CONVERGENCE OF A SEQUENCE OF SOLUTIONS OF THE STOCHASTIC TWO-DIMENSIONAL EQUATIONS OF SECOND GRADE FLUIDS

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Abstract. We study the limit of the stochastic model for two dimensional second grade fluids subjected to the periodic boundary conditions as the stress modulus tends to zero. We show that under suitable conditions on the data the whole sequence of strong probabilistic solutions (u^α) of the stochastic second grade fluid converges to the unique strong probabilistic solution of the stochastic Navier-Stokes equations.

1. Introduction

Let $D = [0, L]^2 \subset \mathbb{R}^2$, $L > 0$, be a periodic square, $T > 0$ a fixed time. Let α denote a sequence of positive numbers $(\alpha_n)_{n\in\mathbb{N}}$ which converges to zero as n converges to ∞; we shall express this by just writing $\alpha \to 0$. We consider a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with the filtration $\mathcal{F}^t$, $0 \le t \le T$, which is the σ-field generated by a given $\mathbb{R}^m$-valued standard Wiener process $\{\mathcal{W}(s), 0 \le s \le T\}$ and the null sets of $\mathcal{F}$. In this paper we investigate the behavior of the sequence (u^α) of strong probabilistic solutions of the following problems:

$$\begin{cases} d(u^\alpha - \alpha\Delta u^\alpha) + (-\nu\Delta u^\alpha + \operatorname{curl}(u^\alpha - \alpha\Delta u^\alpha) \times u^\alpha + \nabla\mathfrak{P})dt = F\,dt + Gd\mathcal{W} \\ \text{in } \Omega \times (0, T] \times D, \\ \operatorname{div} u^\alpha = 0 \text{ in } \Omega \times (0, T] \times D, \\ \int_D u^\alpha dx = 0 \text{ in } \Omega \times (0, T], \\ u^\alpha(0) = u_0 \text{ in } \Omega \times D, \end{cases} \tag{1}$$

when $\alpha \to 0$. The system (1), which is to be understood in the sense of distributions, is the equations of motion for an incompressible second grade fluid driven by random external forces. Here u^α is the velocity of the fluid, $\mathfrak{P}$ is a modified pressure given by

$$\mathfrak{P} = -\tilde{p} - (1/2)|u^\alpha|^2_{\mathbb{R}^2} + \alpha u^\alpha.\Delta u^\alpha + (\alpha/4)\operatorname{tr}(\nabla u^\alpha + (\nabla u^\alpha)^t).$$

Throughout we assume that (1) is subject to the periodic boundary condition. We refer to [23] and [17] for further reading on fluid of complexity two and on second grade fluids. The interest in the investigation of mathematical and physical problems related to second grade fluids arises from the fact that they describe a large class of Non-Newtonian fluids such as dilute polymeric solutions (solution of swollen gel or oil polyols), industrial fluids (oils,...), slurry flows; just to cite a few. Second grade fluids are also connected to Turbulence Theory. Indeed the discussion on the relation between Non-Newtonian fluids, especially fluids of differential type, and Turbulence Theory started with the work of Rivlin [28]. It was rediscovered recently (see, for example, [18] and [12]) that the flow of second grade fluids can be used as a basis for a turbulence closure model.

In the deterministic case, i.e when $G(t, x) \equiv 0$, existence and uniqueness results are given in [14], [13] for instance. It is known from [21] that under general assumption on the data the weak solution (in the partial differential equations sense) of second grade fluids equations converges weakly to the weak solution of the Navier-Stokes equations. We also refer to [7] for interesting discussions related to their relationship with other fluid models. Although there are lots of papers dealing with stochastic partial differential equations and hydrodynamics (see, for instance, [1], [2], [4],[5], [6], [9], [8], [10], [15], [16] ,[20],[22], [24],[29], [30],[31]), there are only few known results for the stochastic version of second grade fluids. The existence of weak probabilistic (or martingale) solution was recently proved in [26]. In this paper, we show that we can construct a sequence (u^{α_j}) of strong stochastic solutions of (1) that converges in a certain sense (see Theorem 3.5 and Remark 3.6) to the stochastic weak solution of the stochastic Navier-Stokes equations (SNSE) as $\alpha_j \to 0$. This result was first established in [25]. We also prove that the whole sequence (u^α) converges in probability to the unique stochastic strong solution of the stochastic Navier-Stokes equations in the topology of $L^2(0,T;\mathbb{H})$ as $\alpha \to 0$ (see Theorem 3.7). Our proof, which is inspired by the papers [2] and [21], relies on deriving estimates (independent of α) for the velocity u^α in Sobolev $\mathbb{H}^1$ norm. Unfortunately, (1) contains a very highly nonlinear term (see the curl-term) and using only the information on the $\mathbb{H}^1$ norm of u^α is not sufficient to pass to the limit in this term. To overcome this difficulty, we proved an interesting and technical mean-type estimate (see Lemma 4.4) which allows to use the deep compactness results of Prokhorov and Skorokhod. This approach is a probabilistic refinement of the idea in [21]. The convergence of the whole sequence to the strong probabilistic solution of the Stochastic Navier-Stokes equations is obtained by using a technical lemma (Lemma 5.1) which originated in [20]. The present paper generalizes the deep result obtained by Iftimie in [21]. Our work also emphasizes the theory of Rivlin in [28].

The layout of this paper is as follows. In addition to the current introduction this article consists of four other sections. In Section 2 we give some notations, necessary backgrounds of probabilistic or analytical nature. We formulate the hypotheses relevant for the paper and our main results in Section 3. The fourth section is devoted to the proof of the first main result. Finally, we prove in the last section that the whole sequence of the strong probabilistic solution for the stochastic model for second grade fluids converges to that of the stochastic Navier-Stokes equations in dimension two.

2. Preliminaries-Notations

For a Banach space X we denote by $\mathbb{X}$ the space of $\mathbb{R}^2$-valued functions such that each component is an element of X. We denote by $\mathbb{H}^1_0(D)$ the space of functions u that belong to the Sobolev space of periodic functions $\mathbb{H}^1(D)$ and satisfying

$$\int_D u(x)dx = 0.$$

We also introduce the spaces

$$\begin{aligned}
\mathcal{V} &= \left\{ u \in \mathcal{C}^\infty_{per}(D) : \operatorname{div} u = 0 \text{ and } \int_D u dx = 0 \right\} \\
\mathbb{V} &= \text{ closure of } \mathcal{V} \text{ in } \quad \mathbb{H}^1_0(D) \\
\mathbb{H} &= \text{ closure of } \mathcal{V} \text{ in } \quad \mathbb{L}^2(D),
\end{aligned}$$

where $\mathcal{C}^\infty_{per}(D)$ denotes the space of infinitely differentiable periodic function with period L.

We denote by $(\cdot,\cdot)$ and $|\cdot|$ the inner product and the norm induced by the inner product and the norm in $\mathbb{L}^2(D)$ on $\mathbb{H}$, respectively. Thanks to Poincaré's inequality, we can endow $\mathbb{V}$ with the gradient scalar product (resp. the norm) $\big((.,.)\big)$ (resp. $||.||$). In the space $\mathbb{V}$, the latter norm is equivalent to the norm generated by the following scalar product

$$(u,v)_\mathbb{V} = (u,v) + \alpha((u,v)), \text{ for any } u \,\text{and}\, v \in \mathbb{V}.$$

More precisely we have

$$(\mathcal{P}+\alpha)^{-1}|v|_\mathbb{V}{}^2 \le ||v||^2 \le (\alpha)^{-1}|v|_\mathbb{V}{}^2, \text{ for any } v \in \mathbb{V}. \tag{2}$$

We also introduce the following space

$$\mathbb{W} = \left\{ u : \operatorname{div} u = 0, \int_D u dx = 0, \text{ and } \operatorname{curl}(u - \alpha\Delta u) \in L^2(D) \right\}.$$

We provide this space with the norm $|.|_{\mathbb{W}}$ generated by the scalar product

$$(u, v)_{\mathbb{W}} = (\operatorname{curl}(u - \alpha\Delta u), \operatorname{curl}(v - \alpha\Delta v)).$$

This norm is equivalent to the usual $\mathbb{H}^3(D)$-norm on $\mathbb{W}$. For any Banach space X, $p, r \geq 1$, we set $L^{p,r}(0, T, \Omega, X) = L^p(\Omega, \mathcal{F}, P; L^r(0, T; X))$. Next we give some results on which most of proofs in forthcoming sections rely.

Theorem 2.1. *[see [32]] Let X, B, Y three Banach spaces such that the following embedding are continuous*

$$X \subset B \subset Y.$$

Moreover, assume that the embedding $X \subset B$ is compact, then the set $\mathfrak{F}$ consisting of functions $v \in L^q(0, T; B) \cap L^1_{loc}(0, T; X)$, $1 \leq q \leq \infty$ such that

$$\sup_{0 \leq h \leq 1} \int_{t_1}^{t_2} |v(t + h) - v(t)|_Y dt \to 0, \; as\, h \to 0,$$

for any $0 < t_1 < t_2 < T$ is compact in $L^p(0, T; B)$ for any $p < q$.

We also need the following product formulas, we refer to [11] for their proof in the case of the whole space (see [19] for the case of periodic condition).

Theorem 2.2. *Let D be a n-dimensional periodic box and let $\beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma > 0$, $\beta < \frac{n}{2}$, $\gamma < \frac{n}{2}$. If $u \in \mathbb{H}^\gamma(D)$ and $v \in \mathbb{H}^\beta(D)$, then there exists a positive constant C such that*

$$|uv|_{\mathbb{H}^{\gamma+\beta-\frac{n}{2}}} \leq C|u|_{\mathbb{H}^\gamma}|v|_{\mathbb{H}^\beta}.$$

If $|\gamma| < \frac{n}{2}$, then

$$|uv|_{\mathbb{H}^{-\frac{n}{2}-\varepsilon}} \leq C'|u|_{\mathbb{H}^\gamma}|v|_{\mathbb{H}^{-\gamma}}, \tag{3}$$

for any $u \in \mathbb{H}^\gamma(D)$, $v \in \mathbb{H}^{-\gamma}(D)$ and $\varepsilon > 0$.

3. Hypotheses and the main results

In this section we will recall briefly the previous results on the problem (1) and will formulate our main results.

3.1. **The hypotheses.** We assume that

(I) $F = F(t, x)$ is a $\mathbb{V}$-valued function defined on $[0, T] \times D$ such that the following holds for any $2 \leq p < \infty$

$$\int_0^T |F(t, x)|^p_{\mathbb{V}} < \infty.$$

(II) $G = G(t, x)$ is a $\mathbb{V}^{\otimes m}$-valued function defined on $[0, T] \times D$ such that the following holds

$$\int_0^T |G(t, x)|^p_{\mathbb{V}^{\otimes m}} < \infty,$$

for any $2 \leq p < \infty$.

(III) We further assume that $u_0 \subset \mathbb{V} \cap \mathbb{H}^3$ is nonrandom and that there exists a positive constant C independent of α such that $|u_0|_{\mathbb{V}} < C$. Suppose also that $\nu > 0$.

We continue with the definition of the concept of the strong probabilistic solution for the problem (1).

Definition 3.1. By a strong probabilistic solution of the system (1), we mean a stochastic process u^α such that

(1) $u^\alpha \in L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;\mathbb{W})) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;\mathbb{V}))$ with $2 \le p < \infty$,
(2) For almost all t, $u^\alpha(t)$ is $\mathcal{F}^t$-measurable,
(3) P-a.s the following integral equation holds

$$(u^\alpha(t) - u^\alpha(0), \phi)_{\mathbb{V}} + \int_0^t [\nu((u^\alpha, \phi)) + (\text{curl}(u^\alpha(s) - \alpha\Delta u^\alpha(s)) \times u^\alpha(s), \phi)]\, ds$$
$$= \int_0^t (F, \phi) ds + \int_0^t (G, \phi) d\mathcal{W}(s)$$

for any $t \in (0, T]$ and $\phi \in \mathcal{V}$.

Remark 3.2. In the above definition the quantity $\int_0^t (G, \phi) d\mathcal{W}(s)$ should be understood as

$$\int_0^t (G, \phi) d\mathcal{W}(s) = \sum_{k=1}^m \int_0^t (G_k, \phi) d\mathcal{W}_k(s),$$

where G_k and $\mathcal{W}_k$ denote the k-th component of G and $\mathcal{W}$, respectively.

We recall now two results from the paper [27].

Theorem 3.3. *Under the assumptions (I), (II) and (III) the problem* (1) *has a solution in the sense of the above definition. Moreover, almost surely the paths of the solution are* $\mathbb{W}$*-valued weakly continuous.*

Let u_1^α *and* u_2^α *be two strong probabilistic solutions of the problem (1) defined on the stochastic basis* $(\Omega, \mathcal{F}, \mathcal{F}^t, P)$. *If we set* $U^\alpha = u_1^\alpha - u_2^\alpha$, *then we have* $U^\alpha = 0$ *almost surely.*

3.2. **Statement of the main theorems.** Before we proceed to the statement of our main theorem we introduce the SNSE

$$\begin{cases} dv + (-\nu\Delta v + (v.\nabla v) + \nabla \mathcal{P})dt = Fdt + Gd\bar{\mathcal{W}} \\ \text{in } \bar{\Omega} \times (0, T] \times D, \\ \text{div}\, v = 0 \text{ in } \bar{\Omega} \times (0, T] \times D, \\ \int_D v dx = 0 \text{ in } \bar{\Omega} \times (0, T], \\ v(0) = u_0 \text{ in } \bar{\Omega} \times D. \end{cases} \tag{4}$$

and recall the concept of a weak probabilistic solution of the problem.

Definition 3.4. By a weak probabilistic solution of (4), we mean a system

$$(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, \bar{\mathcal{F}}^t, \bar{\mathcal{W}}, v),$$

where

(1) $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ is a complete probability space, $\bar{\mathcal{F}}^t$ is a filtration on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$,
(2) $\bar{\mathcal{W}}(t)$ is an m-dimensional $\bar{\mathcal{F}}^t$-standard Wiener process,
(3) $v(t) \in L^p(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;\mathbb{V})) \cap L^p(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^\infty(0,T;\mathbb{H}))$, $\forall \quad 2 \le p < \infty$,
(4) For almost all t, $v(t)$ is $\bar{\mathcal{F}}^t$-measurable,
(5) P-a.s the following integral equation holds

$$\begin{aligned} &(v(t) - v(0), \phi) + \int_0^t [\nu((v, \phi)) + < v.\nabla v, \phi >]\, ds \\ &= \int_0^t (F, \phi) ds + \int_0^t (G, \phi) d\bar{\mathcal{W}}(s) \end{aligned} \tag{5}$$

for any $t \in (0, T]$ and $\phi \in \mathcal{V}$.

In this article we prove the following result.

Theorem 3.5. *Under the hypotheses (I)-(III) there exist a probability space* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$*, a family of probability measures* (Π^{α_j})*, a probability measure* Π*, and stochastic processes* $(\mathcal{W}^{\alpha_j}, u^{\alpha_j})$*,* $(\bar{\mathcal{W}}, v)$ *such that the law of* $(\mathcal{W}^{\alpha_j}, u^{\alpha_j})$ *(resp.* $(\bar{\mathcal{W}}, v)$*) is* Π^{α_j} *(resp.* Π*) and* $\mathcal{W}^{\alpha_j} \to \bar{\mathcal{W}}$ *uniformly* $\bar{P}$*-a.s. when* $j \to \infty$ $(\alpha_j \to 0)$*. The pair* $(\mathcal{W}^{\alpha_j}, u^{\alpha_j})$ *satisfies* $\bar{P}-$*a.s.* (1) *in the sense of distribution and as* $j \to \infty$ $(\alpha_j \to 0)$

$$u^{\alpha_j} \rightharpoonup v, \text{ weakly in } L^p(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;\mathbb{V})), \tag{6}$$

$$u^{\alpha_j} \rightharpoonup v \text{ weakly-* in } L^p(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^\infty(0,T;\mathbb{H})), \tag{7}$$

for all $2 \le p < \infty$ *and* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}, v, \bar{\mathcal{W}})$ *is a weak probabilistic solution of (4).*

Remark 3.6. Since we are in 2-D then it is known that under our hypotheses (I)-(III) the problem (4) has a strong probabilistic solution which is unique, see for example [22]. This implies that the process v of the above theorem is a strong probabilistic solution of the Stochastic Navier Stokes Equations (4).

The convergence of the whole sequence (u^α) to the (either strong or weak probabilistic) solution of the stochastic Navier-Stokes equations could not be proved in [25]. The second main goal of this work is to prove the following convergence result.

Theorem 3.7. *For the given stochastic system* $(\Omega, \mathcal{F}, P), \mathcal{F}^t, \mathcal{W}$ *defined in Introduction, we have that the whole sequence* u^α *converges in probability to* v *in the topology of* $L^2(0,T;\mathbb{H})$*, i.e* $||u^\alpha - v||_{L^2(0,T;\mathbb{H}}$ *converges to zero in probability. Here* v *is the unique strong probabilistic solution of the stochastic Navier-Stokes equations.*

4. Proof of Theorem 3.5

This section is devoted to the proof of our first main result.

4.1. **Uniform a priori estimates.** In this subsection we derive some estimates uniform in α. These inequalities do not follow from previous works (see [24]) which explode when $\alpha \to 0$. Throughout C denotes unessential positive constant independent of α, and which may change from one line to the next. Since we $\alpha \to 0$ we may assume that $\alpha = (\alpha_n) \subset [0,1)$, at least for large n .

Lemma 4.1. *For* $\alpha \in (0,1)$ *we have*

$$E \sup_{0 \le s \le T} (|u^\alpha(s)|^2 + \alpha ||u^\alpha(s)||^2) + E\int_0^T ||u^\alpha(s)||^2 ds \le C, \tag{8}$$

$$E \sup_{0 \le s \le T} (|u^\alpha(s)|^2 + \alpha ||u^\alpha(s)||^2)^{\frac{p}{2}} + E\left(\int_0^T ||u^\alpha(s)||^2 ds\right)^{\frac{p}{2}} \le C, \tag{9}$$

for any $2 \le p < \infty$*.*

Before we prove this result it is important to make the following remark.

Remark 4.2. We recall that the continuous linear operator $(I + \alpha A)^{-1}$, where A is the usual Stokes operator, establishes a bijective correspondence between the spaces $\mathbb{H}^l(D) \cap \mathbb{V}$ (resp. $\mathbb{H}$) and $\mathbb{H}^{l+2}(D) \cap \mathbb{V}$, $l > 1$ (resp. $l = 0$). Furthermore for any $w \in \mathbb{V}$, and $f \in \mathbb{H}^l(D)$, $l \ge 0$,

$$((I + \alpha A)^{-1} f, w)_{\mathbb{V}} = (f, w), \tag{10}$$

$$|(I + \alpha A)^{-1} f|_{\mathbb{V}} \le C|f| \tag{11}$$

Proof. We start the proof of our lemma by proving (8). Since u^α is a solution of (1) then

$$du^\alpha + (I + \alpha A)^{-1} A u^\alpha dt + (I + \alpha A)^{-1} \widehat{B}(u^\alpha, u^\alpha) dt = (I + \alpha A)^{-1} F dt + (I + \alpha A)^{-1} G d\mathcal{W},$$

holds P-a.s. for any $t \in [0,T]$. Here we have set

$$\widehat{B}(u^\alpha, u^\alpha) = \mathrm{curl}(u^\alpha - \alpha\Delta u^\alpha) \times u^\alpha.$$

For the rest of this section and the paper we write

$$(I+\alpha A)^{-1}F = \widehat{F},$$
$$(I+\alpha A)^{-1}G = \widehat{G}.$$

Ito's formula implies that

$$\begin{aligned} &d|u^\alpha|^2_{\mathbb{V}} + 2((I+\alpha A)^{-1}Au^\alpha, u^\alpha)_{\mathbb{V}}dt + 2((I+\alpha A)^{-1}\widehat{B}(u^\alpha,u^\alpha), u^\alpha)_{\mathbb{V}}dt \\ &= (\widehat{F}, u^\alpha)_{\mathbb{V}}dt + |\widehat{G}|^2_{\mathbb{V}\otimes m}dt + 2(\widehat{G}, u^\alpha)_{\mathbb{V}}d\mathcal{W}. \end{aligned}$$

By the relationship (10) in the above remark and the equation

$$(\widehat{B}(u^\alpha, u^\alpha), u^\alpha) = 0,$$

we obtain that

$$d|u^\alpha|^2_{\mathbb{V}} + 2||u^\alpha||^2 dt = 2(F, u^\alpha)dt + |\widehat{G}|^2_{\mathbb{V}\otimes m}dt + 2(\widehat{G}, u^\alpha)_{\mathbb{V}}d\mathcal{W}.$$

This relation combined with Cauchy-Schwarz's inequality and (11) imply that

$$d|u^\alpha|^2_{\mathbb{V}} + 2||u^\alpha||^2 dt \le (|F|^2 + |u^\alpha|^2)dt + |\widehat{G}|^2_{\mathbb{V}\otimes m}dt + 2(\widehat{G}, u^\alpha)_{\mathbb{V}}d\mathcal{W}.$$

Recalling the definition of $|.|^2_{\mathbb{V}}$ we deduce that

$$d|u^\alpha|^2_{\mathbb{V}} + 2||u^\alpha||^2 dt \le |F|^2_{\mathbb{V}} + |u^\alpha|^2_{\mathbb{V}}dt + |\widehat{G}|^2_{\mathbb{V}\otimes m}dt + 2(\widehat{G}, u^\alpha)_{\mathbb{V}}d\mathcal{W}. \tag{12}$$

Taking the sup over $0 \le s \le t$, $t \in [0,T]$ and passing to the mathematical expectation yield

$$E\sup_{0\le s\le t}|u^\alpha|^2_{\mathbb{V}} + 2E\int_0^t ||u^\alpha||^2 ds \le C + E\int_0^t |u^\alpha|^2_{\mathbb{V}}ds + 2E\sup_{0\le s\le t}\left|\int_0^s (\widehat{G}, u^\alpha)_{\mathbb{V}}d\mathcal{W}\right|,$$

where the assumptions on F and G were used. Burkhölder-Davis-Gundy's inequality implies

$$E\sup_{0\le s\le t}|u^\alpha|^2_{\mathbb{V}} + 2E\int_0^t ||u^\alpha||^2 ds \le C + E\int_0^t |u^\alpha|^2_{\mathbb{V}}ds + 6E\left(\int_0^t (\widehat{G}, u^\alpha)^2_{\mathbb{V}}ds\right)^{\frac{1}{2}}.$$

Cauchy's inequality implies

$$E\sup_{0\le s\le t}|u^\alpha|^2_{\mathbb{V}} + 2E\int_0^t ||u^\alpha||^2 ds \le C + E\int_0^t |u^\alpha|^2_{\mathbb{V}}ds + \frac{1}{2}E\sup_{0\le s\le t}|u^\alpha(s)|^2_{\mathbb{V}} + CE\int_0^t |\widehat{G}|^2_{\mathbb{V}\otimes m}ds,$$

or

$$E\sup_{0\le s\le t}|u^\alpha|^2_{\mathbb{V}} + 4E\int_0^t ||u^\alpha||^2 ds \le C + CE\int_0^t |u^\alpha|^2_{\mathbb{V}}ds.$$

Here we have used (11) and the assumption on G. It follows from Gronwall's inequality that

$$E\sup_{0\le s\le t}|u^\alpha|^2_{\mathbb{V}} + 2E\int_0^t ||u^\alpha||^2 ds < C,$$

for any $t \in [0,T]$. This completes the proof of (8).

We continue with the proof of (9). For $2 \le p < \infty$ and $t \in [0,T]$ the following holds:

$$\begin{aligned} |u^\alpha|^p_{\mathbb{V}} + p\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}}||u^\alpha||^2 ds &= |u_0|^p_{\mathbb{V}} + p\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}}(\widehat{F}, u^\alpha)ds + \frac{p}{2}\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}}|\widehat{G}|^2_{\mathbb{V}}ds \\ &+ \frac{(p-2)p}{2}\int_0^t |u^\alpha|^{p-4}_{\mathbb{V}}(\widehat{G}, u^\alpha)_{\mathbb{V}}ds + p\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}}(\widehat{G}, u^\alpha)_{\mathbb{V}}d\mathcal{W}. \end{aligned}$$

Owing to (11) and the above estimate we see that

$$|u^\alpha|^p_{\mathbb{V}} \le |u_0|^p_{\mathbb{V}} + p\int_0^t |u^\alpha|^{p-1}_{\mathbb{V}}|F|ds + \frac{p}{2}C\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}}|G|^2ds + \frac{(p-2)p}{2}C\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}}|G|^2ds$$
$$+ p\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}}(\widehat{G}, u^\alpha)_{\mathbb{V}}d\mathcal{W}.$$

We derive from this by using Young's inequality that

$$|u^\alpha|^p_{\mathbb{V}} \le |u_0|^p_{\mathbb{V}} + C\int_0^t |u^\alpha|^p_{\mathbb{V}}ds + C\int_0^t |F|^pds + C\int_0^t |G|^pds + p\int_0^t |u^\alpha|^{p-2}_{\mathbb{V}}(\widehat{G}, u^\alpha)_{\mathbb{V}}d\mathcal{W}.$$

Taking the sup over $0 \le s \le t$, passing to the mathematical expectation and using the assumptions on F and G imply

$$E\sup_{0\le s\le t}|u^\alpha|^p_{\mathbb{V}} \le |u_0|^p_{\mathbb{V}} + CE\int_0^t |u^\alpha|^p_{\mathbb{V}}ds + pE\sup_{0\le s\le t}\left|\int_0^s |u^\alpha|^{p-2}_{\mathbb{V}}(\widehat{G}, u^\alpha)_{\mathbb{V}}d\mathcal{W}\right|.$$

Invoking the Martingale inequality yields

$$E\sup_{0\le s\le t}|u^\alpha|^p_{\mathbb{V}} \le CE\int_0^t |u^\alpha|^p_{\mathbb{V}}ds + pE\left(\int_0^s t|u^\alpha|^{2p-2}_{\mathbb{V}}|\widehat{G}|^2_{\mathbb{V}}ds\right)^{\frac{1}{2}}.$$

We infer from this estimate, Young's inequality, (11) along with the assumption on G and Gronwall's inequality that

$$E\sup_{0\le s\le t}|u^\alpha(s)|^p_{\mathbb{V}} < C, \tag{13}$$

for any $t \in [0,T]$ and $2 \le p < \infty$. We deduce from (12) with the help of this last estimate that

$$E\left(\int_0^t ||u^\alpha||^2ds\right)^{\frac{p}{2}} \le C + CE\left|\int_0^t (\widehat{G}, u^\alpha)_{\mathbb{V}}d\mathcal{W}\right|^{\frac{p}{2}}.$$

We obtain from this with the help of the Martingale inequality and (13) that

$$E\left(\int_0^t ||u^\alpha||^2ds\right)^{\frac{p}{2}} \le C.$$

And this completes the proof of (9), hence the lemma. □

Remark 4.3. For $1 \le p < \infty$, the following estimates are valid

$$E\sup_{0\le s\le T}(|u^\alpha(s)|^2 + \alpha||u^\alpha(s)||^2)^{\frac{p}{2}} + E\left(\int_0^T ||u^\alpha(s)||^2ds\right)^{\frac{p}{2}} < C, \tag{14}$$

We will need the following key estimate.

Lemma 4.4. *For any $\delta \in (0,1)$ we have*

$$E\sup_{|\theta|\le\delta}\int_0^{T-\delta} |u^\alpha(t+\theta) - u^\alpha(t)|^2_{\mathbb{H}^{-4}} \le C\delta.$$

Proof. In what follows we set

$$\frac{\partial}{\partial x_i} = \partial_i, \text{ for any } i,$$

and we rewrite the first equation in (1) as follows (see [21] for the details)

$$\begin{aligned}&\frac{\partial}{\partial t}(u^\alpha - \alpha\Delta u^\alpha) - \nu\Delta u^\alpha + u^\alpha.\nabla u^\alpha - \alpha\sum_{j,k}\partial_j\partial_k(u^\alpha_j\partial_k u^\alpha) + \alpha\sum_{j,k}\partial_j(\partial_k u^\alpha_j\partial_k u^\alpha)\\ &= \alpha\sum_{j,k}\partial_k(\partial_k u^\alpha_j\nabla u^\alpha_j) - \nabla\mathfrak{P}^\sharp + F + G\frac{d\mathcal{W}}{dt},\end{aligned} \tag{15}$$

where

$$\nabla\mathfrak{P}^\sharp = \frac{1}{2}\nabla(|u^\alpha|^2+\alpha|\nabla u^\alpha|^2)+\nabla\mathfrak{P}.$$

We set $\Phi=\mathbb{P}(u^\alpha-\alpha\Delta u^\alpha)$ where $\mathbb{P}$ is the Leray projector. We see from (15) that

$$\begin{aligned}&d\Phi+\{\nu Au^\alpha+\mathbb{P}(u^\alpha.\nabla u^\alpha)-\alpha\sum_{j,k}\mathbb{P}(\partial_j\partial_k(u^\alpha_j\partial_k u^\alpha))+\alpha\sum_{j,k}\mathbb{P}(\partial_j(\partial_k u^\alpha_j\partial_k u^\alpha))\}dt\\ &=\alpha\sum_{j,k}\mathbb{P}(\partial_k(\partial_k u^\alpha_j\nabla u^\alpha_j))dt+Fdt+Gd\mathcal{W}.\end{aligned}$$

This implies

$$\begin{aligned}\Phi(t+\theta)-\Phi(t)=&\int_t^{t+\theta}\{-\nu Au^\alpha+\alpha\sum_{j,k}\left(\mathbb{P}(\partial_j\partial_k(u^\alpha_j\partial_k u^\alpha))-\mathbb{P}(\partial_j(\partial_k u^\alpha_j\partial_k u^\alpha))\right)\}ds\\ &-\int_t^{t+\theta}\mathbb{P}(u^\alpha.\nabla u^\alpha)ds+\int_t^{t+\theta}\{\alpha\sum_{j,k}\mathbb{P}(\partial_k(\partial_k u^\alpha_j\nabla u^\alpha_j))+F\}ds+\int_t^{t\theta}Gd\mathcal{W},\end{aligned}$$

for any $\theta>0$. We infer from this that

$$\begin{aligned}|\Phi(t+\theta)-\Phi(t)|^2_{\mathbb{H}^{-4}}\leq\ &2\left(\int_t^{t+\theta}\left\{+|u^\alpha.\nabla u^\alpha|_{\mathbb{H}^{-4}}+\alpha\sum_{j,k}|\partial_j\partial_k(u^\alpha_j\partial_k u^\alpha)|_{\mathbb{H}^{-4}}\right\}ds\right)^2\\ &+4\left(\int_t^{t+\theta}\left\{\alpha\sum_{j,k}[|\partial_j(\partial_k u^\alpha_j\partial_k u^\alpha)|_{\mathbb{H}^{-4}}+|\partial_k(\partial_k u^\alpha_j\nabla u^\alpha_j)|_{\mathbb{H}^{-4}}]+|F|_{\mathbb{H}^{-4}}\right\}dst\right)^2\\ &+2\int_t^{t+\theta}\nu|Au^\alpha|_{\mathbb{H}^{-4}}ds+2\left|\int_t^{t+\theta}Gd\mathcal{W}\right|^2_{\mathbb{H}^{-4}},\end{aligned}$$

which implies

$$\begin{aligned}|\Phi(t+\theta)-\Phi(t)|^2_{\mathbb{H}^{-4}}\leq\ &C\theta\int_t^{t+\theta}\left\{+|u^\alpha.\nabla u^\alpha|^2_{\mathbb{H}^{-4}}+\alpha^2\sum_{j,k}|\partial_j\partial_k(u^\alpha_j\partial_k u^\alpha)|^2_{\mathbb{H}^{-4}}\right\}dt\\ &+C\theta\int_t^{t+\theta}\left\{\alpha^2\sum_{j,k}[|\partial_j(\partial_k u^\alpha_j\partial_k u^\alpha)|^2_{\mathbb{H}^{-4}}+|\partial_k(\partial_k u^\alpha_j\nabla u^\alpha_j)|^2_{\mathbb{H}^{-4}}]+|F|^2\right\}dt\\ &+C\theta\int_t^{t+\theta}\nu|Au^\alpha|^2_{\mathbb{H}^{-4}}ds+2\left|\int_t^{t+\theta}Gd\mathcal{W}\right|^2_{\mathbb{H}^{-4}}.\end{aligned}$$

It is not hard to see that

$$|Au^\alpha|^2_{\mathbb{H}^{-4}}\leq C|u^\alpha|^2. \tag{16}$$

For $n=2$ Theorem 2.2 implies that

$$|u^\alpha.\nabla u^\alpha|^2_{\mathbb{H}^{-4}}\leq C|u^\alpha|^2|\nabla u^\alpha|^2, \tag{17}$$

$$\alpha^2|\partial_j\partial_k(u^\alpha_j\partial_k u^\alpha)|^2_{\mathbb{H}^{-4}}\leq C|u^\alpha_j\partial_k u^\alpha|^2_{\mathbb{H}^{-2}}\leq C|u^\alpha|^2|\nabla u^\alpha|^2. \tag{18}$$

From the same theorem we have that

$$|\partial_k u^\alpha_j\partial_k u^\alpha|_{\mathbb{H}^{-2}}\leq C|\nabla u^\alpha|^2\quad\forall k,j,$$

from which we derive that

$$\alpha^2|\partial_j(\partial_k u^\alpha_j\partial_k u^\alpha)|^2_{\mathbb{H}^{-4}}\leq\alpha C\alpha|\nabla u^\alpha|^2|\nabla u^\alpha|^2. \tag{19}$$

A similar argument can be used to show that

$$\alpha^2|\partial_k(\partial_k u^\alpha_j\nabla u^\alpha_j)|^2_{\mathbb{H}^{-4}}\leq\alpha C\alpha|\nabla u^\alpha|^2|\nabla u^\alpha|^2. \tag{20}$$

The estimates (16)-(20) along with (9) allow us to write

$$E\int_0^{T-\delta}\sup_{0\le\theta\le\delta}|\Phi(t+\theta)-\Phi(t)|^2_{\mathbb{H}^{-4}}dt \le C\delta^2+C\delta+C\delta E\int_0^{T-\delta}\int_t^{t+\theta}\alpha|\nabla u^\alpha|^2|\nabla u^\alpha|^2 dsdt$$
$$+CE\int_0^{T-\theta}\sup_{0\le\theta\le\delta}\left|\int_t^{t+\theta}Gd\mathcal{W}\right|^2_{\mathbb{H}^{-4}}dt.$$

But (9) implies that

$$E\sup_{0\le t\le T}\alpha^{\frac{p}{2}}|\nabla u^\alpha(t)|^p+\nu E\left(\int_0^T|\nabla u^\alpha(t)|^2dt\right)^{\frac{p}{2}}\le C,\ 2\le p<\infty.$$

From which we deduce that

$$E\int_0^{T-\delta}\sup_{0\le\theta\le\delta}|\Phi(t+\theta)-\Phi(t)|^2_{\mathbb{H}^{-4}}dt \le C\delta^2+C\delta+C\delta+CE\int_0^{T-\delta}\sup_{0\le\theta\le\delta}\left|\int_t^{t+\theta}Gd\mathcal{W}\right|^2_{\mathbb{H}^{-4}}dt.$$

By making use of the Martingale inequality, the assumption on G we obtain that

$$E\int_0^{T-\delta}\sup_{0\le\theta\le\delta}|\Phi(t+\theta)-\Phi(t)|^2_{\mathbb{H}^{-4}}dt\le C\delta.$$

For almost all $(t,\omega)\in[0,T]\times\Omega$ we have

$$u^\alpha(t+\theta)-u^\alpha(t)=(I+\alpha A)^{-1}(\Phi(t+\theta)-\Phi(t)),$$

which implies that

$$|u^\alpha(t+\theta)-u^\alpha(t)|^2_{\mathbb{H}^\beta}<|\Phi(t+\theta)-\Phi(t)|^2_{\mathbb{H}^\beta},\forall\beta\in\mathbb{R}.$$

Indeed for any $\phi\in\mathbb{H}^\beta(D)$ such that $\operatorname{div}\phi=0$ and $\int_D\phi(x)dx=0$ we have

$$|\phi|^2_{\mathbb{H}^\beta}=\sum_{j=1}^\infty|\phi_j|^2\lambda_j^{2\beta}<\sum_{j=1}^\infty(1+\alpha\lambda_j)|\phi_j|^2\lambda_j^{2\beta},$$

that is,

$$|\phi|^2_{\mathbb{H}^\beta}<|\phi+\alpha A\phi|^2_{\mathbb{H}^\beta},$$

where $\phi=\sum_{j=1}^\infty\phi_je_j$, and $Ae_j=\lambda_je_j,\ j=1,2,...$; the e_j-s are the eigenfunctions of the operator A and the λ_j-s are the corresponding eigenvalues. It follows from this remark that

$$E\int_0^{T-\delta}\sup_{0\le\theta\le\delta}|u^\alpha(t+\theta)-u^\alpha(t)|^2_{\mathbb{H}^{-4}}dt\le C\delta.$$

A similar argument can be carried out to proving the same estimate for the case $\theta<0$. □

4.2. **Compactness result and passage to the limit.** The following compactness result plays a crucial role in the proof of the tightness of the probability measures generated by the sequence $(u^\alpha)_{\alpha\in[0,1)}$.

Lemma 4.5. *Let μ_n, ν_n two sequences of positive real numbers which tend to zero as $n\to\infty$, the injection of*

$$D_{\nu_n,\mu_n}=\left\{q\in L^\infty(0,T;\mathbb{H})\cap L^2(0,T;V);\sup_n\frac{1}{\nu_n}\sup_{|\theta|\le\mu_n}\left(\int_0^T|q(t+\theta)-q(t)|^2_{\mathbb{H}^{-4}}\right)^{1/2}<\infty\right\}$$

in $L^2(0,T;\mathbb{H})$ is compact.

The proof, which is similar to the analogous result in [1], follows from the application of Lemmas 2.1, 5.2 and 4.4. The space D_{ν_n,μ_n} is a Banach space with the norm

$$||q||_{D_{\nu_n,\mu_n}} = \text{ess}\sup_{0\leq t\leq T} |q(t)|+\left(\int_0^T ||q(t)||^2\right)^{1/2} +\sup_n \frac{1}{\nu_n} \sup_{|\theta|\leq\mu_n} \left(\int_0^T |q(t+\theta)-q(t)|^2_{\mathbb{H}^{-4}}\right)^{1/2}.$$

Alongside D_{ν_n,μ_n}, we also consider the space X_{p,ν_n,μ_n}, $1\leq p<\infty$, of random variables ζ endowed with the norm

$$E||\zeta||_{X_{p,\nu_n,\mu_n}} = E\text{ess}\sup_{0\leq t\leq T} |\zeta(t)|^p + E\left(\int_0^T ||\zeta(t)||^2\right)^{p/2}$$
$$+ E\sup_n \frac{1}{\nu_n} \sup_{|\theta|\leq\mu_n} \left(\int_0^T |\zeta(t+\theta)-\zeta(t)|^2_{\mathbb{H}^{-4}}\right)^{1/2};$$

X_{p,ν_n,μ_n} is a Banach space.
Combining (14) and the estimates in Lemma 4.4 we have

Proposition 4.6. *For any real number $p\in[1,\infty)$ and for any sequences ν_n,μ_n converging to 0 such that the series $\sum_n \frac{\sqrt{\mu_n}}{\nu_n}$ converges, the sequence $(u^\alpha)_{\alpha\in[0,1)}$ is bounded uniformly in α in X_{p,ν_n,μ_n} for all n.*

Next we consider the space $\mathfrak{S} = C(0,T;\mathbb{R}^m)\times L^2(0,T;\mathbb{H})$ equipped with the Borel σ-algebra $\mathcal{B}(\mathfrak{S})$. For $\alpha\in[0,1)$, let Φ_α be the measurable $\mathfrak{S}$-valued mapping defined on $(\Omega,\mathcal{F},P)$ by

$$\Phi_\alpha(\omega) = (\mathcal{W}(\omega), u^\alpha(\omega)).$$

For each α we introduce a probability measure Π^α on $(\mathfrak{S};\mathcal{B}(\mathfrak{S}))$ defined by

$$\Pi^\alpha(S) = P(\Phi_\alpha^{-1}(S)),\ \text{for any } S\in\mathcal{B}(\mathfrak{S}).$$

Theorem 4.7. *The family of probability measures $\{\Pi^\alpha : \alpha\in[0,1)\}$ is tight in $(\mathfrak{S};\mathcal{B}(\mathfrak{S}))$.*

Proof. For $\varepsilon>0$ we should find compact subsets

$$\Sigma_\varepsilon\subset C(0,T;\mathbb{R}^m); Y_\varepsilon\subset L^2(0,T;\mathbb{H}),$$

such that

$$P\left(\omega : \mathcal{W}(\omega,.)\notin\Sigma_\varepsilon\right)\leq\frac{\varepsilon}{2}, \tag{21}$$
$$P\left(\omega : u^\alpha(\omega,.)\notin Y_\varepsilon\right)\leq\frac{\varepsilon}{2}, \tag{22}$$

for all α.

The quest for Σ_ε is made by taking into account some facts about Wiener process such as the formula

$$E|\mathcal{W}(t)-\mathcal{W}(s)|^{2j} = (2j-1)!(t-s)^j, j=1,2,.... \tag{23}$$

For a constant $L_\varepsilon>0$ depending on ε to be fixed later and $n\in\mathbb{N}$, we consider the set

$$\Sigma_\varepsilon = \{\mathcal{W}(.)\in C(0,T;\mathbb{R}^m) : \sup_{\substack{t,s\in[0,T]\\ |t-s|<\frac{1}{n^6}}} n|\mathcal{W}(s)-\mathcal{W}(t)|\leq L_\varepsilon\}.$$

The set Σ_ε is relatively compact in $C(0,T;\mathbb{R}^m)$ by Arzela-Ascoli's theorem. Furthermore Σ_ε is closed in $C(0,T;\mathbb{R}^m)$, therefore it is compact in $C(0,T;\mathbb{R}^m)$. Making use of Markov's inequality

$$P(\omega;\zeta(\omega)\geq\beta)\leq\frac{1}{\beta^k}E[|\zeta(\omega)|^k],$$

for any random variable ζ and real numbers k we get

$$
\begin{aligned}
P\left(\omega : \mathcal{W}(\omega) \notin \Sigma_\varepsilon\right) &\leq P\left[\cup_n \left\{\omega : \sup_{\substack{t,s\in[0,T] \\ |t-s|<\frac{1}{n^6}}} |\mathcal{W}(s)-\mathcal{W}(t)| \geq \frac{L_\varepsilon}{n}\right\}\right], \\
&\leq \sum_{n=1}^{\infty}\sum_{i=0}^{n^6-1}\left(\frac{n}{L_\varepsilon}\right)^4 E \sup_{\frac{iT}{n^6}\leq t\leq \frac{(i+1)T}{n^6}} |\mathcal{W}(t)-\mathcal{W}(iTn^{-6}|^4, \\
&\leq C\sum_{n=1}^{\infty}\sum_{i=0}^{n^6-1}\left(\frac{n}{L_\varepsilon}\right)^4 (Tn^{-6})^2 n^6 = \frac{C}{L_\varepsilon^4}\sum_{n=1}^{\infty}\frac{1}{n^2},
\end{aligned}
$$

where we have used (23). Since the right hand side of (23) is independent of α, then so is the constant C in the above estimate. We take $L_\varepsilon^4 = \frac{1}{2C\varepsilon}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{-1}$ and get (21).

Next we choose Y_ε as a ball of radius M_ε in D_{ν_n,μ_m} centered at 0 and with ν_n, μ_n independent of ε, converging to 0 and such that the series $\sum_n \frac{\sqrt{\mu_n}}{\nu_n}$ converges, from Lemma 4.5, Y_ε is a compact subset of $L^2(0,T;\mathbb{H})$. Furthermore, we have

$$
\begin{aligned}
P\left(\omega : u^\alpha(\omega) \notin Y_\varepsilon\right) \leq & P\left(\omega : ||u^\alpha||_{D_{\nu_n,\mu_m}} > M_\varepsilon\right) \\
\leq & \frac{1}{M_\varepsilon}\left(E||u^\alpha||_{D_{\nu_n,\mu_m}}\right), \\
\leq & \frac{1}{M_\varepsilon}\left(E||u^\alpha||_{X_{1,\nu_n,\mu_n}}\right), \\
\leq & \frac{C}{M_\varepsilon},
\end{aligned}
$$

where $C>0$ is independent of α (see Proposition 4.6 for the justification.)

Choosing $M_\varepsilon = 2C\varepsilon^{-1}$, we get (22). From the inequalities (21)-(22) we deduce that

$$
P\left(\omega : \mathcal{W}(\omega) \in \Sigma_\varepsilon; u^\alpha(\omega) \in Y_\varepsilon\right) \geq 1-\varepsilon,
$$

for all $\alpha \in [0,1)$. This proves that for all $\alpha \in [0,1)$

$$
\Pi^\alpha(\Sigma_\varepsilon \times Y_\varepsilon) \geq 1-\varepsilon,
$$

from which we deduce the tightness of $\{\Pi^\alpha : \alpha \in [0,1)\}$ in $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$. $\square$

Prokhorov's compactness result enables us to extract from (Π^α) a subsequence (Π^{α_j}) such that

$$
\Pi^{\alpha_j} \text{ weakly converges to a probability measure } \Pi \text{ on } \mathfrak{S}.
$$

Skorokhod's Theorem ensures the existence of a complete probability space $(\bar\Omega, \bar{\mathcal{F}}, \bar P)$ and random variables $(\mathcal{W}^{\alpha_j}, u^{\alpha_j})$ and $(\bar{\mathcal{W}}, v)$ defined on $(\bar\Omega, \bar{\mathcal{F}}, \bar P)$ with values in $\mathfrak{S}$ such that

$$
\text{The probability law of } (\mathcal{W}^{\alpha_j}, u^{\alpha_j}) \text{ is } \Pi^{\alpha_j}, \tag{24}
$$

$$
\text{The probability law of } (\bar{\mathcal{W}}, v) \text{ is } \Pi, \tag{25}
$$

$$
\mathcal{W}^{\alpha_j} \to \bar{\mathcal{W}} \text{ in } C(0,T;\mathbb{R}^m)\ \bar P-\text{a.s.}, \tag{26}
$$

$$
u^{\alpha_j} \to v \text{ in } L^2(0,T;\mathbb{H})\ \bar P-\text{a.s..} \tag{27}
$$

We let $\bar{\mathcal{F}}^t$ be the σ-algebra generated by $(\bar{\mathcal{W}}(s), v(s)), 0 \leq s \leq t$ and the null sets of $\bar{\mathcal{F}}$. We will show that $\bar{\mathcal{W}}$ is an $\bar{\mathcal{F}}^t$-adapted standard $\mathbb{R}^m$-valued Wiener process. To fix this, it is sufficient to show that for any $0 < t_1 < t_2 < \ldots < t_m = T$, the increments process

$\bar{\mathcal{W}}(t_j)-\bar{\mathcal{W}}(t_{j-1}))$ are independent with respect to $\bar{\mathcal{F}}^{t_{j-1}}$, distributed normally with mean 0 and variance t_j-t_{j-1}. That is, to show that for any $\lambda_j\in\mathbb{R}^m$ and $i^2=-1$

$$\bar{E}\exp\left(i\sum_{j=1}^m\lambda_j(\bar{\mathcal{W}}(t_j)-\bar{\mathcal{W}}(t_{j-1}))\right)=\prod_{j=1}^m\exp\left(-\frac{1}{2}\lambda_j^2(t_j-t_{j-1})\right). \tag{28}$$

The equation (28) will follow if we have

$$\bar{E}\left[\exp\left(i\lambda(\bar{\mathcal{W}}(t+\theta)-\bar{\mathcal{W}}(t))\right)/\,\bar{\mathcal{F}}^t\right]=\exp\left(-\frac{|\lambda|^2\theta}{2}\right). \tag{29}$$

We rely on the fact that for any random variables X and Y on any probability space $(\bar{\Omega},\bar{\mathcal{F}},\bar{P})$ such that X is $\bar{\mathcal{F}}$-measurable and $\bar{E}|Y|<\infty$, $\bar{E}|XY|<\infty$, we have

$$\bar{E}(XY/\bar{\mathcal{F}})=X\bar{E}(Y/\bar{\mathcal{F}}),\quad \bar{E}\bar{E}(Y/\bar{\mathcal{F}})=\bar{E}(Y),$$

that is,

$$\bar{E}(XY)=\bar{E}(X\bar{E}(Y/\bar{\mathcal{F}})). \tag{30}$$

Now, let us consider an arbitrary bounded continuous functional $\vartheta_t(\mathcal{W},v)$ on $\mathfrak{S}$ depending only on the values of $\mathcal{W}$ and v on $(0,T)$. Owing to the independence of $\mathcal{W}(t)$ to $\vartheta_t(\mathcal{W},v)$ and the fact that $\mathcal{W}$ is a Wiener process, we have

$$\begin{aligned}&E\left[\exp\left(i\lambda(\mathcal{W}(t+\theta)-\mathcal{W}(t))\right)\vartheta_t(\mathcal{W},v)\right]\\&=E\left[\exp\left(i\lambda(\mathcal{W}(t+\theta)-\mathcal{W}(t))\right)\right]E\left[\vartheta_t(\mathcal{W},v)\right]\\&=\exp\left(-\frac{|\lambda|^2\theta}{2}\right)E\left[\vartheta_t(\mathcal{W},v)\right].\end{aligned}$$

In view of (24)-(25), this implies that

$$\begin{aligned}&\bar{E}\left[\exp\left(i\lambda(\mathcal{W}^{\alpha_j}(t+\theta)-W^{\alpha_j}(t))\right)\vartheta_t(\mathcal{W}^{\alpha_j},v)\right]\\&=\bar{E}\left[\exp\left(i\lambda(\mathcal{W}^{\alpha_j}(t+\theta)-\mathcal{W}^{\alpha_j}(t))\right)\right]\bar{E}\left[\vartheta_t(\mathcal{W}^{\alpha_j},v)\right]\\&=\exp\left(-\frac{|\lambda|^2\theta}{2}\right)\bar{E}\left[\vartheta_t(\mathcal{W}^{\alpha_j},v)\right].\end{aligned}$$

Now, the convergence (26) and the continuity of ϑ allow us to pass to the limit in this latter equation and obtain

$$\bar{E}\left[\exp\left(i\lambda(\bar{\mathcal{W}}(t+\theta)-\bar{\mathcal{W}}(t))\right)\vartheta_t(\bar{\mathcal{W}},v)\right]=\exp\left(-\frac{|\lambda|^2\theta}{2}\right)\bar{E}\left[\vartheta_t(\bar{\mathcal{W}},v)\right],$$

which, in view of (30), implies (29). The choice of the above filtration implies then that $\bar{\mathcal{W}}$ is a $\bar{\mathcal{F}}^t$-standard m-dimensional Wiener process.

By a similar method as used in [2] (see also [26]), we can prove the following result.

Theorem 4.8. *For any $j\geq 1$, $\phi\in\mathcal{V}$, for all $t\in[0,T]$ the following holds almost surely*

$$\begin{aligned}(u^{\alpha_j},\phi)_{\mathbb{V}}+\int_0^t\{(\nu Au^{\alpha_j}+B(u^{\alpha_j},u^{\alpha_j}),\phi)\}dt=(u_0,\phi)_{\mathbb{V}}+\int_0^t(R(u^{\alpha_j})+F(u^{\alpha_j}),\phi)dt\\+\int_0^t(G,\phi)d\mathcal{W}^{\alpha_j},\end{aligned} \tag{31}$$

where

$$\begin{aligned}B(u^{\alpha_j},u^{\alpha_j})&=\mathbb{P}(u^{\alpha_j}.\nabla u^{\alpha_j}),\\R(u^{\alpha_j})&=\alpha\sum_{i,k}\mathbb{P}\left(\partial_i\partial_k(u_i^{\alpha_j}\partial_k u^{\alpha_j})+\partial_i(\partial_k u_i^{\alpha_j}\partial_k u^{\alpha_j})-\partial_k(\partial_k u_i^{\alpha_j}\nabla u_i^{\alpha_j})\right)\end{aligned}$$

To back the main theorem we have to pass to the limit in the equation (31). Since u^{α_j} satisfies (31) then u^{α_j} satisfies the estimates in Lemma 5.2. Consequently, we can extract from (u^{α_j}) a subsequence denoted by the same symbol such that

$$\begin{aligned}
&u^{\alpha_j} \rightharpoonup v \text{ weak-}* \text{ in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^\infty(0,T;\mathbb{H})),\\
&u^{\alpha_j} \rightharpoonup v \text{ weakly in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;\mathbb{V})).
\end{aligned} \tag{32}$$

We derive from (18)-(20) that

$$R(u^{\alpha_j}) \to 0 \text{ in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;\mathbb{H}^{-4})).$$

Thus

$$(R(u^{\alpha_j}), \phi) \to 0 \text{ in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T)), \forall \phi \in \mathcal{V}.$$

Since A is linear and strongly continuous then owing to (32) we have

$$Au^{\alpha_j} \rightharpoonup Av \text{ weakly in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;\mathbb{H}^{-1})).$$

Hence

$$< Au^{\alpha_j}, \phi > \rightharpoonup < Av, \phi > \text{ weakly in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T)) \text{ for any } \phi \in \mathcal{V}.$$

We derive from (27), the estimate (9) of Lemma 5.2 and Vitali's Theorem that

$$u^{\alpha_j} \to v \text{ strongly in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;\mathbb{H})). \tag{33}$$

For any element $\zeta \in L^\infty(\bar{\Omega} \times [0,T], d\bar{P} \otimes dt)$ and for any $\phi \in \mathcal{V}$ we have

$$E\int_0^T < B(u^{\alpha_j}, u^{\alpha_j}), \zeta\phi > dt = -\sum_{i,k} E\int_{D\times[0,T]} u_i^{\alpha_j} \partial_i\phi_k \zeta u_k^{\alpha_j} dx \otimes dt.$$

Owing to (33)

$$\zeta \partial_i \phi_k u_k^{\alpha_j} \to \zeta \partial_i \phi_k v_k \text{ strongly in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^2(0,T;\mathbb{H})).$$

This and (33) again imply that

$$\begin{aligned}
-\sum_{i,k} E\int_{D\times[0,T]} u_i^{\alpha_j} \partial_i\phi_k \zeta u_k^{\alpha_j} dx \otimes dt \to &-\sum_{i,k} E\int_{D\times[0,T]} v_i \partial_i \phi_k \zeta v_k dx \otimes dt\\
&= E\int_0^T < B(v,v), \zeta\phi > dt.
\end{aligned}$$

That is

$$< u^{\alpha_j}.\nabla u^{\alpha_j}, \phi > \rightharpoonup < v.\nabla v, \phi > \text{ weakly in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^\infty(0,T)) \text{ for any } \phi \in \mathcal{V}.$$

We readily have that

$$\int_0^t (G,\phi) d\mathcal{W}^{\alpha_j} \rightharpoonup \int_0^t (G,\phi) d\bar{\mathcal{W}} \text{ weakly in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^\infty(0,T)) \text{ for any } \phi \in \mathcal{V}.$$

We have that

$$\begin{aligned}
(u^{\alpha_j}, \phi)_{\mathbb{V}} &= (u^{\alpha_j} - \alpha_j \Delta u^{\alpha_j}, \phi)\\
&= (u^{\alpha_j}, \phi) + \alpha_j((u^{\alpha_j}, \phi)).
\end{aligned}$$

This implies that

$$(u^{\alpha_j} - \alpha_j \Delta u^{\alpha_j} - v, \phi) = (u^{\alpha_j} - v, \phi) + \alpha_j((u^{\alpha_j}, \phi)).$$

It follows from Lemma 5.2 and (33) that

$$(u^{\alpha_j} - \alpha_j \Delta u^{\alpha_j} - v, \phi) \to 0 \text{ strongly in } L^2(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}; L^\infty(0,T)) \text{ for any } \phi \in \mathcal{V}.$$

Using all these convergences we can derive from (31) that the following holds almost surely

$$(v,\phi) + \nu\int_0^t \{((v,\phi)) + (\mathbb{P}(v.\nabla v), \phi)\} ds = (u_0, \phi) + \int_0^t (F(v), \phi) ds + \int_0^t (G,\phi) d\bar{\mathcal{W}},$$

for any $\phi \in \mathcal{V}$ and $t \in [0,T]$. That is the system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}^t, \bar{P}); (\bar{\mathcal{W}}, v)$ is a weak solution of the stochastic Navier-Stokes equations.
Owing to the estimates

$$\bar{E} \sup_{0\leq s\leq T} |v(s)|^{\frac{p}{2}} + \nu \bar{E}\left(\int_0^T ||v(s)||^2 ds\right)^{\frac{P}{2}} < \infty,$$

the $\mathbb{H}$-valued process $v(.)$ has almost surely a weak-continuous modification. This ends the proof of Theorem 3.5.

5. Proof of Theorem 3.7

The main ingredients of the proof of Theorem 3.7 are the pathwise uniqueness for the two-dimensional stochastic Navier-Stokes equations and the following lemma whose proof can be found in [20].

Lemma 5.1. *Let X be a Polish space. A sequence of a X-valued random variables $\{x_n; n \geq 0\}$ converges in probability if and only if for every subsequence of joint probability laws, $\{\nu_{n_k,m_k}; k \geq 0\}$, there exists a further subsequence which converges weakly to a probability measure ν such that*

$$\nu\left(\{(x,y) \in X \times X; x = y\}\right) = 1.$$

Now, let

$$\mathfrak{S} = L^2(0,T;\mathbb{H}) \times C(0,T;\mathbb{R}^m),$$
$$\mathfrak{S}^{\mathbb{H}} = L^2(0,T;\mathbb{H}), \quad \mathfrak{S}^{\mathcal{W}} = C(0,T:\mathbb{R}^m),$$

and

$$\mathfrak{S}^{\mathbb{H},\mathbb{H}} = L^2(0,T;\mathbb{H}) \times L^2(0,T;\mathbb{H}).$$

For any $S \in \mathcal{B}(\mathfrak{S}^{\mathbb{H}})$ we set $\Pi^\alpha(S) = P(u^\alpha \in S)$, and $\Pi^\alpha_{\mathcal{W}}(S) = P(\mathcal{W} \in S)$ for $S \in \mathcal{B}(\mathfrak{S}^{\mathcal{W}})$. Next, we define the joint probability laws

$$\Pi^{\alpha,\beta} = \Pi^\alpha \times \Pi^\beta,$$
$$\nu^{\alpha,\beta} = \Pi^\alpha \times \Pi^\beta \times \Pi^\alpha_{\mathcal{W}}.$$

The following tightness property holds.

Lemma 5.2. *The collection $\nu^{\alpha,\beta}$ (and hence any subsequence $\{\nu^{\alpha_j,\beta_j}\}$) is tight on $\mathfrak{S}^{\mathbb{H},\mathbb{H}} \times \mathfrak{S}^{\mathcal{W}}$.*

Proof. The proof is very similar to Theorem 4.7. For any $\varepsilon > 0$ we choose the sets $\Sigma_\varepsilon, Y_\varepsilon$ exactly as in the proof of Theorem 4.7 with appropriate modification on the constants $M_\varepsilon, L_\varepsilon$ so that $\Pi^\alpha(Y_\varepsilon) \geq 1 - \frac{\varepsilon}{4}$ and $\Pi_{\mathcal{W}}(\Sigma_\varepsilon) \geq 1 - \frac{\varepsilon}{2}$ for every $\alpha \in (0,1)$. Now let us take $K_\varepsilon = Y_\varepsilon \times Y_\varepsilon \times \Sigma_\varepsilon$ which is a compact in $\mathfrak{S}$; it is not difficult to see that $\{\nu^{\alpha,\beta}(K_\varepsilon) \geq (1-\frac{\varepsilon}{4})^2(1-\frac{\varepsilon}{2}) \geq 1-\varepsilon$ for all α, β. This completes the proof of the lemma. □

Lemma 5.2 implies that there exists a subsequence from $\{\nu^{\alpha_j,\beta_j}\}$ still denoted by $\{\nu^{\alpha_j,\beta_j}\}$ which converges to a probability measure ν. By Skorokhod's theorem there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ on which a sequence $(u^{\alpha_j}, v^{\beta_j}, \mathcal{W}^j)$ is defined and converges almost surely in $\mathfrak{S}^{\mathbb{H},\mathbb{H}} \times \mathfrak{S}^{\mathcal{W}}$ to a couple of random variables (u, v, W). Furthermore, we have

$$Law(u^{\alpha_j}, v^{\beta_j}, \mathcal{W}^j) = \nu^{\alpha_j,\beta_j},$$
$$Law(u, v, W) = \nu.$$

Now let $Z^u_j = (u^{\alpha_j}, \mathcal{W}^j)$, $Z^v_j = (v^{\beta_j}, \mathcal{W}^j)$, $Z^u = (u, W)$ and $Z^v = (v, W)$ We can infer from the above argument that $\left(\Pi^{\alpha_j,\beta_j}\right)$ converges to a measure Π such that

$$\Pi(\cdot) = \bar{P}((u,v) \in \cdot).$$

As above we can show that Z_j^u and Z_j^v satisfy Theorem 4.8 and that Z^u and Z^v satisfy the stochastic Navier-Stokes equation (see (3.4)) on the same stochastic system $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}), \bar{\mathcal{F}}^t, W$. Since we are in two-dimensional case then we see that $u(0) = v(0)$ almost surely and $u = v$ in $L^2(0, T; \mathbb{H})$. Therefore

$$\Pi\left(\{(x, y) \in \mathfrak{S}^{\mathbb{H},\mathbb{H}}; x = y\}\right) = \bar{P}\left(u = v \text{ in } L^2(0, T; \mathbb{H})\right) = 1.$$

This fact together with Lemma 5.1 imply that the original sequence (u^α) defined on the original probability space $(\Omega, \mathcal{F}, P)$ converges in probability to an element v in the topology of $\mathfrak{S}^{\mathbb{H}}$. By a passage to the limits argument as in the previous section it is not difficult to show that v is the unique solution of the stochastic Navier-Stokes Equations (on the original probability system $(\Omega, \mathcal{F}, \mathbb{P}), \mathcal{F}^t, W)$. This ends the proof of Theorem 3.7.

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