closed if the operator $\langle A, B\rangle : x \in \mathfrak{D} \to \langle Ax, Bx\rangle \in Y \times Y$ is closed. This is the same as saying that if $\{x_n\} \subset \mathfrak{D}$, $x_n \to x$ in $X$, $Ax_n \to y_A$ and $Bx_n \to y_B$, then $x \in \mathfrak{D}$ and $Ax = y_A$, $Bx = y_B$. It takes some refection to conclude that if $\lambda B - A$ is closed for two distinct values of $\lambda$, then $A$ and $B$ is jointly closed. Thus we have

**Theorem 8.** *The generator of an empathy is closed.*

We note that if the operators $A$ and $B$ are both closed, the operator $\langle A, B\rangle$ is closed. The converse is not true.

## 3. Applications

To apply empathy theory to concrete problems, sufficient conditions for an operator pair to be the generator of an empathy are needed. We say that the empathy $\langle S(t), E(t)\rangle$ is *uniformly bounded* if for all $t > 0$ there are constants $M$ and $N$ such that $\|S(t)\| \leq M$ and $\|E(t)\| \leq N$.

Let $A, B : \mathfrak{D} \subset X \to Y$ be given linear operators, and suppose that $P(\lambda) := (\lambda B - A)^{-1}$ exists for every $\lambda > 0$. In accordance with Theorem 5 we let $R(\lambda) := BP(\lambda)$. The Hille–Yosida-type theorem is

**Theorem 9.** *Suppose that the space $Y$ has the Radon–Nikodým property and the operators $P(\lambda)$ and $R(\lambda)$ as defined above, are bounded. Then the operator pair $\langle A, B\rangle$ is the generator of a bounded empathy if and only if there exist constants $M$ and $N$ such that for all $\lambda > 0$ and $k = 1, 2 \ldots$*

$$\|\lambda P(\lambda)\| \leq M \quad \text{and} \quad \|\lambda^k R^k(\lambda)\| \leq N.$$

*$Y_E$, the closure of $\mathfrak{D}_E$ in $Y$, is an invariant subspace of $E(t)$.*

This theorem is still far removed from applications. The Radon–Nikodým property for the space $Y$ is a concern, although the reflexive Banach space all have it. Of deeper concern though, is the requirement imposed by the condition that $P(\lambda) = (\lambda B - A)^{-1}$ is bounded for all $\lambda$ which implies that $A$ and $B$ should be jointly closed. For situations based on dissipative phenomena such as heat and diffusion, the evolution equations are often framed in a Hilbert space setting (*e.g.* $L^2$ and embedded Sobolev spaces) and the problem is to extend the operator pair $\langle A, B\rangle$ to be (jointly) closed. An analogue of the Friedrichs extension had to be found. This was done for the first time to deal with dynamic boundary conditions for the Navier–Stokes equations, and later extended to cover a large class of problems including the so-called Sobolev equations (misnamed after Sobolev) and pseudo-parabolic equations.

A telling example that shows the strength of the approach, is the heat equation in a domain $\Omega \subset \mathbb{R}^3$ with boundary $\Gamma$. The equation is

$$u_t(x,t) - \Delta u(x,t) = 0; \quad x \in \Omega;\ t > 0,$$

with $\Delta$ is the Laplacian.

If the boundary $\Gamma$ is considered as heat-transferring medium with it own thermal properties, the boundary condition is, in a very simple model

$$\partial_t[\gamma_0 u(y,t)] + k\gamma_1 u(y,t) \quad y \in \Gamma.$$

Here $\gamma_0$ denotes the boundary trace operator and $\gamma_1$ the normal derivative. If we take $X = L^2(\Omega)$, $Y = L^2(\Omega) \times L^2(\Gamma)$ and $\mathfrak{D} = W^{2,2}(\Omega)$, the system of equations can be expressed in the form

$$\partial_t \langle u, \gamma_0 u\rangle = \langle \Delta u, -k\gamma_1 u\rangle = 0 \in Y.$$

We use the notation $\langle\, .\, ,\, .\,\rangle$ to denote elements of $Y$. Thus with $Bu = \langle u, \gamma_0 u\rangle$ and $Au = \langle \Delta u, -k\gamma_1 u\rangle$, this becomes a respectable implicit equation. But the operator $B$ is not closeable. Fortunately the joint extension exists and the problem can be handled.

## 4. Holomorphic Empathies

If the semigroup $E(t)$ is *holomorphic*, in the sense that $R(\lambda)$ can be extended to a sector $\Sigma_\phi = \{-(\phi + \pi/2) < \operatorname{Arg}(\lambda) < \pi + \pi/2 : 0 < \phi < \pi/2\}$ of the complex plane such that

$$\|R(\lambda)\| \leq \tfrac{M}{|\lambda|} \text{ for } \lambda \in \Sigma_\phi,$$

the operators $P(\lambda)$ can also be extended and satisfy

$$\|P(\lambda)\| \leq \tfrac{N}{|\lambda|} \text{ for } \lambda \in \Sigma_\phi.$$

In this case $S(t)$ and $E(t)$ can be represented by familiar contour integrals and the associated Cauchy-problem

$$\left.\begin{aligned} &\frac{d}{dt}[Bu(t)] = Au(t), \\ &\lim_{t\to 0+}[Bu(t)] = y, \end{aligned}\right\}$$

can be solved for $y \in Y_E$. The holomorphic case therefore allows a larger class of initial states.

The extension procedure mentioned in Section 3 allows for the extension of an operator pair $\langle A, B\rangle$ to be the generator of a holomorphic empathy. In particular this leads to holomorhic empathies associated with dynamic boundary conditions for heat transfer and diffusion such as the one discussed above where the initial state can be in $L^2(\Omega) \times L^2(\Gamma)$. For the Sobolev-type of Cauchy-problem

$$\left.\begin{aligned} &\partial_t[Mu(x,t)] = Lu(x,t); \quad x \in \Omega, \\ &\lim_{t\to 0+}[Mu(y,t)] = y; \quad y \in \Gamma, \end{aligned}\right\}$$

with $L$ and $M$ strongly elliptic partial differential operators of order $2\ell$ and $2m$ respectively, the underlying empathy is holomorphic. The joint (Friedrichs) extension of $\langle L, M\rangle$ leads to a much wider class of initial states than obtained by extending $L$ and $M$ separately.

Within the framework of holomorhic empathy there is a happy idea of studying perturbations. This is based on comparison within a fixed solution space $X$ and using the complex plane as another steady entity while allowing the operators $A$ and $B$ to have different range spaces. The framework is as follows: For $n = 1, 2, \ldots$ let $A_n, B_n : \mathfrak{D}_n \subset X \to Y_n$ and consider the sequence of Cauchy-problems

$$\left.\begin{aligned} &\frac{d}{dt}[B_n u(t)] = A_n u(t); \\ &\lim_{t\to 0+} u(t) = y_n \in Y_n. \end{aligned}\right\}$$

Let $\langle S(t), E(t)\rangle$ be holomorphic empathy in the spaces $X$ and $Y$ with sectorial domain $\Sigma_{\phi_0}$. If there is a sectorial domain $\Sigma_\psi$ such that

$$\bigcap_{n=0}^{\infty} \Sigma_{\phi n} \supset \Sigma_\psi,$$

and for $\lambda \in \Sigma_\psi$,

$$\lim_{n\to\infty} \|P_n(\lambda)y_n - P(\lambda)y\| = 0,$$

then $S_n(t)y_n \to S(t)y$ in $X$. This result can be used to study singular perturbations of dynamic boundary conditions and Sobolev-type equations.

## 5. Integrated Empathy

The requirement in Theorem 9 that the space $Y$ have the Radon–Nikodým property excludes cases where spaces of continuous functions under the supremum norm are concerned. For semigroups this has been overcome by introducing the notion of *integrated* semigroups (see [2, Chapter 3, p. 124]). In the framework of a double family $\langle S(t), E(t)\rangle$ of evolution operators, this concept is adapted by replacing the causal relation (4) by

$$S(t)E(s)y = \int_0^t [S(s+\sigma) - S(\sigma)]y\, d\sigma. \tag{14}$$

A pair $\langle S(t), E(t)\rangle$ satisfying (14) is called an *integrated empathy.* Analogous to (5) and (6) we introduce the Laplace transforms $p(\lambda)$ and $r(\lambda)$ by

$$p(\lambda)y = \int_0^\infty \exp\{-\lambda t\} S(t)y\, dt,$$

$$r(\lambda)y = \int_0^\infty \exp\{-\lambda t\} E(t)y\, dt$$

and let $P(\lambda) := \lambda p(\lambda)$, $R(\lambda) := \lambda r(\lambda)$. The analysis now follows the same pattern as in Section 1 with some deviations. The representations (11) and (12) are different and one needs to replace $\lim_{t\to 0+} E(t)y$ and $\lim_{t\to 0+} S(t)y$ in Theorem 5 by $\lim_{t\to 0+} t^{-1}E(t)y$ and $\lim_{t\to 0+} t^{-1}S(t)y$.

The implicit evolution equation in Theorem 7 satisfied by $u(t) = S(t)y$; $y \in \mathfrak{D}_E$ is replaced by the implicit integral equation

$$Bu(t) = y + A\int_0^t u(s)\,ds.$$

For a domain smaller than $\mathfrak{D}_E$ the original implicit differential equation is still satisfied.

Within this setting it is possible to study wave motion described by the system of equations

$$u_{1,t}(x,t) - u_{2,x}(x,t) = 0,$$
$$u_{2,t}(x,t) - u_{1,x}(x,t) = 0,$$

for $0 < x < 1$, under the dynamic boundary condition

$$\frac{d}{dt}[\lim_{x\to 1} u_1(x,t)] + \lim_{x\to 1} u_2(x,t) = 0,$$

in the space $C([0,1])$ of continuous functions. The boundary condition at $x = 0$ can be one of many.

## 6. The Kisyński Construction

The Hille–Yosida-like generation Theorem 9 is based on considering, for given (unbounded) linear operators $A$ and $B$ from $\mathfrak{D} \subset X$ to $Y$ that define, for $\lambda > 0$ the operators

$$P(\lambda) = (\lambda B - A)^{-1}; \tag{15}$$
$$R(\lambda) = BP(\lambda), \tag{16}$$

and assuming that they exist and are bounded. Moreover, the result, (Theorem 9), leans on the Widder inversion theorem which can only hold if the space $Y$ has the Radon–Nikodým property. The question is, can these assumptions be weakened in some sense?

An answer to this question can be found in Kisyński's algebraic approach to the Hille–Yosida theorem for semigroups [4, Theorem 12.5]. But there is a price to be paid.

The point of departure is to consider two families $P_\lambda : Y \to X$ and $R_\lambda : Y \to Y$ of bounded linear operators defined for $\lambda > 0$ that satisfy the *entwined* pseudo-resolvent equations

$$\left.\begin{aligned} R_\lambda - R_\mu &= (\mu - \lambda) R_\lambda R_\mu, \\ P_\lambda - P_\mu &= (\mu - \lambda) P_\lambda R_\mu, \end{aligned}\right\} \tag{17}$$

analogous to the expressions (8), (10). They replace the operators $P(\lambda)$ and $R(\lambda)$ associated with Laplace transforms. In this way we free ourselves from the assumption that the space $Y$ has the Radon–Nikodým property which is needed for the Widder inversion theorem to hold if the Hille–Yosida inequalities in Theorem 9 are invoked. Instead, we assume that the pseudo-resolvent $R_\lambda : Y \to Y; \lambda > 0$ satisfies the *strong Widder growth condition* [5]

$$\sup\{\|[\lambda R_\lambda]^k\| : \lambda > 0; k \in \mathbb{N}\} < \infty. \tag{18}$$

In addition it is assumed that the operators $R_\lambda$ are invertible, in contrast to the (invertibility) assumption that $P_\lambda$ be invertible.

We define the class of "initial states" $\mathscr{D}_E := R_\lambda[Y]$ analogous to $\mathfrak{D}_E$. The growth condition (18) ensures the existence of a unique bounded Banach algebra representation $T : r_\lambda \mapsto R_\lambda$ on the convolution algebra $Z := \langle L^1(0,\infty), \circledast\rangle$ of integrable scalar functions. The family of exponentials $\{r_\lambda : \lambda > 0\}$ with $r_\lambda(x) = \exp\{-\lambda x\}$ for $x \geq 0$ is a canonical pseudo-resolvent on $Z$.

Following Kisýnski, we use the representation $T$ to *algebraically* construct a semigroup $\{E(t) : t \geq 0\}$ of class $C_0$ on the subspace $\Delta_K := \{y \in Y : \lim_{\lambda\to\infty} \|\lambda R_\lambda y - y\| = 0\}$. This analytic object has the algebraic reconstruction

$$\Delta_K = \overline{\mathscr{D}}_E = \bigcup_{\phi\in Z} T(\phi)[Y],$$

which is called the *T-regularity space*. Now a *unique* $C_0$-semigroup $E(t)$ on $\Delta_K$ is constructed from the canonical $C_0$-semigroup of right-shifts $R^t$ on $Z$ by letting $E(t)y := [T(R^t(\phi))]y_\phi$ for $y = T(\phi)y_\phi \in \Delta_K$. The (vector) subspace $\mathscr{D}_E$ is precisely the domain of the infinitesimal generator of $E(t)$.

As in Section 1 the domain $\mathscr{D} := P_\lambda[Y]$ is free of the choice of $\lambda$, and assume that that there is, in analogy to (16), a linear operator $B : \mathscr{D} \subset X \to Y$ such that $R_\lambda = BP_\lambda$. Thus we replace the invertibility of $P_\lambda$ by the weaker condition that $BP_\lambda$ is invertible.

The problem at hand is to define a class of "initial states" $y$ and a double family $\langle S(t), E(t)\rangle_{t>0}$ intertwined by the empathy relation such that $u(t) = S(t)y$ satisfies (13).

Towards this goal we define the class $\Delta^2_K := R_\lambda[\Delta_K]$ and call it the $T^2$*-regularity space*. Then $\Delta^2_K$ is a dense subspace (like $\mathscr{D}_E$) of $\Delta_K$. We use the operator $C := P(\lambda)R^{-1}(\lambda) : \Delta^2_K \to \mathscr{D}$ to construct a map $T^2 := CT$ on the algebra $Z$. Note that $T^2$ is not an algebra representation and need not even be closed. Then we construct, (a) the semigroup $E(t) : \Delta_K \to \Delta_K$ by the bounded Banach algebra representation $T$ and (b) the (family of) operators $S(t) : \Delta^2_K \to \mathscr{D}$ by the map $T^2$ as follows:

$$S(t)[R_\lambda T(\phi)] = T^2(r_\lambda \circledast R^t\phi) = P(\lambda)[E(t)T(\phi)].$$

The bounded operator $S(t)$ constructed on the dense subspace $\Delta^2_K$ can be extended by continuity to the Banach space $\Delta_K$. Moreover, the map $T^2$ that generates the operators $S(t)$ underscores the fact that empathy relation (4) is more than a mild generalization of semigroup theory. Then from the relation $R_\lambda = BP_\lambda$ and the commutativity of $E(t)$ and $R_\lambda$ we have the following result:

**Theorem 10.** *Let* $\langle P_\lambda, R_\lambda\rangle$ *be an entwined pseudo-resolvent* (17) *and* $B : \mathscr{D} \subset X \to Y$ *be a linear operator such that* $R_\lambda = BP_\lambda$. *If* $R_\lambda$ *satisfies the growth condition* (18), *then* $u(t) = S(t)y;\ y \in \Delta^2_K$ *solves the implicit Cauchy problem* (13) *with* $A = A_E B$. *Here* $A_E$ *is the infinitesimal generator of* $E(t)$.

If the operators $P(\lambda)$ have the form (15) then a direct calculation shows that $B$ is invertible and $C = B^{-1}$. This is at the core of the theory of empathy.

## 7. Implicit Fokker–Planck Equations

Causal relations explain how a state $u(t)$ at time $t$ evolves to a state $u(t+s)$ at a later time $t+s$ where $s>0$. The semigroup relation $E(t+s)=E(s)E(t)$ is a causal relation that is based on the assumption that the effect of the state $u(t)$ on any later state $u(t+s)$ is determined solely by the the time difference $s$. In many real world situations, one only knows the probability of transition from one state to another in a given time $s$. This is captured in the notion of a stochastic kernel $Q_s(x,B)$ of a Markov process $\mathbf{X}$. The Chapman–Kolmogorov equation

$$Q_{t+s}(x,B)=\int_{y\in\mathbb{R}} Q_t(x,\{dy\})Q_s(y,B) \quad \text{for all } s,t>0, \tag{19}$$

is a causal relation that follows from the assumption of memory-less transitions. For a homogeneous Markov process $\mathbf{X}$ with a transition function $Q=\{Q_s(x,B)\}_{s>0}$ intertwined by the Chapman–Kolmogorov equation, $Q$ has an operator representation $\{E_s : Y\to Y\}_{s>0}$ on a suitable space $Y$ of measurable functions and $E_{t+s}=E_sE_t$. The question is, what causal relation similar to (19) is described by the empathy relation $S(t+s)=S(s)E(t)$?

An answer to this question can be found in a dynamic boundary condition approach to diffusion with an absorbing barrier when the boundary is a object in its own right, interacting meaningfully with the system as another body. Two distinct intensities arise: the absorbing barrier is seen as a distinct collection of states with zero intensity, while the system is a typical pseudo-Poisson process with uniform intensity $a>0$. Also, two distinct state spaces arise: $S_X:=\{1,\ldots,m\}$, the set of $m$ safe (life) states that all have the same intensity $a$, and $S_Y:=\{\overline{1},\ldots,\overline{n}\}$, the set of $n$ death states that all have the same intensity 0. This assumption can be regarded as a stochastic model of a jump diffusion coefficient, which is prohibited in classical diffusion equations. The bar notation distinguishes between a life state and a death state.

The pair of state spaces $\langle S_X,S_Y\rangle$ gives rise to the pair $\langle q_{ij}^{(n)}, r_{i\bar{j}}^{(n)}\rangle_{n\geq 1}$ of $n$-step transition functions,

$$\left.\begin{aligned} q_{ij}^{(n)} &= \sum_{k=1}^{|S_X|} q_{ik}^{(n-1)}p_{kj} \quad \text{with } q_{ij}^{(1)}=p_{ij};\\ r_{i\bar{j}}^{(n)} &= \sum_{k=1}^{|S_X|} q_{ik}^{(n-1)}s_{kj} \quad \text{with } r_{i\bar{j}}^{(1)}=s_{i\bar{j}}, \end{aligned}\right\} \tag{20}$$

where $p_{ij}$ denotes the one step transition $P(X_{n+1}=j|X_n=i)$ and $s_{i\bar{j}}$ denotes the one step transition $P(Y_{n+1}=\bar{j}|X_n=i)$. For the continuous-time version of the discrete Markov chains (20), we partition the event of a transition in the time interval $(0,t]$ into mutually exclusive events determined by the number of transitions in the time interval $(0,t]$:

$$\left.\begin{aligned} Q_t(i,\Gamma=\{j\}) &= P(X_t=j|X_0=i)=e^{-at}\sum_{n=0}^{\infty} q_{ij}^{(n)}\frac{(at)^n}{n!};\\ R_t(i,\overline{\Gamma}=\{\bar{j}\}) &= P(Y_t=\bar{j}|X_0=i)=e^{-at}\sum_{n=0}^{\infty} r_{i\bar{j}}^{(n)}\frac{(at)^n}{n!}. \end{aligned}\right\}$$

Now, the pair $\langle Q, R\rangle := \langle Q_t(i,\Gamma), R_t(i,\overline{\Gamma})\rangle_{t>0}$ is intertwined by a countable version of the *backward extended Chapman–Kolmogorov equation*:

$$Q_{t+s}(x,\Gamma) = \int\limits_{y\in\mathbb{R}} Q_t(x,\{dy\})Q_s(y,\Gamma); s,t>0;$$

$$R_{t+s}(x,\overline{\Gamma}) = \int\limits_{y\in\mathbb{R}} Q_t(x,\{dy\})R_s(y,\overline{\Gamma});\ s,t>0. \tag{21}$$

Their operator representation $\langle S_t : X \to Y, E_t : Y \to Y\rangle_{t>0}$ produces a pair of function spaces $X := BM(S_Y)$ and $Y := BM(S_X)$ of bounded measurable functions on $S_X$ and $S_Y$, respectively. The two-state space uni-directional transition (21) forces the two state spaces $X$ and $Y$ to be disjoint. Indeed, the pair $\langle S_t, E_t\rangle_{t>0}$ has exponential (pencil-operator) representations $S_t = e^{-at}e^{atK_R}$ and $E_t = e^{-at}e^{atK_Q}$ where $\langle K_R, K_Q\rangle$ are the operator representations of the pair of defective transition matrices $\langle [s_{i,\overline{j}}]_{i\in X,\overline{j}\in\overline{Y}}, [p_{ij}]_{i,j\in X}\rangle$, respectively. Moreover,

$$e^{a(t+s)K_R} = e^{atK_Q}e^{asK_R}. \tag{22}$$

By (22), the pair $\langle S_t, E_t\rangle_{t>0}$ is intertwined by the *reverse empathy relation* $S_{t+s} = E_tS_s; s,t > 0$. From the adjoint of the defective transition matrices, we then construct an empathy $\langle \overline{S}_t : Y \to X, \overline{E}_t : Y \to Y\rangle$. The machinery of empathy theory then yields an implicit Fokker–Planck equation.

## 8. K-Bounded Semigroups and Implicit Evolution Equations

Very often the existence of the semigroup $\{E(t)\}_{t\geq 0}$ solving (1) is established in a non-constructive way and then very little quantitative information on the evolution is available. On the other hand, there may exist an operator $K$ such that $t \to KE(t)$ can be calculated constructively yielding some information about the evolution. In other words, it may be possible to look at the evolution through the "lens" of another operator and filter out the information we are not able to quantify. This idea led to the following definition:

**Definition 1.** *Let $X$ and $Y$ be Banach spaces and let $L : \mathfrak{D}_L \subset Y \to Y$ and $K : \mathfrak{D}_K \subset Y \to X$ be linear operators. Suppose that $\mathfrak{D}_L \subset \mathfrak{D}_K$ and for some $\omega \in \mathbb{R}$ the resolvent set of $L$ satisfies*

$$\rho(L) \supset (\omega,\infty). \tag{23}$$

*A family of operators $\{Z(t)\}_{t\geq 0}$ from $\mathfrak{D}_K$ to $X$, which satisfies*
*1) for every $t \geq 0$ and $f \in \mathfrak{D}_K$*

$$\|Z(t)f\|_X \leq M\exp\{\omega t\}\|Kf\|_X, \tag{24}$$

*2) for every $f \in \mathfrak{D}_K$ the function $t \to Z(t)f \in C([0,\infty), X)$,*
*3) for $f \in \mathfrak{D}_{K(L)} := \{f \in \mathfrak{D}_L \cap \mathfrak{D}_K : Lf \in \mathfrak{D}_K\}$*

$$Z(t)f = Kf + \int\limits_0^t Z(s)Lf\,ds; \quad t \geq 0,$$

*is called a $K$-bounded semigroup generated by $L$.*

For an operator $A$ we write $A \in \mathcal{G}(M,\omega,Y)$ if $A$ generates a $C_0$-semigroup in $Y$, that satisfies estimate such as (24). If there is a need to indicate that a semigroup is generated by $A$ we write $\{E_A(t)\}_{t\geq 0}$. Similarly, we write $L \in K-\mathcal{G}(M,\omega,Y,X)$ if $L$ generates a $K$-bounded semigroup as in Definition 1.

Let $Y_K$ be the completion of the quotient space $\mathfrak{D}_K/\ker K$ with respect to the seminorm $\|f\|_K = \|Kf\|_X$. Then $\mathfrak{D}_K/\ker K$ is isometrically isomorphic to a dense subspace of $Y_K$, say $\mathcal{Y}$. The canonical injection of $Y$ into $Y_K$ (and onto $\mathcal{Y}$) will be denoted by $\mathfrak{p}$. In a standard way, $K$ can be extended by density to an isometry $\mathfrak{K} : Y_K \to X$.

An important observation is that if $L$ generates a $K$-bounded semigroup, then $L$ preserves cosets of $\mathfrak{D}_K/\ker K$ and therefore it can be defined to act from $\mathfrak{p}\mathfrak{D}_{K(L)} \subset \mathcal{Y}$ into $\mathcal{Y}$. We denote by $L_K$ the part of $L$ in $\mathfrak{D}_K$, *i.e.* $L_K = L|_{\mathfrak{D}_{K(L)}}$. It can be also proved that if $L \in K-\mathcal{G}(M,\omega,Y,X)$, then the shift of $L$ to $\mathcal{Y}$ is closeable in $Y_K$; we denote its closure by $\mathfrak{L}$.

To simplify the notation we will use the same notation for the operators $L$ and $K$ and their shifts to $\mathcal{Y}$. With this convention the injection $\mathfrak{p}$ becomes the identity (or more precisely projection) and for any operator $\mathfrak{C}$ defined in $Y_K$ and $f \in \mathfrak{D}_K$, the symbol $\mathfrak{C}f$ is to be understood as $\mathfrak{C}\mathfrak{p}f$, if the latter is defined.

We can provide a complete characterization of $K$-bounded semigroups.

**Theorem 11.** *If $L \in K-\mathcal{G}(M,\omega,Y,X)$ and $\overline{K[\mathfrak{D}_{K(L)}]}^X = X_K$, then $\mathfrak{L} \in \mathcal{G}(M,\omega,Y_K)$. Conversely, if there is $\mathcal{L} \supset L$ such that $\mathcal{L} \in \mathcal{G}(M,\omega,Y_K)$, then $\mathcal{L} = \mathfrak{L}$ and $L \in K-\mathcal{G}(M,\omega,Y,X)$.*

*The $K$-bounded semigroup $\{Z(t)\}$ for $f \in \mathfrak{D}_K$ is given by*

$$Z(t)f = E_{\mathfrak{K}\mathfrak{L}\mathfrak{K}^{-1}}(t)Kf = \mathfrak{K}E_{\mathfrak{L}}(t)f. \tag{25}$$

The assumption that $K[\mathfrak{D}_{K(L)}]$ is dense in $X_K$ can be discarded if $X$ (and consequently $X_K$) are reflexive spaces.

Since the space $Y_K$ is in many cases rather difficult to handle, Theorem 11 is most often used in the following version.

**Theorem 12.** *Let the operators $L$ and $K$ satisfy the conditions of Definition 1. Then $L$ is the generator of a $K$-bounded semigroup satisfying*

$$Z(0)f = Kf \quad \textit{for all } f \in \mathfrak{D}_K, \tag{26}$$

*if and only if the following conditions hold:*

1) *$K[D_K(L)]$ is dense in $X_K$,*

2) *there exist $M > 0$ and $\omega \in \mathbb{R}$ such that for every $f \in \mathfrak{D}_K$, $\lambda > \omega$ and $n \in \mathbb{N}$:*

$$\|K(\lambda I - L)^{-n}f\|_X \leq \frac{M}{(\lambda-\omega)^n}\|Kf\|_X.$$

*If we do not require $\{Z(t)\}$ to satisfy* (26)*, then condition* 1. *is sufficient but not necessary.*

We observe that it is not necessary for $L$ to generate a semigroup in $Y$ and $Y$ need not have any topological structure. Let us adopt the following assumptions on $L$ and $Y$.

(2.1′) The space $Y$ is a linear space and the operator $L_K$ is closeable in $Y_K$. Denoting $\mathfrak{L} = \overline{L_K}^{Y_K}$, we assume further that there exist subspaces: $\mathfrak{Y}$ satisfying $\mathfrak{D}_K \subseteq \mathfrak{Y} \subseteq Y_K$, and $\mathfrak{D}$ such that $\mathfrak{D}_{K(L)} \subset \mathfrak{D} \subset \mathfrak{X} \cap \mathfrak{D}_{\mathfrak{L}}$ such that $(\lambda - \mathfrak{L}|_{\mathfrak{D}}) : \mathfrak{D} \to \mathfrak{Y}$ is bijective for all $\lambda > \omega$.

**Theorem 13.** *Let the operators $L$ and $K$ satisfy the conditions of Definition 1 with assumption (23) replaced by* 2.1′. *Then $L \in K - \mathcal{G}(M, \omega, Y, X)$ and (26) holds if and only if the following conditions are satisfied:*

1) *$K[\mathfrak{D}]$ is dense in $X_K$,*

2) *there exist $M > 0$ and $\omega \in \mathbb{R}$ such that for every $\mathfrak{y} \in \mathcal{Y}$, $\lambda > \omega$ and $n \in \mathbb{N}$:*

$$\|\mathfrak{K}(\lambda I - \mathfrak{L}|_{\mathfrak{D}})^{-n}\mathfrak{y}\|_X \leq \frac{M}{(\lambda - \omega)^n}\|\mathfrak{K}\mathfrak{y}\|_X.$$

*If* (26) *is not assumed, the assumption* 1. *is sufficient but not necessary. In both cases the $K$ bounded semigroup is given again by the expression* (25).

Let us consider again the original Cauchy problem (2), (3). It is often the case that the original spaces $X$ and $Y$ are not the most convenient spaces from the mathematical point of view. We are usually interested to keep the values of the solution in the original space which may be related to some physical properties like the finite total energy space, finite mass, etc., but for (2) to hold in the strict sense may be too restrictive and often it is enough that it holds in some other Banach (or even linear topological) space $\tilde{Y}$ with $B$ and $A$ replaced by appropriate extensions $\tilde{B}$ and $\tilde{A}$ acting from $X$ to $\tilde{Y}$.

To be able to link $B$ and $A$ with $\tilde{B}$ and $\tilde{A}$ we restrict these extensions to the closures of respective operators. In other words, $\mathfrak{D}_A$ and $\mathfrak{D}_B$ are required to be cores for $\tilde{A}$ and $\tilde{B}$, respectively.

As we mentioned above, in general, thanks to Theorem 13, we don't need any topological structure in $Y$ and therefore there is no need to introduce any topological assumptions on $B$ and $A$ separately – as we shall see, these will be replaced by appropriate assumptions imposed on either $AB^{-1}$ or $B^{-1}A$.

Let us introduce the following definition.

**Definition 2.** *Let $Y \subset \tilde{Y}$ and $\tilde{A} = \overline{A}^{\tilde{Y}}$, $\tilde{B} = \overline{B}^{Y}$. An $X$-valued function $t \to u(t)$ is called a $\tilde{Y}$-solution of the problem* (2) *if it is a classical solution of the problem*

$$\left.\begin{aligned} &\frac{d}{dt}[\tilde{B}u(t)] = \tilde{A}u \\ &\lim_{t\to 0+}[\tilde{B}u(t)] = \mathring{u}, \end{aligned}\right\} \tag{27}$$

*that is, $t \to \tilde{B}u(t)$ is continuously differentiable in $\tilde{Y}$, the differential equation holds for all $t > 0$ in $\tilde{Y}$, and the initial condition holds as a limit in the topology of $\tilde{Y}$.*

This definition suggests an alternative way of approaching implicit problems (2) with a "bad" operator $B$ – to move the problem to a space in which $B$ has desirable properties. The theory of $K$-bounded semigroups offers tools for such an approach and, as the following theorem shows, the space $Y_{B^{-1}}$ turns out to be a good choice as on this space a suitable extension of $B$ is an isomorphism (and even an isometry).

**Theorem 14.** *Let us suppose that we are given operators* $B : \mathfrak{D}_B \to Y$ *and* $A : \mathfrak{D}_A \to Y$ *with* $\mathfrak{D}_A, \mathfrak{D}_B \subset X$*, where* $X$ *is a Banach and* $Y$ *is a linear space. Assume that* $B$ *is a densely defined, one-to-one operator. Define* $L = AB^{-1}$ *with the natural domain* $\mathfrak{D}_L = B[\mathfrak{D}_A \cap \mathfrak{D}_B]$ *and* $K = B^{-1}$*. If* $L \in K - \mathcal{G}(M, \omega, Y, X)$*, then for* $\mathring{u} \in \mathfrak{D}_{K(L)} = \{u \in B[\mathfrak{D}_A \cap \mathfrak{D}_B] : AB^{-1}u \in B[\mathfrak{D}_B]\}$*, the function* $t \to Z(t)\mathring{u}$*, where* $\{Z(t)\}_{t\geq 0}$ *is the* $K$*-bounded semigroup generated by* $L$*, is an* $Y_K$*-solution of the problem* (27)*.*

Assumption (23), specified to the present conditions, means that the operator $(AB^{-1}, \mathfrak{D}_{AB^{-1}})$ satisfies $\rho(AB^{-1}) \supset [\omega, \infty)$ for some $\omega \in \mathbb{R}$. If this assumption is satisfied, we can combine Theorems 14 and 12 to obtain the following result specified to the holomorphic case.

**Theorem 15.** *Assume that*

1) *the set* $\mathfrak{D}_{B^{-1}A} = \{y \in \mathfrak{D}_A \cap \mathfrak{D}_B : Ay \in ImB\}$ *is dense in* $\overline{\mathfrak{D}_B}^X$*,*

2) *for* $f \in \mathfrak{D}_B$
$$\|(\lambda I - B^{-1}A)^{-1}f\|_X \leq \frac{M}{|\lambda - \omega|}\|f\|_X, \tag{28}$$
*for* $\lambda$ *in some sector with the opening greater than* $\pi$*, then for any* $\mathring{u} \in \mathfrak{D}_{K(L)}$*, the function* $t \to Z(t)\mathring{u}$ *is an* $Y_K$*-solution to* (27)*.*

*In reflexive spaces Assumption 1 is superfluous.*

We illustrate the approach by considering the same example as in Section 3 where, for simplicity, we consider $\Omega = (0, 1)$ and $k = 1$. Let us begin with the setting of Section 3; that is, with $X = L_2(0, 1)$. The boundary operators are $\gamma_0 u = \langle u(0), u(1)\rangle$ and $\gamma_1 u = \langle -u_x(0), u_x(1)\rangle$, for sufficiently regular $u$. Let $\mathfrak{D}_A = W^{2,2}(0, 1)$ and, on $\mathfrak{D}_A$,
$$Au = \langle u_{xx}, -\gamma_1 u\rangle = \langle u_{xx}, u_x(0), -u_x(1)\rangle$$
and $\mathfrak{D}_B = W^{1,2}(0, 1)$ with
$$Bu = \langle u, \gamma_0 u\rangle = \langle u, u(0), u(1)\rangle.$$

However, we see that with such a domain we cannot achieve $A[\mathfrak{D}_A] \subset ImB$ as $u_{xx}$ does not have traces at $x = \pm 1$. Let us now change the setting and take $X = W^{1,2}(0, 1)$. Then $B$ is well-defined and bounded on $X$ with $Im\,B = \langle u, u(0), u(1)\rangle \subset Y := X \times \mathbb{C}^2$. Since this set is closed in $Y$, $B$ is an isomorphism onto its image. According to Theorem 14 we should define $A$ on such a domain that $Au = \langle u_{xx}, u_x(0), -u_x(1)\rangle \in Im\,B$; that is,
$$\mathfrak{D}_A = \{u \in W^{3,2}(0, 1) : u_{xx}(0) - u_x(0) = 0, u_{xx}(1) + u_x(1) = 0\}.$$

Now, $B^{-1}A = u_{xx}$ and (28) with $\omega = 0$ can be proved by standard Hilbert space methods. Since we are working in Hilbert (reflexive) spaces, we get the density of $\mathfrak{D}_{B^{-1}A}$ in $X$ and therefore the problem is solvable. Here $X_K = X$, the semigroup $t \to E_{B^{-1}A}(t)$ acts in

$X$ and the solution operator is defined as $t \to E_{B^{-1}A}(t)B^{-1}$, see (25). Thus the solution operator acts on the subspace of $W^{1,2}(0,1)\times\mathbb{C}^2$ of initial conditions satisfying compatibility conditions, $\langle \mathring{u}, \mathring{u}(0), \mathring{u}(1)\rangle$, in contrast to the solution in [3] where a weaker regularity of the solution does not require such a restriction.

## Notes and Remarks

For the heuristic underpinning of evolution equations by semigroups, consult S.G. Krein [6].

The results of Sections 1, 2 and 3 are based on the paper [7] where complete proofs and discussions can be found. Of historical interest is that the notion of empathy, in a primitive form was introduced in an earlier paper [8]. That was preceded by the notion of *B-evolution* introduced in [3] which involves only the solution operators $S(t)$, pre-supposes the operator $B$ and uses the causal relation $S(t+s) = S(t)BS(s)$ on the space $Y$. If one sets $E(t) = BS(t)$ this is in line with the same expression derived in Section 2 (valid only on the domain $\mathfrak{D}_E$). The example of a non-closeable operator $B$ can be found in [9].

Equations of Sobolev-type were studied by Showalter and Ting in [10, 11]. Within the framework of B-evolutions this was studied by Van der Merwe [12]. Also of note, are the studies of Favini–Yagi where the vantage point is that the operators $A$ and $B$ are assumed to be closed [13].

The Friedrichs extension of an operator-pair was first introduced in [24] and effectively used in a study of dynamic boundary conditions for the Navier–Stokes equations [25]. This was extended in [26] where applications to dynamic boundary conditions and equations of Sobolev-type were treated as examples. An example to illustrate that the empathy-approach gives better results for Sobolev equations can be found in [27].

The idea of using B-evolution theory to study perturbations of evolution problems came from Alna van der Merwe [28] who formulated it in a Hilbert space context. Extension to empathy theory in Banach spaces is without difficulty.

Integrated semigroups were introduced by Arendt in [29] although it was anticipated in earlier work of Favini [30]. Adaptation to double families can be found in [31]. The system of equations in Section 5 is closer to the physics of the problem than the wave equation would be, and the dynamic boundary condition is physically and mathematically "realistic".

The Kisyński-result has been proved to be equivalent to the classical Hille–Yosida theorem (Chojnacki, [32]). A complete account of the discussion of Section 6 can be found in [33].

Very recently an algebraic-analytic approach to causal relations was pioneered in [34]. In this work a generalized convolution-type algebra, which extends the notion of convolution algebra in abstract harmonic analysis, is developed. The basic idea is the consider linear operators from so-called *test spaces* of uniformly bounded continuous functions to a Banach space as homomorphisms and define a *convolution product* of such homomorphisms. The convolution product, in turn, induces a dualism map back to the test space which can be implemented to describe, in a different way, a number of known causal relations such as semigroups, integrated semigroups, empathy and integrated empathy. It also provides a new understanding of linear operators associated with probability measures and the semigroups associated with stochastic processes that satisfy the Chapman–Kolmogorov relation.

In probability theory the occurrence of implicit Fokker–Planck equations is unknown. It was recently introduced in [35] where complete details of the discussion in Section 7 can be found.

The concept of $K$-bounded semigroups (precisely, $B$-bounded semigroups but we had to change the name to avoid the conflict of notation) was introduced by Belleni-Morante, [14, 15]. The generation theorems can be found in [16, 17, 19, 20]. The latter work also contains a comparison of $K$-bounded semigroups and $C$-semigroups. Definition 2 is based on [21], while the idea of using $K$-bounded semigroups to solve implicit evolution equations was developed in [18,22,23]; [22] also contains a comparison of $K$-bounded semigroups and empathy. It is also worthwhile to note that $K$-bounded semigroups are closely related to the lumpability theory, [23], where one seeks an operator that can aggregate the states of the system, decreasing its complexity without changing salient aspects of its dynamics — the idea possibly being the closest to what originally Belleni-Morante had in mind.

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# ПРИЧИННО-СЛЕДСТВЕННЫЕ ОТНОШЕНИЯ В НЕЯВНЫХ ЭВОЛЮЦИОННЫХ УРАВНЕНИЯХ

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Данная статья является кратким изложением подходов в динамических системах, которые составляют основу для линейных неявных эволюционных уравнений, ведущих к интересным приложениям. Примеры в бесконечномерных диссипативных системах и стохастических процессах иллюстрируют фундаментальные понятия, лежащие в основе использования двойных семейств эволюционных уравнений, связанных отношением эмпатии. Эквивалентная формулировка Кисиньского теоремы Хилле – Иосиды подчеркивает существенные различия между теорией полугрупп и теорией эмпатии. В обзоре представлено понятие $K$-ограниченных полугрупп, являющихся более естественным подходом к неявным уравнением с одной стороны и отношением эмпатии с другой.

*Ключевые слова: неявные уравнения; теория эмпатии; полугруппы.*

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