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Fixed point results of Edelstein-Suzuki type multivalued mappings on b-metric spaces with applications

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Abstract

We obtain Edelstein-Suzuki type theorems for multivalued mappings in compact b-metric spaces. Moreover, we prove the existence of coincidence and common fixed points of a hybrid pair of mappings that satisfies Edelstein-Suzuki type contractive condition. We present some examples along with a comparison with results in existing literature. In the end, we present some corollaries in the metric spaces with applications in best approximation theory.

Keywords: Edelstein-Suzuki, metric space, multivalued mapping, best approximations, fixed point.
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1. Introduction

Let (X, d) be a metric space and $CB(X)$ a collection of nonempty closed and bounded subsets of X. The Hausdorff metric H on $CB(X)$ induced by the metric d on X is defined as follows:

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

for all $A, B \in CB(X)$, where $d(x, B) = \inf_{y\in B} d(x, y)$.

It is well-known that $(CB(X), H)$ is a complete metric space, if (X, d) is complete metric space.

The collection of nonempty closed subsets of X is denoted by $Cl(X)$.

A self-mapping f on X is called contraction, if there exists a real number r in $[0, 1)$ such that

$$d(fx, fy) \leqslant rd(x, y),$$

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for all $x, y \in X$. One of the basic and the most widely applied result in metric fixed point theory is "Banach (or Banach- Caccioppoli) contraction principle" due to Banach [3]. It states that if (X, d) is a complete metric space and $f : X \to X$ a contraction mapping, then f has a unique fixed point.

To establish the existence and uniqueness of solution of an operator equation $f(x) = x$, particularly to prove the existence of solution of differential or integral equations, Banach contraction principle guarantees the convergence of a sequence of successive approximations of a required solution. Due to its applications in mathematics and other related disciplines, this principle has been generalized in many directions (see [6, 8, 12, 16, 27, 28]).

A self-mapping f on X is called strictly contraction, if

$$d(fx, fy) < d(x, y),$$

for all $x, y \in X$ with $x \neq y$. A strictly contraction mapping defined on a complete metric space X need not to have a fixed point. However, if strictly contraction mapping has a fixed point, then it is always unique.

To prove the existence of a fixed point of strictly contraction mapping, Edelstein [12] imposed a restriction on the domain of a mapping and proved the following result.

Theorem 1.1 ([12]). *Let (X, d) be a compact metric space and $f : X \longrightarrow X$ a strictly contraction mapping. Then f has a unique fixed point.*

Suzuki [27] presented an interesting extension of a contraction mapping and employed it to characterize the completeness of domain of such mapping. This result is remarkable in the sense that existence of a fixed point of contraction mapping does not characterize the completeness of domain of contraction mapping [25].

Suzuki [27] proved a variant of Edelstein result as follows.

Theorem 1.2. *Let (X, d) be a compact metric space and $f : X \longrightarrow X$. If for any $x, y \in X$ with $\frac{1}{2}d(x, fx) < d(x, y)$ implies that $d(fx, fy) < d(x, y)$, then f has a unique fixed point.*

Recently, Doric et al. [11] generalized above theorem as follows.

Theorem 1.3 ([11]). *Let (X, d) be a compact metric space and $f : X \longrightarrow X$. If for any $x, y \in X$*

$$\frac{1}{2}d(x, fx) < d(x, y),$$

implies that

$$d(fx, fy) < Ad(x, y) + Bd(x, fx) + Cd(y, fy) + Dd(x, fy) + Ed(y, fx),$$

where $A, B, C, D, E \geqslant 0$, $A + B + C + 2D = 1$ and $C \neq 1$, then f has a fixed point. Moreover, f has a unique fixed point, if $E \leqslant B + C + D$.

Popescu [21] proved the following generalization of Theorem 1.2.

Theorem 1.4 ([21]). *Let (X, d) be a compact metric space and $f : X \longrightarrow X$. If for any $x, y \in X$, $ad(x, fx) + bd(y, fx) < d(y, x)$ implies that $d(fx, fy) < d(x, y)$, where $a, b > 0$ and $2a + b < 1$, then f has a unique fixed point.*

Karapinar [17, 18] obtained the following Edelstein-Suzuki type theorem.

Theorem 1.5 ([17]). *Let (X, d) be a compact metric space and $f : X \longrightarrow X$. If for any $x, y \in X$,*

$$\frac{1}{2}d(x, fx) < d(x, y),$$

implies that

$$d(fx, fy) < \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy)}{2}, \frac{d(y, fx)}{2}\right\},$$

then f has a unique fixed point.

Nadler [20] proved the multivalued version of a Banach contraction principle as follows.

Theorem 1.6 ([20]). *Let* (X, d) *be a complete metric space. If a multivalued mapping* $T : X \longrightarrow CB(X)$ *satisfies*

$$H(Tx, Ty) \leqslant kd(x, y),$$

for all $x, y \in X$ *and for some* $k \in [0, 1)$, *then* $F(T) = \{x \in X : x \in Tx\}$ *is nonempty.*

Shaddad et al. [22] obtained the following result:

Theorem 1.7 ([22]). *Let* (X, d) *be a compact metric space and* $T : X \longrightarrow Cl(X)$. *If for any* $x, y \in X$, *there exists some* $r \in \left[0, \frac{1}{2}\right)$ *such that* $\frac{1}{2}d(x, Tx) < d(x, y)$ *implies that*

$$H(Tx, Ty) < r \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\},$$

then $F(T)$ *is nonempty.*

Due to the contractive constant $r \in \left[0, \frac{1}{2}\right)$, above theorem is not an exact multivalued version of Theorem 1.2.

Beg and Aleomraninejad [4] proved the following result in this direction.

Theorem 1.8 ([4]). *Let* (X, d) *be a compact metric space and* $T : X \longrightarrow CB(X)$. *If for any* $x, y \in X$, *there exists some* $r \in \left(0, \frac{1}{2}\right]$ *such that* $rd(x, Tx) < d(x, y)$ *implies that* $H(Tx, Ty) < d(x, y)$, *then* $F(T)$ *is nonempty.*

On the other hand, concept of a metric has been generalized in many directions [14].

In 1993, Czerwik [8] introduced the notion of b-metric spaces as follows:

Definition 1.9. Let X be a nonempty set and $b \geqslant 1$ a real number. A mapping $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X, if for any $x, y, z \in X$, the following conditions hold:

(a_1) $d(x, y) = 0$, if and only if $x = y$;

(a_2) $d(x, y) = d(y, x)$;

(a_3) $d(x, y) \leqslant b\left(d(x, z) + d(z, y)\right)$.

Every metric is b-metric for $b = 1$, but converse does not hold in general [8, 24].

A number of results dealing with existence of fixed point of operators satisfying certain contractive conditions in the framework of b-metric spaces have been obtained [2, 7–10, 15, 17, 18, 24].

We now state the following lemmas from [8–10, 24] needed in the sequel.

Lemma 1.10. *For any* b*-metric space* X, $x, y \in X$ *and* $A, B \in CB(X)$, *the following hold:*

b_1) *If* (X, d) *is* b*-metric space, then so is* $(CB(X), H)$.

b_2) $d(x, B) \leqslant d(x, y)$ *for all* $y \in B$.

b_3) $d(x, B) \leqslant H(A, B)$ *for all* $x \in A$.

b_4) $d(x, A) \leqslant b\left(d(x, y) + d(y, A)\right)$.

b_5) *For* $h > 1$ *and* $a \in A$, *there is a* $b \in B$ *such that* $d(a, b) \leqslant hH(A, B)$.

b_6) *For* $h > 0$ *and* $a \in A$, *there is a* $b \in B$ *such that* $d(a, b) \leqslant H(A, B) + h$.

b_7) $d(x, A) = 0$ *if and only if* $x \in \bar{A} = A$.

b_8) *For any sequence* $\{u_n\}$ *in* X,

$$d(u_0, u_n) \leqslant bd(u_0, u_1) + ... + b^{n-1}d(u_{n-2}, u_{n-1}) + b^{n-1}d(u_{n-1}, u_n).$$

An et al. [2] studied some useful topological properties of b-metric spaces and showed that every b-metric space is a semi-metrizable space. They also proved Stone-type theorem on b-metric spaces.

They stated [2, Example 3.9] the following useful facts about b-metric spaces.

c_1) d is not continuous in each variable.

c_2) A b-metric is not necessarily a metric.

c_3) If d is continuous in one variable then d is continuous in other variable.

c_4) An open ball in b-metric space is not necessarily an open set. An open ball is open if d is continuous in one variable.

Corollary 1.11 ([2, 19])**.** *If* (X, d) *is a* b*-metric space and* $A \subset X$. *Then the following statements are equivalent.*

c_5) $x \in \overline{A}$.

c_6) *For every* $\varepsilon > 0$, $B(x, \varepsilon) \cap A$ *is nonempty, where* $B(x, \varepsilon)$ *is an open ball centered at* x *with radius equal to* ε.

c_7) *There exists a sequence* $\{x_n\}$ *in* A *such that* $\lim_{n\to\infty} x_n = x$.

Corollary 1.12 ([2, 19])**.** *Let* (X, d) *be a* b*-metric space and* $A \subset X$. *Then the following hold.*

c_8) A *is closed if and only if* $x \in A$ *for any sequence* $\{x_n\} \subset A$ *with* $\lim_{n\to\infty} x_n = x$.

c_9) *For any* $x \in \overline{A}$ *and* $\varepsilon > 0$, $B(x, \varepsilon) \cap A$ *is nonempty.*

c_{10}) A *is compact if and only if* A *is sequentially compact.*

c_{11}) *If* A *is compact, then* A *is totally bounded.*

Definition 1.13. Let (X, d) be a b-metric space. The b-metric $d : X \times X \to \mathbb{R}^+$ is continuous, if we have $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$ whenever $\{x_n\}$, $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Note that if b-metric function $d : X \times X \to \mathbb{R}^+$ is continuous, then it is continuous in both the variables. Throughout this paper, we assume the continuity of a b-metric $d : X \times X \to \mathbb{R}^+$.

2. Main results

In this section, we prove Edelstein-Suzuki variant of Hardy-Rogers type fixed point theorem for multivalued mappings in compact b−metric spaces.

Let $\Phi = \left\{ \varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} : \varphi(s, t) \leqslant \frac{1}{2b}s - t \right\}$.

Theorem 2.1. *Let* (X, d) *be a compact* b*-metric space and* $T : X \longrightarrow Cl(X)$. *If for any* $x, y \in X$

$$\varphi\left(d(x, Tx), d(x, y)\right) < 0,$$

implies that

$$H(Tx, Ty) < Ad(x, y) + Bd(x, Tx) + Cd(y, Ty) + \frac{D}{b}d(x, Ty) + Ed(y, Tx), \tag{2.1}$$

where $\varphi \in \Phi$, $A, B, C, D, E \geqslant 0$ *such that* $A + B + C + 2D = \frac{1}{b}$ *and* $C \neq \frac{1}{b}$, *then* T *has a fixed point.*

Proof. If $\beta = \inf_{x\in X} d(x, Tx)$, then there exists a sequence $\{x_n\}$ in X such that $\beta = \lim_{n\to\infty} d(x_n, Tx_n)$. As Tx_n is closed in (X, d) for each $n \in \mathbb{N}$, Tx_n is compact and hence for each $n \in \mathbb{N}$, there exists $y_n \in Tx_n$ such that

$$\beta = \lim_{n\to\infty} d(x_n, y_n).$$

Since (X, d) is compact, it is sequentially compact [2, 19]. Without loss of generality, we assume that $\{x_n\}$ and $\{y_n\}$ converge to v and w, respectively. Thus we have

$$\begin{aligned}\beta &= d(v, w) = \lim_n d(x_n, y_n),\\ \beta &= \lim_n d(x_n, w) = d(v, w),\\ \beta &= \lim_n d(v, y_n) = d(v, w),\\ \lim_n d(v, Tx_n) &\leqslant \lim_n d(v, y_n) = d(v, w) = \beta.\end{aligned}$$

Consequently,

$$\lim_n d(v, Tx_n) \leqslant \lim_n d(v, y_n) = \lim_n d(x_n, w) = \lim_n d(x_n, y_n) = \lim_n d(x_n, Tx_n) = d(v, w) = \beta.$$

We now claim that $\beta = 0$. If $\beta > 0$, then there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, we have

$$\frac{2\beta}{3} < d(x_n, w) < \frac{4\beta}{3} \text{ and } \frac{2\beta}{3} < d(x_n, Tx_n) < \frac{4\beta}{3}.$$

Note that

$$\varphi\left(d(x_n, Tx_n), d(x_n, w)\right) \leqslant \frac{1}{2b} d(x_n, Tx_n) - d(x_n, w) < \frac{2\beta}{3b} - \frac{2\beta}{3} \leqslant 0,$$

for all $n \geqslant n_0$. Now by (2.1), we have

$$\begin{aligned}\frac{1}{b} d(y_n, Tw) &\leqslant d(y_n, Tw) \leqslant H(Tx_n, Tw)\\ &< Ad(x_n, w) + Bd(x_n, Tx_n) + Cd(w, Tw) + \frac{D}{b} d(x_n, Tw) + Ed(w, Tx_n)\\ &\leqslant Ad(x_n, w) + Bd(x_n, y_n) + Cd(w, Tw) + \frac{D}{b} d(x_n, Tw) + Ed(w, y_n),\end{aligned}$$

for all $n \geqslant n_0$. On taking limit as $n \to \infty$ on both sides of above inequality, we have

$$\begin{aligned}\frac{1}{b} d(w, Tw) &\leqslant Ad(v, w) + Bd(v, w) + Cd(w, Tw) + \frac{D}{b} d(v, Tw)\\ &\leqslant Ad(v, w) + Bd(v, w) + Cd(w, Tw) + Dd(v, w) + Dd(w, Tw)\\ &\leqslant (A + B + D)\, d(v, w) + (C + D)\, d(w, Tw),\end{aligned}$$

and hence

$$\left(\frac{1}{b} - C - D\right) d(w, Tw) \leqslant (A + B + D)\, d(v, w). \tag{2.2}$$

Obviously, $\frac{1}{b} - C - D \neq 0$. If $\frac{1}{b} - C - D = 0$, then $A + B + C + 2D = \frac{1}{b}$ gives $A + B + D = \frac{1}{b} - C - D = 0$ and hence $A = B = D = 0$ and $C = \frac{1}{b}$, a contradiction. By (2.2), we obtain that $d(w, Tw) \leqslant d(v, w) = \beta$. Hence, $d(w, Tw) = \beta$. Since Tw is nonempty and compact, for every minimizing sequence $\{w_n\} \in Tw$, there exists a subsequence $\{w_{n_k}\}$ that converges to a point w_0 in Tw. That is, $w_0 = \lim_{k\to\infty} w_{n_k}$. From $\lim_{k\to\infty} d(w, w_{n_k}) = \beta$, we have $d(w, w_0) = \beta$. If $w = w_0$, then $d(w, w_0) = \beta = 0$, a contradiction to our supposition that $\beta > 0$. Let $w \neq w_0$. Then, we have

$$\varphi\left(d(w, Tw), d(w, w_0)\right) \leqslant \frac{1}{2b} d(w, Tw) - d(w, w_0)$$

$$\begin{aligned} &\leqslant \frac{1}{2b} d(w, w_0) - d(w, w_0) \\ &< d(w, w_0) - d(w, w_0) = 0. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{b} d(w_0, Tw_0) \leqslant d(w_0, Tw_0) &\leqslant H(Tw, Tw_0) \\ &< Ad(w, w_0) + Bd(w, Tw) + Cd(w_0, Tw_0) + \frac{D}{b} d(w, Tw_0) + Ed(w_0, Tw) \\ &\leqslant Ad(w, w_0) + Bd(w, w_0) + Cd(w_0, Tw_0) + \frac{D}{b} d(w, Tw_0) + Ed(w_0, w_0) \\ &\leqslant Ad(w, w_0) + Bd(w, w_0) + Cd(w_0, Tw_0) + Dd(w, w_0) + Dd(w_0, Tw_0) \\ &\leqslant (A + B + D)\, d(w, w_0) + (C + D)\, d(w_0, Tw_0). \end{aligned}$$

As $\frac{1}{b} - C - D \neq 0$,

$$(\frac{1}{b} - C - D) d(w_0, Tw_0) < (A + B + D)\, d(w, w_0),$$

implies that $d(w_0, Tw_0) < d(w, w_0) = \beta$, a contradiction to the definition of β. Hence $\beta = 0$. We now prove that T has a fixed point. If not, then $d(x_n, Tx_n) > 0$, for all $n \in \mathbb{N}$. Obviously, $\frac{1}{2b} d(x_n, Tx_n) < d(x_n, Tx_n) \leqslant d(x_n, y_n)$. Note that

$$\varphi\,(d(x_n, Tx_n), d(x_n, y_n)) \leqslant \frac{1}{2b} d(x_n, Tx_n) - d(x_n, y_n) < 0.$$

This further implies that

$$\begin{aligned} \frac{1}{b} d(y_n, Ty_n) \leqslant d(y_n, Ty_n) &\leqslant H(Tx_n, Ty_n) \\ &< Ad(x_n, y_n) + Bd(x_n, Tx_n) + Cd(y_n, Ty_n) + \frac{D}{b} d(x_n, Ty_n) + Ed(y_n, Tx_n) \\ &\leqslant Ad(x_n, y_n) + Bd(x_n, y_n) + Cd(y_n, Ty_n) + \frac{D}{b} d(x_n, Ty_n) \\ &\leqslant Ad(x_n, y_n) + Bd(x_n, y_n) + Cd(y_n, Ty_n) + Dd(x_n, y_n) + Dd(y_n, Ty_n), \end{aligned}$$

for all $n \in \mathbb{N}$. That is

$$\left(\frac{1}{b} - C - D\right) d(y_n, Ty_n) < (A + B + D)\, d(x_n, y_n),$$

for all $n \in \mathbb{N}$. If $\frac{1}{b} - C - D = 0$, then $A + B + D = \frac{1}{b} - C - D = 0$, that is $A = B = D = 0$ and hence $C = \frac{1}{b}$, a contradiction. Thus $\frac{1}{b} - C - D \neq 0$. Consequently

$$d(y_n, Ty_n) < d(x_n, y_n). \tag{2.3}$$

For each n, there exists $z_n \in Ty_n$ such that $d(y_n, Ty_n) \leqslant d(y_n, z_n) < d(y_n, Ty_n) + \frac{1}{n}$. From (2.3), we have $\lim_n d(y_n, z_n) = \lim_n d(y_n, Ty_n) = 0$, which implies that $\lim_n d(v, z_n) \leqslant b \left(\lim_n d(v, y_n) + d(y_n, z_n)\right) = 0$. Hence

$$\lim_n d(v, z_n) = 0.$$

Suppose that there exists some $n_1 \in \mathbb{N}$ such that

$$\varphi\,(d(x_{n_1}, Tx_{n_1}), d(x_{n_1}, v)) \geqslant 0,$$

and

$$\varphi\left(d(y_{n_1}, Ty_{n_1}), d(y_{n_1}, v)\right) \geqslant 0.$$

Then we have

$$\frac{1}{2b}d(x_{n_1}, Tx_{n_1}) \geqslant d(x_{n_1}, v), \quad \text{and} \quad \frac{1}{2b}d(y_{n_1}, Ty_{n_1}) \geqslant d(y_{n_1}, v). \tag{2.4}$$

Using (2.3) and (2.4), we have

$$\begin{aligned} d(x_{n_1}, y_{n_1}) &\leqslant b\left(d(x_{n_1}, v) + d(y_{n_1}, v)\right) \\ &\leqslant \frac{1}{2}d(x_{n_1}, Tx_{n_1}) + \frac{1}{2}d(y_{n_1}, Ty_{n_1}) \\ &< \frac{1}{2}d(x_{n_1}, y_{n_1}) + \frac{1}{2}d(x_{n_1}, y_{n_1}) = d(x_{n_1}, y_{n_1}), \end{aligned}$$

a contradiction. Hence for all $n \in \mathbb{N}$, either $\varphi\left(d(x_n, Tx_n), d(x_n, v)\right) < 0$ or $\varphi\left(d(y_n, Ty_n), d(y_n, v)\right) < 0$. If $\varphi\left(d(x_n, Tx_n), d(x_n, v)\right) < 0$, then we have

$$\begin{aligned} \frac{1}{b}d(y_n, Tv) &\leqslant d(y_n, Tv) \leqslant H(Tx_n, Tv) \\ &< Ad(x_n, w) + Bd(x_n, Tx_n) + Cd(v, Tv) + \frac{D}{b}d(x_n, Tv) + Ed(v, Tx_n). \end{aligned}$$

On taking limit as $n \to \infty$ we obtain that

$$\frac{1}{b}d(v, Tv) \leqslant \left(C + \frac{D}{b}\right) d(v, Tv) \leqslant (C + D)\, d(v, Tv).$$

If $\varphi\left(d(y_n, Ty_n), d(y_n, v)\right) < 0$, then we have

$$\begin{aligned} \frac{1}{b}d(z_n, Tv) &\leqslant d(z_n, Tv) \leqslant H(Ty_n, Tv) \\ &< Ad(y_n, v) + Bd(y_n, Ty_n) + Cd(v, Tv) + \frac{D}{b}d(y_n, Tv) + Ed(v, Ty_n). \end{aligned}$$

On taking limit as $n \to \infty$ we obtain that

$$\frac{1}{b}d(v, Tv) \leqslant \left(C + \frac{D}{b}\right) d(v, Tv) \leqslant (C + D)\, d(v, Tv).$$

Thus we have $\frac{1}{b}d(v, Tv) \leqslant (C + D)\, d(v, Tv)$. Note that $C + D \neq \frac{1}{b}$. Otherwise, we have $A + B + D = 0$, that is, $A = B = D = 0$ and hence $C = \frac{1}{b}$, a contradiction. So $d(v, Tv) = 0$ and $v \in Tv$, a contradiction to the assumption that T has no fixed point. Hence the result follows. □

Remark 2.2. If in Theorem 2.1, $\varphi(s, t) = rs - t$ with $r \in \left[0, \frac{1}{2}\right)$ and $A = \frac{1}{b}$, $B = C = D = E = 0$, then $\varphi \in \Phi$ and we obtain an extension of Theorem 1.8 to b-metric space.

Corollary 2.3. *Let (X, d) be a compact b-metric space and $T : X \longrightarrow Cl(X)$. If for any $x, y \in X$, $\varphi\left(d(x, Tx), d(x, y)\right) < 0$ implies $H(Tx, Ty) < \frac{1}{b}d(x, y)$, where $\varphi \in \Phi$. Then T has a fixed point in X.*

Corollary 2.4. *Let (X, d) be a compact b-metric space and $f : X \longrightarrow X$. If for any $x, y \in X$,*

$$\varphi\left(d(x, fx), d(x, y)\right) < 0,$$

implies that

$$d(fx, fy) < Ad(x, y) + Bd(x, fx) + Cd(y, fy) + \frac{D}{b}d(x, fy) + Ed(y, fx),$$

where $\varphi \in \Phi$, $A, B, C, D, E \geqslant 0$ such that $A + B + C + 2D = \frac{1}{b}$ and $C \neq \frac{1}{b}$. Then f has a fixed point. Moreover f has a unique fixed point provided that $E < B + C + D$.

Proof. Existence of fixed point of f follows from Theorem 2.1. Let v and u be two fixed points of f such that $v \neq u$. Then

$$\varphi\left(d(v, fv), d(v, u)\right) \leqslant \frac{1}{2b} d(v, fv) - d(v, u) = -d(v, u) < 0.$$

Hence

$$\begin{aligned}
\frac{1}{b} d(v, u) \leqslant d(v, u) &= d(fv, fu) \\
&< Ad(v, u) + Bd(v, fv) + Cd(u, fu) + \frac{D}{b} d(v, fu) + Ed(u, fv) \\
&\leqslant Ad(v, u) + Bd(v, v) + Cd(u, u) + \frac{D}{b} d(v, u) + Ed(u, v) \\
&\leqslant \left(A + \frac{D}{b} + E\right) d(v, u) \leqslant (A + D + E)\, d(v, u).
\end{aligned}$$

As $E < B + C + D$, so $A + D + E < A + B + C + 2D = \frac{1}{b}$ implies that $d(v, u) < d(v, u)$, a contradiction. Hence f has a unique fixed point. □

The above corollary generalizes and extends various comparable results in the existing literature.

Remark 2.5. If in Corollary 2.4, $\varphi(s, t) = \dfrac{1}{2b} s - t$, then:

1. We obtain Theorem 1.3 in framework of b-metric space.
2. We have Theorem 1.2 in the setup of a b-metric space provided that $A = \frac{1}{b}$ and $B = C = D = E = 0$.
3. We obtain Edelstein-Suzuki type version of Chatterjea fixed point result [6] in the setup of b-metric space provided that $A = B = 0$, $D = \dfrac{1}{2}$.

Corollary 2.6. *Let (X, d) be a compact b-metric space and $f : X \longrightarrow X$. If for any $x, y \in X$, $\dfrac{1}{2b} d(x, fx) < d(x, y)$ implies that $d(fx, fy) < \frac{1}{2b} d(x, fy) + Ed(y, fx)$, where $E \geqslant 0$, then f has a fixed point in X. Further, if $E \leqslant \dfrac{1}{2}$, then f has a unique fixed point in X.*

Corollary 2.7. *Let (X, d) be a compact b-metric space and $f : X \longrightarrow X$. If for any $x, y \in X$,*

$$\varphi\left(d(x, fx), d(x, y)\right) < 0,$$

implies that

$$d(fx, fy) < Bd(x, fx) + Cd(y, fy),$$

where $\varphi \in \Phi$, $B, C \geqslant 0$ such that $B + C = \frac{1}{b}$ and $C \neq \frac{1}{b}$, then f has a unique fixed point in X.

If in the above Corollary, $\varphi(s, t) = \dfrac{1}{2b} s - t$, we obtain the following:

Corollary 2.8. *Let (X, d) be a compact b-metric space and $f : X \longrightarrow X$. If for any $x, y \in X$, $\dfrac{1}{2b} d(x, fx) < d(x, y)$ implies that $d(fx, fy) < Bd(x, fx) + Cd(y, fy)$ where $B, C \geqslant 0$ with $B + C = \frac{1}{b}$ and $C \neq \frac{1}{b}$, then f has a unique fixed point in X.*

Corollary 2.9. *Let (X, d) be a compact b-metric space and $f : X \longrightarrow X$. If for any $x, y \in X$, $\varphi\left(d(x, fx), d(x, y)\right) < 0$ implies that $d(fx, fy) < \frac{1}{b} d(x, y)$, where $\varphi \in \Phi$, then f has a unique fixed point in X.*

Example 2.10. Let $X = \{a_1, a_2, a_3\}$ and $d : X \times X \to \mathbb{R}^+$ be defined as $d(a_1, a_2) = 2$, $d(a_2, a_3) = \dfrac{1}{2}$, $d(a_1, a_3) = 1$, $d(x, y) = d(y, x)$ and $d(x, x) = 0$ for all $x, y \in X$. Then (X, d) is not a metric space, because $2 = d(a, b) \nleqslant d(a, c) + d(c, b) = \dfrac{3}{2}$. For $b = \dfrac{4}{3}$, d is a b-metric. Let $\varphi(s, t) = \dfrac{1}{2b} s - t$. Define $T : X \longrightarrow Cl(X)$ as follows:

$$Tx = \begin{cases} \{a_1, a_2\}, & \text{when } x \neq a_2, \\ \{a_1\}, & \text{when } x = a_2. \end{cases}$$

Note that for all $x, y \in X$ such that $x \neq y$, we have $\frac{1}{2b} d(x, Tx) < d(x, y)$. Hence $\varphi(d(x, Tx), d(x, y)) < 0$ for all $x, y \in X$ such that $x \neq y$. Further $H(Tx, Ty) = 0$, for all $x, y \in \{a_1, a_3\}$ and $H(Ta_2, Ta_3) = H(Ta_2, Ta_1) = H(\{a_1\}, \{a_1, a_3\}) = 1$. Let $A = \frac{1}{2}, B = \frac{1}{4}, C = D = 0, E = \frac{1}{2}$, then $A + B + C + 2D = \frac{3}{4} = \frac{1}{b}$. Hence

$$H(Ta_2, Ta_3) = 1 < Ad(a_2, a_3) + Bd(a_2, Ta_2) + Ed(a_3, Ta_2) = \frac{5}{4},$$
$$H(Ta_2, Ta_1) = 1 < Ad(a_2, a_1) + Bd(a_2, Ta_2) + Ed(a_1, Ta_2) = \frac{3}{2}.$$

Therefore $\varphi\,(d(x, Tx), d(x, y)) < 0$ implies that

$$H(Tx, Ty) < Ad(x, y) + Bd(x, Tx) + Cd(y, Ty) + \frac{D}{b} d(x, Ty) + Ed(y, Tx),$$

holds for all $x, y \in X$, $A, B, C, D, E \geqslant 0$ such that $A + B + C + 2D = \frac{1}{b}$ and $\varphi \in \Phi$. So all the conditions of Theorem 2.1 are satisfied. Here, a_1 and a_2 are fixed points of T.

Remark 2.11. Consider the b-metric d on $X = \{a_1, a_2, a_3\}$ and mapping T same as in Example 2.10. Let $x = a_2, y = a_3$, then $H(Ta_2, Ta_3) = 1 \nless \frac{3}{8} = \frac{1}{b} d(a_2, a_3)$ and hence Corollary 2.3 is not applicable in this case. Note that Corollary 2.3 is generalization of Theorem 1.1 and Theorem 1.2 for multivalued mappings in the context of b-metric space.

3. Edelstein-Suzuki type coincidence and common fixed point result for a hybrid pair of mappings

Let (X, d) be a b-metric space, $g : X \to X$ and $T : X \to Cl(X)$. A point x in X is called

(i) a coincidence point of hybrid pair (g, T), if $gx \in Tx$;

(ii) a common fixed point of hybrid pair (g, T), if $x = gx \in Tx$.

Denote $C(g, T)$ and $F(g, T)$ by the set of all coincidence and common fixed points of hybrid pair (g, T). In consistent with [1, 13], we need the following definitions and result in the sequel.

Definition 3.1. A hybrid pair (g, T) is called *w*-compatible, if $g(Tx) \subseteq T(gx)$, for all $x \in C(g, T)$.

The mapping g is called T-weakly commuting at some point $x \in X$, if $g^2(x) \in T(gx)$.
Haghi et al. [13] proved the following lemma by using axiom of choice.

Lemma 3.2 ([13])**.** *Let X be a nonempty set and $g : X \to X$. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.*

We now prove the following result.

Theorem 3.3. *Let (X, d) be a b-metric space and (g, T) a hybrid pair of mappings. If for any $x, y \in X$*

$$\varphi\,(d(gx, Tx), d(gx, gy)) < 0,$$

implies that

$$H(Tx, Ty) < Ad(gx, gy) + Bd(gx, Tx) + Cd(gy, Ty) + \frac{D}{b} d(gx, Ty) + Ed(gy, Tx),$$

where $\varphi \in \Phi$, $A, B, C, D, E \geqslant 0$ with $A + B + C + 2D = \frac{1}{b}$ and $C \neq \frac{1}{b}$, then $C(g, T)$ is nonempty provided that $T(X) \subseteq g(X)$ and $g(X)$ is compact. Further $F(g, T)$ is nonempty if any of the following conditions hold:

d_1- *The hybrid pair* (g, T) *is* w*-compatible,* $\lim_{n\to\infty} g^n(x) = u$*, for some* $u \in X$ *,* $x \in C(g, T)$ *and* g *is continuous at* u*.*

d_2- *The mapping* g *is* T*-weakly commuting at some* $x \in C(g, T)$ *and* $g^2x = gx$*.*

d_3- *The mapping* g *is continuous at some* $x \in C(g, T)$ *and* $\lim_{n\to\infty} g^n(u) = x$*, for some* $u \in X$*.*

Proof. By Lemma 3.2, there is a set $E \subseteq X$ such that $g : E \to X$ is one-to-one and $g(E) = g(X)$. Then a mapping $\mathcal{T} : g(E) \to Cl(X)$ defined as $\mathcal{T}(gx) = T(x)$ for all $g(x) \in g(E)$ is well-defined because g is one-to-one. Also

$$\begin{cases} \varphi\left(d(gx, Tx), d(gx, gy)\right) < 0, \quad \text{implies that} \\ H(Tx, Ty) < Ad(gx, gy) + Bd(gx, Tx) + Cd(gy, Ty) + \frac{D}{b}d(gx, Ty) + Ed(gy, Tx). \end{cases}$$

Thus

$$\begin{cases} \varphi\left(d(gx, \mathcal{T}(gx)), d(gx, gy)\right) < 0, \quad \text{implies that} \\ H(\mathcal{T}(gx), \mathcal{T}(gy)) < Ad(gx, gy) + Bd(gx, \mathcal{T}(gx)) + Cd(gy, \mathcal{T}(gy)) \\ \quad + \dfrac{D}{b}d(gx, \mathcal{T}(gy)) + Ed(gy, \mathcal{T}(gx)), \end{cases}$$

for all $gx, gy \in g(E)$. As $g(E) = g(X)$ is compact, $\mathcal{T}$ satisfies all the conditions of Theorem 2.1 with mapping $\mathcal{T}$ on $g(E)$. There exists a point $u \in g(E)$ such that $u \in \mathcal{T}u$. Since $T(X) \subseteq g(X)$, there is a point x in X such that $gx = u$. This implies that $gx \in \mathcal{T}gx = Tx$. That is $x \in C(g, T)$. Now we prove that $F(g, T)$ is nonempty. Let (C_1) hold. As the pair (g, T) is w-compatible and $\lim_{n\to\infty} g^n(x) = u$ for some $u \in X$, the continuity of g at u implies that $gu = u$ and the w-compatibility of the pair (g, T) implies that $g^n(x) \in T\left(g^{n-1}(x)\right)$, that is $g^n(x) \in C(g, T)$ for all $n \in \mathbb{N}$. Note that $g^n(x) \neq g(u)$ for all n, if $g^n(x) = g(u)$ for some n, then we have $u = gu = g^n(x) \in T(g^{n-1}(x)) = T(u)$ and the proof is done. So let $g^n(x) \neq g(u)$ for all n, we further get

$$\begin{aligned} \varphi\left(d(g^n(x), T\left(g^{n-1}(x)\right)), d(gg^{n-1}(x), gu)\right) &\leqslant \frac{1}{2b}d(g^n(x), T\left(g^{n-1}(x)\right)) - d(gg^{n-1}(x), gu) \\ &= 0 - d(gg^{n-1}(x), gu) < 0. \end{aligned}$$

This implies that

$$\begin{aligned} \frac{1}{b}d(g^n(x), Tu) \leqslant d(g^n(x), Tu) &\leqslant H(T\left(g^{n-1}(x)\right), Tu) \\ &< Ad(g^n(x), gu) + Bd(g^n(x), T\left(g^{n-1}(x)\right)) \\ &\quad + Cd(gu, Tu) + \frac{D}{b}d(g^n(x), Tu) + Ed(gu, T\left(g^{n-1}(x)\right)) \\ &\leqslant Ad(g^n(x), gu) + Bd(g^n(x), g^n(x)) \\ &\quad + Cd(gu, Tu) + \frac{D}{b}d(g^n(x), Tu) + Ed(gu, g^n(x)). \end{aligned}$$

On taking limit as $n \to \infty$, we obtain $\frac{1}{b}d(gu, Tu) \leqslant Cd(gu, Tu) + \frac{D}{b}d(gu, Tu) \leqslant (C + D)d(gu, Tu)$. That is, $\left(\frac{1}{b} - C - D\right) d(gu, Tu) \leqslant d(gu, Tu)$. If $\frac{1}{b} - C - D = 0$, then $A + B + D = 0$, consequently $A = B = D = 0$, that is $C = \frac{1}{b}$, a contradiction, hence $d(gu, Tu) = 0$ implies that $u = gu \in Tu$. To prove $F(g, T)$ is nonempty, let (C_2) hold. Thus for some $x \in C(g, T)$, $g^2x = gx$. Since g is T-weakly commuting, therefore $gx = g^2x \in T(gx)$. Hence $gx \in F(g, T)$. If (C_3) holds, then $\lim_{n\to\infty} g^n(u) = x$ for some $u \in X$ and $x \in C(g, T)$. Using continuity of g we get $x = gx \in Tx$. Hence $F(g, T)$ is nonempty. □

Corollary 3.4. *Let* (X, d) *be a* b*-metric space and* (g, T) *be a hybrid pairs of mappings satisfying* $T : X \longrightarrow Cl(X)$ *be a multivalued mapping satisfying* $\frac{1}{2b}d(gx, Tx) < d(gx, gy)$ *implies that* $H(Tx, Ty) < d(gx, gy)$ *for all* $x, y \in X$*. Then* $C(g, T)$ *is nonempty provided that* $T(X) \subseteq g(X)$ *and* $g(X)$ *is compact. Further* $F(g, T)$ *is nonempty if* (d_1)*-*(d_3) *hold as given in Theorem 3.3.*

4. Fixed point theorems in metric spaces with application in best approximation theory

If we take $b = 1$ in Theorem 2.1, we get the following result in metric spaces.

Theorem 4.1. *Let* (X, d) *be a compact metric space and* $T : X \longrightarrow Cl(X)$*. If for any* $x, y \in X$

$$\varphi\left(d(x, Tx), d(x, y)\right) < 0,$$

implies that $H(Tx, Ty) < Ad(x, y) + Bd(x, Tx) + Cd(y, Ty) + Dd(x, Ty) + Ed(y, Tx)$*, where* $\varphi \in \Phi$*,* $A, B, C, D, E \geqslant 0$ *such that* $A + B + C + 2D = 1$ *and* $C \neq 1$*. Then* T *has a fixed point.*

Corollary 4.2. *Let* (X, d) *be a compact metric space and* $f : X \longrightarrow X$*. If for any* $x, y \in X$*,*

$$\varphi\left(d(x, fx), d(x, y)\right) < 0,$$

implies that

$$d(fx, fy) < Ad(x, y) + Bd(x, fx) + Cd(y, fy) + Dd(x, fy) + Ed(y, fx),$$

where, $\varphi \in \Phi$*,* $A, B, C, D, E \geqslant 0$ *such that* $A + B + C + 2D = 1$ *and* $C \neq 1$*. Then* f *has a fixed point. Moreover* f *has a unique fixed point provided that* $E \leqslant B + C + D$*.*

Corollary 4.3. *Let* (X, d) *be a compact metric space and* $f : X \longrightarrow X$ *be a mapping satisfying* $\frac{1}{2}d(y, fx) < d(x, y)$ *implies that* $d(fx, fy) < d(x, y)$ *for all* $x, y \in X$*. Then* f *has a unique fixed point in* X*.*

Corollary 4.4. *Theorem 1.1.*

Example 4.5. Let $X = \{a, b, c\}$ and $d : X \times X \to \mathbb{R}^+$ be defined as

$$d(a, b) = 4 = d(b, c), d(a, c) = 2,$$
$$d(x, y) = d(y, x), \quad \text{and} \quad d(x, x) = 0, \quad \forall x, y \in X.$$

Then (X, d) is a metric space. Let $\varphi(s, t) = \frac{1}{2}s - t$. Define $f : X \longrightarrow Cl(X)$ as follows:

$$fx = \begin{cases} b, & \text{when } x \neq c, \\ c, & \text{when } x = c. \end{cases}$$

For $x = a, y = c$, $\frac{1}{2}d(a, fa) = \frac{1}{2}d(a, b) = 2 = d(a, c)$. Note that $\frac{1}{2}d(x, fx) - d(x, y) < 0$ holds for all $x, y \in X$ and $x \neq a$ and $y \neq c$. Further $d(fa, fc) = d(fb, fc) = d(b, c) = 4$, $d(fa, fb) = d(b, b) = 0$. Let $A = B = \frac{1}{4}, E = \frac{2}{3}, D = \frac{1}{4}$ and $C = 0$. Then $A + B + C + 2D = 1$. Hence

$$\begin{aligned} d(fa, fc) = 4 &< Ad(a, c) + Bd(a, fa) + Cd(c, fc) + Dd(a, fc) + Ed(c, fa) \\ &= Ad(a, c) + Bd(a, b) + Cd(c, c) + Dd(a, c) + Ed(c, b) \\ &= 2A + 4B + 2D + 4E = \frac{14}{3}, \\ d(fb, fc) = 4 &< Ad(b, c) + Bd(b, fb) + Cd(c, fc) + Dd(b, fc) + Ed(c, fb) \\ &= Ad(b, c) + Bd(b, b) + Cd(c, c) + Dd(b, c) + Ed(c, b) \\ &= 4A + 4D + 4E = \frac{14}{3}. \end{aligned}$$

Therefore

$$\varphi\left(d(x, fx), d(x, y)\right) < 0,$$

implies

$$d(fx, fy) < Ad(x, y) + Bd(x, fx) + Cd(y, fy) + Dd(x, fy) + Ed(y, fx),$$

for all $x, y \in X$ such that $A, B, C, D, E \geqslant 0$ and $A + B + C + 2D = 1$. So all the conditions of Corollary 4.2 are satisfied. Here, b and c are fixed points of f.

Remark 4.6. Consider the metric d on $X = \{a, b, c\}$ and mapping f same as in Example 2.10.

- If $x = b, y = c$, then $d(fb, fc) = d(b, c) = 4$ and hence Theorem 1.1 is not applicable in this case.
- If $x = b, y = c$, then $\frac{1}{2}d(b, fb) = 0 < 4 = d(b, c)$ holds but $d(fb, fc) = d(b, c) = 4$. Hence Theorem 1.2 is not applicable in this case.
- If $x = b, y = c$, then we have $rd(b, fb) + sd(c, fb) = rd(b, b) + sd(c, b) < d(c, b)$ for any $s < 1$ and $2r + s < 1$. As $d(fb, fc) = d(b, c) = 4$ and $d(b, c) = 4$, $d(b, fb) = 0$, $d(c, fc) = 0$, $\frac{d(b,fc)}{2} = \frac{d(c,fb)}{2} = 2$. Hence
$$4 = d(fb, fc) \not< 4 = \max\left\{d(b, c), d(b, fb), d(c, fc), \frac{d(b, fc)}{2}, \frac{d(c, fb)}{2}\right\}.$$
Thus Theorem 1.4 and Theorem 1.5 are not applicable in this case.
- If $x = b, y = c$, then $d(b, c) = 4$, $d(b, fb) = 0$, $d(c, fc) = 0$, $\frac{1}{2}(d(b, fc) + d(c, fb)) = 4$. Hence
$$4 = d(fb, fc) \not< 4r = r\max\left\{d(b, c), d(b, fb), d(c, fc), \frac{d(b, fc) + d(c, fb)}{2}\right\},$$
for any $r \in \left[0, \frac{1}{2}\right)$. So Theorem 1.7 is not applicable in this case.
- If $x = b, y = c$, then $rd(b, fb) < d(c, b)$ holds for any $r \in \left(0, \frac{1}{2}\right]$ but $d(fb, fc) = d(b, c) = 4$ implies that Theorem 1.8 is not applicable in this case.

4.1. *Application in best approximation*

Let $(X, \|\cdot\|)$ be a normed linear space. A mapping $f : X \to X$ is called nonexpansive on X, if $\|fx - fy\| \leqslant \|x - y\|$ for all $x, y \in X$. A subset C of X is said to be f-invariant, if $f(C) \subseteq C$. The set $F(f) = \{x \in X : x = fx\}$ is a fixed point set of f and the set
$$P_C(\tilde{x}) = \left\{y \in C \subseteq X : \|y - \tilde{x}\| = \inf_{z \in C} \|z - \tilde{x}\|\right\},$$
is a set of best C-approximations of $\tilde{x}$. A subset C of X is called a starshaped with respect to $q \in C$, if for all x in C and $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)q \in C$. Note that a convex set C is starshaped with respect to every q in C. Brosowski [5] proved the following theorem in approximation theory.

Theorem 4.7 ([5])**.** *Let f be a linear, nonexpansive mapping on a normed linear space X and C an f-invariant subset of X and $x \in F(f)$. If $P_C(x)$ is nonempty, compact and convex then $P_C(x) \cap F(f)$ is nonempty.*

Singh [23] improved the Brosowski theorem by relaxing the linearity of the mapping f and the convexity of the subset C.

Theorem 4.8 ([23])**.** *Let f be a nonexpansive mapping on a normed linear space X and C an f-invariant subset of X and $x \in F(f)$. If $P_C(x)$ is nonempty, compact and starshaped then $P_C(x) \cap F(f)$ is nonempty.*

Suzuki [26] introduced the concept of generalized nonexpansive mappings on a normed linear space. Let f be a mapping on a normed linear space X that satisfies
$$\frac{1}{2}\|x - fx\| \leqslant \|x - y\|, \quad \text{implies that} \quad \|fx - fy\| \leqslant \|x - y\|, \tag{4.1}$$
for all $x, y \in X$. This condition on mappings is known as condition (C) which is weaker than nonexpansiveness but stronger than quasi-nonexpansiveness.

Now we prove the following theorem for generalized nonexpansive mappings (that satisfy condition (C)) on a normed linear spaces.

Theorem 4.9. *Let* f *be a mapping on a normed linear space* X *that satisfies condition (C) for all* $x, y \in X$ *and* C *an* f*-invariant subset of* X *and* $\tilde{x} \in F(f)$*. If* $P_C(\tilde{x})$ *is nonempty, compact and starshaped then* $P_C(\tilde{x}) \cap F(f)$ *is nonempty provided that* $\|y - fx\| \leqslant \|x - y\|$ *holds for all* $x, y \in P_C(\tilde{x})$*.*

Proof. Consider $f : P_C(\tilde{x}) \to P_C(\tilde{x})$. Let $y \in P_C(\tilde{x})$. Then, we have

$$\frac{1}{2}\|\tilde{x} - f\tilde{x}\| = \frac{1}{2}\|\tilde{x} - \tilde{x}\| = 0 \leqslant \|\tilde{x} - y\|,$$

for all $y \in P_C(\tilde{x})$. This implies that $\|fy - \tilde{x}\| = \|fy - f\tilde{x}\| \leqslant \|\tilde{x} - y\|$. Consequently $fy \in P_C(\tilde{x})$, that is f is $P_C(\tilde{x})$-invariant. Fix $p \in P_C(\tilde{x})$ such that

$$\lambda p + (1 - \lambda)x \in P_C(\tilde{x}), \tag{4.2}$$

for all $x \in P_C(\tilde{x})$ and $\lambda \in [0, 1]$. Let $\{k_n\} \in [0, 1)$ such that $\lim_{n\to\infty} k_n = 1$. Define $f_n : P_C(\tilde{x}) \to P_C(\tilde{x})$ as $f_n x = k_n fx + (1 - k_n)p$ for all $x \in P_C(\tilde{x})$. Since f is $P_C(\tilde{x})$-invariant, therefore by (4.2) f_n is $P_C(\tilde{x})$-invariant. Moreover we have

$$\frac{1}{2}\|x - fx\| \leqslant \frac{1}{2}\|x - y\| + \frac{1}{2}\|y - fx\| \leqslant \frac{1}{2}\|x - y\| + \frac{1}{2}\|x - y\| = \|x - y\|,$$

for all $x, y \in P_C(\tilde{x})$. By (4.1) we obtain

$$\|f_n x - f_n y\| = k_n \|fx - fy\| \leqslant k_n \|x - y\| < \|x - y\|,$$

for all $x, y \in P_C(\tilde{x})$ and for all $n \in \mathbb{N}$. Since $P_C(\tilde{x})$ is compact, therefore by Corollary 4.4, for all $n \in \mathbb{N}$, f_n has a unique fixed point, say x_n. Thus $f_n x_n = x_n$ for all $n \in \mathbb{N}$. The compactness of $P_C(\tilde{x})$ yields a convergent subsequence x_{n_i} converging to $\bar{x} \in P_C(\tilde{x})$ (say). Hence $x_{n_i} = f_{n_i} x_{n_i} = k_{n_i} f x_{n_i} + (1 - k_{n_i})p$. On taking limit as $i \to \infty$, we obtain $\bar{x} = f\bar{x}$. □

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