Existence result and conservativeness for a fractional order non-autonomous fragmentation dynamics

Emile. F. Doungmo Goufo[a][*], **Morgan Kamga P.**[a] and **Jeanine. N. Mwambakana**[b]

[a] Department of Mathematical Sciences, University of South Africa, Florida, 0003 South Africa.

[b] Department of Science, Mathematics and Technology Education
University of Pretoria, Pretoria, South Africa.

Abstract

We use the subordination principle together with an equivalent norm approach and semigroup perturbation theory to state and set conditions for a non-autonomous fragmentation system to be conservative. The model is generalized with the Caputo fractional order derivative and we assume that the renormalizable generators involved in the perturbation process are in the class of quasi-contractive semigroups, but not in the class $\mathcal{G}(1,0)$ as usually assumed. This, thenceforth, allows the use of admissibility with respect to the involved operators, Hermitian conjugate, Hille-Yosida's condition and the uniform boundedness to show that the operator sum is closable, its closure generates a propagator (evolution system) and, therefore, a C_0 semigroup, leading to the existence result and conservativeness of the fractional model. This work brings a contribution that may lead to the full characterization of the infinitesimal generator of a C_0 semigroup for fractional non-autonomous fragmentation and coagulation dynamics which remain unsolved.

Keywords: Evolution system, propagator; semigroup perturbation; renormalization; fractional non-autonomous fragmentation; conservativeness.

1   Introduction and useful definitions

The dynamical behavior of a system that can break up to produce smaller particles can be generalized to give the integro-differential system:

$$\begin{cases} D_t^\alpha u(t,x) = -a(t,x)u(t,x) + \int_x^\infty a(t,y)b(t,x,y)u(t,y)dy \\ u(\tau,x) = u_\tau(x) \qquad\qquad\qquad 0 \le \tau < t \le T, \ x > 0, \end{cases} \tag{1}$$

where

$$^C_0 D_t^\alpha u(x,t) = \frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t (t-r)^{-\alpha} \frac{\partial}{\partial r} u(x,r)dr \tag{2}$$

with $0 \le \alpha < 1$ is the fractional derivative of $u(x,t)$ in the sense of Caputo [4], with Γ the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-1} dt, \quad z \in \mathbb{C}.$$

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[*]Correspondence: dgoufef@unisa.ac.za

For reasons of simplicity we note $_0^C D_t^\alpha = D_t^\alpha$. Moreover, u is the particle mass distribution function ($u(\tau, x) = u_\tau(x)$ is the mass distribution at some fixed time $\tau \geq 0$) with respect to the mass x, $b(t, x, y)$ is the distribution of particle masses x spawned by the fragmentation of a particle of mass y, at the time $t \leq T \in \mathbb{R}$ and $a(t, x)$ is the evolutionary time-dependent fragmentation rate, that is, the rate at which mass x particles break up at a time t. The first term on the right-hand side of (1) describe the reduction in the number of particles in the mass range $(x; x + dx)$ due to the fragmentation of particles in the same range. The second term describes the increase in the number of particles in the range due to fragmentation of larger particles.

The idea here is to analyse the equation (1) in the Banach space $L_1(\mathfrak{I}, X_1)$ where $\mathfrak{I} = [0, T]$ and

$$X_1 = L_1([0, \infty), x dx) = \left\{ \psi: \; \|\psi\|_{X_1} := \int_0^\infty x |\psi(x)| \, dx < \infty \right\},$$

using the theory of evolution semigroup.

Throughout we will consider the following regularity assumptions:

$(t, x) \to a(t, x) \in L_1([0, T'], L_\infty([k, l]))$, for any $0 < k < l < \infty$ and $T' \in (0, T)$

$b(t, x, y)$ is a positive measurable function with $b(t, x, y) = 0$ for all $x \geq y$ and $0 \leq t \leq T$,

(3)

and the local conservative law

$$\int_0^y x b(t, x, y) dx = y \tag{4}$$

for all $y \geq 0$ and $0 \leq t \leq T$.

Up to now, existence results and honesty have been proved for number of fragmentation (autonomous or non-local) models, see for e.g. [6, 8], where the authors used various methods including the substochastic semigroup theory. But models with time dependent coefficients (non-autonomous) remain barely touched and there are still only few papers devoted to their analysis (well-posedness, conservativeness, honesty, etc.) In [7], the authors used techniques of truncation to prove existence, uniqueness and mass conservation for a model of type (1). The authors in [16] use evolution semigroups approach which allows them to build on the substochastic semigroup theory and obtain an equivalent result as in [15]. In the analysis of the book by Tosio Kato [9] and later improved by Da Prado et al. [5], it is generally assumed that the generators $A(t)$ and $B(t)$ involved in the perturbation are of class $\mathcal{G}(1, 0)$, but this condition is modified in this paper as we will see later in our analysis.

We begin by recasting (1) as the non-autonomous abstract Cauchy problem in X_1:

$$\begin{cases} D_t^\alpha u(t) = Q(t) u(t) & 0 \leq \tau < t \leq T \\ u(\tau) = u_\tau, \end{cases} \tag{5}$$

where $Q(t)$ is defined by $Q(t) = \mathcal{Q}(t)$ and represents the realization of $\mathcal{Q}(t)$ on the domain $D(Q(t)) = \{u \in X_1; \; \mathcal{Q}(t) u(t) \in X_1\}$, with $(\mathcal{Q}u)$ defined as

$$(\mathcal{Q}u)(t, x) = (\mathcal{Q}u)(t)(x) = -a(t, x) u(t, x) + \int_x^\infty a(t, y) b(t, x, y) u(t, y) dy,$$

$\mathcal{Q}(t)$ is seen as the pointwise operation

$$\psi(t,x) \longmapsto -a(t,x)\psi(t,x) + \int_x^\infty a(t,y)b(t,x,y)\psi(t,x)dy$$

defined on the set of measurable functions. $\mathcal{Q}(t)$ indeed defines various operators. The aim here is to analyze the problem by rephrasing it in abstract form (abstract Cauchy Problem (ACP)) as an ordinary differential equation.

Let us start with some thing simple and come back to the abstract Cauchy Problem (5); To analyze and show existence for this system, we'll need a two parameter family. We consider that for $0 \le t \le T$, $Q(t)$ is a bounded linear operator in X_1 and that $t \longrightarrow Q(t)$ is continuous in the uniform operator topology. We have the following definitions:

Definition 1.1. *(**Solution operator for a fractional model**)*
Consider an operator Q applying in the fractional model

$$D_t^\alpha(u(x,t)) = Qu(x,t), \quad 0 < \alpha < 1, \ x, \ t > 0,\tag{6}$$

subject to the initial condition

$$u(x,0) = f(x) \quad x > 0\tag{7}$$

and defined in a Banach space X_1. A family $(G_Q(t))_{t>0}$ of bounded operators on X_1 is called a solution operator of the fractional Cauchy problem (6)-(12) if

(i) : $G_Q(0) = I_{X_1}$;
(ii) : $G_Q(t)$ is strongly continuous for every $t \ge 0$;
(iii) : $QG_Q(t)v = G_Q(t)Qv$ for all $v \in D(Q)$;
(iv) : $G_Q(t)D(Q) \subset D(Q)$;
(v) : $G_Q(t)v$ is a (classical) solution of the model $(6) - (12)$ for all $v \in D(Q), t \ge 0$.

It is well known [3, 6] that an operator $\widetilde{Q} \in \mathcal{G}(M,\omega)$ means $\widetilde{Q}$ generates a C_0-semigroup $(G_{\widetilde{Q}}(t))_{t>0}$ so that there exists $M > 0$ and ω such that

$$\|G_{\widetilde{Q}}(t)\| \le Me^{\omega t}.\tag{8}$$

Whence, by analogy if the fractional Cauchy problem (6)-(12) has a solution operator $(G_Q(t))_{t>0}$ verifying (8), then we say that $Q \in \mathcal{G}^\alpha(M,\omega)$. The solution operator $(G_Q(t))_{t>0}$ is positive if

$$G_Q(t) \ge 0$$

and contractive if

$$\|G_Q(t)\|_{X_1} \le 1,\tag{9}$$

and we say $Q \in \mathcal{G}^\alpha(1,0)$.

Definition 1.2. *(**Evolution system [16] or propagator [13]**)*
A two parameter family of bounded linear operators $U(t,\tau)$, $0 \leq \tau < t \leq T$, is called propagator or evolution system if the following conditions are respected:
(i) $U(\tau,\tau) = I$.
(ii) $U(t,r)U(r,\tau) = U(t,\tau)$ for $0 \leq \tau \leq r \leq t \leq T$.
(iii) $(t,\tau) \longrightarrow U(t,\tau)$ is strongly continuous for $0 \leq \tau \leq t \leq T$.

Next, we will find the propagator $U(t,\tau)$ associated with (5) such that $u(t) = U(t,\tau)u_\tau$ is in some sense a solution of (5) satisfying the initial condition $u(\tau) = u_\tau$. For that we need the following principle

1.1   Subordination principle [3, 6]

Let us consider the order α as in (5). **Subordination principle** summarizes as follows: The same operator Q guarantees better properties of the solution of (5) if we consider another order γ such that $\gamma < \alpha$. In order words, the subordination principle states that if Q generates a solution operator for the model (5) with the order $\alpha > 0$, then it also generates a solution operator for the model (5) with any order $\gamma > 0$ such that $\gamma < \alpha$. Hence, making use of this principle we just need to consider the model (5):

$$\begin{cases} D_t^\alpha u(t) = Q(t)u(t) & 0 \leq \tau < t \leq T \\ u(\tau) = u_\tau, \end{cases} \tag{10}$$

with $\alpha = 1$.

Lemma 1.3. Let $Q(t)$ be a bounded linear operator in X_1 for $0 \leq t \leq T$. If the function $t \longrightarrow Q(t)$ is continuous in the uniform operator topology, then for every $u_\tau \in X_1$, the abstract Cauchy Problem (10) has a unique classical solution u given by the relation:

$$u(t) = u_\tau + \int_\tau^t Q(\varsigma)u(\varsigma) \, d\varsigma. \tag{11}$$

Proof. see [16, Theorem 5.1, Chapter 5], the proof is done in a Banach space X which is also true in X_1. □

Theorem 1.4. There is a propagator $U(t,\tau)$ associated with the initial value problem (10) such that $U(t,\tau)u_\tau$ is its solution satisfying the initial condition $u(\tau) = u_\tau$.

Proof. From the Lemma (1.3), we already have the existence and uniqueness of the solution. Let $u(t)$ be this solution. We define the so called solution operator of (10) by

$$U(t,\tau)u_\tau = u(t) \qquad\qquad for\ 0 \leq \tau \leq t \leq T. \tag{12}$$

• For every $u_\tau \in X_1$, $U(\tau,\tau)u_\tau = u(\tau) = u_\tau$ then $U(\tau,\tau) = I$ (condition (i)).
• For every $u_\tau \in X_1$, we have $U(t,\tau)u_\tau = u(t)$ and $U(t,r)U(r,\tau)u_\tau = U(t,r)u(r) = u(t)$, then condition (ii) follows from the uniqueness of the solution of (10).
• It is obvious that $U(t,\tau)$ is a linear operator defined in all X_1 since (10) is linear.

The relation (11) implies $\|u(t)\| \leq \|u_\tau\| + \int_\tau^t \|Q(\varsigma)\| \|u(\varsigma)\| \, d\varsigma$ and from Gronwall's inequality we also have $\|u(t)\| \leq \|u_\tau\| \exp\left(\int_\tau^t \|Q(\varsigma)\| \, d\varsigma\right)$. Then, (12) yields $\|U(t,\tau)u_\tau\| = \|u_\tau\| \exp\left(\int_\tau^t \|Q(\varsigma)\| \, d\varsigma\right)$,

leading to

$$\|U(t,\tau)\| = \exp\left(\int_\tau^t \|Q(\varsigma)\| \, d\varsigma\right). \tag{13}$$

Whence, $U(t,\tau)$ is bounded and, therefore, strongly continuous. This concludes the proof.
$\square$

The fact that $Q(t)$ is bounded actually makes this existence result easier to obtain. Unfortunately, $Q(t)$ is not always bounded and then, we use, in the following section, a different approach to obtain an equivalent result.

2 Equivalent norm approach

Let's come back to the equation (10) and split it to have (1) written in the abstract form:

$$\begin{cases} D_t^\alpha u(t) = A(t)u(t) + B(t)u(t) & 0 \leq \tau < t \leq T \\ u(\tau) = u_\tau, \end{cases} \tag{14}$$

where $A(t)$ is defined as $A(t) = \mathcal{A}(t)$ and represents the realization of $\mathcal{A}(t)$ on the domain $D(A(t)) = \{u \in X_1; \ \mathcal{A}(t)u \in X_1\}$, with

$$[\mathcal{A}(t)u_\tau](x) = -a(t,x)u_\tau(x) \tag{15}$$

and $B(t)$ is defined as $B(t) = \mathcal{B}(t)$ and represents the realization of $\mathcal{B}(t)$ on the domain $D(B(t))$ with

$$[\mathcal{B}(t)u_\tau](x) = \int_x^\infty a(t,y)b(t,x,y)u_\tau(x)dy. \tag{16}$$

Making use of the assumptions (3) and (4), $A(t)$ is bounded (then the previous theorem and lemma apply) and it is easy to show that (see [6] or [8]) for any $u \in X_1$, $B(t)u \in X_1$, so we can take $D(B(t)) = D(A(t))$ and $(A(t) + B(t), D(A(t)))$ is well-posed operator. Let us put

$$\mathcal{X}_1 = L_1(\mathfrak{I}, X_1) := \left\{\psi : [0,T] \times \mathbb{R} \ni (\sigma, x) \to u(\sigma, x), \ \ \|\psi\|_1 := \int_0^T \int_0^\infty x|\psi(\sigma,x)|d\sigma dx < \infty\right\},$$

in the following sections the subscript t in A_t means the operator A depends on time t but is defined in $\mathcal{X}_1$ instead of X_1. Our aim here is to set some conditions in $\mathcal{X}_1$ under which the operator sum K_t:

$$K_t\psi = A_t\psi + B_t\psi, \text{ on } D(A_t) \cap D(B_t) = D(A_t) \tag{17}$$

is closable, its closure generates a propagator and therefore a C_0 semigroup. We rely on the following theorem which was originally proved by Tosio Kato [9] and later improved by Da Prado et al. [5].

Theorem 2.1. Consider in $\mathcal{X}_1$ the operators A_t and B_t be generators both belonging to the class $\mathcal{G}(1,0)$. If $D(A_t) \cap D(B_t)$ is dense in $\mathcal{X}_1$ and $\mathrm{ran}(A_t + B_t + \xi)$ is dense in $\mathcal{X}$ for some $\xi < 0$, then K_t is closable and its closure $\bar{K}_t$ is a generator from the class $\mathcal{G}(1,0)$.

The condition A_t and $B_t \in \mathcal{G}(1,0)$ is dropped in this paper to provide a stronger results. Let us treat the integral operator in (14) as a perturbation of the much easier operator of multiplication by a on X_1

$$[A(t)u_\tau](x) = -a(t,x)u_\tau(x). \tag{18}$$

Recall that (Theorem 1.4 $A(t)_{t\in\mathfrak{I}}$ ($\mathfrak{I} = [0,T]$) is a family of generators of C_0 semigroups in X_1, then, for any fixed $t \in \mathfrak{I} = [0,T]$, $A(t)$ generates a propagator $U(t,\tau), 0 \leq \tau < t \leq T$ and this propagator defines a C_0 semigroup $(S_{A_t}(s))_{s\geq 0}$ in $\mathcal{X}_1$ by the relation:

$$
\begin{aligned}
[S_{A_t}(s)u_\tau](\sigma) &= \chi_\mathfrak{I} U(\sigma, \sigma - s) u_\tau(\sigma - s) \\
&= \chi_\mathfrak{I} \exp\left(-\int_{\sigma-s}^{\sigma} a(\xi,.)d\xi\right) u_\tau(\sigma - s), \quad u_\tau \in \mathcal{X}_1,
\end{aligned} \tag{19}
$$

where $\chi_\mathfrak{I}$ is the characteristic function of $\mathfrak{I}$ and $\sigma \in \mathfrak{I}$. Then, when we say A is the generators of C_0-semigroups in $\mathcal{X}_1$, we mean A generates a propagator which defines a C_0-semigroups in $\mathcal{X}_1$ satisfying the relation (19). In the following definition, we assume that Y is a subspace of $\mathcal{X}_1$ which is closed with respect to the norm $\|.\|_Y$, not necessary in the norm $\|.\|_1$ (hence Y is it self a Banach space).

Definition 2.2. Let $S_{A_t}(s)_{s\geq 0}$ be a C_0-semigroup and A_t its infinitesimal generator. A subspace Y of $\mathcal{X}_1$ is called A_t−admissible if it is an invariant subspace of $S_{A_t}(s), s \geq 0$ i.e. $S_{A_t}(s)Y \subseteq Y$, and the restriction of $S_{A_t}(s)$ to Y (i.e. $S_{\breve{A}_t}(s) := S_{A_t}(s)_{|_Y}, s \geq 0$) is a C_0-semigroup in Y (i.e. it is strongly continuous in the norm $\|.\|_Y$).
If $T : Y \longrightarrow \mathcal{X}_1$ is the embedding operator of Y into $\mathcal{X}_1$, we have

$$S_{A_t}(\alpha)Tu = TS_{\breve{A}_t}(\alpha)u, \ u \in Y,$$

which gives

$$A_t Tu = T\breve{A}_t u, \ u \in D(\breve{A}_t),$$

with

$$D(\breve{A}_t) = \{u \in D(A_t) \cap Y : A_t u \in Y\}. \tag{20}$$

Recall that the adjoint A_t^* of A_t is a linear operator from $D(A_t^*) \subset \mathcal{X}_1^*$ into $\mathcal{X}_1^*$ (the dual of $\mathcal{X}_1$) and is defined as follows: $D(A_t^*)$ is the set of all elements $x^* \in \mathcal{X}_1^*$ for which there is a $y^* \in \mathcal{X}_1^*$ such that

$$\langle x^*, A_t x\rangle = \langle y^*, x\rangle \ \text{ for all } x \in D(A_t) \tag{21}$$

and if $x^* \in D(A_t^*)$ then $y^* = A_t^* x^*$ where y^* is the element of $\mathcal{X}_1^*$ satisfying (21). With in mind the the assumptions (3) and (4), we can state the following lemma:

Lemma 2.3. Let $\check{A}_t$ and $\check{B}_t$ two operators defined by (20) and satisfying, for all $\lambda \in (0,\infty) \subset \rho(\check{A}_t)$ and $\kappa \in (0,\infty) \subset \rho(\check{B}_t)$,

$$\|(\lambda I - \check{A}_t)^{-1}\|_Y \leq \frac{1}{\lambda}, \tag{22}$$

$$\|(\kappa I - \check{B}_t)^{-1}\|_Y \leq \frac{1}{\kappa}, \tag{23}$$

in the Banach space Y. If either $\check{A}_t^*$ or $\check{B}_t^*$ are densely defined in Y^*, then for any $\eta < 0$ we have the inequality:

$$|\eta|\|v\|_{Y^*} \leq \|\check{A}_t^* v + \check{B}_t^* v + \eta v\|_{Y^*}, \ v \in D(\check{A}_t^*) \cap D(\check{B}_t^*). \tag{24}$$

Proof. We suppose that $D(\check{B}_t^*)$ is dense in Y^* and define the sum

$$\check{K}_{t,\varepsilon} := \check{A}_t u + \check{B}_t(I + \varepsilon\check{B}_t)^{-1} u, \ u \in D(\check{K}) = D(\check{A}_t), \ \varepsilon < 0.$$

It's obvious that $\check{K}_{t,\varepsilon}$ also satisfies the relations (22) or (23) since $\check{A}_t$ and $\check{B}_t$ do. Then the relation (22) yields

$$\begin{aligned}
\|(\lambda I - \check{K}_{t,\varepsilon})^{-1} u\|_Y &\leq \|(\lambda I - \check{K}_{t,\varepsilon})^{-1}\|_Y \|u\|_Y \\
&\leq \frac{1}{\lambda}\|u\|_Y, \ u \in Y, \lambda > 0, \varepsilon < 0,
\end{aligned}$$

leading to

$$\begin{aligned}
\|u\|_Y &\leq \frac{1}{\lambda}\|(\lambda I - \check{K}_{t,\varepsilon})u\|_Y, \ u \in Y, \lambda > 0, \varepsilon < 0 \\
&\leq \frac{1}{\lambda}\|(\check{K}_{t,\varepsilon} - \lambda I)u\|_Y, \ u \in Y, \lambda > 0, \varepsilon < 0 \\
&\leq \frac{1}{|\eta|}\|(\check{K}_{t,\varepsilon} + \eta I)u\|_Y, \ u \in Y, \varepsilon < 0, \ \text{where} \ \text{we} \ \text{have set} \ -\lambda = \eta < 0
\end{aligned}$$

or

$$\|(\check{K}_{t,\varepsilon} + \eta I)u\|_Y \geq |\eta|\|u\|_Y, \ u \in D(\check{K}_{t,\varepsilon}) = D(\check{A}_t), \varepsilon < 0, \eta < 0.$$

Immediate properties of Hermitian conjugate give

$$\|(\check{K}^*_{t,\varepsilon} + \eta I)v\|_{Y^*} \geq |\eta|\|v\|_{Y^*}, \ v \in D(\check{K}^*_{t,\varepsilon}) = D(\check{A}_t^*), \varepsilon < 0, \eta < 0, \tag{25}$$

and

$$\check{K}^*_{t,\varepsilon} v = \check{A}_t^* v + \check{B}_t^*(I + \varepsilon\check{B}_t^*)^{-1} v, \ v \in D(\check{K}^*_{t,\varepsilon}) = D(\check{A}_t^*), \varepsilon < 0. \tag{26}$$

Since $\check{B}_t^*$ is densely defined in Y^* we have

$$(I + \varepsilon\check{B}_t^*)^{-1} \longrightarrow I \ as \ \varepsilon \nearrow 0$$

and then,

$$\check{K}^*_{t,\varepsilon} v \longrightarrow \check{A}_t^* v + \check{B}_t^* v \ as \ \varepsilon \nearrow 0.$$

Substituting the latter relation in (25) yields (24).

The approach is the same if we consider that it is rather $\check{A}_t^*$ which is densely defined in Y^*. □

Corollary 2.4. Let A_t and B_t two closed and densely defined operators satisfying, for all $\lambda \in (0, \infty) \subset \rho(A_t)$ and $\kappa \in (0, \infty) \subset \rho(B_t)$,

$$\|(\lambda I - A_t)^{-1}\|_1 \leq \frac{1}{\lambda}, \tag{27}$$

$$\|(\kappa I - B_t)^{-1}\|_1 \leq \frac{1}{\kappa} \tag{28}$$

on $\mathcal{X}_1$, let $Y \hookrightarrow \mathcal{X}_1$ be admissible with respect to A_t and B_t and let the operator B_t verify

$$Y \subseteq D(B_t). \tag{29}$$

We assume that the induced generators $\check{A}_t$ and $\check{B}_t$, given by (20), are closed, densely defined and satisfy the relations (22) and (23) respectively. If $D(B_t^*)$ is dense in $\mathcal{X}_1^*$ then

$$|\eta|\|v\|_{Y^*} \leq \|\check{A}_t^{*}v + \check{B}_t^{*}v + \eta v\|_{Y^*}, \quad v \in D(\check{A}_t^{*}) \cap T^*\mathcal{X}_1^*, \ \eta < 0, \tag{30}$$

where $T : Y \longrightarrow \mathcal{X}_1$ is the embedding operator.

Proof. Let $v \in D(\check{A}_t^{*}) \cap T^*\mathcal{X}_1^*$, then there is $w \in \mathcal{X}_1^*$ such that $v = T^*w$. we also have $T^*\mathcal{X}_1^* \subset D(\check{B}_t^{*})$ thanks to the condition (29). Therefore, the relation (26) of the previous lemma applies to $v = T^*w$ as:

$$\check{K}^*_{t,\varepsilon}T^*w = \check{A}_t^* T^*w + \check{B}_t^*(I + \varepsilon \check{B}_t^*)^{-1}T^*w, \quad \varepsilon < 0.$$

Since T is the embedding operator of Y into $\mathcal{X}_1$, we have

$$\check{K}^*_{t,\varepsilon}T^*w = \check{A}_t^* T^*w + \check{B}_t^* T^*(I + \varepsilon B_t^*)^{-1}w, \quad \varepsilon < 0,$$

which is well posed since the operator $B_t T : Y \to \mathcal{X}_1$ is bounded thanks to (29). Since B_t^* is densely defined in $\mathcal{X}_1^*$, we have

$$(I + \varepsilon B_t^*)^{-1} \longrightarrow I \quad as \ \varepsilon \nearrow 0$$

and then,

$$\check{K}^*_{t,\varepsilon}T^*w = \check{A}_t^* T^*w + \check{B}_t^* T^*(I + \varepsilon B_t^*)^{-1}w \longrightarrow \check{A}_t^* T^*w + \check{B}_t^* T^*w \quad as \ \varepsilon \nearrow 0.$$

Substituting the latter relation in (25) with $v = T^*w$ yields (30). $\qquad \square$

Remark 2.5. *It is in general possible to find in the Banach Space $\mathcal{X}_1$ a new norm $\|.\|$, which is equivalent to its natural norm*

$$\|u\|_1 := \int\limits_0^T \int\limits_0^\infty x|u(\sigma, x)| \, d\sigma dx,$$

such that the operator A_t becomes a generator of the contraction semigroups on $\mathcal{X}_1$.

Indeed, since A_t is the generator of a C_0 semigroup, let's say $(S_{A_t}(s))_{s\geq 0}$, there is $M > 0$ and ω such that $\forall s \geq 0$, $\|(S_{A_t}(s)\|_1 \leq Me^{\omega s}$.

We have

$$\begin{aligned} \|(S_{A_t}(s)u\|_1 &\leq Me^{\omega s}\|u\|_1, \ \forall u \in \mathcal{X}_1 \\ &\leq M_{A_t}e^{\omega s}. \end{aligned} \tag{31}$$

We set

$$\|u\| = (MM_{A_t})^{-1}\sup_{s\geq 0}e^{-\omega s}\int_0^T\int_0^\infty x|S_{A_t}(s)u(\sigma,x)|\,d\sigma dx$$

Simple calculations show that

$$\int_0^T\int_0^\infty x|u(\sigma,x)|\,d\sigma dx = \|u\|_1 \leq MM_{A_t}\|u\| \leq M^2M_{A_t}\|u\|_1, \ \forall u \in \mathcal{X}_1 \tag{32}$$

which proves that the norm $\|.\|$ is equivalent to $\|u\|_1$.

On the other hand we have

$$\|S_{A_t}(\varsigma)u\| = (MM_{A_t})^{-1}e^{\omega\varsigma}\sup_{s\geq 0}e^{-\omega(s+\varsigma)}\int_0^T\int_0^\infty x|(S_{A_t}(s)S_{A_t}(\varsigma)u(\sigma,x)|\,d\sigma dx$$

$$\|S_{A_t}(\varsigma)u\| \leq (MM_{A_t})^{-1}e^{\omega\varsigma}\sup_{s\geq 0}e^{-\omega(s+\varsigma)}MM_{A_t}e^{\omega\varsigma}e^{\omega s}$$

which gives

$$\|S_{A_t}(\varsigma)u\| \leq e^{\omega\varsigma}$$

This proves that the semigroup $S_{A_t}(s)_{s\geq 0}$ is in the class $\mathcal{G}(1,\omega)$ of quasi-contractive semigroups in the Banach space $\mathcal{X}_1$ equipped with the norm $\|.\|$. Next we extend this result and characterize the existence of an equivalent norm in $\mathcal{X}_1$ for the pair of generators $\{A_t, B_t\}$.

Definition 2.6. Let A_t and B_t be the generators of C_0-semigroups $S_{A_t}(s)_{s\geq 0}$ and $S_{B_t}(s)_{s\geq 0}$ in $\mathcal{X}_1$. The pair $\{A_t, B_t\}$ is called *renormalizable* with constants ω and ν if for any sequences $\{\alpha_k\}_{k=1}^N$, $\alpha_k \geq 0$ and $\{\delta_k\}_{k=1}^N$, $\delta_k \geq 0$, $n \in \mathbb{N}$, one has

$$\sup_{\substack{\alpha_1\geq 0,...,\alpha_n\geq 0 \\ \delta_1\geq 0,...,\delta_n\geq 0 \\ n\in\mathbb{N}}} e^{-\omega\Sigma\alpha_k}e^{-\nu\Sigma\delta_k}\|S_{A_t}(\alpha_1)S_{B_t}(\delta_1)...S_{A_t}(\alpha_n)S_{B_t}(\delta_n)u\|_1 < \infty \tag{33}$$

for each $u \in \mathcal{X}_1$.

Lemma 2.7. Let A_t and B_t two generators of C_0-semigroups in $\mathcal{X}_1$ equipped with its natural norm

$$\|u\|_1 := \int\limits_0^T \int\limits_0^\infty x|u(\sigma,x)|\,d\sigma dx.$$

The pair $\{A_t, B_t\}$ is renormalizable with constants ω and ν if and only if there is an equivalent norm $\|.\|$ in $\mathcal{X}_1$ such that A_t and B_t are closed, densely defined and we have $(\omega,\infty) \subset \rho(A_t)$ and $(\nu,\infty) \subset \rho(B_t)$, so that for all $\lambda > \omega$, $\kappa > \nu$,

$$\|(\lambda I - A_t)^{-1}\| \leq \frac{1}{\lambda - \omega}, \tag{34}$$

$$\|(\kappa I - B_t)^{-1}\| \leq \frac{1}{\kappa - \nu}, \tag{35}$$

with

$$\rho(A_t) = \{\lambda \in \mathbb{C}, \lambda I - A_t : D(A_t) \to \mathcal{X}_1 \text{ invertible}\}$$

and

$$\rho(B_t) = \{\lambda \in \mathbb{C}, \lambda I - B_t : D(A_t) \to \mathcal{X}_1 \text{ invertible}\}$$

the resolvent sets of A_t and B_t respectively.

Proof. Let's suppose that there is an equivalent norm $\|.\|$ in $\mathcal{X}_1$ such that A_t and B_t are closed, densely defined and satisfy the relations (34) and (35), then using Hille-Yosida's condition, there are ω and ν such that $\forall \alpha, \delta \geq 0$,

$$\|S_{A_t}(\alpha)u\| \leq \|u\|e^{\omega\alpha}, \ \|S_{B_t}(\delta)u\| \leq \|u\|e^{\nu\delta}, \ for\,all\,u \in \mathcal{X}_1.$$

Since $\|.\|$ and $\|.\|_1$ are equivalent, there are $M \geq 0$ and $N \geq 0$ such that

$$\|S_{A_t}(\alpha)u\|_1 \leq M\|S_{A_t}(\alpha)u\| \leq Me^{\omega\alpha}$$

and

$$\|S_{B_t}(\delta)u\|_1 \leq N\|S_{B_t}(\delta)u\| \leq Ne^{\nu\delta},$$

leading to

$$e^{-\omega\alpha}\|S_{A_t}(\alpha)u\|_1 \leq M_{A_t} < \infty, \forall \alpha \geq 0$$

and

$$e^{-\nu\delta}\|S_{B_t}(\delta)u\|_1 \leq N_{B_t} < \infty, \forall \delta \geq 0 \,\text{and}\, u \in \mathcal{X}_1.$$

We see that for any sequences $\{\alpha_k\}_{k=1}^N$, $\alpha_k \geq 0$, and $\{\delta_k\}_{k=1}^N$, $\delta_k \geq 0$, $n \in \mathbb{N}$, one has

$$\sup_{\substack{\alpha_1 \geq 0,\ldots,\alpha_n \geq 0 \\ \delta_1 \geq 0,\ldots,\delta_n \geq 0 \\ n \in \mathbb{N}}} e^{-\omega\Sigma\alpha_k}e^{-\nu\Sigma\delta_k}\|S_{A_t}(\alpha_1)S_{B_t}(\delta_1)...S_{A_t}(\alpha_n)S_{B_t}(\delta_n)u\|_1 < \infty$$

and the pair $\{A_t, B_t\}$ is renormalizable with constants ω and ν. Conversely, we consider the pair $\{A_t, B_t\}$ renormalizable with constants ω and ν. Then,

$$M := \sup_{\substack{\alpha_1 \geq 0, \ldots, \alpha_n \geq 0 \\ \delta_1 \geq 0, \ldots, \delta_n \geq 0 \\ n \in \mathbb{N}, \|u\|_1 \leq 1}} \mathrm{e}^{-\omega \Sigma \alpha_k} \mathrm{e}^{-\nu \Sigma \delta_k} \int_0^T \int_0^\infty x |S_{A_t}(\alpha_1) S_{B_t}(\delta_1) \ldots S_{A_t}(\alpha_n) S_{B_t}(\delta_n) u(\sigma, x)| \, d\sigma dx < \infty.$$

(36)

Now we use the uniform boundedness principle showed in [9] and define in $\mathcal{X}_1$ the norm:

$$\|u\| := M^{-2} \sup_{\substack{\alpha_1 \geq 0, \ldots, \alpha_n \geq 0 \\ \delta_1 \geq 0, \ldots, \delta_n \geq 0 \\ n \in \mathbb{N}}} \mathrm{e}^{-\omega \Sigma \alpha_k} \mathrm{e}^{-\nu \Sigma \delta_k} \int_0^T \int_0^\infty x |S_{A_t}(\alpha_1) S_{B_t}(\delta_1) \ldots S_{A_t}(\alpha_n) S_{B_t}(\delta_n) u(\sigma, x)| \, d\sigma dx.$$

Using the fact that

$$\int_0^T \int_0^\infty x |S_{A_t}(\alpha_1) S_{B_t}(\delta_1) \ldots S_{A_t}(\alpha_n) S_{B_t}(\delta_n) u(\sigma, x)| \, d\sigma dx \leq M \mathrm{e}^{\omega \Sigma \alpha_k} \mathrm{e}^{\nu \Sigma \delta_k} \int_0^T \int_0^\infty x |u(\sigma, x)| \, d\sigma dx,$$

(37)

it is clear that

$$\|u\| \leq M^{-2} M \int_0^T \int_0^\infty x |u(\sigma, x)| \, d\sigma dx \ and \ \int_0^T \int_0^\infty x |u(\sigma, x)| \, d\sigma dx \leq M^2 \|u\| \ \ for \ u \in \mathcal{X}_1. \quad (38)$$

Then,

$$\|u\| \leq M^{-1} \|u\|_1 \ and \ \|u\|_1 \leq M^2 \|u\| \ \ for \ u \in \mathcal{X}_1. \quad (39)$$

Hence the norms $\|.\|$ and $\|.\|_1$ are equivalent. Moreover (37), (39) and the fact that $A_t \in \mathcal{G}(M, \omega)$ also yield

$$\begin{aligned}
\|S_{A_t}(\varsigma) u\| \ &\leq \ M^{-2} \sup_{\substack{\alpha_1 \geq 0, \ldots, \alpha_n \geq 0 \\ \delta_1 \geq 0, \ldots, \delta_n \geq 0 \\ n \in \mathbb{N}}} \mathrm{e}^{-\omega \Sigma \alpha_k} \mathrm{e}^{-\nu \Sigma \delta_k} \int_0^T \int_0^\infty x |S_{A_t}(\alpha_1) S_{B_t}(\delta_1) \ldots S_{A_t}(\alpha_n) S_{B_t}(\delta_n) S_{A_t}(\varsigma) u(\sigma, x)| \, d\sigma dx \\
&\leq \ M^{-2} \sup_{\substack{\alpha_1 \geq 0, \ldots, \alpha_n \geq 0 \\ \delta_1 \geq 0, \ldots, \delta_n \geq 0 \\ n \in \mathbb{N}}} \mathrm{e}^{-\omega \Sigma \alpha_k} \mathrm{e}^{-\nu \Sigma \delta_k} \|S_{A_t}(\alpha_1) S_{B_t}(\delta_1) \ldots S_{A_t}(\alpha_n) S_{B_t}(\delta_n) S_{A_t}(\varsigma) u\|_1 \\
&\leq \ M^{-2} \sup_{\substack{\alpha_1 \geq 0, \ldots, \alpha_n \geq 0 \\ \delta_1 \geq 0, \ldots, \delta_n \geq 0 \\ n \in \mathbb{N}}} \mathrm{e}^{-\omega \Sigma \alpha_k} \mathrm{e}^{-\nu \Sigma \delta_k} \|S_{A_t}(\alpha_1) S_{B_t}(\delta_1) \ldots S_{A_t}(\alpha_n) S_{B_t}(\delta_n)\|_1 \|S_{A_t}(\varsigma) u\|_1 \\
&\leq \ M^{-2} \sup_{\substack{\alpha_1 \geq 0, \ldots, \alpha_n \geq 0 \\ \delta_1 \geq 0, \ldots, \delta_n \geq 0 \\ n \in \mathbb{N}}} \mathrm{e}^{-\omega \Sigma \alpha_k} \mathrm{e}^{-\nu \Sigma \delta_k} \|S_{A_t}(\alpha_1) S_{B_t}(\delta_1) \ldots S_{A_t}(\alpha_n) S_{B_t}(\delta_n)\|_1 M \mathrm{e}^{\omega \varsigma} \\
&\leq \ M^{-2} \mathrm{e}^{\omega \varsigma} \|u\|_1 \\
&\leq \ \mathrm{e}^{\omega \varsigma} \|u\|
\end{aligned}$$

(40)

We have
$$\|S_{A_t}(\varsigma)u\| \leq e^{\omega\varsigma}\|u\|, \ u \in \mathcal{X}_1.$$

On the same way we have
$$\|S_{B_t}(\varsigma)u\| \leq e^{\nu\varsigma}\|u\|, \ u \in \mathcal{X}_1.$$

Which means that the generators $A_t \in \mathcal{G}(1,\omega)$ and $B_t \in \mathcal{G}(1,\nu)$ in the Banach space $\mathcal{X}_1$ endowed with the norm $\|.\|$. Hence, A_t and B_t are closed, densely defined and satisfy the relations (34) and (35) in $(\mathcal{X}_1, \|.\|)$. $\hfill\square$

Actually we have in hands all the essential elements allowing us to state the following perturbation theorem:

Theorem 2.8. Let A_t and B_t be a renormalizable pair of generators of C_0-semigroups on $\mathcal{X}_1$ and the induced generators $\check{A}_t$ and $\check{B}_t$ be closed, densely defined and satisfy the relations (22) and (23) respectively. Further, let the Banach space $Y \hookrightarrow \mathcal{X}_1$ be admissible with respect to operators A_t and B_t so that $Y \subseteq D(B_t)$. If either $\check{A}_t^*$ or $\check{B}_t^*$ is densely defined in Y^*, or only B_t^* is densely defined in $\mathcal{X}_1^*$, then the closure $\overline{K_t}$ of the operator sum K_t:
$$K_t\psi = A_t\psi + B_t\psi, \ \text{on} \ D(A_t) \cap D(B_t) = D(A_t) \tag{41}$$

exists and $\overline{K_t}$ is the generator of a C_0-semigroup.

Proof. We just have to prove that the range of $(K_t + \eta)$ for some $\eta < 0$ ie dense in $\mathcal{X}_1$ and apply the theorem (2.1). Let T be the embedding operator of corollary 2.4, we have by the definition 2.2 that
$$A_t Tu = T\check{A}_t u, \quad for \ u \in D(\check{A}_t)$$

and
$$B_t Tu = T\check{B}_t u, \quad for \ u \in D(\check{B}_t).$$

We also have $D(\check{B}_t^*) \supseteq T^*\mathcal{X}_1^*$ since $D(B_t) \supseteq Y$. Therefore $D(K_t)$ is dense in $\mathcal{X}_1$ and we obtain $D(K_t) \supseteq TD(\check{A}_t)$ since $\check{A}_t$ is closed, densely defined in Y which is itself densely embedded in $\mathcal{X}_1$.
Now let $v \in D(K_t^*) \subseteq \mathcal{X}_1^*$, then we obtain
$$\langle K_t Tu, v \rangle = \langle A_t Tu, v \rangle + \langle B_t Tu, v \rangle$$

or
$$\langle u, K_t^* T^* v \rangle = \langle T\check{A}_t u, v \rangle + \langle u, B_t^* T^* v \rangle.$$

Then,
$$\begin{aligned} \langle \check{A}_t u, T^* v \rangle &= \langle u, K_t^* T^* v \rangle - \langle u, B_t^* T^* v \rangle \\ &= \langle u, A_t^* T^* v \rangle, \quad u \in D(\check{A}_t), \end{aligned}$$

which means $T^* v \in D(\check{A}_t^*)$ and, then, $T^* D(K_t^*) \subseteq D(\check{A}_t^*)$. Since $D(\check{B}_t^*) \supseteq T^*\mathcal{X}_1^*$, we have
$$T^* D(K_t^*) \subseteq D(\check{A}_t^*) \cap D(\check{B}_t^*). \tag{42}$$

Assuming now by contradiction that $\operatorname{ran}(K_t + \eta)$ is not dense in $\mathcal{X}_1$ for some $\eta < 0$, then there is $v \in \mathcal{X}_1^*$ such that

$$\langle (K_t + \eta)u, v \rangle = 0, \quad u \in D(K_t),$$

which means

$$v \in D(K_t^*) \text{ and } (K_t^* + \eta)v = 0.$$

Hence,

$$T^* v \in D(\check{A}_t^{\,*}) \cap D(\check{B}_t^{\,*}), \text{ since } T^* D(K_t^*) \subseteq D(\check{A}_t^{\,*}) \cap D(\check{B}_t^{\,*}).$$

If B_t^* is densely defined in $\mathcal{X}_1^*$ then we apply the corollary (2.4) and find that $T^* v = 0$. If either $\check{A}_t^{\,*}$ or $\check{B}_t^{\,*}$ is densely defined in Y^*, then we apply the lemma (2.3) to also find that $T^* v = 0$. Therefore, we obtain $v = 0$, which is impossible. Hence, $\operatorname{ran}(K_t + \eta)$ is dense in $\mathcal{X}_1$ for all $\eta < 0$. Because A_t and B_t are a renormalizable pair of generators of C_0-semigroups on $\mathcal{X}_1$, we can use the Lemma (2.7) and Hille-Yosida theorem to say that A_t and B_t are of class $\mathcal{G}(1,0)$. Therefore the operator $K_t = A_t + B_t$ is closable and the relation

$$|\eta| \|u\|_1 \le \|K_t u + \eta u\|_1, \quad u \in D(K_t), \; \eta < 0,$$

yields the existence of the closure $\overline{K_t}$ of K_t. The theorem (2.1) completes the proof. $\square$

Corollary 2.9. Let the operators $A(t) = A$ and $B(t) = B$, independent of t and satisfying the conditions of Theorem 2.8, then the closure $\overline{K(t)} = \overline{K}$ given as

$$\overline{K}\psi = \overline{A\psi + B\psi}, \text{ on } D(A) \cap D(B) = D(A) \tag{43}$$

exists and is the generator of a C_0-semigroup.

Proof. In concrete applications, $A(t)$, $t \in \mathfrak{I}$ is often a measurable family of generators or generators belonging uniformly to the class $\mathcal{G}(M,\omega)$, for some constants M and ω, and, since we are in one dimensional case, one can easily verify, as shown in [16], that, in this case the induced multiplication operator A is an anti-generator or generator in $Lp(\mathfrak{I}, X_1)$, for some $p \in [1,\infty)$ with $\mathfrak{I} \subseteq \mathbb{R}_+$. This reduces the problem to finding certain conditions for the operator sum

$$K\psi = A\psi + B\psi, \text{ on } D(A) \cap D(B) = D(A) \tag{44}$$

to be closable and its closure generates a C_0 semigroup and Theorem 2.8 ends the proof. $\square$

Remark 2.10. *By the relation (19), it follows that the closure of $A(t) + B(t)$ generates a propagator.*

This allows us to consider the following conservativeness result.

Theorem 2.11. *(a) The C_0 semigroup $(S_{\overline{K_t}}(s))_{s \ge 0}$ generated by $\overline{K_t} = \overline{A_t + B_t}$ is conservative if and only if the associated propagator $U(t,\tau), 0 \le \tau < t \le T$, is conservative.*
(b) If the operators A_t and B_t, satisfy the conditions of Theorem 2.8, then the model (14) is conservative.

Proof. (*a*) We make use of the relation (19) and properties of U given in the Definition 1.2. (*b*) The second part of the proof follows from (*a*) and based on [7, Theorem 6.13]. □

Before concluding, it is important to add that alternative and similar analysis to this work can be done using the recently introduced definitions of fractional derivatives with and without singular kernel [1, 2, 12] and this may yield an analogue result.

2.1 Concluding Remarks

We have exploit the subordination principle, set conditions on the generators involved in the semigroup perturbation and used the renormalization method, which is different from the preceding ones, to analyze the fractional models of type (14). We dropped the class $\mathcal{G}(1,0)$ for the class $\mathcal{G}(1,\nu)$ of quasi-contractive semigroups in $\mathcal{X}_1 = L_1([0,T] \times [0,\infty), \, xd\sigma dx)$, and showed existence results and conservativeness for the fractional non-autonomous fragmentation model (14), therefore, giving a stronger result than [5, 9], where the model was autonomous and not generlized with coefficients independent of time. The result obtained here can lead to the full characterization of the infinitesimal generator for the fractional non-autonomous fragmentation model (14) and later for fractional non-autonomous fragmentation-coagulation or non-autonomous transport-fragmentation-coagulation models, which remain open problems.

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