Explicit formulae for limit periodic flows on networks

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Abstract

In recent papers it was shown that, under certain conditions, the C_0-semigroup describing a flow on a network (metric graph) that contains terminal strong (ergodic) components converges to the direct sum of periodic semigroups generated by the flows on these components. In this note we shall provide an explicit description of these limit semigroups in terms of the components of the adjacency matrix of the line graph of the network. The result is based on the Frobenius–Perron theory and the estimates of long term behaviour of iterates reducible matrices.

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1. Introduction

Dynamical problems on networks have been considered recently in many papers, see e.g. [1, 3, 5, 6, 9, 13, 15] and also the monograph [16]. Here we focus on transport on networks, modelled by a system of first order transport equations on the edges of the digraph representing the network. The equations are coupled by Kirchhoff's conditions at the vertices of the diagraph. The problem lends itself to a semigroup theoretical approach which, in particular, allows for proving elegant results on the long term behaviour of the flow on the network, see e.g. [1, 9, 14, 17]. The main result of these papers is that, under certain assumptions on the structure of the network and on the speeds of the flow along each edge, the flow is asymptotically periodic. More precisely, the semigroup generated by the flow converges exponentially to the direct sum of periodic semigroups on strong terminal (ergodic) components of the network. While theoretically very interesting, such a result is somehow incomplete from the practical point of view as it does not provide any explicit formulae for the limit semigroups. In other words, to be able to use the limit periodic semigroups as an approximation of the original flow, one needs to provide the appropriate initial conditions for them and this requires precisely knowing how the material from the transient parts enters the ergodic part of the network.

In this paper we complete the theory by deriving the explicit formulae for the action of the limit semigroups by expressing them by the coefficients of the weighted adjacency matrix of the line graph of the original network.

The approach is based on the proof of asymptotic periodicity given in [1] that uses the representation of the flow as the composition of the iterates of the adjacency matrix of the line graph and the solutions of the transport equations on the edges. Then limit periodic semigroups are then defined by the spectral decomposition of this matrix. If the adjacency matrix in question is primitive, then the relation between its iterates and its spectral decomposition is well-known. However, in general, the adjacency matrix related to the flow is reducible with imprimitive irreducible components and its spectral decomposition becomes too complicated for practical purposes. Though for such matrices there is a well-developed theory of Cesàro limits, see e.g. [19, Section 8.4] or [12, Section 8.6], it only provides information about the averaged behaviour of the iterates and thus it is not immediately useful to analyse the finer details of the limit periodic structure of the flow.

We observe that the structure of spectra of imprimitive reducible matrices recently has come under scrutiny, e.g. in [18], but the goals of the *op. cit.* are different from ours. To the author's knowledge there have been few papers dealing with iterates of reducible or imprimitive matrices. In [8] the author focused on reducible matrices with primitive diagonal blocks. The reference [20] specifically deals with Leslie matrices which might be imprimitive and, in Section 3, the author uses an idea that is similar to ours but stops short of finding the formulae for the limit dynamics. None of these results can be directly applied to qualitatively describe the limit periodic semigroups of a network flow which is the main aim of this note.

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2. Transport on networks

We consider a network represented by a simple (no loops or multiple edges) directed metric graph

$$G = (V(G), E(G)) = (\{v_1, \dots, v_r\}, \{e_1, \dots, e_m\})$$

with r vertices $v_1, \dots, v_r$ and m edges (arcs), $e_1, \dots, e_m$. It is assumed that G is connected but not necessarily strongly connected. To shorten notation, we let $\mathcal{M} = \{1, \dots, m\}$. If the vertex v_i has more than one outgoing edge, we place a weight $w_{ij} > 0$ on the outgoing edge e_j such that, for each v_i,

$$\sum_{e_j \text{outgoing from } v_i} w_{ij} = 1.$$

The line graph Q of G is defined by $Q = (V(Q), E(Q)) = (E(G), E(Q))$, where

$$E(Q) \quad = \quad \{uv;\ u, v \in E(G), \text{the head of } u \text{ coincides with the tail of } v\}.$$

By $\mathbb{B}$ we denote the weighted adjacency matrix for the line graph whose coefficients are given by

$$b_{ij} = \begin{cases} w_{ki} & \text{if } \exists_k e_j \stackrel{v_k}{\to} e_i, \\ 0 & \text{otherwise.} \end{cases}$$

If there is an outgoing edge at each vertex, then $\mathbb{B}$ is column stochastic.

A vertex v will be called a source if there are no incoming edges towards it and a sink if there are no edges outgoing from it.

We are interested in a flow on a closed network G. We parameterise the edges of G so as to identify each of them with the interval $[0,1]$, where 0 is the tail and 1 is the head. Then we write any function on G as $\mathbf{u}(x) = (u_j(x))_{j\in\mathcal{M}}, x \in [0,1]$. The flow on G is described by $\mathbf{u}(x,t) = (u_j(x,t))_{j\in\mathcal{M}}$, where $u_j(x,t)$ is the density of particles at position $x \in [0,1]$ and time $t \geq 0$, moving along edge e_j from the tail 0 to the head 1 with a continuous velocity $c_j(x) > 0$. The standard assumption is that the flow satisfies the Kirchhoff law at each vertex v_i,

$$\sum_{\text{incoming edges}} c_j(1)u_j(1,t) = \sum_{\text{outgoing edges}} c_j(0)u_j(0,t), \quad t > 0, \tag{1}$$

which in this context expresses mass conservation at each vertex. Let $\mathbb{C}(x) = \operatorname{diag}(c_j(x))_{j\in\mathcal{M}}$ and $\mathbb{K} = (k_{ij})_{i,j\in\mathcal{M}} = \mathbb{C}^{-1}(0)\mathbb{B}\mathbb{C}(1)$. Then the above transport problem can be written as

$$\begin{cases} \partial_t \mathbf{u}(x,t) + \partial_x(\mathbb{C}(x)\mathbf{u}(x,t)) = 0, \quad x \in (0,1), \quad t \geq 0, \\ \mathbf{u}(0,t) = \mathbb{K}\mathbf{u}(1,t), \\ \mathbf{u}(x,0) = \mathbf{f}(x), \end{cases} \tag{2}$$

if (and only if) G does not contain a sink, [1]. In short, we consider in $\mathbf{X} = (L_1([0,1]))^m$ the abstract Cauchy problem

$$\mathbf{u}_t = A\mathbf{u}, \qquad \mathbf{u}(0) = \mathbb{K}\mathbf{u}(1), \qquad \mathbf{u}(0) = \mathbf{f}, \tag{3}$$

where A is the realization of the expression $\mathsf{A}\mathbf{u} = (-\partial_x(c_j u))_{j\in\mathcal{M}}$ on the domain $D(A) = \{\mathbf{u} \in (W_1^1([0,1]))^m;\ \mathbf{u}(0) = \mathbb{K}\mathbf{u}(1)\}$. There is a strongly continuous semigroup $(T_A(t))_{t\geq 0}$ solving (3), (even if $\mathbb{K}$ is an arbitrary matrix, [3]).

This semigroup has nontrivial long term asymptotic behaviour if the speeds c_j, $j \in \mathcal{M}$, are somehow commensurate. Here we assume that

$$\exists_{c\in\mathbb{R}} \forall_{j\in\mathcal{M}} \ \ c\int_0^1 \frac{ds}{c_j(s)} = l_j \in \mathbb{N}. \tag{4}$$

For constant speeds, this condition is the same as in [1, 14], but it is less general than in [17, Definition 4.8]. Under (4), we rescale time as $\tau = ct$ and, for each $j \in \mathcal{M}$, make the change $v_j = c_j u_j$ and introduce the spatial variable

$$y_j = c\int_0^x \frac{ds}{c_j(s)}.$$

This converts (2) into an analogous problem but with the unit speed on each edge e_j, the matrix $\mathbb{K}$ changed into $\mathbb{B}$ but with the edge e_j having length l_j. However, since each l_j is a natural number, we can subdivide each interval $(0, l_j)$ into l_j intervals of unit length. This will create a new graph, say $\mathcal{G}$, with $\chi(m)$ new edges, where the dividing points become new vertices. Then a function $\mathbf{u}$ on G will be transformed into a vector of $\chi(m)$ functions which are defined on the new edges and are assumed continuous across the new vertices. This continuity requirement defines the Kirchhoff conditions at the new vertices, while the Kirchhoff conditions at the old vertices are left unchanged, see [1]. Since the problem on $\mathcal{G}$ has the same structure as the original one on G, there is a semigroup $(\mathcal{T}_{\mathcal{A}}(t))_{t\geq 0}$ solving it. It can be proved that the above construction provides an isomorphism $\mathbb{S} : \mathbf{X} \to \mathcal{X} := (L_1([0,1]))^{\chi(m)}$ such that

$$T_A(t)\mathbf{u} = \mathbb{S}^{-1}\mathcal{T}_{\mathcal{A}}(ct)\mathbb{S}\mathbf{u}, \qquad \mathbf{u} \in \mathbf{X}. \tag{5}$$

The motivation behind the representation (5) is [1, Eq. (17)] (which is an extension of [10, Proposition 20]) which states that for any $\mathbf{U} \in \mathcal{X}$

$$([T_{\mathcal{A}}(t)]\mathbf{U})(x) = [\mathcal{K}^n\mathbf{U}](n + x - t), \quad n \in \mathbb{N}, \quad 0 \leq n + x - t < 1, \tag{6}$$

(with $([T_{\mathcal{A}}(t)]\mathbf{U})(x) = \mathbf{U}(x - t)$ for $t \leq x < 1$) where, with some abuse of notation, we use the same symbol $\mathcal{K}$ to denote both the adjacency matrix of the line graph of $\mathcal{G}$ and the operator on $\mathcal{X}$ induced by the pointwise multiplication of a function $\mathbf{f} \in \mathcal{X}$ by the matrix $\mathcal{K}$.

Hence, in what follows we will assume that in (2) we have $c_j = 1$, $j \in \mathcal{M}$. Therefore $\mathbb{K} = \mathbb{B}$ and hence the analysis of the long term behaviour of $(T_A(t))_{t\geq 0}$ reduces to that of the iterates of $\mathbb{B}$ (see [1] for the precise relation between the quantitative properties of $(T_A(t))_{t\geq 0}$ and the model with unit speeds).

If G is connected, but not strongly connected, then one can subdivide G, and so the line graph $Q = L(G)$, into the acyclic part, the part containing transient strong components and the ergodic (terminal) strong components, [4]. Strong components of G correspond in a one-to-one manner to strong components of Q, [1]. Then, by simultaneous permutation of rows and columns, the adjacency matrix of Q, $\mathbb{B}$, can be put in the so-called normal form, [11, p. 90]. Moreover, by the topological sorting of Q (or [7, Theorem 9-16]), we see that in the part of $\mathbb{B}$ corresponding to Q_1, the acyclic part of Q, the only nonzero entries can occur below the diagonal. Thus, reordering the vertices so that the sources in Q_1 appear first, followed by other vertices from Q_1, and noting that the sources correspond to zero rows, we obtain the representation

$$\mathbb{B} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbb{B}_{2,1} & \mathbb{B}_2 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{B}_3 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ \mathbb{B}_{g-1,1} & \cdots & \cdots & \mathbb{B}_{g-1,g-2} & \mathbb{B}_{g-1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbb{B}_{g,1} & \cdots & \cdots & \mathbb{B}_{g,g-2} & \mathbb{B}_{g,g-1} & \mathbb{T}_g & \mathbf{0} & \dots & \mathbf{0} \\ \mathbb{B}_{g+1,1} & \cdots & \cdots & \mathbb{B}_{g+1,g-2} & \mathbb{B}_{g+1,g-1} & \mathbf{0} & \mathbb{T}_{g+1} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{B}_{s,1} & \cdots & \cdots & \mathbb{B}_{s,g-2} & \mathbb{B}_{s,g-1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbb{T}_s \end{pmatrix}. \quad (7)$$

Since $\mathbb{B}$ is column stochastic, it has a positive left eigenvector which can be taken to be $\mathbf{1} = (1, \dots, 1)$. Using [11, Theorem 6, Chapter III], we find that for some $g > 2$ the spectral radii $\rho(\mathbb{T}_l)$ satisfy $\rho(\mathbb{T}_l) = 1$ for $g \leq l \leq s$, $\rho(\mathbb{B}_l) < 1$ for $2 < l < g$ and $\mathbb{B}_{l,k} = \mathbf{0}$ for $g+1 \leq l \leq s$ and $g \leq k \leq l-1$ (and certainly, $\mathbb{B}_{l,k} = \mathbf{0}$ above the (block) diagonal). The matrix $\mathbb{B}_2$ is nilpotent, the matrices $\mathbb{B}_l$ for $3 \leq l \leq g-1$, called transient, are either irreducible matrices of dimension $n_l \times n_l$, or zero matrices of dimension 1×1 (scalars). The matrices $\mathbb{T}_l$ for $g \leq l \leq s$ are called ergodic. Since we assumed that there are no loops in G, all diagonal entries in $\mathbb{B}$ are zero.

Due to the block triangular structure of $\mathbb{B}$, we have $\sigma(\mathbb{B}) = \{0\} \cup \sigma(\mathbb{B}_3) \cup \ldots \cup \sigma(\mathbb{B}_{g-1}) \cup \sigma(\mathbb{T}_g) \cup \ldots \cup \sigma(\mathbb{T}_s)$. Let $\sigma_p(\mathbb{B})$ denote the peripheral (of unit modulus) spectrum of $\mathbb{B}$. Eigenvalues in $\sigma_p(\mathbb{B})$ only can come from $\sigma(\mathbb{T}_l)$, $g \leq l \leq s$. Since each $\mathbb{T}_l$ is irreducible, by [19, Chapter 8, p. 696], $\sigma_p(\mathbb{T}_l)$ consists of simple eigenvalues and thus each eigenvalue of $\sigma_p(\mathbb{B})$ is semisimple. Further, if we denote by d_l the index of imprimitivity of $\mathbb{T}_l$; that is, the number of distinct unitary eigenvalues of $\mathbb{T}_l$, then these eigenvalues are of the form $\lambda_l^k = e^{2\pi i k/d_l}$, $k = 0, 1, \dots, d_l - 1$, and each of them is simple, [19, Chapter 8, p. 676]. Let $\mathbf{v}_l^k$ and $\mathbf{w}_l^k$ be, respectively, the right and left eigenvectors of $\mathbb{T}_l$ belonging to λ_l^k, normalized so as $\mathbf{v}_l^k \cdot \mathbf{w}_l^k = 1$. It is always possible as $\mathbf{w}_l^k$ is orthogonal to any right (associated) eigenvector of $\mathbb{T}_l$ belonging to $\lambda \in \sigma(\mathbb{T}_l) \setminus \{\lambda_l^k\}$ and, since λ_l^k is simple, if also $\mathbf{v}_l^k \cdot \mathbf{w}_l^k = 0$, then $\mathbf{w}_l^k = 0$. Let us define

$$\{\mathbf{V}_l^k\}_{g \leq l \leq s, 0 \leq k \leq d_l - 1} = \{(0, \dots, 0, \mathbf{v}_l^k, 0, \dots, 0)\}_{g \leq l \leq s, 0 \leq k \leq d_l - 1}, \quad (8)$$

where $\mathbf{v}_l^k$ occupies coordinates from $n_{l-1} + 1$ to n_l, corresponding to the indices of the block row of $\mathbb{T}_l$. Then $\{\mathbf{V}_l^k\}_{g \leq l \leq s, 0 \leq k \leq d_l - 1}$ spans all eigenspaces of $\mathbb{B}$ corresponding to $\sigma_p(\mathbb{B})$. If $\{\mathbf{W}_l^k\}_{g \leq l \leq s, 0 \leq k \leq d_l - 1}$ is the corresponding set of left eigenvectors of $\mathbb{B}$, then $\mathbf{W}_l^k$ contains $\mathbf{w}_l^k$ at the coordinates from $n_{l-1} + 1$ to n_l (the construction of $\mathbf{W}_l^k$ will be provided later) and hence $\mathbf{V}_l^k \cdot \mathbf{W}_l^k = 1$.

Theorem 2.1. *[1] There is a decomposition* $\mathbf{X} = \mathbf{X}_g \oplus \ldots \oplus \mathbf{X}_s \oplus \mathbf{Y}_e \oplus \mathbf{Y}_i$ *such that*

(i) the spaces $\mathbf{X}_l$, $l = g, \dots, s$, $\mathbf{Y}_e, \mathbf{Y}_i$, *are invariant under* $(T_A(t))_{t \geq 0}$*;*

(ii) $(T_A(t)|_{\mathbf{X}_l})_{t\geq 0}$ *is periodic with period* d_l, $l = g, \ldots, s$;

(iii) $(T_A(t)|_{\mathbf{Y}_e})_{t\geq 0}$ *is exponentially stable;*

(iv) $(T_A(t)|_{\mathbf{Y}_i})_{t\geq 0}$ *is nilpotent.*

We note that this theorem is valid under a more general condition than (4), that only involves commensurability of average times of traversing the ergodic cycles, [9, 17]. For our purpose we need an explicit expression of $(T_A(t))_{t\geq 0}$, that is the basis of the proof in [1]. It uses the spectral decomposition of $\mathbb{B}$: for any $\mathbf{x} \in \mathbb{R}^m$

$$\mathbb{B}^n\mathbf{x} = \sum_{l=g}^{s}\sum_{k=0}^{d_{l-1}} (\lambda_l^k)^n \mathbb{F}_{\lambda_l^k}\mathbf{x} + \sum_{\lambda\in\sigma(\mathbb{B})\setminus\sigma_p(\mathbb{B})} \lambda^n \mathbf{p}_\lambda(n)\mathbf{x}, \quad n \in \mathbb{N}, \tag{9}$$

where $\mathbf{p}_\lambda(n)$ is a matrix valued polynomial in n of order smaller than the algebraic multiplicity of λ (which does not change for larger n) and $\mathbb{F}_{\lambda_l^k}$ is the spectral projection

$$\mathbb{F}_{\lambda_l^k}\mathbf{x} = (\mathbf{W}_l^k \cdot \mathbf{x})\mathbf{V}_l^k, \quad \mathbf{x} \in \mathbb{R}^m, l = g, \ldots, s, 0 \leq k \leq d_l - 1. \tag{10}$$

Recalling (6) and (9), we have, for $\mathbf{u} \in \mathbf{X}, n \in \mathbb{N},\ 0 \leq n + x - t < 1$,

$$\begin{aligned}[]
[T_A(t)\mathbf{u}](x) &= [\mathbb{B}^n\mathbf{u}](n + x - t) \qquad\qquad (11)\\
&= \sum_{l=g}^{s}\sum_{k=0}^{d_{l-1}} (\lambda_l^k)^n [\mathbb{F}_{\lambda_l^k}\mathbf{u}](n + x - t) + \sum_{\lambda\in\sigma(\mathbb{B})\setminus\sigma_p(\mathbb{B})} \lambda^n [\mathbf{p}_\lambda(n)\mathbf{u}](n + x - t).
\end{aligned}$$

The second term decays exponentially. Let us fix $l = g, \ldots, s$, and denote $\mathcal{R}^l = \{\mathbf{x} \in \mathbb{R}^m;\ (0, \ldots, 0, x_{n_{l-1}+1}, \ldots, x_{n_l}, 0, \ldots, 0)\}$ and $\mathbf{X}_l = L_1([0, 1], \mathcal{R}^l)$ (isomorphic to $L_1([0, 1], \mathbb{R}^{n_l - n_{l-1}})$). Then $\mathbf{X}_l$ is invariant under $(T_A(t))_{t\geq 0}$ and, defining

$$[T_{A,l}(t)\mathbf{u}](x) := \sum_{k=0}^{d_{l-1}} (\lambda_l^k)^n [\mathbb{F}_{\lambda_l^k}\mathbf{u}](n + x - t), \tag{12}$$

$0 \leq n + x - t < 1, n \in \mathbb{N}$, we see that $(T_{A,l}(t))_{t\geq 0}$ restricted to $\mathbf{X}_l$ extends to a periodic group with period equal to the imprimitivity index d_l of $\mathbb{T}_l$. Summarizing, we have

$$\left\| T_A(t)\mathbf{u} - \sum_{l=g}^{s} T_{A,l}(t)\mathbf{u} \right\|_{\mathbf{X}} \leq M_\omega e^{-\omega t}\|\mathbf{u}\|_{\mathbf{X}}, \tag{13}$$

where $\omega = -\ln\alpha$ and $\alpha < 1$ is any number satisfying $\alpha > \max|\sigma(\mathbb{B}) \setminus \sigma_p(\mathbb{B})|$.

3. Explicit formulae for $(T_{A,l}(t))_{t\geq 0}$

In the first step we simplify $\mathbb{B}$. By writing (7) as

$$\mathbb{B} = \left(\begin{array}{c|cccc} \mathcal{B}_0 & \mathbf{0} & \dots & \mathbf{0} \\ \hline \mathcal{B}_g & \mathbb{T}_g & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{B}_s & \mathbf{0} & \cdots & \mathbb{T}_s \end{array}\right),$$

where $\mathcal{B}_l = (\mathbb{B}_{l,1}, \dots \mathbb{B}_{l,g-1})$ for $l = g, \dots, s$, we have

$$\mathbb{B}^n = \left(\begin{array}{c|cccc} \mathcal{B}_0^n & \mathbf{0} & \dots & \mathbf{0} \\ \hline \mathcal{Z}_{g,n} & \mathbb{T}_g^n & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Z}_{s,n} & \mathbf{0} & \cdots & \mathbb{T}_s^n \end{array}\right),$$

where

$$\mathcal{Z}_{l,n} = \sum_{k=0}^{n-1} \mathbb{T}_l^k \mathcal{B}_l \mathcal{B}_0^{n-1-k}, \quad l = g, \dots, s.$$

This shows that each block-row $l = g, \dots, s$ only contains inputs from the block-matrix $\mathcal{B}_0$ (the transient part) and from itself but not from other block-rows of the ergodic part. Hence, it is sufficient to consider separately iterates of

$$\bar{\mathbb{B}} = \left(\begin{array}{c|c} \mathcal{B}_0 & \mathbf{0} \\ \hline \mathcal{B}_l & \mathbb{T}_l \end{array}\right) \tag{14}$$

for $l = g, \dots, s$. Note that, in general, $\bar{\mathbb{B}}$ is no longer column stochastic but, as before $\rho(\mathcal{B}_0) < 1$ and $\rho(\mathbb{T}_l) = 1$ with $\sigma_p(\mathbb{T}_l)$ (and thus $\sigma_p(\bar{\mathbb{B}})$) consisting of simple eigenvalues. The long term behaviour of $\bar{\mathbb{B}}^n$ crucially depends on the matrix $\mathbb{T}_l$. There are two cases:

1. $\mathbb{T}_l$ is primitive;
2. $\mathbb{T}_l$ is imprimitive.

In what follows we shall drop index l from the notation.

3.1. Primitive $\mathbb{T}$

This case is well-known, see [8, 12, 19], but we shall recall it to make the considerations in the reducible case more clear. Here, 1 is a strictly dominant eigenvalue with strictly positive eigenvector $\mathbf{v}$ normalized so as $\mathbf{1}\cdot\mathbf{v} = 1$. Then $\mathbf{V} = (\mathbf{0}, \mathbf{v})$ is the eigenvector of $\bar{\mathbb{B}}$ corresponding to the eigenvalue 1 of $\bar{\mathbb{B}}$. Since, in general, $\bar{\mathbb{B}}$ is no longer column stochastic, its left eigenvector $\mathbf{W} = (\mathbf{w}_1, \mathbf{w}_2)$ is found from

$$\mathbf{w}_1\mathcal{B}_0 + \mathbf{w}_2\mathcal{B} = \mathbf{w}_1, \qquad \mathbf{w}_2\mathbb{T} = \mathbf{w}_2;$$

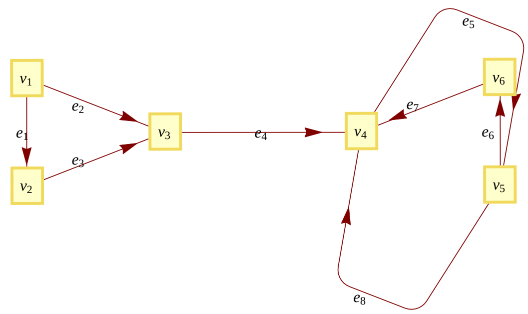

Figure 1: Network in Example 1.

hence $\mathbf{w}_2 = \mathbf{1}$ and, introducing by $\mathcal{R}(\lambda, \mathcal{B}_0) = (\lambda\mathcal{I} - \mathcal{B}_0)^{-1}$ the resolvent of $\mathcal{B}_0$, $\mathbf{w}_1 = [\mathbf{1}\mathcal{B}]\mathcal{R}(1, \mathcal{B}_0)$. Then, for $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$

$$\lim_{n\to\infty} \bar{\mathbb{B}}^n \mathbf{x} = \left(\begin{array}{c|c} \lim\limits_{n\to\infty} \mathcal{B}_0^n & \mathbf{0} \\ \hline \lim\limits_{n\to\infty} \mathcal{Z}_n & \lim\limits_{n\to\infty} \mathbb{T}^n \end{array}\right)\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{v}\otimes\mathbf{u}_1 & \mathbf{v}\otimes\mathbf{1} \end{array}\right)\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = (\mathbf{U}\cdot\mathbf{x})\mathbf{V},$$

where, for vectors $\mathbf{a}, \mathbf{b}$ we define $\mathbf{a}\otimes\mathbf{b} = (a_i b_j)_{i,j}$ and, as it will be established later in a more general situation,

$$\lim_{n\to\infty} \mathcal{Z}_n = \sum_{k=0}^{\infty} \mathbb{T}^k \mathcal{B}\mathcal{B}_0^{n-1-k} = \mathbf{v}\otimes[\mathbf{1}\mathcal{B}](\mathcal{I} - \mathcal{B}_0)^{-1}. \tag{15}$$

Example 1. Consider transport on the network given on Fig. 1, occurring according to the adjacency matrix $\mathbb{B}$ of the line graph given by

$$\mathbb{B} = \left(\begin{array}{c|c} \mathcal{B}_0 & \mathbf{0} \\ \hline \mathcal{B} & \mathbb{T} \end{array}\right) = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \end{array}\right). \tag{16}$$

The characteristic equation of the 4×4 ergodic part $\mathbb{T}$ is $\lambda(\lambda^3 - 0.5\lambda - 0.5) = 0$, with eigenvalues $1, 0, -0.5 \pm 0.5i$, and thus $\mathbb{T}$ is primitive. We have

$$\mathcal{R}(1, \mathcal{B}_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad [\mathbf{1}\mathcal{B}]\mathcal{R}(1, \mathcal{B}_0) = \mathbf{1}.$$

Since the Perron eigenvector of $\mathbb{T}$ is given by $\mathbf{v} = \frac{1}{5}(2,1,1,1)$, we find that the flow on G asymptotically is given by the semigroup

$$[T(t)\mathbf{f}](x) = \sum_{i=1}^{8} f_i(n+x-t)(\mathbf{0},\mathbf{v}), \qquad 0 \le n+x-t < 1, n \in \mathbb{N}.$$

In other words, the flow becomes separated into periodic flows on disconnected edges of the ergodic part of G, weighted by the components of the Perron eigenvectors. Note that there is no periodic flow on the network despite the fact that the ergodic part of G consists of cycles.

3.2. Imprimitive $\mathbb{T}$

If $\mathbb{T}$ is imprimitive then, by the Perron-Frobenius theory, there are additional $d-1$ eigenvalues on the unit circle, given by $\lambda^l = e^{\frac{2\pi i}{d}l}, l = 1,\ldots,d-1$, for some $d \geq 2$. As in (8), we denote by $\mathbf{v}^l$ and $\mathbf{w}^l$ the right and, respectively, left eigenvector of $\mathbb{T}$ belonging to λ^l and by

$$\mathbf{V}^l = (\mathbf{0},\mathbf{v}^l), \qquad \mathbf{W}^l = (\mathbf{w}_1^l,\mathbf{w}_2^l) = ([\mathbf{w}^l\mathcal{B}]\mathcal{R}(\lambda^l,\mathcal{B}_0),\mathbf{w}^l),$$

the corresponding eigenvectors of $\bar{\mathbb{B}}$, normalized so that $\mathbf{U}^l \cdot \mathbf{V}^l = 1$. This allows for the following asymptotic expression

$$\bar{\mathbb{B}}^n\mathbf{x} = (\mathbf{W}\cdot\mathbf{x})\mathbf{V} + e^{\frac{2\pi i}{d}n}(\mathbf{W}^1\cdot\mathbf{x})\mathbf{V}^1 + \ldots + e^{\frac{2\pi i}{d}(d-1)n}(\mathbf{W}^{d-1}\cdot\mathbf{x})\mathbf{V}^{d-1} + O(\alpha^n), \quad (17)$$

where α is an arbitrary number satisfying $\max\{|\sigma(\bar{\mathbb{B}}) \setminus \sigma_p(\bar{\mathbb{B}})|\} < \alpha < 1$ and $\mathbf{V}$ and $\mathbf{W}$ are the Perron eigenvectors of $\bar{\mathbb{B}}$, as in Subsection 3.1.

Our aim is to express the coefficients $(\mathbf{W}\cdot\mathbf{x})\mathbf{V}$ and $(\mathbf{W}^l\cdot\mathbf{x})\mathbf{V}^l$, $l = 1,\ldots,d-1$, explicitly in terms of coefficients of $\bar{\mathbb{B}}$, as then we will have formulae for the spectral projections $\mathbb{F}_{\lambda^l}$ of the limit periodic semigroups in (12).

Theorem 3.1. *Let* $\mathbf{x} \in \mathbb{R}^m$ *and* $r = 0,1,\ldots,d-1$. *Define*

$$\mathbf{F}_r(\mathbf{x}) = \left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \sum\limits_{m=0}^{d-1} \mathbb{T}^{r+m}\mathcal{B}\mathcal{B}_0^{d-m-1}(\mathcal{I}-\mathcal{B}_0^d)^{-1} & \mathbb{T}^r \end{array}\right)\mathbf{x}. \quad (18)$$

Then

$$(\mathbf{W}\cdot\mathbf{x})\mathbf{V} = \frac{1}{d}\sum_{j=0}^{d-1}\mathbf{F}_j(\mathbf{x}), \qquad (\mathbf{W}^r\cdot\mathbf{x})\mathbf{V}^r = \frac{1}{d}\sum_{j=0}^{d-1} e^{-\frac{2\pi i}{d}rj}\mathbf{F}_j(\mathbf{x}). \quad (19)$$

Proof. We have

$$\bar{\mathbb{B}}^n\mathbf{x} = \left(\begin{array}{c|c} \mathcal{B}_0^n & \mathbf{0} \\ \hline \mathcal{Z}_n & \mathbb{T}^n \end{array}\right)\mathbf{x}, \quad (20)$$

where $\mathcal{Z}_n = \sum\limits_{k=0}^{n-1} \mathbb{T}^k\mathcal{B}\mathcal{B}_0^{n-1-k}$ as in (15), but this time neither $\mathbb{T}^n$ nor $\mathcal{Z}_n$ has a limit as $n \to \infty$. We first show that in analysing $\mathcal{Z}_n$ for large n we may assume that $\mathbb{T}$ is periodic; that is, $\sigma(\mathbb{T}) = \{e^{\frac{2\pi i}{d} j}\}_{0\le j\le d-1}$. Indeed, by the spectral decomposition of $\mathbb{T}$,

$$\mathbb{T}^n\mathbf{x} = (\mathbf{1}\cdot\mathbf{x})\mathbf{v} + \sum_{j=1}^{d-1} e^{\frac{2\pi i}{d} jn}(\mathbf{u}^j\cdot\mathbf{x})\mathbf{v}^j + \sum_{\lambda\in\sigma(\mathbb{T}),|\lambda|<1} \lambda^n\mathbf{p}_\lambda(n)\mathbf{x} = \mathbb{T}_0^n\mathbf{x} + \mathbb{T}_1^n\mathbf{x},$$

where $\|\mathbb{T}_1^n\| \le M_1\alpha^n$ for some M_1. Then, since $\sigma(\mathcal{B}_0) < 1$ yields $\|\mathcal{B}_0^n\| \le M_2\beta^n$ (where M_2 is a constant and we can choose $\alpha \ne \beta < 1$), we obtain

$$\left\|\sum_{k=0}^{n-1}\mathbb{T}_1^k\mathcal{B}\mathcal{B}_0^{n-1-k}\right\| \le M_1M_2\beta^{n-1}\|\mathcal{B}\|\sum_{k=0}^{n-1}\left(\frac{\alpha}{\beta}\right)^k = M_1M_2\beta^{n-1}\frac{\beta^n-\alpha^n}{\beta-\alpha}.$$

Hence, by $\alpha,\beta<1$,

$$\lim_{n\to\infty}\left(\mathcal{Z}_n - \sum_{k=0}^{n-1}\mathbb{T}_0^k\mathcal{B}\mathcal{B}_0^{n-1-k}\right) = 0.$$

Therefore in what follows we assume that $\sigma(\mathbb{T}) = \{e^{\frac{2\pi i}{d} j}\}_{0\le j\le d-1}$.Then we have $\mathbb{T}^{pd+r} = \mathbb{T}^r$. Consider now $\mathcal{Z}_n$. For a given $r = 0, 1, d-1$, and $p \in \mathbb{N}_0$

$$\begin{aligned}
\mathcal{Z}_{pd+r+1} &= \mathcal{B}\mathcal{B}_0^{dp+r} + \ldots + \mathbb{T}^r\mathcal{B}\mathcal{B}_0^{dp} + \sum_{k=r+1}^{dp+r}\mathbb{T}^k\mathcal{B}\mathcal{B}_0^{dp+r-k}\\
&= \mathcal{B}\mathcal{B}_0^{dp+r} + \ldots + \mathbb{T}^r\mathcal{B}\mathcal{B}_0^{dp} + \sum_{l=0}^{dp-1}\mathbb{T}^{l+r+1}\mathcal{B}\mathcal{B}_0^{dp+l-1}.
\end{aligned}$$

Since $\mathbb{T}^i\mathcal{B}\mathcal{B}_0^{dp+r-i}, i = 0, 1, \ldots, r$ converge to 0 as $p\to\infty$, we can focus on the last term. Here, grouping terms within each period

$$\sum_{l=0}^{dp-1}\mathbb{T}^{l+r+1}\mathcal{B}\mathcal{B}_0^{dp-l-1} = \mathbb{T}^r\sum_{m=0}^{d-1}\mathbb{T}^{m+1}\mathcal{B}\mathcal{B}_0^{d-m-1}\sum_{s=0}^{p-1}(\mathcal{B}_0^d)^s,$$

where we used the periodicity of $\mathbb{T}$. Thus

$$\lim_{p\to\infty}\mathcal{Z}_{pd+r+1} = \mathbb{T}^r\sum_{m=0}^{d-1}\mathbb{T}^{m+1}\mathcal{B}\mathcal{B}_0^{d-m-1}\mathcal{R}(1,\mathcal{B}_0^d). \tag{21}$$

Thus, using (18), (20) and (17), we get for $r = 0, 1, \ldots, d-1$

$$\lim_{p\to\infty}\bar{\mathbb{B}}^{pd+r}\mathbf{x} = (\mathbf{W}\cdot\mathbf{x})\mathbf{V} + e^{\frac{2\pi i}{d} r}(\mathbf{W}^1\cdot\mathbf{x})\mathbf{V}^1 + \ldots + e^{\frac{2\pi i}{d}(d-1)r}(\mathbf{W}^{d-1}\cdot\mathbf{x})\mathbf{V}^{d-1} = \mathbf{F}_r(\mathbf{x}).$$

In other words, we have the following system of equations for $\mathbf{C}_0 = (\mathbf{W}\cdot\mathbf{x})\mathbf{V}$ and $\mathbf{C}_r = (\mathbf{W}^r\cdot\mathbf{x})\mathbf{V}^r, r = 1, \ldots, d-1$,

$$\begin{aligned}
\mathbf{C}_0 + \mathbf{C}_1 + \ldots + \mathbf{C}_{d-1} &= \mathbf{F}_0,\\
\mathbf{C}_0 + e^{\frac{2\pi i}{d} r}\mathbf{C}_1 + \ldots + e^{\frac{2\pi i}{d}(d-1)r}\mathbf{C}_{d-1} &= \mathbf{F}_r, \quad r = 1, \ldots, d-1.
\end{aligned}$$

We use the fact that for any $m \in \mathbb{Z}$, which is not 0 or multiple of d, we have

$$\sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}mk} = \frac{1 - e^{\frac{2\pi i}{d}md}}{1 - e^{\frac{2\pi i}{d}m}} = 0.$$

Now, fix $0 \le r \le d-1$ and multiply row j by $e^{-\frac{2\pi i}{d}rj}$. Then the coefficient at $\mathbf{C}_k, k \neq r$, in row j will become $e^{\frac{2\pi i}{d}(k-r)j}$. Since $k-r$ is not a multiple of d, adding together the equations we obtain, for $r = 1, \dots, d-1$,

$$\mathbf{C}_0 = (\mathbf{W} \cdot \mathbf{x})\mathbf{V} = \frac{1}{d}\sum_{j=0}^{d-1} \mathbf{F}_j(\mathbf{x}), \qquad \mathbf{C}_r = (\mathbf{W}^r \cdot \mathbf{x})\mathbf{V}^r = \frac{1}{d}\sum_{j=0}^{d-1} e^{-\frac{2\pi i}{d}lj}\mathbf{F}_j(\mathbf{x}).$$

$\square$

Example 2. Consider the network given on Fig. 2, with the adjacency matrix of the line graph given by

$$\mathbb{B} = \left(\begin{array}{c|c} \mathcal{B}_0 & \mathbf{0} \\ \hline \mathcal{B} & \mathbb{T} \end{array}\right) = \left(\begin{array}{ccc|cc} 0 & \frac{1}{2} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

The eigenvalues of $\mathbb{T}$ are ± 1 and the eigenvalues of $\mathcal{B}_0$ are $\pm 1/\sqrt{2}, 0$ and thus

$$\mathbb{B}^n\mathbf{x} = (\mathbf{W} \cdot \mathbf{x})\mathbf{V} + (-1)^n(\mathbf{W}^1 \cdot \mathbf{x})\mathbf{V}^1 + O\left((\sqrt{2})^{-n}\right).$$

We have

$$\mathcal{R}(1, \mathcal{B}_0^2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and hence the bottom left blocks in $\mathbf{F}_0$ and $\mathbf{F}_1$ are, respectively,

$$\begin{aligned} \lim_{p\to\infty} \mathcal{Z}_{pd} &= (\mathcal{B}\mathcal{B}_0 + \mathbb{T}\mathcal{B})\mathcal{R}(1, \mathcal{B}_0^d) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \\ \lim_{p\to\infty} \mathcal{Z}_{pd+1} &= (\mathbb{T}\mathcal{B}\mathcal{B}_0 + \mathcal{B})\mathcal{R}(1, \mathcal{B}_0^d) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Thus, we have to solve the system

$$\left(\begin{array}{ccc|cc} & \mathbf{0} & & & \mathbf{0} \\ \hline 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \end{array}\right) = (\mathbf{W} \cdot \mathbf{x})\mathbf{V} + (\mathbf{W}^1 \cdot \mathbf{x})\mathbf{V}^1,$$

$$\left(\begin{array}{ccc|cc} & \mathbf{0} & & & \mathbf{0} \\ \hline -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right) = (\mathbf{W} \cdot \mathbf{x})\mathbf{V} - (\mathbf{W}^1 \cdot \mathbf{x})\mathbf{V}^1,$$

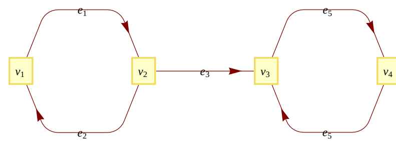

Figure 2: Network in Example 2

getting

$$
\begin{aligned}
(\mathbf{W} \cdot \mathbf{x})\mathbf{V} &= \frac{1}{2} \left(\begin{array}{ccc|cc} & \mathbf{0} & & \multicolumn{2}{c}{\mathbf{0}} \\ \hline -1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 2 & 1 & 1 \end{array} \right) \mathbf{x}, \\
(\mathbf{W}^1 \cdot \mathbf{x})\mathbf{V}^1 &= \frac{1}{2} \left(\begin{array}{ccc|cc} & \mathbf{0} & & \multicolumn{2}{c}{\mathbf{0}} \\ \hline 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 0 & -1 & 1 \end{array} \right) \mathbf{x}.
\end{aligned}
$$

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