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Homogenization of a stochastic nonlinear reaction–diffusion equation with a large reaction term: The almost periodic framework

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ABSTRACT

Homogenization of a stochastic nonlinear reaction–diffusion equation with a large nonlinear term is considered. Under a general Besicovitch almost periodicity assumption on the coefficients of the equation we prove that the sequence of solutions of the said problem converges in probability towards the solution of a rather different type of equation, namely, the stochastic nonlinear convection–diffusion equation which we explicitly derive in terms of appropriate functionals. We study some particular cases such as the periodic framework, and many others. This is achieved under a suitable generalized concept of Σ-convergence for stochastic processes.

1. Introduction

Homogenization theory is an important branch of asymptotic analysis. Since the pioneering work of Bensoussan et al. [1] it has grown very significantly, giving rise to several sub-branches such as the *deterministic* homogenization theory and the *random* homogenization theory. Each of these sub-branches has been developed and deepened. Regarding the deterministic homogenization theory, from the classical periodic theory [1] to the recent general deterministic ergodic theory [2–5], many results have been reported and continue to be published. We refer to some of these results [6,2–5] relating to the deterministic homogenization of deterministic partial differential equations in the periodic framework and in the deterministic ergodic framework in general.

The random homogenization theory is divided into two major subgroups: the homogenization of differential operators with random coefficients, and the homogenization of stochastic partial differential equations. As far as the first subgroup is concerned, many results have been obtained to date; we refer e.g. to [7–17].

In contrast with either the deterministic homogenization theory or the homogenization of partial differential operators with random coefficients, very few results are available in the setting of the homogenization of stochastic partial differential equations (SPDEs). We cite for example the works [18–22] which consider the homogenization problems related to SPDEs with periodic coefficients (only!). Homogenization of SPDEs with non oscillating coefficients was considered in [23] in domains with non periodically distributed holes. The approach in [23] is different and is the stochastic version of Marchenko–Khruslov–Skrypnik's theory developed in [24,25]. It should be noted that unfortunately for SPDEs with oscillating coefficients no results are available beyond the periodic setting.

Given the significance of SPDEs in modeling of physical phenomena, in addition to simple random periodically perturbed phenomena, it is important to think of a theory generalizing that of the homogenization of SPDEs with periodic coefficients. This is one of the objectives of this work.

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More precisely, we discuss the homogenization problem for the following nonlinear SPDE

$$\begin{cases} du_\varepsilon = \left(\operatorname{div}\left(a\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right)Du_\varepsilon\right)+\frac{1}{\varepsilon}g\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},u_\varepsilon\right)\right)dt + M\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},u_\varepsilon\right)dW \quad \text{in } Q_T \\ u_\varepsilon = 0 \quad \text{on } \partial Q\times(0,T) \\ u_\varepsilon(x,0)=u^0(x) \quad \text{in } Q \end{cases} \tag{1.1}$$

in the almost periodic environment, where $Q_T = Q \times (0, T)$, Q being a Lipschitz domain in $\mathbb{R}^N$ with smooth boundary ∂Q, T is a positive real number and W is a m-dimensional standard Wiener process defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The choice of the above problem lies in its application in engineering (see for example [26–28] in the deterministic setting, and [29] in the stochastic framework, for more details). In fact, as in [26], the unknown u_ε may be viewed as the concentration of some chemical species diffusing in a porous medium of constant porosity, with diffusivity $a(y, \tau)$ and reacting with background medium through the nonlinear term $g(y, \tau, u)$ under the influence of a random external source $M(y, \tau, u)dW$ (M is a $\mathbb{R}^m$-valued function and throughout $M(y, \tau, u)dW$ will denote the scalar product of M and dW in $\mathbb{R}^m$). The motivation of this choice is several fold. Firstly, we start from a SPDE of reaction–diffusion type, and we end up, after the passage to the limit, with a SPDE of a convection–diffusion type; this is because of the large reaction's term $\frac{1}{\varepsilon}g(x/\varepsilon, t/\varepsilon^2, u_\varepsilon)$ which satisfies some kind of centering condition; see Section 4 for details. Secondly, the order of the microscopic time scale here is twice that of the microscopic spatial scale. This leads after the passage to the limit, to a rather complicated so-called cell problem, which is besides, a deterministic parabolic type equation, the random variable behaves in the latter equation just like a parameter. Such a problem is difficult to deal with as, in our situation, it involves a microscopic time derivative derived from the semigroup theory, which is not easy to handle. Thirdly, in order to solve the homogenization problem under consideration, we introduce a suitable type of convergence which takes into account both deterministic and random behavior of the data of the original problem. This method is formally justified by the theory of Wiener chaos polynomials [30,31]. In fact, following [30] (see also [31]), any sequence of stochastic processes $u^\varepsilon(x, t, \omega) \in L^2(Q \times (0, T) \times \Omega)$ expresses as follows:

$$u^\varepsilon(x,t,\omega)=\sum_{j=1}^{\infty}u_j^\varepsilon(x,t)\Phi_j(\omega)$$

where the functions Φ_j are the generalized Hermite polynomials, known as the Wiener-chaos polynomials. The above decomposition clearly motivates the definition of the concept of convergence used in this work; see Section 3 for further details. Finally, the periodicity assumption on the coefficients is here replaced by the almost periodicity assumption. Accordingly, it is the first time that an SPDE is homogenized beyond the classical period framework, and our result is thus, new. It is also important to note that in the deterministic framework, i.e. when $M = 0$ in (1.1), the equivalent problem obtained has just been solved by Allaire and Piatnitski [26] under the periodicity assumption on the coefficients, but with a weight function on the derivative with respect to time. Our results therefore generalize to the almost periodic setting, those obtained by Allaire and Piatnitski in [26].

The layout of the paper is as follows. In Section 2 we recall some useful facts about almost periodicity that will be used in the next sections. Section 3 deals with the concept of Σ-convergence for stochastic processes. In Section 4, we state the problem to be studied. We proved there a tightness result that will be used in the next section. We state and prove homogenization results in Section 5. In particular we give in that section the explicit form of the homogenization equation. Finally, in Section 6, we give some applications of the result obtained in the previous section.

Throughout Section 2, vector spaces are assumed to be complex vector spaces, and scalar functions are assumed to take complex values. We shall always assume that the numerical space $\mathbb{R}^m$ (integer $m \geq 1$) and its open sets are each equipped with the Lebesgue measure $dx = dx_1 \dots dx_m$.

2. Spaces of almost periodic functions

The concept of almost periodic functions is well known in the literature. We present in this section some basic facts about it, which will be used throughout the paper. For a general presentation and an efficient treatment of this concept, we refer to [32–34].

Let $\mathcal{B}(\mathbb{R}^N)$ denote the Banach algebra of bounded continuous complex-valued functions on $\mathbb{R}^N$ endowed with the sup norm topology.

A function $u \in \mathcal{B}(\mathbb{R}^N)$ is called a almost periodic function if the set of all its translates $\{u(\cdot + a)\}_{a\in\mathbb{R}^N}$ is precompact in $\mathcal{B}(\mathbb{R}^N)$. The set of all such functions forms a closed subalgebra of $\mathcal{B}(\mathbb{R}^N)$, which we denote by $AP(\mathbb{R}^N)$. From the above definition, it is an easy matter to see that every element of $AP(\mathbb{R}^N)$ is uniformly continuous. It is classically known that the algebra $AP(\mathbb{R}^N)$ enjoys the following properties:

(i) $\overline{u} \in AP(\mathbb{R}^N)$ whenever $u \in AP(\mathbb{R}^N)$, where $\overline{u}$ stands for the complex conjugate of u;
(ii) $u(\cdot + a) \in AP(\mathbb{R}^N)$ for any $u \in AP(\mathbb{R}^N)$ and each $a \in \mathbb{R}^N$.

(iii) For each $u \in AP(\mathbb{R}^N)$ the closed convex hull of $\{u(\cdot + a)\}_{a\in\mathbb{R}^N}$ in $\mathcal{B}(\mathbb{R}^N)$ contains a unique complex constant $\mathfrak{M}(u)$ called the mean value of u, and which satisfies the property that the sequence $(u^\varepsilon)_{\varepsilon>0}$ (where $u^\varepsilon(x) = u(x/\varepsilon)$, $x \in \mathbb{R}^N$) weakly $*$-converges in $L^\infty(\mathbb{R}^N)$ to $\mathfrak{M}(u)$ as $\varepsilon \to 0$. $\mathfrak{M}(u)$ also satisfies the property

$$\mathfrak{M}(u) = \lim_{R\to+\infty} \frac{1}{(2R)^N} \int_{[-R,R]^N} u(y)dy.$$

As a result of (i)–(iii) above we get that $AP(\mathbb{R}^N)$ is an algebra with mean value on $\mathbb{R}^N$ [11]. The spectrum of $AP(\mathbb{R}^N)$ (viewed as $\mathcal{C}^*$-algebra) is the Bohr compactification of $\mathbb{R}^N$, denoted usually in the literature by $b\mathbb{R}^N$, and, in order to simplify the notation, we denote it here by $\mathcal{K}$. Then, as it is classically known, $\mathcal{K}$ is a compact topological Abelian group. We denote its Haar measure by β (as in [3]). The following result is due to the Gelfand representation theory of $\mathcal{C}^*$-algebras.

Theorem 1. *There exists an isometric $*$-isomorphism $\mathcal{G}$ of $AP(\mathbb{R}^N)$ onto $\mathcal{C}(\mathcal{K})$ such that every element of $AP(\mathbb{R}^N)$ is viewed as a restriction to $\mathbb{R}^N$ of a unique element in $\mathcal{C}(\mathcal{K})$. Moreover the mean value $\mathfrak{M}$ defined on $AP(\mathbb{R}^N)$ has an integral representation in terms of the Haar measure β as follows:*

$$\mathfrak{M}(u) = \int_{\mathcal{K}} \mathcal{G}(u)d\beta \quad \text{for all } u \in AP(\mathbb{R}^N).$$

The isometric $*$-isomorphism $\mathcal{G}$ of the above theorem is referred to as the Gelfand transformation. The image $\mathcal{G}(u)$ of u will very often be denoted by $\widehat{u}$.

For $m \in \mathbb{N}$ (the positive integers) we introduce the space $AP^m(\mathbb{R}^N) = \{u \in AP(\mathbb{R}^N) : D_y^\alpha u \in AP(\mathbb{R}^N)$ for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ with $|\alpha| \le m\}$, a Banach space with the norm $|||u|||_m = \sup_{|\alpha|\le m} \sup_{y\in\mathbb{R}^N} |D_y^\alpha u|$, where $D_y^\alpha = \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} \ldots \partial y_N^{\alpha_N}}$. We also define the space $AP^\infty(\mathbb{R}^N) = \cap_m AP^m(\mathbb{R}^N)$, a Fréchet space with respect to the natural topology of projective limit, defined by the increasing family of norms $|||\cdot|||_m$ ($m \in \mathbb{N}$).

Next, let $B_{AP}^p(\mathbb{R}^N)$ ($1 \le p < \infty$) denote the space of Besicovitch almost periodic functions on $\mathbb{R}^N$, that is the closure of $AP(\mathbb{R}^N)$ with respect to the Besicovitch seminorm

$$\|u\|_p = \left(\limsup_{r\to+\infty} \frac{1}{|B_r|} \int_{B_r} |u(y)|^p dy\right)^{1/p}$$

where B_r is the open ball of $\mathbb{R}^N$ of radius r centered at the origin. It is known that $B_{AP}^p(\mathbb{R}^N)$ is a complete seminormed vector space verifying $B_{AP}^q(\mathbb{R}^N) \subset B_{AP}^p(\mathbb{R}^N)$ for $1 \le p \le q < \infty$. From this last property one may naturally define the space $B_{AP}^\infty(\mathbb{R}^N)$ as follows:

$$B_{AP}^\infty(\mathbb{R}^N) = \left\{ f \in \cap_{1\le p<\infty} B_{AP}^p(\mathbb{R}^N) : \sup_{1\le p<\infty} \|f\|_p < \infty \right\}.$$

We endow $B_{AP}^\infty(\mathbb{R}^N)$ with the seminorm $[f]_\infty = \sup_{1\le p<\infty} \|f\|_p$, which makes it a complete seminormed space. We recall that the spaces $B_{AP}^p(\mathbb{R}^N)$ ($1 \le p \le \infty$) are not general Fréchet spaces since they are not separated. The following properties are worth noticing [4,5]:

(1) The Gelfand transformation $\mathcal{G} : AP(\mathbb{R}^N) \to \mathcal{C}(\mathcal{K})$ extends by continuity to a unique continuous linear mapping, still denoted by $\mathcal{G}$, of $B_{AP}^p(\mathbb{R}^N)$ into $L^p(\mathcal{K})$, which in turn induces an isometric isomorphism $\mathcal{G}_1$, of $B_{AP}^p(\mathbb{R}^N)/\mathcal{N} = \mathcal{B}_{AP}^p(\mathbb{R}^N)$ onto $L^p(\mathcal{K})$ (where $\mathcal{N} = \{u \in B_{AP}^p(\mathbb{R}^N) : \mathcal{G}(u) = 0\}$). Moreover if $u \in B_{AP}^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ then $\mathcal{G}(u) \in L^\infty(\mathcal{K})$ and $\|\mathcal{G}(u)\|_{L^\infty(\mathcal{K})} \le \|u\|_{L^\infty(\mathbb{R}^N)}$.

(2) The mean value $\mathfrak{M}$ viewed as defined on $AP(\mathbb{R}^N)$, extends by continuity to a positive continuous linear form (still denoted by $\mathfrak{M}$) on $B_{AP}^p(\mathbb{R}^N)$ satisfying $\mathfrak{M}(u) = \int_{\mathcal{K}} \mathcal{G}(u)d\beta$ ($u \in B_{AP}^p(\mathbb{R}^N)$). Furthermore, $\mathfrak{M}(u(\cdot + a)) = \mathfrak{M}(u)$ for each $u \in B_{AP}^p(\mathbb{R}^N)$ and all $a \in \mathbb{R}^N$, where $u(\cdot + a)(z) = u(z + a)$ for almost all $z \in \mathbb{R}^N$. Moreover for $u \in B_{AP}^p(\mathbb{R}^N)$ we have $\|u\|_p = [\mathfrak{M}(|u|^p)]^{1/p}$.

We refer to [35,36] for the definitions and properties of the vector-valued spaces of almost periodic functions, namely, $AP(\mathbb{R}^N; X)$ and $B_{AP}^p(\mathbb{R}^N; X)$ and the connected spaces $\mathcal{C}(\mathcal{K}; X)$ and $L^p(\mathcal{K}; X)$, where X is a given Banach space. In particular when $X = \mathbb{C}$ we get $AP(\mathbb{R}^N)$ and $B_{AP}^p(\mathbb{R}^N)$ respectively.

Now let $\mathbb{R}_{y,\tau}^{N+1} = \mathbb{R}_y^N \times \mathbb{R}_\tau$ denotes the space $\mathbb{R}^N \times \mathbb{R}$ with generic variables (y, τ). It is known that $AP(\mathbb{R}_{y,\tau}^{N+1}) = AP(\mathbb{R}_\tau; AP(\mathbb{R}_y^N))$ is the closure in $\mathcal{B}(\mathbb{R}_{y,\tau}^{N+1})$ of the tensor product $AP(\mathbb{R}_y^N) \otimes AP(\mathbb{R}_\tau)$ [37]. In what follows, we set $A_y = AP(\mathbb{R}_y^N)$, $A_\tau = AP(\mathbb{R}_\tau)$ and $A = AP(\mathbb{R}_{y,\tau}^{N+1})$. We denote the mean value on A_ζ ($\zeta = y, \tau$) by $\mathfrak{M}_\zeta$.

In the above notations, let $g \in A$ with $\mathfrak{M}_y(g) = 0$. Then arguing as in [11, p. 246] we see that there exists a unique $R \in A$ with $\mathfrak{M}_y(R) = 0$ such that

$$g = \Delta_y R \tag{2.1}$$

where Δ_y stands for the Laplacian operator defined on $\mathbb{R}^N_y$: $\Delta_y = \sum_{i=1}^N \partial^2/\partial y_i^2$. Owing to the hypoellipticity of the Laplacian on $\mathbb{R}^N$ we deduce that the function R is at least of class C^2 with respect to the variable y. The above fact will be very useful in the last two sections of the work.

Next following the theory presented in [38, Chap. B1] (see also [36]), let $1 \le p < \infty$ and consider the N-parameter group of isometries $\{T(y) : y \in \mathbb{R}^N\}$ defined by

$$T(y) : \mathcal{B}^p_{AP}(\mathbb{R}^N) \to \mathcal{B}^p_{AP}(\mathbb{R}^N), \qquad T(y)(u + \mathcal{N}) = \tau_y u + \mathcal{N} \quad \text{for } u \in B^p_{AP}(\mathbb{R}^N)$$

where $\tau_y u = u(\cdot + y)$. Since the elements of $AP(\mathbb{R}^N)$ are uniformly continuous, $\{T(y) : y \in \mathbb{R}^N\}$ is a strongly continuous group in the sense of semigroups: $T(y)(u + \mathcal{N}) \to u + \mathcal{N}$ in $\mathcal{B}^p_{AP}(\mathbb{R}^N)$ as $|y| \to 0$. In view of the isometric isomorphism $\mathcal{G}_1$ we associated to $\{T(y) : y \in \mathbb{R}^N\}$ the following N-parameter group $\{\overline{T}(y) : y \in \mathbb{R}^N\}$ defined by

$$\overline{T}(y) : L^p(\mathcal{K}) \to L^p(\mathcal{K})$$
$$\overline{T}(y)\mathcal{G}_1(u + \mathcal{N}) = \mathcal{G}_1(T(y)(u + \mathcal{N})) = \mathcal{G}_1(\tau_y u + \mathcal{N}) \quad \text{for } u \in B^p_{AP}(\mathbb{R}^N).$$

The group $\{\overline{T}(y) : y \in \mathbb{R}^N\}$ is also strongly continuous. The infinitesimal generator of $T(y)$ (resp. $\overline{T}(y)$) along the ith coordinate direction is denoted by $D_{i,p}$ (resp. $\partial_{i,p}$) and is defined by

$$D_{i,p}u = \lim_{t\to 0} t^{-1}(T(te_i)u - u) \quad \text{in } \mathcal{B}^p_{AP}(\mathbb{R}^N)$$
$$\left(\text{resp. } \partial_{i,p}v = \lim_{t\to 0} t^{-1}(\overline{T}(te_i)v - v) \text{ in } L^p(\mathcal{K})\right),$$

where, we have used the same letter u to denote the equivalence class of an element $u \in B^p_{AP}(\mathbb{R}^N)$ in $\mathcal{B}^p_{AP}(\mathbb{R}^N)$, $e_i = (\delta_{ij})_{1\le j\le N}$ (δ_{ij} being the Kronecker δ). The domain of $D_{i,p}$ (resp. $\partial_{i,p}$) in $\mathcal{B}^p_{AP}(\mathbb{R}^N)$ (resp. $L^p(\mathcal{K})$) is denoted by $\mathcal{D}_{i,p}$ (resp. $\mathcal{W}_{i,p}$). By using the general theory of semigroups [39, Chap. VIII, Section 1], the following result holds.

Proposition 1. *$\mathcal{D}_{i,p}$ (resp. $\mathcal{W}_{i,p}$) is a vector subspace of $\mathcal{B}^p_{AP}(\mathbb{R}^N)$ (resp. $L^p(\mathcal{K})$), $D_{i,p} : \mathcal{D}_{i,p} \to \mathcal{B}^p_{AP}(\mathbb{R}^N)$ (resp. $\partial_{i,p} : \mathcal{W}_{i,p} \to L^p(\mathcal{K})$) is a linear operator, $\mathcal{D}_{i,p}$ (resp. $\mathcal{W}_{i,p}$) is dense in $\mathcal{B}^p_{AP}(\mathbb{R}^N)$ (resp. $L^p(\mathcal{K})$), and the graph of $D_{i,p}$ (resp. $\partial_{i,p}$) is closed in $\mathcal{B}^p_{AP}(\mathbb{R}^N) \times \mathcal{B}^p_{AP}(\mathbb{R}^N)$ (resp. $L^p(\mathcal{K}) \times L^p(\mathcal{K})$).*

In the sequel we denote by ϱ the canonical mapping of $B^p_{AP}(\mathbb{R}^N)$ onto $\mathcal{B}^p_{AP}(\mathbb{R}^N)$, that is, $\varrho(u) = u + \mathcal{N}$ for $u \in B^p_{AP}(\mathbb{R}^N)$. The following properties are immediate. The verification can be found either in [38, Chap. B1] or in [36].

Lemma 1. *Let $1 \le i \le N$. (1) If $u \in AP^1(\mathbb{R}^N)$ then $\varrho(u) \in \mathcal{D}_{i,p}$ and*

$$D_{i,p}\varrho(u) = \varrho\left(\frac{\partial u}{\partial y_i}\right). \tag{2.2}$$

(2) If $u \in \mathcal{D}_{i,p}$ then $\mathcal{G}_1(u) \in \mathcal{W}_{i,p}$ and $\mathcal{G}_1(D_{i,p}u) = \partial_{i,p}\mathcal{G}_1(u)$.

One can naturally define higher order derivatives by setting $D^\alpha_p = D^{\alpha_1}_{1,p} \circ \cdots \circ D^{\alpha_N}_{N,p}$ (resp. $\partial^\alpha_p = \partial^{\alpha_1}_{1,p} \circ \cdots \circ \partial^{\alpha_N}_{N,p}$) for $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ with $D^{\alpha_i}_{i,p} = D_{i,p} \circ \cdots \circ D_{i,p}$, α_i-times. Now, let

$$\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N) = \cap_{i=1}^N \mathcal{D}_{i,p} = \{u \in \mathcal{B}^p_{AP}(\mathbb{R}^N) : D_{i,p}u \in \mathcal{B}^p_{AP}(\mathbb{R}^N)\ \forall 1 \le i \le N\}$$

and

$$\mathcal{D}_{AP}(\mathbb{R}^N) = \{u \in \mathcal{B}^\infty_{AP}(\mathbb{R}^N) : D^\alpha_\infty u \in \mathcal{B}^\infty_{AP}(\mathbb{R}^N)\ \forall \alpha \in \mathbb{N}^N\}.$$

It can be shown that $\mathcal{D}_{AP}(\mathbb{R}^N)$ is dense in $\mathcal{B}^p_{AP}(\mathbb{R}^N)$, $1 \le p < \infty$. We also have that $\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)$ is a Banach space under the norm

$$\|u\|_{\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)} = \left(\|u\|_p^p + \sum_{i=1}^N \|D_{i,p}u\|_p^p\right)^{1/p} \qquad (u \in \mathcal{B}^{1,p}_{AP}(\mathbb{R}^N));$$

this comes from the fact that the graph of $D_{i,p}$ is closed.

The counterpart of the above properties also holds with

$$W^{1,p}(\mathcal{K}) = \cap_{i=1}^N \mathcal{W}_{i,p} \quad \text{in place of } \mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)$$

and

$$\mathcal{D}(\mathcal{K}) = \{u \in L^\infty(\mathcal{K}) : \partial^\alpha_\infty u \in L^\infty(\mathcal{K})\ \forall \alpha \in \mathbb{N}^N\} \quad \text{in that of } \mathcal{D}_{AP}(\mathbb{R}^N).$$

Moreover the restriction of $\mathcal{G}_1$ to $\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)$ is an isometric isomorphism of $\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)$ onto $W^{1,p}(\mathcal{K})$; this comes from [Part (2) of] Lemma 1.

Let $u \in \mathcal{D}_{i,p}$ ($p \geq 1$, $1 \leq i \leq N$). Then the inequality

$$\|t^{-1}(T(te_i)u - u) - D_{i,p}u\|_1 \leq c\|t^{-1}(T(te_i)u - u) - D_{i,p}u\|_p$$

for a positive constant c independent of u and t, yields $D_{i,1}u = D_{i,p}u$, so that $D_{i,p}$ is the restriction to $\mathcal{B}^p_{AP}(\mathbb{R}^N)$ of $D_{i,1}$. Therefore, for all $u \in \mathcal{D}_{i,\infty}$ we have $u \in \mathcal{D}_{i,p}$ ($p \geq 1$) and $D_{i,\infty}u = D_{i,p}u$ $\forall 1 \leq i \leq N$. We will need the following result in the sequel.

Lemma 2. *We have $\mathcal{D}_{AP}(\mathbb{R}^N) = \varrho(AP^\infty(\mathbb{R}^N))$.*

Proof. From (2.2) we have that, for $u \in \varrho(AP^\infty(\mathbb{R}^N))$ and $\alpha \in \mathbb{N}^N$, $D^\alpha_\infty u = \varrho(D^\alpha_y v)$ where $v \in AP^\infty(\mathbb{R}^N)$ is such that $u = \varrho(v)$. This leads at once to $\varrho(AP^\infty(\mathbb{R}^N)) \subset \mathcal{D}_{AP}(\mathbb{R}^N)$. Conversely if $u \in \mathcal{D}_{AP}(\mathbb{R}^N)$, then $u \in \mathcal{B}^\infty_{AP}(\mathbb{R}^N)$ with $D^\alpha_\infty u \in \mathcal{B}^\infty_{AP}(\mathbb{R}^N)$ for all $\alpha \in \mathbb{N}^N$, that is, $u = v + \mathcal{N}$ with $v \in B^\infty_{AP}(\mathbb{R}^N)$ being such that $D^\alpha_y v \in B^\infty_{AP}(\mathbb{R}^N)$ for all $\alpha \in \mathbb{N}^N$, i.e., $v \in AP^\infty(\mathbb{R}^N)$ since, as v is in $L^p_{\text{loc}}(\mathbb{R}^N)$ with all its distributional derivatives, v is of class $\mathcal{C}^\infty$. Hence $u = v + \mathcal{N}$ with $v \in AP^\infty(\mathbb{R}^N)$, so that $u \in \varrho(AP^\infty(\mathbb{R}^N))$. □

From now on, we write $\widehat{u}$ either for $\mathcal{G}(u)$ if $u \in B^p_{AP}(\mathbb{R}^N)$ or for $\mathcal{G}_1(u)$ if $u \in \mathcal{B}^p_{AP}(\mathbb{R}^N)$. The following properties are easily verified (see once again either [38, Chap. B1] or [36]).

Proposition 2. *The following assertions hold.*

(i) $\int_{\mathcal{K}} \partial^\alpha_\infty \widehat{u} d\beta = 0$ *for all $u \in \mathcal{D}_{AP}(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}^N$;*
(ii) $\int_{\mathcal{K}} \partial_{i,p} \widehat{u} d\beta = 0$ *for all $u \in \mathcal{D}_{i,p}$ and $1 \leq i \leq N$.*
(iii) $D_{i,p}(u\phi) = uD_{i,\infty}\phi + \phi D_{i,p}u$ *for all $(\phi, u) \in \mathcal{D}_{AP}(\mathbb{R}^N) \times \mathcal{D}_{i,p}$ and $1 \leq i \leq N$.*

The formula (iii) in the above proposition leads to the equality

$$\int_{\mathcal{K}} \widehat{\phi}\partial_{i,p}\widehat{u} d\beta = -\int_{\mathcal{K}} \widehat{u}\partial_{i,\infty}\widehat{\phi} d\beta \quad \forall (u, \phi) \in \mathcal{D}_{i,p} \times \mathcal{D}_{AP}(\mathbb{R}^N).$$

This suggests us to define the concept of distributions on $\mathcal{D}_{AP}(\mathbb{R}^N)$ and of a weak derivative. Before we can do that, let us endow $\mathcal{D}_{AP}(\mathbb{R}^N) = \varrho(AP^\infty(\mathbb{R}^N))$ with its natural topology defined by the family of norms $N_n(u) = \sup_{|\alpha| \leq n} \sup_{y \in \mathbb{R}^N} |D^\alpha_\infty u(y)|$, integers $n \geq 0$. In this topology, $\mathcal{D}_{AP}(\mathbb{R}^N)$ is a Fréchet space. We denote by $\mathcal{D}'_{AP}(\mathbb{R}^N)$ the topological dual of $\mathcal{D}_{AP}(\mathbb{R}^N)$. We endow it with the strong dual topology. The elements of $\mathcal{D}'_{AP}(\mathbb{R}^N)$ are called *the distributions on $\mathcal{D}_{AP}(\mathbb{R}^N)$*. One can also define the weak derivative of $f \in \mathcal{D}'_{AP}(\mathbb{R}^N)$ as follows: for any $\alpha \in \mathbb{N}^N$, $D^\alpha f$ stands for the distribution defined by the formula

$$\langle D^\alpha f, \phi\rangle = (-1)^{|\alpha|}\langle f, D^\alpha_\infty \phi\rangle \quad \text{for all } \phi \in \mathcal{D}_{AP}(\mathbb{R}^N).$$

Since $\mathcal{D}_{AP}(\mathbb{R}^N)$ is dense in $\mathcal{B}^p_{AP}(\mathbb{R}^N)$ ($1 \leq p < \infty$), it is immediate that $\mathcal{B}^p_{AP}(\mathbb{R}^N) \subset \mathcal{D}'_{AP}(\mathbb{R}^N)$ with continuous embedding, so that one may define the weak derivative of any $f \in \mathcal{B}^p_{AP}(\mathbb{R}^N)$, and it verifies the following functional equation:

$$\langle D^\alpha f, \phi\rangle = (-1)^{|\alpha|}\int_{\mathcal{K}} \widehat{f}\partial^\alpha_\infty \widehat{\phi} d\beta \quad \text{for all } \phi \in \mathcal{D}_{AP}(\mathbb{R}^N).$$

In particular, for $f \in \mathcal{D}_{i,p}$ we have

$$-\int_{\mathcal{K}} \widehat{f}\partial_{i,p}\widehat{\phi} d\beta = \int_{\mathcal{K}} \widehat{\phi}\partial_{i,p}\widehat{f} d\beta \quad \forall \phi \in \mathcal{D}_{AP}(\mathbb{R}^N),$$

so that we may identify $D_{i,p}f$ with $D^{\alpha_i}f$, $\alpha_i = (\delta_{ij})_{1 \leq j \leq N}$. Conversely, if $f \in \mathcal{B}^p_{AP}(\mathbb{R}^N)$ is such that there exists $f_i \in \mathcal{B}^p_{AP}(\mathbb{R}^N)$ with $\langle D^{\alpha_i}f, \phi\rangle = -\int_{\mathcal{K}} \widehat{f_i}\widehat{\phi} d\beta$ for all $\phi \in \mathcal{D}_{AP}(\mathbb{R}^N)$, then $f \in \mathcal{D}_{i,p}$ and $D_{i,p}f = f_i$. We are therefore justified in saying that $\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)$ is a Banach space under the norm $\|\cdot\|_{\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)}$. The same result holds for $W^{1,p}(\mathcal{K})$. Moreover it is a fact that $\mathcal{D}_{AP}(\mathbb{R}^N)$ (resp. $\mathcal{D}(\mathcal{K})$) is a dense subspace of $\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)$ (resp. $W^{1,p}(\mathcal{K})$).

We end this section with the definition of the space of correctors. For that we need the following space:

$$\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)/\mathbb{C} = \{u \in \mathcal{B}^{1,p}_{AP}(\mathbb{R}^N) : \mathfrak{M}(u) = 0\}.$$

We endow it with the seminorm

$$\|u\|_{\#,p} = \left(\sum_{i=1}^N \|D_{i,p}u\|^p_p\right)^{1/p} \quad (u \in \mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)/\mathbb{C}).$$

One can check that this is actually a norm on $\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)/\mathbb{C}$. With this norm $\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)/\mathbb{C}$ is a normed vector space which is unfortunately not complete. We denote by $\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N)$ its completion with respect to the above norm and by J_p the canonical embedding of $\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)/\mathbb{C}$ into $\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N)$. The following properties are due to the theory of completion of uniform spaces (see [40]):

(P_1) The gradient operator $D_p = (D_{1,p}, \dots, D_{N,p}) : \mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)/\mathbb{C} \to (\mathcal{B}^p_{AP}(\mathbb{R}^N))^N$ extends by continuity to a unique mapping $\overline{D}_p : \mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N) \to (\mathcal{B}^p_{AP}(\mathbb{R}^N))^N$ with the properties

$$D_{i,p} = \overline{D}_{i,p} \circ J_p$$

and

$$\|u\|_{\#,p} = \left(\sum_{i=1}^N \|\overline{D}_{i,p} u\|_p^p\right)^{1/p} \quad \text{for } u \in \mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N).$$

(P_2) The space $J_p(\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)/\mathbb{C})$ (and hence $J_p(\mathcal{D}_{AP}(\mathbb{R}^N)/\mathbb{C})$) is dense in $\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N)$.

Moreover the mapping $\overline{D}_p$ is an isometric embedding of $\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N)$ onto a closed subspace of $(\mathcal{B}^p_{AP}(\mathbb{R}^N))^N$, so that $\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N)$ is a reflexive Banach space. By duality we define the divergence operator $\mathrm{div}_{p'} : (\mathcal{B}^p_{AP}(\mathbb{R}^N))^N \to (\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N))'$ ($p' = p/(p-1)$) by

$$\langle \mathrm{div}_{p'} u, v\rangle = -\langle u, \overline{D}_p v\rangle \quad \text{for } v \in \mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N) \quad \text{and} \quad u = (u_i) \in (\mathcal{B}^{p'}_{AP}(\mathbb{R}^N))^N, \tag{2.3}$$

where $\langle u, \overline{D}_p v\rangle = \sum_{i=1}^N \int_{\mathcal{K}} \widehat{u}_i \partial_{i,p} \widehat{v} d\beta$. The operator $\mathrm{div}_{p'}$ just defined extends the natural divergence operator defined in $\mathcal{D}_{AP}(\mathbb{R}^N)$ since $D_{i,p} f = \overline{D}_{i,p}(J_p f)$ for all $f \in \mathcal{D}_{AP}(\mathbb{R}^N)$.

Now if in (2.3) we take $u = D_{p'} w$ with $w \in \mathcal{B}^{p'}_{AP}(\mathbb{R}^N)$ being such that $D_{p'} w \in (\mathcal{B}^{p'}_{AP}(\mathbb{R}^N))^N$ then this allows us to define the Laplacian operator on $\mathcal{B}^{p'}_{AP}(\mathbb{R}^N)$, denoted here by $\Delta_{p'}$, as follows:

$$\langle \Delta_{p'} w, v\rangle = \langle \mathrm{div}_{p'}(D_{p'} w), v\rangle = -\langle D_{p'} w, \overline{D}_p v\rangle \quad \text{for all } v \in \mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N). \tag{2.4}$$

If in addition $v = J_p(\phi)$ with $\phi \in \mathcal{D}_{AP}(\mathbb{R}^N)/\mathbb{C}$ then $\langle \Delta_{p'} w, J_p(\phi)\rangle = -\langle D_{p'} w, D_p \phi\rangle$, so that, for $p = 2$, we get

$$\langle \Delta_2 w, J_2(\phi)\rangle = \langle w, \Delta_2 \phi\rangle \quad \text{for all } w \in \mathcal{B}^2_{AP}(\mathbb{R}^N) \quad \text{and} \quad \phi \in \mathcal{D}_{AP}(\mathbb{R}^N)/\mathbb{C}.$$

The following result is also immediate.

Proposition 3. *For $u \in AP^\infty(\mathbb{R}^N)$ we have*

$$\Delta_p \varrho(u) = \varrho(\Delta_y u)$$

where Δ_y stands for the usual Laplacian operator on $\mathbb{R}^N_y$.

We end this subsection with some notations. Let $f \in \mathcal{B}^p_{AP}(\mathbb{R}^N)$. We know that $D^{\alpha_i} f$ exists (in the sense of distributions) and that $D^{\alpha_i} f = D_{i,p} f$ if $f \in \mathcal{D}_{i,p}$. So we can drop the subscript p and therefore denote $D_{i,p}$ (resp. $\partial_{i,p}$) by $\overline{\partial}/\partial y_i$ (resp. ∂_i). Thus, $\overline{D}_y$ will stand for the gradient operator $(\overline{\partial}/\partial y_i)_{1\le i\le N}$ and $\overline{\mathrm{div}}_y$ for the divergence operator div_p. We will also denote the operator $\overline{D}_{i,p}$ by $\overline{\partial}/\partial y_i$. Since J_p is an embedding, this allows us to view $\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)/\mathbb{C}$ (and hence $\mathcal{D}_{AP}(\mathbb{R}^N)/\mathbb{C}$) as a dense subspace of $\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N)$. $D_{i,p}$ will therefore be seen as the restriction of $\overline{D}_{i,p}$ to $\mathcal{B}^{1,p}_{AP}(\mathbb{R}^N)/\mathbb{C}$. Thus we will henceforth omit J_p in the notation if it is understood from the context and there is no risk of confusion. This will lead to the notation $\overline{D}_p = \overline{D}_y = (\overline{\partial}/\partial y_i)_{1\le i\le N}$ and $\partial_p = \partial = (\partial_i)_{1\le i\le N}$. Finally, we will denote the Laplacian operator on $\mathcal{B}^p_{AP}(\mathbb{R}^N)$ by $\overline{\Delta}_y$.

3. The Σ-convergence method for stochastic processes

In this section we define an appropriate notion of the concept of Σ-convergence adapted to our situation. It is to be noted that it is built according to the original notion introduced by Nguetseng [3]. Here we adapt it to systems involving random behavior. In all that follows Q is an open subset of $\mathbb{R}^N$ (integer $N \ge 1$), T is a positive real number and $Q_T = Q \times (0, T)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The expectation on $(\Omega, \mathcal{F}, \mathbb{P})$ will throughout be denoted by $\mathbb{E}$. Let us first recall the definition of the Banach space of all bounded $\mathcal{F}$-measurable functions. Denoting by $F(\Omega)$ the Banach space of all bounded functions $f : \Omega \to \mathbb{R}$ (with the sup norm), we define $B(\Omega)$ as the closure in $F(\Omega)$ of the vector space $H(\Omega)$ consisting of all finite linear combinations of the characteristic functions 1_X of sets $X \in \mathcal{F}$. Since $\mathcal{F}$ is an σ-algebra, $B(\Omega)$ is the Banach space of all bounded $\mathcal{F}$-measurable functions. Likewise we define the space $B(\Omega; Z)$ of all bounded $(\mathcal{F}, B_Z)$-measurable functions $f : \Omega \to Z$, where Z is a Banach space endowed with the σ-algebra of Borelians B_Z. It is a fact that the tensor product $B(\Omega) \otimes Z$ is a dense subspace of $B(\Omega; Z)$.

This being so, let $A_y = AP(\mathbb{R}^N_y)$ and $A_\tau = AP(\mathbb{R}_\tau)$. We know that $A = AP(\mathbb{R}^{N+1}_{y,\tau})$ is the closure in $\mathcal{B}(\mathbb{R}^{N+1}_{y,\tau})$ of the tensor product $A_y \otimes A_\tau$. We denote by $\mathcal{K}_y$ (resp. $\mathcal{K}_\tau$, $\mathcal{K}$) the spectrum of A_y (resp. A_τ, A). The same letter $\mathcal{G}$ will denote the Gelfand transformation on A_y, A_τ and A, as well. Points in $\mathcal{K}_y$ (resp. $\mathcal{K}_\tau$) are denoted by s (resp. s_0). The Haar measure on the compact group $\mathcal{K}_y$ (resp. $\mathcal{K}_\tau$) is denoted by β_y (resp. β_τ). We have $\mathcal{K} = \mathcal{K}_y \times \mathcal{K}_\tau$ (Cartesian product) and the Haar measure on $\mathcal{K}$ is precisely the product measure $\beta = \beta_y \otimes \beta_\tau$; the last equality follows in an obvious way by the density of $A_y \otimes A_\tau$ in A and by the Fubini's theorem. Points in Ω are as usual denoted by ω.

Unless otherwise stated, random variables will always be considered on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Finally, the letter E will throughout denote exclusively an ordinary sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. In what follow, we use the same notation as in the preceding section.

Definition 1. A sequence $(u_\varepsilon)_{\varepsilon>0}$ of $L^p(Q_T)$-valued random variables $(1 \leq p < \infty)$ is said to weakly Σ-converge in $L^p(Q_T \times \Omega)$ to some $L^p(Q_T; \mathcal{B}^p_{AP}(\mathbb{R}^{N+1}_{y,\tau}))$-valued random variable u_0 if as $\varepsilon \to 0$, we have

$$\int_{Q_T\times\Omega} u_\varepsilon(x,t,\omega) f\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega\right) dxdtd\mathbb{P} \to \iint_{Q_T\times\Omega\times\mathcal{K}} \widehat{u}_0(x,t,s,s_0,\omega)\widehat{f}(x,t,s,s_0,\omega)dxdtd\mathbb{P}d\beta \tag{3.1}$$

for every $f \in B(\Omega; L^{p'}(Q_T; A))$ $(1/p' = 1 - 1/p)$, where $\widehat{u}_0 = \mathcal{G}_1 \circ u_0$ and $\widehat{f} = \mathcal{G}_1 \circ (\varrho \circ f) = \mathcal{G} \circ f$. We express this by writing $u_\varepsilon \to u_0$ in $L^p(Q_T \times \Omega)$-weak Σ.

Remark 1. The above weak Σ-convergence in $L^p(Q_T \times \Omega)$ implies the weak convergence in $L^p(Q_T \times \Omega)$. One can also see from the density of $B(\Omega)$ in $L^{p'}(\Omega)$ (in the case $1 < p < \infty$) that (3.1) obviously holds for $f \in L^{p'}(\Omega; L^{p'}(Q_T; A))$. One can show as in the usual framework of Σ-convergence method [3] that each $f \in L^p(\Omega; L^p(Q_T; A))$ weakly Σ-converges to $\varrho \circ f$ (that we can identified here with its representative f).

As said in the introduction, in the case $p = 2$, our convergence method is formally motivated by the following fact: using the chaos decomposition of u_ε and f we get $u_\varepsilon(x,t,\omega) = \sum_{j=1}^\infty u_{\varepsilon,j}(x,t)\Phi_j(\omega)$ and $f(x,t,y,\tau,\omega) = \sum_{k=1}^\infty f_k(x,t,y,\tau)\Phi_k(\omega)$ where $u_{\varepsilon,j} \in L^2(Q_T)$ and $f_k \in L^2(Q_T; A)$, so that

$$\int_{Q_T\times\Omega} u_\varepsilon(x,t,\omega) f\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},\omega\right) dxdtd\mathbb{P}$$

can be formally written as

$$\sum_{j,k}\int_\Omega \Phi_j(\omega)\Phi_k(\omega)d\mathbb{P}\int_{Q_T} u_{\varepsilon,j}(x,t) f_k\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) dxdt,$$

and by the usual Σ-convergence method (see [5,3]), as $\varepsilon \to 0$,

$$\int_{Q_T} u_{\varepsilon,j}(x,t) f_k\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) dxdt \to \iint_{Q_T\times\mathcal{K}} \widehat{u}_{0,j}(x,t,s,s_0)\widehat{f}_k(x,t,s,s_0)dxdtd\beta.$$

Hence, by setting

$$\widehat{u}_0(x,t,s,s_0,\omega) = \sum_{j=1}^\infty \widehat{u}_{0,j}(x,t,s,s_0)\Phi_j(\omega);\ \widehat{f}(x,t,s,s_0,\omega) = \sum_{k=1}^\infty \widehat{f}_k(x,t,s,s_0)\Phi_k(\omega)$$

we get (3.1). This is of course what formally motivated our definition.

The following result holds.

Theorem 2. *Let $1 < p < \infty$. Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence of $L^p(Q_T)$-valued random variables verifying the following boundedness condition:*

$$\sup_{\varepsilon\in E}\mathbb{E}\|u_\varepsilon\|^p_{L^p(Q_T)} < \infty.$$

Then there exists a subsequence E' from E such that the sequence $(u_\varepsilon)_{\varepsilon\in E'}$ is weakly Σ-convergent in $L^p(Q_T \times \Omega)$.

Proof. Applying [4, Theorem 3.1] with $Y = L^{p'}(Q_T \times \Omega \times \mathcal{K})$ and $X = B(\Omega; L^{p'}(Q_T; \mathcal{C}(\mathcal{K}))) = \mathcal{G}(B(\Omega; L^{p'}(Q_T; A)))$ we are led at once to the result. □

The next result is of capital interest in the homogenization process.

Theorem 3. *Let $1 < p < \infty$. Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence of $L^p(0,T; W_0^{1,p}(Q))$-valued random variables which satisfies the following estimate:*

$$\sup_{\varepsilon\in E}\mathbb{E}\|u_\varepsilon\|^p_{L^p(0,T;W_0^{1,p}(Q))} < \infty.$$

Then there exist a subsequence E' of E, an $L^p(0,T;W_0^{1,p}(Q))$-valued random variable u_0 and an $L^p(Q_T;\mathcal{B}^p_{AP}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y)))$-valued random variable u_1 such that, as $E'\ni\varepsilon\to 0$,

$$u_\varepsilon \to u_0 \quad \text{in } L^p(Q_T\times\Omega)\text{ -weak;}$$

$$\frac{\partial u_\varepsilon}{\partial x_i}\to\frac{\partial u_0}{\partial x_i}+\frac{\overline{\partial} u_1}{\partial y_i}\quad \text{in } L^p(Q_T\times\Omega)\text{-weak }\Sigma,\quad 1\le i\le N. \tag{3.2}$$

Proof. The proof of the above theorem follows exactly the same lines of reasoning as the one of [5, Theorem 3.6]. □

The above theorem will not be used in its present form. In practice, the following modified version will be used.

Theorem 4. *Assume the hypotheses of Theorem 3 are satisfied. Assume further that $p\ge 2$ and that there exist a subsequence E' from E and a random variable u_0 with values in $L^p(0,T;W_0^{1,p}(Q))$ such that, as $E'\ni\varepsilon\to 0$,*

$$u_\varepsilon\to u_0 \quad \text{in } L^2(Q_T)\ \mathbb{P}\text{-almost surely.} \tag{3.3}$$

Then there exist a subsequence of E' not relabeled and a random variable u_1 with values in $L^p(Q_T;\mathcal{B}^p_{AP}(\mathbb{R}_\tau;\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y)))$ such that (3.2) holds as $E'\ni\varepsilon\to 0$.

Proof. Since $\sup_{\varepsilon\in E'}\mathbb{E}\|Du_\varepsilon\|^p_{L^p(Q_T)^N}<\infty$, there exist a subsequence of E' not relabeled and $v=(v_i)_i\in L^p(Q_T\times\Omega;\mathcal{B}^p_{AP}(\mathbb{R}^{N+1}_{y,\tau}))^N$ such that $\frac{\partial u_\varepsilon}{\partial x_j}\to v_j$ in $L^p(Q_T\times\Omega)$-weak Σ. Let $\Phi_\varepsilon(x,t,\omega)=\varphi(x,t)\Psi(x/\varepsilon)\chi(t/\varepsilon^2)\phi(\omega)$ $((x,t,\omega)\in Q_T\times\Omega)$ with $\varphi\in\mathcal{C}_0^\infty(Q_T)$, $\chi\in AP^\infty(\mathbb{R}_\tau)$, $\phi\in B(\Omega)$ and $\Psi=(\psi_j)_{1\le j\le N}\in(AP^\infty(\mathbb{R}^N_y))^N$ with $\overline{\mathrm{div}}_y[\varrho^N_y(\Psi)]=0$ where $\varrho^N_y(\Psi):=(\varrho_y(\psi_j))_{1\le j\le N}$, ϱ_y denoting the canonical mapping of $B^p_{AP}(\mathbb{R}^N_y)$ into $\mathcal{B}^p_{AP}(\mathbb{R}^N_y)$. Clearly

$$\sum_{j=1}^N\int_{Q_T\times\Omega}\frac{\partial u_\varepsilon}{\partial x_j}\varphi\psi_j^\varepsilon\chi^\varepsilon\phi dxdtd\mathbb{P}=-\sum_{j=1}^N\int_{Q_T\times\Omega}u_\varepsilon\psi_j^\varepsilon\frac{\partial\varphi}{\partial x_j}\chi^\varepsilon\phi dxdtd\mathbb{P}$$

where $\psi_j^\varepsilon(x)=\psi_j(x/\varepsilon)$ and $\chi^\varepsilon(t)=\chi(t/\varepsilon^2)$. One can easily see that assumption (3.3) implies the weak Σ-convergence of $(u_\varepsilon)_{\varepsilon\in E'}$ towards u_0, so that, passing to the limit in the above equation when $E'\ni\varepsilon\to 0$ yields

$$\sum_{j=1}^N\iint_{Q_T\times\Omega\times\mathcal{K}}\widehat{v}_j\varphi\widehat{\psi}_j\widehat{\chi}\phi dxdtd\mathbb{P}d\beta=-\sum_{j=1}^N\iint_{Q_T\times\Omega\times\mathcal{K}}u_0\widehat{\psi}_j\frac{\partial\varphi}{\partial x_j}\widehat{\chi}\phi dxdtd\mathbb{P}d\beta$$

or equivalently,

$$\iint_{Q_T\times\Omega\times\mathcal{K}}(\widehat{\mathbf{v}}-Du_0)\cdot\widehat{\Psi}\varphi\widehat{\chi}\phi dxdtd\mathbb{P}d\beta=0,$$

and so, as φ, ϕ and χ are arbitrarily fixed,

$$\int_{\mathcal{K}_y}(\widehat{\mathbf{v}}(x,t,s,s_0,\omega)-Du_0(x,t,\omega))\cdot\widehat{\Psi}(s)d\beta_y=0$$

for all Ψ as above and for a.e. x,t,s_0,ω. Therefore, the existence of a function $u_1(x,t,\cdot,\tau,\omega)\in\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y)$ such that

$$\mathbf{v}(x,t,\cdot,\tau,\omega)-Du_0(x,t,\omega)=\overline{D}_yu_1(x,t,\cdot,\tau,\omega)$$

for a.e. x,t,τ,ω is ensured by a well-known classical result. This yields the existence of a random variable $u_1:(x,t,\tau,\omega)\mapsto u_1(x,t,\cdot,\tau,\omega)$ with values in $\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y)$ such that $\mathbf{v}=Du_0+\overline{D}_yu_1$. □

We will also deal with the product of sequences. For that reason, we give one further.

Definition 2. A sequence $(u_\varepsilon)_{\varepsilon>0}$ of $L^p(Q_T)$-valued random variables $(1\le p<\infty)$ is said to strongly Σ-converge in $L^p(Q_T\times\Omega)$ to some $L^p(Q_T;\mathcal{B}^p_{AP}(\mathbb{R}^{N+1}_{y,\tau}))$-valued random variable u_0 if it is weakly Σ-convergent towards u_0 and further satisfies the following condition:

$$\|u_\varepsilon\|_{L^p(Q_T\times\Omega)}\to\|\widehat{u}_0\|_{L^p(Q_T\times\Omega\times\mathcal{K})}. \tag{3.4}$$

We denote this by $u_\varepsilon\to u_0$ in $L^p(Q_T\times\Omega)$-strong Σ.

Remark 2. (1) By the above definition, the uniqueness of the limit of such a sequence is ensured. (2) By Nguetseng [3] it is immediate that for any $u\in L^p(Q_T\times\Omega;AP(\mathbb{R}^{N+1}_{y,\tau}))$, the sequence $(u^\varepsilon)_{\varepsilon>0}$ is strongly Σ-convergent to $\varrho(u)$.

The next result will be very useful in the last section of this paper. Its proof is copied on the one of [41, Theorem 6]; see also [42].

Theorem 5. *Let $1 < p, q < \infty$ and $r \geq 1$ be such that $1/r = 1/p + 1/q \leq 1$. Assume $(u_\varepsilon)_{\varepsilon\in E} \subset L^q(Q_T \times \Omega)$ is weakly Σ-convergent in $L^q(Q_T \times \Omega)$ to some $u_0 \in L^q(Q_T \times \Omega; \mathcal{B}^q_{AP}(\mathbb{R}^{N+1}_{y,\tau}))$, and $(v_\varepsilon)_{\varepsilon\in E} \subset L^p(Q_T \times \Omega)$ is strongly Σ-convergent in $L^p(Q_T \times \Omega)$ to some $v_0 \in L^p(Q_T \times \Omega; \mathcal{B}^p_{AP}(\mathbb{R}^{N+1}_{y,\tau}))$. Then the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon\in E}$ is weakly Σ-convergent in $L^r(Q_T \times \Omega)$ to $u_0 v_0$.*

The following result will be of great interest in practice. It is a mere consequence of the preceding theorem.

Corollary 1. *Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(Q_T \times \Omega)$ and $(v_\varepsilon)_{\varepsilon\in E} \subset L^{p'}(Q_T \times \Omega) \cap L^\infty(Q_T \times \Omega)$ ($1 < p < \infty$ and $p' = p/(p-1)$) be two sequences such that:*

(i) *$u_\varepsilon \to u_0$ in $L^p(Q_T \times \Omega)$-weak Σ;*
(ii) *$v_\varepsilon \to v_0$ in $L^{p'}(Q_T \times \Omega)$-strong Σ;*
(iii) *$(v_\varepsilon)_{\varepsilon\in E}$ is bounded in $L^\infty(Q_T \times \Omega)$.*

Then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^p(Q_T \times \Omega)$-weak Σ.

Proof. By Theorem 5, the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon\in E}$ Σ-converges towards $u_0 v_0$ in $L^1(Q_T \times \Omega)$. Besides the same sequence is bounded in $L^p(Q_T \times \Omega)$ so that by Theorem 2, it weakly Σ-converges in $L^p(Q_T \times \Omega)$ towards some $w_0 \in L^p(Q_T \times \Omega; \mathcal{B}^p_{AP}(\mathbb{R}^{N+1}))$. This gives as a result $w_0 = u_0 v_0$. $\square$

4. Statement of the problem: a priori estimates and tightness property

4.1. Statement of the problem

Let Q be a Lipschitz domain of $\mathbb{R}^N$ and T a positive real number. By Q_T we denote the cylinder $Q \times (0, T)$. On a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined a prescribed m-dimensional standard Wiener process W whose components are one-dimensional independent, identically distributed Wiener processes. We equip $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration of W. We consider the following stochastic partial differential equation

$$\begin{cases} du_\varepsilon = \left(\operatorname{div}\left(a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) Du_\varepsilon\right) + \frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right)\right) dt + M\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right) dW \quad \text{in } Q_T \\ u_\varepsilon = 0 \quad \text{on } \partial Q \times (0, T) \\ u_\varepsilon(x, 0) = u^0(x) \in L^2(Q). \end{cases} \tag{4.1}$$

We impose on the coefficients of (4.1) the following constraints:

A1 **Uniform ellipticity**. The matrix $a(y, \tau) = (a_{ij}(y, \tau))_{1\leq i,j\leq N} \in (L^\infty(\mathbb{R}^{N+1}))^{N\times N}$ is real, not necessarily symmetric, positive definite, i.e, there exists $\Lambda > 0$ such that

$$\|a_{ij}\|_{L^\infty(\mathbb{R}^{N+1})} < \Lambda^{-1}, \quad 1 \leq i, j \leq N,$$
$$\sum_{i,j=1}^N a_{ij}(y, \tau)\zeta_i\zeta_j \geq \Lambda|\zeta|^2 \quad \text{for all } (y, \tau) \in \mathbb{R}^{N+1},\ \zeta \in \mathbb{R}^N.$$

A2 **Lipschitz continuity**. There exists $C > 0$ such that for any $(y, \tau) \in \mathbb{R}^{N+1}$ and $u \in \mathbb{R}$

$$|\partial_u g(y, \tau, u)| \leq C$$
$$|\partial_u g(y, \tau, u_1) - \partial_u g(y, \tau, u_2)| \leq C|u_1 - u_2|(1 + |u_1| + |u_2|)^{-1}.$$

A3 $g(y, \tau, 0) = 0$ for any $(y, \tau) \in \mathbb{R}^{N+1}$.

A4 **Almost periodicity**. We assume that $g(\cdot, \cdot, u) \in AP(\mathbb{R}^{N+1}_{y,\tau})$ for any $u \in \mathbb{R}$ with $\mathfrak{M}_y(g(\cdot, \tau, u)) = 0$ for all $(\tau, u) \in \mathbb{R}^2$. We see by (2.1) (see Section 2) that there exists a unique $R(\cdot, \cdot, u) \in AP(\mathbb{R}^{N+1}_{y,\tau})$ such that $\Delta_y R(\cdot, \cdot, u) = g(\cdot, \cdot, u)$ and $\mathfrak{M}_y(R(\cdot, \tau, u)) = 0$ for all $\tau, u \in \mathbb{R}$. Moreover $R(\cdot, \cdot, u)$ is at least twice differentiable with respect to y. Let $G(y, \tau, u) = D_y R(y, \tau, u)$. Thanks to A2 and A3 we see that

$$|G(y, \tau, u)| \leq C|u|, \qquad |\partial_u G(y, \tau, u)| \leq C, \tag{4.2}$$

$$|\partial_u G(y, \tau, u_1) - \partial_u G(y, \tau, u_2)| \leq C|u_1 - u_2|(1 + |u_1| + |u_2|)^{-1}. \tag{4.3}$$

We assume that the functions a_{ij} lie in $B^2_{AP}(\mathbb{R}^{N+1}_{y,\tau})$ for all $1 \leq i, j \leq N$. We also assume that the function $(y, \tau) \mapsto M_i(y, \tau, u)$ lies in $B^2_{AP}(\mathbb{R}^{N+1}_{y,\tau}) \cap L^\infty(\mathbb{R}^{N+1}_{y,\tau})$ for all $u \in \mathbb{R}$ and $1 \leq i \leq m$, where $M(y, \tau, u) = (M_i(y, \tau, u))_{1\leq i\leq m}$ satisfies the following hypothesis.

A5 There is a positive constant K such that

$$\sum_{i=1}^{m} |M_i(y,\tau,0)|^2 \leq K,$$
$$|M_i(y,\tau,u_1) - M_i(y,\tau,u_2)| \leq K|u_1-u_2|, \quad 1 \leq i \leq m$$

for any $(y,\tau) \in \mathbb{R}^{N+1}$ and $u_1, u_2 \in \mathbb{R}$. We easily infer from the above inequalities that

$$\sum_{i=1}^{m} |M_i(y,\tau,u)|^2 \leq K(1+|u|^2) \quad \text{for any } u \in \mathbb{R} \text{ and } (y,\tau) \in \mathbb{R}^{N+1}.$$

We recall that the stochastic term $M(x/\varepsilon, t/\varepsilon^2, u_\varepsilon)dW$ is defined as the scalar product

$$\sum_{k=1}^{m} M_k(x/\varepsilon, t/\varepsilon^2, u_\varepsilon)dW_k$$

where W_k is a one-dimensional Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In order to simplify our presentation, we need to make some notations that will be used in the sequel. We denote by $L^2(Q)$ and $H^1(Q)$ the usual Lebesgue space and Sobolev space, respectively. By (u, v) we denote the inner product in $L^2(Q)$, and by $x \cdot y$, the inner product in $\mathbb{R}^N$. Its associated norm is denoted by $|\cdot|$. The space of elements of $H^1(Q)$ whose trace vanishes on ∂Q is denoted by $H_0^1(Q)$. Thanks to Poincaré's inequality we can endow $H_0^1(Q)$ with the inner product $((u,v)) = \int_Q Du \cdot Dv dx$ whose associated norm $\|u\|$ is equivalent to the usual H^1-norm for any $u \in H_0^1(Q)$. The duality pairing between $H_0^1(Q)$ and $H^{-1}(Q)$ is denoted by $\langle \cdot, \cdot \rangle$.

Let X be a Banach space, by $L^p(0,T;X)$ we mean the space of measurable functions $\phi : [0,T] \to X$ such that

$$\begin{cases} \left(\int_0^T \|\phi(t)\|_X^p \right)^{1/p} < \infty & \text{if } 1 \leq 1 < \infty, \\ \operatorname{ess}\sup_{t\in[0,T]} \|\phi(t)\|_X < \infty & \text{if } p = \infty. \end{cases}$$

Similarly we can define the space $L^p(\Omega; X)$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

From the work of [43] for example (see also [29]), the existence and uniqueness of solution u_ε of (4.1) which is subjected to conditions A1–A5 are very well-known.

Theorem 6 *([43]). For any fixed $\varepsilon > 0$, there exists an $\mathcal{F}^t$-progressively measurable process $u_\varepsilon \in L^2(\Omega \times [0,T]; H_0^1(Q))$ such that*

$$(u_\varepsilon(t), v) = (u^0, v) - \int_0^t \int_Q a\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}\right) Du_\varepsilon \cdot Dv dx d\tau + \frac{1}{\varepsilon} \int_0^t \int_Q g\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}, u_\varepsilon\right) v dx d\tau$$
$$+ \int_0^t \int_Q M\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}, u_\varepsilon\right) v dx dW \tag{4.4}$$

for any $v \in H_0^1(Q)$ and for almost all $(\omega, t) \in \Omega \times [0,T]$. Such a process is unique in the following sense:

$$\mathbb{P}(\omega : u_\varepsilon(t) = \overline{u}_\varepsilon(t) \text{ in } H^{-1}(Q)\ \forall t \in [0,T]) = 1$$

for any u_ε and $\overline{u}_\varepsilon$ satisfying (4.4).

4.2. A priori estimates and tightness property of u_ε

We begin this section by obtaining crucial uniform a priori energy estimates for the process u_ε.

Lemma 3. *Under assumptions A1–A5 the following estimates hold true for $1 \leq p < \infty$:*

$$\mathbb{E} \sup_{0\leq t\leq T} |u_\varepsilon(t)|^p \leq C, \tag{4.5}$$

$$\mathbb{E} \left(\int_0^T \|u_\varepsilon(t)\|^2 dt \right)^{p/2} \leq C \tag{4.6}$$

where C is a positive constant which does not depend on ε.

Proof. In what follows, we use the notations:

$$\int_Q M\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}, u_\varepsilon\right) v dx \equiv (M^\varepsilon(u_\varepsilon), v),$$
$$\int_Q a\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}\right) Du_\varepsilon \cdot Dv dx \equiv \left(a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) Du_\varepsilon(t), Dv\right) \quad \text{and}$$
$$\int_Q g\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}, u_\varepsilon\right) v dx \equiv \left(g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon(t)\right), v\right).$$

Then, thanks to [43] or [29] we have that $u_\varepsilon \in \mathcal{C}(0, T; L^2(Q))$ almost surely and $u_\varepsilon \in L^2(\Omega \times [0, T]; H_0^1(Q))$, so that we may apply Itô's formula to $|u_\varepsilon(t)|^2$ and get

$$d|u_\varepsilon(t)|^2 = -2\left(a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) Du_\varepsilon(t), Du_\varepsilon(t)\right) dt + \frac{2}{\varepsilon}\left(g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon(t)\right), u_\varepsilon(t)\right) dt + \sum_{k=1}^m |M_k^\varepsilon(u_\varepsilon(t))|^2 dt + 2(M^\varepsilon(u_\varepsilon(t)), u_\varepsilon(t))dW \tag{4.7}$$

where we also set $M^\varepsilon(u_\varepsilon)(x, t, \omega) = M(x/\varepsilon, t/\varepsilon^2, u_\varepsilon(x, t, \omega))$. Thanks to condition A1 we have

$$d|u_\varepsilon(t)|^2 + 2\Lambda\|u_\varepsilon(t)\|^2 dt \le \frac{2}{\varepsilon}\left(g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon(t)\right), u_\varepsilon(t)\right) dt + \sum_{k=1}^m |M_k^\varepsilon(u_\varepsilon(t))|^2 dt + 2(M^\varepsilon(u_\varepsilon(t)), u_\varepsilon(t))dW. \tag{4.8}$$

To deal with the first term of the right hand side of (4.8), we use the following representation

$$\frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right) = \operatorname{div} G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right) - \partial_u G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right) \cdot Du_\varepsilon \tag{4.9}$$

which can be checked by straightforward computation. From this we see that

$$\frac{1}{\varepsilon}\left(g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon(t)\right), u_\varepsilon(t)\right) = \left(G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right), Du_\varepsilon(t)\right) - \left(\partial_u G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon(t)\right) \cdot Du_\varepsilon(t), u_\varepsilon(t)\right),$$

from which we infer that

$$\frac{2}{\varepsilon}\left(g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon(t)\right), u_\varepsilon(t)\right) \le C|u_\varepsilon(t)|\|u_\varepsilon(t)\| + C|u_\varepsilon(t)|\|u_\varepsilon(t)\|. \tag{4.10}$$

Here we have used the assumptions A2–A4. Thanks to A5 the second term of the right hand side of (4.8) can be estimated as

$$\sum_{k=1}^m |M_k^\varepsilon(u_\varepsilon(t))|^2 \le C(1 + |u_\varepsilon(t)|^2). \tag{4.11}$$

Using (4.10) and (4.11) in (4.8) and integrating over $0 \le \tau \le t$ both sides of the resulting inequality yields

$$|u_\varepsilon(t)|^2 + 2\Lambda \int_0^t \|u_\varepsilon(\tau)\|^2 d\tau \le |u^0|^2 + C\int_0^t |u_\varepsilon(\tau)|\|u_\varepsilon(\tau)\| d\tau + C(T) + C\int_0^t |u_\varepsilon(\tau)|^2 d\tau + 2\int_0^t (M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW. \tag{4.12}$$

By Cauchy's inequality we have

$$|u_\varepsilon(t)|^2 + 2\Lambda \int_0^t \|u_\varepsilon(\tau)\|^2 d\tau \le |u^0|^2 + C(\delta)\int_0^t |u_\varepsilon(\tau)|^2 d\tau + \delta\int_0^t \|u_\varepsilon(\tau)\|^2 d\tau + C(T) + C\int_0^t |u_\varepsilon(\tau)|^2 d\tau + 2\int_0^t (M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW,$$

where δ is an arbitrary positive constant. We choose $\delta = \Lambda$ so that we see from (4.12) that

$$|u_\varepsilon(t)|^2 + \Lambda \int_0^t \|u_\varepsilon(\tau)\|^2 d\tau \le |u^0|^2 + C(T) + C\int_0^t |u_\varepsilon(\tau)|^2 d\tau + 2\int_0^t (M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW. \tag{4.13}$$

In (4.13) we take the sup over $0 \le \tau \le t$ and the mathematical expectation. This procedure implies that

$$\mathbb{E}\sup_{0\le\tau\le t}|u_\varepsilon(\tau)|^2 + \Lambda\mathbb{E}\int_0^t \|u_\varepsilon(\tau)\|^2 d\tau \le |u^0|^2 + C(T) + C\mathbb{E}\int_0^t |u_\varepsilon(\tau)|^2 d\tau + 2\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s (M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW\right|.$$

By Burkhölder–Davis–Gundy's inequality we have that

$$2\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s (M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW\right| \le 6\mathbb{E}\left(\int_0^t (M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))^2 d\tau\right)^{1/2} \le 6\mathbb{E}\left(\sup_{0\le s\le t}|u_\varepsilon(s)|\left(\int_0^t |M^\varepsilon(u_\varepsilon(\tau))|^2 d\tau\right)^{1/2}\right).$$

By Cauchy's inequality,

$$2\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s (M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW\right| \le \frac{1}{2}\mathbb{E}\sup_{0\le s\le t}|u_\varepsilon(s)|^2 + 18\mathbb{E}\int_0^t |M^\varepsilon(u_\varepsilon(\tau))|^2 d\tau.$$

By using condition A5 we see from this last inequality that

$$2\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s (M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW\right| \le \frac{1}{2}\mathbb{E}\sup_{0\le s\le t}|u_\varepsilon(s)|^2 + C(T) + C\mathbb{E}\int_0^t |u_\varepsilon(\tau)|^2 d\tau.$$

From this and (4.13) we derive that

$$\mathbb{E}\sup_{0\le\tau\le t}|u_\varepsilon(\tau)|^2 + \Lambda\mathbb{E}\int_0^t \|u_\varepsilon(\tau)\|^2 d\tau \le C(|u^0|^2, T) + C\mathbb{E}\int_0^t |u_\varepsilon(\tau)|^2 d\tau. \tag{4.14}$$

Now it follows from Gronwall's inequality that

$$\mathbb{E}\sup_{0\le t\le T}|u_\varepsilon(t)|^2 \le C, \tag{4.15}$$

where $C > 0$ is independent of ε. Thanks to this last estimate we derive from (4.14) that

$$\mathbb{E}\int_0^T \|u_\varepsilon(\tau)\|^2 d\tau \le C. \tag{4.16}$$

As above $C > 0$ does not depend on ε. Now let $p > 2$. Thanks to Itô's formula we derive from (4.7) that

$$\begin{aligned} d|u_\varepsilon(t)|^p = {} & -p\left(a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)Du_\varepsilon(t), Du_\varepsilon(t)\right)|u_\varepsilon(t)|^{p-2}dt + \frac{p}{\varepsilon}\left(g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon(t)\right), u_\varepsilon(t)\right)|u_\varepsilon(t)|^{p-2}dt \\ & + \frac{p}{2}|u_\varepsilon(t)|^{p-2}\sum_{k=1}^m |M_k^\varepsilon(u_\varepsilon(t))|^2 dt + \frac{p(p-2)}{2}|u_\varepsilon(t)|^{p-4}(M^\varepsilon(u_\varepsilon(t)), u_\varepsilon(t))^2 dt \\ & + p|u_\varepsilon(t)|^{p-2}(M^\varepsilon(u_\varepsilon(t)), u_\varepsilon(t))dW. \end{aligned}$$

Thanks to A1, (4.9)–(4.11) we have that

$$\begin{aligned} d|u_\varepsilon(t)|^p + p\Lambda|u_\varepsilon(t)|^{p-2}\|u_\varepsilon(t)\|^2 dt \le {} & pC|u_\varepsilon(t)|^{p-1}\|u_\varepsilon(t)\|dt + \frac{p}{2}C|u_\varepsilon(t)|^{p-2}(1 + |u_\varepsilon(t)|^2)dt \\ & + \frac{p(p-2)}{4}|u_\varepsilon(t)|^{p-4}(M^\varepsilon(u_\varepsilon(t)), u_\varepsilon(t))^2 dt \\ & + p|u_\varepsilon(t)|^{p-2}(M^\varepsilon(u_\varepsilon(t)), u_\varepsilon(t))dW. \end{aligned} \tag{4.17}$$

Thanks to A5 we get form easy calculations that

$$|u_\varepsilon(t)|^{p-4}(M^\varepsilon(u_\varepsilon(t)), u_\varepsilon(t))^2 \le C(p)|u_\varepsilon(t)|^p. \tag{4.18}$$

Using (4.18) in (4.17) yields

$$\begin{aligned} d|u_\varepsilon(t)|^p + p\Lambda|u_\varepsilon(t)|^{p-2}\|u_\varepsilon(t)\|^2 dt \le {} & pC|u_\varepsilon(t)|^{p-1}\|u_\varepsilon(t)\|dt + C(p)|u_\varepsilon(t)|^p dt \\ & + p|u_\varepsilon(t)|^{p-2}(M^\varepsilon(u_\varepsilon(t)), u_\varepsilon(t))dW, \end{aligned} \tag{4.19}$$

which is equivalent to

$$|u_\varepsilon(t)|^p + p\Lambda \int_0^t |u_\varepsilon(\tau)|^{p-2}\|u_\varepsilon(\tau)\|^2 d\tau \le p\int_0^t |u_\varepsilon(\tau)|^{p-2}(M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW + |u^0|^p + C(p)\int_0^t |u_\varepsilon(\tau)|^p d\tau + C(p)\int_0^t |u_\varepsilon(\tau)|^{p-1}\|u_\varepsilon(\tau)\| d\tau. \tag{4.20}$$

Due to Cauchy's inequality the second term of the right hand side of (4.20) can be estimated as follows

$$C(p)\int_0^t |u_\varepsilon(\tau)|^{p-1}\|u_\varepsilon(\tau)\|d\tau \le C(p,\delta)\int_0^t |u_\varepsilon(\tau)|^p d\tau + \delta\int_0^t |u_\varepsilon(\tau)|^{p-2}\|u_\varepsilon(\tau)\|^2 d\tau,$$

where $\delta > 0$ is arbitrary. Choosing $\delta = p\Lambda/2$ in the last inequality and using the resulting estimate in (4.20) implies that

$$|u_\varepsilon(t)|^p + (p\Lambda/2)\int_0^t |u_\varepsilon(\tau)|^{p-2}\|u_\varepsilon(\tau)\|^2 d\tau \le |u^0|^p + C(p,\Lambda)\int_0^t |u_\varepsilon(\tau)|^p d\tau + p\int_0^t |u_\varepsilon(\tau)|^{p-2}(M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW.$$

Taking the supremum over $0 \le \tau \le t$ and the mathematical expectation to both sides of this last inequality yields

$$\begin{aligned}&\mathbb{E}\sup_{0\le\tau\le t}|u_\varepsilon(\tau)|^p + (p\Lambda/2)\mathbb{E}\int_0^t |u_\varepsilon(\tau)|^{p-2}\|u_\varepsilon(\tau)\|^2 d\tau \\ &\le |u^0|^p + C(p,\Lambda)\mathbb{E}\int_0^t |u_\varepsilon(\tau)|^p d\tau + p\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s |u_\varepsilon(\tau)|^{p-2}(M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW\right|.\end{aligned} \tag{4.21}$$

Thanks to Burkhölder–Davis–Gundy's inequality we have that

$$\begin{aligned}p\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s |u_\varepsilon(\tau)|^{p-2}(M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW\right| &\le 3p\mathbb{E}\left(\int_0^t |u_\varepsilon(\tau)|^{2p-4}(M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))^2 d\tau\right)^{1/2} \\ &\le 3p\mathbb{E}\left(\sup_{0\le\tau\le t}|u_\varepsilon(\tau)|^{p/2}\int_0^t |u_\varepsilon(\tau)|^{p-2}|M^\varepsilon(u_\varepsilon(\tau))|^2 d\tau\right)^{1/2}\end{aligned}$$

Thanks to Cauchy's inequality and the assumption A5 we get that

$$\begin{aligned}&p\mathbb{E}\sup_{0\le s\le t}\left|\int_0^s |u_\varepsilon(\tau)|^{p-2}M^\varepsilon(u_\varepsilon(\tau), u_\varepsilon(\tau))dW\right| \\ &\quad \le 3p\delta\mathbb{E}\sup_{0\le\tau\le t}|u_\varepsilon(\tau)|^p + C(p,\delta,T) + C(p,\delta,T)\mathbb{E}\int_0^t |u_\varepsilon(\tau)|^p d\tau,\end{aligned} \tag{4.22}$$

where $\delta > 0$ is arbitrary. Using (4.22) in (4.21) yields

$$\begin{aligned}&\mathbb{E}\sup_{0\le\tau\le t}|u_\varepsilon(\tau)|^p + (p\Lambda/2)\mathbb{E}\int_0^t |u_\varepsilon(\tau)|^{p-2}\|u_\varepsilon(\tau)\|^2 d\tau \\ &\quad \le C(p,\Lambda,\delta,T)\mathbb{E}\int_0^t |u_\varepsilon(\tau)|^p d\tau + |u^0|^p + C(\delta,T,p) + 3p\delta\mathbb{E}\sup_{0\le\tau\le t}|u_\varepsilon(\tau)|^p.\end{aligned}$$

It follows from this and by taking $\delta = 1/6p$ that

$$\mathbb{E}\sup_{0\le\tau\le t}|u_\varepsilon(\tau)|^p + p\Lambda\mathbb{E}\int_0^t |u_\varepsilon(\tau)|^{p-2}\|u_\varepsilon(\tau)\|^2 d\tau \le |u^0|^p + C(\delta,T,p) + C(p,\Lambda,\delta,T)\mathbb{E}\int_0^t |u_\varepsilon(\tau)|^p d\tau.$$

Gronwall's Lemma implies that

$$\mathbb{E}\sup_{0\le\tau\le T}|u_\varepsilon(\tau)|^p \le C, \tag{4.23}$$

where $C > 0$ is independent of ε. From (4.13) we see that

$$\int_0^t \|u_\varepsilon(\tau)\|^2 d\tau \le C(|u^0|^2, T, \Lambda) + C(\Lambda)\int_0^T |u_\varepsilon(\tau)|^2 d\tau + C(\Lambda)\int_0^t (M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW.$$

Raising both sides of this inequality to the power $p/2$ and taking the mathematical expectation imply that

$$\mathbb{E}\left(\int_0^t \|u_\varepsilon(\tau)\|^2\right)^{p/2} d\tau \le C(\Lambda, p)\mathbb{E}\left(\int_0^t (M^\varepsilon(u_\varepsilon(\tau)), u_\varepsilon(\tau))dW\right)^{p/2} + C(|u^0|^2, T, \Lambda, p).$$

Here we have used (4.23) to deal with the term $C(\Lambda, p, T)\mathbb{E}\sup_{0\le t\le T}|u_\varepsilon(t)|^p$. It follows from martingale inequality and some straightforward computations that

$$\mathbb{E}\left(\int_0^T \|u_\varepsilon(t)\|^2 dt\right)^{p/2} \le C. \tag{4.24}$$

The estimates (4.15), (4.16), (4.23) and (4.24) complete the proof of the lemma. □

Lemma 4. *There exists a constant $C > 0$ such that*

$$\mathbb{E}\sup_{|\theta|\le\delta}\int_0^T |u_\varepsilon(t+\theta) - u_\varepsilon(t)|^2_{H^{-1}(Q)} dt \le C\delta,$$

for any ε, and $\delta \in (0, 1)$. Here $u_\varepsilon(t)$ is extended to zero outside the interval $[0, T]$.

Proof. Let $\theta > 0$. We have that

$$u_\varepsilon(t+\theta) - u_\varepsilon(t) = \int_t^{t+\theta} \operatorname{div}\left(a\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}\right)Du_\varepsilon(\tau)\right)d\tau + \frac{1}{\varepsilon}\int_t^{t+\theta} g\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}, u_\varepsilon\right)d\tau + \int_t^{t+\theta} M^\varepsilon(u_\varepsilon(\tau))dW,$$

as an equality of random variables taking values in $H^{-1}(Q)$. It follows from this that

$$\begin{aligned} |u_\varepsilon(t+\theta) - u_\varepsilon(t)|_{H^{-1}(Q)} &\le C\theta\int_t^{t+\theta}\left|\operatorname{div}\left(a\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}\right)Du_\varepsilon(\tau)\right)\right|^2_{H^{-1}(Q)} d\tau \\ &\quad + C\theta\int_t^{t+\theta}\left|\frac{1}{\varepsilon}g\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}, u_\varepsilon(\tau)\right)\right|^2_{H^{-1}(Q)} d\tau + \left|\int_t^{t+\theta} M^\varepsilon(u_\varepsilon(\tau))dW\right|^2. \end{aligned} \tag{4.25}$$

Firstly,

$$\begin{aligned} \left|\operatorname{div}\left(a\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}\right)Du_\varepsilon(\tau)\right)\right|_{H^{-1}(Q)} &= \sup_{\substack{\phi\in H_0^1(Q)\\ \|\phi\|=1}}\left|\left\langle \operatorname{div}\left(a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)Du_\varepsilon\right), \phi\right\rangle\right| \\ &= \sup_{\substack{\phi\in H_0^1(Q)\\ \|\phi\|=1}}\left|\int_Q a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)Du_\varepsilon D\phi dx\right| \end{aligned}$$

from which we derive that

$$\left|\operatorname{div}\left(a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)Du_\varepsilon\right)\right|^2_{H^{-1}(Q)} \le C(\Lambda)\|u_\varepsilon\|^2, \tag{4.26}$$

where the assumption A1 was used. Secondly,

$$\left|\frac{1}{\varepsilon}g\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}, u_\varepsilon\right)\right|_{H^{-1}(Q)} = \sup_{\substack{\phi\in H_0^1(Q)\\ \|\phi\|=1}}\left|\int_Q G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right)\cdot D\phi dx + \int_Q\left(\partial_u G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right)\cdot Du_\varepsilon\right)\phi dx\right|.$$

By using the conditions in A4 and Poincaré's inequality we get that

$$\left|\frac{1}{\varepsilon}g\left(\frac{x}{\varepsilon}, \frac{\tau}{\varepsilon^2}, u_\varepsilon\right)\right|_{H^{-1}(Q)} \le \sup_{\phi\in H_0^1(Q), \|\phi\|=1}(C|u_\varepsilon| + C\|u_\varepsilon\||\phi|) \le C|u_\varepsilon| + C\|u_\varepsilon\| \tag{4.27}$$

Using (4.26) and (4.27) in (4.25) yields

$$|u_\varepsilon(t+\theta) - u_\varepsilon(t)|^2_{H^{-1}(Q)} \le C\theta\int_t^{t+\theta}\|u_\varepsilon(\tau)\|^2 d\tau + C\theta\int_t^{t+\theta}|u_\varepsilon(\tau)|^2 d\tau + \left|\int_t^{t+\theta} M^\varepsilon(u_\varepsilon(\tau))dW\right|^2,$$

which implies that

$$\mathbb{E}\int_0^T \sup_{0\le\theta\le\delta} |u_\varepsilon(t+\theta)-u_\varepsilon(t)|^2_{H^{-1}(Q)}dt \le C\delta\mathbb{E}\int_0^T\int_t^{t+\delta}\|u_\varepsilon(\tau)\|^2 d\tau dt + \mathbb{E}\int_0^T \sup_{0\le\theta\le\delta}\left|\int_t^{t+\theta} M^\varepsilon(u_\varepsilon(\tau))dW\right|^2 dt + C\delta\mathbb{E}\int_0^T\int_t^{t+\theta}|u_\varepsilon(\tau)|^2 d\tau dt.$$

Thanks to Lemma 3 we have that

$$\mathbb{E}\int_0^T \sup_{0\le\theta\le\delta} |u_\varepsilon(t+\theta)-u_\varepsilon(t)|^2_{H^{-1}(Q)}dt \le C\delta + \mathbb{E}\int_0^T \sup_{0\le\theta\le\delta}\left|\int_t^{t+\theta} M^\varepsilon(u_\varepsilon(\tau))dW\right|^2 dt.$$

Due to Fubini's theorem and Burkhölder–Davis–Gundy's inequality we see from this last estimate that

$$\mathbb{E}\int_0^T \sup_{0\le\theta\le\delta} |u_\varepsilon(t+\theta)-u_\varepsilon(t)|^2_{H^{-1}(Q)}dt \le C\delta + \mathbb{E}\int_0^T\int_t^{t+\delta} |M^\varepsilon(u_\varepsilon(\tau))|^2 d\tau dt.$$

Assumptions A5 and Lemma 3 yields that

$$\mathbb{E}\int_0^T \sup_{0\le\theta\le\delta} |u_\varepsilon(t+\theta)-u_\varepsilon(t)|^2_{H^{-1}(Q)}dt \le C\delta,$$

where $C > 0$ does not depend on ε and δ. By the same argument, we can show that a similar inequality holds for negative values of θ. This completes the proof of the lemma. □

The following compactness result plays a crucial role in the proof of the tightness of the probability measures generated by the sequence $(u_\varepsilon)_\varepsilon$.

Lemma 5. *Let μ_n, ν_n two sequences of positive real numbers which tend to zero as $n\to\infty$, the injection of*

$$D_{\nu_n,\mu_n} := \left\{ q\in L^\infty(0,T;L^2(Q))\cap L^2(0,T;H_0^1(Q)) : \sup_n \frac{1}{\nu_n}\sup_{|\theta|\le\mu_n}\left(\int_0^T |q(t+\theta)-q(t)|^2_{H^{-1}(Q)}\right)^{1/2} < \infty \right\}$$

in $L^2(Q_T)$ is compact.

The proof, which is similar to the analogous result in [44], follows from the application of Lemmas 3, 4. The space D_{ν_n,μ_n} is a Banach space with the norm

$$\|q\|_{D_{\nu_n,\mu_n}} = \operatorname*{ess\,sup}_{0\le t\le T}|q(t)| + \left(\int_0^T \|q\|^2 dt\right)^{1/2} + \sup_n \frac{1}{\nu_n}\sup_{|\theta|\le\mu_n}\left(\int_0^T |q(t+\theta)-q(t)|^2_{H^{-1}(Q)}\right)^{1/2}.$$

Alongside D_{ν_n,μ_n}, we also consider the space X_{p,ν_n,μ_n}, $1\le p<\infty$, of random variables ζ endowed with the norm

$$\mathbb{E}\|\zeta\|_{X_{p,\nu_n,\mu_n}} = \left(\mathbb{E}\operatorname*{ess\,sup}_{0\le t\le T}|\zeta(t)|^p\right)^{1/p} + \left(\mathbb{E}\left(\int_0^T \|\zeta(t)\|^2\right)^{p/2}\right)^{2/p} + \mathbb{E}\sup_n\frac{1}{\nu_n}\sup_{|\theta|\le\mu_n}\left(\int_0^T|\zeta(t+\theta)-\zeta(t)|^2_{H^{-1}}\right)^{1/2};$$

X_{p,ν_n,μ_n} is a Banach space.

Combining Lemma 3 and the estimates in Lemma 4 we have

Proposition 4. *For any real number $p\in[1,\infty)$ and for any sequences ν_n, μ_n converging to 0 such that the series $\sum_n \frac{\sqrt{\mu_n}}{\nu_n}$ converges, the sequence $(u_\varepsilon)_\varepsilon$ is bounded uniformly in ε in X_{p,ν_n,μ_n} for all n.*

Next we consider the space $\mathfrak{S} = \mathcal{C}(0,T;\mathbb{R}^m)\times L^2(Q_T)$ equipped with the Borel σ-algebra $\mathcal{B}(\mathfrak{S})$. For $0<\varepsilon<1$, let Φ_ε be the measurable $\mathfrak{S}$-valued mapping defined on $(\Omega,\mathcal{F},\mathbb{P})$ by

$$\Phi_\varepsilon(\omega) = (W(\omega), u_\varepsilon(\omega)).$$

For each ε we introduce a probability measure Π^ε on $(\mathfrak{S};\mathcal{B}(\mathfrak{S}))$ defined by

$$\Pi^\varepsilon(S) = \mathbb{P}(\Phi_\varepsilon^{-1}(S)), \quad \text{for any } S\in\mathcal{B}(\mathfrak{S}).$$

Theorem 7. *The family of probability measures $\{\Pi^\varepsilon : 0<\varepsilon<1\}$ is tight in $(\mathfrak{S};\mathcal{B}(\mathfrak{S}))$.*

Proof. For $\delta > 0$ we should find compact subsets

$$\Sigma_\delta \subset \mathcal{C}(0,T;\mathbb{R}^m);\ Y_\delta \subset L^2(Q_T),$$

such that

$$\mathbb{P}(\omega : W(\cdot,\omega) \notin \Sigma_\delta) \le \frac{\delta}{2}, \tag{4.28}$$

$$\mathbb{P}(\omega : u_\varepsilon(\cdot,\omega) \notin Y_\delta) \le \frac{\delta}{2}, \tag{4.29}$$

for all ε.

The quest for Σ_δ is made by taking into account some facts about Wiener process such as the formula

$$\mathbb{E}|W(t) - W(s)|^{2j} = (2j-1)!(t-s)^j, \quad j = 1,2,\dots. \tag{4.30}$$

For a constant $L_\delta > 0$ depending on δ to be fixed later and $n \in \mathbb{N}$, we consider the set

$$\Sigma_\delta = \left\{ W(\cdot) \in \mathcal{C}(0,T;\mathbb{R}^m) : \sup_{\substack{t,s\in[0,T] \\ |t-s|<\frac{1}{n^6}}} n|W(s) - W(t)| \le L_\delta \right\}.$$

The set Σ_δ is relatively compact in $\mathcal{C}(0,T;\mathbb{R}^m)$ by Ascoli–Arzela's theorem. Furthermore Σ_δ is closed in $\mathcal{C}(0,T;\mathbb{R}^m)$, therefore it is compact in $\mathcal{C}(0,T;\mathbb{R}^m)$. Making use of Markov's inequality

$$\mathbb{P}(\omega; \zeta(\omega) \ge \beta) \le \frac{1}{\beta^k}\mathbb{E}[|\zeta(\omega)|^k],$$

for any random variable ζ and real numbers k we get

$$\begin{aligned}
\mathbb{P}\left(\omega : W(\omega) \notin \Sigma_\delta\right) &\le \mathbb{P}\left[\cup_n \left\{ \omega : \sup_{\substack{t,s\in[0,T] \\ |t-s|<\frac{1}{n^6}}} |W(s) - W(t)| \ge \frac{L_\delta}{n} \right\}\right], \\
&\le \sum_{n=1}^{\infty}\sum_{i=0}^{n^6-1}\left(\frac{n}{L_\delta}\right)^4 \mathbb{E} \sup_{\frac{iT}{n^6} \le t \le \frac{(i+1)T}{n^6}} |W(t) - W(iTn)^{-6}|^4, \\
&\le C\sum_{n=1}^{\infty}\sum_{i=0}^{n^6-1}\left(\frac{n}{L_\delta}\right)^4 (Tn^{-6})^2 n^6 = \frac{C}{L_\delta^4}\sum_{n=1}^{\infty}\frac{1}{n^2},
\end{aligned}$$

where we have used (4.30). Since the right hand side of (4.30) is independent of ε, then so is the constant C in the above estimate. We take $L_\delta^4 = \frac{1}{2C\varepsilon}(\sum_{n=1}^{\infty}\frac{1}{n^2})^{-1}$ and get (4.28).

Next we choose Y_δ as a ball of radius M_δ in D_{ν_n,μ_m} centered at 0 and with ν_n, μ_n independent of δ, converging to 0 and such that the series $\sum_n \frac{\sqrt{\mu_n}}{\nu_n}$ converges, from Lemma 5, Y_δ is a compact subset of $L^2(Q_T)$. Furthermore, we have

$$\begin{aligned}
\mathbb{P}(\omega : u_\varepsilon(\omega) \notin Y_\delta) &\le \mathbb{P}(\omega : \|u_\varepsilon\|_{D_{\nu_n,\mu_m}} > M_\delta) \\
&\le \frac{1}{M_\delta}(\mathbb{E}\|u_\varepsilon\|_{D_{\nu_n,\mu_m}}) \\
&\le \frac{1}{M_\delta}(\mathbb{E}\|u_\varepsilon\|_{X_{1,\nu_n,\mu_n}}) \\
&\le \frac{C}{M_\delta}
\end{aligned}$$

where $C > 0$ is independent of ε (see Proposition 4 for the justification.)

Choosing $M_\delta = 2C\delta^{-1}$, we get (4.29). From the inequalities (4.28)–(4.29) we deduce that

$$\mathbb{P}(\omega : W(\omega) \in \Sigma_\delta; u_\varepsilon(\omega) \in Y_\delta) \ge 1 - \delta,$$

for all $0 < \varepsilon \le 1$. This proves that for all $0 < \varepsilon \le 1$

$$\Pi^\varepsilon(\Sigma_\delta \times Y_\delta) \ge 1 - \delta,$$

from which we deduce the tightness of $\{\Pi^\varepsilon : 0 < \varepsilon \le 1\}$ in $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$. $\square$

Prokhorov's compactness result enables us to extract from (Π^{ε}) a subsequence (Π^{ε_j}) such that

Π^{ε_j} weakly converges to a probability measure Π on $\mathfrak{S}$.

Skorokhod's theorem ensures the existence of a complete probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and random variables $(W^{\varepsilon_j}, v_{\varepsilon_j})$ and $(\bar{W}, u_0)$ defined on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with values in $\mathfrak{S}$ such that

$$\text{The probability law of } (W^{\varepsilon_j}, v_{\varepsilon_j}) \text{ is } \Pi^{\varepsilon_j}, \tag{4.31}$$

$$\text{The probability law of } (\bar{W}, u_0) \text{ is } \Pi, \tag{4.32}$$

$$W^{\varepsilon_j} \to \bar{W} \quad \text{in } \mathcal{C}(0, T; \mathbb{R}^m)\ \bar{\mathbb{P}}\text{-a.s.}, \tag{4.33}$$

$$v_{\varepsilon_j} \to u_0 \quad \text{in } L^2(Q_T)\ \bar{\mathbb{P}}\text{-a.s.}. \tag{4.34}$$

We can see that $\{W^{\varepsilon_j} : \varepsilon_j\}$ is a sequence of m-dimensional standard Brownian Motions. We let $\bar{\mathcal{F}}^t$ be the σ-algebra generated by $(\bar{W}(s), u_0(s))$, $0 \leq s \leq t$ and the null sets of $\bar{\mathcal{F}}$. We can show by arguing as in [44] (see also [45–47]) that $\bar{W}$ is an $\bar{\mathcal{F}}^t$-adapted standard $\mathbb{R}^m$-valued Wiener process. By the same argument as in [48,45–47] we can show that

$$v_{\varepsilon_j}(t) = u^0 + \int_0^t \operatorname{div}\left(a\left(\frac{x}{\varepsilon_j}, \frac{\tau}{\varepsilon_j^2}\right) Dv_{\varepsilon_j}(\tau)\right) d\tau + \frac{1}{\varepsilon_j}\int_0^t g\left(\frac{x}{\varepsilon_j}, \frac{\tau}{\varepsilon_j^2}, v_{\varepsilon_j}\right) d\tau + \int_0^t M^{\varepsilon_j}(v_{\varepsilon_j}(\tau))dW^{\varepsilon_j}, \tag{4.35}$$

holds (as an equation in $H^{-1}(Q)$) for almost all $(\bar{\omega}, t) \in \bar{\Omega} \times [0, T]$.

5. Homogenization results

We assume in this section that all vector spaces are real vector spaces, and all functions are real-valued. We keep using the same notation as in the previous sections.

5.1. Preliminary results

Let $1 < p < \infty$. It is a fact that the topological dual of $\mathcal{B}^p_{AP}(\mathbb{R}_\tau; \mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y))$ is $\mathcal{B}^{p'}_{AP}(\mathbb{R}_\tau; [\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y)]')$; this can be easily seen from the fact that $\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y)$ is reflexive (see Section 2) and $\mathcal{B}^p_{AP}(\mathbb{R}_\tau; \mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y))$ is isometrically isomorphic to $L^p(\mathcal{K}_\tau; \mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y))$. We denote by $\langle, \rangle$ (resp. $[,]$) the duality pairing between $\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y)$ (resp. $\mathcal{B}^p_{AP}(\mathbb{R}_\tau; \mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y))$) and $[\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y)]'$ (resp. $\mathcal{B}^{p'}_{AP}(\mathbb{R}_\tau; [\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y)]')$). For the above reason, we have, for $u \in \mathcal{B}^{p'}_{AP}(\mathbb{R}_\tau; [\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y)]')$ and $v \in \mathcal{B}^p_{AP}(\mathbb{R}_\tau; \mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y))$,

$$[u, v] = \int_{\mathcal{K}_\tau} \langle \widehat{u}(s_0), \widehat{v}(s_0)\rangle d\beta_\tau(s_0).$$

For a function $\psi \in \mathcal{D}_{AP}(\mathbb{R}^N_y)/\mathbb{R}$ we know that ψ expresses as follows: $\psi = \varrho_y(\psi_1)$ with $\psi_1 \in AP^\infty(\mathbb{R}^N_y)/\mathbb{R}$ where ϱ_y denotes the canonical mapping of $B^p_{AP}(\mathbb{R}^N_y)$ onto $\mathcal{B}^p_{AP}(\mathbb{R}^N_y)$; see Section 2. We will refer to ψ_1 as the representative of ψ in $AP^\infty(\mathbb{R}^N_y)/\mathbb{R}$. Likewise we define the representative of $\psi \in \mathcal{D}_{AP}(\mathbb{R}_\tau) \otimes [\mathcal{D}_{AP}(\mathbb{R}^N_y)/\mathbb{R}]$ as an element of $AP^\infty(\mathbb{R}_\tau) \otimes [AP^\infty(\mathbb{R}^N_y)/\mathbb{R}]$ satisfying a similar property.

With all this in mind, we have the following.

Lemma 6. *Let $\psi \in B(\bar{\Omega}) \otimes \mathcal{C}^\infty_0(Q_T) \otimes (\mathcal{D}_{AP}(\mathbb{R}_\tau) \otimes [\mathcal{D}_{AP}(\mathbb{R}^N_y)/\mathbb{R}])$ and ψ_1 be its representative in $B(\bar{\Omega}) \otimes \mathcal{C}^\infty_0(Q_T) \otimes [AP^\infty(\mathbb{R}_\tau) \otimes (AP^\infty(\mathbb{R}^N_y)/\mathbb{R})]$. Let $(u_\varepsilon)_{\varepsilon \in E}$, E' and (u_0, u_1) be either as in Theorem 3 or as in Theorem 4. Then, as $E' \ni \varepsilon \to 0$*

$$\int_{Q_T \times \bar{\Omega}} \frac{1}{\varepsilon} u_\varepsilon \psi_1^\varepsilon dxdtd\bar{\mathbb{P}} \to \int_{Q_T \times \bar{\Omega}} [u_1(x, t, \omega), \psi(x, t, \omega)]dxdtd\bar{\mathbb{P}}.$$

Proof. We recall that for ψ_1 as above, we have

$$\psi_1^\varepsilon(x, t, \omega) = \psi_1\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega\right) \quad \text{for } (x, t, \omega) \in Q_T \times \bar{\Omega}.$$

This being so, since $\psi_1(x, t, \cdot, \tau, \omega) \in AP^\infty(\mathbb{R}^N_y)/\mathbb{R} = \{u \in AP^\infty(\mathbb{R}^N_y) : \mathfrak{M}_y(u) = 0\}$, there exists a unique $\phi \in B(\bar{\Omega}) \otimes \mathcal{C}^\infty_0(Q_T) \otimes [AP^\infty(\mathbb{R}_\tau) \otimes (AP^\infty(\mathbb{R}^N_y)/\mathbb{R})]$ such that $\psi_1 = \Delta_y \phi$. We therefore have

$$\begin{aligned}\int_{Q_T \times \bar{\Omega}} \frac{1}{\varepsilon} u_\varepsilon \psi_1^\varepsilon dxdtd\bar{\mathbb{P}} &= \int_{Q_T \times \bar{\Omega}} \frac{1}{\varepsilon} u_\varepsilon (\Delta_y \phi)^\varepsilon dxdtd\bar{\mathbb{P}} \\ &= -\int_{Q_T \times \bar{\Omega}} Du_\varepsilon \cdot (D_y \phi)^\varepsilon dxdtd\bar{\mathbb{P}} - \int_{Q_T \times \bar{\Omega}} u_\varepsilon (\operatorname{div}_x(D_y\phi))^\varepsilon dxdtd\bar{\mathbb{P}}.\end{aligned}$$

Passing to the limit in the above equation as $E' \ni \varepsilon \to 0$ we are led to

$$\int_{Q_T\times\bar{\Omega}} \frac{1}{\varepsilon} u_\varepsilon \psi_1^\varepsilon dxdtd\bar{\mathbb{P}} \to -\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} (Du_0 + \partial\widehat{u}_1)\cdot\partial\widehat{\phi} dxdtd\bar{\mathbb{P}}d\beta - \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} u_0 \,\mathrm{div}_x(\partial\widehat{\phi})dxdtd\bar{\mathbb{P}}d\beta$$
$$= -\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \partial\widehat{u}_1\cdot\partial\widehat{\phi} dxdtd\bar{\mathbb{P}}d\beta$$

since $\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} u_0 \mathrm{div}_x(\partial\widehat{\phi})dxdtd\bar{\mathbb{P}}d\beta = -\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} Du_0\cdot\partial\widehat{\phi}dxdtd\bar{\mathbb{P}}d\beta$. But

$$-\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \partial\widehat{u}_1\cdot\partial\widehat{\phi} dxdtd\bar{\mathbb{P}}d\beta$$
$$= \int_{Q_T\times\bar{\Omega}} \left[\int_{\mathcal{K}_\tau} \left(-\int_{\mathcal{K}_y} \partial\widehat{u}_1(x,t,s,s_0,\omega)\cdot\partial\widehat{\phi}(x,t,s,s_0,\omega)d\beta_y \right) d\beta_\tau \right] dxdtd\bar{\mathbb{P}}.$$

Recalling the definition of the Laplacian $\overline{\Delta}_y$ in Section 2, we deduce from (2.4) and Proposition 3 that

$$-\int_{\mathcal{K}_y} \partial\widehat{u}_1(x,t,s,s_0,\omega)\cdot\partial\widehat{\phi}(x,t,s,s_0,\omega)d\beta_y = \langle \overline{\Delta}_y \varrho_y(\widehat{\phi}(x,t,\cdot,s_0,\omega)), \widehat{u}_1(x,t,\cdot,s_0,\omega)\rangle$$
$$= \langle \varrho_y(\Delta_y\widehat{\phi}(x,t,\cdot,s_0,\omega)), \widehat{u}_1(x,t,\cdot,s_0,\omega)\rangle$$
$$= \langle \widehat{\varrho_y(\Delta_y\phi)}(x,t,\cdot,s_0,\omega), \widehat{u}_1(x,t,\cdot,s_0,\omega)\rangle$$
$$= \langle \widehat{\psi}(x,t,\cdot,s_0,\omega), \widehat{u}_1(x,t,\cdot,s_0,\omega)\rangle$$

where from the first of the above series of equalities, the hat $\widehat{\cdot}$ stands for the Gelfand transform with respect to $AP(\mathbb{R}_\tau)$ and so, does not act on Δ_y and ϱ_y. The lemma therefore follows from the equalities

$$\int_{\mathcal{K}_\tau} \left(-\int_{\mathcal{K}_y} \partial\widehat{u}_1(x,t,s,s_0,\omega)\cdot\partial\widehat{\phi}(x,t,s,s_0,\omega)d\beta_y \right) d\beta_\tau$$
$$= \int_{\mathcal{K}_\tau} \langle \widehat{\psi}(x,t,\cdot,s_0,\omega), \widehat{u}_1(x,t,\cdot,s_0,\omega)\rangle d\beta_\tau(s_0)$$
$$= [\psi(x,t,\cdot,\cdot,\omega), u_1(x,t,\cdot,\cdot,\omega)]. \quad \square$$

For $u \in \mathcal{B}^p_{AP}(\mathbb{R}_\tau)$ we denote by $\overline{\partial}/\partial\tau$ the temporal derivative defined exactly as its spatial counterpart $\overline{\partial}/\partial y_i$. We also put $\partial_0 = \mathcal{G}_1(\overline{\partial}/\partial\tau)$. $\overline{\partial}/\partial\tau$ and ∂_0 enjoy the same properties as $\overline{\partial}/\partial y_i$ (see Section 2). In particular, they are skew adjoint. Now, let us view $\overline{\partial}/\partial\tau$ as an unbounded operator defined from $\mathcal{V} = \mathcal{B}^p_{AP}(\mathbb{R}_\tau; \mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y))$ into $\mathcal{V}' = \mathcal{B}^{p'}_{AP}(\mathbb{R}_\tau; [\mathcal{B}^{1,p}_{\#AP}(\mathbb{R}^N_y)]')$. Proceeding as in [49, pp. 1243–1244], it gives rise to an unbounded operator still denoted by $\overline{\partial}/\partial\tau$ with the following properties:

(P)$_1$ The domain of $\overline{\partial}/\partial\tau$ is $\mathcal{W} = \{v \in \mathcal{V} : \overline{\partial}v/\partial\tau \in \mathcal{V}'\}$;
(P)$_2$ $\overline{\partial}/\partial\tau$ is skew adjoint, that is, for all $u, v \in \mathcal{W}$,

$$\left[u, \frac{\overline{\partial}v}{\partial\tau} \right] = -\left[\frac{\overline{\partial}u}{\partial\tau}, v \right].$$

(P)$_3$ The space $\mathcal{E} = \mathcal{D}_{AP}(\mathbb{R}_\tau) \otimes [\mathcal{D}_{AP}(\mathbb{R}^N_y)/\mathbb{R}]$ is dense in $\mathcal{W}$.

The above operator will be useful in the homogenization process. This being so, the preceding lemma has a crucial corollary.

Corollary 2. *Let the hypotheses be those of Lemma 6. Assume moreover that $u_1 \in \mathcal{W}$. Then, as $E' \ni \varepsilon \to 0$,*

$$\int_{Q_T\times\bar{\Omega}} \varepsilon u_\varepsilon \frac{\partial\psi_1^\varepsilon}{\partial t} dxdtd\bar{\mathbb{P}} \to -\int_{Q_T\times\bar{\Omega}} \left[\frac{\overline{\partial}u_1}{\partial\tau}(x,t,\omega), \psi(x,t,\omega) \right] dxdtd\bar{\mathbb{P}}.$$

Proof. We have

$$\int_{Q_T\times\bar{\Omega}} \varepsilon u_\varepsilon \frac{\partial\psi_1^\varepsilon}{\partial t} dxdtd\bar{\mathbb{P}} = \varepsilon \int_{Q_T\times\bar{\Omega}} u_\varepsilon \left(\frac{\partial\psi_1}{\partial t} \right)^\varepsilon dxdtd\bar{\mathbb{P}} + \frac{1}{\varepsilon} \int_{Q_T\times\bar{\Omega}} u_\varepsilon \left(\frac{\partial\psi_1}{\partial\tau} \right)^\varepsilon dxdtd\bar{\mathbb{P}}.$$

Since $\frac{\partial\psi_1}{\partial\tau}$ is a representative of some function in $B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T) \otimes (\mathcal{D}_{AP}(\mathbb{R}_\tau) \otimes [\mathcal{D}_{AP}(\mathbb{R}_y^N)/\mathbb{R}])$, we infer from Lemma 6 that, as $E' \ni \varepsilon \to 0$,

$$\int_{Q_T\times\bar{\Omega}} \varepsilon u_\varepsilon \frac{\partial\psi_1^\varepsilon}{\partial t} dxdtd\bar{\mathbb{P}} \to \int_{Q_T\times\bar{\Omega}} \left[\int_{\mathcal{K}_\tau} \langle\widehat{u}_1(x,t,\cdot,s_0,\omega), \partial_0\widehat{\psi}(x,t,\cdot,s_0,\omega)\rangle d\beta_\tau(s_0)\right] dxdtd\bar{\mathbb{P}}.$$

But

$$\begin{aligned}\int_{\mathcal{K}_\tau} \langle\widehat{u}_1(x,t,\cdot,s_0,\omega), \partial_0\widehat{\psi}(x,t,\cdot,s_0,\omega)\rangle d\beta_\tau(s_0) &= \left[u_1(x,t,\cdot,\cdot,\omega), \frac{\overline{\partial}\psi}{\partial\tau}(x,t,\cdot,\cdot,\omega)\right] \\ &= -\left[\frac{\overline{\partial}u_1}{\partial\tau}(x,t,\cdot,\cdot,\omega), \psi(x,t,\cdot,\cdot,\omega)\right],\end{aligned}$$

the last equality coming from the fact that $\overline{\partial}/\partial\tau$ is skew adjoint. □

We will also need the following

Lemma 7. *Let $g : \mathbb{R}_y^N \times \mathbb{R}_\tau \times \mathbb{R}_u \to \mathbb{R}$ be a function verifying the following conditions:*

(i) $|\partial_u g(y,\tau,u)| \leq C$
(ii) $g(\cdot,\cdot,u) \in AP(\mathbb{R}_{y,\tau}^{N+1})$.

Let $(u_\varepsilon)_\varepsilon$ be a sequence in $L^2(Q_T \times \bar{\Omega})$ such that $u_\varepsilon \to u_0$ in $L^2(Q_T \times \bar{\Omega})$ as $\varepsilon \to 0$ where $u_0 \in L^2(Q_T \times \bar{\Omega})$. Then, setting $g^\varepsilon(u_\varepsilon)(x,t,\omega) = g(x/\varepsilon, t/\varepsilon^2, u_\varepsilon(x,t,\omega))$ we have, as $\varepsilon \to 0$,

$$g^\varepsilon(u_\varepsilon) \to g(\cdot,\cdot,u_0) \quad \text{in } L^2(Q_T \times \bar{\Omega})\text{-weak } \Sigma.$$

Proof. Assumption (i) implies the Lipschitz condition

$$|g(y,\tau,u) - g(y,\tau,v)| \leq C|u-v| \quad \text{for all } y,\tau,u,v. \tag{5.1}$$

Next, observe that from (ii) and (5.1), the function $(x,t,y,\tau,\omega) \mapsto g(y,\tau,u_0(x,t,\omega))$ lies in $L^2(Q_T \times \bar{\Omega}; AP(\mathbb{R}_{y,\tau}^{N+1}))$, so that by Remark 1, we have $g^\varepsilon(u_0) \to g(\cdot,\cdot,u_0)$ in $L^2(Q_T \times \bar{\Omega})$-weak Σ as $\varepsilon \to 0$. Now, let $f \in B(\bar{\Omega}; L^2(Q_T; AP(\mathbb{R}_{y,\tau}^{N+1})))$; then

$$\begin{aligned}&\int_{Q_T\times\bar{\Omega}} g^\varepsilon(u_\varepsilon) f^\varepsilon dxdtd\bar{\mathbb{P}} - \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{g}(\cdot,\cdot,u_0)\widehat{f} dxdtd\bar{\mathbb{P}}d\beta \\ &= \int_{Q_T\times\bar{\Omega}} (g^\varepsilon(u_\varepsilon) - g^\varepsilon(u_0)) f^\varepsilon dxdtd\bar{\mathbb{P}} + \int_{Q_T\times\bar{\Omega}} g^\varepsilon(u_0) f^\varepsilon dxdtd\bar{\mathbb{P}} - \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{g}(\cdot,\cdot,u_0)\widehat{f} dxdtd\bar{\mathbb{P}}d\beta.\end{aligned}$$

Using the inequality

$$\left|\int_{Q_T\times\bar{\Omega}} (g^\varepsilon(u_\varepsilon) - g^\varepsilon(u_0)) f^\varepsilon dxdtd\bar{\mathbb{P}}\right| \leq C\|u_\varepsilon - u_0\|_{L^2(Q_T\times\bar{\Omega})} \|f^\varepsilon\|_{L^2(Q_T\times\bar{\Omega})}$$

in conjunction with the above convergence results leads at once to the result. □

Remark 3. From the Lipschitz property of the function g above we may get more information on the limit of the sequence $g^\varepsilon(u_\varepsilon)$. Indeed, since $|g^\varepsilon(u_\varepsilon) - g^\varepsilon(u_0)| \leq C|u_\varepsilon - u_0|$, we deduce the following convergence result:

$$g^\varepsilon(u_\varepsilon) \to \widetilde{g}(u_0) \quad \text{in } L^2(Q_T \times \bar{\Omega}) \text{ as } \varepsilon \to 0$$

where $\widetilde{g}(u_0)(x,t,\omega) = \int_{\mathcal{K}} \widehat{g}(s,s_0,u_0(x,t,\omega))d\beta$, so that we can derive the existence of a subsequence of $g^\varepsilon(u_\varepsilon)$ that converges a.e. in $Q_T \times \bar{\Omega}$ to $\widetilde{g}(u_0)$.

We will need the following spaces:

$$\mathbb{F}_0^1 = L^2(\bar{\Omega} \times (0,T); H_0^1(Q)) \times L^2(Q_T \times \bar{\Omega}; \mathcal{W})$$

and

$$\mathcal{F}_0^\infty = [B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T)] \times [B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T) \otimes \mathcal{E}]$$

where $\mathcal{W} = \{v \in \mathcal{V} : \overline{\partial}v/\partial\tau \in \mathcal{V}'\}$ with $\mathcal{V} = \mathcal{B}_{AP}^2(\mathbb{R}_\tau; \mathcal{B}_{\#AP}^{1,2}(\mathbb{R}_y^N))$, and $\mathcal{E} = \mathcal{D}_{AP}(\mathbb{R}_\tau) \otimes [\mathcal{D}_{AP}(\mathbb{R}_y^N)/\mathbb{R}]$. $\mathbb{F}_0^1$ is a Hilbert space under the norm

$$\|(u_0,u_1)\|_{\mathbb{F}_0^1} = \|u_0\|_{L^2(\bar{\Omega}\times(0,T);H_0^1(Q))} + \|u_1\|_{L^2(Q_T\times\bar{\Omega};\mathcal{W})}.$$

Moreover, since $B(\bar{\Omega})$ is dense in $L^2(\bar{\Omega})$, it is an easy matter to check that $\mathcal{F}_0^\infty$ is dense in $\mathbb{F}_0^1$.

5.2. Global homogenized problem

Let (v_{ε_j}) be the sequence determined in Section 4 and satisfying Eq. (4.35). It therefore satisfies the a priori estimates (4.5)–(4.6), so that, by the diagonal process, one can find a subsequence of $(v_{\varepsilon_j})_j$ not relabeled, which weakly converges in $L^2(\bar{\Omega}; L^2(0,T; H_0^1(Q)))$ to u_0 determined by the Skorokhod's theorem and satisfying (4.34). From Theorem 4, we infer the existence of a function $u_1 \in L^2(\bar{\Omega}; L^2(Q_T; \mathcal{B}^2_{AP}(\mathbb{R}_\tau; \mathcal{B}^{1,2}_{\#AP}(\mathbb{R}^N_y))))$ such that the convergence results

$$v_{\varepsilon_j} \to u_0 \quad \text{in } L^2(Q_T) \text{ almost surely} \tag{5.2}$$

and

$$\frac{\partial v_{\varepsilon_j}}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i} \quad \text{in } L^2(Q_T \times \bar{\Omega})\text{-weak } \Sigma \ (1 \le i \le N) \tag{5.3}$$

hold when $\varepsilon_j \to 0$. The following result holds.

Proposition 5. *The couple $(u_0, u_1) \in \mathbb{F}_0^1$ determined above solves the following variational problem*

$$\begin{cases} -\int_{Q_T\times\bar{\Omega}} u_0 \psi_0' dxdtd\bar{\mathbb{P}} + \int_{Q_T\times\bar{\Omega}} \left[\frac{\overline{\partial} u_1}{\partial \tau}, \psi_1\right] dxdtd\bar{\mathbb{P}} \\ \quad = -\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{a}(Du_0 + \partial\widehat{u}_1)\cdot(D\psi_0 + \partial\widehat{\psi}_1) dxdtd\bar{\mathbb{P}}d\beta \\ \quad\quad + \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{g}(s, s_0, u_0)\widehat{\psi}_1 dxdtd\bar{\mathbb{P}}d\beta - \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{G}(s, s_0, u_0)\cdot D\psi_0 dxdtd\bar{\mathbb{P}}d\beta \\ \quad\quad - \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} (\widehat{\partial_u G}(s, s_0, u_0)\cdot(Du_0 + \partial\widehat{u}_1))\psi_0 dxdtd\bar{\mathbb{P}}d\beta \\ \quad\quad + \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{M}(s, s_0, u_0)\psi_0 d\bar{W} dxd\bar{\mathbb{P}}d\beta \quad \text{for all } (\psi_0, \psi_1) \in \mathcal{F}_0^\infty. \end{cases} \tag{5.4}$$

Proof. In what follows, we omit the index j momentarily from the sequence ε_j. So we will merely write ε instead of ε_j. With this in mind, we set

$$\Phi_\varepsilon(x,t,\omega) = \psi_0(x,t,\omega) + \varepsilon\psi\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \omega\right), \quad (x,t,\omega) \in Q_T \times \bar{\Omega}$$

where $(\psi_0, \psi_1) \in \mathcal{F}_0^\infty$ with ψ being a representative of ψ_1. Using Φ_ε as a test function in the variational formulation of (4.35) we get

$$\begin{aligned} -\int_{Q_T\times\bar{\Omega}} u_\varepsilon \frac{\partial \Phi_\varepsilon}{\partial t} dxdtd\bar{\mathbb{P}} &= -\int_{Q_T\times\bar{\Omega}} a^\varepsilon Du_\varepsilon \cdot D\Phi_\varepsilon dxdtd\bar{\mathbb{P}} + \frac{1}{\varepsilon}\int_{Q_T\times\bar{\Omega}} g^\varepsilon(u_\varepsilon)\Phi_\varepsilon dxdtd\bar{\mathbb{P}} \\ &\quad + \int_{Q_T\times\bar{\Omega}} M^\varepsilon(u_\varepsilon)\Phi_\varepsilon dxdW^\varepsilon d\bar{\mathbb{P}} \end{aligned} \tag{5.5}$$

where here and henceforth, we use the notation $a^\varepsilon = a(x/\varepsilon, t/\varepsilon^2)$, $\psi^\varepsilon = \psi(x,t,x/\varepsilon,t/\varepsilon^2,\omega)$, $M^\varepsilon(u_\varepsilon) = M(x/\varepsilon, t/\varepsilon^2, u_\varepsilon)$ and $g^\varepsilon(u_\varepsilon) = g(x/\varepsilon, t/\varepsilon^2, u_\varepsilon)$. We will consider the terms in (5.5) respectively.

We have

$$\begin{aligned} \frac{1}{\varepsilon}\int_{Q_T\times\bar{\Omega}} g^\varepsilon(u_\varepsilon)\Phi_\varepsilon dxdtd\bar{\mathbb{P}} &= \frac{1}{\varepsilon}\int_{Q_T\times\bar{\Omega}} g^\varepsilon(u_\varepsilon)\psi_0 dxdtd\bar{\mathbb{P}} + \int_{Q_T\times\bar{\Omega}} g^\varepsilon(u_\varepsilon)\psi^\varepsilon dxdtd\bar{\mathbb{P}} \\ &= I_\varepsilon^1 + I_\varepsilon^2. \end{aligned}$$

Lemma 7 and convergence result (5.2) imply

$$I_\varepsilon^2 \to \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{g}(s, s_0, u_0)\widehat{\psi}_1 dxdtd\bar{\mathbb{P}}d\beta$$

since $\widehat{\psi}_1 = \mathcal{G}_1 \circ \psi_1 = \mathcal{G}_1 \circ (\varrho(\psi)) = \mathcal{G} \circ \psi = \widehat{\psi}$. For I_ε^1, we know from assumption A4 that

$$\frac{1}{\varepsilon} g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right) = \operatorname{div}_x\left[G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right)\right] - \partial_u G\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, u_\varepsilon\right)\cdot Du_\varepsilon,$$

in such a way that

$$I_\varepsilon^1 = -\int_{Q_T\times\bar{\Omega}} G^\varepsilon(u_\varepsilon)\cdot D\psi_0 dxdtd\bar{\mathbb{P}} - \int_{Q_T\times\bar{\Omega}} \left[\partial_u G\left(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2},u_\varepsilon\right)\cdot Du_\varepsilon\right]\psi_0 dxdtd\bar{\mathbb{P}}.$$

Once again, owing to assumption A4 (see the inequalities (4.2) and (4.3) therein) we deduce from Lemma 7, convergence results (5.2) and (5.3) that

$$I_\varepsilon^1 \to -\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{G}(s,s_0,u_0)\cdot D\psi_0 dxdtd\bar{\mathbb{P}}d\beta - \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} [\widehat{\partial_u G}(s,s_0,u_0)\cdot(Du_0+\partial\widehat{u}_1)]\psi_0 dxdtd\bar{\mathbb{P}}d\beta.$$

Next, we have

$$\int_{Q_T\times\bar{\Omega}} u_\varepsilon \frac{\partial\Phi_\varepsilon}{\partial t} dxdtd\bar{\mathbb{P}} = \int_{Q_T\times\bar{\Omega}} u_\varepsilon \frac{\partial\psi_0}{\partial t} dxdtd\bar{\mathbb{P}} + \int_{Q_T\times\bar{\Omega}} \varepsilon u_\varepsilon \frac{\partial\psi^\varepsilon}{\partial t} dxdtd\bar{\mathbb{P}}$$

which, from Corollary 2 leads to

$$\int_{Q_T\times\bar{\Omega}} u_\varepsilon \frac{\partial\Phi_\varepsilon}{\partial t} dxdtd\bar{\mathbb{P}} \to \int_{Q_T\times\bar{\Omega}} u_0 \frac{\partial\psi_0}{\partial t} dxdtd\bar{\mathbb{P}} - \int_{Q_T\times\bar{\Omega}} \left[\frac{\bar{\partial} u_1}{\partial\tau}(x,t,\omega),\psi_1(x,t,\omega)\right] dxdtd\bar{\mathbb{P}}.$$

It is an easy exercise to see, using Corollary 1 that

$$\int_{Q_T\times\bar{\Omega}} a^\varepsilon Du_\varepsilon\cdot D\Phi_\varepsilon dxdtd\bar{\mathbb{P}} \to \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{a}(Du_0+\partial\widehat{u}_1)\cdot(D\psi_0+\partial\widehat{\psi}_1) dxdtd\bar{\mathbb{P}}d\beta.$$

Next, owing to Remark 3, assumption A5 and the convergence result (4.33) we get

$$\int_{Q_T\times\bar{\Omega}} M^\varepsilon(u_\varepsilon)\Phi_\varepsilon dxdW^\varepsilon d\bar{\mathbb{P}} \to \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{M}(s,s_0,u_0)\psi_0 dxd\bar{W}d\bar{\mathbb{P}}d\beta.$$

Hence letting $\varepsilon\to 0$ in (5.5) we end up with (5.4), thereby completing the proof. □

The problem (5.4) is called the *global homogenized* problem for (4.1).

5.3. Homogenized problem

The problem (5.4) is equivalent to the following system:

$$\begin{cases} -\int_{Q_T\times\bar{\Omega}} \left[\frac{\bar{\partial} u_1}{\partial\tau},\psi_1\right] dxdtd\bar{\mathbb{P}} - \iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{a}(Du_0+\partial\widehat{u}_1)\cdot\partial\widehat{\psi}_1 dxdtd\bar{\mathbb{P}}d\beta \\ +\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{g}(s,s_0,u_0)\widehat{\psi}_1 dxdtd\bar{\mathbb{P}}d\beta = 0 \\ \text{for all } \psi_1\in B(\bar{\Omega})\otimes\mathcal{C}_0^\infty(Q_T)\otimes\mathcal{E}, \end{cases} \tag{5.6}$$

and

$$\begin{cases} -\int_{Q_T\times\bar{\Omega}} u_0\psi_0' dxdtd\bar{\mathbb{P}} = -\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{a}(Du_0+\partial\widehat{u}_1)\cdot D\psi_0 dxdtd\bar{\mathbb{P}}d\beta \\ -\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{G}(s,s_0,u_0)\cdot D\psi_0 dxdtd\bar{\mathbb{P}}d\beta \\ -\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} (\widehat{\partial_u G}(s,s_0,u_0)\cdot(Du_0+\partial\widehat{u}_1))\psi_0 dxdtd\bar{\mathbb{P}}d\beta \\ +\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{M}(s,s_0,u_0)\psi_0 d\bar{W}dxd\bar{\mathbb{P}}d\beta \quad \text{for all } \psi_0\in B(\bar{\Omega})\otimes\mathcal{C}_0^\infty(Q_T). \end{cases} \tag{5.7}$$

The following uniqueness result is highlighted.

Proposition 6. *The solution of the variational problem (5.6) is unique.*

Proof. Taking in (5.6) $\psi_1(x,t,y,\tau,\omega)=\phi(\omega)\varphi(x,t)w(y,\tau)$ with $\phi\in B(\bar{\Omega})$, $\varphi\in\mathcal{C}_0^\infty(Q_T)$ and $w\in\mathcal{E}$, we obtain after mere computations

$$-\left[\frac{\overline{\partial}u_1}{\partial\tau}(x,t,\omega),w\right]-\int_{\mathcal{K}}\widehat{a}(Du_0(x,t,\omega)+\partial\widehat{u}_1(x,t,\omega))\cdot\partial\widehat{w}d\beta+\int_{\mathcal{K}}\widehat{g}(u_0(x,t,\omega))\widehat{w}d\beta=0$$
$$\text{for all } w\in\mathcal{E}. \tag{5.8}$$

So, fixing (x,t,ω), if $u_1=u_1(x,t,\omega)$ and $u_2=u_2(x,t,\omega)$ are two solutions to (5.8), then $u=u_1-u_2$ is solution to

$$\left[\frac{\overline{\partial}u}{\partial\tau},w\right]=-\int_{\mathcal{K}}\widehat{a}\partial\widehat{u}\cdot\partial\widehat{w}d\beta\quad\text{for all } w\in\mathcal{E}. \tag{5.9}$$

By the density of $\mathcal{E}$ in $\mathcal{W}$, (5.9) still holds for $w\in\mathcal{W}$. So taking there $w=u$ and using the fact that $\overline{\partial}/\partial\tau$ is skew adjoint (which yields $[\overline{\partial}u/\partial\tau,u]=0$) we get

$$\int_{\mathcal{K}}\widehat{a}\partial\widehat{u}\cdot\partial\widehat{u}d\beta=0.$$

But, since

$$\int_{\mathcal{K}}\widehat{a}\partial\widehat{u}\cdot\partial\widehat{u}d\beta\geq\Lambda\|u\|^2_{\mathcal{B}^2_{AP}(\mathbb{R}_\tau;\mathcal{B}^{1,2}_{\#AP}(\mathbb{R}^N_y))},$$

we are led to $u=0$. Whence the uniqueness of the solution of (5.6). □

Let us now deal with some auxiliary equations connected to (5.6).

Let $\chi\in(\mathcal{W})^N$ and $w_1=w_1(\cdot,\cdot,r)$ (for fixed $r\in\mathbb{R}$) be determined by the following variational problems:

$$\left[\frac{\overline{\partial}\chi}{\partial\tau},\phi\right]=-\int_{\mathcal{K}}\widehat{a}\partial\widehat{\chi}\cdot\partial\widehat{\phi}d\beta-\int_{\mathcal{K}}\widehat{a}\cdot\partial\widehat{\phi}d\beta\quad\forall\phi\in\mathcal{W}; \tag{5.10}$$

$$\left[\frac{\overline{\partial}w_1}{\partial\tau},\phi\right]=-\int_{\mathcal{K}}\widehat{a}\partial\widehat{w}_1\cdot\partial\widehat{\phi}d\beta-\int_{\mathcal{K}}\widehat{G}(\cdot,\cdot,r)\cdot\partial\widehat{\phi}d\beta\quad\text{for all }\phi\in\mathcal{W}. \tag{5.11}$$

Eqs. (5.10) and (5.11) are respectively equivalent to the following equations:

$$\frac{\overline{\partial}\chi}{\partial\tau}-\overline{\mathrm{div}}_y(a\overline{D}_y\chi)=\overline{\mathrm{div}}_y a\quad\text{in }\mathcal{W}',\ \chi\in(\mathcal{W})^N,$$

and

$$\frac{\overline{\partial}w_1}{\partial\tau}-\overline{\mathrm{div}}_y(a\overline{D}_y w_1)=g(\cdot,\cdot,r)\quad\text{in }\mathcal{W}',\ w_1\in\mathcal{W}.$$

The existence of χ and $w_1(\cdot,\cdot,r)$ is ensured by a classical result [50] since $\overline{\partial}/\partial\tau$ is a maximal monotone operator [49] (see also [51] or [52]) and further the uniqueness of χ and $w_1(\cdot,\cdot,r)$ follows the same way of reasoning as in the proof of Proposition 6.

Now, taking $r=u_0(x,t,\omega)$ in (5.11), it is easy to verify that the function

$$(x,t,y,\tau,\omega)\mapsto\chi(y,\tau)\cdot Du_0(x,t,\omega)+w_1(y,\tau,u_0(x,t,\omega))$$

solves Eq. (5.6), so that, by the uniqueness of its solution, we are led to

$$u_1(x,t,y,\tau,\omega)=\chi(y,\tau)\cdot Du_0(x,t,\omega)+w_1(y,\tau,u_0(x,t,\omega)). \tag{5.12}$$

For fixed $r\in\mathbb{R}$, and set as in [26]

$$F_1(r)=\int_{\mathcal{K}}\widehat{a}\partial\widehat{w}_1(s,s_0,r)d\beta;\,F_2(r)=\int_{\mathcal{K}}\widehat{\partial_u g}(s,s_0,r)\widehat{\chi}d\beta$$
$$F_3(r)=\int_{\mathcal{K}}\widehat{\partial_u g}(s,s_0,r)\widehat{w}_1(s,s_0,r)d\beta;\,\widetilde{M}(r)=\int_{\mathcal{K}}\widehat{M}(s,s_0,r)d\beta.$$

With this in mind, we have following.

Lemma 8. *The solution u_0 to the variational problem (5.7) solves the following boundary value problem:*

$$\begin{cases} du_0 = (\mathrm{div}(bDu_0) + \mathrm{div}F_1(u_0) - F_2(u_0)\cdot Du_0 - F_3(u_0))dt + \widetilde{M}(u_0)d\bar{W} \quad \text{in } Q_T \\ u_0 = 0 \quad \text{on } \partial Q \times (0,T) \\ u_0(x,0) = u^0(x) \quad \text{in } Q. \end{cases} \tag{5.13}$$

Proof. We replace in Eq. (5.7) u_1 by the expression (5.12); we therefore get

$$\begin{cases} -\int_{Q_T\times\bar{\Omega}} u_0\psi_0' dxdtd\bar{\mathbb{P}} = -\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{G}(s,s_0,u_0)\cdot D\psi_0 dxdtd\bar{\mathbb{P}}d\beta \\ -\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{a}(Du_0 + \partial\widehat{\chi}\cdot Du_0 + \partial\widehat{w}_1(s,s_0,u_0))\cdot D\psi_0 dxdtd\bar{\mathbb{P}}d\beta \\ -\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} (\widehat{\partial_u G}(s,s_0,u_0)\cdot(Du_0 + \partial\widehat{\chi}\cdot Du_0 + \partial\widehat{w}_1(s,s_0,u_0)))\psi_0 dxdtd\bar{\mathbb{P}}d\beta \\ +\iint_{Q_T\times\bar{\Omega}\times\mathcal{K}} \widehat{M}(s,s_0,u_0)\psi_0 d\bar{W}dxd\bar{\mathbb{P}}d\beta \quad \text{for all } \psi_0 \in B(\bar{\Omega})\otimes\mathcal{C}_0^\infty(Q_T). \end{cases}$$

In particular, for $\psi_0 = \phi\otimes\varphi$ with $\phi \in B(\bar{\Omega})$ and $\varphi \in \mathcal{C}_0^\infty(Q_T)$, we obtain

$$\begin{cases} -\int_{Q_T} u_0\varphi' dxdt = -\iint_{Q_T\times\mathcal{K}} \widehat{a}([I+\partial\widehat{\chi}]\cdot Du_0)\cdot D\varphi dxdtd\beta \\ -\iint_{Q_T\times\mathcal{K}} \widehat{a}(\partial\widehat{w}_1(s,s_0,u_0)\cdot D)\varphi dxdtd\beta - \iint_{Q_T\times\mathcal{K}} \widehat{G}(s,s_0,u_0)\cdot D\varphi dxdtd\beta \\ -\iint_{Q_T\times\mathcal{K}} (\widehat{\partial_u G}(s,s_0,u_0)\cdot(Du_0 + \partial\widehat{\chi}\cdot Du_0 + \partial\widehat{w}_1(s,s_0,u_0)))\varphi dxdtd\beta \\ +\iint_{Q_T\times\mathcal{K}} \widehat{M}(s,s_0,u_0)\varphi d\bar{W}dxd\beta \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(Q_T), \end{cases} \tag{5.14}$$

where I stands for the unit $N\times N$ matrix, and $\mathrm{div}_y G(y,\tau,u) = g(y,\tau,u)$ as in Section 4. Let

$$b = \int_{\mathcal{K}} \widehat{a}(I+\partial\widehat{\chi})d\beta$$

be the homogenized tensor. Since we have

$$\begin{aligned} -\iint_{Q_T\times\mathcal{K}} \widehat{G}(s,s_0,u_0)\cdot D\varphi dxdtd\beta &= \iint_{Q_T\times\mathcal{K}} (\widehat{\partial_u G}(s,s_0,u_0)\cdot Du_0)\varphi dxdtd\beta, \\ &\quad -\iint_{Q_T\times\mathcal{K}} (\widehat{\partial_u G}(s,s_0,u_0)\cdot\partial\widehat{w}_1(s,s_0,u_0))\varphi dxdtd\beta \\ &= \iint_{Q_T\times\mathcal{K}} \widehat{\partial_u g}(s,s_0,u_0)\widehat{w}_1(s,s_0,u_0)\varphi dxdtd\beta \end{aligned}$$

and

$$-\iint_{Q_T\times\mathcal{K}} (\widehat{\partial_u G}(s,s_0,u_0)\cdot(\partial\widehat{\chi}\cdot Du_0))\varphi dxdtd\beta = \iint_{Q_T\times\mathcal{K}} \widehat{\partial_u g}(s,s_0,u_0)(\widehat{\chi}\cdot Du_0)\varphi dxdtd\beta,$$

Eq. (5.14) becomes

$$\begin{cases} -\int_{Q_T} u_0\varphi' dxdt = -\int_{Q_T} (bDu_0)\cdot D\varphi dxdt \\ -\iint_{Q_T\times\mathcal{K}} \widehat{a}\partial\widehat{w}_1(s,s_0,u_0)\cdot D\varphi dxdtd\beta \\ -\iint_{Q_T\times\mathcal{K}} \widehat{\partial_u g}(s,s_0,u_0)(\widehat{\chi}\cdot Du_0 + \widehat{w}_1(s,s_0,u_0))\varphi dxdtd\beta \\ +\iint_{Q_T\times\mathcal{K}} \widehat{M}(s,s_0,u_0)\varphi d\bar{W}dxd\beta \text{ for all } \varphi \in \mathcal{C}_0^\infty(Q_T), \end{cases} \tag{5.15}$$

which is the variational form of (5.13). □

As in [26], it can be checked straightforwardly that the functions F_i $(1 \le i \le 3)$ are Lipschitz continuous functions. As in [26] again, we can show that $F_2(u)$ is uniformly bounded, that is, there exists C_{F_2} such that $|F_2(u)| \le C_{F_2}$ for any $u \in \mathbb{R}$. Likewise, following the same way of reasoning, it can also be proved that the function $\widetilde{M}$ is Lipschitz continuous.

Proposition 7. *Let u_0 and $u_0^\#$ be two solutions of (5.13) on the same probabilistic system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{W}, \bar{\mathcal{F}}^t)$ with the same initial condition u^0. We have that $u_0 = u_0^\#$ almost surely.*

Proof. Let $w(t) = u_0(t) - u_0^\#(t)$. From Itô's formula it is easily seen that w satisfies:

$$\begin{aligned} d|w(t)|^2 &= -2(bDw(t), Dw(t))dt + 2\Bigg[(F_1(u_0(t)) - F_1(u_0^\#(t)), Dw) \\ &\quad - (F_2(u_0(t)) \cdot Du_0(t) - F_2(u_0^\#(t)).Du_0^\#(t), w(t)) \\ &\quad - \ (F_3(u_0(t)) - F_3(u_0^\#(t)), w(t)) + \frac{1}{2}|\widetilde{M}(u_0(t)) - \widetilde{M}(u_0^\#(t))|^2\Bigg] dt \\ &\quad + 2(\widetilde{M}(u_0(t)) - \widetilde{M}(u_0^\#(t)), w(t))d\bar{W}. \end{aligned}$$

Let $\sigma(t)$ a differentiable function on $[0, T]$. Thanks again to It ô's formula we have that

$$d(\sigma(t)|w(t)|^2) = \sigma'(t)|w(t)|^2 dt + \sigma(t)d|w(t)|^2.$$

By using the Lipschitzity of $F_1, F_3, \widetilde{M}$ and some elementary inequalities we see that

$$\begin{aligned} d(\sigma(t)|w(t)|^2) \le\ & (\sigma'(t)|w(t)|^2 + \sigma(t)[-2(bDw(t), Dw(t)) + \delta|Dw(t)|^2 + C_\delta|w(t)|^2])dt \\ & + (|F_2(u_0(t)).Du_0(t)| + |F_2(u_0^\#)(t).Du_0^\#(t)|)\sigma(t)|w(t)|dt \\ & + C\sigma(t)|w(t)|^2 dt + 2\sigma(t)(\widetilde{M}(u_0(t)) - \widetilde{M}(u_0^\#(t)), w(t))d\bar{W}, \end{aligned}$$

where $\delta > 0$ is arbitrary. Integrating over $[0, t]$ and taking the mathematical expectation yields

$$\begin{aligned} \bar{\mathbb{E}}(\sigma(t)|w(t)|^2) \le\ & -2\bar{\mathbb{E}}\int_0^t \sigma(s)(bDw(s), Dw(s))ds + C\bar{\mathbb{E}}\int_0^t \sigma(s)|w(s)|^2 ds \\ & + \bar{\mathbb{E}}\int_0^t (|F_2(u_0).Du_0| + |F_2(u_0^\#).Du_0^\#|)\sigma(s)|w(s)|ds \\ & + \delta\bar{\mathbb{E}}\int_0^t \sigma(s)|Dw(s)|^2 ds + \bar{\mathbb{E}}\int_0^t \sigma'(s)|w(s)|^2 ds. \end{aligned}$$

Choosing $\delta > 0$ so that $\bar{\mathbb{E}}\int_0^t \sigma(s)[(bDw, Dw) - \delta|Dw|^2]ds > 0$, we infer from the last estimate that

$$\bar{\mathbb{E}}(\sigma(t)|w(t)|^2) \le C\bar{\mathbb{E}}\int_0^t \sigma(s)|w(s)|^2 ds + \bar{\mathbb{E}}\int_0^t (|Du_0| + |Du_0^\#|)C_{F_2}\sigma(s)|w(s)|ds + \bar{\mathbb{E}}\int_0^t \sigma'(s)|w(s)|^2 ds, \tag{5.16}$$

where we have used the fact that F_2 is uniformly bounded. By choosing

$$\sigma(t) = \exp\left(-\int_0^t \frac{(|Du_0(s)| + |Du_0^\#(s)|)C_{F_2}}{|w(s)|}ds\right),$$

we deduce from (5.16) that

$$\bar{\mathbb{E}}(\sigma(t)|w(t)|^2) \le C\bar{\mathbb{E}}\int_0^t \sigma(s)|w(s)|^2 ds,$$

from which we derive by using Gronwall's lemma that $|u_0(t) - u_0'(t)| = 0$ almost surely for any $t \in [0, T]$. This completes the proof of the pathwise uniqueness. □

Remark 4. The pathwise uniqueness result in Proposition 7 and Yamada–Watanabe's Theorem (see, for instance, [53]) implies the existence of a unique strong probabilistic solution of (5.13) on a prescribed probabilistic system $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}^t, W)$.

The aim of the rest of this section is to prove the following homogenization result.

Theorem 8. *Assume* A1–A5 *hold. For each $\varepsilon > 0$ let u_ε be the unique solution of (1.1) on a given stochastic system $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}^t, W)$ defined as in Section 4. Then the whole sequence u_ε converges in probability to u_0 as $\varepsilon \to 0$, in the topology of $L^2(Q_T)$ (i.e $\|u_\varepsilon - u_0\|_{L^2(Q_T)}$ converges to zero in probability) where u_0 is the unique strong probabilistic solution of (5.13).*

The main ingredients for the proof of this theorem are the pathwise uniqueness for (5.13) and the following criteria for convergence in probability whose proof can be found in [54].

Lemma 9. *Let X be a Polish space. A sequence of a X-valued random variables $\{x_n; n \geq 0\}$ converges in probability if and only if for every subsequence of joint probability laws, $\{\nu_{n_k,m_k}; k \geq 0\}$, there exists a further subsequence which converges weakly to a probability measure ν such that*

$$\nu(\{(x, y) \in X \times X; x = y\}) = 1.$$

Let us set $\mathfrak{S}^{L^2} = L^2(Q_T)$, $\mathfrak{S}^W = \mathcal{C}(0, T : \mathbb{R}^m)$, $\mathfrak{S}^{L^2,L^2} = L^2(Q_T) \times L^2(Q_T)$, and finally $\mathfrak{S} = L^2(Q_T) \times L^2(Q_T) \times \mathfrak{S}^W$. For any $S \in \mathcal{B}(\mathfrak{S}^{L^2})$ we set $\Pi^\varepsilon(S) = \mathbb{P}(u_\varepsilon \in S)$ and $\Pi^W = \mathbb{P}(W \in S)$ for any $S \in \mathcal{B}(\mathfrak{S}^W)$. Next we define the joint probability laws:

$$\Pi^{\varepsilon,\varepsilon'} = \Pi^\varepsilon \times \Pi^{\varepsilon'}$$
$$\nu^{\varepsilon,\varepsilon'} = \Pi^\varepsilon \times \Pi^{\varepsilon'} \times \Pi^W.$$

The following tightness property holds.

Lemma 10. *The collection $\{\nu^{\varepsilon,\varepsilon'} : \varepsilon, \varepsilon' \in E\}$ (and hence any subsequence $\{\nu^{\varepsilon_j,\varepsilon'_j} : \varepsilon_j, \varepsilon'_j \in E'\}$) is tight on $\mathfrak{S}$.*

Proof. The proof is very similar to Theorem 7. For any $\delta > 0$ we choose the sets Σ_δ, Y_δ exactly as in the proof of Theorem 7 with appropriate modification on the constants M_δ, L_δ so that $\Pi^\varepsilon(Y_\delta) \geq 1 - \frac{\delta}{4}$ and $\Pi^W(\Sigma_\delta) \geq 1 - \frac{\delta}{2}$ for every $\varepsilon \in E$. Now let us take $K_\delta = Y_\delta \times Y_\delta \times \Sigma_\delta$ which is a compact in $\mathfrak{S}$; it is not difficult to see that $\{\nu^{\varepsilon,\varepsilon'}(K_\delta) \geq (1 - \frac{\delta}{4})^2(1 - \frac{\delta}{2}) \geq 1 - \delta$ for all $\varepsilon, \varepsilon'$. This completes the proof of the lemma. □

Proof of Theorem 8. Lemma 10 implies that there exists a subsequence from $\{\nu^{\varepsilon_j,\varepsilon'_j}\}$ still denoted by $\{\nu^{\varepsilon_j,\varepsilon'_j}\}$ which converges to a probability measure ν. By Skorokhod's theorem there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ on which a sequence $(u_{\varepsilon_j}, u_{\varepsilon'_j}, W^j)$ is defined and converges almost surely in $\mathfrak{S}^{L^2,L^2} \times \mathfrak{S}^W$ to a couple of random variables $(u_0, v_0, \bar{W})$. Furthermore, we have

$$Law(u_{\varepsilon_j}, u_{\varepsilon'_j}, W^j) = \nu^{\varepsilon_j,\varepsilon'_j} \quad \text{and} \quad Law(u_0, v_0, \bar{W}) = \nu.$$

Now let $Z_j^{u_\varepsilon} = (u_{\varepsilon_j}, W^j)$, $Z_j^{u_{\varepsilon'}} = (u_{\varepsilon'_j}, W^j)$, $Z^{u_0} = (u_0, \bar{W})$ and $Z^{v_0} = (v_0, \bar{W})$. We can infer from the above argument that $(\Pi^{\varepsilon_j,\varepsilon'_j})$ converges to a measure Π such that

$$\Pi(\cdot) = \bar{\mathbb{P}}((u_0, v_0) \in \cdot).$$

As above we can show that $Z_j^{u_\varepsilon}$ and $Z_j^{u_{\varepsilon'}}$ satisfy (4.35) and that Z^u and Z^v satisfy (5.13) on the same stochastic system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \bar{\mathcal{F}}^t, \bar{W})$, where $\bar{\mathcal{F}}^t$ is the filtration generated by the couple $(u_0, v_0, \bar{W})$. Since we have the uniqueness result above, then we see that $u^0 = v^0$ almost surely and $u_0 = v_0$ in $L^2(Q_T)$. Therefore

$$\Pi(\{(x, y) \in \mathfrak{S}^{L^2,L^2}; x = y\}) = \bar{\mathbb{P}}(u_0 = v_0 \text{ in } L^2(Q_T)) = 1.$$

This fact together with Lemma 9 imply that the original sequence (u_ε) defined on the original probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}^t, W)$ converges in probability to an element u_0 in the topology of $\mathfrak{S}^{L^2}$. By a passage to the limit's argument as in the previous subsection it is not difficult to show that u_0 is the unique solution of (5.13) (on the original probability system $(\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}^t, W)$. This ends the proof of Theorem 8. □

6. Some applications

In this subsection we provide some applications of the results obtained in the previous sections to some special cases.

6.1. Example 1

The first application is related to the periodicity hypothesis stated as follows:

A6 $g(\cdot, \cdot, u) \in \mathcal{C}_{\text{per}}(Y \times Z)$ for all $u \in \mathbb{R}$ with $\int_Y g(y, \tau, u)dy = 0$ for all $\tau, u \in \mathbb{R}$; $a_{ij}, M_i(\cdot, \cdot, u) \in L^\infty_{\text{per}}(Y \times Z)$ for all $1 \leq i, j \leq N$; $M_i(\cdot, \cdot, u) \in L^\infty_{\text{per}}(Y \times Z)$ for each $1 \leq i \leq m$ and for all $u \in \mathbb{R}$,

where $Y = (0, 1)^N$ and $Z = (0, 1)$ and, $\mathcal{C}_{\text{per}}(Y \times Z)$ and $L^\infty_{\text{per}}(Y \times Z)$ denote the usual spaces of $Y \times Z$-periodic functions.

As the periodic functions are part of almost periodic functions, all the results of the previous sections apply to this case. We have the following result.

Theorem 9. *Assume hypotheses* A1–A5 *are satisfied with the almost periodicity therein being replaced by the periodicity hypothesis* A6. *For each* $\varepsilon > 0$ *let* u_ε *be uniquely determined by* (1.1). *Then as* $\varepsilon \to 0$,

$$u_\varepsilon \to u_0 \text{ in } L^2(Q \times (0, T)) \text{ almost surely}$$

and

$$\frac{\partial u_\varepsilon}{\partial x_j} \to \frac{\partial u_0}{\partial x_j} + \frac{\partial u_1}{\partial y_j} \quad \text{in } L^2(Q \times (0, T) \times \bar{\Omega})\text{-weak } \Sigma \quad (1 \le j \le N)$$

where $(u_0, u_1) \in L^2(\bar{\Omega} \times (0, T); H_0^1(Q)) \times L^2(Q_T \times \bar{\Omega}; \mathcal{W})$ *is the unique solution to the variational problem*

$$\begin{cases} -\int_{Q_T\times\bar{\Omega}} u_0 \psi_0' dxdtd\bar{\mathbb{P}} + \int_{Q_T\times\bar{\Omega}} \left[\frac{\partial u_1}{\partial \tau}, \psi_1\right] dxdtd\bar{\mathbb{P}} \\ \quad = -\iint_{Q_T\times\bar{\Omega}\times Y\times Z} a(Du_0 + D_y u_1)\cdot(D\psi_0 + D_y\psi_1) dxdtd\bar{\mathbb{P}}dyd\tau \\ \qquad + \iint_{Q_T\times\bar{\Omega}\times Y\times Z} g(y, \tau, u_0)\psi_1 dxdtd\bar{\mathbb{P}}dyd\tau \\ \qquad - \iint_{Q_T\times\bar{\Omega}\times Y\times Z} G(y, \tau, u_0)\cdot D\psi_0 dxdtd\bar{\mathbb{P}}dyd\tau \\ \qquad - \iint_{Q_T\times\bar{\Omega}\times Y\times Z} \left(\partial_u G(y, \tau, u_0)\cdot(Du_0 + D_y u_1)\right)\psi_0 dxdtd\bar{\mathbb{P}}dyd\tau \\ \qquad + \iint_{Q_T\times\bar{\Omega}\times Y\times Z} M(y, \tau, u_0)\psi_0 d\bar{W} dxd\bar{\mathbb{P}}dyd\tau \quad \text{for all } (\psi_0, \psi_1) \in \mathcal{F}_0^\infty \end{cases}$$

where $\mathcal{W} = \{v \in L^2_{per}(Z; W^{1,2}_\#(Y)) : \partial v/\partial\tau \in L^2_{per}(Z; [W^{1,2}_\#(Y)]')\}$ *with* $W^{1,2}_\#(Y) = \{u \in W^{1,2}_{per}(Y) : \int_Y u(y)dy = 0\}$, *and* $\mathcal{F}_0^\infty = [B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T)] \times [B(\bar{\Omega}) \otimes \mathcal{C}_0^\infty(Q_T) \otimes \mathcal{E}$ *with* $\mathcal{E} = \mathcal{C}^\infty_{per}(Z) \otimes \mathcal{C}^\infty_\#(Y)$ *and* $\mathcal{C}^\infty_\#(Y) = \{u \in \mathcal{C}^\infty_{per}(Y) : \int_Y u(y)dy = 0\}$.

Proof. Theorem 9 is a consequence of the following facts: (1) in the periodic setting, the mean value of a function $u \in L^p_{\text{per}}(Y) = \{u \in L^p_{\text{loc}}(\mathbb{R}^N_y) : u \text{ is } Y\text{-periodic}\}$ is merely expressed as $\mathfrak{M}(u) = \int_Y u(y)dy$ (the same definition for the other spaces); (2) the Besicovitch space corresponding to the periodic functions is exactly the space $L^p_{\text{per}}(Y)$; (3) the derivative $\overline{\partial}/\partial y_i$ (resp. $\overline{\partial}/\partial\tau$) is therefore exactly the usual one in the distribution sense $\partial/\partial y_i$ (resp. $\partial/\partial\tau$). □

Remark 5. The above result extends to the case of stochastic partial differential equations the result obtained by Allaire and Piatnitski [26] in the periodic deterministic setting.

6.2. Example 2

Our purpose in the present example is to study the homogenization problem for (1.1) under the following assumptions, where the indices $1 \le i, j \le N$ and $1 \le l \le m$ are arbitrarily fixed:

$(\text{HYP})_1$ $a_{ij}(\cdot, \tau) \in B^2_{AP}(\mathbb{R}^N_y)$ a.e. in $\tau \in \mathbb{R}$.

$(\text{HYP})_2$ The function $\tau \mapsto a_{ij}(\cdot, \tau)$ from $\mathbb{R}$ to $B^2_{AP}(\mathbb{R}^N_y)$ is piecewise constant in the sense that there exists a mapping $q_{ij} : \mathbb{Z} \to B^2_{AP}(\mathbb{R}^N_y)$ such that

$$a_{ij}(\cdot, \tau) = q_{ij}(k) \quad \text{a.e. in } k \le \tau < k + 1 \ (k \in \mathbb{Z}).$$

We assume further that $q_{ij} \in \mathcal{C}_{\text{per}}(\mathbb{Z}; B^2_{AP}(\mathbb{R}^N_y))$.

$(\text{HYP})_3$ The functions $g(\cdot, \cdot, u) \in AP(\mathbb{R}^{N+1}_{y,\tau})$ with $\mathfrak{M}_y(g(\cdot, \cdot, u)) = 0$, and $M_l(\cdot, \cdot, u) \in \mathcal{C}_{\text{per}}(Y \times Z)$ for all $u \in \mathbb{R}$.

Then arguing as in [55] we are led to the homogenization of (1.1) with in A3–A5 the almost periodicity replaced by $(\text{HYP})_1$–$(\text{HYP})_3$ above. Indeed the above assumptions lead to the almost periodicity of the involved functions with respect to y and τ.

6.3. Example 3

Our concern here is the study of the homogenization of (1.1) under the following assumptions, the indices $1 \le i, j \le N$ and $1 \le l \le m$ being arbitrarily fixed:

(1) The function $\tau \mapsto a_{ij}(\cdot, \tau)$ maps continuously $\mathbb{R}$ into $L^2_{\text{loc}}(\mathbb{R}^N_y)$ and is Z-periodic ($Z = (0, 1)$).
(2) For each fixed $\tau \in \mathbb{R}$, the function $a_{ij}(\cdot, \tau)$ is Y_τ-periodic, where $Y_\tau = (0, c_\tau)^N$ with $c_\tau > 0$.
(3) $g(\cdot, \cdot, u) \in \mathcal{C}_{\text{per}}(Y \times Z)$ with $\int_Y g(y, \tau, u)dy = 0$ for all $\tau, u \in \mathbb{R}$, and $M_l(\cdot, \cdot, u) \in B^2_{AP}(\mathbb{R}^{N+1}_{y,\tau})$ for all $u \in \mathbb{R}$.

Hypothesis (1) and (2) imply that $a_{ij} \in \mathcal{C}_{\text{per}}(Z; B^2_{AP}(\mathbb{R}^N_y)) \subset B^2_{AP}(\mathbb{R}^{N+1}_{y,\tau})$, such that the homogenization of (1.1) under the above hypotheses is solvable.

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