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EULERIAN DERIVATION OF NON-INERTIAL NAVIER-STOKES EQUATIONS FOR COMPRESSIBLE FLOW IN CONSTANT, PURE ROTATION

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ABSTRACT

This paper presents an Eulerian Derivation of the non-inertial Navier-Stokes equations for compressible flow in constant, pure rotation. The method that is presented is an extension of work done by Kageyama & Hyodo [4] where the momentum equation in incompressible conditions was derived. It is shown that no fictitious effects are present in the non-inertial mass and energy conservation equations. The origin of the Coriolis and centrifugal forces in the momentum equation is shown.

INTRODUCTION

This paper extends the application of the method of Kageyama & Hyodo [4] and to the full set of Navier-Stokes equations in compressible flow for constant rotation conditions.

The method was first presented to only derive the non-inertial momentum equation in incompressible flow. It is used here to show that non-inertial effects are not present in the conservation of mass and energy equations respectively.

During a literature study numerous instances were observed where fictitious effects were added to the conservation of energy equation in the same manner as what it is added to the momentum equation [3,5]. Since mass and energy are scalar quantities it should follow logically that the equations should remain unchanged between inertial and non-inertial frames, but it had to be shown in a manner that is mathematically rigorous. The method is of general use since it can be extended for application in any transformation between scalar values. It is also clear, so that no misconceptions with regards to the form of the final equations can occur.

The equations in the inertial frame are transformed to the rotational frame with the use of two operations on the vector fields. The first is a modified Galilean transformation that transforms the vector fields from the inertial frame to an orientation preserving frame. The second is a rotational transformation that transforms the vectors from the orientation preserving frame to the rotational frame.

FRAME TRANSFORMATION

Assume that three (3) frames exist; O, O’ and Ô as indicated in Figure 1. Frame O is the stationary, inertial frame. Frame O’ is an orientation preserving frame ($\mathbf{i}$ and $\mathbf{i}$’has the same orientation), which can be either inertial or non-inertial depending on the cases analyzed. This frame shares an origin with the rotational frame Ô. Frame Ô is the non-inertial, rotational frame and is therefore not orientation preserving.

Now consider a point P which can be observed from all the frames. Point P is rotating around the origin of frame O, but it is stationary in frames O’ and Ô. The set of equations will be developed to describe the motion of point P in the rotational frame Ô.

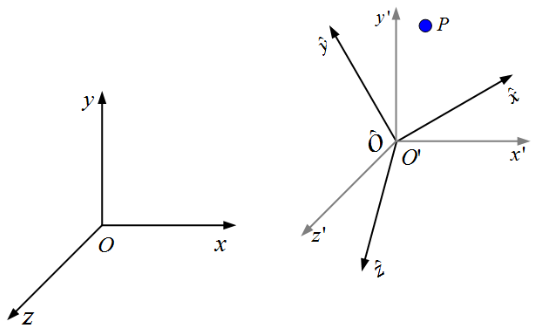

Figure 1. Frame Descriptions

This point is described in frame O from where a modified Galilean transformation, G^{M}, will be used to describe it in frame O’. The rotational transform, $R^{\Omega t}$, will then be used to transform the resulting equations (as described in frame O’) to the rotational frame Ô. In order to simplify this derivation the assumption will be made that for this specific case all the frames share a common origin.

Modified Galilean Transformation

The standard Galilean transform is limited in its application to constant translation in the x-direction. Kageyama & Hyodo [4] modified it to accommodate constant rotational conditions.

The Galilean transform is used to transform between two reference frames that only differ by a constant vector of motion. In Figure 2 such a motion is described between frame O and O'.

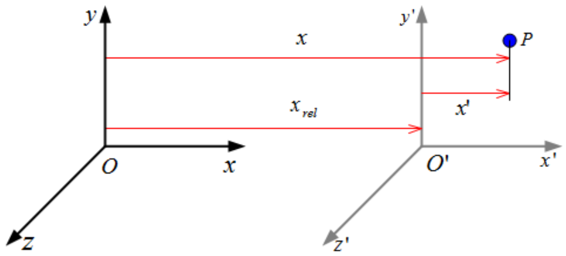

Figure 2. Galilean Transformation between Frames

Assume that the origins of the two frames intersect at time $t=0$ and that frame O' is moving at a constant velocity $\mathbf{V}$ in the x-direction. At time t = Δt, the frames O and O' are then distance x_{rel} from each other. The relationship between the co-ordinates points for this single event between frames O and O' is described by Equation 1. This is known as the standard Galilean transform.

$$\begin{aligned} x' &= x - V\Delta t \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{1}$$

Lets assume at this point that the constant motion need not be in the x-direction alone and that it can be presented as a vector of motion as shown in Figure 3. Lets further assume that it can be used to described constant motion in rotation as well.

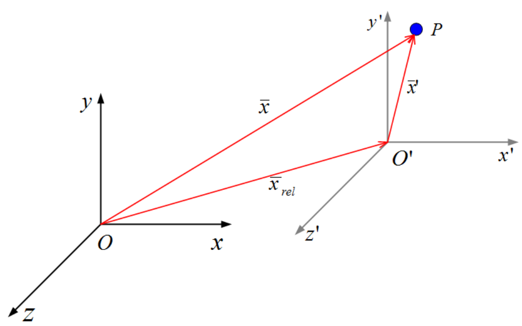

Figure 3. Modified Galilean Transformation between Frames

In order to simplify this case let all the frames share the same origin and let the point P be stationary in the rotational frame Ô. Therefore point P is rotating with a constant angular velocity around the origin or the inertial frame O.

The $\mathbf{x_{rel}}$ component can then be described as:

$$\mathbf{x}_{\mathrm{rel}} = \mathbf{V}\Delta t \tag{2}$$

where

$$\mathbf{V} = \mathbf{\Omega} \wedge \mathbf{x} \tag{3}$$

The modified Galilean transform operator is introduced such that any vector observed from the inertial frame O can be related to the vector observed from the orientation preserving frame O' as:

$$\mathbf{u}'(\mathbf{x}',t) = G^{\mathrm{M}}\mathbf{u}(\mathbf{x},t) \tag{4}$$

This definition will lead to a mathematical description to directly relate the vector fields in the inertial frame O, to the vector fields in the orientation preserving frame O':

$$\begin{aligned} \mathbf{u}'(\mathbf{x}',t) &= G^{\mathrm{M}}\mathbf{u}(\mathbf{x},t) \\ &= \mathbf{u}(\mathbf{x},t) - \mathbf{\Omega} \wedge \mathbf{x} \end{aligned} \tag{5}$$

Rotational Transform

Since frame Ô shares an origin with the frame O' the vector components in Ô is related to O' by defining a rotational transform, $R^{\Omega t}$. Equation 6 can be used to describe a vector as seen from frame Ô in relation to a vector in frame O.

$$\begin{aligned} \hat{\mathbf{u}}(\hat{\mathbf{x}},t) &= R^{\Omega t}\mathbf{u}'(\mathbf{x}',t) \\ &= R^{\Omega t} G^{\Omega \wedge x}\mathbf{u}(\mathbf{x},t) \end{aligned} \tag{6}$$

$R^{\Omega t}$ is therefore the rotational transform that operates on $\mathbf{x}'$ to obtain the $\hat{\mathbf{x}}$ co-ordinates in the rotational frame.

From Equation 5 and 6 it can be derived that for the velocity vector the following relation holds:

$$\hat{\mathbf{u}}(\hat{\mathbf{x}},t) = R^{\Omega t}\{\mathbf{u}(\mathbf{x},t) + \mathbf{x} \wedge \mathbf{\Omega}\} \tag{7}$$

Lets assume, for convenience sake, that the rotation is around the z-axis of frame O. The vector $\mathbf{\Omega}$ is then described as $\mathbf{\Omega} = (0,0,\Omega)$. The rotational transform in this case will be described by the following tensor:

$$R^{\Omega t} = \begin{bmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Now that the modified Galilean invariance and the rotational transform has been described for constant rotational conditions, both can be used in the derivation of the non-inertial Navier-Stokes equations for constant rotation.

TRANSFORMATION OF THE NAVIER STOKES EQUATIONS

In this section the non-inertial Navier-Stokes equations for constant rotation in compressible flow will be derived. Each term in the equation will be transformed separately. Once this has been completed the summation of the transformed terms will lead to the final equation in the rotational frame.

Continuity Equation

The conservation of mass in the inertial frame takes the form [7]:

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \mathbf{u}) = 0 \tag{9}$$

The first term represents the temporal change in density due to compressibility of the flow. The second term is the divergence of density and velocity which represents the residual mass flux of a given control volume.

Scalars, such as density and mass flux, are invariant under Galilean transformation. The first term in the inertial frame can therefore be directly equated to the non-inertial frame:

$$\frac{\partial \hat{\rho}}{\partial t} = R^{\Omega t} \frac{\partial \rho}{\partial t} \tag{10}$$

The second term of the continuity equation will be affected by both frame transformations since it contains the velocity vector:

$$\begin{aligned}(\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}}) &= R^{\Omega t} G^{\mathbf{\Omega} \wedge \mathbf{x}} (\nabla \cdot \rho \mathbf{u}) \\ &= R^{\Omega t} [\nabla \cdot \rho (G^{\mathbf{\Omega} \wedge \mathbf{x}} \mathbf{u})]\end{aligned} \tag{11}$$

Equation 7 is used to complete the modified Galilean transformation, and the equation becomes:

$$(\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}}) = R^{\Omega t} \{\nabla \cdot \rho (\mathbf{u} + \mathbf{x} \wedge \mathbf{\Omega})\} \tag{12}$$

The second term of the equation above falls away since the divergence of the cross product of distance and rotation is zero. The equation thus becomes:

$$\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = R^{\Omega t} (\nabla \cdot \rho \mathbf{u}) \tag{13}$$

The addition of Equation 10 and Equation 13 leads to a relation between the continuity equation in the inertial and rotational frames:

$$\frac{\partial \hat{\rho}}{\partial t} + \hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = R^{\Omega t} \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u}\right) \tag{14}$$

The right hand side of the equation is equal to zero since this represents the continuity equation in the inertial frame (Equation 9):

$$\frac{\partial \hat{\rho}}{\partial t} + \hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} = 0 \tag{15}$$

Conservation of Momentum Equation

The incompressible form of the momentum equation as shown in Kageyama and Hyodo [4], made the assumption that the change in density is negligible. Therefore, the equation could be simplified by dividing density into all the terms as there are no temporal or spatial gradients in density. The diffusion term could be simplified in a manner that would facilitate easy transformation where the divergence of the gradient of velocity yields the same result as taking the laplacian of the velocity. This is not the case when compressibility has to be accounted for. The compressible Navier-Stokes Equation in the inertial frame will take the form:

$$\frac{\partial}{\partial t} \rho \mathbf{u} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) \quad = -\nabla p + \nabla \cdot [\mu (\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \lambda (\nabla \cdot \mathbf{u}) \mathbf{I}] \tag{16}$$

First consider the unsteady term in the rotational frame and apply the product rule for partial derivatives. This operation will result in two terms that was not considered during the incompressible case:

$$\frac{\partial}{\partial t} (\hat{\rho}\hat{\mathbf{u}}) = \hat{\rho} \frac{\partial \hat{\mathbf{u}}}{\partial t} + \hat{\mathbf{u}} \frac{\partial \hat{\rho}}{\partial t} \tag{17}$$

With the aid of Equations 6 (which defined the transformation between the frames) and further manipulation (10) the above (shown in [1]) becomes:

$$\frac{\partial}{\partial t} (\hat{\rho}\hat{\mathbf{u}}) = R^{\Omega t} G^{\mathbf{\Omega} \wedge \mathbf{x}} [\rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{\Omega} \wedge \mathbf{x}) \cdot \nabla \mathbf{u} - \rho \mathbf{\Omega} \wedge \mathbf{u} + \mathbf{u} \frac{\partial \rho}{\partial t}] \tag{18}$$

The product rule is then used to combine the terms $\rho \frac{\partial \mathbf{u}}{\partial t}$ and $\mathbf{u} \frac{\partial \rho}{\partial t}$ so that the equation simplifies to:

$$\frac{\partial}{\partial t} (\hat{\rho}\hat{\mathbf{u}}) = R^{\Omega t} [\frac{\partial}{\partial t} (\rho) + \rho (\mathbf{\Omega} \wedge \mathbf{x}) \cdot \nabla - \rho \mathbf{\Omega} \wedge] G^{\mathbf{\Omega} \wedge \mathbf{x}} \mathbf{u} \tag{19}$$

Manipulation of this equation [1,4] will lead to the final form of the temporal transformed term:

$$\frac{\partial}{\partial t} (\hat{\rho}\hat{\mathbf{u}}) = R^{\Omega t} [\frac{\partial}{\partial t} (\rho) + \rho (\mathbf{\Omega} \wedge \mathbf{x}) \cdot \nabla - \rho \mathbf{\Omega} \wedge] \mathbf{u} \tag{20}$$

The relation between the frames for the advection term in the compressible Navier-Stokes momentum equation, and expanded using Equation 5, is:

$$\begin{aligned}\hat{\nabla} \cdot (\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) \quad &= R^{\Omega t} \{\nabla \cdot \rho [(\mathbf{u} + \mathbf{x} \wedge \mathbf{\Omega}) \otimes (\mathbf{u} + \mathbf{x} \wedge \mathbf{\Omega})]\} \\ &= R^{\Omega t} \{(\nabla \cdot \rho \mathbf{u}) \otimes \mathbf{u} + (\nabla \cdot \rho \mathbf{u}) \otimes (\mathbf{x} \wedge \mathbf{\Omega}) \\ &\quad + [\nabla \cdot \rho (\mathbf{x} \wedge \mathbf{\Omega})] \otimes \mathbf{u} + [\nabla \cdot \rho (\mathbf{x} \wedge \mathbf{\Omega})] \otimes (\mathbf{x} \wedge \mathbf{\Omega})\}\end{aligned} \tag{21}$$

The identity below can be used to simplify the equation.

$$(\nabla \cdot \mathbf{a}) \otimes (\mathbf{x} \wedge \boldsymbol{\Omega}) = \mathbf{a} \wedge \boldsymbol{\Omega} \tag{22}$$

This will lead to the following expression for relating the diffusion term in the rotational frame to the terms in the inertial frame:

$$\begin{aligned}\hat{\nabla} \cdot (\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) &= R^{\Omega t}[\nabla \cdot \rho\mathbf{u} \otimes \mathbf{u} + \rho\mathbf{u} \wedge \boldsymbol{\Omega} \\ &\quad + (\nabla \cdot \rho(\mathbf{x} \wedge \boldsymbol{\Omega}) \otimes \mathbf{u} + (\rho\mathbf{x} \wedge \boldsymbol{\Omega}) \wedge \boldsymbol{\Omega}]\end{aligned} \tag{23}$$

The pressure gradient term in the momentum equation is transformed in and this part of the equation remain invariant since it is a scalar.

$$\begin{aligned}\hat{\nabla}\hat{p} &= R^{\Omega t} G^{\boldsymbol{\Omega}\wedge\mathbf{x}} \nabla p \\ \hat{\nabla}\hat{p} &= R^{\Omega t} \nabla p\end{aligned} \tag{24}$$

In the transformation of the diffusion term the difference between the compressible and incompressible cases must be noted. The divergence of the velocity vector is not equal to zero, therefore the completed diffusion term must be accounted for. The expression for relating the diffusion term between the frames hence becomes:

$$\begin{aligned}&\hat{\nabla} \cdot [\hat{\mu}(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T) + \hat{\lambda}(\hat{\nabla} \cdot \hat{\mathbf{u}})\hat{\mathbf{I}}] \\ &= R^{\Omega t} G^{\boldsymbol{\Omega}\wedge\mathbf{x}} \nabla \cdot [\mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I}]\end{aligned} \tag{25}$$

With the implementation of Equation 7, the right hand side of the equations becomes:

$$\begin{aligned}R^{\Omega t} \nabla \cdot \{\mu[\nabla(\mathbf{u} + \mathbf{x} \wedge \boldsymbol{\Omega}) &+ \nabla(\mathbf{u} + \mathbf{x} \wedge \boldsymbol{\Omega})^T] \\ &+ \lambda(\nabla \cdot (\mathbf{u} + \mathbf{x} \wedge \boldsymbol{\Omega}))\mathbf{I}\}\end{aligned} \tag{26}$$

If it is considered that,

$$\nabla(\mathbf{x} \wedge \boldsymbol{\Omega}) + \nabla(\mathbf{x} \wedge \boldsymbol{\Omega})^T = 0 \tag{27}$$

and

$$\nabla \cdot (\mathbf{x} \wedge \boldsymbol{\Omega}) = 0 \tag{28}$$

It can be shown that the diffusion component of the momentum equation is invariant for constant rotation conditions:

$$\begin{aligned}&\hat{\nabla} \cdot [\hat{\mu}(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T) + \hat{\lambda}(\hat{\nabla} \cdot \hat{\mathbf{u}})\hat{\mathbf{I}} \\ &= R^{\Omega t} \nabla \cdot [\mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I}]\end{aligned} \tag{29}$$

Summation of the transformed parts of the momentum equation lead to the final form of the compressible, non-inertial momentum equation:

$$\begin{aligned}\frac{\partial \hat{\rho}\hat{\mathbf{u}}}{\partial t} + \hat{\nabla} \cdot (\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) &= -\hat{\nabla}\hat{p} + \hat{\nabla} \cdot [\hat{\mu}(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T) + \hat{\lambda}(\hat{\nabla} \cdot \hat{\mathbf{u}})\hat{\mathbf{I}}] \\ &\quad + 2\rho\hat{\mathbf{u}} \wedge \boldsymbol{\Omega} - \rho\hat{\mathbf{x}} \wedge \boldsymbol{\Omega} \wedge \boldsymbol{\Omega}\end{aligned} \tag{30}$$

Conservation of Energy Equation

The general energy equation in the inertial frame takes the following form:

$$\frac{\partial \rho\varepsilon}{\partial t} + (\nabla \cdot \rho\varepsilon\mathbf{u}) = -p(\nabla \cdot \mathbf{u}) + \nabla \cdot (k\nabla T) + \Phi \tag{31}$$

The time dependant term is transformed in a similar manner to the pressure term in the momentum equation since internal energy is a scalar. It was already discussed that scalars are invariant under transformation. The first term is therefore transformed though the following operation:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} = R^{\Omega t} \frac{\partial \rho\varepsilon}{\partial t} \tag{32}$$

The convective term is transformed between the frames with the use of the rotational transform, modified Galilean transform and by substitution of Equation 7

$$\begin{aligned}(\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) &= R^{\Omega t} G^{\boldsymbol{\Omega}\wedge\mathbf{x}} (\nabla \cdot \rho\varepsilon\mathbf{u}) \\ &= R^{\Omega t}[\nabla \cdot \rho\varepsilon\mathbf{u} + \nabla \cdot \rho\varepsilon(\mathbf{x} \wedge \boldsymbol{\Omega})]\end{aligned} \tag{33}$$

It was already shown in Equation 28 that the second term on the right hand side is equal to zero. Therefore the transformed equation becomes:

$$(\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) = R^{\Omega t} (\nabla \cdot \rho\varepsilon\mathbf{u}) \tag{34}$$

The terms that represents the rate of work done by the normal pressure forces is transform between the frames and Equation 7 is inserted:

$$\begin{aligned}-\hat{p}(\hat{\nabla} \cdot \hat{\mathbf{u}}) &= R^{\Omega t} G^{\boldsymbol{\Omega}\wedge\mathbf{x}}[-p(\nabla \cdot \mathbf{u})] \\ &= R^{\Omega t}[-p\nabla \cdot \mathbf{u} + -p\nabla \cdot (\mathbf{x} \wedge \boldsymbol{\Omega})]\end{aligned} \tag{35}$$

It can be shown that this terms is also invariant under transformation by the insertion of Equation 28:

$$-\hat{p}(\hat{\nabla} \cdot \hat{\mathbf{u}}) = R^{\Omega t}[-p\nabla \cdot \mathbf{u}] \tag{36}$$

The diffusion is invariant under transformation since the heat transfer coefficient (k) and temperature (T) are scalars. The transformation between the frames then becomes:

$$\begin{aligned}\hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) &= R^{\Omega t} G^{\boldsymbol{\Omega}\wedge\mathbf{x}}[\nabla \cdot (k\nabla T)] \\ &= R^{\Omega t}[\nabla \cdot (k\nabla T)]\end{aligned} \tag{37}$$

The parameter Φ is a scalar value that represents the rate at which mechanical energy is expended in the process of deformation of the fluid due to viscosity [6]. The dissipation function is a scalar and therefore invariant under transformation:

$$\hat{\Phi} = R^{\Omega t} G^{\Omega \wedge x} \Phi = R^{\Omega t} \Phi \tag{38}$$

All the transformed terms of the energy equation is summed to obtain the equation below.

$$\begin{aligned} &\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} + (\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) + \hat{p}(\hat{\nabla} \cdot \hat{\mathbf{u}}) - \hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) + \hat{\Phi} = \\ &R^{\Omega t}[\frac{\partial \rho\varepsilon}{\partial t} + (\nabla \cdot \rho\varepsilon\mathbf{u}) + p(\nabla \cdot \mathbf{u}) - \nabla \cdot (k\nabla T) + \hat{\Phi}] \end{aligned} \tag{39}$$

The right hand side of the equation is equal to zero, as shown in Equation 31. The energy equation in the non-inertial frame for constant rotation is invariant under transformation in this specific case:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} + (\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) = -\hat{p}(\hat{\nabla} \cdot \hat{\mathbf{u}}) + \hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) + \hat{\Phi} \tag{40}$$

The lack of fictious work and energy terms in the non-inertial frame was confirmed by Diaz et. al [2] by using a point mass model.

INCOMPRESSIBLE EQUATIONS AS ASPECIAL CASE OF THE COMPRESSIBLE EQUATIONS

The assumption of incompressibility can be made when the flow velocity is substantially small (below Mach 0.3) so that the temporal changes in mass flux is close to zero. This is a special case of compressible flow that assumes that no temporal changes in density occurs. As such, if the compressible Navier-Stokes equations where derived correctly, applying the incompressible assumptions to it, should lead to the incompressible Navier-Stokes equations.

The compressible continuity equation in the rotational frame was determined to be:

$$\frac{\partial \hat{\rho}}{\partial t} + \hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} \quad = 0 \tag{41}$$

The applied assumption of incompressibility will result in the transient change in density being zero:

$$\frac{\partial \hat{\rho}}{\partial t} \quad = 0 \tag{42}$$

The equation therefore becomes:

$$\hat{\nabla} \cdot \hat{\rho}\hat{\mathbf{u}} \quad = 0 \tag{43}$$

This equation has the same form as it would have for incompressible flow in the inertial frame [1].

The derived, compressible momentum equation in the rotational frame is:

$$\begin{aligned} \frac{\partial \hat{\rho}\hat{\mathbf{u}}}{\partial t} + \hat{\nabla} \cdot (\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) \quad &= -\hat{\nabla}\hat{p} + \hat{\nabla} \cdot [\hat{\mu}(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T) + \hat{\lambda}(\hat{\nabla} \cdot \hat{\mathbf{u}})\hat{\mathbf{I}}] \\ &+ 2\rho\hat{\mathbf{u}} \wedge \mathbf{\Omega} - \rho\hat{\mathbf{x}} \wedge \mathbf{\Omega} \wedge \mathbf{\Omega} \end{aligned} \tag{44}$$

The first term that must be simplified to account from incompressibility is the diffusion term:

$$\hat{\nabla} \cdot \hat{\mu}\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla} \cdot [(\hat{\mu} + \hat{\lambda})(\hat{\nabla} \cdot \hat{\mathbf{u}})] \tag{45}$$

Lets consider the x-momentum components of the divergence of the deviatoric stress tensor. Assume in this instance that the dynamic viscosity μ is a constant. Simplify the relation below to obtain the form as shown below:

$$\begin{aligned} &\frac{\partial}{\partial x}(2\mu\frac{\partial u}{\partial t} + \lambda\nabla \cdot \mathbf{u}) + \frac{\partial}{\partial y}(\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})) + \frac{\partial}{\partial z}(\mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x})) \\ = \quad &\mu(2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x}) + \frac{\partial}{\partial x}(\lambda\nabla \cdot \mathbf{u}) \\ &= \mu\nabla^2 u + \mu\frac{\partial}{\partial x}(\nabla \cdot \mathbf{u}) + \frac{\partial}{\partial x}(\lambda\nabla \cdot \mathbf{u}) \end{aligned} \tag{46}$$

This relation can be written in the vector form to account for all the components of the diffusive momentum if it is assumed that the second viscosity, λ, is constant (Stokes Hypothesis [7]):

$$\hat{\nabla} \cdot \hat{\mu}\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla} \cdot [(\hat{\mu} + \hat{\lambda})(\hat{\nabla} \cdot \hat{\mathbf{u}})] \tag{47}$$

The second term in the relation above will be equal to zero if the incompressible continuity equation is substituted into the relation. This result in the following equation:

$$\begin{aligned} \frac{\partial \hat{\rho}\hat{\mathbf{u}}}{\partial t} \quad &+ \hat{\nabla} \cdot (\hat{\rho}\hat{\mathbf{u}} \otimes \hat{\mathbf{u}}) = -\hat{\nabla}\hat{p} + \hat{\nabla} \cdot \hat{\mu}\hat{\nabla}\hat{\mathbf{u}} \\ &+ 2\rho\hat{\mathbf{u}} \wedge \mathbf{\Omega} - \rho\hat{\mathbf{x}} \wedge \mathbf{\Omega} \wedge \mathbf{\Omega} \end{aligned} \tag{48}$$

Since density is constant in the equation above, it can be divided into the equation, which will lead to the non-inertial momentum equation:

$$\frac{\partial \hat{\mathbf{u}}}{\partial t} + (\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}} = -\hat{\nabla}\hat{\psi} + \nu\hat{\nabla}^2\hat{\mathbf{u}} + 2\hat{\mathbf{u}} \wedge \mathbf{\Omega} - \hat{\mathbf{x}} \wedge \mathbf{\Omega} \wedge \mathbf{\Omega} \tag{49}$$

This equation above is identical to the momentum equation presented in [1, 4].

The conservation of energy equation in the rotational frame for compressible flow is described by:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} + (\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) = -\hat{p}(\hat{\nabla} \cdot \hat{\mathbf{u}}) + \hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) + \Phi \tag{50}$$

If the continuity equation is applied to this equation it will result in:

$$\frac{\partial \hat{\rho}\hat{\varepsilon}}{\partial t} + (\hat{\nabla} \cdot \hat{\rho}\hat{\varepsilon}\hat{\mathbf{u}}) = \hat{\nabla} \cdot (\hat{k}\hat{\nabla}\hat{T}) + \hat{\Phi}_I \tag{51}$$

The energy equation above has the same form as it would have in the inertial frame for incompressible flow conditions as shown in [1].

It is therefore shown that there are no observed discrepancies between the derived equations for the compressible and incompressible cases in the rotational frame.

CONCLUSION

This paper presented an Eulerian derivation of the non-ineritial Navier-Stokes equations for compressible flow in pure rotational conditions.

It was shown that the continuity equation and the energy equation is invariant under transformation in all cases. Some instances have been observed in the literature where the fictitious effects were added to the energy equation due to misconception that arise when using the fluid parcel (Lagrangian) approach. This work indicates that no fictitious effects are present in the energy equation.

In the derivation of the non-inertial momentum equation the origin of the fictitious forces was shown. The Coriolis force originates from the transformation of both the transient and the adjective terms. The centrifugal force originates from the transformation of the advection term and is also present in all rotation cases.

The incompressible equations was derived as a special case of the compressible equation set. The incompressible momentum equation that was arrived at is the same as in shown by [1] and [4]. The continuity and energy equations, as expected, is of the same form in the incompressible inertial and non-inertial frames [2].

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NOMENCLATURE

Super Scripts and Sub Scripts

‘	Orientation preserving frame
^	Rotational frame
rel	Relative conditions
t	Time
Δt	Change in time

Alphanumeric Symbols

$\mathbf{a}$	Acceleration vector
$\mathbf{b}$	Vector
k	Heat transfer coefficient
p	Pressure per unit mass
t	Time
$\mathbf{u}$	Veloctiy vector
x	Distance in x-direction
$\mathbf{x}$	Position vector
y	Distance in y-direction
z	Distance in z-direction
G	Galilean operator
$\mathbf{I}$	Identity matrix
O	Frame designations
R	Rotational transform operator
T	Temperature
V	Velocity in x-direction
$\mathbf{V}$	Velocity vector
$\mathbf{X}$	Position vector

Greek Symbols

ε	Internal energy
λ	Second viscosity
μ	Dynamic viscosity
ν	Kinematic viscosity
ρ	Density
ψ	Specific pressure
Ω	Rotational speed around the z-axis
$\mathbf{\Omega}$	Rotational speed vector

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