Exploring Zambian Mathematics student teachers' content knowledge of functions and trigonometry for secondary schools

by

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DECLARATION

I declare that the thesis, which I hereby submit for the degree PhD in Mathematics Education at the University of Pretoria, is my own work and has not previously been submitted by me for a degree at this or any other tertiary institution.

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ABSTRACT

This was a qualitative case study that explored Zambian mathematics student teachers' content knowledge of functions and trigonometry in secondary schools. The students were in their final year of study and had studied advanced university mathematics and had completed Mathematics Education courses. Content knowledge was investigated as Common Content Knowledge (CCK) and Specialised Content Knowledge (SCK). Data was collected in two phases: Phase 1 utilised a mathematics test to gather CCK and SCK data from 22 conveniently chosen University of Zambia student teachers majoring in mathematics. Phase 2 used semi-structured interviews to collect SCK data from a sub-sample of six purposefully selected students.

Descriptive statistics and qualitative techniques were used to analyse the test data and content analysis to analyse the interviews. Although the students achieved a mean score of about 52% in the CCK of functions, an item by item analysis suggested that they were not proficient therein. They had a shallow understanding of composite functions, domains and ranges, extreme values, and turning points. They also had a superficial understanding of the definitions of concepts. While the students managed to identify functions, the majority could not coherently explain concepts and justify their reasoning. The students showed a limited understanding of the Cartesian plane representations of functions and the algebraic representation of quadratic functions.

The students achieved a mean score of approximately 53% in the CCK of trigonometry, and 68% of the sample achieved scores above 50%. An item by item analysis suggests that most of the students were proficient in CCK. However, the students could not comprehensively explain concepts and justify their reasoning. While most of the students could apply rules and formulas, they could not coherently explain and prove these. Similarly, they could not translate algebraic trigonometric functions to the Cartesian plane. Generally, there seemed to be a disconnection between the students' CCK and SCK of trigonometry.

These findings suggested that the study of advanced mathematics does not automatically result in students' comprehensive understanding of school mathematics. While the students had studied advanced UNZA mathematics, it was found out that they had not acquired an in-depth understanding of the functions and trigonometry required at secondary school level.

Keywords: Content Knowledge; Common Content Knowledge; Specialised Content Knowledge; Secondary schools; Student teachers; Scores; Mathematics major student teachers; Advanced university mathematics; Functions; Trigonometry.

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LIST OF ABBREVIATIONS

AIEMS	- Action to Improve English, Mathematics and Science
CCK	- Common Content Knowledge
CPD	- Continuing Professional Development
EAPS	- Educational Administration and Policy Studies
EEC	- European Economic Commission
EPPS	- Educational Psychology, Sociology and Special education
HCK	- Horizon Content Knowledge
KCC	- Knowledge of Content and Curriculum
KCS	- Knowledge of Content and Students
KCT	- Knowledge of Content and Teaching
LMT	- Learner Mathematical Thinking
MCK	- Mathematical Content Knowledge
ME	- Mathematics Education
MKT	- Mathematical Knowledge for Teaching
MOE	- Ministry of Education
MSE	- Mathematics and Science Education
MT	- Mathematics Test
SCK	- Specialised Content Knowledge
SE	- School of Education
SNS	- School of Natural Sciences
SPSS	- Statistical Package for Social Sciences
TE	- Teacher Education
UNZA	- University of Zambia
ZAMSTEP	- Zambia Mathematics and Science Teacher Education Project
ZCDC	- Zambian Curriculum Development Centre
ZECF	- Zambia Education Curriculum Framework

CHAPTER 1 INTRODUCTION

1.1 INTRODUCTION

In Zambia, the Ministry of Education (MOE) has for a long time recognised an acute shortage of degree holding teachers of mathematics (Ministry of Education, 1992). Consequently, the MOE is continuously engaged in efforts to alleviate the shortage of graduate teachers of mathematics. In this context, several initiatives have been undertaken (Ministry of Education, 1996), including the continued awarding of bursaries to secondary school mathematics student teachers to train at the University of Zambia (UNZA), and the engagement of UNZA to train graduate teachers of mathematics through what is called a ‘fast track’ arrangement. The latter is a distance education setup where practicing teachers who already hold a secondary school teachers’ diploma in mathematics are being trained with a view of upgrading to a degree level status. It is hoped that that in so doing, the deficiencies in the teaching of secondary school mathematics will be addressed, and consequently the problem of poor pupil performance in Grade 12 mathematics national examinations will have been solved (Ministry of Education, 2013b).

Apart from valuing the critical role that the UNZA was expected to play in teacher training in the post-independence era (Ministry of Education, 1977), the MOE has been involved in projects to improve mathematics education in the country. A typical example is a project called the Zambia Mathematics and Science Teacher Education Project (ZAMSTEP), which was undertaken by the MOE and the European Economic Commission (EEC) (Haambokoma, 2002). This project was established due to the shortage of graduate mathematics teachers who could teach senior secondary school mathematics in Zambia. The main aim of ZAMSTEP was to upgrade the content knowledge and teaching skills of diploma holding secondary school mathematics teachers. It was envisaged that the teachers whose content knowledge and teaching skills would be upgraded would be in a position to effectively teach senior secondary school mathematics.

Likewise, the Ministry of Education undertook another project between 1994 and the year 2000, Action to Improve English, Mathematics and Science (AIEMS), partly aimed at improving mathematics education in Zambia. In this regard, Teacher Resource Centres were established around the country with a view to facilitating decentralised in-service education of teachers. The idea was to promote a realisation among teachers that they needed to take a lead in their own Continuing Professional Development (CPD) in their subject areas (British Council, 1997).

Regardless of the successes that were achieved through these projects, it is argued that they had setbacks such as a loss of momentum when the donors' funding ended (Haambokoma, 2002). Unfortunately, instead of the teachers themselves identifying the professional development activities that need to be implemented, it was the Government agencies that identified these for the teachers (Haambokoma, 2002). This contributed to the exploration of alternative ways of enhancing sustainable teachers' CPD within secondary school mathematics and science. The intention was to involve mathematics and science subject associations in the country to promote teachers' CPD (Haambokoma, 2002). Notwithstanding, this approach focused on the practising teachers and did not directly include consideration of how thoroughly mathematics student teachers understand secondary school mathematics by the end of their training.

Currently, mathematics is one of the core subjects in the Zambian secondary school curriculum. Learners have continued to perform poorly in the Grade 12 national examinations in this subject (Ministry of Education, 2013b). The MOE (2013b) reports that "cumulatively, one-third of boys, and two-thirds of girls, have registered complete fail in mathematics since 2005, while only half of the boys and one-fifth of the girls have managed to obtain a pass or better" (p. 4). This failure rate could be attributed to several factors that may include teachers' lack of comprehensive understanding of the mathematics subject matter taught at secondary school level.

Although there is no direct relationship between teachers' qualifications and effective teaching (Watson & Harel, 2013), research has shown that teachers' mathematical content knowledge has an influence on the teaching of mathematics and student achievement (Ball, 1990; Ball, Thames, & Phelps, 2008; Campbell, Nishio, Smith, Clark, Conant, Rust & Choi, 2014; Hill, Rowan, & Ball, 2005; Ogbonnaya & Mogari, 2014). At the same time, a study involving primary school teachers showed that the majority of these lacked in-depth understanding of the mathematics intended for the levels that they taught (Venkat & Spaull, 2015). Another study indicated that mathematics educators who likely possessed metacognitive skills could not satisfactorily implement these skills in their classrooms (van der Walt & Maree, 2007). These two studies seem to be in line with the views that teacher knowledge is complex and that subject matter knowledge is not a sufficient requirement for effective teaching (Fennema, 1992). However, teachers require a thorough understanding of the subject matter for them to assist learners to acquire an understanding of the subject matter (Even, 1990; Shulman, 1986). Thus, the effective teaching of mathematics requires that teachers should acquire an in-depth understanding of the subject matter that they are supposed to teach (Ball, 1990; Ball et al., 2008; Nyikahadzoyi, 2013; Steele, Hillen, & Smith, 2013). This is consistent with ideas advanced by Bennett (1993), who argues that teachers ought to have "enough understanding of the subject to know which ideas are central, which are peripheral, how different ideas relate to one another, and how these ideas can be represented to the uninitiated" (p. 10).

According to the MOE (1996), the Zambian national policy on education attaches significance to teacher training and states that "the essential competencies required in every teacher are mastery of the material that is to be taught" (p. 108). By implication, the MOE recognises the necessity of secondary school mathematics teachers having sufficient understanding of the mathematics topics that they teach in schools. This is consistent with the aspiration that is enshrined in the Zambia Education Curriculum Framework (ZECF), which suggests that Teacher Education (TE) programmes should produce teachers who are competent in the subject matter that is taught (Ministry of Education, 2013b), for example, teacher educators can engage in research that investigates and describes how well impending graduate mathematics teachers know secondary school mathematics. In this regard, the present study assessed and explored UNZA's mathematics student teachers' content knowledge (herein also referred to as subject matter knowledge) of two secondary school mathematics topics, namely: functions and trigonometry. The reason for selecting these two topics will later be discussed in Section 1.2. The following section presents a discussion of the rationale and background to the study.

1.2 RATIONALE AND BACKGROUND TO THE STUDY

This section discusses the rationale of and background to the study, and begins with an overview relating to the researcher's personal interest in the research. The overview is followed by a discussion of the reasons that led to the research carried out in this study. The section ends with the researcher's justifications for the choice of the two topics under analysis: functions and trigonometry.

1.2.1 Personal interest in the study

The student's interest in the topic of this research developed when he was an undergraduate student in the UNZA mathematics education programme. It had always been of interest how knowledgeable the student teachers are in secondary school mathematics by the end of their university studies. As a lecturer of mathematics education at UNZA, the researcher has participated in the training and assessment of student teachers during their teaching practice. These opportunities deepened the researcher's interest in assessing and exploring student teachers' content knowledge of the topics they are supposed to teach. In the next sub-section, an explanation is given regarding the reasons why a content knowledge study was conducted on UNZA's student teachers.

1.2.2 Why conduct a content knowledge study involving UNZA's student teachers?

UNZA is the oldest Government owned institution that prepares graduate (degree holders) secondary school teachers in Zambia (Ministry of Education, 1996). It is also one of the universities in the country with a long history of preparing secondary school graduate mathematics teachers. It is reported that practising teachers who trained at UNZA have found gaps between the subject matter they studied during training and what is taught at secondary school (Masaiti & Manchishi, 2011). Nevertheless, there is a lack of research literature focusing on UNZA mathematics student teachers' content knowledge of secondary school mathematics. The closest literature available on the subject relates to two studies that were conducted in the Zambian context: (1) a qualitative case study that analysed the extent to which Learner Mathematical Thinking (LMT) is considered in the area of algebra at teacher education level (Nalube, 2014), and (2) a comparative study of classroom practices involving degree and diploma holding mathematics teachers; this study was conducted in selected schools on the

Copper belt, one of the ten provinces of Zambia (Maliwatu, 2011). Although mathematics student teachers formed part of the sample of Nalube's (2014) study; its focus was not on the exploration of the student teachers' content knowledge of functions and trigonometry at secondary school level. Similarly, Maliwatu's (2011) research was premised on a comparison of classroom practices and not an exploration of the student teachers' content knowledge. The findings of the research indicate that degree holding mathematics teachers had more competency in content knowledge than their counterparts who held secondary teachers' diplomas. Moreover, Maliwatu's study involved mathematics teachers and not mathematics student teachers. The situation highlighted above encouraged the researcher to make a contribution towards the growth of mathematics education literature in Zambia.

Another reason why UNZA student teachers were selected for this study is that the Government of the Republic of Zambia views the University of Zambia as a vitally important institution. Government contends that the knowledge that student teachers acquire during training at this institution would enable them to implement the school curriculum effectively (Ministry of Education, 2013b). This view motivated the researcher to investigate what student teachers trained at that university knew and how well they understood secondary school mathematics. The researcher wanted to find out whether, by the end of their training in mathematics, student teachers actually acquired a thorough understanding of functions and trigonometry, which they are supposed to teach upon graduation. In Section 1.2.3, the justifications for selecting functions and trigonometry for secondary schools as the topic under study are presented.

1.2.3 Justifying the selection of the topics functions and trigonometry

In order to gain an in-depth understanding of the student teachers' content knowledge, this study used two secondary school mathematics topics: functions and trigonometry. The idea of focusing on only two topics in the secondary school curriculum is consistent with Ball's (1990) suggestion that, "Examining a specific topic also makes more vivid the contrast between some key characteristics of what pre-service teachers have learned as students and what they need to know as teachers" (p. 451). The reasons why functions and trigonometry were specifically chosen for this study are explained below.

The concept of functions is considered as one of the most important topics in the mathematics curriculum and is viewed as a unifying concept (Dubinsky & Wilson, 2013; Even & Tirosh, 1995; Nyikahadzoyi, 2013). Some scholars have posited that the concept of functions is fundamentally significant in mathematics and that a strong understanding of functions contributes greatly to the further study of mathematics and related subjects (Even, 1998; Watson & Harel, 2013). Functions are relevant in topics such as trigonometry, and central to the study of pre-calculus, calculus, and subjects such as Physics (Akkoc, 2008), and are considered to be foundational in secondary school mathematics. However, research has shown that students struggle to understand the concept of functions (Bloch, 2003; Even, 1993; Spyrou & Zagorianakos, 2010). In Spyrou and Zagorianakos's (2010) study, 16 out of 17 students had problems in viewing 'many to one' relations as functions. These considerations influenced the selection of functions as a topic worth studying.

The choice of trigonometry was premised on the understanding that this is also a vital topic in school mathematics. Trigonometry enhances learners' reasoning skills, and offers wide problem solving and visual representation opportunities (Abdulkadir, 2013; Fi, 2003). Furthermore, trigonometry has applications in life areas such as navigation and in topics such as differentiation and integration, and affords the transition from algebra to geometry (Abdulkadir, 2013; Fi, 2003). In spite of its purported significance, scholars contend that trigonometry has not received much attention in mathematics education research (Akkoc, 2008; Fi, 2003, 2006). Moreover, some of the most recent research articles on the topic only deal with components of trigonometry such as degrees and radian measures, and trigonometric functions (Abdulkadir, 2013; Hiebert, 2013; Moore, 2012, 2013; Moore, LaForest, & Kim, 2012; Ogbonnaya & Mogari, 2014).

In an academic report, Zambian mathematics teachers disclosed that they found trigonometry in three dimensions difficult to teach, while pupils found graphs of functions and trigonometry in three dimensions difficult to learn (Haambokoma, 2002). A search on the UNZA research database shows that there has been no reported follow-up research conducted to specifically explore university mathematics student teachers' content knowledge of these two secondary school topics. There was therefore a compelling need to explore what mathematics student teachers at UNZA know of the subject matter of these school topics. Section 1.3 outlines the problem statement for this research.

1.3 PROBLEM STATEMENT

Although UNZA's student teachers in the mathematics education programme are training to be teachers of mathematics at secondary school level of education, secondary school mathematics is not offered as a course in this programme. The Mathematics Education (ME) courses offered by the department of Mathematics and Science Education (MSE) in the School of Education (SE), with the exception of MSE 131 (see Section 2.2.1 for details), deal with the generic aspects of teaching mathematics. In other words, secondary school mathematics topics are not specifically outlined in the ME courses, and are mainly considered indirectly during peer teaching. From the researcher's experience, individual mathematics teacher educators select secondary school mathematics topics to tackle, and the areas to be emphasised in those topics. Notwithstanding, student teachers study advanced mathematics courses that are purely content based and cover subject matter that goes beyond what is taught in mathematics at secondary school level (see Appendix 8). Advanced mathematics courses are offered by the School of Natural Sciences (SNS), although this school does not strictly exist to train mathematics teachers. Thus, the mathematics courses offered to student teachers by the SNS are the same as those that mathematicians in training study.

It would appear that the tacit view held by the designers of mathematics education degree programmes is that the mathematics content offered by the SNS is sufficient for the student teachers' understanding of secondary school mathematics. However, some scholars argue that secondary school mathematics has its 'own life', which is different from the mathematics that is taught in universities (Bromme, 1994; Bryan, 1999). Bromme (1994) posits that:

> The contents of learning mathematics are not just simplifications of mathematics as it is taught in universities. The school subjects have a 'life of their own' with their own logic; that is, the meaning of the concepts taught cannot be explained simply from the logic of the respective scientific disciplines. Or, in student terms: mathematics and 'math' are not the same (p. 74).

Bromme's (1994) argument suggests that knowledge of advanced university mathematics is not a guarantee that a teacher will have in-depth understanding of the content and pedagogy of secondary school mathematics. The assumption that a teacher who has studied high level mathematics automatically understands secondary school mathematics and can teach it effectively does not always hold (Akkoc, 2008; Ball, 1990; Even, 1990, 1993; Fi, 2003, 2006).

Other research has shown that student teachers' subject matter knowledge of secondary school mathematics lacks conceptual depth (Bryan, 1999; Hiebert, 2013). To be specific, a student teacher who may have studied several UNZA mathematics courses in the SNS and ME courses may still lack in-depth understanding of the subject matter required in secondary school mathematics topics. Nevertheless, what should be acknowledged is that knowledge of advanced mathematics can create a foundation for the comprehension of school mathematics (Cooney, 2003). According to Cooney (2003), this foundation is not a guarantee that a student teacher will know school mathematics because student teachers' "experiences with school mathematics are often limited to their mathematical experiences as teenagers, with all the immaturity that implies" (p. 804). Even and Tirosh (1995) suggest that teacher education "should explicitly refer to topics included in the high school curriculum, such as functions and undefined mathematical operations" (p. 18). They conclude that it is not safe to assume that teachers have sufficiently comprehensive and articulated content knowledge simply because they have studied those topics during their secondary school years. They contend, for example that teachers may sometimes know how to solve an algorithm (know content), but may not be able to explain why certain procedures are used (know why).

In respect to mathematics student teachers trained at UNZA, the concern relates to whether and how well they are able to demonstrate in-depth understanding of functions and trigonometry required at secondary school level by the end of their training. Lack of research literature addressing this concern has created a gap in our knowledge, and thus motivated this study. In summary, this study sought to assess, explore, and ultimately describe UNZA's mathematics student teachers' content knowledge of functions and trigonometry for secondary schools. In the absence of such a study, it was not known how effective the UNZA mathematics education programmes were in facilitating student teachers' acquisition of in-depth understanding of these two topics, which they are expected to teach upon graduation. The following section outlines the aims and objectives of the study.

1.4 AIMS AND OBJECTIVES OF THE STUDY

The aim of this study was to describe how UNZA's mathematics student teachers understood the content knowledge of functions and trigonometry at secondary school level. The specific objectives were to describe the proficiency of student teachers in Common Content Knowledge (CCK). At the same time, the study endeavoured to explore the Specialised Content Knowledge (SCK) that the student teachers held. In this regard, explorations of the student teachers' abilities to explain and justify concepts, conceptual relationships and differences, as well as their abilities to work with different representations were conducted. Furthermore, the study intended to elicit explanations and clarifications from the student teachers regarding their solutions to particular test items. Section 1.5 presents a summary of the significance of the study.

1.5 SIGNIFICANCE OF THE STUDY

This study was positioned to make a contribution to the development and growth of the UNZA mathematics education literature based on student teachers' content knowledge of functions and trigonometry at secondary school level. Apart from enriching the researcher's practice as a mathematics educator, results of the study provide information for stakeholders such as the Zambian Curriculum Development Centre (ZCDC) regarding the UNZA mathematics student teachers' understandings of secondary school mathematics. Particularly, the study provides information to stakeholders on whether UNZA mathematics student teachers do acquire proficiency in the CCK of functions and trigonometry at secondary school level by the end of their training. Furthermore, this study informs the mathematics education community with regard to the nature of the SCK held by UNZA mathematics student teachers. Hopefully, the results of the study will promote further research in Zambia relating to the content knowledge of student teachers in several other secondary school mathematics topics. In Section 1.6, the research questions will be outlined and explained.

1.6 RESEARCH QUESTIONS

The questions in this section were generated through the reading of the relevant literature and were also derived from the researcher's personal interest in the research topic. The main question that guided the research is: **How can the University of Zambia's mathematics student teachers' content knowledge of functions and trigonometry at secondary school level be described?** The following sub-questions contributed in answering the main question:

Sub-question 1: How proficient are student teachers in the Common Content Knowledge of functions and trigonometry at secondary school level?

Sub-question 2: What Specialised Content Knowledge of functions and trigonometry at secondary school level is held by the student teachers?

Sub-question 1 served the purpose of assessing whether student teachers had a thorough understanding of the functions and trigonometry, as understood by non-teacher professionals who have in-depth content knowledge of the functions and trigonometry to be taught at secondary schools. Put differently, the question required the provision of a description regarding how well the student teachers understood functions and trigonometry as prescribed in the Zambian secondary school curriculum. In order to answer this question, some of the following content areas listed below were assessed in the mathematics test (MT) that was developed for this study (see Section 3.4.3.3, Tables 3.3 and 3.4 for specific areas assessing CCK):

- Definitions of a function, relation, inverse of a function, one-to-one functions, and composite functions; graphs of linear functions defined on discrete domains and those defined on continuous domains; graphs of quadratic functions defined on continuous domains; representation of functions and relations such as formulae, Cartesian graphs, sets of ordered pairs, and arrow diagrams; domains and ranges of relations and functions; completing the square for quadratic functions; turning points of quadratic functions; extreme values of quadratic functions; calculation of inverses of linear and quadratic functions; and composite functions.

- Simple trigonometric equations; trigonometric ratios (sine, cosine and tangent); signs of trigonometric ratios in specific quadrants; computation of values of sine, cosine and tangents of angles without the use of calculators; angles associated with parallel lines; characteristics of isosceles triangles and right angled triangles; sum of angles at a point; computation and locating of positions of three-figure bearings; use of sine and cosine rules to calculate angles and lengths in plane figures and solids; computation of the area of right-angled and non right-angled triangles; identification and calculation of shortest distances in plane figures and solids, use of area of triangles and perpendicular lines to calculate shortest distances; ratios of sides and the calculation of angles of non right-angled triangles; acute angles and their corresponding obtuse angles in the second quadrant; computation of lengths in plane figures using Pythagoras' theorem; graphs of the sine, cosine, and tangent functions; and periods, ranges, and maximum and minimum values of trigonometric functions.

Sub-question 2 was intended to generally explore student teachers' abilities to explain and justify their reasoning, recognise and explain conceptual differences and relationships, and understand different representations of the concepts of functions and trigonometry at secondary schools. Some of these aspects were assessed in the test (see Section 3.4.3.3, Tables 3.5, 3.6 and 3.7). With regard to interviews, Sub-question 2 provided a basis for eliciting explanations and justifications from the student teachers concerning their understanding of the concepts and the procedures that they used to solve test items. It equally allowed for the probing of the student teachers in order to understand why they employed specific methods when solving particular items. Furthermore, this sub-question formed the basis for exploring the student teachers' understanding of different representations of the functions and trigonometry concepts. An intensive exploration of the student teachers' abilities with respect to these aspects was done through semi-structured interviews (see Appendix 5 and 6 for specific content areas explored). Section 1.7 highlights the methodological and data analysis concerns of this study.

1.7 METHODOLOGICAL AND DATA ANALYSIS CONCERNS

This is a qualitative case study for which data was collected in two phases. Phase 1 included descriptive quantitative data, which was collected using a mathematics test from a conveniently selected sample (Gravetter & Forzano, 2012). A document analysis provided content areas from which to develop the test items. The test results were initially analysed using descriptive statistics, and subsequently, student teachers' answers to the test items were analysed qualitatively.

Phase 2 involved qualitative data that was gathered using semi-structured interviews from a purposefully selected sub-sample (Creswell, 2012; Gay, 2011; Merriam, 2009). The interviews were transcribed and then analysed using the components of the study's conceptual framework (see Section 2.7). Furthermore, the interview data was explored for possible emergent themes.

1.8 DEFINITIONS OF TERMS

Common Content Knowledge: For this study, this term refers to the mathematical knowledge required to teach functions and trigonometry at secondary schools, which can be understood and used by both teachers of mathematics and individuals who are in non-teaching professions.

Specialised Content Knowledge: In this study, the term relates to the mathematical knowledge required to teach functions and trigonometry at secondary schools, which enables teachers to 'unpack' concepts. Furthermore, this knowledge is not necessarily required or utilised by other non-teaching professionals who use mathematics.

Content knowledge: Refers to subject matter content knowledge, which is investigated as Common Content Knowledge and Specialised Content Knowledge.

Graduate mathematics teachers: This term is used to describe teachers who were trained to teach mathematics from Grades 8 to 12, and hold a degree qualification in mathematics education.

Diploma holders: In this study, the term refers to individuals who were trained in Education Colleges to teach secondary school mathematics from Grades 8 to 9 (junior secondary school level).

Practising teachers: This refers to individuals who are employed and serving as mathematics teachers in secondary schools. In addition to this, these individuals must be holders of a secondary teachers' diploma or degree in mathematics education (or both).

Student teachers: This term is generally used to describe individuals who are studying at an institution in order to acquire certification as graduate secondary school mathematics teachers. In the context of the University of Zambia, it refers to both Pre-service and In-service individuals who are studying for a degree in mathematics education.

Students: In this study, the term is used to refer to student teachers who are training as graduate teachers of secondary school mathematics. However, in research literature, the term 'student' is used interchangeably to refer to student teachers and to learners in secondary schools. The term 'learner' is used to refer to pupils in secondary schools. Therefore, in the literature review, the term 'students' refers to both student teachers and learners.

Finalist student teachers: These are UNZA mathematics student teachers who are in their last phase of their training as teachers of secondary school mathematics.

Scores: This term refers to the marks that were awarded to student teachers during the marking of their test scripts. It also relates to the marks distributed across test items in the mathematics test administered during Phase 1 of the study.

Proficiency: In this study, this term refers to the ability to correctly and accurately solve problems, use rules, formulas, theorems, notations, and properties of concepts, provide valid definitions and properties of concepts, and identify examples and non-examples of concepts.

Mathematical understanding: This statement is used to either refer to instrumental or relational understanding as espoused by Skemp (2006).

Advanced university mathematics: This refers to the purely content based mathematics subject matter offered by the School of Natural Sciences at UNZA. Furthermore, this subject matter is beyond what is taught at secondary school level in Zambia.

1.9 SUMMARY OF THE CHAPTER AND STRUCTURE OF THE STUDY

Chapter 1 provided discussions on the rationale and background to the study, problem statement, the aims and objectives of the study, significance of the study, followed by the research questions that guided the study. An outline of methodological and data analysis concerns was then presented followed by the definitions of a list of terms that are used in this study. Chapter 2 discusses the context in which the study was conducted, as well as the relevant literature that was reviewed for the study. Thereafter, a discussion of secondary school mathematics teacher education in the Zambian context is provided. An overview of the relevant frameworks of teacher knowledge as espoused by different scholars is then presented. This is linked with a review of literature focusing on students' content knowledge of functions and trigonometry. How students' content knowledge can be investigated is discussed, while the study's conceptual framework closes Chapter 2. Chapter 3 focuses on the research design and methodology. Included here are discussions on the philosophical orientation of the study, research approach and design, data collection, data analysis methods, and ethical considerations. The data collection methods are then presented, followed by a summary that closes the chapter. Chapter 4 deals with the presentation and analysis of the data from the test and interviews based on the topic of functions. Chapter 5 provides the analysis of the test and interview data concerning the topic of trigonometry. In Chapter 6, a synthesis of the study's findings is presented, followed by Chapter 7, which provides the conclusions of the study.

CHAPTER 2 CONTEXT OF THE STUDY AND LITERATURE REVIEW

2.1 INTRODUCTION

This chapter articulates the context and theoretical foundation upon which the present study was conducted. The chapter begins with a general discussion of secondary school mathematics teacher education in Zambia. This is followed by a discourse on how secondary school mathematics teachers are trained at UNZA, which is the location where this study was conducted. Then, an outline is provided concerning the teaching and learning outcomes of functions and trigonometry for secondary schools in Zambia. Next, a review of the relevant frameworks of teacher knowledge is provided. This is followed by a discussion of teacher content knowledge as a category of teacher knowledge. As part of the justification for this study, an integrated overview of research findings on students' content knowledge of functions and trigonometry is presented thereafter. An outline of some of the ways in which student teachers' content knowledge could be investigated is also provided. In the last section, the study's conceptual framework is presented and elaborated on.

2.2 SECONDARY SCHOOL MATHEMATICS TEACHER EDUCATION IN ZAMBIA

In Zambia, teachers of secondary school mathematics are trained in Education Colleges and Universities. These institutions provide training to both Pre-service and In-service mathematics student teachers. Specifically, Education Colleges award secondary teachers' diplomas in mathematics education, while universities confer degree qualifications in mathematics education. Whereas Diploma programmes are designed to be studied for three years, degree programmes are generally intended to be completed in four years (Ministry of Education, 2013b). Degree holding mathematics teachers are supposed to teach Grades 8 to 12, whereas teachers who hold a secondary teachers' diploma are expected to teach Grades 8 and 9. However, diploma holders teach senior secondary school mathematics in schools where there is a shortage of degree holders (Haambokoma, 2002; Kelly, 1991; Ministry of Education, 1996).

The available literature suggests that in 1925, missionaries operated two institutions that trained teachers in Zambia (Snelson, 1990). However, before the independence of Zambia, there were no Zambian institutions that trained secondary school mathematics teachers (Mwanakatwe, 1968). After 1964, which is the year Zambia became independent, there was an increase in the enrolment numbers of secondary school learners, and this led to the need for secondary school teacher training institutions in the country. In 1966, UNZA was opened and subsequently embarked on the training of mathematics secondary school teachers. Another institution was established in 1967 whose purpose was to train secondary school teachers at diploma level (Mwanakatwe, 1968).

Around the year 2000, there were five institutions in Zambia that trained secondary school mathematics teachers. Among those institutions, only UNZA trained graduate teachers of mathematics (Haambokoma, 2002). Presently, another government university, and private universities, are now training graduate teachers of secondary school mathematics. Moreover, two Colleges of Education have since been transformed by the government into universities and are now offering degree programmes in mathematics education. Notwithstanding, the University of Zambia remains the oldest and largest institution that prepares graduate teachers of secondary school mathematics in the country.

Generally, diploma awarding institutions offer mathematics content that is similar to the content that prospective graduate teachers study during their first year at university. The mathematics course content in the Colleges of Education and the diploma certificates are mostly underwritten by UNZA. Apart from that, lecturers who teach mathematics in the diploma offering institutions are expected to hold at least a first degree qualification in mathematics education. While the ZECF suggests that Teacher Education should familiarise the student teachers with the school curriculum (Ministry of Education, 2013b), the practice is that universities determine their own programmes and mathematics course contents. Additionally, universities require that lecturers should hold at least a Master’s degree qualification. The next sub-section focuses on secondary school mathematics teacher training at the University of Zambia.

2.2.1 Secondary school mathematics teacher training at the University of Zambia

There are three mathematics education degree qualifications that are attainable at UNZA: (1) Bachelor of Arts with Education, (2) Bachelor of Science with Education, and (3) Bachelor of Education in Mathematics and Science. The first two are intended for both In-service and Pre-service student teachers. The Bachelor of Education in Mathematics and Science programme is planned for student teachers who already hold a secondary teachers' diploma in mathematics education. While the Bachelor of Arts and the Bachelor of Science are four years programmes, the Bachelor of Education in Mathematics and Science is a three year programme. Student teachers pursuing these degree programmes study advanced university mathematics courses in the School of Natural Sciences (Appendix 8). The number of courses studied varies depending on whether the student teacher is studying mathematics as a single subject major, major, or minor. Those who do a single subject major or a major in mathematics education usually take core and other optional mathematics courses, while their counterparts who take mathematics as a minor usually study fewer courses.

All mathematics student teachers study ME courses from the MSE department in the School of Education. These courses are generally methodology oriented and are intended to teach the student teachers generic aspects of mathematics teaching (Appendix 7). An exception to this is the Foundation Mathematics for Teachers (MSE 131), which is intended to blend content mathematics with pedagogic skills. Analysis of the contents of MSE 131 suggests that it mainly comprises of topics that are similar to those offered in the first year mathematics courses in the School of Natural Sciences. Apart from this, MSE 131 is only studied by In-service student teachers who follow the three year Bachelor of Education in Mathematics and Science programme. Mathematics student teachers also study other non-mathematics orientated courses from the following departments in the School of Education: Educational Administration and Policy Studies (EAPS), Educational Psychology, and Sociology and Special education (EPSS). The student teachers who take mathematics as a minor study other teaching subjects outside of mathematics.

Furthermore, mathematics student teachers undertake school teaching experience in schools of their choice in the first term of their third year of study. School teaching experience affords the student teachers an opportunity to experience the actual school environment (Ministry of Education, 2013b). This is in addition to the peer teaching that they conduct at university. During peer teaching, student teachers are given an opportunity to obtain preliminary experience and practice in teaching within a college campus (Olaitan, 1981). At UNZA, student teachers are required to design lesson plans and teach fellow students in an artificial classroom situation. One notable feature of peer teaching is that shorter teaching durations are permitted for students than the time allowed to practicing teachers in schools. At the end of those lessons, a mathematics education lecturer reflects on the lesson with the student teacher emphasising areas where improvement is necessary, as well as what went well in the teaching of the lesson. Section 2.3 outlines the teaching and learning outcomes of functions and trigonometry for secondary schools in Zambia.

2.3 TEACHING AND LEARNING OUTCOMES: FUNCTIONS AND TRIGONOMETRY

In the Zambian context, functions and trigonometry occupy a central position in the secondary school mathematics syllabus (Ministry of Education, 2013a). The purpose of teaching functions and trigonometry at secondary school is encapsulated in the following general outcomes:

- Provision of clear mathematical thinking and expression in pupils;
- Development of the pupils’ mathematical knowledge and skills;
- Enrichment of the pupils’ understanding of mathematical concepts in order to facilitate further study of the discipline;
- Building up an appreciation of mathematical concepts so that pupils can apply these for problem solving in everyday life; and
- Enabling the pupils to represent, interpret and use data in a variety of forms.

A summary of the learning outcomes for the topics of functions and trigonometry, as intended in the Zambian mathematics secondary school curriculum, is provided in Table 2.1 below.

Table 2.1: Summary of learning outcomes on functions and trigonometry for secondary schools

TOPIC	LEARNING OUTCOMES
Functions	Show understanding of different types of relations; Demonstrate understanding of the idea of a function; Exhibit understanding of a domain and a range; Relate members of two sets using a given rule; Determine relationships that are functions; Use function notation; Evaluate functions expressed in algebraic form; Find the inverse of a one-to-one function; Draw and interpret graphs of linear functions; Solve problems involving linear functions; Draw and interpret graphs of quadratic functions (including intercepts, minimum, and maximum values, turning points); and Simplify composite functions.
Trigonometry	Identify the sides of right-angled triangles; Describe the sine, cosine, and tangent ratios on a right-angled triangle; Determine signs of the three trigonometric ratios in the four quadrants; Work with the two special triangles; Work with special angles $(30^\circ, 45^\circ, 60^\circ)$; Evaluate trigonometric ratios using special angles; Calculate sides and angles of a right-angled triangle using the three trigonometric ratios; Use Pythagoras' theorem to solve problems involving right-angled triangles; Find sides and angles of non right-angled triangles using the sine and cosine rules; Calculate areas of right-angled and non right-angled triangles; Draw graphs for the sine, cosine and tangent functions; Solve trigonometric equations; Apply trigonometry to solve questions involving three dimensions figures; Determine three-figure bearings (including sketching diagrams to represent position and direction).

Although learners are taught aspects of functions and other concepts connected to trigonometry at Grades 8 and 9 (junior secondary school level), most of the concepts in these topics are taught from Grades 10 to 12 (senior secondary school level). Linear functions and their graphs, Pythagoras' theorem, right angled triangles and non right-angled triangles, areas of triangles, and bearings are taught at junior secondary school level. These concepts are also taught at senior secondary school level. The concept of a function and the different types of relations, and composite functions are introduced at senior secondary school. Quadratic functions and their graphs, quadrants, trigonometric ratios, trigonometric functions and their curves, sine, and cosine rules, special triangles and angles, and the trigonometric formula for the area of a triangle are only taught at senior secondary school level (see Section 3.4.3.1 for results of the document analysis). In the following section, a discussion of the relevant frameworks of teacher knowledge is provided.

2.4 RELEVANT FRAMEWORKS OF TEACHER KNOWLEDGE

Teacher knowledge has been articulated by researchers through various frameworks (Ball et al., 2008; Holmes, 2012; Nyikahadzoyi, 2013; Shulman, 1986, 1987). Shulman (1987) suggests that a teacher knowledge base should have the following categories: general pedagogical knowledge; knowledge of learners and their characteristics; knowledge of educational contexts; knowledge of educational ends, purposes, and values, content knowledge; pedagogical content knowledge (PCK); and curricular knowledge. These seven categories can be grouped into two broad domains, namely: (1) one that emphasises generic teacher knowledge such as general pedagogical knowledge, knowledge of learners and their characteristics, knowledge of educational contexts, and knowledge of educational ends, purposes and values, and (2) a domain that embodies the important role of content knowledge in the task of teaching.

According to Shulman (1986), the categories that constitute content knowledge are subject matter knowledge, curricular knowledge, and PCK. These categories help us to explain content specific issues (Ball et al., 2008); this categorisation is shown in Figure 2.1.

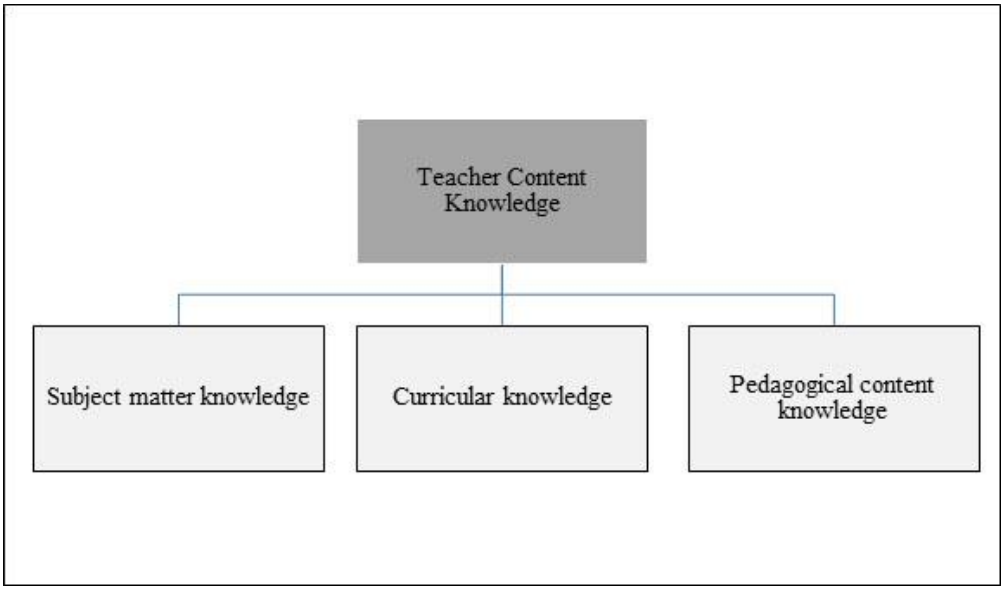


Figure 2.1: Shulman's (1986) categorisation of teacher content knowledge

It would appear that subject matter content knowledge was later referred to as content knowledge in the categorisation of a teacher knowledge base (Shulman, 1987). General pedagogical knowledge relates to broad principles and strategies that go beyond subject matter. Shulman (1987) indicates that knowledge of educational contexts could range from "the workings of the group or classroom, the governance and financing of school districts, to the character of the communities and cultures" (p. 8). Shulman (1986) explains curricular knowledge as teachers' knowledge that includes several teaching materials and "familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them" (p. 10). The argument here is that teachers require knowledge of the programmes that are specifically tailored to the teaching of given subjects and topics at particular grade levels (Shulman, 1986).

Pedagogical content knowledge is subject matter knowledge for teaching, which distinguishes teachers from other professionals who may, for example, have knowledge of mathematics, but not necessarily for teaching purposes. Shulman (1986) conceptualises PCK to be a blend of pedagogy and subject matter. He explains that for most regularly taught topics, PCK includes "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). What is contended here is that teachers' knowledge should not only consist of generic knowledge of teaching, but it should be a combination of generic aspects of teaching and subject matter specific knowledge, for example, teachers should not only have knowledge of how to organise a classroom, but should be able to explain the subject matter to the pupils with clarity (refer to Section 2.4.1).

In their quest to answer the question of what teachers need to understand to be effective teachers, Ball et al. (2008) provide a framework for mathematical knowledge for teaching (a diagram representing this framework is reproduced as Figure 2.2). This framework is composed of six categories: Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), Specialised Content Knowledge (SCK), Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC). The authors did not principally place emphasis on teachers, but on the utilisation of knowledge in and for teaching. Ball et al.'s (2008) idea was to "determine what else teachers need to know about mathematics and how and where teachers might use such mathematical knowledge in practice" (p. 395).

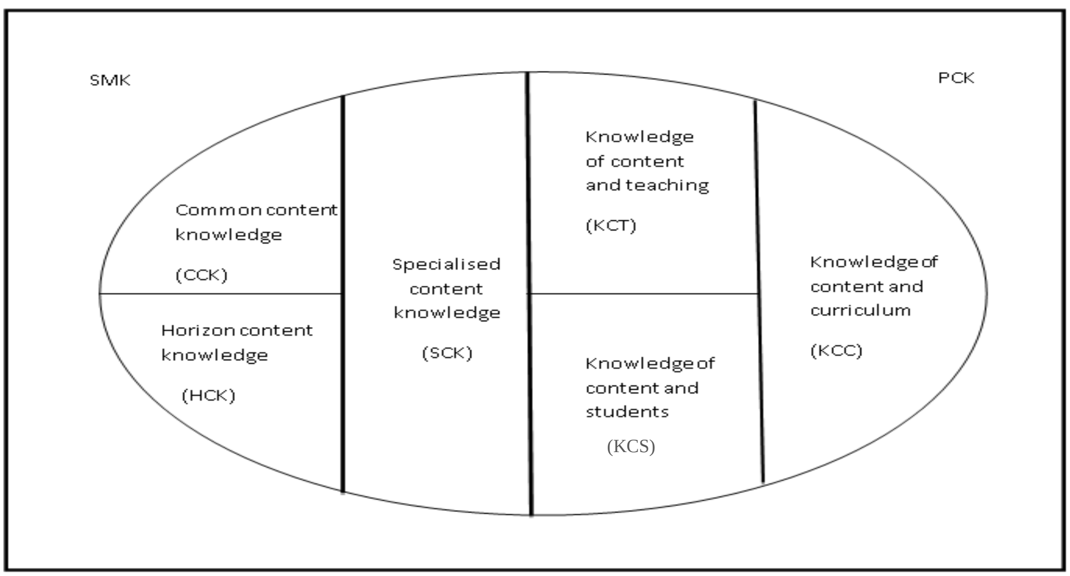


Figure 2.2: Ball et al.'s (2008) framework for Mathematical Knowledge for Teaching

The mathematical knowledge for teaching framework (MKT) can be organised into two broad divisions, namely: the subject matter knowledge orientated domain, which is composed of sub-categories such as Common Content Knowledge, Horizon Content Knowledge, and Specialised Content Knowledge on the one side (see Section 2.4.1); and on the other side, PCK inclined sub-categories consisting of KCC, KCS, and KCT (Ball et al., 2008; Holmes, 2012). Knowledge of Content and Curriculum refers to teachers' understanding of mathematical content with regard to materials and programmes that are employed in the task of teaching (Bair & Rich, 2011). In contrast, Knowledge of Content and Students is defined as knowledge that empowers a teacher to have an idea of what is likely to be problematic or easy for students as they learn a particular content area. At the core of Knowledge of Content and Students is teachers' knowledge of common conceptions and misconceptions regarding specific mathematics content (Ball et al., 2008). The last strand, KCT, is described as knowledge that combines a teacher's knowledge about teaching with knowledge about mathematics. Having discussed the teacher knowledge frameworks above, teacher content knowledge will now be presented, as articulated in the literature.

2.4.1 Teacher content knowledge

Shulman (1986) describes the emphasis on generic knowledge of teaching without any reference to content knowledge as a 'missing paradigm' in research on the topic. Shulman argues that, in addition to the significance attached to generic knowledge of teaching, such as classroom management practices, motivation strategies, and knowledge of pupils' characteristics, teachers are supposed to acquire competence in the subject matter to teach effectively. Consequently, teacher content knowledge was proposed as one of the categories of a teacher knowledge base (Shulman, 1987). Over the years, researchers in mathematics education have generally acknowledged the requirement for mathematics teachers to have in-depth content knowledge of the mathematics topics that they are expected to teach (Ball, 1990; Ball et al., 2008; Bennett, 1993; Bromme, 1994; Bryan, 1999; Nyikahadzoyi, 2013; Steele et al., 2013). As stated in Section 2.4, Ball et al. (2008) depict Shulman's (1987) category of teacher content knowledge as an overarching category of subject matter knowledge. Clearly, in the MKT framework subject matter knowledge has been expanded and sub-divided into CCK, HCK and SCK while Shulman's PCK was divided into KCC, KCT, and KCS (Ball et al., 2008).

Holmes (2012) describes subject matter knowledge as "the knowledge of the content of the specified discipline […which] is broken down into two categories: (a) substantive and (b) syntactical knowledge" (p. 64). Substantive knowledge encompasses comprehension and the capacity to explain mathematical facts, concepts and principles (Holmes, 2012). According to Holmes (2012), syntactical knowledge "is the ability to 'speak' of the structures underlying the mathematical concepts; it requires understanding the grammar, rules, and proofs of mathematics that underlie the particular topic under engagement" (p. 64). Holmes' view is that a mathematics teacher's knowledge of the subject matter should not only constitute knowledge of facts, concepts and skills, but should also include an understanding of whether, when and why such facts, concepts and skills are correct or incorrect (Shulman, 1986).

Common Content Knowledge is mathematical knowledge that is not unique to teachers of mathematics, but is also known and used by other professionals, like engineers and economists, to find answers to problems (Bair & Rich, 2011; Ball et al., 2008). Holmes (2012), adapts this definition and describes CCK as "the mathematics knowledge that any educated professional would know" (p. 65). Thus, Common Content Knowledge can be used in similar ways by both teachers and other individuals who are in non-teaching professions that nevertheless use mathematics. Horizon Content Knowledge is described by Bair and Rich (2011) as knowledge that entails "how mathematical topics are related over the span of mathematics included in the curriculum, across grade levels" (p. 294). It involves teachers' knowledge of how mathematics topics are related and enables teachers to provide explanations in terms of how the topics contribute to mathematical goals and purposes (Nyikahadzoyi, 2013).

Specialised Content Knowledge has been conceptualised as knowledge that has a relationship with the practice of teaching, but which is, however, distinctively mathematical (Schilling, Hill, & Ball, 2008). Furthermore, Ball et al. (2008) argue that SCK is:

> The mathematical knowledge and skill unique to teaching […] mathematical knowledge not typically needed for purposes other than teaching. In looking for patterns in student errors or in sizing up whether a non-standard approach would work in general […] teachers have to do a kind of mathematical work that others do not. This work involves an uncanny kind of unpacking of mathematics that is not needed or even desirable in settings other than teaching (p. 400).

The quotation above implies that SCK is only required by people engaged in teaching mathematics. It includes mathematics teachers' ability to make explanations and representation of mathematics concepts, as well as their ability to examine and comprehend solutions that are non-routine (Ball, Hill, & Bass, 2005). Arguably, mathematics teachers require specialised knowledge, for instance, for them to be able to assess pupils' understanding of concepts in mathematics effectively.

One of the characteristics of a teacher with SCK would be his or her ability to present mathematical knowledge in a manner that provides meaning to the pupils other than teaching pupils mathematical procedures alone. Procedural knowledge is mainly composed of formal language and algorithms, and can be contrasted with conceptual knowledge, which is rich in relationships and has to be learned meaningfully (Hiebert, 2013). These views about SCK are consistent with Bair and Rich (2011), who posit that it is "the unique mathematical content knowledge needed for teaching mathematics with understanding" (p. 295). These authors believe that there are certain mathematical demands, such as 'decompressing' of the subject matter (Ball et al., 2008), which are unique to teachers of mathematics.

2.4.2 Summary of relevant teacher knowledge frameworks

Although there are many frameworks available, Section 2.4 has presented a review of the literature on relevant teacher knowledge frameworks with a particular focus on teacher content knowledge. Shulman's proposed framework of a teacher knowledge base has been discussed. Also, his conceptualisation of teacher content knowledge comprising subject matter knowledge, curricular knowledge, and pedagogical content knowledge has been discussed. Ball et al.'s (2008) MKT framework, which builds on Shulman's idea, highlights that teacher content knowledge is considered to be an overarching category of subject matter knowledge composed of CCK, HCK and SCK (Ball et al., 2005; Ball et al., 2008; Holmes, 2012; Nyikahadzoyi, 2013; Steele et al., 2013). Ball et al.'s (2008) ideas of subject matter knowledge are used in this study because their MKT framework relates to what teachers should understand to be effective. For the purpose of this study, only CCK and SCK will be investigated. HCK will not be investigated as this study does not aim to explore the relationships of mathematics topics in a curriculum (Bair

& Rich, 2011). The next section presents an overview of the literature review on students' content knowledge of functions and trigonometry (see Section 1.2.3).

2.5 STUDENTS' CONTENT KNOWLEDGE OF FUNCTIONS AND TRIGONOMETRY

This section provides an overview of the research findings on students' content knowledge of functions and trigonometry. Firstly, a review of the literature on students' content knowledge of functions is presented. Afterwards, the research findings on students' content knowledge of trigonometry are presented. This is followed by a review of the literature on student teachers' content knowledge of functions and trigonometry in the Zambian context, after which a summary closes the section. In Section 2.5.1, the students' content knowledge of functions is dealt with.

2.5.1 Overview of students' content knowledge of functions

Scholars have conducted studies that range from teachers' and students' knowledge of functions to those that sought to develop theoretical frameworks and content-focused method courses on the function concept (Chesler, 2012; Even, 1990, 1993, 1998; Even & Tirosh, 1995; Gerson, 2008; Leinhardt, Zaslavsky, & Stein, 1990; Nyikahadzoyi, 2013; Spyrou & Zagorianakos, 2010; Steele et al., 2013; Watson & Harel, 2013). Of particular interest in the current study were those studies that investigated students, teachers, and student teachers' understanding of functions. Such studies suggest that there are aspects of functions that students still find difficult to understand (Bayazit, 2011; Chesler, 2012; Clement, 2001; Dede & Soybaş, 2011; Dubinsky & Wilson, 2013; Evangelidou, Spyrou, Elia, & Gagatsis, 2004; Even, 1993; Even & Tirosh, 1995; Gerson, 2008; Hitt, 1998; Spyrou & Zagorianakos, 2010; Tall & Bakar, 1992). In the following discussion, there will be no distinction between CCK and SCK because in the literature there are various views and opinions of concepts such as CCK, SCK, and Mathematical Content Knowledge.

The concept of function

Even (1993) discusses the interrelation of student teachers' subject matter knowledge and PCK with regard to the concept of function. The findings of that study indicate that most of the student teachers in question had a limited understanding of a modern concept of functions, which in turn had an influence on their pedagogical reasoning. Other researchers report that, although prospective teachers use the univalence condition to define a function, and to identify functions and non-functions, their understandings of the univalence property of functions is shallow (Even & Tirosh, 1995). For example, one of the respondents in Even and Tirosh's (1995) study posited that a graph of a circle is a function. The prospective teacher demonstrated a conflict in understanding by acknowledging that a circle does not satisfy the vertical line test, and yet, alternatively, he considered a circle to be a function. That conflict led the prospective teacher to confine the vertical line test to linear functions.

Definition of functions

Earlier studies suggest that students have the misconception that for a relation to be a function, it has to meet the one-to-one correspondence condition (Leinhardt et al., 1990; Markovits, Eylon, & Bruckheimer, 1986). Moreover, there is research evidence that students face the challenge of discriminating between the definition of a one-to-one function and the property that qualifies ordinary relations to be functions (Dubinsky & Harel, 1992). Recently, a qualitative study was conducted in the context of definitions of functions in an undergraduate course (Chesler, 2012). Student teachers were asked to perform the following tasks: select and use the definitions of functions, and make evaluations as to whether the definitions were equivalent or not. They were also asked to give an interpretation and critique of a definition of a specific type of function from a secondary school mathematics text book. The findings of this study suggest that mathematics student teachers have problems reasoning with and about mathematical definitions of functions. With respect to the third task, research has shown that some mathematics textbooks lack appropriate definitions of mathematics concepts (Harel & Wilson, 2011).

Compartmentalised knowledge

Gerson (2008) provides another example of a study that illustrates students' lack of in-depth understanding of functions. The findings of this study show that the students' knowledge was compartmentalised with regard to functions, function notation, and periodicity (Gerson, 2008). Compartmentalisation was analysed at two levels: compartmentalisation within representations, and compartmentalisation within concepts. The former is a situation where a teacher possesses an image of a concept of functions in two representations, but is still unable to connect them, while compartmentalisation within concepts is where a teacher cannot connect two concepts that are related.

An example of compartmentalisation within representations is where a teacher gives the correct period of a function when the function is presented in graph form, but fails to give the period when the function is represented by a table of values (Gerson, 2008).

Relations and functions, and composite functions

Spyrou and Zagorianakos'(2010) study investigated students' understanding of the difference between functions and relations. A questionnaire and interviews were used as data collection instruments and the results show that most of the students experienced difficulty in distinguishing functions from relations. These challenges were exhibited in both their examples and application of the definition of the function concept. Almost all of the interviewed students only gave examples of one-to-one functions and had problems in conceiving many-to-one relations as a type of functions. Other than the difficulties that students have in articulating the definition of a function, distinguishing functions and ordinary relations, research has also shown that students do not exhibit a comprehensive understanding of composite functions (Ayers, Davis, Dubinsky, & Lewin, 1988; Jojo, Brijlall, & Maharaj, 2011).

Explanations of concepts and justifications of reasoning

Even and Tirosh (1995) discuss a student teacher who correctly determined the characteristics of the graph of the the function $f(x) = ax^2 + bx + c$ when $a > 0$ or $a < 0$. The student declared that the graph of f opens upwards when $a > 0$, but opens downwards if $a < 0$. Notwithstanding, that student teacher was unable to provide a justifications for his viewpoint. One of the studies that typifies the preceding scenario was conducted by Bryan (1999). This scholar reports the students' understanding concerning the graphical importance of m and b in the formula $y = mx + b$. Specifically, he posits that while all the students involved in the study recognised that $\frac{1}{2}$ and -3 are gradient and *y*-intercept respectively in the linear function $y = \frac{1}{2}x - 3$, only two students provided justifications for their assertions. He adds that none of the respondents had an explanation of why the coefficient of the variable x is the gradient in $y = mx + b$.

Besides this, none of the students showed the ability to derive the quadratic formula. This was despite the fact that six students correctly recited the formula, and all nine students appropriately mentioned its use. By implication, students were unable, for instance, to use the method of completing the square for $f(x) = ax^2 + bx + c$ to derive the quadratic formula.

Results such as those reported by Even and Tirosh (1995), and Bryan (1999) point to students' lack of ability to provide explanations and justifications for mathematical ideas that they may appear to understand. These results indicate that while students may know how to use procedures and methods, they may not understand the reasons why those procedures and methods work. Thus, students can be familiar with concepts without understanding why those concepts apply. It is such possibilities that justify the need to probe what student teachers 'know' so as to discover how knowledgeable they are in the topics that they are trained to teach.

Shulman (1986) refers to such aspects when he contends that teachers "need not only understand that something is so; the teacher must further understand why it is so" (p. 9). He emphasises that teachers ought to have an understanding that combines 'knowing that' with 'knowing why'. Similarly, two terms, 'instrumental' and 'relational', are used in literature to distinguish two kinds of understanding (Skemp, 2006). Instrumental understanding relates to the knowledge of algorithms, rules, and procedures, for example, without necessarily knowing why these work or are used. Then, relational understanding involves a knowledge of the algorithms, rules, and procedures as well as why those algorithms, rules, and procedures work or are used.

Representation of functions

Research has shown that students and teachers have limited understanding in the area of representation of functions (Akkoç & Tall, 2002, 2003, 2005; Carlson, 1998; Clement, 2001; Jones, 2006; Nyikahadzoyi, 2006; Schwarz, Dreyfus, & Bruckheimer, 1990; Sierpinska, 1992; Thompson, 1994). It has been discovered, for example, that students struggle in the area of changing the representations of functions (Schwarz et al., 1990). Other researchers have reported that pupils in schools find it easier to change from the algebraic to the graphical representation than from the graphical to the algebraic representation (Markovits et al., 1986; Zaslavsky, 1997).

Likewise, students have difficulties identifying examples of functions and non-examples of functions when they are represented as sets of ordered pairs (Nyikahadzoyi, 2006). There is also research evidence that university students have problems accepting that algebraic or graphical constant functions are functions, while graphs of circles are erroneously taken as functions (Tall & Bakar, 1992).

Hilt's (1998) study considered teachers' and secondary school students' difficulties in relation to different representations in the area of functions. The results of the study indicate that mathematics teachers faced difficulties in preserving meaning when translating from one form of function representation to another. Secondary school students experienced problems in articulating some of the forms of representations. Among the specific results were the following: (1) mathematics teachers demonstrated problems in the area of identifying the domain and range sets of functions when dealing with graphic representations with Cartesian axes, (2) teachers did not favour a definition of a function that is connected to the idea of a variable, but preferred one that relates to the rule of correspondence, and (3) generally, the activity of identifying functions did not pose a challenge for teachers.

The studies discussed in this section suggest that students still find the function concept, and representation of functions, difficult to understand. Similarly, some of the research findings seem to suggest that students have difficulty explaining concepts and justifying their reasoning. The function concept and representation of functions in different forms are central aspects in the Zambian secondary school curriculum (Section 2.3 and Section 3.4.3.1). It is therefore important for student teachers to show an understanding in these areas. Teachers could assist learners with translation from an algebraic representation to the graphical one, and vice versa, if they are not only able to demonstrate that ability, but are equally able to explain, and justify their reasoning. A review of the literature on students' content knowledge of trigonometry follows.

2.5.2 Overview of students' content knowledge of trigonometry

There is research evidence of a significant relationship between teachers' content knowledge and pupils' achievement in trigonometric functions (Ogbonnaya & Mogari, 2014). Although trigonometric functions do not represent the entire topic of trigonometry, this relationship, nonetheless, is suggestive of the significant role that teachers' content knowledge of trigonometry plays in teaching. Moreover, the research results generally indicate that mathematics student teachers do not exhibit in-depth understanding of concepts in trigonometry (Abdulkadir, 2013; Akkoc, 2008; Bryan, 1999; Chinnappan, 1996; Fi, 2003, 2006).

Fi (2003) conducted a study involving mathematics secondary school student teachers, which assessed student teachers' subject matter content knowledge, PCK, and envisioned pedagogy in the area of trigonometry. The study utilised card sorting, a test on trigonometry, concept maps, and semi-structured interviews as data collection instruments. The findings of the study show that the participants lacked in-depth understanding of trigonometry in the following areas, among others: radian measure of angles, inverse trigonometric functions, and reciprocal functions. Another study by Fi suggests that student teachers lack in-depth understanding of periodicity, radian measure, one-to-one functions, and identities (Fi, 2006). Fi (2006) concludes that such findings confirm views that "pre-service teachers' knowledge of school mathematics may not be sufficiently robust to support meaningful instruction on some key trigonometric ideas" (p. 833).

Akkoc's (2008) study investigated the concept images of radian held by student teachers, as well as the sources of those concept images. A questionnaire was initially used to collect data and subsequently, a sample of student teachers was selected and subjected to interviews as a way of exploring their concept images of radian. The specific findings of the study were as follows: (1) student teachers were mainly familiar with the concept images of degree; (2) when dealing with trigonometric functions, student teachers were more comfortable with using degrees other than real numbers as objects; and (3) student teachers had two different images of π where it was viewed either as an angle in radian or as an irrational number. These findings suggest that the student teachers did not have in-depth understanding of the radian concept. Recently, Abdulkadir (2013) utilised quantitative and qualitative methods in assessing student teachers' knowledge levels of the degree and radian concepts. The research also intended to establish whether student teachers understood the relationship between the degree and radian concepts. The researcher points out that only 40% of the participants were able to provide a correct definition of the degree concept. In addition to this, an estimated 90% of the participants could not correctly define the radian concept. These results indicate that the student teachers in question did not have an in-depth understanding of the degree and radian measure of angles.

Bryan (1999) investigated, among many other concepts, student teachers' conceptual understanding of the Pythagorean identity and area of a triangle. The results of this study show that the student teachers lacked conceptual understanding of some of the vital concepts in trigonometry. During the interviews, some of the student teachers found it difficult to explain the meaning of a radian. Others could not comprehensively explain what, for instance, $\sin x$ represented. While all nine students involved presented $\frac{1}{2} \times base \times height$ as a formula that is used to calculate the area of a triangle, four students failed to provide any justification for the basis of the formula. Three out of the five students who gave justifications only managed to provide explanations that relate the formula to a right angled triangle. In other words, out of nine students, there were only two students that gave explanations that suggested why the formula works for all triangles.

Chinnappan's (1996) initial investigation included an assessment of a student teacher's mathematical knowledge of concepts related to trigonometric ratios and Pythagoras' theorem. This investigation showed that the participant's content knowledge base of concepts lacked substantial organisation and integration, for example, the student teacher could not explicitly state the relationship between sine, cosine and tangent ratios.

Other research studies have shown that students hold misconceptions in the area of trigonometry (Orhun, 2004; Weber, 2005). Some of the student misconceptions that appear in research literature are the following: (1) $\sin 40 - \sin 10 = \sin 30$, (2) $\frac{\cos(60 \times x)}{x} = \cos\left(\frac{60 \times x}{x}\right)$, (3) $\cos^{-1} x = \frac{1}{\cos x}$, (4) $\sin 2x = 2\sin x$, (5) $\sin x \cos x = \sin x \Rightarrow \cos x = 1$, and (6) $\pi < x < \frac{3\pi}{2}, \tan x = \frac{3}{4} \Rightarrow \cos x = \frac{4}{5}$ (Tuna, 2013). Tuna's experimental study utilised the 5E learning model to determine its effect on the elimination of the above students' misconceptions. The 5E model involves the stages of engaging, exploring, explaining, elaborating, and evaluating during the process of students' learning (Tuna, 2013). The findings of the study showed that most of the students in the control group held on to the misconceptions stated above, while only a few students from the experimental group still had misconceptions. In other words, the results suggested that the 5E model significantly facilitated the elimination of the students' misconceptions in trigonometry. However, methods such as the 5E can be effectively utilised when the teacher himself or herself has in-depth content knowledge of trigonometry.

From the foregoing discussion, it is apparent that student teachers struggle with certain concepts in trigonometry. Apart from Fi's (2003, 2006) studies, most of the researchers investigated student teachers' content knowledge of sub-topics in trigonometry, and not the entire topic of trigonometry as prescribed in a secondary school curriculum. Furthermore, student teachers' understanding of concepts, such as signs of trigonometric ratios in the four quadrants, were not explicitly explored in these studies. Other concepts connected to trigonometry that have not received much attention in research are bearings. In Section 2.5.3, a synopsis of the literature review is presented that focuses on the Zambian context.

2.5.3 Student teachers' content knowledge of functions and trigonometry: The Zambian context

Online journal searches did not yield any results on Zambian mathematics student teachers' content knowledge of functions and trigonometry. Similarly, the library records at the University of Zambia did not present any research that has been conducted on the topic. None of the masters and doctoral dissertations posted on the UNZA database deal with the issue of university mathematics student teachers' content knowledge of functions and trigonometry. More specifically, no study exploring UNZA's mathematics student teachers' content knowledge of functions and trigonometry to be taught at secondary school level was found.

One survey report exists that mentions that practising mathematics secondary school teachers stated that they had difficulties teaching trigonometry in three dimensions, while leaners had difficulties learning the graphs of functions and trigonometry in three dimensions (Haambokoma, 2002). In that study, the practising teachers were administered a questionnaire where they were required to indicate which secondary school mathematics topics they found difficult to teach.

2.5.4 Summary of students' content knowledge of functions and trigonometry

Although Tuna's (2013) study was experimental and Fi's (2003) study included a quantitative data analysis component, most of the international studies considered in Section 2.5 employed qualitative case study methodologies. This means that even when the findings from these studies can be related to other contexts, they cannot be generalised. A common finding of these studies is that the sampled students lacked in-depth understanding of particular aspects of functions and trigonometry. With regard to functions, students lacked comprehensive understanding of the function concept. They did not exhibit an understanding of the modern definition of a function, composite functions, and differences between relations and functions. The findings of some of these studies suggest that students' knowledge is compartmentalised and that students cannot provide comprehensive explanations of concepts and justifications for their reasoning. Apart from this, the literature review showed that students have difficulties with the representation of functions. In the case of trigonometry, the studies reviewed have shown that student teachers have difficulties understanding concepts such as the degree and radian measures of angles. Furthermore, students have misconceptions like $\sin 40 - \sin 10 = \sin 30$, $\cos^{-1} x = \frac{1}{\cos x}$, and $\frac{\cos(60 \times x)}{x} = \cos\left(\frac{60 \times x}{x}\right)$. A review of the literature in the Zambian context suggests that there is a lack of documented literature on student teachers' content knowledge functions and trigonmetry at secondary school level. The absence of a Zambian context study that either corroborates or disproves the findings of the international research justifies the current study.

2.6 INVESTIGATING STUDENT TEACHERS' CONTENT KNOWLEDGE

What student teachers need to know of secondary school mathematics topics may be established through an examination of the secondary school mathematics curriculum, and its prescribed teaching outcomes. This is consistent with Ball et el. (2008), who point out that "an understanding of the mathematics in student curriculum plays a critical role in planning and carrying out instruction" (p. 399). However, the content knowledge that student teachers actually possess of secondary school mathematics topics can be investigated in several ways, for example, categories of knowledge such as CCK and SCK derived from teacher knowledge frameworks can be used to explore mathematics student teachers' content knowledge (Ball et al., 2008). Bair and Rich (2011) developed a framework that involves four interconnected components believed to be at the centre of progression in SCK. The components comprise student teachers' ability to: (1) explain and justify their work, (2) use multiple representations, (3) recognise, use and generalise relationships among conceptually similar problems, and (4) pose problems. Each of the identified components is characterised by five levels for the development of deep and connected SCK to teach algebraic reasoning and the number theory. The following are the five levels that characterise each component of the framework: Level 0 (Entry level), Level 1 (Emerging), Level 2 (Developing), Level 3 (Maturing) and Level 4 (Deep and connected MKT) (Bair & Rich, 2011). Figure 2.3 is an adapted diagrammatic representation of the four components included in Bair and Rich's (2011) framework.

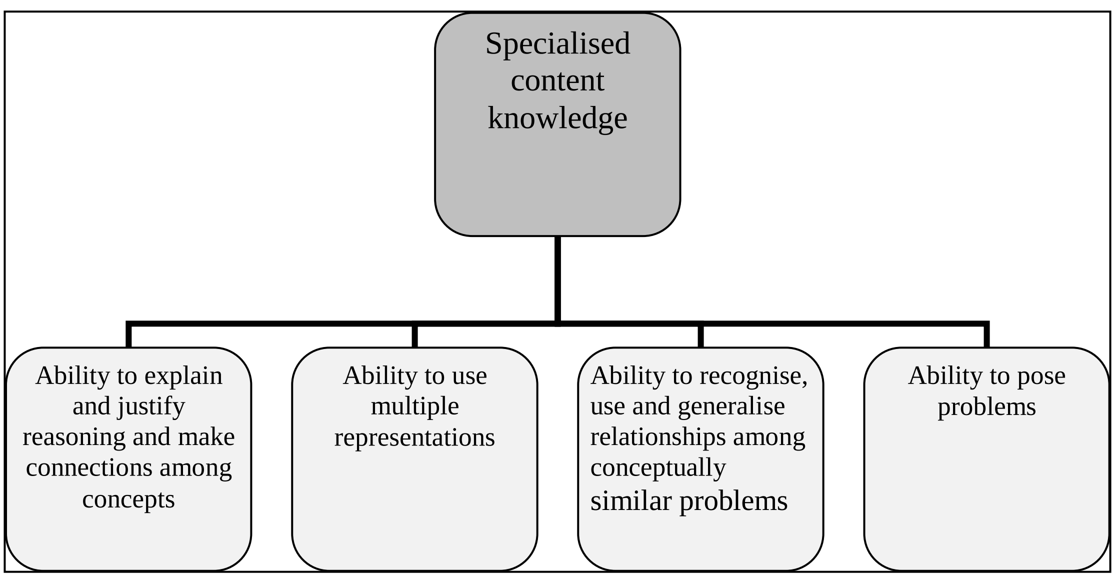


Figure 2.3: Components of SCK adapted from Bair and Rich's (2011) framework.

In addition to categories of teacher knowledge, analytical frameworks can assist in exploring and describing teachers' content knowledge of specific mathematics topics (Even, 1990; Nyikahadzoyi, 2013). Even (1990) proposes an analytical framework that is composed of seven components: essential features; different representations; alternative ways of approaching; the strength of the concept; basic repertoire; knowledge and understanding of a concept; and knowledge about mathematics. The essential features component focuses on concept images (Vinner, 1983) and advances the view that there should be a balance between teachers' understanding of concepts, and concept definitions. Thus, mathematics teachers are expected to differentiate examples from non-examples of given concepts.

The component on different representations emphasises the need for mathematics teachers to understand concepts in several representations. In this regard, a complete understanding of concepts requires that teachers demonstrate an ability to translate concepts across representations and formulate connections among concepts. However, it does not follow that a teacher who understands a concept in one representation will exhibit an understanding of the same concept when it is expressed in another representation (Even, 1990). 'Alternative ways of approaching' relates to the different ways in which a concept is used in varied branches of mathematics and even other subject areas, while 'the strength of the concept' deals with teachers' comprehension of the characteristics that make a concept special in its way of providing rare opportunities. 'Basic repertoire' points to teachers' ease of understanding and access to specific examples of the concept. Even (1990) adds that "acquiring the basic repertoire gives insights into a deeper understanding of general and more complicated knowledge" (p. 525).

Knowledge and understanding of a concept component is premised on the need for teachers to have both procedural and conceptual understanding of the concepts. Procedural and conceptual knowledge should be connected if teachers are to find answers to questions and at, the same time, be able to understand what they are doing. The last component, knowledge about mathematics, is suggested to transcend procedural and conceptual knowledge. According to Even (1990), this component relates to "general knowledge about a discipline which guides the construction and use of conceptual and procedural knowledge" (p. 527). In Section 2.7, the conceptual framework is presented and elaborated on.

2.7 CONCEPTUAL FRAMEWORK

This study assesses and explores mathematics student teachers' content knowledge of two specific secondary school topics: functions and trigonometry. Arising from the literature study, a conceptual framework was developed based on the works of Bair and Rich (2011), Ball et al. (2008), Dubinsky and Wilson (2013), Even (1990), Lloyd, Beckmann, Zbiek, and Cooney (2010), Nyikahadzoyi (2013), Shulman (1986, 1987), and Steele, Hillen, and Smith (2013). The framework is heavily influenced by Ball et al.'s (2008) domain map (Figure 2.2) that allows subject matter knowledge to be investigated as Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialised Content Knowledge (SCK). However, only CCK and SCK are integrated in the framework as these categories of knowledge embody the focus of this study. Notwithstanding that content knowledge includes HCK (Section 2.4), this category is excluded from the framework because it is not the aim of the current study to explore how mathematics topics are related in a curriculum (Bair & Rich, 2011). The conceptual framework is presented in Figure 2.4 below, where after it is explained with special reference to CCK and SCK.

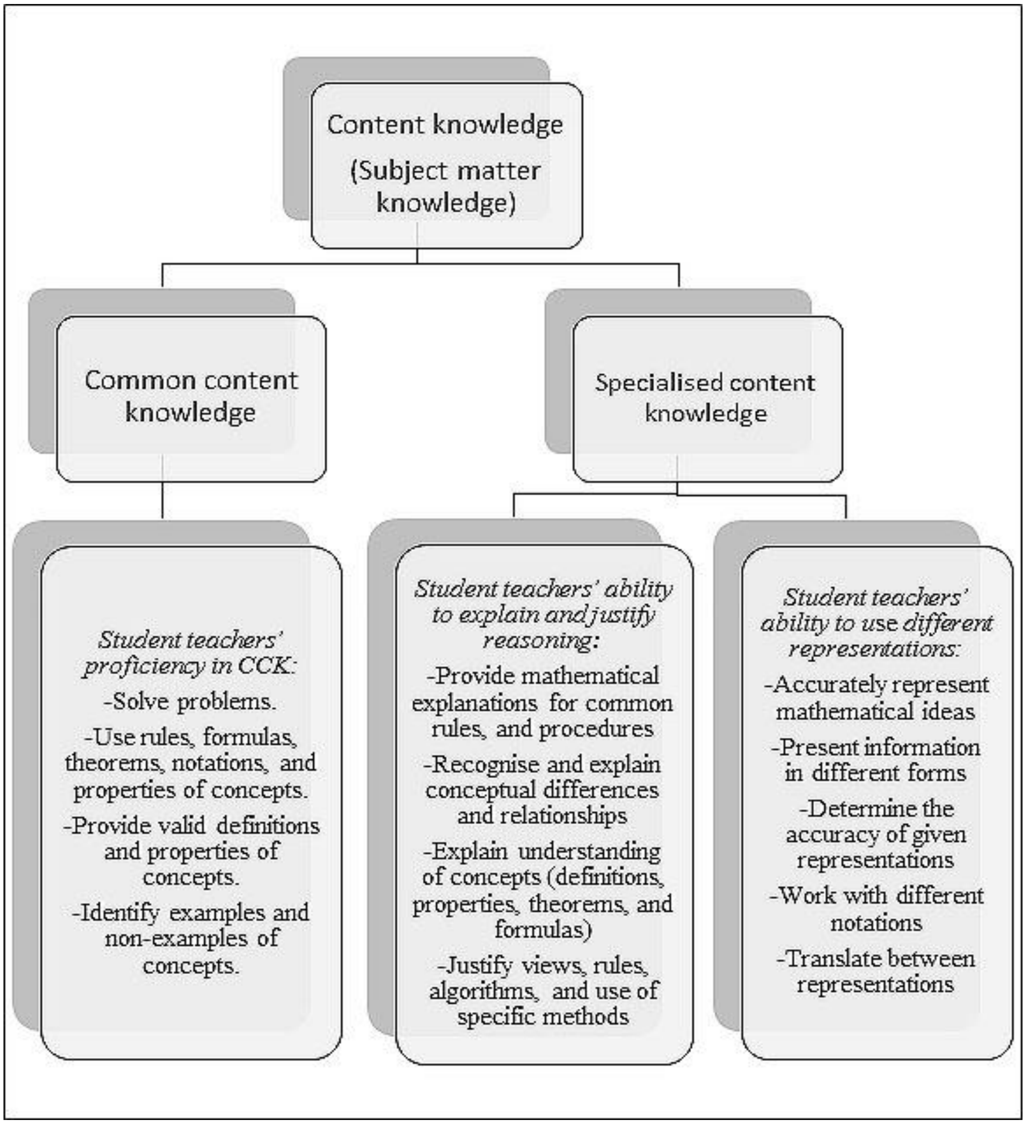


Figure 2.4: Mathematics student teachers' content knowledge (Adapted from the works of Bair & Rich (2011), Ball et al. (2008), Dubinsky & Wilson (2013), Even (1990), Lloyd et al. (2010), Nyikahadzoyi (2013), Shulman (1986, 1987), & Steele et al. (2013))

2.7.1. General explanation of the conceptual framework

This conceptual framework facilitated the research in describing the University of Zambia's mathematics student teachers' content knowledge of functions and trigonometry to be taught at secondary schools. In this framework, content knowledge based on functions and trigonometry, as prescribed in the Zambian secondary school mathematics curriculum, is also referred to as subject matter knowledge. As indicated in Figure 2.4 above, content knowledge is categorised into Common Content Knowledge and Specialised Content Knowledge which are two constructs adopted from Ball et al. (2008). Common Content Knowledge is described through a list of descriptors, while Specialised Content Knowledge is broken down into two components: (1) student teachers' ability to explain and justify their reasoning, and (2) student teachers' ability to use different representations. Each of these two components is also described by a list of descriptors. It is these specific descriptors of Common Content Knowledge and those of the two components of Specialised Content Knowledge that were used to assess and explore the student teachers' content knowledge. The categories of content knowledge (inclusive of their descriptors) are explained in Sections 2.7.2 and 2.7.3.

2.7.2. Common Content Knowledge

Ball, Hill and Bass (2005) suggest that CCK entails "knowledge that is used in the work of teaching in ways common with how it is used in many other professions or occupations that also use mathematics" (p. 377). CCK is mathematical in nature and, given that it is used by other professions, does not lessen its necessity in the teaching profession. If anything, teachers require this knowledge for them to find solutions to questions on mathematical topics in similar ways to non-teaching individuals who have studied these topics.

Some researchers have provided descriptors of CCK with respect to the function concept, for example, it is suggested that CCK includes being able to provide accurate definitions and properties of functions, and being able to identify examples and non-examples of functions (Nyikahadzoyi, 2013; Steele et al., 2013). In defining functions, mathematics teachers are expected to exhibit an understanding of the essential features of a function such as univalence and arbitrariness of functions (Even, 1990; Lloyd, Beckmann, Zbiek, & Cooney, 2010; Nyikahadzoyi, 2013; Steele et al., 2013). More than this, it is suggested that mathematics teachers who have the CCK of secondary school functions should be able to correctly solve questions on the topic, and determine answers and definitions that are valid (Nyikahadzoyi, 2013).

The literature review did not yield specific descriptors of CCK associated with the topic of trigonometry. Nonetheless, the previous discussion on the function concept, as well as Ball et al.'s (2005), and Ball et al.'s (2008) definitions formed the basis for the adaptation of descriptors that generally characterised CCK in this study. Furthermore, the works of Dubinsky and Wilson

(2013) provided ideas that were incorporated in the adapted descriptors. In this regard, the following list comprises descriptors for the CCK category in the conceptual framework:

- Correctly solving questions;
- Provision of valid definitions and properties of concepts;
- Accurate identification of examples and non-examples of concepts; and
- Correct use of rules, formulas, theorems, notations, and properties of concepts.

The descriptors above were instrumental in the formulation of a data collection instrument that sought answers to research Sub-question 1, which reads: *How proficient are student teachers in Common Content Knowledge of functions and trigonometry at secondary school level?* The initial intention of this question was to assess what the student teachers knew of functions and trigonometry, as prescribed in the Zambian secondary school curriculum. Ultimately, the goal of the question was to provide a description relating to how comprehensively the student teachers understood functions and trigonometry.

2.7.3. Specialised Content Knowledge

Specialised Content Knowledge refers to unique mathematical knowledge that enables mathematics teachers to 'unpack' the subject matter (Ball et al., 2008). The 'unpacking' of subject matter helps mathematics teachers to effectively conduct the business of teaching and distinguishes them from other professionals (not necessarily teachers) who use mathematics in its compressed form (Bair & Rich, 2011; Nyikahadzoyi, 2013). Moreover, whereas other users of mathematics can sufficiently operate with CCK, a mathematics teacher requires more than CCK to proficiently teach mathematics (Ball et al., 2008).

For the purposes of this study, Specialised Content Knowledge is conceptualised into two components: the ability to explain and justify reasoning, and the ability to use different representation. These components were adapted from Bair and Rich (2011), and Even (1990), while the descriptors of the components are an adaptation from the works of Bair and Rich (2011), Ball et al. (2008), Dubinsky and Wilson (2013), Even (1990), Lloyd et al. (2010), Nyikahadzoyi (2013), and Steele et al. (2013).

The two components were intended to address research Sub-question 2 which reads: *What Specialised Content Knowledge of functions and trigonometry at secondary school level is held by the student teachers?* This question is elaborated on in Section 1.6. The explanations of the two components of SCK in relation to functions and trigonometry are presented next.

2.7.3.1. Ability to explain and justify reasoning

The ability to explain and justify reasoning provides a basis for the exploration of the concept images and concept definitions held by teachers (Vinner, 1983, 1991; Vinner & Dreyfus, 1989). This component can prevent making pupils wait for an explanation that addresses their concerns until the teacher has completed solving the question (Ball et al., 2008). In this study, the ability to explain and justify reasoning implies that student teachers should go beyond the computation and identification of correct solutions to questions by providing clear mathematical explanations of concepts, common rules and procedures (Ball et al., 2005).

At the same time, student teachers should give a justification for particular methods and procedures (Bair & Rich, 2011), and should exhibit proficiency in recognising and explaining conceptual differences and relationships. They should equally demonstrate an ability to determine, provide and justify alternative definitions that are valid and /or equivalent (Nyikahadzoyi, 2013; Steele et al., 2013). 'Ability to explain and justify reasoning' provided the ground for assessment and exploration of either student teachers' explanations or justifications on various aspects of functions and trigonometry. The following are specific descriptors of this component as used in this regard.

- Provide mathematical explanations for common rules and procedures;
- Recognise and explain conceptual differences and relationships;
- Explain understanding of concepts (definitions, properties, theorems, and formulas); and
- Justify rules, algorithms, and use of specific methods.

2.7.3.2. *Ability to use different representations*

This component is concerned with whether student teachers have an in-depth understanding of concepts in several representations. It should be noted that some scholars use the phrase ‘multiple representations’ instead of ‘different representations’ (Bair & Rich, 2011; Lloyd et al., 2010; Steele et al., 2013). However, in this study the two phrases ‘different representations’ and ‘multiple representations’ are used interchangeably. Lloyd et al. (2010) assert that mathematics teachers’ ability to use multiple representations and translate between representations is a significant aspect of SCK. The ‘ability to use different representations’ helps to explore teachers’ capacity to determine the accuracy of given representations and appropriately translate between representations (Bair & Rich, 2011). It can also assist in assessing whether student teachers’ content knowledge is compartmentalised within concepts (see Section 2.5.1).

The ability to translate can be explained as the capacity to change the representation in which data is presented (Gerson, 2008). Gerson (2008) contends that translation includes skills such as “plotting points from a table of data, finding an equation for a graph, and creating a table of data from an equation” (p. 28). Gerson adds that one notable feature of translation is that as it occurs, the mathematical features of the original representation are carried along. Even (1990) shares similar views to those of Gerson (2008), and points out that “the most common representations of functions are formulae and Cartesian graphs […] arrow diagrams, tables, sets of ordered pairs, and situations from everyday life or disciplines” (p. 532-533). Some of these forms of representation are consistent with Markovits et al. (1986), who summarise the representations of a function and its components in a table, which is reproduced and presented here as Table 2.2 below.

Table 2.2: Representations of a function and its components (Reproduced from the works of Markovits et al. (1986))

Sub-concepts	**Forms of representations**			
	Verbal	**Arrow diagrams**	**Algebraic formula**	**Graphical**
Domain	Verbal or mathematical notation	A curve enclosing the members of the domain	Verbal or mathematical notation	The horizontal x-axis or parts thereof
Range	Verbal or mathematical notation	A curve enclosing the members of the range	Verbal or mathematical notation	The vertical y-axis or parts thereof
Rule of correspondence	Verbal	Arrows	Formula	A set of points in the Cartesian plane

In the present study, the ability to use different representations created a basis on which to assess student teachers' proficiency in working with different forms of representations for given function and trigonometry problems. Additionally, it formed the basis upon which student teachers' capacity to determine the accuracy of given representations was investigated. Furthermore, it provided grounds for exploration of student teachers' understandings of different representations. The list below constitutes descriptors of this component of SCK as used in the study's conceptual framework.

- Accurately representing mathematical ideas;
- Presenting information in different forms;
- Determining the accuracy of given representations;
- Working with different notations; and
- Translating between representations.

2.8. SUMMARY OF CHAPTER 2

Chapter 2 described the context in which the study was conducted. In this regard, an overview of the secondary school mathematics teacher education in Zambia has been presented. Special focus was placed on the training of secondary school mathematics teachers at the University of Zambia, which is the site of this research. Since the study explored the student teachers' content knowledge of functions and trigonometry to be taught at Zambian secondary schools, an articulation of the teaching and learning outcomes based on the two topics was made. At the same time, frameworks of teacher knowledge that are relevant to this study have been discussed. Teacher content knowledge was particularly discussed as a sub-category of teacher knowledge, which includes Common Content Knowledge, Horizon Content Knowledge, and Specialised Content Knowledge. Subsequently, a synthesised review of international research literature regarding teachers and student teachers' content knowledge of functions and trigonometry was presented. This was followed by a synopsis of reviewed research literature regarding teachers and student teachers' content knowledge of functions and trigonometry as it relates to the Zambian context. Furthermore, a discussion was presented regarding the different ways in which student teacher content knowledge could be investigated. The chapter ended with the presentation and elaboration of the study's conceptual framework. Chapter 3 provides a discourse on the research design and methodology of this study.

3. RESEARCH DESIGN AND METHODOLOGY

3.1. INTRODUCTION

This study was mainly motivated by two aspects: (1) a lack of literature focusing on the Zambian context, and specifically on the University of Zambia's mathematics student teachers' content knowledge of functions and trigonometry at secondary school level, and (2) the researcher's personal interest in the topic. As explained in Section 1.4, this study's aim was to determine and describe the University of Zambia's mathematics student teachers' content knowledge of functions and trigonometry to be taught at secondary school level. In this endeavour, the study investigated the Common Content Knowledge (CCK) and the Specialised Content Knowledge (SCK) of the student teachers with respect to functions and trigonometry. It is hoped that this study will contribute to the development and growth of Zambian orientated mathematics education literature on university student teachers' subject matter knowledge of secondary school mathematics topics.

In terms of organisation, Chapter 3 begins with a discussion of the philosophy that guided the study, after which the research approach and design are explained. This is followed by a section that focuses on the data collection. Then, a presentation of the procedures and techniques that were used to analyse the data is made. Subsequently, a section regarding the administration of the data collection instruments is presented, followed by a closing summary. Table 3.1 below provides a synopsis of the research design and methodology, as well as the research questions.

Table 3.1 Summary of research design and methodology

Research topic	Exploring Zambian Mathematics student teachers' content knowledge of functions and trigonometry for secondary schools.
Research approach	The study follows a qualitative approach (Creswell, 2012; Gay, 2011; Ivankova, 2014; Merriam, 2009).
Research design	This study is a single intrinsic case study in which data was collected in two phases (Merriam, 2009).
Philosophical orientations	This study is situated in a qualitative interpretivist paradigm (Holden & Lynch, 2004; Leitch, Hill, & Harrison, 2009; Nieuwenhuis, 2014a). Ontological assumption: reality is a social construction which is not independent of the researcher. Epistemological assumptions: reality is known through explorations of people's understandings concerning a particular phenomenon (Ferguson, 2007; Nieuwenhuis, 2014a).
Aim of the study	The aim of this study was to determine and describe how UNZA's mathematics student teachers understood the subject matter knowledge of functions and trigonometry for secondary schools
Research main question	How can the University of Zambia's mathematics student teachers' content knowledge of functions and trigonometry at secondary school level be described?
Sub-question 1	How proficient are the student teachers in the Common Content Knowledge of functions and trigonometry at secondary school level?
Sub-question 2	What Specialised Content Knowledge of functions and trigonometry at secondary school level is held by the student teachers?
Research site	University of Zambia.
Samples, and sampling techniques	Phase 1: 22 UNZA's mathematics major student teachers in their final year of study, chosen using a combination of convenience and purposeful sampling techniques. Phase 2: Six mathematics major student teachers purposefully chosen from among those who wrote the first phase test (three were recruited from among those whose composite test scores were high and the other three where chosen among those who scored low marks).
Development of data collection instruments	Phase 1: Conducted document analysis, consulted with mathematics experts and supervisors, piloted the draft test involving university student teachers in Zambia, and developed final version of the test (after seeking expert judgement). The test contained six questions on functions and six questions on trigonometry. Phase 2: Developed a draft semi-structured interview schedule based on functions and trigonometry, piloted and refined it. Subsequently developed two final versions (one on functions and the other one on trigonometry) which included ideas from preliminary analysis of the test data.
Research questions and data collection instruments	Sub-question 1-mathematics test. Sub-question 2-mathematics test and semi-structured interviews
Data analysis	Phase 1: Test data analysed using descriptive statistics (mean, mode, median, range, and standard deviation). Data then qualitatively analysed according to: (1) proficiency in CCK, and regarding the SCK, (2) ability to explain and justify reasoning, and (3) ability to use different representations. Phase 2: Interviews audio recorded, and transcribed. Deductive-inductive approach using content analysis (firstly using the categories of the conceptual framework, and then explored emergent themes).

In the following section, the philosophical alignment of the study is discussed. In this regard, the ontological and epistemological assumptions that guided the study are highlighted.

3.2. PHILOSOPHICAL ORIENTATION TO THE STUDY

This study is situated in a qualitative interpretivist paradigm (Holden & Lynch, 2004; Leitch et al., 2009; Morgan, 2013; Sefotho, 2015; Sklar, 2013). The perspective is that phenomena are understood through the meanings which people ascribe. However, the researcher does not hold extremist views that suggest that either reality only exists in one's mind or that it is completely objective, and can only be discovered. The researcher's outlook is generally motivated by Holden and Lynch's (2004) disclosure that "very few researchers today make such extreme assumptions" (p. 7). Notwithstanding, the researcher's inclination is not to be detached from what is being researched. In line with this preference, the ontological assumption that underpins this study's world view is that reality is a social construction that is not independent of the researcher (Nieuwenhuis, 2014a).

In terms of epistemology, this study's assumption is that reality is known through explorations of people's understanding concerning particular phenomena (Nieuwenhuis, 2014a). Here, the intention of this research study is to acquire an understanding of how, in a specific context, mathematics student teachers make sense of phenomena. This is done without any intention to generalise the findings to other institutions and situations. This is consistent with the views by Nieuwenhuis (2014a) that "the stories, experiences and voices of the respondents are the mediums through which we explore and understand reality" (p. 55). The researcher upholds social constructivism, as it recognises social interaction and communication as avenues through which personal understanding is constructed (Adams, 2006; Orton, 1994). Moreover, social constructivism allows an exploration and understanding of the research participants' ideas (Ferguson, 2007). The researcher therefore agrees with the view that social interactions have the potential of promoting learning chances (Cobb, Wood, & Yackel, 2002). Learning is viewed as a process in which learners are actively involved and are not passive recipients of information from teachers. Based on these perspectives, the researcher interacted with the mathematics student teachers and questioned them with regard to how they construct reality.

3.3. RESEARCH APPROACH AND DESIGN

This study was primarily qualitative in approach, but included descriptive quantitative data. In terms of design, this was a single intrinsic case study. Merriam (2009) explains that a case study involves detailed descriptions and analysis of systems that are bounded. At the same time, a case study is compatible with several data collection methods such as testing, observations, and interviews. In this regard, a case study is purported to include the search for meaning, insight, understanding and interpretation, and that during data collection, the researcher acts as the primary data collection instrument (Merriam, 2009). Nieuwenhuis (2014b) posits similar views and contends that a case study allows for the involvement of a small sample as well as the use of multiple techniques when collecting data. A researcher who utilises a case study design can decide before hand the type of information to collect and how to analyse that data in the quest to find answers to a research question (Nieuwenhuis, 2014b). There are different types of case studies, an intrinsic case study is distinguished from an instrumental case study in that the former is concerned with the researcher's harboured interest in a particular case (Merriam, 2009).

A case study design was deemed suitable for the current study as the intention was to gain insight, interpret, and describe the student teachers' understanding of the concepts of functions and trigonometry. The feature of bounded systems was typified by the finite number of the student teachers from a specific mathematics education programme. In addition to this, the researcher was involved in the data collection process, thereby making himself a primary data collection instrument. Moreover, assessments and explorations were conducted concerning topics that are already well defined in the Zambian secondary school curriculum. As a result, a case study allowed for the determination of the concepts in advance to assess and explore functions and trigonometry as taught in secondary schools. An intrinsic case study design was specifically utilised because the interest was to assess and explore the content knowledge of mathematics student teachers who were studying at a particular Zambian university.

The aim of this study was to provide a comprehensive description of the University of Zambia mathematics student teachers' content knowledge of functions and trigonometry at secondary school level. This was done by gathering data in two phases. The first phase was intended to assess: (1) the student teachers' proficiency in the CCK of functions and trigonometry, and (2) the student teachers' SCK. In order to allow for the description, in a natural setting, of the student teachers' content knowledge, the first phase was principally descriptive (Gravetter & Forzano, 2012). Phase 2 allowed for an elucidation of Phase 1's data and provided an opportunity to gain further insight into the student teachers' SCK using a sub-sample.

3.4. DATA COLLECTION

3.4.1. Introduction

This section provides a discussion focusing on the data collection stage of this research. Firstly, an explanation is given regarding the research site. Secondly, descriptions of the sample and sampling techniques that were employed are provided, and lastly, the data collection instruments are discussed. Seabi's (2013) suggestion that research questions must serve as an important guide was used in this study. In this regard, document analysis, a test, and interviews were used to gather the research data. Document analysis revealed the content areas from which to prepare the test, while the test allowed the student teachers to show both their methods in general and specific mathematical calculations as they provided answers. The opportunity to see methods employed and calculations made assisted to a great extent in determining the proficiency of the student teachers in functions and trigonometry. The interviews allowed the collection of rich and descriptive data (Nieuwenhuis, 2014b; Seabi, 2013) regarding the SCK that was held by the student teachers. These also assisted in seeking clarification from the student teachers concerning their answers to specific test items.

3.4.2. Research site and sampling

3.4.2.1. Research site

A research site should be amenable to sampling as is the case with a sample (Creswell, 2012). This study was conducted at the University of Zambia, and this site was chosen both conveniently (Gravetter & Forzano, 2012), and on purpose (Nieuwenhuis, 2014b) for the following reasons: (1) the motivation was to conduct a study involving student teachers of a public university accessed by the majority of the citizens of Zambia, (2) the idea was to involve student teachers of an institution that has been training graduate teachers of mathematics for a longer period as compared to other institutions in Zambia, and (3) both the university and the mathematics student teachers were easily accessible to the researcher (see Section 1.2.2).

3.4.2.2. *Sampling for Phase 1 of the study*

The population of the study involved all the UNZA finalist mathematics student teachers. The sample for Phase 1 consisted of 22 University of Zambia mathematics major student teachers who were in their final year of study. These student teachers had completed the core courses and optional courses (except for the final year courses) in the following areas, as offered in the mathematics education programmes: Mathematics methods, Algebra, Real analysis, Statistics, and Mathematics Education (including peer teaching). They had also studied courses from the following departments in the School of Education: Educational Administration and Policy Studies (EAPS), and Educational Psychology, Sociology and Special education (EPSS). None of the sample had undergone school teaching experience while enrolled at the university, but were yet to do so after completing their studies. During this study, the UNZA was in the process of transforming from the semester system to the three term calendar system and, prior to this change, student teachers used to undergo school experience for one month after they had completed their studies (see Apendix 7 and 8 for the UNZA mathematics education and mathematics courses respectively).

The involvement of final year mathematics student teachers was considered an appropriate decision as such students had reached the last phase of university teacher preparation. Thus, they would most likely be employed as graduate teachers of secondary school mathematics in the near future. Additionally, student teachers majoring in mathematics were suitable in that they would have studied the University of Zambia core courses that are deemed necessary for graduate secondary school teachers of mathematics. Moreover, mathematics major student teachers would have completed more university mathematics courses in comparison to their counterparts who minored in mathematics. This seemingly comparative advantage made the mathematics major student teachers a suitable group for this study.

The actual selection of the student teachers was conducted using convenience sampling techniques (see Section 3.7 for student recruitment). Gravetter and Forzano (2012) describe convenience sampling as a non-probability method that is characterised by the choice of participants that are available, easy to obtain, and willing to participate in a study. These features of convenience sampling have both advantages and disadvantages, for example, due to its reliance on available and willing participants, this method has disadvantages, which include sample bias and sometimes lack of representativeness. Gravetter and Forzano (2012) suggest that one of the ways of minimising the disadvantages of convenience sampling is “to provide a clear description of how the sample was obtained and who the participants are” (p. 152). The advantages of this method are premised on the fact that it is easier to implement, and less expensive as compared to probability methods, which require random processes (Gravetter & Forzano 2012).

Notwithstanding the weaknesses associated with convenience sampling, it was used because it comprehensively served the purpose of the first phase of the study. Possible sample bias and lack of representativeness were minimised by providing an elaborate description of the sample, and through the manner in which the participants were selected. It had been decided in advance that the sample should be composed of final year mathematics major student teachers who had successfully completed the core courses offered in the mathematics education degree programme. In this respect, the convenience sampling method allowed for the recruitment of available and willing student teachers who met the set criteria.

3.4.2.3. *Sampling for Phase 2 of the study*

The sample for Phase 2 was composed of six University of Zambia mathematics student teachers who were in their final year of study. This sample was purposefully selected (Creswell, 2012; Gay, 2011; Merriam, 2009; Nieuwenhuis, 2014b) from among the participants of the first phase of the study. The student teachers were chosen using a selection criterion that builds on the extreme case sampling strategy (Creswell, 2012). The extreme sampling strategy involves the selection of a case or cases that exhibit extreme features with a view to describing such a case or cases. In this context, three of the six student teachers were selected from among those whose content knowledge was considered high and the other three were recruited from among those whose content knowledge was deemed low. The idea was to seek clarification from these student teachers regarding their performance on specific items in Phase 1. Most importantly, it was the study's intention to explore the SCK which was held by student teachers who were considered to be a source of enlightening and rich information (Creswell, 2012; Merriam, 2009).

Practically, the student teachers who participated in Phase 1 were placed into two groups after the preliminary analysis of the data. One group comprised student teachers whose total scores in both functions and trigonometry were below the 50% pass mark. The other group constituted those whose total scores in the two topics were above the 50% pass mark. Then, an average score was calculated for each student teacher in the two groups. It should be mentioned that only students who attempted items on both functions and trigonometry had their average scores computed. Student teachers with high average scores were considered to have high content knowledge, while student teachers with low content knowledge were those whose average scores were low. In the next section, the instruments that were used to gather the data are discussed. To start with, the document analysis that was conducted is highlighted, followed by an overview of how the mathematics test was developed.

3.4.3. Data collection instruments

3.4.3.1. Document analysis

To develop the test instrument, a complete document analysis was required. The rationale of conducting a document analysis was to ensure that the test items reflected the contents of functions and trigonometry, as taught in Zambian secondary schools. It was also important to prepare the test items using mathematical materials that were familiar to the student teachers in the Zambian context.

The document analysis included a review of the Zambian secondary school mathematics curriculum, and mathematics text books (Bostock, 2000; Buckwell, 1996; Channon, 1994, 1996; Fuller, 1986; Kalimukwa, 1995; Laridon, 1995; Nkhata, 1995; Redspot, 2013; Talbert, 1995). An analysis of these documents was rigorously carried out after permission was applied for and granted by the Zambian Curriculum Development Centre (ZCDC) to make use of the secondary school mathematics curriculum (see Appendix 4 for the application letter to the ZCDC).

For the topic of functions, it was established that at secondary school level, the following concepts are taught: relations, function, domain, range, one-to-one functions, inverse functions, linear functions, quadratic functions, and composite functions. Learners are expected to evaluate functions, calculate inverse functions, and simplify composite functions. They are supposed to identify the inverse of a function, represent composite functions, and even solve problems involving linear functions. They are also expected to identify relations that are functions and those that are non-functions.

Learners are also expected to demonstrate an understanding of the algebraic representations of linear and quadratic functions. In this regard, they are supposed to be taught the standard linear equation $y = mx + c$ where m is the gradient and c is the *y*-intercept. Regarding quadratic functions, they learn the general form $f(x) = ax^2 + bx + c$ where $a, b,$ and c are constants and $a \neq 0$. Apart from that, secondary school learners are supposed to know how to draw graphs of linear and quadratic functions, and should be able to interpret those graphs. In this context, learners are expected to compute and explain the intercepts of the quadratic curves and linear graphs. They should show an understanding of the maximum and minimum values, as well as the turning points of quadratic functions. An additional aspect relates to the notations of functions, inverse functions, and composite functions. In this regard, notations like $y = f(x)$, $f: x \to y$, $f^{-1}(x), (f \circ g)(x)$ and the set builder notations are taught at secondary school level.

In terms of trigonometry, it was discovered that learners are supposed to learn the definitions of the sine, cosine, and tangent ratios in relation to the right angled triangle. This aspect goes hand in hand with the identification of the sides of a right angled triangle (hypotenuse, adjacent, and opposite sides). The signs of the trigonometric ratios in the four quadrants are also taught together with the two special triangles and the special angles: $30^\circ, 45^\circ, 60^\circ$. The calculation of the lengths of sides and sizes of angles of two and three dimensional figures using the trigonometric ratios, as well as the sine and cosine rules, is a prominent aspect at secondary school level in Zambia.

At the same time, learners are taught the following formulas for the computation of areas of triangles: $\frac{1}{2} \times base \times perpendicular\ height$ and $\frac{1}{2} ab \sin C$. Furthermore, they are taught Pythagoras' theorem, basic trigonometric equations, curves of the sine, cosine, and tangent functions, and their characteristics. Three figure bearings are also taught, including the sketching of diagrams to represent position and direction. In the following sub-section, a discussion regarding how the test was developed is provided.

3.4.3.2. Development of the test

During the development process of the mathematics test, generic ideas from Creswell (2012) were utilised. Creswell posits that it is important to be clear about the purpose of the instrument during its development. He also recommends a review of literature before engaging in item writing. Apart from that, he advises that piloting an instrument is necessary. The purpose of the mathematics test in this study was to assess the abilities of the student teachers in an already defined content area of functions and trigonometry at secondary school level. In the following segments, the manner in which Creswell's (2012) ideas concerning the development of a data collection instrument were implemented in this study are discussed.

The draft test development

The findings of the literature review (Chapter 2) and the document analysis (Section 3.4.3.1) facilitated the preparation of the draft test items. The draft items for the test were designed in relation to the learning outcomes as espoused in the secondary school curriculum (see Table 2.1). The items were also developed to be consistent with the descriptors of CCK and SCK in the study's conceptual framework (Section 2.7). Some of the items from the mathematics textbooks and the Zambian Grade 12 national past examinations papers were adapted for the test.

Eventually, a complete draft test was developed consisting of items that required the award of partial credit and gathering of non-dichotomous data. Thus, the draft test did not comprise objective types of items. Next, an item by item analysis of the draft test was conducted to ascertain the extent to which its contents reflected functions and trigonometry, as intended in the Zambian secondary school curriculum. During this process, Zambian mathematics experts such as mathematics and mathematics education lecturers, curriculum specialists, and secondary school mathematics teachers were consulted and requested to comment on the mathematical suitability of the draft test items. The reviewers were not asked to relate the draft items to CCK or SCK as these constructs were generally not familiar in the Zambian context. However, the researcher ensured that the draft paper had items which assessed CCK and SCK since he understood the definitions of these constructs. The comments made by these experts culminated in a refined draft test, which was then forwarded to the research supervisors for their input. The supervisors' feedback was acted on and this led to further revision of the draft test.

Piloting of the draft test

Questions that are poorly designed, and participants misunderstanding questions can result in invalid scores (Creswell, 2008). In order to minimise the possibility of this, the revised version of the draft test was piloted in Zambia in December, 2013. The pilot study was intended to assess whether the draft test items were clear, well phrased, and measured the purported secondary school mathematics content. Beyond that, the pilot assisted in determining the viability of the marking rubric, and equally provided a chance to estimate how long it would take the sample to complete the final version of the test (Bell, 2014).

Creswell (2012) advises that participants whose characteristics are similar to the sample will ensure successful pilot results; this was the case for this study. In this regard, the pilot involved university mathematics student teachers who were not necessarily in their final year of study. As a way of promoting confidentiality of the draft items, the researcher administered the pilot test, and collected all the question papers and answer sheets at the end of the session. The student teachers' answers to the pilot test items led to the discovery of items that had ambiguity, and those that lacked clarity.

Furthermore, some draft items were not solved by some of the student teachers in spite of the initial 90 minute time restriction being removed during the test administration. This occurrence suggested that the items proved difficult for the participants. This was confirmed by some of the student teachers during the informal interviews. The results of the pilot study also revealed that the numbering that had been utilised in the draft test was inappropriate in some cases. From the results highlighted above, some of the draft items were dropped, while those that proved ambiguous were rephrased, and the numbering of test items was changed. These changes automatically resulted in the altering of the marking rubric in terms of mark distribution.

Expert judgement on the draft test

Principally, expert judgement was sought to ensure that the final version of the test had content and face validity. Content validity refers to the extent to which a data collection instrument assesses the entire content area that it is intended for, while face validity is concerned with whether an instrument appears to measure what it purports to measure (Pietersen, 2014a). Since the test was based on specific content areas, it was important to ensure that it adequately assessed the functions and trigonometry as prescribed in the Zambian secondary school mathematics curriculum. The improved version of the draft test, together with the learning outcomes (Table 2.1) were presented to a professor of mathematics in the department of Mathematics and Applied Mathematics at the University of Pretoria.

The professor was informed of the purpose of the test, and was requested to solve the draft items, and give his expert judgement with respect to the following: (1) the clarity of the items, (2) mathematical correctness of the items, and (3) content and face validity. Following this request, the professor analysed the test and reported that the test items were representative of the content areas of functions and trigonometry as prescribed in Zambian secondary school curriculum. He commented that the items were mathematically correct and that the test was generally suitable for the intended purpose. Nonetheless, the professor suggested the rephrasing of some of the items as a way of promoting clarity.

After implementing the professor’s suggestions, a different professor of mathematics in the department of Mathematics and Applied Mathematics at the University of Pretoria was consulted for additional expert judgement on the draft test. He was also provided with the learning outcomes and content areas based on functions and trigonometry, informed of the purpose of the test, requested to solve the items in the draft test, and finally, to give his judgement in line with the following:

- Recommend an estimate of appropriate duration for the test.
- Indicate the aspects of functions and trigonometry each test item assesses.
- Identify items that are vague, and suggest how such items could be phrased to avoid ambiguity and misinterpretation.
- Suggest any items that should be included in the test.
- Comment on the mathematical correctness and appropriateness of the items, and suggest any improvement that may be necessary.
- Comment, in your view, whether the test has content and face validity with respect to functions and trigonometry as prescribed in the Zambian secondary school curriculum. If not, suggest what should be done to enhance these two types of validity.

Subsequently, the researcher and the second professor met in the latter's office to discuss and agree on ways of strengthening the test. The professor approved the mathematical correctness of the items, but proposed minor amendments to the wording of some of the items. After all the items had been mutually reviewed and revisions suggested, the professor commented that the test had content and face validity. Although he admitted that the test proved challenging, overall, he indicated that the draft test was suitable and appropriate for the intended purpose.

In addition to this, the professor revealed that it took him more than 2 hours to solve the test items, and consequently recommended that the duration of the test be 3 hours. In summary, the version of the test that was administered reflects improvement based on the contribution of teachers of secondary school mathematics, lecturers of mathematics and mathematics education, the research supervisors, the pilot study, and two mathematics professors. Section 3.4.3.3 below provides a description of the final version of the test.

3.4.3.3. Description of the final version of the test

The final version of the mathematics test contained 12 questions, and was organised into Sections A and B. Six of the questions in Section A assessed subject matter knowledge of functions, while the other six in Section B examined subject matter knowledge of trigonometry.

Each of the 12 questions consisted of open-ended items that merited partial credit when marking. The total number of items per question ranged from two to five, giving a total of 42 test items. Furthermore, each item was assigned marks that were deemed consistent either with the anticipated rigour of the work involved to attain the correct answer, or with the complexity of the concept assessed. The composite mark for Section A was 76, while that of Section B was 60. Table 3.2 shows the distribution of marks across the test items (see Section 3.5.1.2 for further details).

Table 3.2: Distribution of marks across test items (Q = Question number, M = Total mark)

Q	1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	2(c)	2(d)	3(a)	3(b)	3(c)	3(d)	3(e)	4(a)
M	**2**	**3**	**3**	**2**	**5**	**12**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**2**
Q	4(b)	5(a)	5(b)	5(c)	5(d)	5(e)	6(a)	6(b)	6(c)	6(d)	7(a)	7(b)	7(c)	7(d)
M	**6**	**2**	**2**	**3**	**1**	**2**	**3**	**2**	**3**	**2**	**3**	**3**	**6**	**3**
Q	8(a)	8(b)	9(a)	9(b)	9(c)	9(d)	10(a)	10(b)	11(a)	11(b)	11(c)	12(a)	12(b)	12(c)
M	**6**	**4**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**3**	**2**	**3**

All of the test items were aligned with the components of the study's conceptual framework (see Sections 2.7.2 and 2.7.3). Tables 3.3 and 3.4 show the test items that assessed proficiency in the CCK of functions and trigonometry respectively. Table 3.5 shows the test items that assessed the ability to explain and justify reasoning in functions. Finally, Tables 3.6 and 3.7 indicate the test items that investigated the use of different representations in functions and trigonometry respectively. In other words, Tables 3.3 and 3.4 are based on the CCK category of the conceptual framework, while Tables 3.5, 3.6, and 3.7 relate to the SCK category of the conceptual framework (the final version of the test appears in Appendix 3).

Table 3.3: Test items assessing student teachers' proficiency in CCK of functions

Features of Common Content Knowledge assessed	**Item number in the test**	**Marks**
Valid definition of a relation	1(a)	2
Provision of a valid definition of a function	2(a)	2
Identification of figures representing examples and non-examples of functions	2(b)	4
Appropriately defining a one-to-one function	4(a)	2
Valid definition of an inverse function	5(b)	2
Determining the range of a quadratic function	5(d)	1
Calculation of images of inverse functions given objects and vice versa	6(a)	3
Finding solutions of an equation using the graph of a quadratic function	3(c)	3
Completing the square of a quadratic function and computing a turning point	3(d)	3
Stating an extreme value of a quadratic function	3(e)	1
Identification of examples and non-examples of Cartesian graphs of one-to-one functions	4(b)	2
Computing the inverse of a function	5(c)	3
Composition of functions	6(c)	3
Evaluating composite functions	6(d)	2
Total mark		33

Table 3.4: Test items assessing student teachers' proficiency in CCK of trigonometry

Features of Common Content Knowledge assessed	**Item number in the test**	**Possible mark per item**
Solutions of simple trigonometric equations	7(a)	3
Finding values of trigonometric ratios using special angles	7(b)	3
Evaluating trigonometric expressions	7(d)	3
Calculation of triangle lengths using trigonometric ratios	8(a)	3
Identification and computation of three figure bearings	8(b)	4
Calculation of triangle lengths using the sine rule	9(a)	3
Calculation of triangle lengths using the cosine rule	9(b)	3
Calculation of the area of a non-right angled triangle	9(c)	3
Finding the shortest distance from a point to a line	9(d)	3
Calculation of two possible values of an angle of a triangle using the sine rule, knowledge of sum of angles in a triangle, and supplementary angles	10(a)	3
Calculation of the smallest of the angles of a triangle using the cosine rule	10(b)	3
Using Pythagoras' theorem to calculate lengths in a three dimensional figure	11(a)	3
Calculation of an angle of a triangle using trigonometric ratios	11(b)	3
Using the sine, cosine or tangent ratios to calculate lengths in three dimensional figure	11(c)	3
Determining the range and period of a trigonometric function	12(b)	2
Total marks in the category		45

Table 3.5: Test items assessing student teachers' ability to explain and justify reasoning in functions

Features of the 'ability to explain and justify reasoning' component assessed	Item number in the test	Possible mark per item
Recognising and explaining relationships among domains and ranges of two relations	1(d)	2
Explaining the difference between a general relation and a function	2(a)	3
Providing justifications why specific figures are examples or non-examples of functions	2(b)	8
Confirming the existence or explaining the lack of other functions whose graphs pass through two given points on a Cartesian plane	2(c)	1
Explaining why a formula (with given domain) represents a function or not	2(d)	2
Explaining how an extreme value relates to the range of a function	3(e)	2
Explaining why particular graphs are examples or non-examples of one-to-one functions	4(b)	4
Recognising differences between functions	5(a)	2
Recognising and explaining a relationship between the range of a function and the domain of an inverse of another function	5(e)	2
Recognising and justifying domains for which a given function has an inverse	6(b)	2
Total marks in the category		28

Table 3.6: Test items assessing student teachers' ability to use different representations in functions

Features of the 'ability to use different representations' component	Item number in the test	Possible mark
Representing a relation as a set of ordered pairs given a set and a 'rule'	1(b)	3
Translating a relation from ordered pairs to a graph on a Cartesian plane	1(c)	3
Sketching graphs of functions passing through two given points on a Cartesian plane	2(c)	2
Determining the accuracy of a given symbolic representation of a function	2(d)	1
Drawing a graph of a function when a formula and discrete domain are provided	3(a)	3
Translating to a graph given a quadratic function in formula form, and a table of values	3(b)	3
Total marks in the category		15

Table 3.7: Test items assessing student teachers' ability to use different representations in trigonometry

Features of the 'ability to use different representations' component	**Item number in the test**	**Possible mark**
Accurately drawing and correctly labelling two special triangles used to evaluate trigonometric expressions	7(c)	6
Translating information relating to bearings onto a diagram	8(a)	3
Translating the sine function from a symbolic representation to a graphical one given a specified domain	12(a)	3
Changing representation of a tangent function from a symbolic representation to a graphical one when a specific domain is provided	12(c)	3
Total mark in the category		15

As indicated in Section 3.4.1, interviews were another data collection instrument that was utilised in this study. In this regard, the next section provides a discussion on the interviews.

3.4.3.4. Interviews

According to Nieuwenhuis (2014b), an interview is "a two-way conversation in which the interviewer asks the participant questions to collect data and to learn about the ideas, beliefs, views, opinions and behaviours of the participants" (p. 87). Generally, interviews are distinguished on the basis of how structured or unstructured they are (Gay, 2011; Merriam, 2009; Nieuwenhuis, 2014b; Seabi, 2013). This study used face-to-face semi-structured interviews, which are a combination of structured and unstructured interviews. The decision to use semi-structured interviews was motivated by the exploratory nature of Phase 2 of the study in which follow ups, probing, and prompting would be necessary (Bell, 2014).

Athough the interview questions were predetermined, the second phase did not impose predetermined answers on the respondents, but explored understanding from the student teachers' perspectives (Nieuwenhuis, 2014a, 2014b). In this regard, explanations and justifications of reasoning, as well as understanding of different representations of functions and trigonometry concepts were elicited from the student teachers. Semi-structured interviews allowed for flexibility in the wording of questions and order of questioning (Merriam, 2009), and allowed for corroboration of the data from the test (Nieuwenhuis, 2014b). Probing and promptings were utilised to acquire insight into the student teachers' understanding of functions and trigonometry concepts. Likewise, follow ups were made of their explanations.

3.4.3.5. Interview schedules

Development of the draft interview schedule

Based on the findings of the document analysis (Section 3.4.3.1), and the literature review (Chapter 2), a draft semi-structured interview schedule was developed containing questions on functions and trigonometry. The following questions, which were aligned with the SCK components of the study's conceptual framework, formed the basis of the semi-structured interview questions that were included in the draft interview schedule: (1) How do the student teachers explain their understanding of concepts? (2) What justifications do the student teachers provide for their reasoning? (3) How well do the student teachers demonstrate an understanding of the different representations of concepts? and (4) How accurately do the student teachers translate between different representations?

The draft schedule was subjected to the expert judgement of the research supervisor and one lecturer of Education Management at the University of Pretoria. It was suggested by the supervisor that the schedule should include questions that would elicit explanations as to why the respondents solved specific test items in a particular way. The Education Management lecturer advised that some of the interview questions should be combined as a way of reducing the quantity. Consequently, the draft schedule was revised by including follow-up questions that would probe respondents' understanding and reasoning for the test items. Additionally, some of the questions were combined while others were dropped.

Piloting of the draft and development of the final schedule

The revised version of the schedule was then piloted involving a PhD student of Mathematics Education. The objectives of the pilot were to assess the clarity of the questions, and determine the estimated time it would take to interview the mathematics student teachers. Since the draft schedule included follow-up questions to the test items, the pilot student was requested to solve the test items in his own time in readiness for the interviews. When he had completed this task, a date and place for the pilot interview were fixed and arranged respectively.

The pilot study revealed that there were a few vague questions, and that the schedule still had too many questions, which could cause the respondents to become tired if posed in one session. These findings prompted the rephrasing of vague questions. It was also decided on to split the interview schedule into two: one section focusing on functions, and another section focusing on trigonometry (see Appendix 5 and Appendix 6). In Section 3.5, the methods of data analysis as they relate to Phase 1 and Phase 2 of this study are explained.

3.5. METHODS USED TO ANALYSE THE DATA

This section discusses the methods that were used to analyse the data that was collected during Phase 1 and 2 of this study. The phase 1 analysis procedures will be generally explained followed by a discussion on how the test data was quantitatively analysed. A description of the marking rubric that was used to score the test items will then be given. Subsequently, details will be provided concerning how Phase 1's data was qualitatively analysed. The section closes with a discussion regarding how Phase 2's data was analysed.

3.5.1. Data analysis methods for Phase 1

Phase 1 sought to find answers to Sub-questions 1 and 2 which read respectively: *How proficient are student teachers in Common Content Knowledge of functions and trigonometry at secondary school level?* and *What Specialised Content Knowledge of functions and trigonometry at secondary school level is held by the student teachers?* (Section 1.6 gives a detailed explanation of these questions). In an attempt to acquire comprehensive answers to these questions, the test data were first of all analysed quantitatively and thereafter qualitatively. Quantitative data analysis involves scoring, tabulation, and the use of statistics. Among the factors that determine the kind of statistical techniques to use when dealing with quantitative data are a research design and the kind of data that is to be analysed (Gay, Mills & Airasian, 2011).

Depending on the research design and data, either descriptive or both descriptive and inferential statistics can be used to analyse data (McMillan, 2006). However, a researcher who uses inferential statistics requires prior availability of descriptive statistics to make conclusions, based on the data from a sample, about a population. Although qualitative data analysis is primarily inductive (Merriam, 2009), to a larger extent, there are no specifically prescribed procedures to follow (McMillan, 2006). A researcher may decide to inductively analyse data in which case categories emerge from the data after a coding process, or may resolve to use predetermined categories. The latter is perceived as an easier option in instances where the researcher is quite knowledgeable of the research questions or topics (McMillan, 2006).

3.5.1.1. *Quantitative analysis techniques for Phase 1 data*

Phase 1 data analysis commenced with the scoring of the test items for correctness using an adapted marking rubric (Section 3.5.1.2). Gay, Mills and Airasian (2011) suggests that a researcher who uses a rubric to score open-ended items should ensure consistency by allowing more than one person to score the scripts; this was applied in this study. The researcher also went through the marked answer scripts several times to cross check the scoring activity to ensure that there was consistency. Partial credit was awarded for the necessary calculations, explanations, justifications, and translations. Furthermore, credit was given for appropriate steps even when the final answer was either incomplete or inaccurate.

After scoring the test items, the student teachers' answers to test items were organised for analysis according to the following categories, which were aligned with the conceptual framework: (1) items assessing proficiency in CCK, and regarding the SCK, (2) items assessing the ability to explain and justify reasoning, and (3) items assessing the ability to use different representations. Thereafter, descriptive statistics like the mean, mode, median, range and standard deviation were generated using the SPSS software (see Appendix 9, 10, 11, 14, and 15). Descriptive statistics were used to provide summaries of the test data with a view to understanding the meaning of the scores (McMillan, 2006; Pietersen, 2014b).

It must be pointed out that descriptive statistics were calculated independently for each of the sets of data that were derived from the categorisation of the student teachers' answers according to points 1, 2, and 3 above. The measures of central tendency that were computed per test item and topic gave an indication of the proficiency and competence of the student teachers in CCK and SCK respectively. More than this, the measures of spread provided an opportunity to assess whether scores per item or topic were closer to the computed mean or not. This assisted in determining how suitable the mean was as a representative measure of the scores. Table 3.8 below reflects the total marks for each category of analysis for functions and trigonometry.

Table 3.8 Total marks for each category of analysis

	Total marks	
Category	**Functions**	**Trigonometry**
Items assessing student teachers' proficiency in CCK	33	45
Items assessing student teachers' competency in provision of explanations or justifications	28	Not applicable
Items assessing student teachers' competency in the use of different representations	15	15

In order to address research Sub-question 1, the data analysis focused on the scores of test items and answers to items that assessed CCK (Tables 3.3 and 3.4). With regard to Sub-question 2, only the scores and answers to items that were aligned with the two components of SCK were considered for analysis (Tables 3.5, 3.6, and 3.7). Microsoft Excel 2010 was used to organise the student teachers' scores into tables and to compute separate composite scores per student teacher in functions and trigonometry. The Statistical Package for Social Sciences (SPSS) version 22 software was also used to generate frequencies per test item for functions and trigonometry. In the following section, the rubric that was used to score the test items is explained.

3.5.1.2. *Explaining the test scoring rubric*

This section presents and elaborates on the rubric that was utilised in scoring the test items (Table 3.9). The rubric was rigidly applied when scoring test items whose maximum possible mark was at least 3. For items whose maximum possible mark was less than 3, only points 1, 2 and 4 of the rubric applied. In instances where an item's maximum possible mark was 1, only points 1, and 4 of the rubric applied.

Table 3.9: Marking rubric for scoring test items (Adapted from the works of Fi (2003))

S/N	Description of areas/reasons for awarding or denying marks	Description of marks allocated
1	Appropriate methods or procedures used to solve test items. Answers complete, accurate, and correct. Provision of valid definitions, properties, rules, and theorems, notations and applying these correctly. Mathematically appropriate explanations or justifications of concepts. Correct/accurate identification of examples and non-examples of concepts, conceptual relationships and differences. Appropriately working with different representations, providing accurate representations (including accurate interpretation of scales), and making accurate translations.	Maximum possible mark
2	Similar to descriptions above. However, a few minor mistakes and errors spotted in at least one area of the representation, definition, property, theorem, rule, calculation or final answer.	One mark less than the maximum possible mark
3	Major errors and mistakes noted in the calculations or final answer to an item. Lack of thorough understanding of the representation including scales, definition, theorem, properties, notations, and rules noted although generally showing a bit of comprehension of what is being assessed.	Two marks less than the maximum possible mark (where applicable)
4	Displays lack of understanding of what the item is assessing. The calculation process and answer out of context: not relevant to what is being assessed. Failure to explain or justify concepts, conceptual relationships and differences. Failure to provide relevant representations, definitions, theorems, rules, and properties. Lack of understanding of different representations, definitions, theorems, rules, properties, notations.	Zero marks (0)

A maximum possible mark of 3 per item was awarded when scoring items that required a display of the process (calculations) leading to the final answer. Furthermore, a mark of 3 per item was awarded for items that required the student teachers' provision of different representations, and translating between representations (see Table 3.2 for marks distribution per item). In the category that required calculations were items such as: 1(b), 1(c), 3(a), 3(b), 3(c), 3(d), 5(c), 6(a), 6(c), 7(a), 7(b), 7(d), second part of 8(b), 9(a), 9(b), 9(c), 9(d), 10(a), 10(b), 11(a), 11(b), 11(c), 12(a), and 12(c) (see Tables 3.3, 3.4, 3.5, 3.6, and 3.7 for specific aspects assessed by each item). Nonetheless, question 6(d), which required an evaluation, was awarded a total of 2 marks. This is because it was felt that student teachers could still provide the correct answer without any calculations.

Furthermore, item 8(a) had two parts: the first assessed the student teachers' ability to present information on bearings in a diagram and the second part required them to compute the distance from one point to another. Each of these parts had a maximum possible score of 3, giving a total of 6 marks for item 8(a). Items that elicited definitions, explanations or justifications of concepts were awarded a maximum possible mark of 2 each. Such items included the following: 1(a), and aspects of 2(a), 2(d), and 3(e), 4(a), part of 4(b), 5(b), and 5(e).

Graphs of functions passing through two given points, identification of graphs passing through two given points, examples and non-examples of concepts, conceptual relationships and differences, determination of accuracy of representations, and location of bearings were awarded 1 mark each. In this category were items such as 1(d), aspects of 2(b), 2(c), 2(d), and the first part of 8(b). There were exceptions such as item 2(b), which was composed of four figures. The four figures required identification of whether they were examples or non-examples of functions, as well as the student teachers' justification of their reasoning. As a result, each figure was awarded a possible mark of 3, giving a maximum possible mark of 12 for item 2(b).

Other exceptions included items 4(b) and 7(c), in which case 4(b) was composed of two figures and candidates were required to identify which one of the two figures was an example or non-example of a one-to-one function. In addition to this, item 4(b) required the student teachers to provide an explanation/justification for their reasoning. To this effect, the correct identification and appropriate explanation/justification were awarded a total of 3 marks each. Since two figures were involved in 4(b), it was consequently awarded a total of 6 marks.

Item 7(c) assessed the candidates' ability to represent the two special triangles used in trigonometry to evaluate expressions. Each accurate drawing was awarded 3 marks, giving a total of 6 marks. The areas of items that required the differences or determination of specific values to be stated were awarded 1 mark for each correct answer. In this context, item 5(a), which required student teachers to provide two differences between two functions, was awarded a total of 2 marks. Similarly, item 12(b), which assessed the determination of the range and period of the function, was awarded 2 marks.

Item 6(b), which required the student teachers to give two domains on which a given function had an inverse, was also allocated a total score of 2 marks. Lastly, areas of 5(d), which asked for the determination of the range of a given function, had a maximum possible score of 1 mark. A discourse on the qualitative analysis procedures of Phase 1 data follows.

3.5.1.3. *Qualitative analysis procedures of data from Phase 1*

Qualitative analysis of the data from Phase 1 was conducted in relation to the corresponding descriptors of CCK and those of the two components of SCK in the conceptual framework (Section 2.7). Research sub-question 1 was qualitatively explored through the student teachers' answers to items aligned with the descriptors of CCK. Research Sub-question 2 was explored using answers to the items that assessed the relevant descriptors of the components of SCK. Practically, the student teachers' calculations, definitions, identifications, explanations, justifications, and use of rules, properties, theorems, and different representations were explored for any emergent patterns or regularities which could assist in providing descriptive accounts (Merriam, 2009).

Furthermore, a classification of the answers, which was adapted from Bryan (1999), was used when assessing aspects of CCK and SCK such as calculations, definitions, explanations, justifications or use of different representations. Cases where the student teachers failed to provide any relevant calculations, definitions, explanations, justifications, or to identify conceptual relationships or differences, or use different representations were categorised as "No answer". Whenever a student teacher attempted to provide a partially correct, inaccurate or incomplete calculation, definition, explanation, justification, or representation, the categorisation was deemed "flawed". Lastly, if a student teacher presented a correct, complete, valid, and accurate calculation, definition, explanation or justification, or accurately identified conceptual relationships, differences, or made an appropriate and accurate translation, the categorisation was accordingly labelled "sound".

Items that were subjected to a qualitative analysis were identified during the scoring process. At this stage, common or particular student teachers' answers to items were effectively noted down. Additionally, the computed frequencies of scores per item in each of the three categories of analysis provided another criterion for choosing items that were to be analysed qualitatively. In this regard, a qualitative analysis was centred on test items that were either answered incorrectly or correctly by the majority of the student teachers. The next section discusses the procedures that were used to analyse Phase 2's data.

3.5.2. Data analysis procedures for the interview data

The interview data analysis followed a Deductive-inductive approach. The use of an uppercase 'D' indicates that preference and prominence was accorded to the deductive mode of analysis and that the inductive mode was only used afterwards. The idea was to provide answers to research Sub-question 2, which reads: *What Specialised Content Knowledge of functions and trigonometry at secondary school level is held by the student teachers?* Merriam (2009) describes qualitative data analysis as a process in which the goal is to make sense of the data so that, ultimately, the answers to research questions can be provided. Merriam (2009) also posits that this process "involves consolidating, reducing, and interpreting what people have said and what the researcher has seen and read-it is the process of making meaning" (p. 175-176). The implication of this definition is that answers to research questions are generated on the basis of the interviewees' responses.

Two predetermined components of the SCK category of the conceptual framework (Section 2.7) were used to analyse the interview data. The predetermined categories were employed because the interviews were used to explore the student teachers' content knowledge in an area in which the researcher is knowledgeable. Audio-recordings of the interviews were transcribed, giving a total of 12 transcripts. Thereafter, the transcripts were read several times in order to acquire a sense of the data. Following this process, a content analysis was used to explore the transcripts in relation to the descriptors of the two components of SCK.

Merriam (2009) describes content analysis as a technique that involves the search for themes and patterns of meaning that frequently appear in a data set. In the context of this study, content analysis was employed to identify segments of data in the transcripts that would provide answers to specific interview questions. Transcripts were also explored for commonalities and differences in the students' responses to interview questions.

3.6. TRUSTWORTHINESS

One of the ways in which the trustworthiness of qualitative research can be enhanced is through the use of multiple data collection methods (Nieuwenhuis, 2014b). In this regard, this study's data were gathered through a mathematics test in addition to interviews. Some of the aspects that were assessed in the test were explored further during the interviews. The interviews provided an opportunity for the verification of the student teachers' understanding, as revealed by the test (see Section 3.4.3.2 for details on development of the test).

To enhance the trustworthiness of the interviews, the draft interview schedule was piloted to facilitate the clarity of the interview questions (Section 3.4.3.5). During the administration of the final versions of the interview schedule, the student teachers were asked similar questions. In order to avoid bias, the researcher tried to minimise subjectiveness (Nieuwenhuis, 2014b), and did not allow his knowledge of functions and trigonometry to influence the student teachers' explanations. Thus, the student teachers were encouraged to explain their perspectives without any suggestions on the researcher's part whether they were right or wrong. At the beginning of each interview session, the student teachers were encouraged to express their viewpoints during the interview without any fear. The respondents were also assured that the interview records would have no effect on their academic performance at the University of Zambia. Apart from this, the researcher aimed to facilitate an informal atmosphere during the interviews. These steps were undertaken to minimise the Hawthorne effect (Cohen, Manion, & Morrison, 2000), a situation where the researcher's presence would influence the student teachers' explanations.

3.7. ETHICAL CONSIDERATIONS

Ethically, it is required that participants be well informed of the objectives of the study. It is also important that the best practices of confidentiality and anonymity, as well as the participants' right to voluntarily give consent to participate in the research are respected (Creswell, 2012; Gay et al., 2011). In this regard, the ethical guidelines of the Faculty of Education at the University of Pretoria were followed. After successfully defending the study's research proposal at a meeting called by the relevant committee of the Faculty of Education at the University of Pretoria, an application for ethical clearance was submitted. When the application was approved, an application to access the site was sent to the Registrar of the University of Zambia (see Appendix 1). Upon approval of the application by the Registrar of UNZA, the researcher entered the site to recruit participants for the study.

Participants for Phase 1 were recruited from a class of mathematics education student teachers during a regular laboratory session at the University of Zambia. With the sanctioning of the Head of Mathematics and Science Education department, an initial list of the names of registered final year student teachers majoring in mathematics was compiled as prospective participants. Subsequently, permission was sought from one mathematics education lecturer to speak to the mathematics student teachers, and consequently, a meeting was arranged.

During the meeting, the student teachers were given a request letter (see Appendix 2), and verbally informed of the objectives of the PhD study. They were also informed that it was their voluntarily participation which was being sought. It was explained to them that the consenting participants would write a mathematics test based on functions and trigonometry at secondary school level, and that some of them would subsequently be requested to participate in semi-structured interviews. Their right to withdraw at any time from the research was emphasised, and at the same time, they were assured of confidentiality and anonymity in the study. The prospective participants were equally given time to raise any concerns that they had, and these were then clarified and addressed by the researcher. By the end of the interaction, 3 female and 19 male student teachers agreed to participate in the study, and each one of these 22 signed the informed consent form (Appendix 2).

Since the sample for Phase 2 came from among the participants of Phase 1, only those student teachers who met the sampling criteria (see Section 3.4.2.3) were individually invited and requested to participate in the semi-structured interviews. The purpose of the interviews was explained to the student teachers. They were equally reminded that they had the right to withdraw at any time during the research process if they so desired. Although some of the student teachers did not turn up (without giving reasons), others accepted and finally, 6 students were interviewed in a secluded and quiet mathematics education conference room on different days and at different times.

3.8. ADMINISTRATION OF THE DATA COLLECTION INSTRUMENTS

3.8.1. Administration of the test

Administration of the test was conducted in May, 2014, at the UNZA in the mathematics education laboratory on a day that was mutually agreed upon by the researcher and the student teachers. The student teachers were allowed to use calculators when solving questions on trigonometry, except for Question 7(b) in which it was strictly emphasised that a calculator should not be used. This was done because calculators are allowed in the Zambian Grade 12 national mathematics paper two examinations. The researcher administered the test and provided the student teachers with writing paper and sheets of graph paper. At the end of the three hour session, the answer sheets were collected and kept confidentially by the researcher for scoring and analysis. In the following section, an articulation of the administration of the interviews is provided.

3.8.2. Administration of the interviews

The final versions of the interview schedules included ideas that emanated from the preliminary analysis of the test data. After mutually agreeing on a time and place, 6 student teachers were individually interviewed at the University of Zambia in the mathematics education conference room (see Section 3.4.2.3 for a description of the sample). The interviews were conducted in the months of June and July 2014.

Each student teacher was interviewed twice where the first interview was based on functions, while the second interview focused on trigonometry. In order to acquire ‘rich’ data from the respondents, leading questions and interruption of the respondents as they gave their explanations were avoided. Furthermore, follow-up questions, probing, and prompts were a prominent feature in the interviews (McMillan, 2006).

The interviews were audio recorded and each student teacher was requested to keep the interview details confidential. This would help to avoid a situation where the student teachers could acquire prior knowledge of the common interview questions. During the interview sessions, respondents were provided a copy of the Phase 1 test paper, a copy of their answers to specific test items (marks were hidden), and writing paper. The answers to the test items were intended to remind the student teachers of how they solved the items. The writing paper allowed them to solve cases where they needed to do so as they were being interviewed. At the end of each interview session, the audio recording was played in the presence of the student teachers so that they could confirm its accuracy. Figure 3.1 below encapsulates how the data for Phase 1 and 2 were collected.

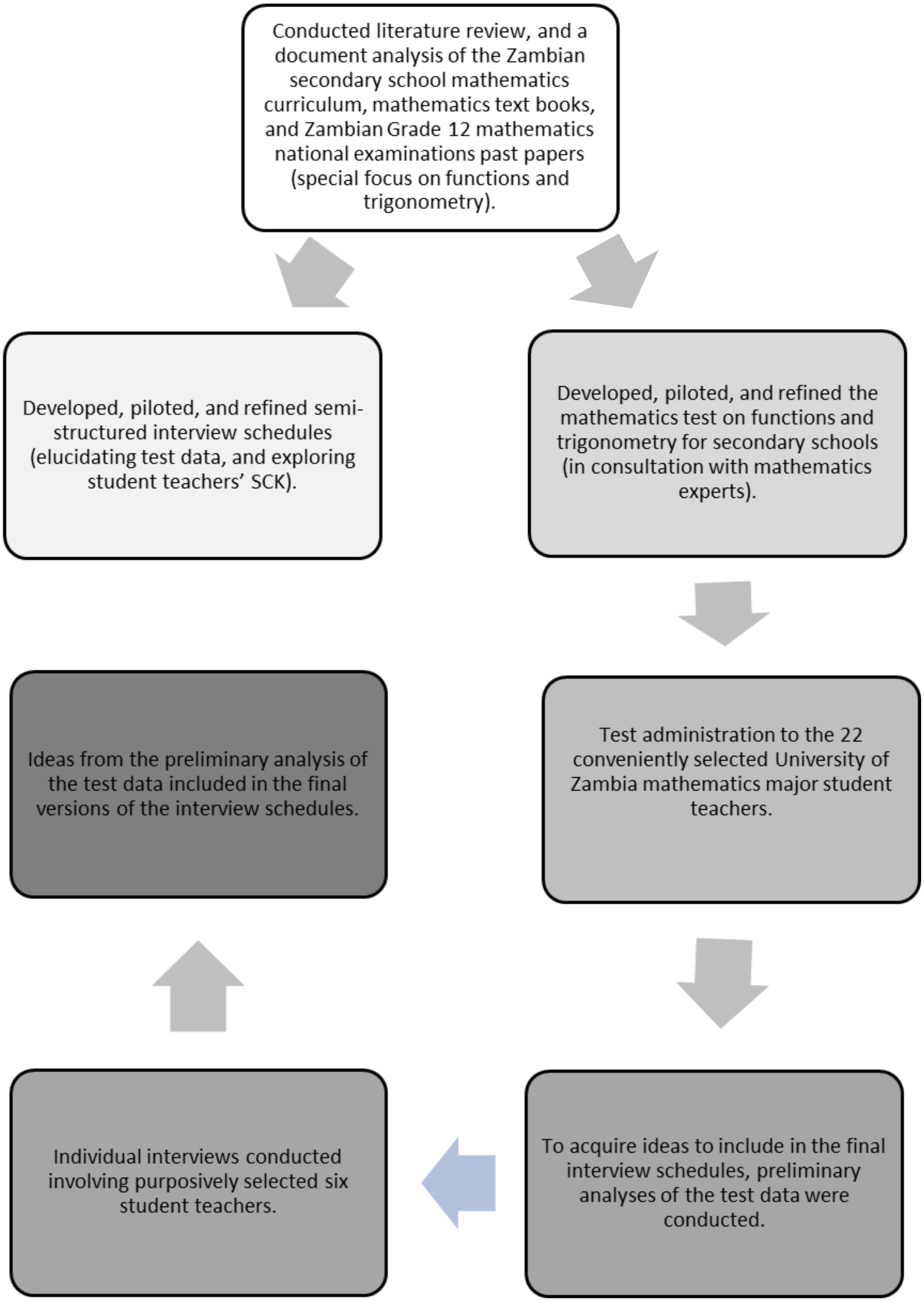


Figure 3.1: Data collection process.

3.9. SUMMARY OF CHAPTER 3

Chapter 3 discussed the paradigm as well as the ontological and epistemological assumptions that underpin this study. It has been explained that this study followed a qualitative approach, and was a single intrinsic case study. Issues relating to the data collection, such as development of the data collection instruments, have been discussed. In addition to this, samples for Phases 1 and 2 of the study were explained including the sampling techniques that were employed. At the same time, this chapter highlighted the methods that were utilised to analyse the test and interview data.The ethical issues, as well as the administration of the data collection instruments, have been discussed. In the next chapter, the presentation and analysis of the results from the test and interviews for the topic of functions are provided.

4. ANALYSIS OF THE DATA ON FUNCTIONS

4.1. INTRODUCTION

This chapter presents and analyses the data concerning functions, which was collected during Phase 1 and 2 of the research. These two phases of the study were intended to assess and explore the CCK and SCK of UNZA's final year mathematics student teachers in functions and trigonometry at secondary school level. The data analysis was intended to provide answers to the research questions and was conducted using the categories of the conceptual framework. The research questions are reproduced below for easy reference. While the research questions include the topic of trigonometry, the analysis of the data based on trigonometry will only be presented in Chapter 5.

The main question that guided the research is: **How can the University of Zambia's mathematics student teachers' content knowledge of functions and trigonometry at secondary school level be described?** The two sub-questions that contributed in answering the main question are:

How proficient are student teachers in the Common Content Knowledge of functions and trigonometry at secondary school level?

What Specialised Content Knowledge of functions and trigonometry at secondary school level is held by the student teachers?

Chapter 4 is organised into two main sections, namely, Section 4.2 and Section 4.3. Section 4.2 presents and analyses data from Phase 1 (the test), while Section 4.3 presents and analyses the data from Phase 2 (the interviews). Section 4.2 is designed to answer Sub-questions 1 and 2, while Section 4.3 answers Sub-question 2. Section 4.2 provides a specific analysis of the test data using the three categories of the conceptual framework. Section 4.3 provides the Deductive-inductive analysis of the data that was gathered through semi-structured interviews. The categories of SCK were used to analyse the interview data, and subsequently, the transcripts were explored for any emergent themes. A summary of how the research data was analysed is presented in Table 4.1 below. What follows thereafter is a presentation and analysis of the data from the mathematics test.

Table 4.1: Summary of organisation of Chapter 4: presentation and analysis of data on functions

Section 4.2	**Research questions 1 and 2:**	
Phase 1	• How proficient are student teachers in the Common Content Knowledge of functions at secondary school level? • What Specialised Content Knowledge of functions at secondary school level is held by the student teachers?	Presentation of results and analysis using the three categories of the conceptual framework (student teachers' proficiency in CCK, ability to explain and justify reasoning, and ability to use different representations).
Section 4.3	**Research questions 2:**	
Phase 2	• What Specialised Content Knowledge of functions at secondary school level is held by the student teachers?	Deductive analysis of interview data using the two categories of SCK in the conceptual framework (student teachers' ability to explain and justify reasoning, and student teachers' ability to use different representations). Exploration of the interview data for any emergent themes.

4.2. RESULTS AND ANALYSIS OF STUDENT TEACHERS' PERFORMANCE IN THE TEST

The data presented and analysed in this section was collected using a mathematics test that was administered to 22 UNZA's final year student teachers majoring in mathematics (Section 3.4). These student teachers had completed core courses in the mathematics education programmes (Section 3.4.2.2), and were recruited using convenience sampling techniques. Furthermore, all the student teachers wrote the test at the same time in May, 2014 in the mathematics education laboratory at the UNZA. For the purpose of ensuring the anonymity of the student teachers, each of them was assigned a pseudonym.

Each of the questions in the test consisted of items that assessed the different concepts of the three categories of analysis (Section 3.5.1.1). It should be recalled that the objectives of the test were to: (1) assess the proficiency of the student teachers in CCK, (2) assess how competent student teachers were at explaining and justifying their reasoning, and (3) assess the student teachers' ability to use different representations. For these objectives to be comprehensively addressed, there was a need to specifically analyse the performance of the students in each one of the three categories above. In this regard, Sections 4.2.1, 4.2.2, and 4.2.3 explain these three categories independently.

4.2.1. Proficiency of student teachers in the CCK of functions

The mathematics test consisted of 14 items that assessed the student teachers' proficiency in the CCK of functions. These items totalled 33, and each one assessed a particular aspect of functions, as found in the Zambian secondary school mathematics curriculum. The student teachers' total scores in the CCK category of functions are summarised in Figure 4.1 below.

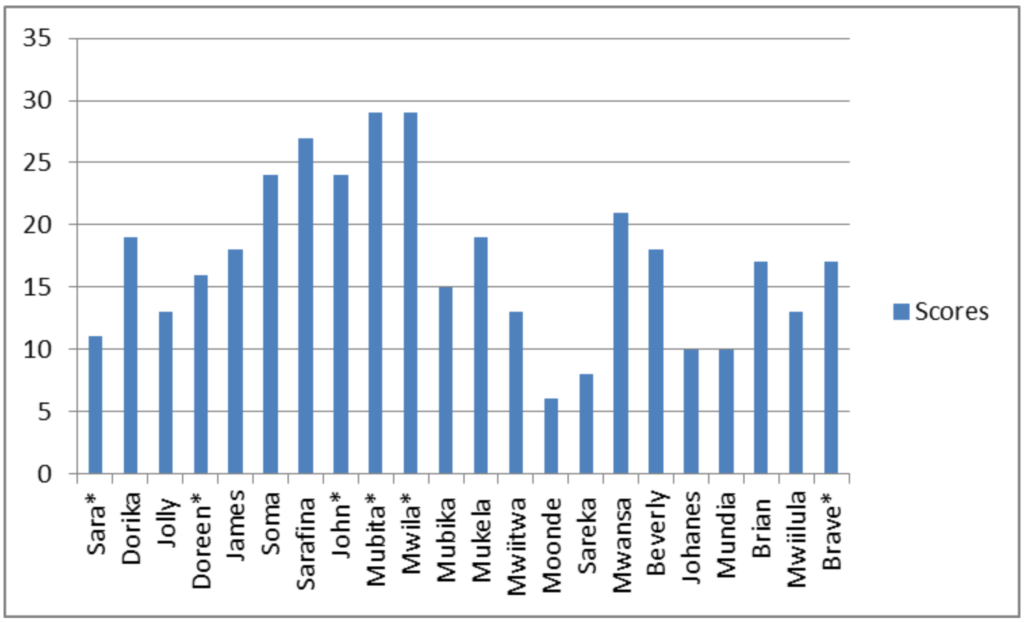


Figure 4.1: Student teachers' total scores in the CCK category of functions

* represents student teachers who agreed to be interviewed.

Figure 4.1 shows that there were 12 student teachers whose total scores in the CCK category were above the 50% pass mark, while ten student teachers achieved total scores that were below 50%. None of the 22 student teachers scored a zero total mark. The descriptive statistics based on the student teachers' performance in these test items are presented below in Table 4.2. Specifically, Table 4.2 provides the descriptive statistics such as the overall mean score, range, and standard deviation for items in the CCK category.

Table 4.2: Summary of descriptive statistics: proficiency in the CCK of functions

Items	Number of student teachers	Range	Minimum score	Maximum score	Mean score	Standard deviation
1a to 6d	22	23.00	6.00	29.00	17.0909	6.54588

Table 4.2 shows that out of a possible total mark of 33 for the items in the CCK category, the student teachers' mean score was 17, translating to approximately 52%. This achievement would suggest a satisfactory performance, especially since the mean score was beyond 50%. Thus, it would appear that most of the student teachers were proficient in the CCK of functions. Nonetheless, when other statistics are taken into account, a different picture seems to emerge, for example, the minimum score in the CCK category was six, while the maximum score was 29. This gives a range of 23, which is a substantial figure considering the fact that the total mark in the CCK category was 33.

Additionally, a standard deviation of approximately 7 may be indicative that the student teachers' scores were not very close to the mean. In order to acquire a clear picture, it is necessary to analyse the student teachers' performance item by item in the CCK category. A summary of the statistics such as the range, mean, and standard deviation per item is provided in Appendix 9. Individual items will be explained first, followed by the student teachers' performance results. Where necessary, a sample answer will be presented and analysed as a way of providing further insight.

Item 1(a)

Define what a relation is as you would teach it to secondary school pupils.

This item required the student teachers to provide a valid definition of a 'relation' as they would teach it to secondary school pupils. Generally, the student teachers were assessed on their ability to clarify that a relation is a 'rule' of some kind that associates/connects/links each member of a first set to at least one member of a second set. The student teachers' definitions were expected to not be restricted to regular formulas, but to include all possible forms that a relation can take.

Seven student teachers did not present any definition of a relation, while ten students provided definitions that were flawed. Each one of the last five students presented definitions that were considered sound (see Section 3.5.1.3 for an explanation of these categorisations). Although almost half of the total number of student teachers had an idea of what a relation is, their definitions were not sufficient to be considered as valid. One student teacher defined a relation as 'A formula which maps members of the domain called objects to members of the range called images'.

This definition is generally acceptable as it shows that the student teacher was knowledgeable about the concepts of domain, range, objects and images. However, this definition is restrictive in the sense that it confines relations to formulas. Thus, it excludes other cases where relations may not necessarily be represented by a standard/regular formula.

The other student teachers defined relations as 'mappings that are one-to-one' or as 'functions mapping elements from the domain to the range'. These definitions suggest that the student teachers lacked an understanding of the difference between general relations and functions. In summary, the student teachers' performances on this item indicate that the majority (77%) were not proficient at providing valid definitions of a relation.

Item 2(a)

Give a definition of what a function is.

This item required the student teachers to provide a valid definition of a function. A definition was considered valid if it made reference to the characteristic that distinguishes functions from ordinary relations without being restrictive. The student teachers were expected to demonstrate the understanding that for a relation to be a function, each object is supposed to be mapped onto one image. This was to be done without necessarily confining the definition to a regular rule or formula.

12 of the student teachers did not give any definition. This group represents 55% of the sample. Five students provided definitions that were not comprehensive and consequently, these were deemed flawed. The other five student teachers gave definitions that were considered 'sound'. The results suggest that over 70% of the student teachers could not provide a valid definition of a function. One of the five student teachers who provided a flawed definition defined a function as 'a relation which is both one-to-one and many-to many'. Although this definition correctly

indicated that a relation that is one-to-one is a function, it also exposed the student teacher's superficial understanding. This is obvious in the student teacher's claim that many-to-many relations are functions. While the univalence property is implied in one-to-one relations, the inclusion of a many-to-many relation in the definition of a function brought the student teacher's understanding of this property into question.

Two other examples of flawed definitions are as follows: 'a function is a relation that maps one-to-one' and 'a function is a mapping of one element to the other by the use of a given rule'. The first definition suggests that this student teacher thought of functions as a one-to-one correspondence. This definition is not valid because it is incomplete and inaccurate. The second definition seems to show this student teacher's lack of understanding of both the univalence and the arbitrariness of functions. In this definition, there is no indication of each object being mapped onto only one image. Similarly, it seems that the student teacher's conception of functions involved only cases where a rule had to be provided.

Item 2(b)

In each of the cases below, state whether the figure represents a function or not.

This item assessed the student teachers' ability to identify an example or non-example of a function. Four figures were presented and in each case, the student teachers were required to determine whether each of those figures represented a function or a non-function. Each of these four figures is now presented and briefly explained thereafter.

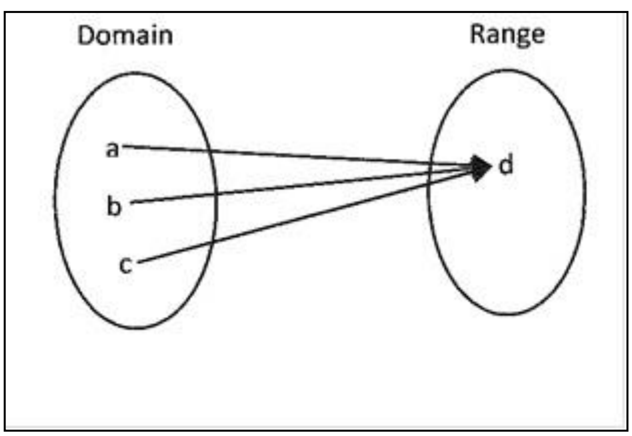


Figure 4.2: A many-to-one relation

Figure 4.2 is a many-to-one relation presented in an arrow diagram form. This figure does not have any regular rule or description associating members of the domain with those of the range. The idea was to assess whether the student teachers understood the univalence and arbitrariness properties of functions.

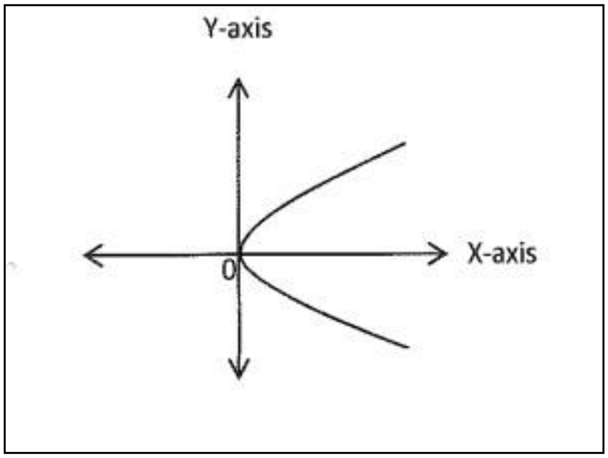


Figure 4.3: A one-to-many graph

Figure 4.3 is a one-to-many graph, and is not a function. Although, it can be a function if the y-axis is considered as the domain and the x-axis as the range, it was presented in a manner that disqualified it as a function.

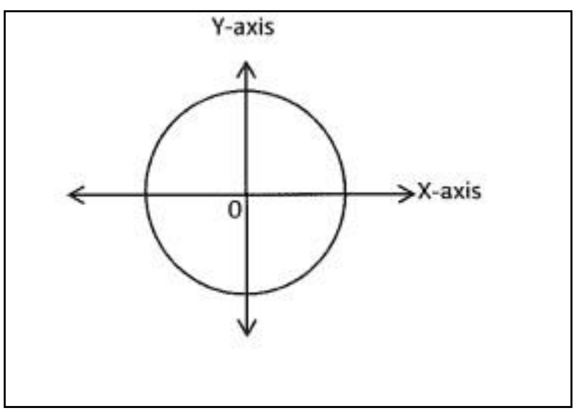


Figure 4.4: Cartesian circle

Figure 4.4 is a Cartesian circle, which is a non-function. The idea was to assess whether the student teachers understood that a Cartesian circle is a non-function.

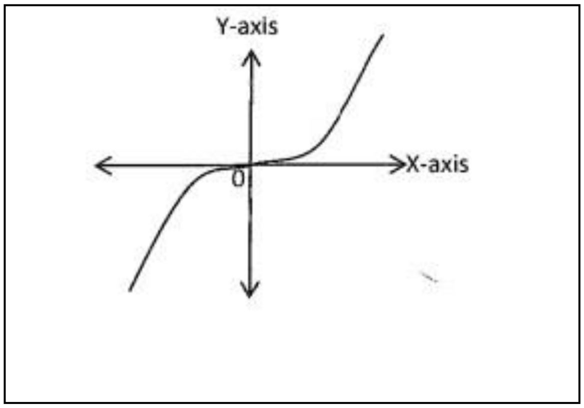


Figure 4.5: One-to-one Cartesian graph

Figure 4.5 is a one-to-one Cartesian graph. The intention was to assess whether the student teachers understood that a graph of a one-to-one relation is an example of a function.

The student teachers' results are provided in Table 4.3. Of the 88 possible identifications, nine (10%) fall under the 'no identification provided' category. 12 answers (approximately 14%) were flawed identifications, while 67 (76%) of the identifications were sound.

Table 4.3: Results of student teachers' identification of examples or non-examples of functions

Figures	NIP*	FIP*	SIP*	Totals
4.2	1	3	18	22
4.3	5	4	13	22
4.4	2	5	15	22
4.5	1	0	21	22
Total	9	12	67	88

*NIP means 'no identification provided', FIP implies 'flawed identification provided', and SIP means 'sound identification provided' (see Section 3.5.1.3).

The results per figure show that 18 student teachers, representing approximately 82% of the sample, soundly identified Figure 4.2. Three student teachers (approximately 14% of the sample) made flawed identifications, and one student teacher did not indicate a position. These results suggest that the majority of the student teachers considered a many-to-one relation as a function.

In terms of Figure 4.3, 13 student teachers (59%) correctly identified the one-to-many relation. However, five student teachers (23%) did not indicate their position, while four (18%) made flawed identifications. As for Figure 4.4, there were two (9%) student teachers who did not state their position. five students (23%) made flawed identifications as they considered a circle to be a function. 15 student teachers (68%) made sound identifications by taking a circle to be a non-function. 21 student teachers (95%) correctly identified Figure 4.5, while one student teacher (approximately 5%) did not indicate his position. It seemed, therefore, that the majority of the student teachers could identify examples and non-examples of functions.

Item 3(c)

Use the graph of $f(x) = -2x^2 - x + 8$ to solve $f(x) = 2$.

This item assessed the student teachers' ability to use the graph of the function $f(x) = -2x^2 - x + 8$ defined on the domain $-3 \leq x \leq 3$ to find the roots of an equation $f(x) = 2$. This question was used to determine whether the students could apply their knowledge of quadratic graphs to solve equations.

Five student teachers had difficulties solving this item, two of whom completely failed to solve the item, while three provided flawed answers. 17 student teachers correctly solved the equation using the graph of the function *f*. This number, which represents 77% of the sample, suggests that the majority of the student teachers were proficient at this task.

Item 3(d)

Complete the square for $f(x) = -2x^2 - x + 8$ and hence determine the turning point of f.

Here, student teachers were assessed regarding their ability to compute the turning point of the function $f(x) = -2x^2 - x + 8$ using the completing of the square method. 11 student teachers, representing 50% of the sample, could not solve this item. Five student teachers (23%) presented flawed answers, while six student teachers (27%) were able to both complete the square and determine the turning point. These statistics suggest that most of the student teachers were not proficient in the CCK assessed by this item. An example of a flawed answer is presented below in Figure 4.6.

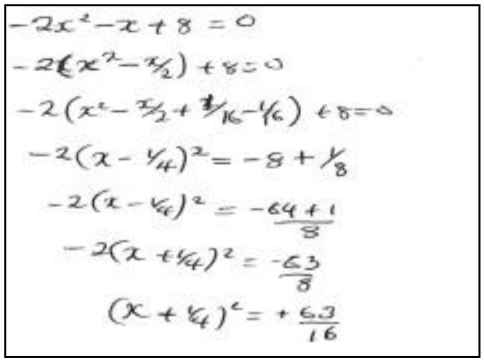


Figure 4.6: Answer A

It appears that this student teacher 'mistook' the roots of a quadratic equation $f(x)=0$ for the turning point of *f*. This is evident in that his first step was to equate the right hand side of $f(x)$ to zero. This process, if correctly followed, would have generated the *x*-intercepts of the function f. While it is possible to use the *x*-intercepts to calculate the axis of symmetry of a quadratic function, this item required the completion of square method.

Item 3(e)

State the maximum value of $f(x)=-2x^2-x+8$.

Item 3(e) assessed the student teachers' proficiency in determining the maximum value of the function $f(x)=-2x^2-x+8$. To determine this extreme value, the student teachers were at liberty to use either the graph of the function or the turning point determined in item 3(d).

The results show that approximately 64% of the student teachers failed to determine the accurate maximum value of the function, while 36% succeeded. By implication, the majority of the student teachers did not know that the maximum value of the function *f* was the *y*-value at its turning point. It appears that the majority of the student teachers did not know how to use the *x*-value at the turning point to evaluate the maximum value of *f*.

Item 4(a)

Give a definition of a *one-to-one*

This item required the student teachers to provide a valid definition of a one-to-one function. They were expected to demonstrate the understanding that in one-to-one functions, not only do objects have unique images in the range, but images equally have unique objects in the domain. The rationale was to determine the student teachers' ability to distinguish one-to-one functions from many-to-one functions.

18 student teachers either did not answer this item or when they did, their definitions were out of context and could only be classified as 'no answer'. This represents approximately 82% of the sample. Moreover, three student teachers provided definitions that were deemed flawed. Only one student teacher presented a definition that was considered sound. By implication, approximately 96% of the student teachers could not demonstrate the ability to give a valid definition of a one-to-one function.

A sample of flawed definitions, which was provided by one of the student teachers, reads as follows: 'a one-to-one function is where all the elements of the domain are mapped exactly to one point (or element) in the range. Also, all the elements of the range have exactly one object from the domain'. This definition lacks clarity and, consequently, was not valid. For example, in one sense, the definition emphasises that in a one-to-one function, 'all' objects have one image, and yet in another sense, it stresses that 'all' images have one object. It is probable that the student teacher wanted to indicate that in a one-to-one function, each object has a unique image and each image has a unique object, however, he failed to demonstrate this aspect.

Another flawed definition reads as follows: 'a one-to-one function is a function that maps each element in the domain to only one image in the range'. This definition shows that the student teacher had knowledge about the feature that distinguishes a function from a general relation. However, this definition does not show that in a one-to-one function, the images have unique objects in the domain. The student teacher's superficial conception of a one-to-one function was demonstrated by two arrow diagrams, which he presented alongside the definition. The arrow diagrams are reproduced below as Sketch B1 and B2.

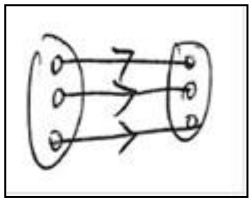


Figure 4.7: Sketch B1

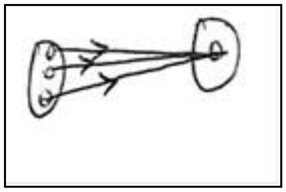
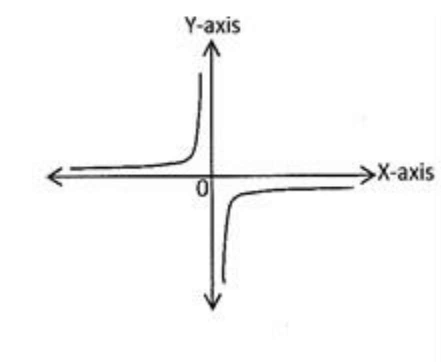


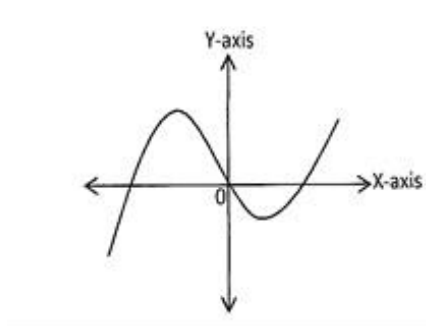
Figure 4.8: Sketch B2

The use of arrow diagrams without any regular rule or formula connecting non-numerical objects and images suggests that the student teacher appreciated the arbitrariness of functions. While Sketch B1 is an accurate representation of a one-to-one function, Sketch B2 is a many-to-one function. It would appear that the student teacher lacked an understanding of the difference between a one-to-one function and a function in general.

Item 4(b)

A mathematics textbook shows the following graphs as examples of one-to-one functions.





Is the textbook correct in this regard, or not?

For this item, the two given figures are purported to be examples of one-to-one functions in a mathematics textbook. The student teachers were required to determine whether one or both of these two graphs are a one-to-one function. In reality, only the first figure is a one-to-one function, while the second was a many-to-one function.

Eight student teachers failed to correctly identify both figures, while eight other students managed to correctly identify one figure. The remaining six student teachers were able to correctly identify the two graphs. These results suggest that the majority of the student teachers (approximately 73%) were not proficient at this task. Most importantly, the results seemed to reinforce what was observed in item 4(a), that most of the student teachers had difficulties providing a valid definition of a one-to-one function.

Item 5(b)

Define an inverse function.

This item required the student teachers to provide the definition of an inverse function. A definition was considered valid if it indicated that an inverse function of a function, say *f*, relates images in the range of *f* to their corresponding objects in the domain while preserving the univalence condition.

17 student teachers completely failed to give a valid definition of an inverse function. This is a very large number, representing 77% of the sample. There were four student teachers who provided definitions that were flawed, while one student teacher gave a definition that was considered sound. This means that 95% of the student teachers had difficulties giving a valid definition of an inverse function.

An example of a flawed definition is as follows: ‘an inverse function is the function that takes back the element in the range back to the domain’. The student teacher who gave this definition may have been thinking in terms of the characteristics of an inverse function. Nevertheless, this definition lacks clarity. At face value, this definition can be interpreted to mean that elements in the range are normally members of the domain, which is of course flawed.

Item 5(c)

Let $f(x) = x^2 + 1$ for $0 \le x \le 2$. Find an expression for $f^{-1}(x)$ and specify the domain of f^{-1}.

This item required the student teachers to find an expression for $f^{-1}(x)$ and to specify the domain of f^{-1}. Thus, the item assessed the student teachers' ability to find a correct expression representing the inverse of a quadratic function whose restricted domain has been provided. In addition to this, the item assessed whether the student teachers understood that the range of f is the domain of f^{-1}. They were expected to show that they could correctly make the variable x the subject of the formula, and ultimately provide an expression for $f^{-1}(x)$ and the domain of f^{-1}.

The results show that four student teachers failed to provide both the correct inverse of f and the domain of f^{-1}. 14 student teachers had an idea of what to do, except their answers were either incomplete or had errors and consequently, those answers were classified as 'flawed'. The other four students were successful at providing complete and accurate answers, which suggests that their knowledge of this item was sound. From these results, it can be argued that the student teachers performed moderately well. Most of the student teachers were able to make the independent variable the subject of the formula, but struggled to calculate the domain of f^{-1}.

Item 5(d)

Let $h(x) = x^2 + 1$ for $-2 \le x \le 2$. Determine the range of h.

The student teachers' ability to determine the range of the quadratic function h was determined in this question. The possibility existed that students would merely substitute -2 and 2 into the function as a way of obtaining images, but this strategy may not have resulted in an accurate range. Two approaches that assist in the generation of accurate ranges for functions in terms of h are: (1) making a sketch of the graph of the function, and (2) determining the turning point of the graph of the function.

14 student teachers did not give the correct range of h, which constitutes approximately 64% of the sample. Nonetheless, eight student teachers were able to correctly determine the required range of the function h. These results seem to suggest that the majority of the student teachers struggled with the concept of range as it relates to quadratic functions.

Item 6(a)

Given that $g(x)=\dfrac{3}{2x+1}$, find the value of $g^{-1}(-5)$.

This item assessed the student teachers' ability in two areas: (1) they were expected to demonstrate that they were capable of finding an expression for the inverse of a linear function g, and (2) they were expected to show the ability to calculate the image of an inverse function g^{-1} when an object is provided.

14 student teachers correctly solved this item. 6 other student teachers provided calculations that were flawed. The last two student teachers completely failed to provide answers for the item. These results indicate that the majority of the student teachers were proficient at calculating inverse functions of linear algebraic functions. Additionally, the results suggest that most of the student teachers could evaluate linear functions.

Item 6(c)

Given that $g(x)=\dfrac{3}{2x+1}$ and $z: x \to x^2-2x$, find an expression for $(z \circ g)(x)$ where $z \circ g$ denotes the composite function of z and g.

This item assessed the student teachers' ability to find a function that is a composition of the functions z and g. The item also assessed whether they understood that in the composition of z and g, the function g is supposed to operate as an object of the function z.

The results indicate that eight student teachers (36%) were unable to provide any answers. Four student teachers (18%) provided answers that were flawed, while ten students managed to provide correct and complete answers. Although about 46% of the students managed to correctly solve the item, a significant number of the students had no idea regarding the composition of functions. The majority of the student teachers (54%) lacked proficiency in finding the correct composition of two functions. An example of the student teachers' flawed answers is presented here in Figure 4.9.

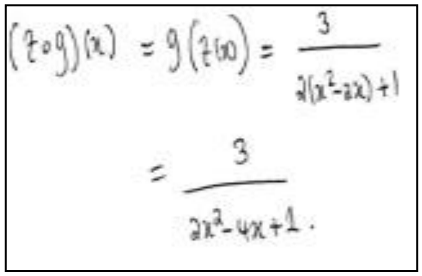


Figure 4.9: Answer C

Answer C suggests that the student teacher did not know how to correctly find the expression for $(z \circ g)(x)$. In as much as it seems he knew the process of composing functions, the calculation shows that he lacked proficiency when it comes to the order of composing functions. Instead of letting the function g act as an object for the function z, the student teacher took z as the object, which resulted in the wrong expression for the composite function of *z* and *g*.

Item 6(d)

Given that $g(x) = \dfrac{3}{2x+1}$, evaluate $(g \circ g^{-1})(-5)$ where -5 belongs to the domain of g^{-1}.

Firstly, this question assessed the students' ability to find a composite function of the function g with its inverse g^{-1}. They were then required to demonstrate proficiency by finding the image of the composite function when an object had been provided. The student teachers could decide to use their knowledge and state the answer without calculations, or they could initially find an expression for $(g \circ g^{-1})(x)$ and then evaluate $(g \circ g^{-1})(-5)$.

The number of students who failed to evaluate the composite function $(g \circ g^{-1})(-5)$ was almost equal to the number of those who successfully evaluated it. Specifically, nine student teachers failed to provide the correct answer, while the remaining 13 student teachers evaluated the composite function. In other words, approximately 41% of the student teachers showed a lack of proficiency, while the majority (59%) demonstrated proficiency in evaluating a composite function.

4.2.2. Student teachers' ability to explain and justify reasoning in functions

This category of analysis involved ten items that assessed the student teachers' ability to explain and justify their reasoning. Included here are items that assessed their ability to recognise and explain or justify conceptual relationships and differences. The possible total marks per item in this category ranged from one to eight, and the possible total score was 28. Figure 4.10 below shows the student teachers' total scores in the 'ability to explain and justify reasoning' category.

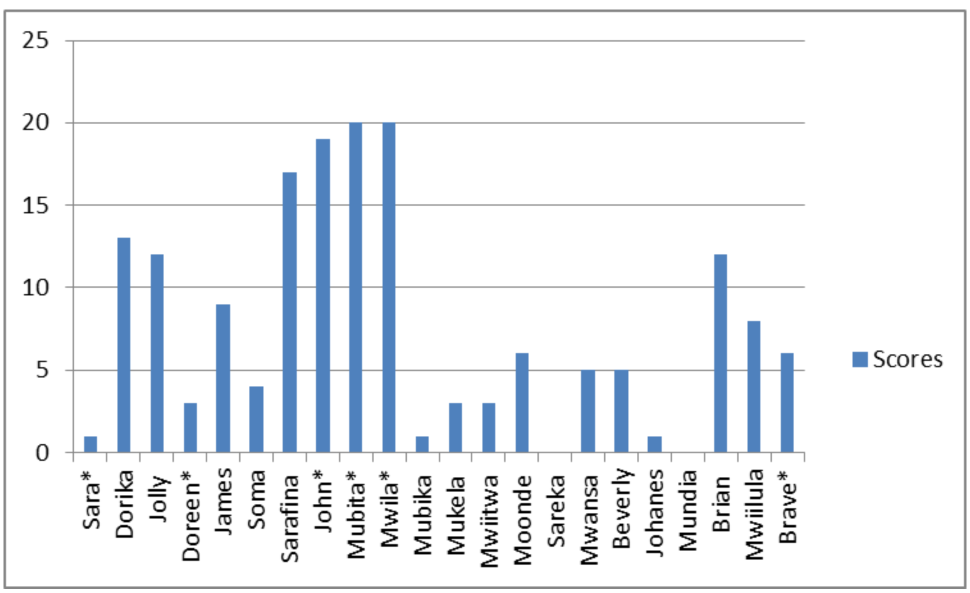


Figure 4.10: Student teachers' total scores in the 'ability to explain and justify reasoning' category

* Interviewees in Phase 2.

The minimum score among the student teachers' composite scores was a zero, while the maximum score was 20. This gives a range of 20, which suggests a large disparity in the students' performance. An analysis of the data indicates that 18 student teachers (approximately 82%) scored below the 50% pass mark, while only four student teachers (18%) managed to obtain scores that were above 50%. The mean mark representing the student teachers' achievement in this category is almost eight (27%).

Furthermore, these results suggest that the majority of the student teachers found it difficult to explain and justify their reasoning for the topic of functions. Similarly, the results imply that most of the student teachers lacked the competence to recognise and explain conceptual relationships and differences. This perspective seems to reinforce the conclusion that emerged in relation to the student teachers' ability to provide valid definitions of concepts in the CCK category (Section 4.2.1). A summary of the achievement of each student teacher per item and in the entire category that assessed the student teachers' ability to explain and justify reasoning is provided as Appendix 10.

Item 1(d)

Explain the relationships of the domains and ranges of R and R_1.

This item required the student teachers to recognise and explain the relationships that existed among the domains and ranges of R and R_1. The relation R was defined on set $X = \{3,4,5,6\}$ by the rule 'is greater than', while R_1 was defined on set X by the rule 'is less than'. The student teachers were expected to recognise and explain that the domain of R is equal to the range of R_1, while the range of R is equal to the domain of R_1.

13 student teachers (59%) could neither recognise the relationships nor explain a relationship of any kind among the domains and ranges of R and R_1. Alternatively, nine student teachers (41%) recognised and appropriately explained the relationships. Although the number of student teachers who were able to recognise and explain the relationships is almost equal to that of the students who were unable to do so, this result is indicative that the majority of the students had difficulty answering the item.

Item 2(a)

Is there a difference between a *relation* and a *function*? Explain your view.

In this item the student teachers were asked to confirm whether or not they thought there was a difference between a relation and a function, and then provide an explanation in support of their view.

Ten student teachers (45%) neither stated their positions nor provided explanations. Five student teachers (about 23%) correctly stated that a difference exists between a general relation and a function without providing a reason to support their view. Furthermore, there were five student teachers (about 23%) who correctly found that a difference exists between a relation and a function, but provided explanations that were ‘flawed’. The remaining two student teachers correctly stated that a difference exists and also supported their position with sound explanations.

While 12 student teachers (about 55%) stated that a difference exists between a relation and a function, only two (17%) of these students provided sound explanations regarding the purported difference. Moreover, 20 student teachers (about 91%) failed to present a sound explanation. These findings suggest that the majority of the student teachers lacked competence in explaining their reasoning.

Item 2(b)

In each of the cases below, state whether the figure represents a function or not. Justify your answers.

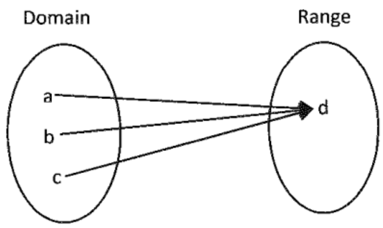


Figure 1

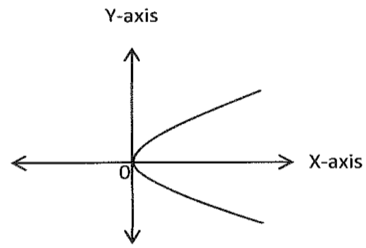


Figure 2

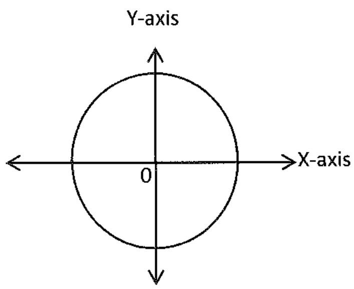


Figure 3

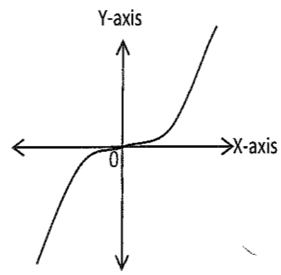


Figure 4

The student teachers were required to provide a justification as to why they thought a particular figure represented a function or a non-function. A summary of the results per figure is provided in Table 4.4 below.

Table 4.4: Summary of the results per figure for item 2(b)

Figures	**NJP***	**FJP**	**SJP**	**Totals**
1	13	0	9	22
2	13	2	7	22
3	11	5	6	22
4	13	4	5	22
Total	50	11	27	88

*NJP represents 'no justification provided', FJP implies 'flawed justification provided', and SJP means 'sound justification provided' (see Section 3.5.1.3 for a detailed explanation of this key).

In Table 4.4, it can generally be seen that 50 (57%) cases are categorised as ‘no justification provided’, while 11 (13%) are considered to be flawed. Only in 27 (31%) instances did the student teachers give a justification that was sound. In terms of the results per figure, there were 13 student teachers who provided ‘no justification’ for Figure 1; none of the student teachers provided flawed justifications, and nine students provided sound justifications. This result shows that the majority of the student teachers (59%) could not justify why a many-to-one relation is a function. Apart from this result, it seems to indicate that most of the student teachers lacked an understanding of what qualified a many-to-one relation as a function.

With respect to Figure 2, 13 instances (59%) are categorised as ‘no justification provided’, while in two instances (9%), the students provided flawed justifications. Only in seven instances (32%) did the student teachers provide a sound justification. These results suggest that the majority of the student teachers were not competent in providing a sound justification as to why a one-to-many graph is not a function.

For Figure 3, the student teachers provided no justification in 11 instances (50%), while five instances (23%) were categorised as flawed justifications. Only in six instances (27%) did the student teachers give a sound justification. Most of the student teachers (73%) were either unable to justify why a Cartesian graph of a circle is not a function or, when they did, their justifications were not accurate or comprehensive.

In the case of Figure 4, there were 13 instances (59%) in which the student teachers could not provide a justification as to why a one-to-one relation was a function. In four instances (18%), flawed justifications were provided, while only in five instances (23%) did the student teachers provide a sound justification. It would appear from these results that the majority of the student teachers were not competent in justifying why a one-to-one graph is a function.

Item 2(c)

Are there other functions whose graphs pass through the two points X and Y? If yes, draw the graph of such a function and if no such other function exists, explain why.

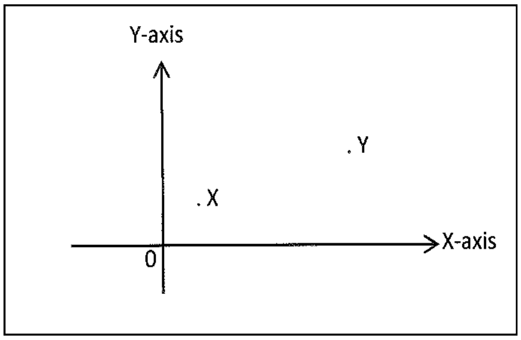


Diagram D

In the context of the 'ability to explain and justify reasoning' category, the student teachers were required to confirm the existence or explain the lack of alternative functions whose graphs passed through the points X and Y shown in Diagram D above. The aspects assessed here were a follow-up of what was initially assessed in item 2(c), as discussed in Section 4.2.3.

18 student teachers (82%) were unable to confirm the existence of alternative functions whose graphs pass through X and Y. At the same time, these student teachers could not explain the lack of such functions. Nevertheless, four student teachers (18%) did confirm that there were other functions whose graphs equally passed through X and Y in the graph above. This result suggests that the majority of the student teachers were not certain that several other functions exist whose graphs pass through the two points X and Y. This uncertainty could point to the student teachers' limited understanding of the graphs of functions.

It is also possible that most of the student teachers thought that only graphs of linear functions pass through two points on a Cartesian plane. Thus, the understanding that other functions, such as quadratic functions, could have their graphs pass through these two points was probably far-fetched. The majority of the students drew a straight line graph when they were initially asked to draw a graph of a function passing through the points X and Y (Section 4.2.3 on item 2c).

Item 2(d)

A secondary school pupil gave $y^2 = x + 9$ with domain $\{x : 0 \le x < 2 \text{ and } x \in Z\}$ as an example of a function. Is the pupil right or wrong? Explain.

A formula $y^2 = x + 9$ with domain $\{x : 0 \le x < 2 \text{ and } x \in Z\}$ was purported to have been provided by a secondary school pupil as an example of a function. The student teachers were assessed on their ability to explain why they thought this formula and the given domain was either an example or non-example of a function.

13 student teachers (59%) were unable provide any explanation of why they either considered the given formula and its domain to be an example or a non-example of a function. Two student teachers (9%) attempted to provide an explanation in support of their view, but those explanations were flawed. The other seven student teachers (about 32%) were able to provide sound explanations why the given formula was a non-example of a function. One common characteristic of the sound explanations that were provided by the student teachers was that for each member of the domain $\{x : 0 \le x < 2 \text{ and } x \in Z\}$ the relation $y^2 = x + 9$ would produce more than one image. From the student teachers' performance results it would appear that they mostly lacked the capacity to explain their position.

Item 3(e)

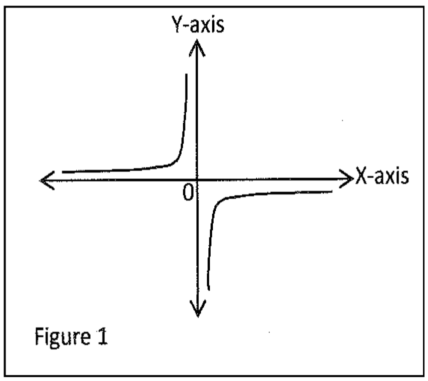
State the maximum value of $f(x) = -2x^2 - x + 8$ and explain how that value relates to the range of the function f.

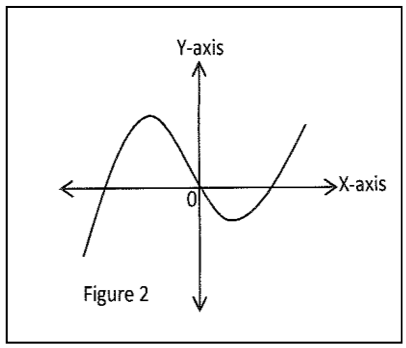
The explanation part of this item, which lies in the 'explain and justify' category, required the student teachers to recognise and explain the relationship that exists between the maximum value and the range of the function $f(x) = -2x^2 - x + 8$. This item assessed the student teachers' ability to recognise and explain that the maximum value of the function f is the largest value in the range of f.

15 student teachers (68%) could neither recognise nor explain the relationship that exists between the maximum value of f and its range. Six student teachers (27%) provided flawed explanations, while only one student teacher (approximately 5%) was able to present an explanation that was considered sound. These results seem to suggest that the majority of the student teachers (95%) were incompetent in explaining the relationship that exists between the maximum value of f and its range.

Item 4(b)

A mathematics textbook shows the following graphs as examples of one-to-one functions. Is the textbook correct in this regard, or not? Explain.





For the category 'explain and justify reasoning', the student teachers were required to provide an explanation as to why they considered each of the two figures to be either an example or non-example of a one-to-one function. Table 4.5 below provides a summary of the results for item 4(b) per figure.

Table 4.5: Summary of results per figure for item 4(b)

Figures	**NEP***	**FEP**	**SEP**	**Totals**
1	18	4	0	22
2	18	4	0	22
Total	36	8	0	44

*NEP: 'no explanation provided', FEP: 'flawed explanation provided', and SEP: 'sound explanation provided'.

Table 4.5 shows that there were 36 instances in which the participating student teachers failed to provide an explanation as to why they thought each of the two figures was either an example or non-example of a one-to-one function. There were eight occasions on which the student teachers' explanations were considered to be flawed. In no instance did any of the student teachers provide an explanation that was deemed sound.

With respect to specific figures, 18 student teachers (about 82%) did not provide any explanation to support their view for Figure 1, while four students (18%) presented explanations that were deemed flawed. None of the 22 student teachers provided sound explanations for Figure 1. Most notably, this scenario repeated itself with Figure 2. These results appear to suggest that the majority of the student teachers lacked competence in explaining why a Cartesian graph (Figure1) represents a one-to-one function and why a Cartesian graph of a many-to-one relation (Figure 2) is a non-example of a one-to-one function. This result seems to be consistent with the findings in Section 4.2.1 in which most of the student teachers had difficulties providing a valid definition of a one-to-one function as well as identifying a one-to-one function.

Item 5(a)

Let $h(x) = x^2 + 1 \text{ for } -2 \leq x \leq 2$ and $f(x) = x^2 + 1 \text{ for } 0 \leq x \leq 2$. State two differences between h and f.

The student teachers were required to recognise and state two differences between h and f. Some of the obvious differences between these two functions are such that whereas h is a many-to-one function, f is a one-to-one function. To this effect, h is not invertible for $[-2,2]$, whereas f is invertible for $[0,2]$. The student teachers were not restricted to the above mentioned differences, but were at liberty to recognise and state any two appropriate differences that exist between h and f.

Six student teachers (27%) failed to recognise and state any difference between the two functions. However, 13 student teachers (59%) did recognise and state one of the required differences each between the two functions h and f. Lastly, only three student teachers (approximately 14%) were successful at recognising and stating two appropriate differences between h and f. The three student teachers were able, for instance, to recognise and state that h was an example of a many-to-one function, while f was a one-to-one function. Based on the performance results in item 5(a), it would appear that the majority of the student teachers performed fairly well in the area of recognising conceptual differences between functions.

Item 5(e)

Describe the relationship between the range of h and the domain of f^{-1}.

This item was a build-up of the functions $h(x) = x^2 + 1 \text{ for } -2 \leq x \leq 2$ and $f(x) = x^2 + 1 \text{ for } 0 \leq x \leq 2$. This item presupposed the student teachers' appropriate knowledge of the range of the function h and the domain of f^{-1}. The specific purpose of the item was the assessment of the student teachers' ability to recognise and describe the relationship that exists between the range of h and the domain of f^{-1}. In other words, the student teachers were assessed on their ability to recognise and describe that the range of the function h is equal to the domain of f^{-1}.

An analysis of the results for this item indicates that 15 student teachers did not provide any description of the relationship between the two functions. One student teacher provided a description that was flawed. Furthermore, six student teachers were able to present descriptions that were considered to be sound. These results suggest that most of the student teachers (approximately 73%) were not competent in recognising and describing the relationship between the range of h and the domain of f^{-1}.

Item 6(b)

State, with justification, two domains on which the function $z : x \rightarrow x^2 - 2x$ has an inverse.

For this item, the student teachers were required to state and justify two of the domains in which the function $z : x \rightarrow x^2 - 2x$ has an inverse. The idea was to assess whether they could relate inverse functions to one-to-one functions where the latter are the only type of functions that have inverse functions. There are several restricted domains such as $[1, \infty)$, $(-\infty, 1]$, $[1, 2]$, and $[0, 1]$ in which the function z assumes a one-to-one status, and consequently becomes invertible. However, the student teachers were only required to provide and justify two such domains. 20 student teachers neither stated any domain nor provided a justification. Two student teachers managed to state one appropriate domain each without giving any justification. This means that there were no students who stated two domains and gave justifications. These results seem to suggest a lack of competence of the majority of the student teachers (approxiamtely 91%) in recognising and justifying restricted domains for which quadratic functions are one-to-one.

4.2.3. Ability of the student teachers to use different representations in functions

This category presents and analyses the data on the student teachers' ability to work with different representations. There were six items in the mathematics test that assessed the students' ability in this category. The possible total mark for all the items in this category was 15. The student teachers' total scores are shown in Figure 4.11 below.

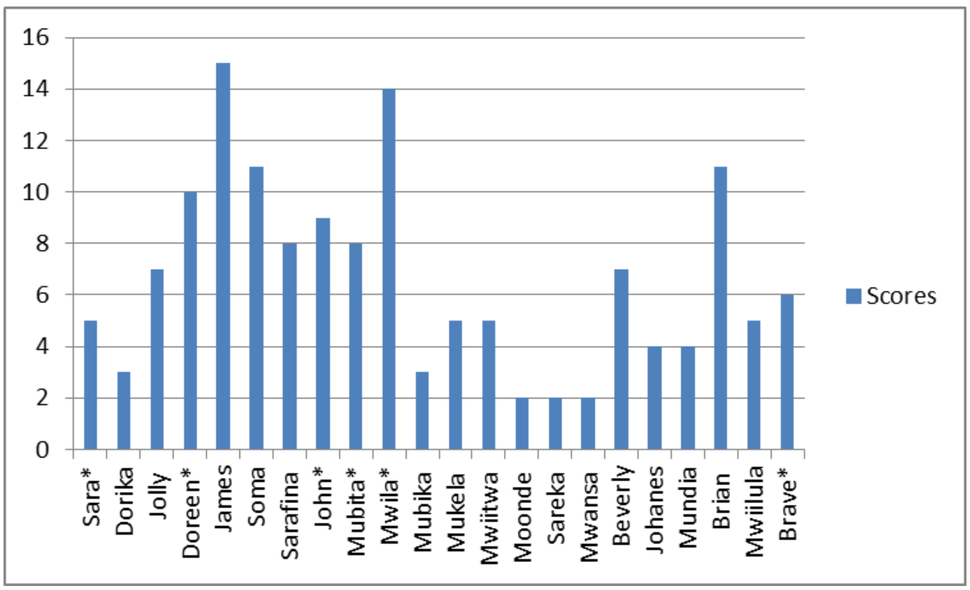


Figure 4.11: Student teachers' total scores in the 'ability to use different representations' category

* Interviewees in Phase 2.

Using the 50% pass mark as a point of reference, 14 student teachers (approximately 64%) achieved total scores that were below 50%. A further analysis indicates that only eight of the 22 student teachers (36%) obtained total scores that were above 50%. Out of the possible total score of 15, the mean mark was 7, representing a 44% average performance. This mean seems to suggest that, generally, the student teachers' performance was not satisfactory. Two student teachers scored 14 and 15 respectively. However, the mode of the total scores is 5, which translate into 33%. The student teachers' individual scores per item, total scores, and the overall mean score representing the student teachers' performance in the category 'ability to use different representations' are presented as Appendix 11.

Based on the preceding analysis, it would appear that the majority of the student teachers were incompetent in working with different representations in the topic of functions. Nonetheless, it is necessary to analyse their performance item by item in order to acquire a clearer picture; this follows below.

Item 1(b)

The relation *R* on the set $X = \{3,4,5,6\}$ is defined by the rule 'is greater than'. Express *R* as a set of ordered pairs.

This item involved the relation *R*, which was defined on the set $X = \{3,4,5,6\}$ by the rule 'is greater than'. The students were required to express this relation as a set of ordered pairs. A representation would be considered accurate if all six ordered pairs were accurately presented in the set *R*. The student teachers needed to have an understanding of what a set of ordered pairs entails and how to use the given rule on the set *X* to determine the domain and range. Thus, the appropriate application of the rule to identify elements of the domain and range, as well as an accurate representation of the elements of *R* in a set in the form of ordered pairs was necessary for this item.

The data of this item indicates that 13 student teachers scored a mark of zero. This means that either these student teachers had no idea how to express *R* as a set of ordered pairs, or when they attempted to do so, their representations were not relevant. One student teacher attempted to express *R* as a set of ordered pairs except the representation had a minor error. Instead of presenting six correct ordered pairs in the set *R*, he presented seven ordered pairs among which six were correct. The other eight student teachers appropriately applied the rule 'is greater than' to relate members of set *X*, and successfully expressed the relation *R* as a set of ordered pairs. It would appear that the majority of the student teachers were incapable of expressing the relation *R* as a set of ordered pairs. One of the student teachers presented the following answer: $(6,5)>(5,4)>(4,3)$. He failed to accurately use the rule 'is greater than' to relate members of the domain to the corresponding members of the range. After generating a few correct ordered pairs, he then wrongly applied the rule by associating ordered pairs. This misapplication of the rule and apparent failure to provide an accurate set of ordered pairs exposed his lack of in-depth understanding of sets of ordered pairs as a form of representation.

Another student teacher who seemed to lack in-depth understanding of sets of ordered pairs depicted *R* as follows: $4>3, 5>4, 5>6, 4\times 3, 5\times 4, 5\times 6$. While this answer suggests that the student teacher partly appreciated the rule 'is greater than', it equally shows that he did not comprehend what a set of ordered pairs is. At the same time, it appears that the student teacher did not realise that 5>3, 6>3, and 6>4. Moreover, the involvement of out of context elements such as $4\times 3, 5\times 4, 5\times 6$ reinforces the view that this student did not know how to express the relation *R* as a set of ordered pairs. Figure 4.12 and Figure 4.13 below are examples of other answers that were provided by student teachers who did not know how to express the relation *R* as a set of ordered pairs.

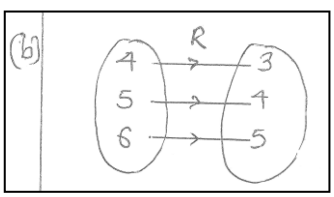


Figure 4.12: Sketch C1

Figure 4.12 indicates a lack of understanding of the difference between a set of ordered pairs and an arrow diagram. The student teacher correctly applied the rule 'is greater than' to link the elements shown in the arrow diagram. Nevertheless, seeing that expressing *R* as an arrow diagram was the answer required, his answer would still be incomplete. There are three other ordered pairs: (5, 3), (6, 3), and (6, 4) that are not implied in the student's diagram, and yet they are supposed to be members of the relation *R*.

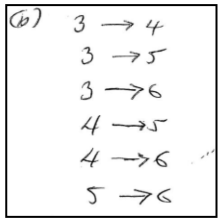


Figure 4.13: Sketch C2

Figure 4.13 shows that this student teacher lacked an understanding of what a set of ordered pairs is. Moreover, this answer suggests that the student teacher did not understand the rule 'is greater than'. All the elements in what is implied by the arrows to be the domain are 'less than' the elements that they are linked to in what is implied as the range.

Item 1(c)

If R_1 is another relation on the set $X = \{3,4,5,6\}$ defined by the rule 'is less than', express R_1 as a graph in the Cartesian plane.

The student teachers were assessed on their ability to determine the relation R_1 and translate it to a graph. To accurately translate the relation to the Cartesian plane, the student teachers would need to know the ordered pairs composing R_1 and then plot these points. An important aspect of the relation R_1 is that it has a discrete domain. In this regard, the student teachers were not expected to connect the plotted points.

15 student teachers (68%) did not accurately translate the relation R_1 to a Cartesian graph. This is a large number considering that only seven student teachers (32%) demonstrated the ability to correctly translate R_1 to the Cartesian plane. These results suggest that the majority of the student teachers were not competent in translating a relation to a Cartesian plane.

An example of an answer from the student teachers who were unable to translate the relation R_1 to a graph is: $3<4, 4<5, 5<6$. The student teacher recognised that the elements in the domain were supposed to be 'less than' their corresponding images. Notwithstanding, the student teacher did not indicate that 3<5, 3<6, and 4<6. Since no attempt was made to draw a Cartesian plane, the answer suggests that the student teacher did not know how to translate to a Cartesian plane. Another answer that characterises student teachers' inability to translate to the Cartesian plane is reproduced as Figure 4.14.

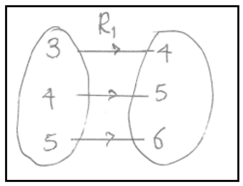


Figure 4.14: Sketch C3

Figure 4.14 portrays this student teacher's inability to distinguish between an arrow diagram and a Cartesian graph. Nonetheless, the student teacher who provided this answer understood that elements in the domain should be 'less than' their corresponding images in the range. It seems that this student teacher's understanding was dominated by one-to-one relations, especially in that he left the three necessary pairs: 3<5, 3<6, and 4<6. An insightful answer is replicated below as Figure 4.15.

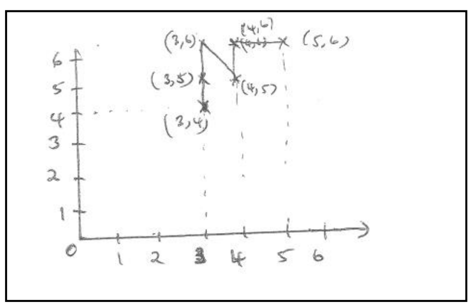
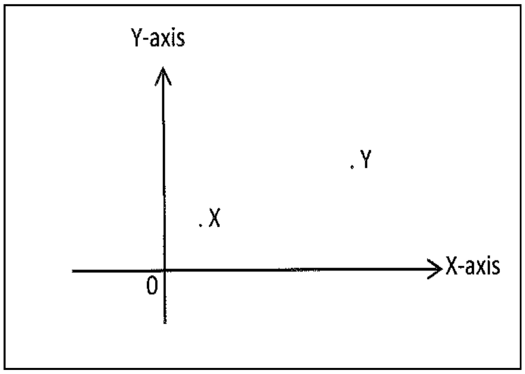


Figure 4.15: Sketch C4

After accurately translating to a Cartesian plane, the student teacher used straight lines to join the plotted points. What the student teacher did not seem to understand is that the relation R_1 is defined on a discrete domain, which does not permit the connection of the plotted points. It appears that this student teacher's conception of a Cartesian graph is one where the plotted points are always connected. Furthermore, the use of straight lines and not curved lines to connect the points suggests that the student teacher's understanding of Cartesian graphs was confused with straight line graphs.

Item 2(c)

Draw a graph of a function that passes through the points X and Y in the following figure:



Are there other functions whose graphs pass through the two points X and Y? If yes, draw the graph of such a function.

For the category 'ability to use different representations', item 2(c) had a segment that assessed two aspects. The student teachers were initially required to draw a graph of a function that they thought passed through the two points X and Y. If the student teachers thought that there were other functions whose graphs pass through the same points X and Y, they were then supposed to draw another graph representing one of these functions.

These two aspects assessed the student teachers' ability to apply their knowledge of graphs of functions. It was expected that they would think of linear and quadratic functions graphs, which are common in the secondary school mathematics curriculum. However, they were at liberty to draw graphs of other functions that they were familiar.

11 student teachers showed competence by appropriately drawing two different graphs. Among these 11 student teachers, seven students neither confirmed the existence nor explained the lack of other functions whose graphs pass through X and Y (Section 4.2.2 on item 2c). Ten student teachers drew a straight line graph each for the first part of item 2(c). One student teacher did not draw any graph of a function passing through the two points X and Y. These results suggest that the majority of the student teachers were at least able to draw a graph of a function passing through two points on a Cartesian plane. 21 student teachers drew a graph, as required by the first part of the item. Each one of these student teachers drew a straight line graph as depicted by Figure 4.16.

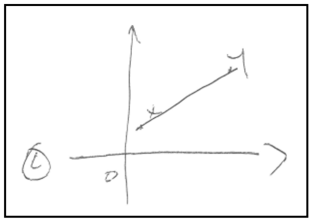


Figure 4.16: Sketch C5

The fact that 21 student teachers drew a graph similar to Figure 4.16 suggests that their understanding of functions passing through two points was dominated by linear functions. It is possible that the ten student teachers who did not draw an alternative graph, but drew one similar to Figure 4.16, thought that only straight lines can pass through two points on a Cartesian plane.

Item 2(d)

A secondary school pupil gave $y^2 = x + 9$ with domain $\{x : 0 \le x < 2 \text{ and } x \in Z\}$ as an example of a function. Is the pupil right or wrong?

The aspect of item 2(d) that assessed the student teachers' ability to use different representations required them to determine the accuracy of the representation. In this regard, they were expected to indicate clearly whether $y^2 = x + 9$ with domain $\{x : 0 \le x < 2 \text{ and } x \in Z\}$ as an example of a function was right or wrong.

14 student teachers (approximately 64%) correctly stated that the formula $y^2 = x + 9$ with domain $\{x : 0 \le x < 2 \text{ and } x \in Z\}$ is not a function. Strangely, 13 student teachers did not provide any explanation for their viewpoint (see the results in Section 4.2.2 on item 2c). This suggests

that some of the student teachers who correctly determined the inaccuracy of the formula merely guessed. The other eight student teachers (36%) displayed a lack of ability by stating that the formula $y^2 = x + 9$ with domain $\{x : 0 \leq x < 2 \text{ and } x \in Z\}$ is a function. It would appear from these results that most of the student teachers were capable of determining the accuracy of a representation expressed in formula form.

Item 3(a)

Represent $g(x) = \lvert x \rvert$ whose domain is $\{x : -3 \leq x \leq 2 \text{ and } x \in Z\}$ on a Cartesian plane.

The student teachers were required to represent this function on a Cartesian plane. The purpose was to assess the student teachers' ability to change from the algebraic representation of this absolute value function to a graphical one. Since the function had a discrete domain, it assessed whether the student teachers understood that they were not supposed to connect the plotted points.

The results show that 16 student teachers (approximately 73%) were completely unable to translate the function *g* to the graphical representation. Two student teachers (9%) accurately translated from the formula representation to the graphical representation, while four student teachers (18%) presented flawed answers. These results suggest that most of the student teachers could not change the representation of the linear function from the algebraic form to the Cartesian plane. A representative answer of those that were flawed is reproduced below as Figure 4.17.

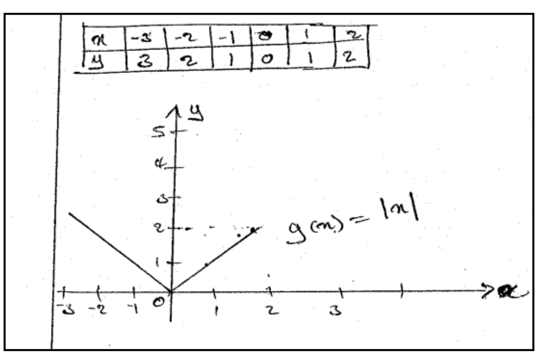


Figure 4.17: Sketch C6

The four student teachers who provided answers like Figure 4.17 effectively used the formula and the elements of the domain to generate the images of the function. Likewise, they could plot the ordered pairs. These students also knew that, since the item involved an absolute value function, all the images are positive. Nonetheless, the figure above reveals that the answers were flawed because the plotted points were connected. The act of joining the points suggests that the students were not familiar with discrete domains.

Item 3(b)

Answer this question on the sheet of graph paper provided. The table of values below shows corresponding values of the objects and images of the function $f(x) = -2x^2 - x + 8$.

x	-3	-2	-1	-0.5	0	0.5	1	2	3
$f(x)$	-7	2	7	8	8	7	5	-2	-13

Taking 2cm to represent 1 unit on the x-axis for $-3 \le x \le 3$ and 1cm to represent 2 units on the y-axis, draw the graph of $f(x) = -2x^2 - x + 8$.

For this item, the student teachers were expected to correctly use the given horizontal and vertical scales to draw the XY plane. They were also required to use the table of values to accurately plot the points and then draw a smooth curve for the function *f*. The parabola was supposed to be drawn on graph paper.

The results indicate that five student teachers did not translate *f* from the table of values/algebraic representation to the Cartesian plane. Four students presented graphs that were flawed. 13 student teachers successfully translated from the table of values/algebraic representation to the graphical representation. The majority of the student teachers (59%) could draw the graph of a quadratic function when a table of values was provided. However, the overall percentage of student teachers who struggled was 41%.

A common finding relating to flawed graphs was the student teachers' inability to correctly interpret and apply the given scales, for example, one student teacher utilised 1cm to represent 1 unit on the y-axis instead of 1cm to represent 2 units. Another flaw concerned the student teachers' failure to accurately plot the points on the graph paper. In Section 4.3, the results and analysis of the semi-structured interviews are presented.

4.3. RESULTS AND ANALYSIS OF THE INTERVIEWS ABOUT FUNCTIONS

This section presents the results and analysis of the semi-structured interviews that were conducted with six UNZA final year student teachers majoring in mathematics. The student teachers are categorised in two groups: high content knowledge and low content knowledge groups (see Section 3.4.2.3 for sampling details). More than six student teachers were purposefully selected to participate in the interviews, but the other students did not show up without giving a reason. Table 4.6 identifies with a cross (X) which student teachers are from the high content knowledge bracket and those who are from the low content knowledge category. In order to preserve anonymity and confidentiality, pseudonyms are used for each of the six students.

Table 4.6: Groups of the interviewees: high content knowledge and low content knowledge

Pseudonyms	High content knowledge	Low content knowledge
Sara		X
Doreen		X
Brave		X
Mubita	X	
John	X	
Mwila	X	

This section is organised into Sections 4.3.1 and 4.3.2 according to the categories of analysis that were identified in advance in the study's conceptual framework. Section 4.3.1 provides an analysis based on the student teachers' ability to explain and justify their reasoning. Section 4.3.2 exclusively provides an analysis of the student teachers' ability to use different representations in functions. In the subsequent section, the analyses of the student teachers' ability to explain and justify their reasoning are presented.

4.3.1. Analysis of the student teachers' ability to explain and justify their reasoning regarding functions

Two questions guide the analyses under the category 'ability to explain and justify reasoning in functions': (1) How do the student teachers explain their understanding of concepts, and (2) What justifications do the student teachers provide for their reasoning? A summary of the concepts that comprised the investigations is shown in Table 4.7 below.

Table 4.7: Concepts comprising 'ability to explain and justify reasoning in functions'

Guiding questions	Concepts
How do the student teachers explain their understanding of concepts?	Student teachers' understanding of: • A function; • Domains and ranges of relations and functions; • One-to-one functions; • Inverse functions; and • Composite functions.
What justifications do the student teachers provide for their reasoning?	Justifications based on: • Examples and non-examples of functions; • Examples and non-examples of Cartesian graphs of one-to-one functions; • Procedures for the calculation of algebraic inverses of functions; • Vertical and horizontal lines drawn on a Cartesian plane; • Graphs of $f(x) = ax^2 + bx + c$, and the signs of the coefficients of x^2.

How do the student teachers explain their understandings of concepts?

Student teachers' understanding of a function

Sara, a student teacher from among the low content knowledge group, considered a function to be a 'relationship' between two sets. She was able to mention the link between the first and second sets. However, she did not explain that in a function the purported 'relationship' of the first and second sets should be such that all the members of the first set are 'connected' to unique members of the second set. It seemed that Sara had no in-depth understanding of the univalence property.

When asked why she described a function as a 'relationship' between two sets, Sara asserted that there is need for a 'pattern' that connects two sets. She contended that the elements of a second set can only be generated through the use of a 'pattern'. This viewpoint was restrictive as it seemed to eliminate the arbitrariness property of functions. Interestingly, Sara declared that Figure 1 of item 2(b) of the Phase 1 test was a function (see Appendix 3), and yet the many-to-one arrow diagram had no accompanying rule or 'pattern' associating elements in the domain to the single element in the range. There was thus a disconnection between Sara's claim that a 'pattern' must be used to generate images, and her view that Figure 1 is a function. This finding suggested that Sara's understanding of the arbitrariness of functions was conflicted.

Brave's notion of a function was similar to that of Sara. He described a function in the context of a relationship between the domain and the range. The discussion with Brave developed as follows:

> **Interviewer (Int):** I would like for you to define what a function is.
>
> **Brave:** Aah, I would actually define a function as a relation that maps elements in the (pauses), elements in the domain. They map elements aah in the domain onto the range.
>
> **Int:** Okay.
>
> **Brave:** That is to say, only one element should map an element in the range.

Brave's declaration that 'only one element should map an element in the range' was ambiguous. His description of a function did not include the requirement for elements in the domain to be connected to unique elements in the range. In addition to this, he did not indicate that a relation can qualify as a function even in the absence of a regular 'rule' or algebraic formula. Moreover, he showed no knowledge of what the univalence and the arbitrary properties of functions entail.

Doreen, the third student teacher from the low content knowledge bracket, defined a function as a mapping that 'connects' two sets. She firmly declared that only the one-to-one and the many-to-one types of relations qualify as functions, however, she did not know why. While the univalence condition was implied in her description, she did not directly make reference to it. Doreen was, therefore, probed regarding this important condition:

Int: What of the univalence of functions, have you ever heard of something like that phrase?

Doreen: Yes, but though I haven't paid much attention to univalence of functions, yes I have heard of it.

Int: Okay what do you know about univalence of functions?

Doreen: Mmm I haven't paid much attention to it.

Even though Doreen made the claim of having heard of the univalence condition, she did not thoroughly understand it.

Mubita included the concepts of domain and range in his definition of a function in the same manner as Brave. The discussion provided hereafter presents Mubita's idea of what a function is:

Int: I would like for you to describe for me what a function is, a function in mathematics.

Mubita: Yes when we talk about a function in mathematics, eeh the way I understand it, aah it is some special kind of relation in which members of one set, the one we refer to as the domain, each of those members is mapped onto a unique image from the other set which I may refer to now as the range.

Mubita emphasised that the elements in the domain are supposed to be related to unique elements in the range. He understood the phrase 'unique' to mean only one, and this suggested that Mubita was familiar with the univalence property of functions. It was apparent that his understanding of a function was founded on the difference that exists between ordinary relations and functions. In contrast to Doreen, he confidently declared that there is a difference between ordinary relations and functions. When requested to explain a relation, he was able to do so:

Int: Okay. You are saying that you defined it (function) that way so that you distinguish it from any other relation, what is a relation then?

Mubita: A relation there will mean an association of elements from one set with elements (pauses), basically an association or should I say a relation can be a rule that associates members of one set to members of another set.

Against the background of an appropriate definition of a relation, Mubita was requested to provide an example of a relation, to which he responded as follows:

Int: Are you able to give me an example of a relation?

Mubita: Yes, I can give an example. If one set is a set of let's say fathers, and the other set is a set of their children. So there is a relationship there from this set containing fathers to this set containing their children. So that is an association, these will be called 'father to', so that is the way a relation from the first set to the second set is.

Mubita validated his understanding of a general relation through a thoughtful example. This example was used to explore his conception of a function:

> **Int:** So now if that relationship is to be a function, again what should happen?
>
> **Mubita:** In that case, only one father should be associated with a particular child in the other set.
>
> **Int:** Okay.
>
> **Mubita:** No (stammers), no father should have maybe in my example here that is, if each father had a son there. So each particular father can only be associated with that son in that set. He may not be associated with a son of another father.

Mubita's explanation regarding how the relation 'father to' can be a function reinforced his familiarity with the univalence property of functions. However, his example did not allow for a many-to-one function.

Mwila originally asserted that a function is 'an operation' that 'changes one variable into another'. He then related a function to a hammer mill that grinds maize corn into mealie meal. In this regard, he perceived a function to be a machine in which inputs provide outputs. Lastly, he argued that a function is 'simply a rule which is going to assign each element of the domain to exactly one element of the output which you call the range'. The prominence attached to a 'rule' suggested that he viewed all functions as relations that are accompanied by either algebraic formulas or descriptive statements. Moreover, Mwila admitted that he did not know what the arbitrariness of functions entails.

John confined his explanation of a function to the aspect of a one-to-one correspondence between the elements of sets. His description of a function is presented below:

> **Int:** Would you describe for me what a function is?
>
> **John:** A function, in my view, is a one-to-one relationship between sets of one object to sets of another. There must be one-to-one correspondence between elements of one set to the elements of the other.
>
> **Int:** So in other words you are saying that for a function there must be one-to-one correspondence.
>
> **John:** One-to-one correspondence, yah (yes).
>
> **Int:** So, if there is no one-to-one correspondence then it means it's not a function?
>
> **John:** It's not really a function.

John recognised that a function relates the elements of two sets. What he did not do was to identify the two sets whose elements are related with names such as domain and range. He made a restrictive claim that in a function, there is always a one-to-one correspondence among objects and images. Subsequently, John was reminded of his definition in the test and asked to explain its meaning. He had defined a function as 'a rule which assigns each element of one set to exactly one element in another set'. Despite the direct request for an explanation, he only recited the definition.

In a further discussion, John correctly affirmed that there is a difference between an ordinary relation and a function. He argued that all functions are relations, while not all relations are functions. Nonetheless, he contradicted himself by claiming that for a relation there is no one-to-one correspondence. He accurately indicated that elements in the domains of ordinary relations can be mapped onto one or more elements in the range. A prominent feature, however, was John's continued assertion that the only requirement for a relation to be a function is a one-to-one correspondence. He seemed not to know that even a many-to-one relation is a type of function. His lack of understanding in this regard is amplified by the following excerpt (the interviewer drew an arrow diagram):

> **Int:** Okay, so like when you talk of a many-to-one relation, suppose you have an illustration like this one where you have two sets. Maybe in set A you have elements 'a, b, c', and in the next set B you have maybe elements 'd' and 'e'. And you have maybe 'a' related (linked) to 'd', 'b' related to 'd' and 'c' going to 'e'. What type of relation would this one be, would this be a many-to-one relation?
>
> **John:** Mmm this one can't be a many-to-one because these three elements in set A are not all going to one element. It is only two elements that go to one element and this other element is going to another element, so you can say that (pauses), you can't say that this is specifically a many-to-one relation.

John was unable to recognise that the example which was cited by the interviewer is a many-to-one relation. His viewpoint was that in a many-to-one relation, all the elements of the domain must be linked to the same element in the range. Apart from this, he seemed to consider a many-to-one relation as one in which there is a single image in the range. It is this narrow conception that explains John's assertion that only relations that fulfil the one-to-one correspondence property qualify as functions.

Student teachers' understanding of the domains and ranges of relations and functions

Brave, Mwila, and Mubita directly mentioned the concepts of domain and range in their explanation of a function, whereas Sara and John only implied these concepts in their descriptions. Although Doreen had not originally included these concepts in her definition of a function, she did refer to them at a later stage. One of the student teachers who could not accurately explain the concepts of the domain and range of functions is Sara:

> **Int:** Okay. How would you define the domain of a function?
>
> **Sara:** The domain of a function are those elements which when you substitute in the same function it will give you aah a range; a different number.

Sara viewed a domain as a collection of elements that a function operates on to yield the elements in a range. This would be a reasonable interpretation as long as Sara's use of the phrase, 'are those elements' referred to an entire collection of the objects of a function. Nonetheless, Sara's description of a domain was limited as it excluded domains whose elements are not numbers, which can be substituted into algebraic functions. She also seemed to think of the range as a number. Her conception of the range is again shown in the extract below:

> **Int:** So, in your view a range is; how would you define a range?
>
> **Sara:** A range is, it is more like (pauses), it corresponds to, to (pauses again).
>
> **Int:** Ehe.
>
> **Sara:** To what's this, what I said aah (pauses).
>
> **Int:** To the domain?
>
> **Sara:** to the domain, yes.
>
> **Int:** Okay, okay.
>
> **Sara:** It is the outcome, more like the outcome when you put the domain you come up with the answer, gives you what's this (pauses), the range.

Sara intimated that there is a relationship between the domain and the range of any given function. She also claimed that a range is 'the outcome of putting the domain'. It was inexplicit how the domain is 'put', and 'where' in order to generate the range. Her perspective, however, suggested that she perceived a domain as a number that can be substituted into an algebraic function. She seemed not to appreciate the difference between the elements in a domain, and the domain itself. What is accurate is that numerical elements of a domain, and not the set itself, are individually substituted for the independent variable in an algebraic function to generate images.

Brave's explanation of the concept of domain corresponded with Sara's. He viewed a domain to be synonymous with individual objects, and not as an entire collection of the objects. At the same time, he provided a description that confined domains to sets whose elements are numbers to the exclusion of domains whose elements are not numbers. With respect to a range, Brave posited that it is 'the output'. Previously, he had explained that an output is what is obtained when an element from the domain is acted upon by a function. However, his explanation of a range as 'the output' did not indicate that there is a difference between images (individual outputs) in the range and the range itself (collection of outputs).

Doreen explained that a domain 'is the set from which a relation comes', and that a range is 'the set of elements into which the relation goes'. These descriptions required unpacking, and consequently, the student teacher was requested to clarify this answer. However, Doreen failed to elucidate her description of a domain. Her recourse was to point out that the learners would have been introduced to 'mappings', but she did not explain how the concept would be presented to the learners. It appeared that she was merely trying to conceal her lack of ability to explain the concept of domain. This was validated by Doreen's corresponding inability to clarify her idea of a range. When asked to explain her definition of a range, she could not do so.

Mwila explained the concept of a domain with respect to relations that can be represented in formula form, ordered pairs or the Cartesian plane. This was evident in his reference to the aspects of independent and dependent variables. His conception of a range showed familiarity with ranges of functions that are represented by standard formulas. This was evident when he mentioned the idea of substituting x-values in order to acquire the y-values. Mubita espoused views that are similar to those articulated by Mwila:

> **Int:** Okay. Uh now I would like to find out from you how you would define a domain. I notice that you have used it in your definition of a function, what is a domain?
> **Mubita:** Uh a domain here I would say it is a set on which a function is being defined. The (stammers), the elements that we are supposed to (stammers again), to use in a rule for us to get to the range, those elements are coming from that set.

Mubita described a domain in connection with a 'rule' that is supposed to link the elements of two sets. This viewpoint suggests that his understanding did not include the domains of functions that are not defined by specific 'rules'. With regard to the sub-concept of range, he provided a generic explanation by contending that it is a set that 'contains the images'.

John presented his understanding of a domain in the following extract:

> **Int:** Okay. Uh what do you understand by the phrase: domain of a function?
>
> **John:** The domain of a function; the domain of a function, that is the set where those elements, those elements we input into the function, that's where (pauses), the set where those elements we are supposed to pick and put into the given function, where we pick them from, we call it a domain.

John's explanation of the concept of domain resonated with the descriptions that were provided by Mwila and Mubita, for example, he indicated that elements in a domain are supposed to be 'put' in a function in order to generate images. In this regard, his viewpoint was confined to the domains of functions that are defined by standard formulas. Thus, his description did not include the domains of functions that are expressed without a formula. John's description of a range showed an understanding that was premised on what a range consists of, and not exactly on what it is:

> **Int:** Okay, and the range itself what is it?
>
> **John:** The range is where those elements; the set which is formed after 'imputing' a certain element from the domain we 'impute' it and solve it so that element, that final answer we find is an element which is found in the range of a given function.

John placed emphasis on the elements that are found in a range, and how they are computed. The aspect of 'solving' suggested that he understood ranges to be characterised by sets that consist of numbers. Again, his standpoint was consistent with the viewpoints of Mwila and Mubita.

Student teachers' understanding of one-to-one functions

Doreen described a one-to-one function as 'a function whose elements or objects are mapped only onto one image in the range'. The explanation was vague as it could imply that each object maps onto a unique image or that the range has a single image. Doreen was therefore probed using an example that consisted of three objects and a single image:

Int: So that I understand your explanation; supposing you have two sets where you have in the first set, let's call them say set A maybe you have elements 'a, b and c' and then in the second set which we can call B you have an element, maybe let me call it 'f'. Then from set A, 'a' is going to 'f', b is going to 'f' and c is going to 'f', would you call this a one-to-one function because every element in the first set is going to one image?

Doreen: No.

Doreen correctly indicated that the interviewer's example was not a one-to-one function. When asked for reasons why the cited example was not a one-to-one function, she posited that elements from the domain were 'sharing' the same image. She was then asked to clarify the idea of a one-to-one function:

Int: So what should happen then in a one-to-one function?

Doreen: What should happen in a one-to-one there should be, they [there] are three elements here, they [there] should be three corresponding images there (in the range), so that each one of these elements only has one image.

This extract suggested that the conception of a one-to-one function that Doreen had in mind was different from the definition she had verbally provided. It seemed that she had the understanding that in a one-to-one function, objects have unique images, and equally, images are linked to unique objects in the domain. This apparent contradiction between the definition provided and the message she had intended to convey suggested her lack of capacity to comprehensively explain a one-to-one function. Doreen was reminded of the definition she had presented in the test, and asked to explain its meaning:

Int: Okay, okay. Now in the test that I administered to you; you said a one-to-one function: 'simply means for every object there is only one image'. Would you explain the meaning of this definition?

Doreen: What I have just been saying is actually (pauses and laughs).

Int: Yes (laughs also)?

Doreen: (Laughs again, and then says): what I meant to say, there should not be (pauses), much as there should be one image, that image is not supposed to be shared by the other elements in the domain.

Doreen's explanation of the definition she provided in the test was consistent with the explanation she provided when requested to clarify the phrase 'only onto one image'. However, her inability to competently express ideas on the one-to-one function was still obvious. Apart from the definition of a one-to-one function presented in the test, and the other one verbalised during the interview, Doreen said that she did not have an alternative definition.

John restricted his definition of a function to the one-to-one correspondence property. It was, therefore, interesting to hear his views concerning the definition of a one-to-one function:

> **Int:** Okay. In your definition of what a function is, I heard you talk of one-to-one correspondence property, but I guess there is a function called a one-to-one function. How would you define a one-to-one function?
> **John:** A one-to-one function is a function which maps all the elements in the range [domain], first of all the elements in the domain are mapped into specific (stammers), into the elements in the range then there is no element which is not mapped into any element. For example, if there are three elements in set A which is the domain and three elements in set B which is the range, so each element in A is mapped to a specific element in B. There are no other elements which are not mapped then you can say that this is a one-to-one function.

John emphasised that in a one-to-one function, 'all' elements of the [domain] are linked to 'specific' elements of the range. While he correctly indicated that every object must be connected to an image, the sense in which he used the term 'specific' was unclear. It was not obvious whether John was implying the feature of 'one and only one', or whether he intended to underscore that no object should share an image. Considering his example involving sets A and B, it would appear that he had a general understanding of the concept of a one-to-one function. This view is plausible, especially since he utilised the same number of elements in both the domain and the range. In subsequent discussions, John suggested the following analytic definition: if $f(a)=f(b)\Rightarrow a=b$ for all $a,b\in D(f)$ then f is a one-to-one function. Nonetheless, his explanations suggested that he had a limited capacity to comprehensively explain the idea of a one-to-one function.

Mwila's definition of a one-to-one function was clearer regarding the characteristics of one-to-one functions than those expressed by Doreen and John. The discussion with Mwila was conducted in the following manner:

> **Int:** You had given me the definition of a one-to-one function, probably would you like to give me again that definition that you gave me, what is a one-to-one function?
> **Mwila:** A one-to-one function is simply a kind of a function in which every element, exactly every element of the domain is mapped to exactly one element of the range uniquely. Each element from the domain is mapped to uniquely one element in the range such that there are no two elements of the domain which are going to have the same output. It is a one-to-one relation meaning one-to-one correspondence. No two elements are going to have the same output and no two elements in the range are going to have the same input again.

Mwila's definition alluded to what differentiates a function from ordinary relations. This is depicted in the statement: 'exactly every element of the domain is mapped to exactly one element of the range uniquely'. The provision of the preceding statement alone would have included the many-to-one function as well. However, as an indication of understanding, Mwila added that 'such that there are no two elements of the domain which are going to have the same output' and 'no two elements in the range are going to have the same input'. This statement was sufficient to exclude the many-to-one type of functions.

Brave explained that for a one-to-one relation 'one element in the range maps onto only one element in the range'. When asked to elaborate, he changed his explanation and indicated that a one-to-one relation maps 'an element in the domain ... onto an element in the range'. Nevertheless, his rewording did not highlight the necessary characteristics of a one-to-one relation. It appeared that Brave did not know that in a one-to-one relation, each element of the domain needs to map onto a unique image in the range, and that each image must be linked to a unique object in the domain.

The researcher cited an example that involved three arbitrary objects and two arbitrary images and asked Brave to confirm if it was a one-to-one function or not. Two objects were linked to the same image and the last object was associated with the second image. Initially, Brave indicated that this was not a one-to-one function. However, after being probed in relation to his earlier definition, he changed his position and indicated that the example was a one-to-one relation. The sudden change in perspective exposed Brave's lack of conviction and clearly showed his lack of in-depth understanding. This is because the example that the interviewer had presented was a many-to-one function. With a view to validate the findings, Brave was reminded of the definition he had presented in the test, and requested to explain its meaning:

Int: You said 'a one-to-one function is a function in which one element in the domain maps to one and only one element in the range'. Just clarify to me the meaning of this definition.

Brave: Okay, the meaning of this definition is that aah mmm because when you talk of a one-to-one function (pauses), there should be elements in the domain and elements in the range which are outputs. After substituting in the function, the given expression, so once aah I fuse in probably the one element in the expression.

Int: Ehe.

Brave: Then the outcome should only be one. So, meaning that I'm not supposed to have more than two elements in the range. So there should only be (pauses), one element in the domain should map onto one element in the range or the output should only be one.

There is consistency between Brave's definition in the test and the definition he provided during the interview. It would appear that in both definitions, he tried to express a characteristic that qualifies a relation as a function, and not define a one-to-one function. He claimed that the 'name' one-to-one was the reason for the presented definition. His alternative definition was consistent with the flawed definitions provided in the test and during the interview.

Mubita used a practical example in his description of a one-to-one function. He involved the heads of countries to represent elements in a domain, and flags of their countries to represent elements in the range. He explained that since each head of state is connected to one flag of a country, and each flag is associated with a particular head of state, there is a one-to-one correspondence between the two sets. Although Mubita did not provide a definition, he seemed to have adequately demonstrated a one-to-one function. Mubita was then requested to comment on the analytical definition of a one-to-one function that he had presented in the test. This definition was similar to the one that John attempted to present during the interview. The conversation in that regard proceeded as follows:

Int: Okay, I know in the test that you defined a one-to-one function as one of course that has a one-to-one correspondence, but you said: you gave an example of elements for instance 'a' and 'b' belonging to the domain of a function *f* and you said if *f* of 'a' is equal to *f* of 'b', and if that implies that 'a' is equal to 'b', then *f* is a one-to-one function, would you just clarify what this definition means?

Mubita: Yes, referring to that if 'a' and 'b' are coming from that set [domain], and then we realise that if (stammers), if 'a' is mapped to a particular image which is also the image for 'b', then in a one-to-one correspondence 'a' must be 'b'.

Mubita did not explain what the definition means, but merely recounted it. He did not, for instance, clarify why for all $a, b \in D(f)$ if $f(a) = f(b) \Rightarrow a = b$ means that f is a one-to-one function. His replication of the definition did not suggest why the function f can only be one-to one when $a = b$. The definition was correct, and showed Mubita's competence at accurately reciting an analytical definition of a one-to-one function. But what was required of him was to show an understanding of the definition by providing a comprehensive explanation. Mubita's approach in this context corroborated his earlier approach when instead of defining a one-to-one function, he provided an illustration.

In the test, Sara answered that a one-to-one function is a relation 'that maps an element from one set to the other'. This definition suggested that, as long as a function mapped 'one element' from one set to the other, it qualified as a one-to-one function. Firstly, all types of functions map elements from one set to another. Secondly, there was nothing in Sara's definition that distinguished ordinary relations from functions. Thirdly, her description provided no area of distinction between a one-to-one function and a many-to-one function. Sara was, therefore, requested to explain the meaning of her definition:

> **Int:** What did you mean by this definition?
>
> **Sara:** Oh! what I meant was when (pauses), like I have said already when you have two sets we are saying when aah eeh a one-to-one function is where you have (pauses again), a situation where you have one element being mapped onto one and only one element in the other set.

Sara's explanation seemed to have been an attempt to emphasise a feature that distinguishes relations from functions. This is because with functions, each element from the domain is supposed to map onto only one element in the range. Her explanation did not capture the essence of a one-to-one function. Sara declared that the definition that she provided in the test was the only one she knew. In an endeavour to acquire supplementary insight, she was requested to explain one area of distinction between many-to-one and one-to-one functions:

> **Int:** Okay, are you able to explain to me one difference that exists between a many-to-one function and a one-to-one function?
>
> **Sara:** Okay, a one-to-one function is a function where one element is being mapped onto one and only one whereas a many-to-one, you have many elements being mapped onto one element in the range.

This extract reinforced the observation that Sara did not have the correct understanding of the concept of one-to-one functions. Her knowledge in this regard was centred on a flawed understanding of the univalence property of functions. This finding is consistent with her earlier struggle to comprehensively explain what a function entails. Similarly, Sara lacked an understanding of the difference between a one-to-one function and a many-to-one function.

Student teachers' understanding of inverse functions

Initially, Doreen explained that an inverse function is 'simply a reverse function that will get you back to the original function where you started from'. It would appear that her view was that the role of an inverse function is to generate a function for which it is an inverse. This perspective did not clarify that an inverse function operates on the elements of the range of the function for which it is an inverse, and relates them to the corresponding objects in the domain of the original function. Subsequently, Doreen implicitly alluded to this feature as the following excerpt demonstrates.

> **Int:** So when you say it is 'a reverse function' shed a bit of light. What does it mean when you say it is 'a reverse function'?
>
> **Doreen:** Say for example to get to (pauses), given an equation, you have say for example aah $2x + y$ as your relation or your function. Whatever you have in set A to get the members of set B you are going to apply this [relation], you are going to 'fix' in this equation or this expression put as an expression.
>
> **Int:** Ehe.
>
> **Doreen:** Now to get back to the initial number that you had at first which I had put in this expression, you need another expression. That expression that you need to get back to the initial number that you had is what we call an inverse function.

This excerpt suggests that Doreen's idea of inverse functions was characterised by algebraic formulas. This is evident in her illustration involving two sets A and B, and her emphasis on 'to get back to the initial number … you need another expression'. It seems that her idea of 'reversing' related to an algebraic formula that acts on images of a function to generate the corresponding objects. In the test, she described an inverse function as 'simply an [a] function that will map the range back to the domain, in short it's a reverse function'. Doreen's explanation of this definition confirmed that her conception of inverse functions was confined to formula representations.

Brave explained that an inverse function is a process of 'reverting' or 'reversing' to the original expression. This viewpoint was consistent with Doreen's initial idea of 'getting back to the original function'. The 'process' that was alluded to by Brave was implied in Doreen's use of the term 'reverse function'. As one 'reversed' or 'mapped back' the range onto the domain, a process would have to be undertaken. Brave explained the idea of reversing in the following manner:

> **Int:** What do you reverse in that case?
>
> **Brave:** You aah you just, okay when you are finding the inverse, there are certain procedures that we follow.
>
> **Int:** Ehe.
>
> **Brave:** There are certain methods (pauses), procedures that we follow to arrive at the inverse of a function.
>
> **Int:** Ehe.
>
> **Brave:** So now what we reverse are the same procedures, we reverse them back to the original expression.

This extract suggests that Brave's understanding of an inverse function was dominated by the procedures that are involved in computing the inverse of a function. He placed emphasis on the methods and procedures that needed to be followed to arrive at an inverse function. He gave an example of a linear function $y = 2x + 3$ and explained that its inverse function would be derived from making x the subject of the formula. It then became obvious that what he meant by 'reversing the procedures' was essentially the process of calculating the inverse function of an algebraic linear function.

Mubita explained an inverse function through its distinctive feature of 'reversing' the action of another function. His understanding was premised on the perceived 'actions' that are 'performed' by the function for which it is an inverse. Mubita's views were similar to those of Brave when he pointed out that an inverse function 'reverses procedures'. His arguments were expressed in the following conversation:

> **Int:** Probably what do you mean by reversing the operations?
>
> **Mubita:** Uh, uh I would say to reverse here will mean to un-do. To un-do what was done by the other function, but I don't know if I sufficed (laughs).
>
> **Int:** (Laughs also) you could still clarify again what you mean by 'un-doing'.
>
> **Mubita:** Laughs again, and says: aah (pauses).

Mubita argued that the notion of 'reversing the operations' was in line with the characteristic of an inverse function 'un-doing' what is done by another function, an idea which was also shared by Doreen. However, the uncertainty with which Mubita explained the idea of 'reversing' prompted the researcher to ask him to give an illustration:

> **Int:** Would you maybe illustrate your idea?
>
> **Mubita:** Yes, I can uh if a relation for example was $2x+3$, is defined by the formula $2x+3$.
>
> **Int:** '$2x+3$', yes?
>
> **Mubita:** Yes, so the reverse of this would be 'x' subtract 'three' then divide by 'two', because here the 'x' is being multiplied by 'two' then we add 'three', so the last action here is to add 'three'. So in the reverse that will be the first action, we shall subtract 'three' and then divide by 'two'.

Mubita's illustration suggests that by 'un-doing' or 'reversing', he implied going backwards in terms of operations from the range to the domain. Just like Doreen and Brave, it seems that Mubita could only successfully explain the concept of inverse functions in the context of the computation of an inverse function of an algebraic function. He was successful to the point of providing a correct expression $\frac{x-3}{2}$ as an inverse of the function $y=2x+3$.

John explained the concept of an inverse function from the perspective of its characteristic to operate on elements of the range of another function. Specifically, he contended that an inverse function is 'a function which maps elements of the range into elements of the domain or it is the reverse of a given function'. Furthermore, John communicated the idea of 'the reverse of a given function', which was articulated by Doreen, Brave and Mubita. Similarly, Mwila alluded to an inverse function as being a 'reverse function'.

Mwila's notion of an inverse function as one that 'undoes what the original function did' was consistent with Mubita's explanations. Primarily, Mwila seemed to narrow his view of inverse functions down to the symbolic representations. This was noted from his emphasis on 'what the original function did to the given value'. Notwithstanding, he described a function using an example of a hammer mill in which maize corn is ground to produce mealie meal. Therefore, the case of a machine in which 'you can put mealie meal' to get back 'maize corn' seems to have illustrated the idea of an inverse function 'un-doing' or 'reversing' that which the original function had done.

Sara also explained an inverse function from the perspective of it being the ‘reverse’ of a function. However, she added another dimension to her definition by asserting that an inverse function is ‘the opposite side of the function’. When Sara was asked to elucidate the definition presented, she could not do so, but opted to cite an example, $f(x) = 2x + 1$. Similarly, she could not proceed, but decided to change this example after several promptings:

> **Sara:** Example you have $y = 2x + 4$.
>
> **Int:** Ehe.
>
> **Sara:** Yah this is, this (stammers), this function, if I make *x* the subject of the formula, I’m reversing then latter I will change the *x* into *y*.
>
> **Int:** Ehe.
>
> **Sara:** This is ‘the opposite’ of this function because if I was to re-do this process, I will get back to the original function.

Just like the other five student teachers, Sara’s understanding of an inverse function was aligned with algebraic formulas. She understood inverse functions from the perspective of the process that is undergone in the computation of an inverse of a function. What Sara meant by the terms ‘reverse’ or ‘opposite’ was the ‘method’ of making the independent variable subject of the formula when computing the inverse of a function.

Student teachers’ understanding of composite functions

Brave described a composite function as a function that ‘consist[s] of two or more functions’. He explained that composing functions involves combining functions with a view to coming up with one function. To demonstrate his understanding, he cited $f(x) = 2x + 3$ and $g(x) = 4x + 5$ as two examples of functions. He then indicated that the composite function of g and f denoted $(g \circ f)(x)$ is found by firstly letting $y = g(x)$, and successively making x the subject of the formula. According to Brave, the resultant expression $\frac{y-5}{4}$ is supposed to be substituted into the function *f*.

The notion of making x the subject of the formula exposed Brave's superficial understanding of how two functions can be combined to find an expression for a composite function. It appeared that he confused the process of finding a composite function with that of calculating the inverse of a function. He was, therefore, probed as follows:

> **Int:** Ehe, but I notice you started by making x subject, so is it always that you have to start by making x subject?
> **Brave:** No, actually it depends with how the functions have been given.
> **Int:** Okay, but the way you had given those two functions, you started by making x subject?
> **Brave:** (Long silence).

Brave revealed a misconception when he claimed that the way in which functions are presented is what determines whether x has to be made subject of the formula or not. Subsequently, he explained that the composite function $(g \circ f)(x)$ is obtained by substituting the function g into the function f. This view confirmed that he lacked the understanding that a combination of the functions g and f is not always commutative. Moreover, when reminded of his assertion of making x the subject of the formula for the function g, Brave responded with extended silence, an indication of uncertainty.

Doreen explained that the process of combining functions involves 'bringing the functions together'. She appeared to know that in the process of finding the composition of two functions, one function is supposed to operate as an object of another function. Nevertheless, she portrayed a lack of understanding in that the order is important when combining functions into a single function. This was evident when she posited that f acting first followed by g typified the composite function of f and g. While there are instances when a combination of functions is commutative, it was incorrect to postulate that commutativity generally holds.

Sara explained that the composition of a function by itself refers to a situation where a function is 'contained in another function'. The next excerpt shows her attempt to clarify what was meant by this description:

Int: Okay, what do you mean by a 'function is contained in another function'?

Sara: Mmm let's say I have *g* of *x*. *f* of *g* of *x* maybe then in brackets in the (pauses), where normally we have *f* of *x* like that (pauses), now where they, because if *f* of *x* (stammers), if I substitute *g* of *x* and *g* of *x* is a function.

Int: Ehe.

Sara: This is called a composite function. That is why I have said that it is a function contained in another function.

Sara appeared to have a basic idea of the process involved in obtaining a single expression for a combination of two functions. Her explanation suggested that $(f \circ g)(x)$ is attained by substituting *g(x)* into the function *f*. Nevertheless, she did not indicate that the order is important when combining functions. Apart from this, the initial interview question required her to explain the combination of a function, say *f* by itself, and not the combination of two different functions, *f* and *g*. By trying to answer a question that was not even asked, Sara showed that her understanding was limited.

Mwila's description of a composite function was not varied from the definitions that were provided by Brave, Doreen, and Sara. He showed a considerable understanding of the process involved in finding composite functions. Notwithstanding, his explanation was not coherent, and he struggled to indicate that in a composite function, one of the functions acts as an object. Moreover, he did not allude to the fact that composite functions are generally not commutative. Similarly, John appeared to have a basic understanding of the process involved in combining functions. He, as was the case with the other student teachers, did not understand that when combining functions, the order is important. John gave two functions $f(x) = x + 1$ and $g(x) = 2x + 4$, and claimed that the composite function of *f* and *g* is the same as $(g \circ f)(x)$. What he did not realise is that the composite function of *f* and *g* referred to $(f \circ g)(x)$, whereas $(g \circ f)(x)$ denoted the composite function of *g* and *f*.

In contrast to the other student teachers, Mubita demonstrated an understanding that the order is not generally commutative when combining functions. The following extract provides evidence in this regard:

> **Int:** Okay, okay. Let's shift to composite functions. Describe for me what a composite function is.
> **Mubita:** A composition of function uh aah I would maybe explain a composition in this way: if *f* is one function and *g* is another function. Then the composition of *f* and *g* will mean aah *g* uh *g* of (pauses), *f* will operate on *g*. So, *g* will act, after *g* has acted then *f* is going to operate on *g*.

Mubita explained that the composite function of *f* and *g* implied that the function *g* was supposed to act first, followed by the function *f*. This suggested that he equally understood that $(g \circ f)(x)$ is not always equal to $(f \circ g)(x)$.

What justifications do the student teachers provide for their reasoning?

Justifications based on examples and non-examples of functions

For this section, investigations were primarily conducted through item 2(b), as presented in the test. This item consisted of four figures and required the student teachers to indicate whether they deemed each of the figures to be either a function or a non-function. The student teachers were also expected to provide a justification for their position (see Sections 4.2.1 and 4.2.2 for Phase1 results on this item). To facilitate the interpretation of the interview results, the four figures of item 2(b) are reproduced below as Figure 4.18.

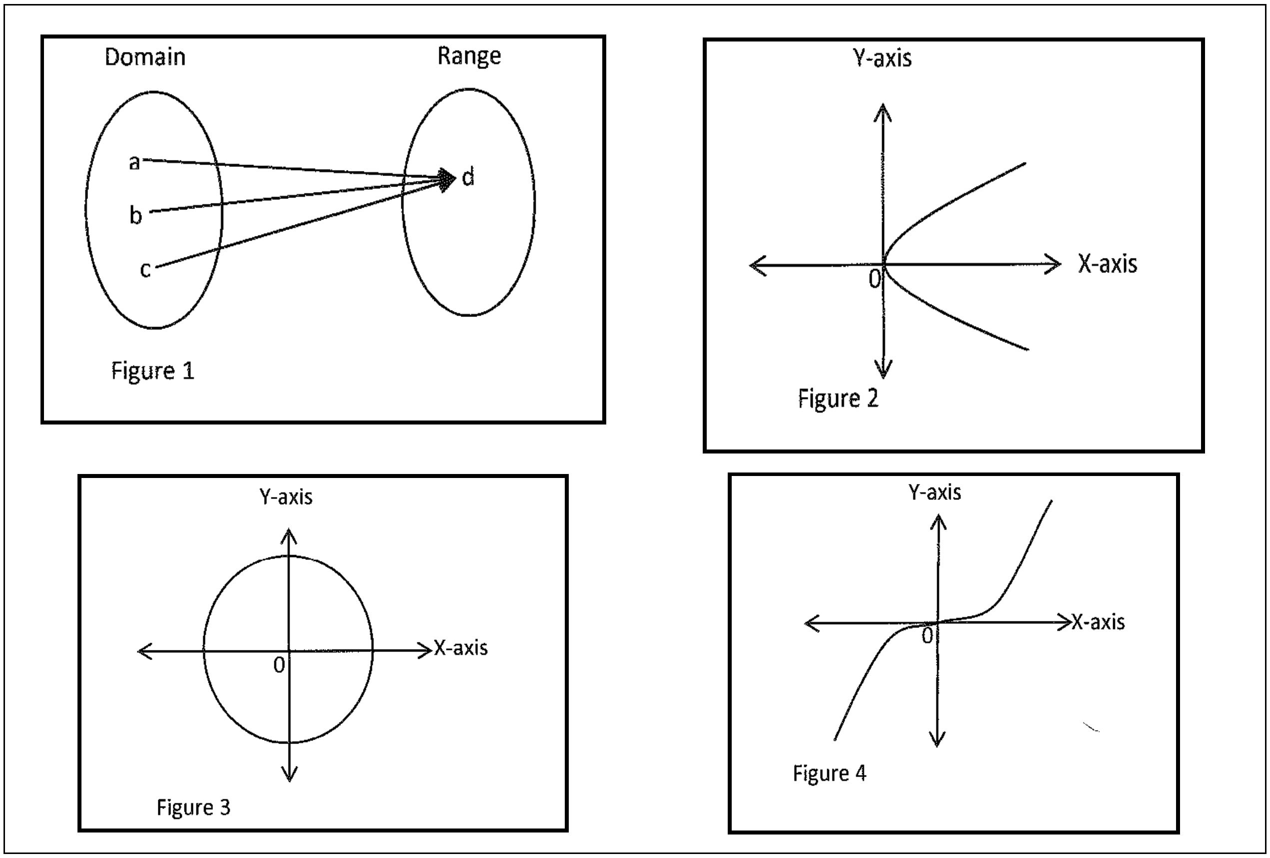


Figure 4.18: Illustration of figures in Item 2(b)

Brave exhibited a conflicted understanding with regard to the identification of Figure 1. At first, he indicated that this figure is a function, but changed his point of view after being requested to provide a justification. He defended his change of position by claiming that Figure 1 cannot be a function since all the elements in the domain are linked to one element, 'd', in the range. This view corroborated his lack of ability to appropriately define a function.

Nonetheless, Brave correctly identified Figures 2, 3, and 4 in Figure 4.18. He explained that Figure 2 is not a function because some of the *x*-values have two values of *y*. Additionally, he indicated that a vertical line would intersect the Cartesian graph at two points thereby causing an *x*-value to have two *y*-values. This argument suggests that he used the vertical line test to identify Figure 2 as a non-function. He also asserted that Figure 3 is not a function since there are *x*-values, which are associated with more than one value of *y*. While this explanation did not suggest that he knew that a circle is a many-to-many relation, it was a relevant reason. Furthermore, Brave explained that Figure 4 is a function since a vertical line would only intersect the graph at one point. He successfully used the vertical line test to ascertain that Figure 2 is a non-function, and that Figure 4 is a function. Nonetheless, it should be mentioned that a vertical line test in itself is not a justification, but a strategy that is applied to identify graphs that are functions.

Sara and Doreen struggled to identify and justify examples and non-examples of functions. Doreen posited that Figure 1 is not a function. This suggested that she did not know that the arrow diagram is a many-to-one function, or that she did not know that a many-to-one relation is a function. Her justification is articulated in the following excerpt:

> **Int:** Why would you take that view that Figure 1 is not a function?
>
> **Doreen:** Like I said aah much as they all have images, they have an image in the range.
>
> **Int:** Ehe.
>
> **Doreen:** But they are mapped onto one image. There should be just one corresponding image for every object in the domain.

According to Doreen, Figure 1 could only be a function if each object in the domain had a different image. This perspective is similar to that of Brave. However, Doreen's views contradicted her definition of a function. She described a function as a relation that is either one-to-one or many-to-one. It would appear that when she was faced with a many-to-one arrow diagram, she became confused. This inconsistency in understanding provided additional confirmation of Doreen's limited understanding of the univalence property of functions. She did not seem to realise that each object in the domain was connected to a unique image, 'd', in the range.

Doreen asserted that Figure 2 is a function when in reality the one-to-many Cartesian graph is not a function. When probed for her reasoning, she claimed that 'for every point that you take on the graph, there will only be one corresponding *x*-value there'. This explanation was not satisfactory in that, apart from $x = 0$, each of the *x*-values is connected to more than one *y*-value on the y-axis. Again, this exposed her lack of comprehensive understanding of the requirement for a relation to be a function. In spite of this, she appeared to think that there is a difference between a relation and a function:

Int: So I guess what you are saying is that there is a difference between a relation and a function, so would you share with me what that difference is?

Doreen: Silent then says: what I'm saying is there isn't (is no) much difference (laughs).

She hesitantly indicated that Figure 3 is not a function. Afterwards, Doreen changed her perspective and claimed that there is a possibility that the circle is a function. She attempted to relate the circle to an implied, but inaccurate, algebraic equation before positing that 'no, this one is debatable'. A clarification was then sought from her:

Int: This one is debatable, okay.

Doreen: Because when you look at it, it can be, but how can I call it (pauses), anyway I'm not sure (then laughs).

Int: You are not sure, okay.

Doreen: Laughs still.

This extract undoubtedly confirms that Doreen was not certain if the Cartesian circle is a function or a non-function. Doreen indicated with a degree of uncertainty that Figure 4 is a function because 'whatever point you take on the graph, it will have a corresponding *x*-value, *x*-value just like that'. This explanation was not comprehensive and furthermore, her reference to a point that is an ordered pair weakened the justification.

Sara posited that Figure 1 is a function 'because all the elements' in the domain are mapped onto 'one and only one element'. It appears that Sara was motivated to suggest that it is a function by the same reason that made Brave and Doreen view this figure as a non-function. Sara argued that although several images can exist in a range, the objects from the domain need to map onto 'only one' image. In order to ascertain the meaning of this explanation, she was queried through the use of an example:

Int: Okay, supposing I gave you something like in set A maybe I have 'a, b, c' and in the second set B I have maybe 'd and e' and maybe 'a' goes to 'd', 'b' goes to 'd', and 'c' goes to 'e', would this be an example of a function?

Sara: Silent then says: yes.

After Sara correctly declared that the interviewer's example represented a function, she was requested to provide a justification. Initially, she was very hesitant to provide a justification for her viewpoint:

> **Int:** Why would this be an example of a function?
>
> **Sara:** Okay, mmm this wouldn't be a function, sorry, it won't be a function. It can't be a function because these are being mapped onto more than one image.

Sara apologetically stated that the example the interviewer had cited was not a function. She then stated that: 'because these are mapped onto more than one'. By implication, her justification for considering Figure 1 as a function is because all elements in the domain are connected to one and the same element in the range. Obviously, such a justification would suggest a narrow conception of the univalence property of functions. Of interest was Sara's change of perspective from the correct answer to an incorrect answer when pressured. To confirm if her change of perspective was based on knowledge, she was probed again:

> **Int:** Ehe, so you are saying if 'a' is being mapped onto'd', 'b' onto 'd' and 'c' onto 'e', you are saying this would not be an example of a function now?
>
> **Sara:** Silent and then says: it is confusing.
>
> **Int:** It is confusing?
>
> **Sara:** Laughs and says: yes. Laughs again and says: ha!
>
> **Int:** Okay (laughs also), you are not too sure?
>
> **Sara:** Laughs, and says: yah, I'm not too sure.

Sara's change of positions did not arise from conviction, but was simply an act of guessing. She seemed frustrated and unsure. Sara's reaction suggests a lack of capacity to comprehensively account for the assertion that Figure 1 is a function. Just like Doreen had done, Sara considered Figure 2 to be a function. She, however, failed to provide a justification for this point of view. She appeared not to know that when any element from the domain is associated with more than one element in the range, the univalence property of functions is violated.

Unlike Doreen, who was uncertain whether a Cartesian circle is a function or a non-function, Sara asserted that Figure 3 is not a function. She argued that in terms of a circle, there will always be elements from the domain that are linked to more than one image in the range. While a circle is a many-to-many relation, it is interesting that Sara did not apply similar reasoning when confronted with Figure 2, a one-to-many relation.

When the interviewer cited the equation $x^2 + y^2 = 1$, Sara originally considered it to be a function. However, after realising that it is an equation of a unit circle, she changed her mind, and said that it is not a function. Sara did not provide any justification for her standpoint other than stating that it is a circle. She argued that Figure 4 is a function because each 'element from the domain has only one element in the range'. While this explanation was relevant, it was obvious that she did not realise that the figure represented a one-to-one function.

Mwila, Mubita, and John accurately identified the four figures represented in Figure 4.18. These student teachers recognised Figures 1 and 4 as functions, and Figures 2 and 3 as non-functions. Nonetheless, the justifications provided by these students varied for particular figures. With respect to Figure 1, Mubita argued that it is a function because every member of the domain is mapped onto a unique image in the range. This explanation suggested that he used the univalence property of functions to justify his stance. John contended that Figure 1 is a function since each element in the domain is mapped onto an element in the range. He did not explain that the justification was not in having a single image for all the objects, but that it was in the uniqueness of the image for each object. He posited that Figure 1 is a function because it is a many-to-one function. This was an interesting revelation since John had earlier confined his definition of a function to one-to-one functions.

Moreover, John exhibited a superficial understanding of a many-to-one function when he did not recognise an example of a many-to-one function that was cited by the interviewer. He had contended at the time that a many-to-one function is supposed to have a single image in the range and that all elements in the domain should be connected to that single image. It appeared, therefore, that his assertion that Figure 1 is a many-to-one function was based on a similarly narrow conception of many-to-one functions. Mwila pointed out that Figure 1 is a function because it is a many-to-one relation. Interestingly, his conception of many-to-one functions appeared as restrictive as John's. When explaining the different types of relations, he suggested that for a many-to-one function, a lot of elements from the domain map onto a single element in the range. This understanding seemed to exclude other examples of many-to-one functions for which several images exist in the range.

Mwila, Mubita, and John gave similar justifications for contending that Figure 2 is not a function. They identified the figure as a one-to-many relation and explained that it was possible to find at least one *x*-value on the x-axis that is associated with more than one *y*-value on the y-axis. This explanation suggests that the student teachers utilised the univalence property of functions to justify their perspectives.

Mwila explained that Figure 3 is a many-to-many relation and argued that apart from the *x*-values at the points where the circle crossed the x-axis, each of the other *x*-values is associated with two values of *y*. In addition to this, he clarified that each of the *y*-values is connected to two *x*-values. Mubita indicated that Figure 3 is not a function because it is possible to have an element from the domain (x-axis) being connected to more than one image on the y-axis. John explained that Figure 3 is not a function because it does not pass the vertical line test. His conception of the vertical line test was articulated as follows:

> **John:** If we draw a straight line to cut that given graph, if it cuts the given graph at more than two points, then you know that that particular value for *x* it has more than two values along the y-axis so therefore it fails to be a function.
>
> **Int:** So what you mean by straight line is a vertical line?
>
> **John:** Is a vertical line.

John was unambiguous in his conviction that the vertical line test is used to determine whether a graph represents a function or not. Additionally, he knew how to utilise the vertical line test to ascertain graphs of functions and non-functions.

Student teachers' justifications of examples and non-examples of one-to-one functions

Mwila and John identified Figure 4 of Figure 4.18 as a one-to-one function, but did not provide satisfactory reasons to support their perspective. They explained that the figure is a one-to-one function because every *x*-value is connected to only one value of *y*. The student teachers did not indicate that, similarly, each of the *y*-values is linked to only one *x*-value. In the absence of this supplementary statement, the student teachers' reasoning was open-ended and could relate to many-to-one functions as well. In other words, Mwila and John's justifications only accessed the univalence condition of functions.

An analysis of the abilities of Mubita, Brave, Sara, and Doreen to identify one-to-one functions and provide justifications are based on Figures 1 and 2 which composed item 4(b) of the test (Appendix 3). For easy interpretation of the results, the two figures are reproduced below as Figure 4.19.

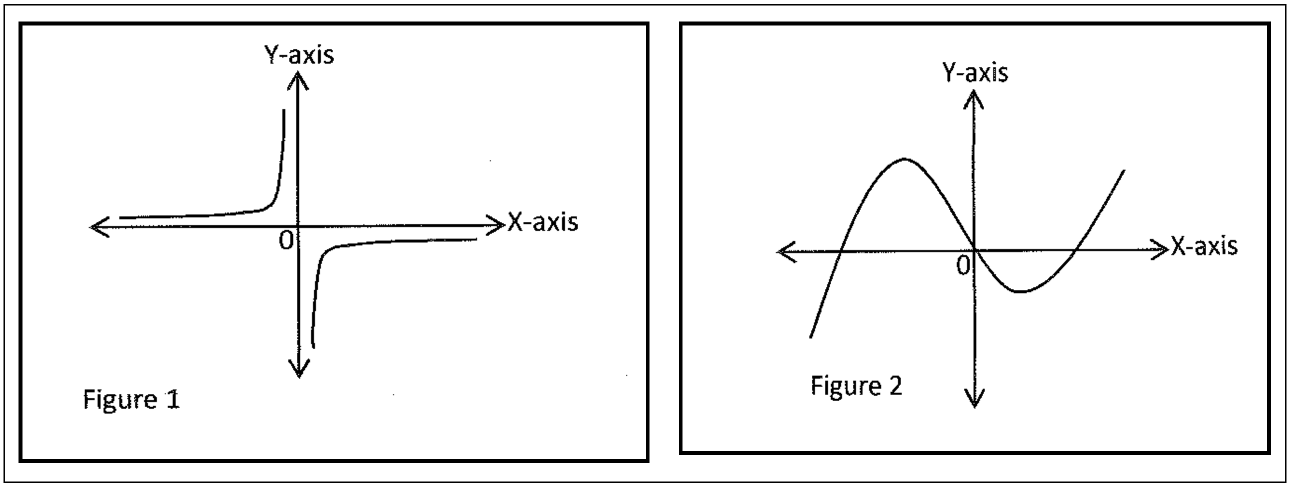


Figure 4.19: Illustration of figures in Item 4(b)

Mubita identified Figure 1 as a one-to-one function and posited that for $x > 0$ and $x < 0$ 'every member of the x-axis is paired with exactly one member of the y-axis'. This explanation only depicted what distinguishes an ordinary relation from a function. To that extent, Mubita showed an inability to justify his point of view.

In the test, Mubita indicated that Figure 2 is not a one-to-one function. During the interview, he explained that the figure is not one-to-one because a horizontal line cuts the graph at two points. While this view may be applicable, it did not amount to a comprehensive justification as it only indicated how the student teacher identified the graph. Moreover, what Mubita said afterwards demonstrated a conflict in understanding:

> **Mubita:** So now I notice that there is one *x* that can be paired to more than one *y* because when *x* is greater than zero I can obtain this point here, and when *x* is less than zero. Okay, I see, no I think I missed that one.
> **Int:** So you want to change; you are saying it is not or it is an example of a one-to-one function?
> **Mubita:** This should be a one-to-one function.
> **Int:** A one-to-one function?
> **Mubita:** Yes.

Showing a lack of in-depth understanding, Mubita changed his mind and incorrectly concluded that Figure 2 is a one-to-one function. This revealed Mubita's superficial understanding of the horizontal line test. Brave explained that Figure 1 is a one-to-one function since there is one value of *y* for every value of *x*. This explanation did not exclude cases where at least two *x*-values would have the same *y*-value (many-to-one functions). Based on this observation, it can be concluded that he was unable to sufficiently justify his affirmation that Figure 1 is a one-to-one function. When asked to provide reasons for stating that Figure 2 is not a one-to-one function, Brave argued as follows:

> **Int:** Why do you think it is not a one-to-one (Figure 2)?
>
> **Brave:** Why it is not a one-to-one function it's aah because this time around if I draw a (stammers), if I draw a parallel line which is (pauses), if I draw a parallel line this time around which is parallel to the x-axis. For example if I; if I was to draw a parallel line somewhere here, then what it will mean is aah (pauses).
>
> **Int:** What would it mean?
>
> **Brave:** It will mean that one *y* will have more than (pauses), *y* will actually have more than two values of (pauses again), will have more than one value of *x* because it will cut the graph at; it will actually cut this graph at three different points.

Brave utilised the horizontal line test to determine that Figure 2 is not a one-to-one function. He explained that a single *y*-value has more than one value of *x*. To an extent, this view clarified why the figure is not a one-to-one function.

Unlike Brave, Sara had no ability to recognise that Figure 1 is a one-to-one function. She justified her flawed stance in the following excerpt:

> **Int:** Why do you take that position that Figure 1 is not one-to-one?
>
> **Sara:** Uh because aah one point is being mapped onto more than (pauses).
>
> **Int:** One point is being mapped onto more than what?
>
> **Sara**: One element in the range.

Sara claimed that Figure 1 is not a one-to-one function because an element from the domain is connected to more than one image. However, the figure does not show any of the elements from the horizontal axis (x-axis) mapping onto more than one image in the range (y-axis). Therefore, this student teacher's assertion was incorrect.

Also, the use of the word ‘point’ instead of ‘element or object’ was ambiguous as it could refer to an ordered pair. Sara added that Figure 1 can be a one-to-one function if each element in the domain is mapped onto ‘one and only one’ element in the range. What she seemed to not understand is that this description is a condition for a relation to be a function, and not a sufficient requirement for one-to-one functions. Sara was thereafter guided to Figure 2, a many-to-one Cartesian graph, and asked to state her position:

> **Int:** What of Figure 2, in your view, is it an example of a one-to-one function or not?
>
> **Sara:** This is a function; it’s a one-to-one function.
>
> **Int:** Why is it a one-to-one function?
>
> **Sara:** Because one element is being mapped onto one and only one element.

Sara erroneously argued that Figure 2 is a one-to-one function because an element was mapping onto only one element. It was revealing to observe that the same justification she utilised to disqualify Figure 1, a real one-to-one function, was applied to justify her incorrect view that Figure 2 is a one-to-one function. Sara’s inability to identify an example and a non-example of one-to-one functions corresponded with her lack of capacity to provide a valid definition of a one-to-one function. Similarly, Doreen incorrectly claimed that Figure 1 is not a one-to-one function, while she declared that Figure 2 is a one-to-one function. Additionally, she could not provide a justification for her incorrect identifications.

Procedures for the calculation of an algebraic inverse function

Mwila, Mubita, John, Doreen, Sara, and Brave understood that calculating the inverse of an algebraic function involves expressing the independent variable as the subject of the formula. To avoid repetition, only the analysis of the justifications articulated by Sara and Brave are reported here. Sara’s explanations characterise the procedures that were performed by the student teachers as they calculated the inverse of the function $f(x) = x^2 + 1, 0 \le x \le 2$. She explained the process leading to an expression for $f^{-1}(x)$ in the following excerpt:

> **Sara:** First I changed *f* of *x* into *y* because *f* of *x* is the same as *y*. Then I solved for *x* meaning I made *x* the subject of the formula. And later I had to change again where there is *x* I had to put *y*, and *y* again *x* and that gave me the inverse of this function.

Despite being able to recite the process leading to the correct inverse function, Sara could not justify the procedures performed. Similarly, Brave explained that the procedure of calculating the inverse of the function $y = 2x + 3$ includes making x the subject of the formula, and replacing y by x in the final expression. He correctly applied this procedure and obtained an accurate expression $\frac{x-3}{2}$. Brave's justification for using this procedure was elicited in the subsequent excerpt:

> **Int:** Do you know why an inverse function is obtained that way?
>
> Brave: Silent.
>
> **Int:** Why would you make *x* the subject of the formula?
>
> **Brave:** The reason why I'm making *x* the subject, its aah because I want to find the inverse, but that's just a method of helping me to get the inverse of a function.

Brave did not have any justification for the overall procedure that seemed to 'work' for him. He perceived the procedure as an inevitable 'process' that needed to be undertaken whenever the inverse of a function is calculated. When asked to explain why he made x the subject in $y = 2x + 3$, his answer was that 'it is just a method helping me to get the inverse of a function'. This answer suggested that he did not understand why the independent variable had to be expressed in terms of the dependent variable. Moreover, he did not have a reason why the dependent variable y is normally replaced by the independent variable x after making x the subject of the formula:

> **Int:** Why should you replace *y* with *x* in the final expression?
>
> **Brave:** Because I don't want to actually distort (pauses).

This shows that he could not actually account for this action. Brave was knowledgeable in the mechanics of computing an inverse function without any in-depth understanding as to why the calculation works. In other words, he had proficiency in the procedures, but could not explain why such procedures are used or work.

Vertical and horizontal lines on a Cartesian plane

Brave indicated that a vertical line is a function, a view that demonstrates his lack of thorough understanding of the function concept. Notwithstanding, he provided a justification for his perspective in the following extract:

> **Int:** Give me the justification; how would you justify your viewpoint?
>
> **Brave:** Aah mmm this is (stammers), this is simply because this is a straight line.
>
> **Int:** Ehe.
>
> **Brave:** Assuming that this line is parallel to (pauses); is parallel to [the] y-axis.
>
> **Int:** Ehe.
>
> **Brave:** What it means here is that aah mmm why I'm saying this one is a function is because if I pick any value or any *x* coordinate value on this line, I will only have one value of *y*.

Brave asserted that a vertical line is a function because for every value of *x* there would be only one corresponding value of *y*. It seemed that he viewed a vertical line in the same manner as the graph of a linear function, which is a slanted straight line. His notion of a vertical line being parallel to the y-axis should have helped him to realise that along such a line, the values of *y* vary while the value of *x* is constant. The implication of this is that a single *x*-value from the domain is linked to at least two values of *y* in the range. In view of this, any line parallel to the vertical axis (vertical line) is an example of a one-to-many relation, which obviously is not a function. At the outset, Brave indicated that a horizontal line in a Cartesian plane is a function. Surprisingly, when asked to provide a justification he changed his mind and argued that a horizontal line is a non-function:

> **Brave:** Okay, I think (stammers), I think for this one actually does not qualify to be a function.
>
> **Int:** Horizontal line?
>
> **Brave:** Because the horizontal line I think does not qualify to be a function in the sense that if I pick a point, any, I pick a point on the x-axis that point will give me an (pauses), I would probably say it will give me so many values of *y*.
>
> **Int:** Okay, so that's what makes it not a function?
>
> **Brave:** Not a function.

Brave argued that a horizontal line drawn on a Cartesian plane is a non-function because any value from the x-axis would be connected to several values of *y*. This perspective is flawed in that for a horizontal line, several values of *x* (from the horizontal axis) are associated with the same value of *y* from the vertical axis. If anything, a horizontal line is a classic example of a constant function. Brave's justification, therefore, suggests that he lacked an in-depth understanding of the 'special' many-to-one relations, which are alternatively called constant functions.

Doreen correctly asserted that vertical lines drawn on a Cartesian plane are non-functions. However, she lacked the ability to provide a valid justification for this point of view. According to Doreen, it is sufficient 'just by looking' to determine that a vertical line is a non-function. Further exploration of her understanding revealed the following:

> **Int:** Other than 'by looking', what other justification would you give?
>
> **Doreen:** Because it just looks like a straight line; [it] is just vertically like this.
>
> **Int:** Ehe.
>
> **Doreen:** Functions! Aah, I'm not sure (then laughs).
>
> **Int:** You are not sure?
>
> **Doreen:** No, I'm not sure (laughs still).

Doreen exclaimed that she was not sure why a vertical line drawn on a Cartesian plane is not a function. At the same time, she cited an algebraic function $f(x)=2$, which depicts a horizontal line, and concluded that since it is 'just a straight line' it is not a function. It appeared that the line that she had in mind was $x=2$, and not $f(x)=2$, which she was grappling with. This was evident when she indicated that 'cutting the x-axis at 2, parallel to the y-axis'. Obviously, Doreen had problems with the algebraic representation of vertical lines.

Doreen's failure to justify why a vertical line is a non-function brought to light her lack of understanding of constant functions. She did not understand why $f(x)=2$ is a function and exclaimed that 'what is 2 being mapped onto ... unless if it has a variable maybe to say 2*x*'. She did not realise that for the function $f(x)=2$, the values of *x* vary and that it is the value of $f(x)$ that is constant. Regardless of her views on the symbolic constant function $f(x)=2$, Doreen was asked to explicitly state whether a horizontal line drawn on a Cartesian plane is a function or a non-function:

Int: Okay, okay. What of a horizontal line drawn in a Cartesian plane; would that represent a function or not?

Doreen: Because horizontal (pauses), vertical it's more like one and the same thing.

Int: Okay.

Doreen: They only take values of (pauses), because it's like the value of the *y* coordinate, it's, it's not moving, it's not changing. What is changing are the values of the *x* coordinate. Maybe they can be functions, but I'm not sure (laughs).

Int: You are not sure?

Doreen: No.

Doreen's final remark was that she was not sure whether a horizontal line drawn on a Cartesian plane is a function or a non-function. She had no justification for this viewpoint, but pointed out that vertical and horizontal lines are 'more like one and the same thing'. Doreen understood that for a horizontal line, the *x*-values vary while the *y*-value is constant. Amazingly, this understanding did not assist her to conclude that horizontal lines are functions. The uncertainty revealed in the statement 'maybe they can be functions, but I'm not sure' corroborates earlier findings regarding her lack of in-depth understanding of the univalence property of functions.

Sara explained that the vertical lines drawn on a Cartesian plane are non-functions because they have no gradients. It is, of course, not possible to compute the gradient of a vertical line as it involves a division by zero. Sara did not clarify that it is this impossibility to 'divide by zero' that she implied. She pointed out that horizontal lines are functions because they have gradients. Despite having the correct understanding that horizontal lines are functions and even citing an appropriate equation $y = 3$, Sara's justification was shallow.

John initially mentioned that a vertical line is not a function. However, after being queried if a vertical line is a straight line, he changed his stance and claimed that vertical lines on a Cartesian plane are also functions. John had a misconception that vertical lines are the same as graphs of linear functions. He argued that a vertical line is a function as it 'crosses the *x*-value at one point', and that at that point $y = 0$. Evidently, he lacked the understanding that on a vertical line the *x*-value is constant, while the *y*-values vary, which violates the univalence condition of functions. John emphasised that horizontal lines drawn on a Cartesian plane are functions. He was, however, quick to add that such lines are not one-to-one functions. His justification is highlighted in the ensuing extract:

Int: Why would a horizontal line in the Cartesian plane be a function?

John: Mmm okay it's a function because it's (pauses), there is one element maybe many elements in the domain will be mapped onto one element in the range which is the value of *y*. So you pick elements along the x-axis, we map them, they will go to one element, so it is an example of a many-to-one function.

John's explanation showed that he understood that on a Cartesian horizontal line, the *x*-values change while the *y*-value is constant. He correctly observed that a Cartesian plane's horizontal line is a many-to-one function. This perspective is consistent with the explanations provided by Mwila and Mubita. Nevertheless, whereas Mwila and Mubita provided comprehensive explanations, John's justification showed a narrow conception of the univalence condition of functions. It suggested that the range of a function must only consist of a single element.

Mwila contended that a vertical line on a Cartesian plane is not a function since a constant *x*-value would be mapped onto several *y*-values. In the same way, Mubita explained that on a Cartesian plane vertical line the *x*-value is constant, while the *y*-values are changing. He explained that the univalence condition is conflicted when a single *x*-value is associated with several *y*-values. This perspective encapsulated Mubita's satisfactory justification. Additionally, Mubita showed that he understood the difference between vertical lines, and the graphs of linear functions. He explained that, unlike vertical lines, graphs of linear functions are slanted straight lines.

Graphs of $f(x) = ax^2 + bx + c$, and the signs of the coefficients of x^2

Doreen gave $f(x) = x^2 + 1$ as an example of a quadratic function, and explained that its graph opens upwards $\vee$ because the coefficient of x^2 is positive. Furthermore, she claimed that the parabola of $f(x) = x^2 + 1$opens upward because 'whatever values we put in there they will all be positives'. This claim exposed her inability to comprehensively justify this viewpoint. Similarly, she failed to explain why parabolas open downwards $\wedge$ when $a < 0$.

John had no explanation for his perspective that the graph of $f(x) = ax^2 + bx + c$ has a minimum turning point when $a > 0$. This student teacher admitted that a reason existed, but he did not know how to express it. His lack of understanding was manifested when he reacted with silence after being encouraged to express the reason in any way possible. John struggled to explain why a parabola has a maximum turning point when $a < 0$. He revealed that 'the graph will first have a positive gradient then go to the maximum and finally have a negative gradient'. This was a description of the characteristic of parabolas, and not a justification. The silence that subsequently characterised his reaction indicated that he did not have a justification. Similarly, Mubita was at a loss to explain why for $a < 0$ quadratic curves have maximum turning points:

> **Mubita:** When '*a*' is less than zero then it is going to have a maximum turning point. Why am I saying that?
>
> **Int:** Ehe.
>
> **Mubita:** Aah because this is a particular feature that (pauses, and laughs). I was taught to say (laughs again) when '*a*' is less than zero then this is what you expect so I have learnt it that way (laughs still).
>
> **Int:** Laughs also. Okay, so you don't have an explanation?
>
> **Mubita:** I don't have an explanation.

Mubita was quick to confess a lack of understanding and indicated that he was merely reproducing what had been taught to him. He had never understood why the parabola opened downwards when $a < 0$. The scenario was not different for the case where $a > 0$:

> **Int:** Okay, so when it opens upwards?
>
> **Mubita:** It means '*a*' is positive.
>
> **Int:** Again there, do you have an explanation for that relationship?
>
> **Mubita:** Silent and then says: aah there could be an explanation, but I'm not aware of that.

Mubita confessed that he did not understand why the parabola opens upwards when $a > 0$. Sara asserted that parabolas open upward when $a > 0$ because $a > 0$. This claim was a mere replication of the condition. Subsequently, she mentioned that the reason is 'all the numbers will be greater than zero' when substituted. Sara's justification for the case where $a < 0$ was as vague as her justification for the case when $a > 0$.

In the same way, Brave showed a lack of capacity to provide a justification as to why parabolas have maximum turning points when $a < 0$. He stated that the graph facing downwards is an inescapable 'pattern'. Brave claimed that the 'pattern' resulted after substituting the values of either *x* or *y* in an algebraic quadratic function. He added that the graph opens downwards when $a < 0$ because it is increasing as one moves from the negative direction to the positive direction on the graph. This explanation was vague as it did not clarify why quadratic functions increase only when $a < 0$. Brave did not provide any explanation as to why a parabola has a minimum turning point when $a > 0$.

Mwila endeavoured to provide a substantive justification, except he got confused in the process. He elaborated the steps involved in completing the square for $f(x) = ax^2 + bx + c$, and stated that the completed square form is of the kind: $f(x) = \left[\frac{4ac - b^2}{4a}\right] + a\left[x + \frac{b}{2a}\right]^2$. He indicated that the first bracket always results in a constant.

Additionally, Mwila indicated that since the second bracket is squared, it has a value of zero for $x = -\frac{b}{2a}$. He pointed out that for the other values of *x*, the second bracket has values that are greater than zero. Notwithstanding, Mwila did not use the 'completed square form' to explain why the parabola opens upwards for $a > 0$, and for $a < 0$ it opens downwards.

4.3.2. Analysis of the student teachers' ability to use different representations in functions

This sub-section focuses on the analysis of the student teachers' ability to use different representations in functions. The following two questions are utilised to present the analysis: How well do the student teachers demonstrate an understanding of the different representations of concepts in functions? and, How accurately do the student teachers translate between the different representations of functions? Table 4.8 below provides an overview of the concepts that were explored in the category 'ability to use different representations'.

Table 4.8: Concepts comprising the ability to use different representations in functions

Guiding questions	**Concepts**
How well do the student teachers demonstrate an understanding of the different representations of concepts in functions?	Student teachers' knowledge of the forms of representations for functions; graphs of linear functions; student teachers' understanding of the formula representation of quadratic functions; characteristic shapes of graphs of quadratic functions; and sketching quadratic graphs (including turning points, maximum and minimum values, and values of discriminants).
How accurately do the student teachers translate between the different representations of functions?	Changing representation from ordered pairs to a Cartesian graph; and translating from algebraic formulas to graphical representations and vice versa.

How well do the student teachers demonstrate an understanding of the different representations of concepts?

Student teachers' knowledge of the forms of representations for functions

John, Mwila, and Mubita explained that the formula, arrow diagrams, Cartesian representation, and sets of ordered pairs are some of the forms of representations of functions. Sara, Doreen, and Brave's explanations revealed diverse understanding. Particularly, Doreen indicated formulas as one form of representing functions. She also said that functions can be represented in a set builder notation, although the example she gave was flawed. Doreen's example of 'a set builder notation' was the set $\{f(x): 2a+b, 1 \leq a \leq 3\}$. This is a set of images of a function *f*, which did not highlight the fact that a function is defined on a domain to give the elements in a range. Moreover, the variable *a* in the interval $1 \leq a \leq 3$ did not correspond with the implied object *x* of the function *f*.

Originally, Sara was of the opinion that functions can only be represented graphically. However, she was reminded of her example of a function $y = 2x + 4$ presented in formula form, and queried whether that was not a form of representation. To this challenge, she consented that formulas are another form of the representation of functions. Similarly, Brave cited the algebraic formulas and Cartesian graphs as forms of representations of functions.
Furthermore, he mentioned what he called 'the set notation' as another form of representing functions. Brave's description of the 'set notation' suggests that he had the arrow diagram

representation in mind. A noteworthy finding was Doreen and Brave's claim that all functions can be represented as formulas. The following discussion shows Doreen's perspective in this regard:

> **Int:** Okay. Now is it possible to represent every function as a formula?
>
> **Doreen:** Silent and then says: yes.
>
> **Int:** All functions can be represented as a formula?
>
> **Doreen:** As a formula, yes.

Brave used an arrow diagram to support his assertion that every function can be represented as a formula. He claimed that arrow diagrams can only be fully presented when they are attended by an algebraic formula, which ought to be employed to generate the elements of the range. The perspective espoused by Brave and Doreen suggests that they had a limited understanding of the different representations of functions. This is because the view that every function can be represented as a formula disregards the arbitrary property of functions. In Doreen's case, her assertion provided extra clarity as to why she posited that the many-to-one arrow diagram is a non-function (Section 4.3.1).

Sara's explanation suggests that she was of the opinion that all functions can be represented as formulas after being required to suggest a different form of repesentation for a many-to-one arrow diagram shown in Figure 1 of item 2(b) (see Figure 4.18).

The arrow diagram was not accompanied by a formula and had three objects 'a, b, and c' which were all linked to a single image 'd'. Sara referred to the aspect of substituting objects into a function *f* to produce the image 'd'. She indicated that an algebraic formula can be used as an alternative form of representation for the many-to-one arrow diagram.

Linear functions

Doreen's conception of linear functions was characterised by algebraic representations. Mwila, Mubita, John, Sara, and Brave were of the common view that a Cartesian graph of a linear function is a straight line. They explained that at least two points are necessary if a graph of a linear function is to be drawn. While the students suggested that any arbitrary two points would suffice, they were particularly clear that the *x*-intercept and the *y*-intercept make it easier to sketch the graph of a linear function. They said that the *x*-intercept is obtained when $y = 0$ in the formula, and the *y*-intercept is obtained by letting $x = 0$.

Nonetheless, Mubita's explanation exposed a conflict in his understanding. He pointed out that when a vertical line is involved 'all I need is the *x*-intercept'. This statement suggests that he took a vertical line to be a graph of a linear function. This view contradicted his earlier assertion that a vertical line drawn on a Cartesian plane is not a function. He had previously stated that Cartesian graphs of linear functions are slanted straight lines.

Five student teachers could explain the process of finding formulas of linear functions when Cartesian graphs are provided. Mubita's explanations are representative of the other four student teachers' ideas in this regard. He explained that what is necessary are the gradient of the graph as well as the *y*-intercept. Additionally, he explained that two points that lie on the graph are necessary for the computation of the gradient. Mubita mentioned that the standard equation $y = mx + c$, the value of *m* (gradient), and either of the initially selected two points can then be utilised to find the value of *c*, which is the *y*-intercept. Ultimately, the values of *m* and *c* can be substituted into the equation $y = mx + c$ to give the required formula. Against this background, Mubita was asked to explain how the equation $y = mx + c$ is derived:

> **Int:** Okay, how do you come up with this equation $y = mx + c$?
>
> **Mubita:** This equation where $y = mx + c$ is the general form for an equation of a straight line.
>
> **Int:** Okay, how does it come about?
>
> **Mubita:** How it comes about aah I think it shows how the *y*-coordinate is related to the *x*-coordinate for all the points that will lie on that particular line.

As much as Mubita could explain the process of the computation of a formula for a linear function represented as a Cartesian graph, he lacked in-depth understanding concerning the derivation of the equation $y = mx + c$. In place of explaining how the equation is derived, he presented its characteristic. Similarly, the other student teachers could not competently explain the basis of the equation $y = mx + c$, for example, when Sara was asked to provide an explanation, she merely asserted that 'aah $y = mx + c$, you get it from the same equation line because even directly without finding this, I can still find the equation of a line'.

Student teachers' understanding of the formula representation of quadratic functions

This section specifically provides the understanding of the student teachers who fall under the low content knowledge classification. Doreen asserted that in an algebraic quadratic function, the greatest power of the variable involved is two. Her attempt to clarify this stance revealed that she had a shallow understanding of algebraic quadratic functions. For instance, she presented $f(x) = x^2 + y$ as an example of a quadratic function, and posited that *x* and *y* are the variables. She contended that the index 2 for the variable *x* is what qualified the cited example as a quadratic function. Her example did not represent a quadratic function. In pursuit of additional insight, Doreen was requested to confirm if a given formula is a quadratic function or a non-quadratic function:

> **Int:** Supposing I wrote something like $y = 3x^2 + 4$, and I have avoided the $f(x)$ notation, would this be a quadratic function?
>
> **Doreen:** Silent.
>
> **Int:** $y = 3x^2 + 4$; is this a quadratic function?
>
> **Doreen:** Because again this *y* is not telling us to say *y* is mapped onto this.
>
> **Int:** Okay, okay.
>
> **Doreen:** Unless if you say *y* is mapped onto this, then that will make it to be a quadratic function.

Doreen did not think that $y = 3x^2 + 4$ is a quadratic function. She appeared to be of the opinion that the notation used was inappropriate. Later on, she suggested that $y = 3x^2 + 4$ can only be a function when the notation $f(x)$ is used. This perspective exposed Doreen's superficial understanding of notations that are used to denote quadratic functions.

Sara demonstrated her understanding of the symbolic representation of quadratic functions in the following excerpt:

> **Int:** Okay. So if I gave you something like maybe $f(x) = x^2 + y^2 + 3x + 5$, would this be an example of a linear function?
>
> **Sara:** No, it is a quadratic function.
>
> **Int:** It is a quadratic function?
>
> **Sara:** Yes.
>
> **Int:** Why would it be a quadratic function?
>
> **Sara:** Because we have at least one variable being raised to the power 2.

This extract demonstrates Sara's superficial understanding of the formula form of a quadratic function. She claimed that $f(x) = x^2 + y^2 + 3x + 5$ is a quadratic function because it has at least one variable being raised to the power of two. Earlier, she had said that the quadratic equation $3x^2 + 5x + 4 = 0$ is a quadratic function. Sara seemed to not appreciate the difference between a quadratic function, and an ordinary quadratic equation.

Brave appropriately presented $f(x) = 3x^2 + 4$ as an example of a quadratic function. To validate his understanding, he was asked to confirm if $y = x^2$ represents a quadratic function:

> **Int:** Supposing I gave you something like $y = x^2$, would this be a quadratic function?
>
> **Brave:** This one uh (pauses).
>
> **Int:** What do you think?
>
> **Brave:** This one actually is not a quadratic function.

Interestingly, after appropriately proposing that $f(x) = 3x^2 + 4$ is a quadratic function, Brave claimed that $y = x^2$ is not a quadratic function. He was asked to explain his reasons for asserting that $y = x^2$ is not a quadratic function:

Int: Why is it not a quadratic function?

Brave: It is not a quadratic function eeh simply because aah I would have two values of (pauses), I will have two values.

Int: Ehe, what do you mean you will have two values?

Brave: Assuming that uh mmm x^2 there is equal to (pauses), assuming that $x^2 = 4$, then after computing at the end of the day I will have a negative and a positive.

Int: So, that is?

Brave: So that's what actually disqualifies this one to be a quadratic function.

Brave did not realise that $y = x^2$ was in the form of the quadratic function $f(x) = ax^2 + bx + c$ and that the only difference is that for $y = x^2$, each one of the constants b and c is equal to zero, and $a = 1$. He also viewed the formula in terms of the computation of solutions for a quadratic equation. This can be concluded from his claim that $x^2 = 4$ would result into two values: one negative, and the other positive. The failure to detect that $y = x^2$ is a many-to-one function suggests that Brave did not have the univalence property of functions in mind, and obviously this suggests that his understanding was limited.

When the notation $f(x)$ was applied, Brave asserted that $f(x) = 3x^2 + 4$ is a quadratic function, but in a situation where $f(x)$ was not used as in $y = x^2$, he claimed that the example is not a quadratic function. This resonated with Doreen's understanding when she intimated that $y = 3x^2 + 4$ can only be a quadratic function if the notation $f(x)$ is utilised. Whereas Brave declared that $f(x) = 3x^2 + 4$ is a quadratic function, Doreen suggested that $y = 3x^2 + 4$ is not a quadratic function. Brave and Doreen had misconceptions regarding the notation that is appropriate to use when representing quadratic functions.

Student teachers' understanding of the characteristic shapes of quadratic functions graphs

While all the six student teachers asserted that quadratic functions have graphs commonly called parabolas, only Mwila, John, and Brave posited that the graphs of $f(x) = ax^2 + bx + c$ have at the most two shapes. These student teachers correctly explained that a parabola can open either upwards $\vee$ or downwards $\wedge$ depending on the sign of the coefficient of x^2. Doreen, Mubita, and Sara advanced a claim that the graphs of quadratic functions can assume two additional forms. The conversation with Doreen in that regard is represented by the subsequent excerpt:

Int: Are you suggesting that there are two forms for shapes of graphs of quadratic functions?

Doreen: Silent then says: no.

Int: How many forms can graphs of quadratic functions take?

Doreen: They can take I think four.

Int: Four?

Doreen: Yes.

Doreen was very explicit in her claim that there are four possible shapes that graphs of quadratic functions can take. With this finding, she was requested to make sketches of the two additional forms that the graphs of quadratic functions take:

Int: Maybe draw the sketch; you can put the drawing somewhere else, ehe.

Doreen: Then there can also be (pauses).

Int: Ehe.

Doreen: It can also turn this way (draws Sketch D1 below).

Int: Ehe.

Doreen: Depending on the type of equation you are given.

Int: Okay.

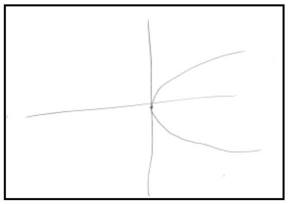


Figure 4.20: Sketch D1

Doreen: It can also turn the other way (draws Sketch D2 shown below).

Int: Okay.

Doreen: So that is why I said it can take four.

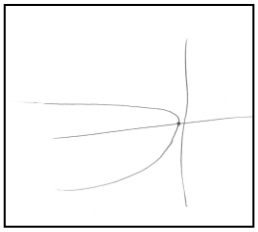


Figure 4.21: Sketch D2

Figure 4.20 and 4.21 confirm that Doreen understood that in addition to the two forms $\vee$ and $\wedge$, parabolas can also open either to the right of the positive x-axis or to the left of the negative x-axis. Correspondingly, Mubita and Sara claimed that parabolas can also appear as shown in Figures 4.20 and 4.21. Mubita particularly explained his view as follows:

> **Int:** Then you are saying there is this one graph which opens to the right.
>
> **Mubita:** Yes.
>
> **Int:** Of the y-axis?
>
> **Mubita:** Yes, and then to the left of the y-axis.
>
> **Int:** To the left, so you are saying there are four shapes that graphs of quadratic functions can take?
>
> **Mubita:** Ja, four shapes.

By indicating that some parabolas open to the right of the y-axis, Mubita presented an idea that is consistent with Figure 4.20. Furthermore, his conception of a graph that opens to the left of the y-axis was in line with Figure 4.21. These findings brought into question the depth of Mubita's understanding of the vertical line test, which he had satisfactorily explained. Figures 4.20 and 4.21 do not pass the vertical line test, and yet Mubita considered them to be graphs of quadratic functions.

Sara explained that at secondary school level, the graphs of quadratic functions open either upwards or downwards. She, however, claimed that at higher levels of education, such as in a university, the quadratic graphs take two additional forms as depicted in Figures 4.20 and 4.21. Doreen, Mubita, and Sara's statements suggest that they lacked a thorough understanding of the forms of the graphs of quadratic functions. While Figures 4.20 and 4.21 can be shapes of curves of quadratic functions if the vertical axes are taken to be domains and the horizontal axes as the ranges, none of the student teachers explained this view.

Sketching quadratic graphs

Mwila and John initially explained that a graph of a quadratic function can be sketched through the development of a table of values of the objects and their corresponding images. They indicated that given a function $y = ax^2 + bx + c$, arbitrary *x*-values can be used to derive the respective *y*-values. It is the generated ordered pairs (x, y) that are then plotted on a Cartesian plane.

Mwila and John were clear that the plotted points are normally joined using a smooth curve. When John was asked to explain why it is necessary to proceed in the manner described, he claimed that 'that is the general rule that is used'. Other student teachers who espoused the method of developing a table of values were Doreen, Brave, and Sara. Brave's perspective was consistent with Doreen's. The latter contended that this method is the 'easiest way' and posited that 'issues of finding the turning points sometimes you forget the formulas; it's a bit trick'. She lacked an understanding of how to calculate the axis of symmetry, and the *y*-value at a turning point of $f(x) = ax^2 + bx + c$. Another probing question was posed regarding the choice of the *x*-values during the development of a table of values:

> **Int:** Now again I would ask you; what will determine the choice of the *x*-values?
>
> **Doreen:** It's the formula that you are given and the number that you want to work with. If you are comfortable working with (pauses), because if it is on the graph paper you can even work with decimals, but if you want to work with whole numbers then you would rather choose numbers that will give you whole numbers.

Doreen explained that 'the formula' is what guides the selection process of the *x*-values. What was not clear is how $f(x) = ax^2 + bx + c$ determines the *x*-values that ought to be utilised. Nevertheless, she added that the choice of the *x*-values, and the number of those values are entirely reliant on the person drawing the graph. Sara's notion was to 'pick on some few points' together with the turning point, and plot them on a Cartesian plane. She was therefore questioned as follows:

> **Int:** How would you pick those other 'few' points?
>
> **Sara:** I will pick strategically, I will look at the equation, the coefficient of my variables in the quadratic function.
>
> **Int:** Okay.
>
> **Sara:** Yes, then those help me to pick on which points.

The approach of selecting points 'strategically' was vague as Sara did not clarify how the coefficients of the variables in a quadratic function help when selecting the ordered pairs to be used. Furthermore, she did not seem to realise, for instance, that there are instances when parabolas intersect the x-axis at two different points, one point, or not at all. The silence of the student teacher concerning these important aspects of sketching a curve of a quadratic function suggests that she had a limited understanding of the procedures involved.

The lack of clarity regarding the choice of the *x*-values can prove to be problematic. For instance, it is not easy to know when sufficient ordered pairs have been attained for the sketching of a quadratic curve. Moreover, it may not be easy to determine the turning point from a table of values. While Doreen indicated that it is difficult to remember formulas, Sara mentioned that turning points are computed through a process of completing the square of a quadratic function. However, she could not complete the square of a quadratic function.

Mwila and John described an alternative approach for drawing graphs of quadratic functions. This strategy, which was explained by Mubita, involves the calculation of a turning point, *x*-intercepts, and the *y*-intercept. While Mwila only specified the necessity of calculating the coordinates of a turning point and the values of the intercepts, Mubita and John included the aspect of determining the nature of a turning point. Mubita and John suggested that it is important to know in advance whether a turning point is a maximum or minimum. This suggestion exposed Mubita's conflicted understanding. During the course of the interview, he had claimed that there are four characteristic shapes of graphs of quadratic functions. At the same time, he indicated that only two types of turning points exist.

Mwila, Mubita, and John shared the view that the turning point (x, y) of $f(x) = ax^2 + bx + c$ where $a, b,$ and c are constants and $a \neq 0$, is obtained using $x = -\frac{b}{2a}$, and $y = f\left(-\frac{b}{2a}\right)$. Mwila and Mubita posited that the coordinates of the turning point are established through a process of completing the square, and could explain the process of completing the square. John, however, could not.

Mwila, Mubita, and John explained that the *x*-intercepts are values of *x* at points where parabolas intersect the x-axis, whereas the *y*-intercept was described as the *y*-value at a point where the curve crosses the y-axis. The following excerpt, which was based on the discussion that was held with Mwila, is similar to the perspectives of Mubita and John regarding the process of calculating the *x*-intercepts as well as the *y*-intercept:

> **Mwila:** Then, apart from getting the coordinates of the turning point, you need also to get the intercepts in this case, we mean where the graph is going to cross the x-axis and the y-axis. The graph is going to cross the y-axis at a point where *x* is zero because we know that every point on the y-axis the value of *x* is zero. So in the given function wherever there is *x* you put zero, then you find the value of *y*, that will be y-intercept. But for the *x*-intercept, that is what you call the zeros of the function, so in that particular case we are saying the value of *y* on the x-axis, we know that for any point on the x-axis the value of *y* is always zero. So you equate that expression to zero then you solve for *x* to determine the *x*-intercepts. Once you have those variables, those points, then you can draw the graph.

This extract shows that Mwila knew how to calculate the *x*-intercepts and *y*-intercept. He mentioned that the *x*-intercepts are obtained by solving $f(x)=0$, while the *y*-intercept is calculated by letting $x=0$ in $f(x)=ax^2+bx+c$. Mwila, Mubita, and John explained that the graphs of quadratic functions do not always intersect the x-axis. They pointed out that the discriminant b^2-4ac helps to determine the position of a parabola relative to the x-axis. The students said that $b^2-4ac<0$ implies that a parabola does not cross the x-axis, while $b^2-4ac>0$ means that the curve intersects the x-axis at two different points. They also said that when $b^2-4ac=0$, it follows that the quadratic curve touches the x-axis at only one point.

How accurately do the student teachers translate between different representations?

Changing representation from ordered pairs to a Cartesian graph

Mwila, John, and Doreen successfully transferred the representation of the relation R_1 on the set $X=\{3,4,5,6\}$ defined by the rule 'is less than' to a graph. Mubita, Sara, and Brave presented flawed Cartesian graphs for the relation R_1. Mubita's graphical representation of R_1 was inaccurate in that it included the following points, which were not supposed to be part of the graph: (4, 7), and (5, 8). However, his Cartesian graph showed that he knew that a relation defined on a discontinuous domain must not have its plotted points connected. Sara presented the graph that is reproduced below as Figure 4.22. Her explanation of how the figure was generated is given in the excerpt that is provided afterwards.

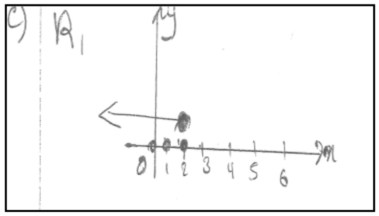


Figure 4.22: Sketch E1

Sara: Yes aah, yes I just drew a Cartesian plane and then I dotted all the numbers which are less than three that[s] why (pauses), yes I had to put an arrow by a dot (pauses again), I shaded on two because two was not part of this, then my arrow was facing on the left direction.

Int: Are you able to share with me why you did what you did?

Sara: Because of this this word I don't know if I got it correct (pauses), it says 'is less than'. So I thought it is all numbers less than three, four, five and six, that's why I did this.

Figure 4.22 and the extract confirm that Sara failed to come up with the correct ordered pairs that should have been accurately plotted on the plane. Additionally, she seemed not to realise that the arrow drawn in the Cartesian plane meant that the domain was infinite. Furthermore, she appeared not to know that the points (0, 0), (1, 0), and (2, 0), which were plotted, implied that the image of the objects 0, 1, and 2 was zero.

Brave expressed the relation R_1 as an arrow diagram before providing a Cartesian representation. That arrow diagram, and the Cartesian representation are provided below as Figure 4.23.

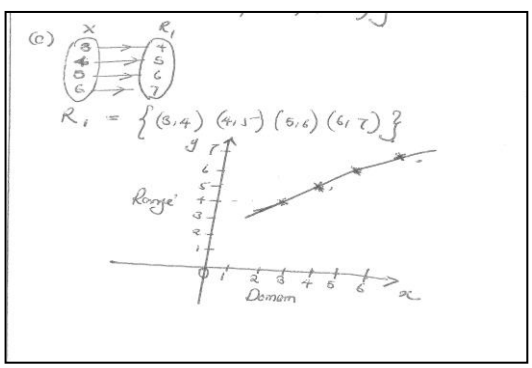


Figure 4.23: Sketch E2

Even though some of the required points such as (3, 4), (4, 5), and (5, 6) were accurately plotted, Brave's Cartesian graph was incorrect. His explanation of the process leading to Sketch E2 in Figure 4.23 is suggested by this excerpt:

> **Int:** That is the R_1 (Sketch E2) you represented in the Cartesian plane. What are some of the factors you considered?
>
> **Brave:** The rule actually that defined uh the elements of set *X*. So one of the things that I took into consideration when attempting this question was uh the rule which was defining the elements in set *X*, which is 'less than'.

Brave pointed out that the rule 'is less than' defined the elements of set *X*. This perspective suggests that either he misconstrued the test item or generally lacked an understanding of the relation, because the rule defined the relation R_1, and not the elements of the set *X*. While Brave claimed to have used the rule to produce Sketch E2, he did not reflect the following necessary points on the Cartesian plane: (3, 5), (3, 6), and (4, 6). Furthermore, instead of showing the plotted points in their unconnected form, as is expected of Cartesian graphs of relations defined on discrete domains, he drew a smooth straight line graph. Brave also erroneously included 6 in the domain of R_1, and 7 in its range.

Translating from algebraic formulas to graphical representations and vice versa

Mwila's Cartesian graph for the function $g(x) = |x|$ for $-3 \leq x \leq 2$ and $x \in Z$ was appropriate, while Mubita, John, Sara, and Brave presented inappropriate Cartesian graphs. The discussion with Doreen revealed that she could not represent the function *g* on a Cartesian plane. She said that 'to me the question wasn't clear', and declared that 'first I don't understand uh what a discontinuous domain is'. With these admissions, it is obvious that she did not have the ability to translate the function *g* to the Cartesian plane. John and Brave had a similar understanding as that of Mubita and Sara regarding the translation of the function *g* to the Cartesian plane. Figure 4.24 below is typical of the Cartesian graphs of the function *g* that were presented by Mubita, John, Sara, and Brave. This sketch is a reproduction of the graph, as presented by John.

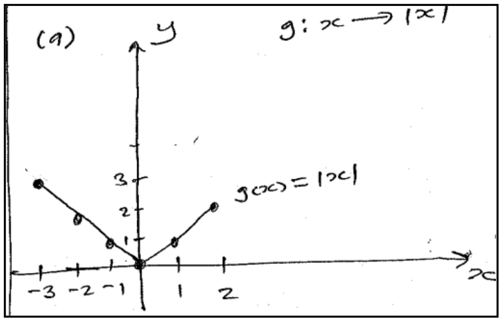


Figure 4.24: Sketch E3

At first, John explained how Sketch E3 in Figure 4.24 was drawn without realising that the graph is an incorrect Cartesian representation of the function *g*. He asserted that the graph was appropriately composed of straight lines. Consequently, John was asked about the domain $-3 \leq x \leq 2$ for $x \in Z$. This probe eventually led him to realise that the sample was inaccurate since the domain is discrete and not continuous. He then explained that the graph is supposed to consist of 'dotted lines', and not a situation where the points are connected using straight lines. Although the use of the phrase 'dotted lines' was misleading, it seemed that John was trying to indicate that the points should not have been connected.

Brave, Mubita, and Sara did not see anything wrong with the Cartesian graphs of the function *g* that are similar to Sketch E3. Mubita argued that the function *g* is continuous 'because it is defined I think for the given range; it is defined everywhere including at the point (0,0)'. When Sara was asked to explain why she had connected the points concerning the Cartesian graph of the function *g*, she merely laughed, which was taken as an indication of a lack of understanding. Regarding the domain $-3 \leq x \leq 2$ for $x \in Z$ of the function *g*, she claimed that 'mmm, this one is continuous'. Brave gave the following defence for presenting a graph similar to Sketch E3:

> **Int:** Do you have a reason why you connected the points?
>
> **Brave:** Uh the reason why is because each time actually you are plotting; each time you are plotting (pauses).
>
> **Int:** You have to connect the points?
>
> **Brave:** You have to connect the points.
>
> **Int:** Is that a rule?
>
> **Brave:** Sometimes it is not always that you can (pauses); it has to be a straight line. So, depending on the aah depending on the points plotted, it can either be a curve or a straight line.

Brave believed that every time a Cartesian graph is drawn, the points have to be connected either using a straight line or curves. This explanation exposed his superficial understanding of discrete and continuous domains. It also demonstrated that he did not have the competence to translate an algebraic function whose domain is separate to a Cartesian graph.

4.4. SUMMARY OF CHAPTER 4

Chapter 4 presented the results and analysis of the study's data on functions, and was organised into two main sections. Whereas the first section focused on the presentation and anlayses of the test data, the second section was based on the interview data. The test data was presented and analysed according to the three categories of the study's conceptual framework namely: (1) the student teachers' proficiency in Common Content Knowledge, (2) the student teachers' ability to explain and justify their reasoning, and (3) the student teachers' ability to use different representations. For each of these categories, descriptive statistics were computed and utilised to summarise the test data, and subsequently, samples of the student teachers' answers were qualitatively analysed. The interview data was deductively analysed according to the two components of the Specialised Content Knowledge in the conceptual framework: the student teachers' ability to explain and justify their reasoning, and their ability to use different representations. The next chapter provides the analysis of the data on trigonometry.

5. ANALYSIS OF THE DATA ON TRIGONOMETRY

5.1. INTRODUCTION

Chapter 5 presents an analysis of the test and interview data on trigonometry. The data analysis seeks to provide answers to the study's research questions (see Table 5.1). The analysis is made according to the categories of the study's conceptual framework (see Section 2.7 for the conceptual framework). This chapter is organised into Sections 5.2 and 5.3. Section 5.2 presents an analysis of the test data, while Section 5.3 provides an analysis of the interview data. Table 5.1 below gives a summary of the organisation of Chapter 5, after which the analyses of the data from the mathematics test are provided.

Table 5.1: Summary of the organisation of Chapter 5: presentation and analysis of the data on trigonometry

Section 5.2	**Research questions 1 and 2:**	
	How proficient are the student teachers in the Common Content Knowledge of trigonometry at secondary school level?	Presentation of results and analysis using the following categories of the conceptual framework: student teachers' proficiency in CCK, and ability to use different representations.
Section 5.3	**Research questions 2:**	
	What Specialised Content Knowledge of trigonometry at secondary school level is held by the student teachers?	Deductive analysis of interview data using the following category of SCK in the conceptual framework: student teachers' ability to explain and justify reasoning. Exploration of the interview data for any emergent themes.

5.2. ANALYSIS OF THE TEST DATA ON TRIGONOMETRY

5.2.1. Proficiency of student teachers in the CCK of trigonometry

For the topic of trigonometry, there were 15 items in the mathematics test that assessed the student teachers' proficiency in the different aspects of CCK. Marks per item ranged from two to four, while the total mark for the category was 45 (See Chapter 3, Section 3.4.3.3). Figure 5.1 below shows the student teachers' total scores in the CCK category of trigonometry.

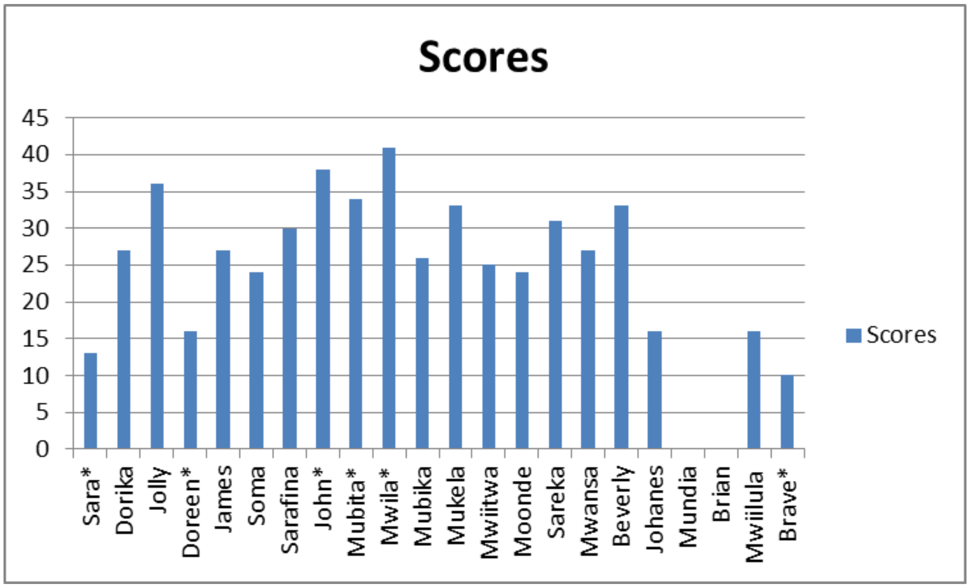


Figure 5.1: Student teachers' total scores in CCK of trigonometry

* represents interviewees in Phase 2.

The performance results show that only seven student teachers had total scores that were below the 50% pass mark, while the other 15 obtained total scores that were above 50%. These data indicate that there were more student teachers who were proficient in the CCK of trigonometry than there were those who were not. The data relating to the student teachers' performance per item is shown as Appendix 12, while each student teacher's total score in the category that assessed the CCK of trigonometry is presented as Appendix 13.

Table 5.2 below indicates that the mean mark in the CCK category was approximately 24 (53%) out of a possible total score of 45. This descriptive statistic seems to suggest that the student teachers were generally proficient in the CCK of trigonometry. Nonetheless, Table 5.2 shows that the minimum total score in the CCK category was zero, while the maximum composite score was 41, giving a range of 41. These statistics point to the existence of outlier scores that might have had an effect on the mean. Furthermore, the range and a standard deviation of 11 show that the student teachers' scores were scattered away from the mean. The substantial disparities in the student teachers' total scores and the large measures of spread suggest that the mean may not be a 'good' measure of central tendency.

Table 5.2: Summary of descriptive statistics: proficiency in the CCK of trigonometry

Items	**Number of student teachers**	**Range**	**Minimum**	**Maximum score**	**Mean score**	**Std. Deviation**
7a to 12b	22	41.00	.00	41.00	23.9545	11.27356

The minimum score in each of the 15 items in this category was zero. Nonetheless, in almost all of the 15 items (except for item 11c), there was at least one student teacher who scored all the possible marks (see Appendix 12). Figure 5.2 below shows the mean scores per item in the CCK category, while Figure 5.3 indicates the standard deviations per item.

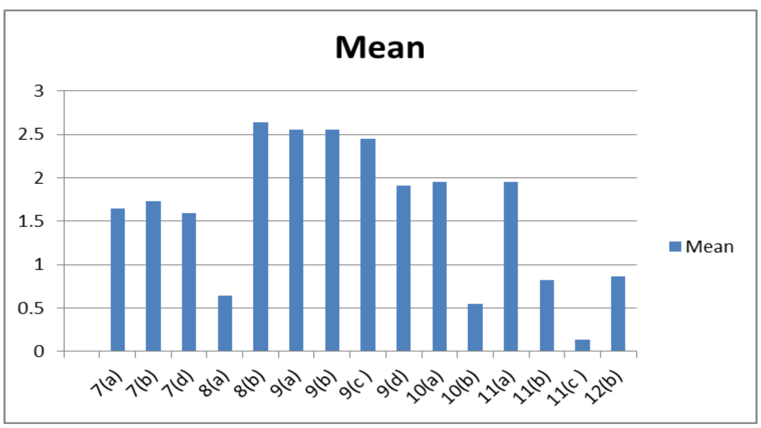


Figure 5.2: Mean scores per item in the CCK category of trigonometry

Figure 5.2 reveals that five items, 8(a), 10(b), 11(b), 11(c), and 12(b), had mean scores that are less than 1. In addition to this, two of these items, 11(c) and 12(b), had standard deviations that are less than 1, while the other three had standard deviations that are approximately 1 (see Figure 5.3 below and Appendix 14 for details). The implication of these statistics is that the student teachers did not perform well in the items mentioned above.

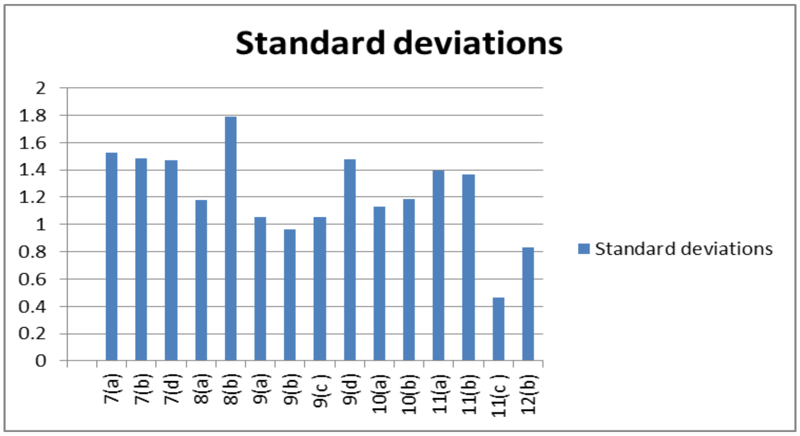


Figure 5.3: Standard deviations per item in the CCK category of trigonometry.

Another observation from Figure 5.2 is that there were six items, 7(a), 7(b), 7(d), 9(d), 10(a), and 11(a), whose mean scores were above 1, but less than 2. Figure 5.3 shows that the standard deviations for these items were also above 1, but less than 2. These statistics suggest that the student teachers' performance in these items was better than their performance in the items discussed in the preceding paragraph. In items: 8(b), 9(a), and 9(b) the mean scores were approximately 3, while for item 9(c), the mean score was above 2 but below 2.5. Items 8(b), 9(a), and 9(c) have standard deviations that are approximately 1, but less than 2. Again, these statistics suggest that the student teachers achieved satisfactory performance in these items. Descriptive statistics, such as the minimum and maximum scores, ranges, mean scores, and standard deviations per item in the CCK category of trigonometry are provided as Appendix 14.

Below is an item by item analysis of the student teachers' performance in the CCK category. In terms of the structure of the analysis, each item is firstly explained. Subsequently, the performance results are provided and discussed, followed by the analysis of the sample answers.

Item 7(a)

Calculate the value of β given that $\cos\beta = -\cos 64^\circ$ for $90^\circ < \beta < 180^\circ$.

For the student teachers to correctly solve this item, they needed to know that the cosine ratio is negative in the second quadrant where β lies. They were equally supposed to know how to find the obtuse angle that corresponds with the acute angle 64°. The student teachers also needed to know that since β lies in the second quadrant, a supplementary angle of β that lies in the second quadrant was required. Alternatively, the students could use a calculator to solve the equation.

Ten student teachers obtained a mark of zero, which indicates that these student teachers completely failed to solve this item. The other 12 student teachers managed to correctly solve this item. These results suggest that the majority of the student teachers were proficient at this task, however, the number of student teachers who could not solve the equation was significant. An answer that characterises the student teachers' inability to correctly solve this item is presented below as Figure 5.4.

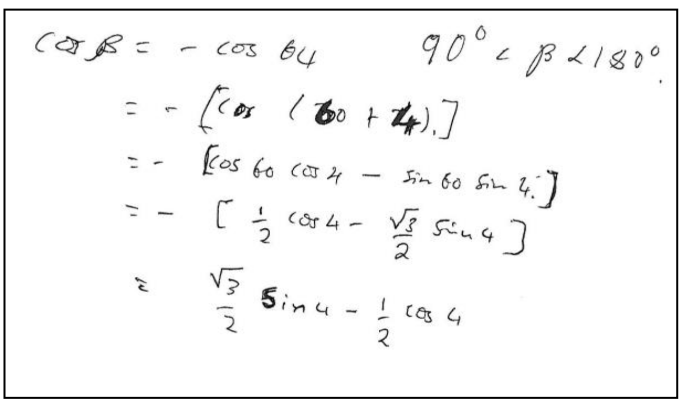


Figure 5.4: Sketch F1

Figure 5.4 above suggests that the student teacher wanted to use the idea of special angles and the 'cosine of a sum' identity to find the value of β. This is implied in the second step where $\cos 64^\circ$ is expressed as $\cos(60+4)$. He correctly expanded the expression $\cos(60+4)$ in step three to obtain $\cos 60 \cos 4 - \sin 60 \sin 4$. However, what the student teacher probably failed to appreciate was that although 60° is a special angle, 4° is not.

In the fourth step, the student teacher correctly replaced $\cos 60^{\circ}$ and $\sin 60^{\circ}$ with $\frac{1}{2}$, and $\frac{\sqrt{3}}{2}$ respectively, but could not proceed further than distributing -1 across the terms of the right hand side expression. His calculation up to this stage suggests that the student teacher did not know how to evaluate $\sin 4^{\circ}$ and $\cos 4^{\circ}$. Regardless, the expression constituting his final answer was still equal to $\cos \beta$, and this confirms that he did not know how to find the value of β.

Item 7(b)

Without the use of a calculator, find the value of $\sin 315^{\circ}$.

For this item, the student teachers were required to find the value of the trigonometric expression $\sin 315^{\circ}$ without using a calculator. One of the ways in which student teachers could have correctly solved this item is by using trigonometric identities and special angles, such as 45°, which corresponds to 315°. In this context, knowledge of the quadrant in which the angle 315° lies, as well as that of the sign of the sine ratio in that quadrant was necessary. Additionally, the student teachers needed to know the special triangles that are used in trigonometry to evaluate trigonometric expressions.

Nine student teachers were unable to provide answers to this item or, when they tried to do so, their answers were not relevant. One student teacher gave an answer that had an error and was consequently considered flawed. 12 student teachers correctly solved the item. Based on these results, it seems that the majority of the student teachers demonstrated proficiency in solving this item. Notwithstanding, the number of student teachers who were not proficient is quite significant, representing approximately 41%. Figure 5.5 below exemplifies the student teachers' flawed answers.

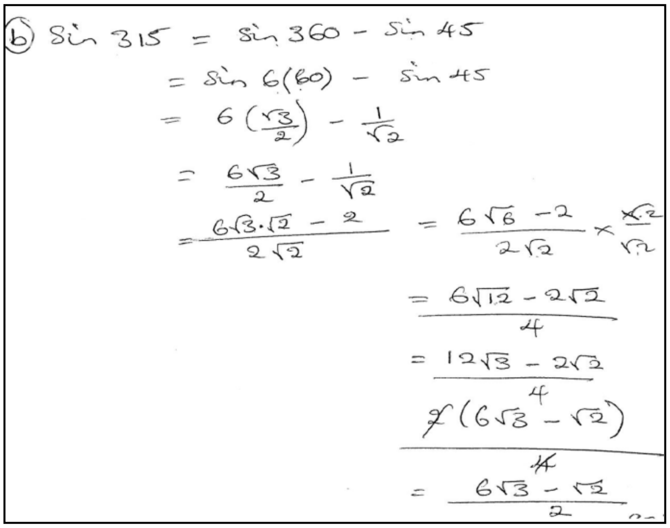


Figure 5.5: Sketch F2

Figure 5.5 suggests that the student teacher lacked a thorough understanding of what was required to evaluate $\sin 315^{\circ}$, for example, he did not to know that $\sin 315^{\circ}$ is not equal to $\sin 360^{\circ} - \sin 45^{\circ}$. It appears that the student teacher thought that trigonometric expressions can be subtracted like numbers. Although the student knew that $\sin 45^{\circ} = \dfrac{1}{\sqrt{2}}$, he did not know that $\sin 6(60)$ is not the same as $6\left(\dfrac{\sqrt{3}}{2}\right) = 6 \times \sin 60^{\circ}$.

Item 7(d)

Given that θ is an acute angle and that $\sin\theta = \dfrac{7}{25}$, find the exact value of $\cos(180^{\circ} - \theta)$.

The student teachers' ability to use their knowledge of the sine, cosine, and tangent ratios to find the value of angle θ was assessed in this item. Additionally, this item assessed the student teachers' ability to use the computed value of θ to evaluate the trigonometric expression $\cos(180^{\circ} - \theta)$. Although trigonometric identities are not a feature in the secondary school mathematics curriculum, the student teachers could use their knowledge of identities and Pythagoras' theorem in order to successfully resolve this item.

Nine student teachers failed to present solutions to this item, while two students provided answers that were flawed. The remaining 11 demonstrated proficiency by correctly solving the item. More students could provide correct solutions than those who could not provide any solution at all. However, the total number of the student teachers who were not proficient was equal to that of those who were proficient. Figure 5.6 below exemplifies the student teachers' flawed answers.

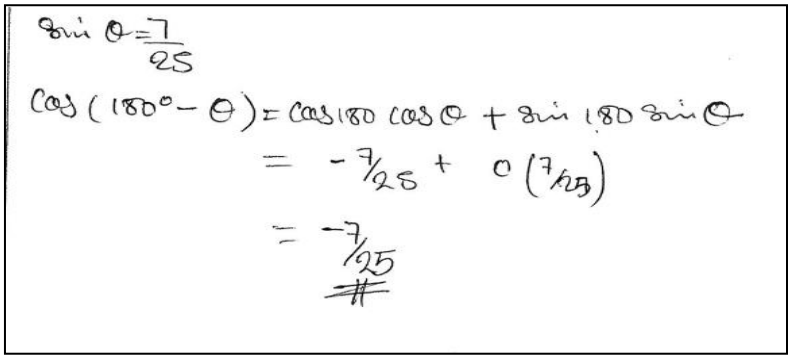


Figure 5.6: Sketch F3

The sketch above shows that the student teacher correctly expanded the expression $\cos(180^\circ - \theta)$. Similarly, he demonstrated the understanding that $\sin 180^\circ$ is equal to zero, and proficiently substituted $\sin\theta$ by $\frac{7}{25}$. Furthermore, the second line suggests that he understood that $\cos 180^\circ = -1$, as the minus sign can be seen. However, this student failed to use $\sin\theta = \frac{7}{25}$ to find the correct value of $\cos\theta$, which should have been substituted into the second step. Another student teacher correctly expanded the expression $\cos(180^\circ - \theta)$, and appropriately applied Pythagoras' theorem to evaluate $\cos\theta$. Thinking that $\cos 180^\circ = +1$ and not -1 categorised his answer as flawed. Knowledge of quadrants or the graph of the cosine function would have assisted him in realising that the value of $\cos 180^\circ$ is negative and not positive.

Item 8(a)

The University of Zambia library denoted L, is 8 metres away on a bearing of 040° from a Mathematics Department office denoted M. Draw a diagram depicting this information and determine how far L is north of M.

The component of item 8(a) belonging to the category 'proficiency in CCK' constituted the following: Determine how far L is north of M. The other aspect, which required the student teachers to draw a diagram depicting the information contained therein, is analysed under the category 'use of different representations' in Section 5.2.2.

For the current category, item 8(a) assessed the student teachers' ability to apply an appropriate trigonometric ratio to calculate the required length on a right angled triangle. The student teachers were also being assessed concerning the definitions of trigonometric ratios as they apply to a right angled triangle.

16 student teachers (approximately 73%) failed to demonstrate an understanding of what was required, and consequently presented no answer. Two student teachers (9%) presented flawed answers. The remaining four student teachers (18%) provided answers that were considered sound. These results suggest that the majority of the student teachers could not use an appropriate trigonometric ratio to calculate the length of a triangle when that triangle has not been drawn for them. The two flawed answers are presented and analysed below as Figure 5.7 and Figure 5.8.

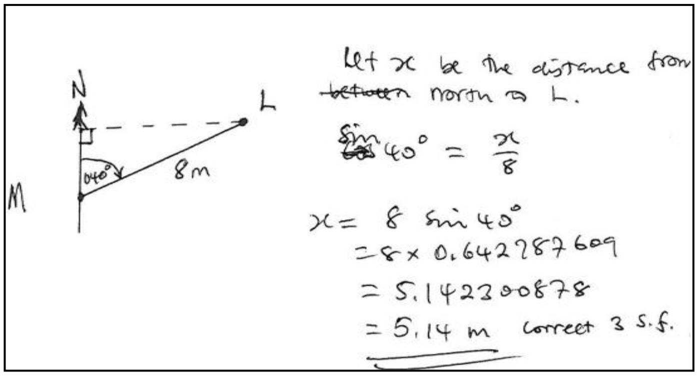


Figure 5.7: Sketch F4

This answer can be analysed from the following two perspectives:

(1) After representing the information on a diagram, the student teacher might have failed to identify the side whose length he needed to calculate. Instead of computing the correct side MN on his diagram, he wrongly calculated NL. With this line of argument, the student appeared to understand that $\sin\theta = \frac{opp}{hyp}$ as he indicated that $\sin 40^\circ = \frac{x}{8}$. It becomes obvious then that his x represented the side NL (opposite side according to his diagram), and the hypotenuse was accurately identified as being a length of 8. It can therefore be speculated from the calculation (including the diagram) that his interpretation of the phrase 'how far L is north of M' is what was flawed. This may probably be a case of 'how far L is east of M'.

(2) Sketch F4 can equally be interpreted as a case of a student teacher who did not know the appropriate trigonometric ratio to use. At first, he had written $\cos 40^\circ = \frac{x}{8}$, and cancelled the cos and replaced it with sin. What is interesting about this is that the cosine ratio was actually the applicable trigonometric ratio, which could have yielded him a correct answer. Had he evaluated $8 \times \cos 40^\circ$, the answer would have been correct. The cancellation suggests doubt or a lack of conviction on the part of the student teacher as to which trigonometric ratio was appropriate in the circumstance. Figure 5.8 is presented below and analysed hereafter.

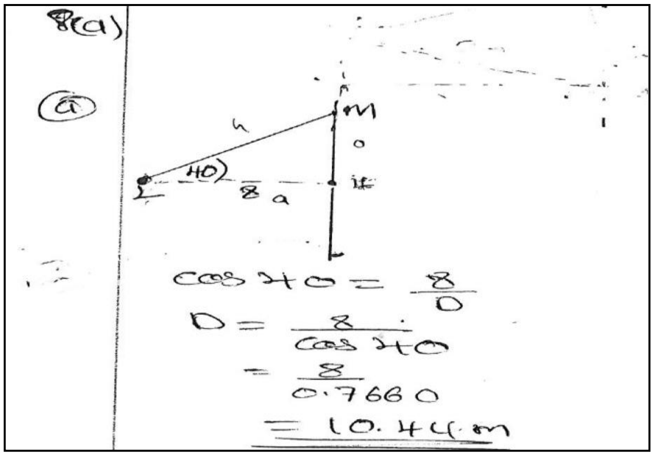


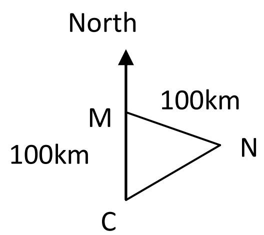
Figure 5.8: Sketch F5

Figure 5.8 presents insight into the student teacher's struggle to use trigonometric ratios to calculate the lengths of right angled triangles. Whereas the diagram drawn to depict the information in item 8(a) is inaccurate (Section 5.2.2 on item 8a), the calculation suggests that the student teacher did not know that $\cos\theta = \frac{adj}{hyp}$. For example, he correctly indicated $\cos 40^\circ$, but wrongly equated it to $\frac{8}{o}$, which clearly is the adjacent side over the opposite side (the opposite side is denoted using the letter O on the diagram).

Another possibility is that the student teacher knew the definition of the cosine ratio and considered O (which is the opposite side on his diagram) as the hypotenuse and 8 as the adjacent side. This perspective points to an incorrect identification of the sides of a right angled triangle.

Item 8(b)

The diagram below shows the town of Choma denoted C, which is 100 km due south of the town of Monze denoted M. Given that Namwala District, denoted N, is 100 km away from Monze on a bearing of 130°, label, on the diagram, the angle that represents the bearing of 130° on which the District of Namwala lies using the letter **x**. Find the 3-figure bearing of the town of Choma from the District of Namwala.



This item assessed the student teachers' ability to identify the position of the angle that represents the described bearing. Additionally, the student teachers were being assessed on their ability to use their knowledge of angles in general to determine a 3-figure bearing of one point from another. The item also assessed whether the student teachers understood that a 3-figure bearing is an angle quoted in three digits, and that it is measured in a clockwise direction from the north up to the line connecting the two points.

13 student teachers (59%) were proficient at this task in that they were able to correctly identify the position of the angle 130° and were able to provide the correct bearing of C from N. Three students (approximately 14%) were only able to identify the location of the angle 130°, but completely failed to determine the bearing of C from N. One student teacher did not indicate the position of the bearing on which Namwala lay, but correctly solved the bearing of C from N. Five student teachers (about 23%) failed to identify the position of the bearing on which Namwala lay as well as to calculate the correct bearing of C from N. Based on these results, it would appear that most of the student teachers were proficient at this task. Figure 5.9 below depicts a student teacher who had no ability to identify the position of angle 130°.

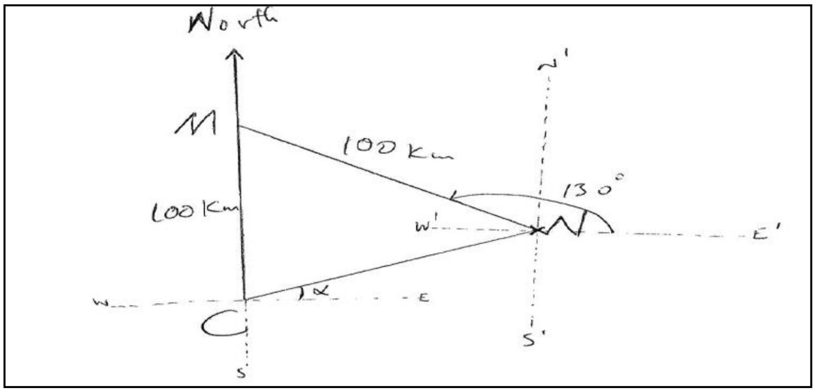
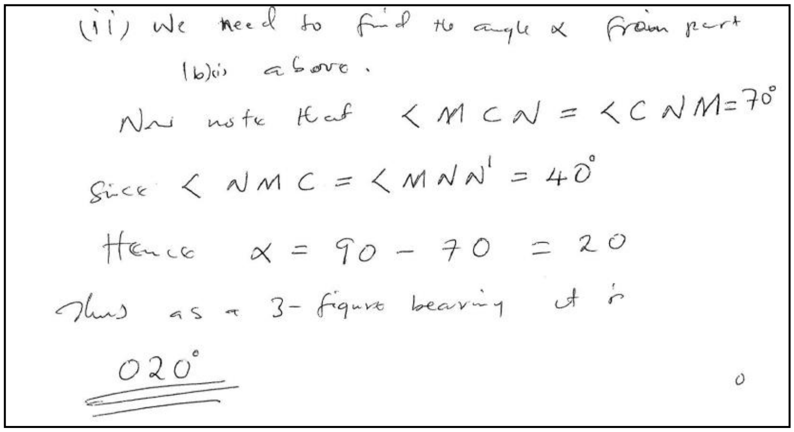


Figure 5.9: Sketch F6

Figure 5.9 shows that the student teacher wrongly marked the angle at the point N. The arrow and the compass points drawn at the point N confirm that he was measuring the bearing in an anti-clockwise direction, and not from the north. This means that the student teacher lacked the understanding that bearings are measured in a clockwise direction from the north. Figure 5.10, which presents Sketch F7, depicts the calculation that the student teacher presented.


(ii) We need to find the angle α from part (b)(i) above.

Now note that < MCN = < CNM = 70°

Since < NMC = < MNN' = 40°

Hence α = 90 − 70 = 20

Thus as a 3-figure bearing it is

020°

Figure 5.10: Sketch F7

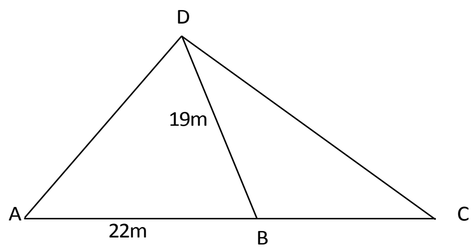
This sample suggests that the student teacher incorrectly identified α in Sketch F6 as the required bearing. Although the student teacher understood that the triangle MCN is an isosceles, he calculated a wrong value for the sizes of the base angles. It seems that his failure to locate the position of 130° contributed to his subsequent inability to realise that $N\hat{M}C = 50^{\circ}$. A correct computation of this angle could have enabled him to obtain the correct value for $M\hat{C}N$ and $C\hat{N}M$. Nonetheless, his final answer would still be wrong because he could not identify the position of the bearing that was required.

Items 9(a), 9(b), 9(c), and 9(d)

The foremost purpose of these four items was to assess the student teachers' ability to correctly apply specific rules such as the sine and the cosine rules. However, the student teachers were at liberty to use other appropriate methods at their disposal in the resolution of each specific item. Secondly, although the four items assessed different aspects of trigonometry, they were all reliant on the following information and diagram:

[Give the answers to 3 significant figures where possible]

The diagram below shows stopcocks connected to some water pipes at the University of Zambia, Great East Road Campus. The stopcocks are represented by points A, B, C, and D while the water pipes are represented by straight lines AB, BC, AD, DC, and BD.



Given that AB=22m, BD=19m, $A\hat{B}D = 60^\circ$ and $B\hat{C}D = 34^\circ$, calculate:

(a) The length of the water pipe from stopcock B to stopcock C.

(b) The length of the water pipe from stopcock A to stopcock D.

(c) The area bounded by the water pipes AB, BD, and AD.

(d) The shortest distance from stopcock B to the water pipe AD.

Item 9(a)

This item assessed the student teachers' ability to apply the sine rule to calculate the length of a side on a non right-angled triangle when one side and two angles are provided. The student teachers' ability to correctly state the sine rule was also assessed. In this context, it was necessary that student teachers calculate $\hat{DBC}$ using its supplementary $\hat{ABD}$. Angles $\hat{DBC}$ and $\hat{BCD}$ (which was provided) would then enable the students to find the value of $\hat{BDC}$. It is the two angles $\hat{BDC}$ and $\hat{BCD}$, together with the length of BD (provided) which would then enable the student teachers to apply the sine rule.

18 student teachers (almost 82%) demonstrated their proficiency by correctly using the sine rule to find the length of the side BC. One student teacher presented an answer that had minor mistakes, while three other student teachers (approximately 14%) completely failed to provide any calculations. These results suggest that the majority of the student teachers were proficient in using the sine rule to calculate the length of a side on a non right-angled triangle.

Item 9(b)

In this item, the student teachers were assessed regarding their ability to correctly state and apply the cosine rule to find the length of AD on a non right-angled triangle when two sides and an included angle are known.

17 student teachers (77%) demonstrated their proficiency in this task in that they were able to both correctly state and use the cosine rule to compute the unknown side of a non right-angled triangle. Two student teachers' calculations suggested that they knew the cosine rule and how to use it, except for the fact that their calculations had minor errors. These student teachers represented 9% of the sample. One student teacher (5%) who had a general idea concerning the cosine rule presented a calculation that had major mistakes, and two student teachers (9%) did not provide any answer. From these results, it seems that most of the student teachers were proficient in using the cosine rule to compute unknown lengths of the sides of non right-angled triangles.

Item 9(c)

This item assessed the student teachers' ability to utilise the appropriate methods to correctly calculate the area of an implied non right-angled triangle ABD.

16 student teachers scored a mark of 3 each out of a possible mark of 3. This means that these student teachers were able to proficiently solve the area bounded by the pipes AB, BD, and AD. Three student teachers provided answers that were relevant, but these answers had minor errors. Consequently, each one of these three student teachers was allocated a mark of 2. The remaining three student teachers did not present any answer for this item. It therefore appears that the majority of the student teachers were proficient in calculating the area of a non right-angled triangle. A common finding of this item was that most of the student teachers applied the trigonometric formula to calculate the required area.

Item 9(d)

This item assessed the student teachers' ability to both identify the shortest distance from the point B to the line AD, and apply trigonometry to calculate that shortest distance. The student teachers' understanding that the shortest distance from a point to a line is actually the perpendicular distance from such a point to a line was also assessed by this item.

14 student teachers (approximately 64%) scored full marks. This indicates that each one of these student teachers proficiently computed the shortest distance from the point B to the line AD. The remaining eight student teachers (36%) provided incorrect, irrelevant or no answers and therefore scored zero. These results indicate that there were more student teachers who were proficient in identifying and calculating the shortest distance from a point to a line than those who were unable to do so. However, the number of students who showed a lack of proficiency in the task was significant (36%).

Item 10(a)

If a triangle XYZ is such that XY=7cm, YZ=6cm, and angle X is equal to 44 degrees, find, to one decimal place, two possible values for angle Z.

This item was an example of an ambiguous case for which more than one possible value for angle Z existed. Principally, the item assessed whether the student teachers knew that the sine rule can be applied to find an acute angle of a non right-angled triangle in which two sides and a non-included angle are provided. Furthermore, the item looked at whether the student teachers understood that there is another appropriate value for angle Z that lies in the second quadrant.

Nine student teachers (41%) correctly applied the sine rule to calculate the smaller of the two values of angle Z, and then correctly solved the supplementary angle. Four student teachers (18%) did not provide any answer, while seven student teachers (32%) applied the sine rule to calculate the acute angle. However, among these seven were student teachers who either did not present any calculation for the other value of angle Z or, when they attempted to do so, their calculations were flawed. One student teacher calculated the size of the third angle of the triangle XYZ instead of subtracting the computed acute angle from 180°. By implication, this student teacher lacked the understanding that the second value of angle Z lies in the second quadrant. Two student teachers (9%) presented calculations that were relevant, although their calculations contained errors. For instance, one of these students indicated that sin Z is equal to 55.1° instead of stating that angle Z is equal to 55.1°.

Moreover, this student teacher assumed that triangle XYZ is right-angled at Y. Consequently, he calculated the second possible value of Z by subtracting the sum of 90° and 44° from 180°. His may have been a case of a student teacher whose knowledge of triangles was limited to right-angled triangles. The results discussed in item 10(a) suggest that the majority of the student teachers understood that the sine rule can be applied to find an unknown angle of a non right-angled triangle when two sides and a non-included angle are provided. What seemed to have been problematic for the student teachers was the computation of the other value of angle Z.

Item 10(b)

Giving the answer to one decimal place, calculate the smallest of the angles of a triangle whose sides are in the ratio 2: 4: 5.

This item assessed the student teachers' ability to apply the cosine rule to find the unknown angles of non-right angled triangles. The item included the assessment of the student teachers' ability to determine that the smallest angle in a triangle is usually opposite to the shortest of the three sides. At the same time, it examined if the student teachers were able to relate the ratios of the sides of a triangle to the lengths of the sides.

Four student teachers (18%) correctly applied the cosine rule to calculate the smallest angle of the triangle. 18 student teachers (82%) presented incorrect answers. These results suggest that the majority of the student teachers did not know that the cosine rule can be used to calculate the required angle. Most of the student teachers provided calculations that are similar to the one replicated below as Figure 5.11.

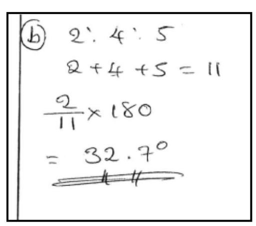
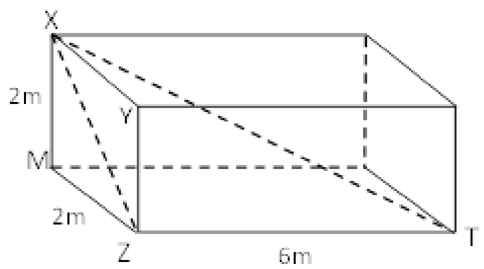


Figure 5.11: Sketch F8

Figure 5.11 demonstrates that the student teachers failed to relate the ratio of the sides to the lengths of the triangle. It seems that they misinterpreted the ratio of the sides of a triangle to represent the ratio of the angles of the triangle. The students did not seem to know that the cosine rule was applicable.

Items 11(a), 11(b), and 11(c)

These three items were all based on the information relating to the 3-dimensional solid below: The figure below shows a petrol tank which is made in the form of a cuboid.



Given that one end of the petrol tank is a square XYZM of side 2 metres, ZT=6 metres, and that XT is a diagonal of the tank, calculate:

(a) The length of XT giving the answer to 1 decimal place.
(b) The angle between XT and the plane XYZM to the nearest degree.
(c) The shortest distance between the plane XTM and YZ to 1 decimal place.

Item 11(a)

This item assessed the student teachers' ability to apply Pythagoras' theorem. For the student teachers to successfully solve this item, they had to identify the appropriate right-angled triangles from the cuboid on which to apply Pythagoras' theorem. For example, they could initially use Pythagoras' theorem on triangle XMZ to find the length of XZ, which they could then use together with length ZT to calculate for XT.

13 student teachers (59%) correctly applied Pythagoras' theorem to calculate the length of XT. Two student teachers (9%) provided answers that suggest that they struggled to identify the relevant right angled triangles from the cuboid. Consequently, their answers were incorrect even though they knew how to apply Pythagoras' theorem. One of these two student teachers committed arithmetical errors, and seven students (about 32%) did not solve this item. Although most of the student teachers demonstrated their proficiency in using Pythagoras' theorem, the number of those who failed to present an answer is important.

Item 11(b)

This item first and foremost assessed the student teachers' ability to locate, on the cuboid, the angle that lies between XT and the plane XYZM. Secondly, the item assessed the students' proficiency in using an appropriate trigonometric ratio (or any other appropriate method) to calculate for the value of the required angle.

16 student teachers (approximately 73%) failed to provide a correct answer to this item. Some of these students did not provide any calculation, while others made an incorrect identification of the required angle, and consequently ended up presenting an incorrect calculation. Alternatively, six student teachers (27%) successfully computed the required angle using the appropriate trigonometric ratios. These results suggest that the majority of the student teachers were not proficient in the identification and calculation of the required angle.

Item 11(c)

This item looked at the student teachers' ability to identify the shortest distance between the plane XTM and YZ in a 3-dimensional figure. Furthermore, the item gauged the student teachers' capacity to apply the appropriate trigonometric ratios to find the shortest distance. Either Pythagoras' theorem and sine ratio, or the tangent and sine ratios could be applied to solve this item.

20 student teachers (91%) were unable to calculate the shortest distance between the plane XTM and YZ. Two student teachers (9%) provided relevant answers, but the calculations of one of these contained a major error. While the second student teacher’s calculations were appropriate, his final answer was incorrectly expressed to the required one decimal place. The performance statistics suggest that most of the student teachers were incapable of recognising opportunities that require the use of trigonometric ratios to compute lengths in solids. Figure 5.12 shows how one of the student teachers solved item 11 (c).

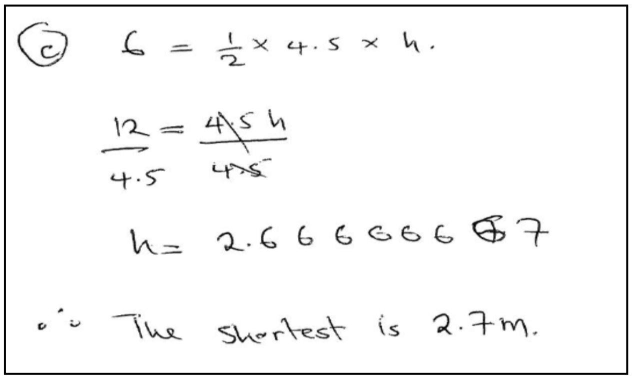


Figure 5.12: Sketch F9

Sketch F9 suggests that the student teacher was thinking in terms of the area of triangle XZT, as seen in the first step of the calculation where the student wrote $6 = \frac{1}{2} \times 4.5 \times h$. By implication, this student teacher seemed to be of the view that the shortest distance is the perpendicular distance to the base XT. What is unclear about Sketch F9 is how the student obtained the number six on the left hand side. Moreover, the final answer does not come close to the correct shortest distance between the plane XTM and YZ. These considerations suggest this student’s lack of aptitude.

Item 12(b)

The table of values below represents the function *f* for $0^{\circ} \leq \theta \leq 360^{\circ}$. State the range and period of the function *f*.

θ	0°	90°	180°	270°	360°
$f(\theta)$	0	1	0	-1	0

This item investigated whether or not the student teachers were capable of determining the range and period of the sine function, which is represented as a table of values. In addition to this, the item assessed whether the student teachers were capable of determining the period of the sine function in degrees consistent with the domain that had been provided. Furthermore, the item looked at the student teachers' conceptions of the range of the sine function.

Nine student teachers (41%) were incapable of stating the correct range and period of the sine function *f*. A total of six student teachers (27%) correctly indicated both the range and the period of the function *f*. Each one of the remaining seven students (32%) gave either the correct range or the correct period. Most of the student teachers presented the period of the function *f* as a multiple of π even when the domain of the function was cited in degrees. This suggests that the student teachers thought that the angles of trigonometric functions must be expressed in terms of π. Based on the performance statistics, it can be concluded that the majority of the student teachers were not familiar either with the range or period of the sine function.

Two of the answers that suggest the student teachers' lack of understanding of the range and period of the sine function are presented and elaborated: 'the range are 0, 1, -1' and 'the period is 180°'. The student teacher's answer suggested that he did not know that the range of the sine function is not discrete, but that it is the continuous interval $-1 \leq f(\theta) \leq 1$. With respect to the period, the student teacher gave the period of the tangent function and not that of the sine function. This shows that either the student did not know that the period of the sine function is 360° or that *f* was the sine function.

The second student teacher indicated that the range of the function f is $0^\circ \leq \theta \leq 360^\circ$ and that its period is 'from -1 to 1'. This student did not know that the range of the sine function is not in degrees or that the period involves the number of degrees or radians. These answers showed the student teacher's serious lack of understanding of the concepts of range and period of trigonometric functions.

5.2.2. The student teachers' ability to use different representations in trigonometry

The test items under the 'use of different representations' category were intended to investigate the student teachers' competence in the use of alternative representation of trigonometric concepts. This category consisted of four items, each of which had a possible mark of 3. An exception to this was item 7(c), which assessed two components, each of which was allocated a possible mark of 3. The total mark for the category was 15. The student teachers' achievement scores are shown in Figure 5.13 below.

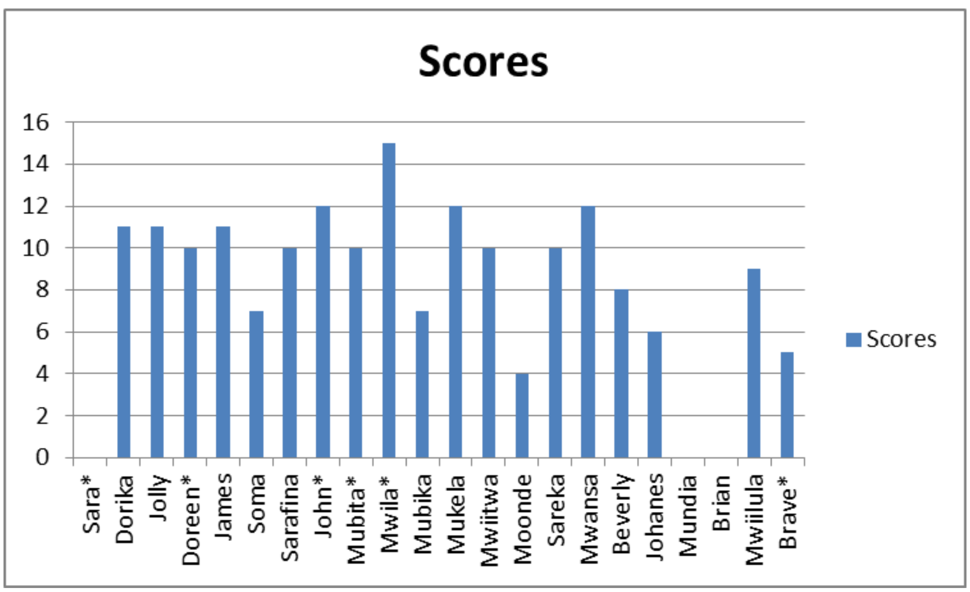


Figure 5.13: Student teachers' total scores in the 'use of different representations' category of trigonometry

* represents students who agreed to be interviewed in Phase 2.

14 student teachers scored above the 50% pass mark, while eight student teachers achieved total scores that were below the 50% mark. The mean score in this category was 8 out of a possible score of 15. This result suggests that, largely, the student teachers attained total scores that were approximated at 53%. It would appear that the majority of the student teachers were competent in working with the different representations that were assessed by the items in this category.

Nevertheless, the minimum composite score of the category is a mark of zero, while the maximum composite score is 15, thereby giving a range of 15. While a range does not include all the scores in its computation, the range of 15 can suggest that there are scores that are spread away from the mean. Moreover, the computed mean is not a good representative of scores such as 12 and 15, which were achieved by some of the student teachers. The performances of each one of the 22 student teachers per item, their composite scores, as well as the category mean score are shown as Appendix 15. In order to provide further insight into the student teachers' ability to use specific representations in the topic of trigonometry, an item by item analysis is discussed hereafter. Items will firstly be presented and explained, followed by a brief highlight of the student teachers' performance. The sample answers will then be presented and analysed.

Item 7(c)

Draw two special triangles which are usually used in trigonometry to calculate values of trigonometric expressions.

This item investigated the student teachers' ability to accurately draw the two triangles. One of the triangles consists of the angles: 45°, 45° and 90°, and the following lengths of the sides: $1, 1, \text{and } \sqrt{2}$. For the second triangle, the student teachers were required to show the following angles: 30°, 30°, 60° and 60°, and the following lengths of sides: $2, 2, 1, 1, \text{and } \sqrt{3}$. However, it was acceptable for the students to indicate the angles: 30°, 60° and 90°, and the lengths: $2, 1, \sqrt{3}$. A triangle was considered accurately drawn if all the angles and sides had their dimensions correctly labelled.

Eight student teachers (36%) accurately drew the two triangles. Five student teachers (approximately 23%) accurately drew one of the two triangles and labelled it accordingly, while for the second triangle, they omitted providing some of the dimensions. Three student teachers (14%) drew the two triangles with some of the dimensions provided, except not all the dimensions were labelled. One student teacher managed to accurately draw and correctly label one of the two triangles, while another student teacher drew one triangle that was not completely labelled. There were four student teachers (18%) who completely failed to provide any drawing whatsoever of the two triangles. These results suggest an average performance in that there were student teachers who showed competence in completing this task, while others demonstrated that they were generally knowledgeable of what was being assessed. Figures 5.14 and 5.15 below depict the student teachers' flawed answers.

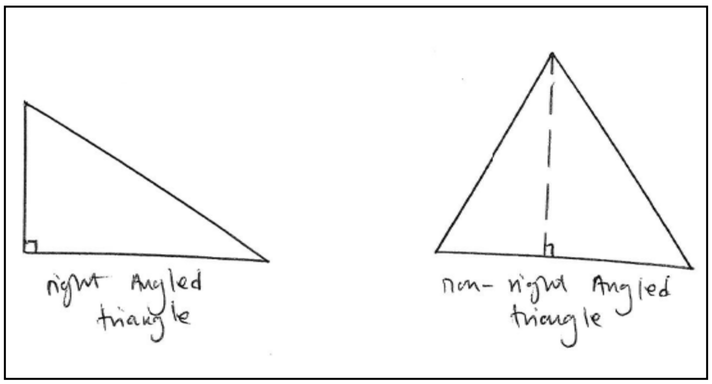


Figure 5.14: Sketch G1

Figure 5.14 shows two triangles whose angles and sides were not labelled by the student teacher. Because of the absence of the dimensions, the triangles were considered incomplete and consequently, flawed. The right angled triangle drawing would have been accurate if the angles 45°, 45° and 90°, as well as the sides $1, 1,$ and $\sqrt{2}$ had been appropriately labelled. Similarly, the second drawing would have been considered accurate if the angles 30°, 30°, 60° and 60° and the sides $2, 2, 1, 1, \sqrt{3}$ had been appropriately labelled. Figure 5.15 is shown below:

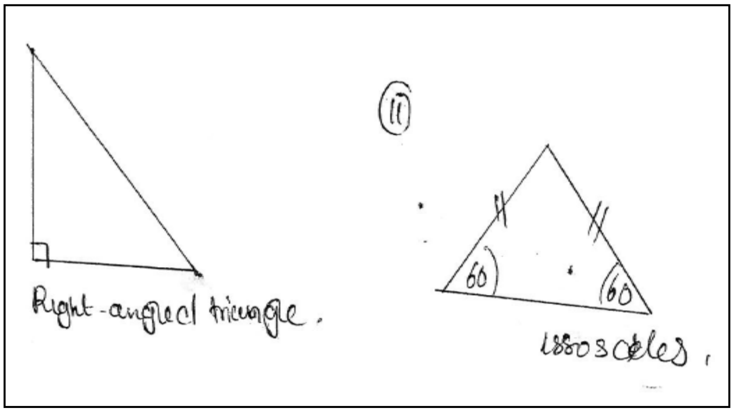


Figure 5.15: Sketch G2

The student teacher's failure to label the angles 45° and 45° and the sides $1, 1,$ and $\sqrt{2}$ made the first diagram incomplete, and therefore inaccurate. While it is implied in the second triangle that the angle that was not labelled is 60°, the student teacher's conflicted understanding is exposed by the label 'isosceles'. The equality symbols on the two sides of the triangle indicate that this was not done in error. It appears that the student teacher lacked the understanding that the second triangle is an equilateral triangle. Furthermore, the incomplete labelling of the second triangle suggests that the student teacher lacked an in-depth understanding of special triangles.

Item 8(a)

The University of Zambia library denoted L is 8 metres away on a bearing of 040° from a Mathematics Department office denoted M. Draw a diagram depicting this information.

This item gauged the student teachers' competence in representing information involving bearings in a diagram. The correct positioning of the labels for the library (L), mathematics department (M), and the bearing 040° were required.

12 student teachers successfully represented the information in a diagram. Nine student teachers did not provide any answer, while one student presented a diagram that was flawed. These results suggest that the majority of the student teachers were competent in representing information on bearings in a diagram. Nonetheless, the number of students who lacked competence is significant (45%). Two examples that suggest their lack of competence are reproduced below as Figures 5.16 and 5.17.

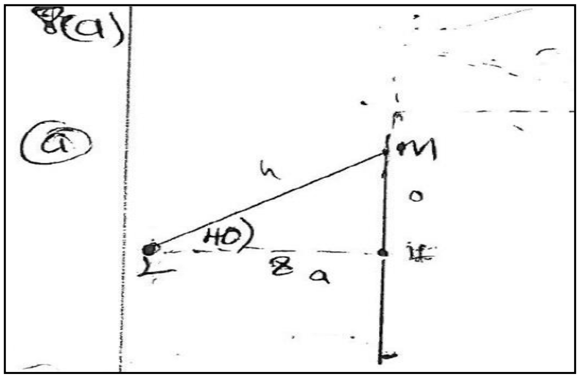


Figure 5.16: Sketch G3

Figure 5.16 suggests that this student teacher did not understand the phrase ‘L is 8 metres away on a bearing of 040° from … M’. The 8 metres is wrongly indicated as horizontal distance from L to the vertical line passing through M. The hypotenuse is indicated to be *h*, yet this is the distance that is supposed to be the length of 8 metres according to the information in the item. In addition to this, the bearing 040° is inappropriately positioned as an angle of elevation when it is supposed to be between the north and the line connecting M and L.

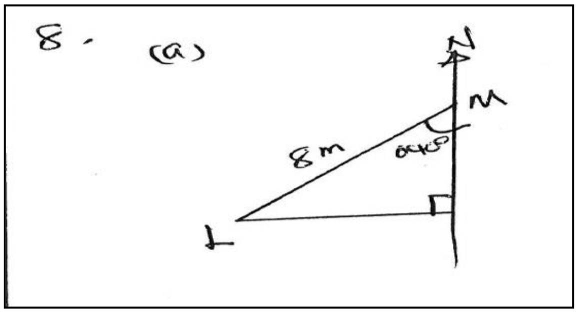


Figure 5.17: Sketch G4

Figure 5.17 above shows the incorrect placement of the L and M, as well as the bearing on which the library (L) lies. Although the hypotenuse is correctly indicated as 8 metres, it is the library (L) that is supposed to be ‘8 metres away from … M’ and not the other way around; the positions of M and L were interchanged. Furthermore, the bearing 040° on which the library is located is supposed to be measured from north to the line connecting L and M. Figure 5.16 and Figure 5.17 suggest that these student teachers could not accurately represent the information using bearings in diagrams.

Item 12(a)

Sketch the graph of the function $y = \sin\theta$ for $0^\circ \leq \theta \leq 360^\circ$.

This item assessed the student teachers' ability to translate the sine function from a symbol representation to a graphical representation when the domain has been specified. It was the intention of this item to also investigate whether the student teachers were able to work consistently with angles quoted in degrees, or whether their notion of the domains of trigonometric functions was dominated by the multiples of π. An accurate translation included the following: complete and correct labelling of the XY plane, plotting of the major points like (0°, 0), (90°, 1), (180°, 0), (270°, -1) and (360°, 0), an accurate sinusoidal curve within the specified domain, and the consistent use of degrees.

Five student teachers (23%) did not present any graphs, and this suggested that they did not know how to translate the symbol sine function to a graphical representation. Seven student teachers (32%) correctly translated the sine function from a symbol to a graphical representation. Ten student teachers (45%) presented Cartesian graphs that had errors/omissions, and this suggests that they had a partial understanding of the graph of the sine function. An example of the graphs that were presented by some of these ten student teachers is given below as Figure 5.18.

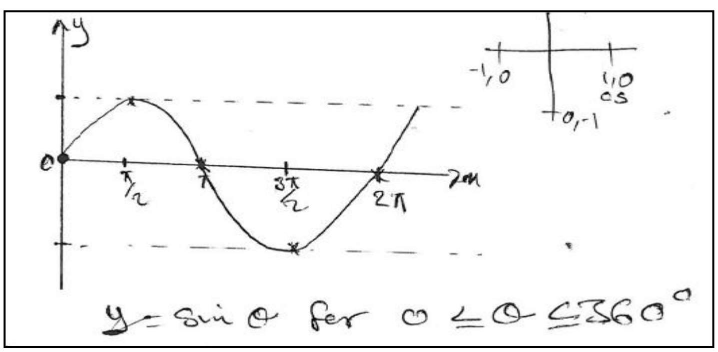


Figure 5.18: Sketch G5

Figure 5.18 depicts the drawing of a student teacher who appeared to know how to represent the algebraic sine function as a graph. However, the following three observations were made regarding this graph: (1) while the item gave the domain of the function in degrees, the horizontal axis of the graph consists of angles that are presented as multiples of π, (2) the graph does not reflect 1 and -1 as the maximum and minimum values respectively of the sine function, and (3) the curve that was drawn went beyond the provided domain $0^\circ \leq \theta \leq 360^\circ$.

The provision of angles in degrees suggested that the student teacher's conception of the domains of trigonometric functions was dominated by angles expressed as multiples of π. Additionally, the absence of the minimum and maximum values of $\sin\theta$ may be a case of omission or that the student was unsure about these. The latter may suggest the student teacher's familiarity with the appearance of the sine curve without any in-depth understanding. The extension of the curve beyond the specified domain suggests the student teacher's failure to pay attention to the domain, and yet a domain is a critical component in the definition of any function.

Item 12(c)

Sketch the graph of $y = \tan\theta$ for $0^\circ \leq \theta \leq 360^\circ$.

This item gauged the students' ability to accurately change the representation from an algebraic form to a graph when a definite domain has been provided. The item also assessed the student teachers' aptitude to work consistently with a domain whose angles are expressed in degrees. An accurate representation involved the following: (1) the correct and complete labelling of the XY plane, (2) the accurate plotting of the major points such as (0°, 0), (180°, 0), (360°, 0), (3) the clear indication of the angles at which the tangent function is undefined, and (4) showing the characteristics of the graph between the angles 0° and 90°, 90°, and 270°, 270° and 360°.

11 student teachers (50%) did not provide any graph of the tangent function. Ten student teachers (45%) drew flawed graphs, and only one student teacher (5%) accurately and completely sketched the tangent function in the specified domain. It therefore appears that the majority of the student teachers lacked an in-depth understanding of the tangent function in the category 'use of different representations'. Figure 5.19 below is a duplicate of a graph that was presented by one of the student teachers.

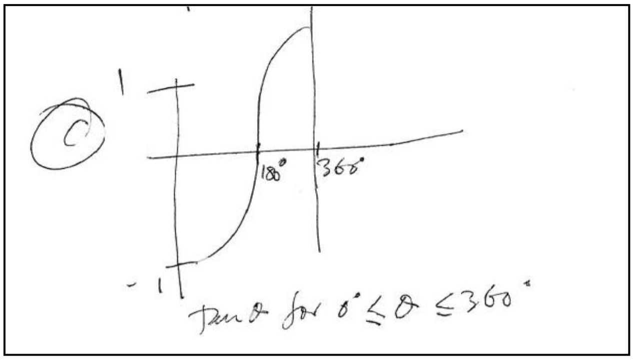


Figure 5.19: Sketch G6

Figure 5.19 shows -1 and 1 on the vertical axis, which implies that the curve has maximum and minimum values, and yet the curve that was drawn exceeds the value of 1. The curve does not meet most of the four areas that the item assessed. The vertical axis drawn through 360° is inexplicable.

5.3. RESULTS AND ANALYSIS OF THE INTERVIEWS ON TRIGONOMETRY

5.3.1. Analysis of the student teachers' ability to explain and justify their reasoning in trigonometry

This section presents the analysis of the student teachers' ability to explain and account for their reasoning concerning concepts in trigonometry. The presentation of the analysis is directed by two questions that encapsulate the essence of the descriptors of the category 'ability to explain and justify reasoning': How do the student teachers explain their understandings of concepts? and, What justifications do the student teachers provide for their reasoning? A summary of the ideas that comprise the analysis is provided in Table 5.3.

Table 5.3 Concepts comprising ability to explain and justify reasoning in trigonometry

Guiding questions	**Concepts**
How do the student teachers explain their understanding of concepts?	Quadrants in which the trigonometric ratios are positive or negative; test item involving knowledge of quadrants and special angles: 30°, 45°, and 60°; student teachers' understanding of the sine rule; student teachers' understanding of the cosine rule; student teachers' understanding of formulas for the computation of the areas of triangles (including the concept of area); student teachers' explanations based on a simple trigonometric equation; explaining methods used to compute an angle of a non-right angled triangle; student teachers' understanding of the periods of the sine, cosine, and tangent functions; student teachers' understanding of the differences between the sine and tangent functions.
What justifications do the student teachers provide for their reasoning?	Why trigonometric ratios are positive or negative in the four quadrants; justifications relating to characteristics of the sine, cosine, and tangent functions (including angles for which trigonometric functions are defined or undefined, and graphs of trigonometric functions); maximum and minimum values of the sine, cosine, and tangent functions;

How do the student teachers explain their understanding of concepts?

Quadrants in which the trigonometric ratios are positive or negative

Mwila, Mubita, John, and Doreen were not only able to identify which trigonometric ratios are positive or negative in each quadrant, but also described the four quadrants. While Brave had a challenge identifying the sign of the cosine ratio in the fourth quadrant where he claimed it is negative, Sara showed a lack of capacity to identify the signs of the sine and cosine ratios in the second and fourth quadrants. For the purposes of highlighting the misconceptions of these two student teachers, this section presents the Sara's perspective.

Sara easily described the first quadrant, but it took several promptings for her to declare that only the sine ratios are positive in that quadrant. This revelation suggests that her viewpoint was that the cosine and the tangent ratios are negative in the first quadrant. This is incorrect since between 0° and 90° the cosine and tangent ratios also give positive values. After an exploratory question was posed, Sara changed her mind and stated that in the first quadrant, the cosine and the tangent ratios are both positive. She was then questioned with regard to the second quadrant:

Int: Now let us go to the second quadrant. Which of the three trigonometric ratios are positive in the second quadrant?

Sara: It is only the cosine ratio which is positive in the second quadrant.

Sara claimed that between 90° and 180° only the cosine ratio is positive. This perspective implied that the sine and the tangent ratios are negative between 90° and 180°. Although this view is applicable to the tangent ratio, it is incorrect for the sine ratio. When requested to indicate which trigonometric ratio is positive in the third quadrant, Sara pointedly explained that it was the tangent ratio. She stated the following with regard to the fourth quadrant:

Sara: The fourth quadrant it's only the sine which is positive.

Int: It is only the sine ratio which is what?

Sara: Which is positive.

Int: Which is positive?

Sara: Umm (meaning, yes).

Int: The rest you are saying are negative?

Sara: Yes.

By positing that the sine ratio is positive in the fourth quadrant and that the cosine and the tangent ratios are negative, Sara exhibited a superficial understanding.

Test item involving knowledge of quadrants and special angles: 30°, 45°, and 60°

In this section, Sara represents the student teachers who did not understand the special triangle and angles that were necessary in the evaluation of sin 315°. Alternatively, Mwila typifies understanding in this regard. The following excerpt shows Sara's explanations when requested to explain how sin 315° can be evaluated without the use of a calculator. She gave her explanation alongside Figure 5.20:

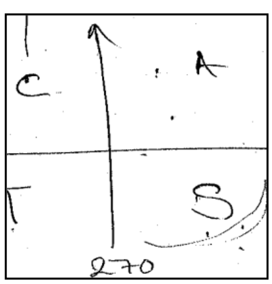


Figure 5.20: Sketch H1

Int: How can you find the value of sin 315° of course without using a calculator?

Sara: Oh without using a calculator, I will use this same property of angles in quadrants. I will say ACTS; it's the 'ka' word associated to these (pauses).

Int: What is that ACTS?

Sara: Uh it means that in the first quadrant it only; it's all the angles positive, then second, C is cosine, then third is the tangent and the last one is the uh sine. That makes ACTS, so now if I'm to find this angle 315° [sin 315°], 315° is uh between 270°, 270° and 300° and what, 60°. So automatically I will know before even solving I will know that my answer has to be positive.

The preceding conversation suggests that Sara had an idea that knowledge of the sign of the sine ratio in the fourth quadrant was necessary to successfully evaluate sin 315°. It also appeared that she possessed knowledge of the quadrant in which 315° lies, which was another necessary aspect. Nevertheless, her misconception that the sine ratio is positive in the fourth quadrant led to the incorrect conclusion. Sara contended that the value of sin 315° is positive based on her misconception that the sine ratio is positive between 270° and 360°. After clarifying that 315° lies in the fourth quadrant and that the sign of the ratio is positive, Sara drew Figure 5.21 with a view to using it in the evaluation of sin 315°.

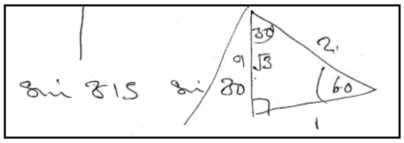


Figure 5.21: Sketch H2

Sara thought that 315° can be expressed using the special angles 30° and 60°, which appear in Sketch H2 above. Without explaining the basis for choosing the divisor, she divided 315° by 3 and obtained 105°. Afterwards, she confessed that her process was inappropriate as it did not provide the sought 'special' angle. Sara had an idea that special angles can be utilised to evaluate sin 315°, but she did not know the applicable angle. Similarly, she did not understand that 315° cannot be expressed as a sum or a difference involving either of the two special angles 30° and 60°. When asked if there is an alternative way in which sin 315° can be evaluated without utilising a calculator, Sara said she did not know.

Subsequently, Sara was asked to confirm which one of sin 30° and sin 315° is greater. She correctly indicated that sin 30° is greater than sin 315°. When queried about her reasoning, she explained:

> **Sara:** That gives me half which means that [sin 30°] it is half; I mean half like that. Now here on the sin 315°, because I have failed to find the answer, so it is very difficult, but I think just by looking I can say this is smaller.
> **Int:** You are saying just by looking?
> **Sara:** Umm (yes).

Sara admitted that she failed to evaluate sin 315°. It was therefore inconclusive for her to claim that she determined that sin 30° is greater than sin 315° 'just by looking'. Lastly, she explained that sin 30° is greater than sin 315° because the former lies in the first quadrant, while the latter lies in the fourth quadrant. This explanation was flawed in the sense that it is the 30° angle that lies in the first quadrant and not sin 30°. Similarly, it is 315° that lies in the fourth quadrant and not sin 315°. Notwithstanding, she was familiar with the definition of the sine ratio as well as with the special triangle using special angles 30° and 60°.

Mwila explained that 45° is the corresponding acute angle to the reflex angle 315°. He explained that the sine ratio is negative in the fourth quadrant, where 315° lies. Furthermore, Mwila utilised the appropriate isosceles triangle consisting of the special angle 45° to successfully evaluate sin 315°.

Student teachers' understanding of the sine rule

Mwila was the only student teacher who explained that the sine rule is a relationship between the sides of a triangle and the sines of the corresponding opposite angles. With the exception of Mwila and Mubita, the other four student teachers could not coherently explain that the sine rule is used to calculate an angle of a non right-angled triangle when two lengths and a non-included angle are known. Again, it was only Mwila and Mubita who clearly explained that this rule is used to calculate the length of a side of a non right-angled triangle when dimensions of two angles and one side are provided. The discussion with Sara regarding the sine rule went as follows:

> **Int:** Are you able to describe for me or to tell me what you know of the sine rule?
>
> **Sara:** Okay, the sine rule yes is the rule which only applies uh which you can only use when you have uh you have two sides and the angle. You have one pair which is complete like it has an angle and a corresponding side.

Sara asserted that the sine rule is solely applicable when two sides and 'the' angle are known. She did not indicate that the angle is not supposed to be included. Furthermore, Sara did not point out that when the dimensions of two sides and an excluded angle are known, the rule is used to find the size of an angle of a non-right angled triangle. It was essential to make reference to a non right-angled triangle because in a right angled triangle, the third angle can easily be computed using basic knowledge of the sum of angles in a triangle. Concerning 'an angle and a corresponding side', Sara drew the triangle presented below as Figure 5.22 and used it to make clarifications:

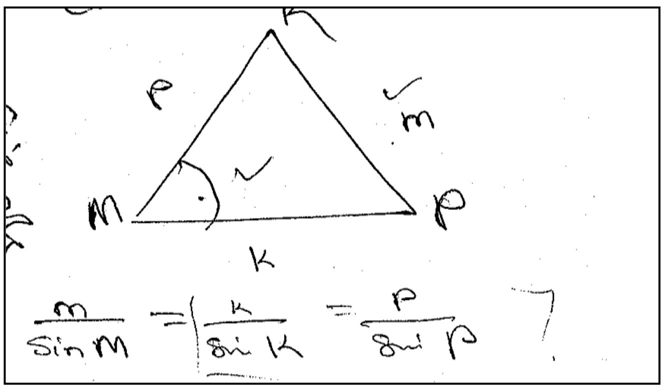


Figure 5.22: Sketch J1

> **Int:** So what is the meaning of that: 'an angle and a corresponding side'?
>
> **Sara:** Yes, for instance, if I have this triangle let's say MKP, this is my small side 'k', I have my 'm' there and 'p', so if I'm given angle M and a side, I have these two; these [this] set is completed.
>
> **Int:** You have angle M and what else?
>
> **Sara:** Angle M and side 'm'.

Sara posited that a 'complete' pair entails a situation where the measurements of one side and one angle are provided (ticked in Sketch J1). Subsequently, she added that there was need for either another side (p or k) or another angle (K or P). But she did not indicate the specific information required for the computation of either an angle or a side. Nonetheless, Sara accurately stated the sine rule appearing in Sketch J1.

Doreen correctly stated the sine rule, but exhibited an inability to comprehensively explain the information required before the rule is utilised. Initially, she indicated that the rule can be used when the 'angle and a corresponding side' are known. She could not explain why, but declared that 'it becomes a bit difficult'. Through probing, it was however established that she had the idea that the sine rule can be applied to find the length of a side when two angles and the length of one side are known. Doreen was then requested to explain the information required for the calculation of angles:

> **Int:** So for angles how do you find them, because the example you have given me is one for finding sides?
>
> **Doreen:** If they have given you angle (pauses), side 'a' and angle A and then they give you side 'b' and they ask you to find angle B it is just a matter of making this sine of angle B the subject.

Doreen understood that the sine rule can be utilised to find an angle when two sides and a non-included angle are known. Like John and Brave, she generally lacked the capacity to coherently explain the two conditions that warrant the use of the sine rule. The conversation with John regarding the sine rule proceeded as follows:

> **Int:** Okay, okay. Are there specific conditions that must exist for you to use the sine rule?
>
> **John:** Yes.
>
> **Int:** Would you share those conditions with me?
>
> **John:** So when you are given the sides, one of the sides then the angle opposite that side must be given and the side to be calculated the angle opposite it must also be given.

John knew that in order to apply the sine rule to compute the length of a side of a non right-angled triangle, the dimensions of two angles and a side opposite to one of the given angles must be known. Surprisingly, he claimed that this same condition is necessary for the computation of angles of non right-angled triangles. He did not seem to realise that in a situation where two angles are known, the third angle can easily be determined by subtracting the sum of the known angles from 180°. Moreover, John did not understand that the sine rule can be applied to calculate the angle of a non right-angled triangle when the dimensions of two sides and a non-included angle are known.

Brave stated the condition of two angles and one side, and mentioned that in such a situation, the sine rule can be utilised to calculate the length of a side of a triangle. What he did not mention is that the given side should be opposite one of the known angles. His explanation was vague with regard to the computation of angles. He asserted that if two sides and one angle are known, then the sine rule can be applied to compute an angle. This explanation was open ended in that it did not specify that the known angle is supposed to be non-included.

An area in which Sara, Doreen, John, and Brave had difficulties involved the proof of the sine rule. When Sara was first questioned regarding the proof of the sine rule, she was unsure:

> **Int:** Now, how do you come up with the sine rule?
>
> **Sara:** The sine rule mmm.
>
> **Int:** Ehe.
>
> **Sara:** Mmm you just get the angle, and put it on top of the (laughs and pauses).
>
> **Int:** Okay.
>
> **Sara:** Yes (laughs again).

The student teacher's assertion that the sine rule is derived from putting the angle 'on top' was not explicit. Similarly, her continued laughter suggested that she did not have an answer to the interview question. Nevertheless, she was again queried:

> **Int:** How did you get that sine rule?
>
> **Sara:** Aah I'm supposed to do some, I'm supposed to use; I think it is an isosceles triangle which we use then we come up with this formula.
>
> **Int:** Okay, is it something that you can show me?
>
> **Sara:** Uh as at now, no. I have forgotten unless I revise.

This time Sara intimated that the sine rule can be derived via an isosceles triangle. However, when asked if she could demonstrate this, she could not, and mentioned the need to revise this topic. Similarly, John also struggled:

> **Int:** So how do you come up with that sine rule?
>
> **John:** This sine rule can be derived from the (pauses), using these 'ma' sine ratios and (pauses again), maybe bisecting one of the sides and doing the calculations they reduce to this.
>
> **Int:** Without computing are you able to explain how that can be done?
>
> **John:** Mmm (pauses).
>
> **Int:** How do you come up with the sine rule?
>
> **John:** Silent and then says: no, but it's a bit complicated.

John had the idea that the sine ratio can be used to prove the sine rule, but could not derive the rule. When pressed to clarify how the sine rule is derived, he appeared helpless and said it was 'a bit complicated'. When queried how the sine rule is derived, Doreen claimed that she had forgotten. Similarly, Brave pointedly stated that he did not know how to derive the sine rule. He declared that he was merely reproducing what was written in a book.

Mwila and Mubita could explain how the sine rule was derived through the use of trigonometric ratios. A representative explanation as provided by Mubita is depicted in Figure 5.23:

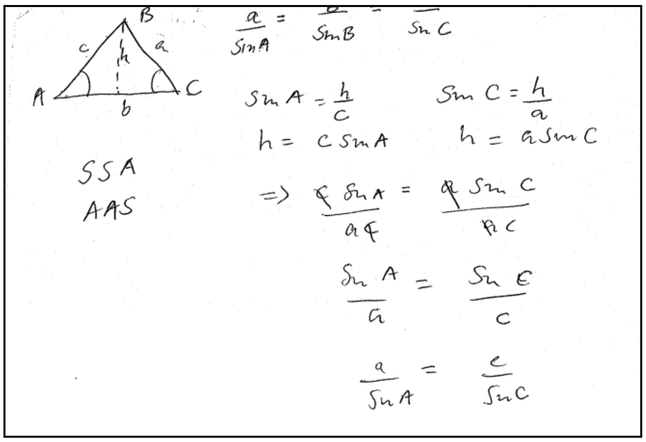


Figure 5.23: Sketch J2

Mubita drew a non right-angled triangle which he labelled ABC, as shown in Figure 5.23. Each of the sides of the triangle ABC was denoted by a small letter of the opposite angle. The side opposite angle A was labelled 'a', the one opposite angle B was denoted as 'b', and the side opposite angle C was denoted as 'c'. He then sketched a dotted perpendicular line from point B to the side AC to represent the height of the triangle ABC, and denoted the line 'h'. After this Mubita showed that the height (h) of the triangle ABC is given by $c\sin A$, or alternatively by $a\sin C$. Equating the two expressions of 'h' resulted into $\frac{a}{\sin A} = \frac{c}{\sin C}$.

Mubita explained that using a trigonometric ratio that involves angle B and changing the position of the height would result in the third ratio of the sine rule. It was clear that he had the competence to show that the ratio $\frac{b}{\sin B}$ is equal to $\frac{a}{\sin A}$, and $\frac{c}{\sin C}$. Nonetheless, Mubita did not prove the sine rule for a situation where a triangle involves an obtuse angle.

Student teachers' understanding of the cosine rule

Since the explanation that was presented by Mubita, John, and Mwila are similar, only Mubita's views are reported and analysed here. The following excerpt encapsulates Mubita's conception of the cosine rule:

> **Int:** Okay, tell me something about the cosine rule.
>
> **Mubita:** The cosine rule is also applied to a triangle which is not right angled, but for the cosine rule eeh it is only applicable, it is only applicable if (stammers), if the two sides are given or should I say if the two sides and the angle are, or should I say if the angle is included between the two sides, then it can be used now to find the third side. In short, I'm saying the cosine rule works when those two conditions for the (pauses), the conditions that we use for the sine law, if those two conditions for the sine law are not the ones that are given, then the only rule that we can use now is the cosine rule and this cosine rule will involve maybe when you know the three sides, which is not one of the conditions for sine law or if you know two sides and the included angle.

Mubita pointed out that the cosine rule is useful in instances where the sine rule cannot be directly applied. He declared that the cosine rule can be utilised to find the length of a side of a triangle when two sides and an included angle are known. He did not mention how the cosine rule can be used in a situation where the lengths of the three sides of a triangle are known. However, by implication, he seemed to be aware that in a scenario where all the lengths of the sides of a triangle are known, the remaining option is calculation of the angles. Mubita appropriately explained the cosine rule using the triangle ABC replicated below as Figure 5.24. He also explained that the letters in the rule $a^2 = b^2 + c^2 - 2bc\cos A$ can be varied to depict the other two representations of the rule: $b^2 = a^2 + c^2 - 2ac\cos B$, and $c^2 = a^2 + b^2 - 2ab\cos C$.

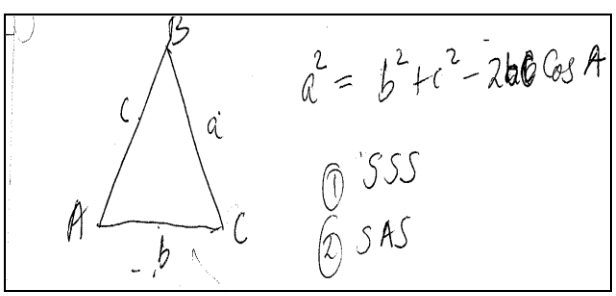


Figure 5.24: Sketch K1

Aside from the ability shown by Mwila, Mubita, and John to correctly state the cosine rule, as well as their capacity to appropriately elucidate when the rule is applicable, they were unable to derive the rule. While John said that he did not know, Mubita and Mwila asserted that the cosine rule is generated using Pythagoras' theorem. Mubita indicated that he could not recall how to prove the rule, but Mwila declared that the rule is never proved at secondary school level. Mwila explained how the cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$ converts into Pythagoras' theorem $a^2 = b^2 + c^2$ when an included angle A is 90° since cos 90° = 0. This explanation did not amount to proof, but was a depiction of the relationship that exists between the cosine rule and right angled triangles. In view of this, Mwila was questioned as follows:

> **Int:** Is it something that you can illustrate here or it is beyond you here?
>
> **Mwila:** Elementary method or just the usual?
>
> **Int:** Whichever way, just the way you come up with the cosine rule?
>
> **Mwila:** It is not very handy as at now, but I believe I can do it if I had time to go through it, I can do it.
>
> **Int:** Okay, go ahead.
>
> **Mwila:** It is not just handy.

Mwila claimed that given 'time to go through it', he could derive the cosine rule. His declaration that the proof of the rule is not 'handy' suggested a lack of ability. What follows next is the analysis of the understanding of the student teachers in the low content knowledge category. To begin with, an analysis of Sara's understanding is presented:

> **Int:** Okay, in your view what is the cosine rule?
>
> **Sara:** The cosine rule is uh it is a rule which can help you to find missing sides or angle[s] especially [in] a situation where you are given just three angles and you want to know the sides or you are given just sides you want to find the angles.

Although Sara did not mention the word 'triangle', she claimed that when three angles are known, the cosine rule can be used to calculate the lengths of the sides of a triangle. She also indicated that if the lengths of the three sides of a triangle are known, it is possible to utilise the cosine rule to find any of its angles. While the second assertion is applicable, the first is a misconception in that the cosine rule is used to find the length of a side of a triangle when two sides and an included angle are known. Despite the misconception, Sara accurately stated the cosine rule using an acute angled triangle, reproduced as Figure 5.25.

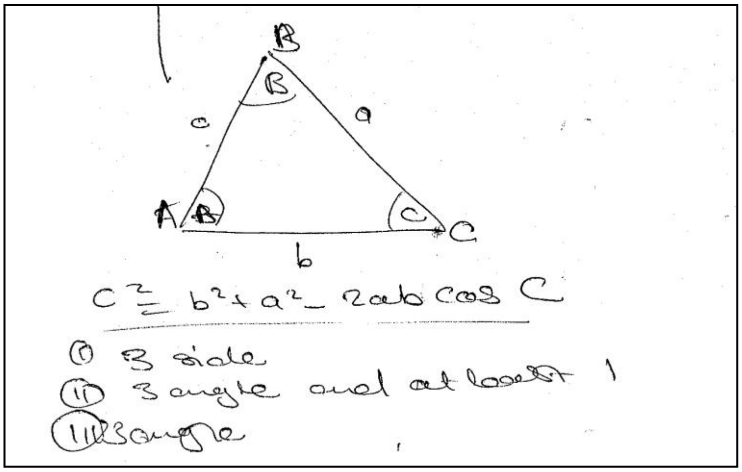


Figure 5.25: Sketch K2

Sara indicated the following three conditions alongside the cosine rule: 3 side(s), 3 angle(s) and at least 1 (side), and 3 angle(s). While she had earlier articulated the other two conditions, the condition involving three angles and at least one (side) appeared to be an addition. This condition can be applicable if the lengths of two sides are known. In the event that it is just 3 angles and one length that are known, it is impossible to employ the cosine rule to calculate the length of a side. The condition was, therefore, vague and suggested a superficial understanding on Sara's part. Moreover, the following excerpt shows her conflicted understanding:

Sara: Then lastly, when you just have three (pauses), the three of them are angles.

Int: When the three of them are angles?

Sara: Eeye (yes).

Int: Okay, you can use the cosine rule to do what?

Sara: You can use it to find the missing side.

By insisting that the rule can be applied to calculate the length in a situation where three angles of a triangle are known, Sara confirmed her shallow understanding. Doreen, like Sara, exhibited misconceptions regarding when to apply the cosine rule. While she appropriately stated the cosine rule, Doreen exhibited a limited understanding concerning the two conditions that are necessary for use of the rule. She indicated that two sides and an included angle must be known, but did not expressly mention that this condition relates to the calculation of sides. She claimed that this condition is sufficient for the calculation of angles and sides. By implication, she lacked the understanding that the computation of angles using the cosine rule requires that the three sides of a non right-angled triangle are known.

Brave described the cosine rule as a formula that depicts a relationship between the angles and sides of a triangle. Unlike Doreen, he effectively articulated the two instances when the cosine rule can be applied. Firstly, he explained that when two sides and an included angle are known in a triangle, the cosine rule can be employed to find the length of the third side. Secondly, Brave mentioned that the cosine rule can be applied to calculate an angle of a triangle whose three sides are known. His version of the cosine rule was presented alongside the triangle replicated below as Figure 5.26. Although it is not mathematically incorrect, this student teacher unconventionally labelled the side opposite to angle C with small letter 'a', the side opposite angle A with small letter 'b', while the side opposite angle B was denoted as small letter 'c'.

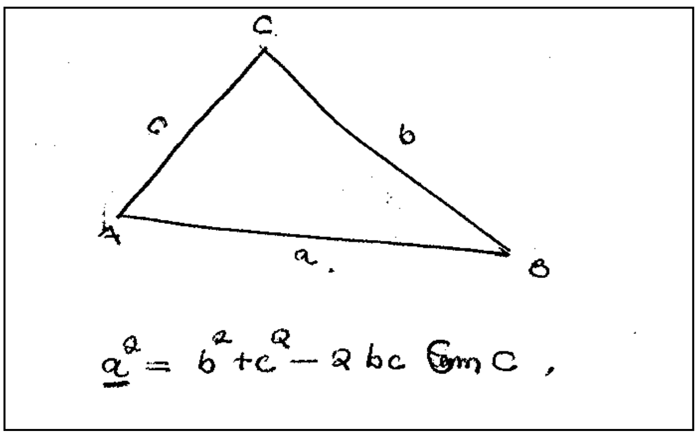


Figure 5.26: Sketch K3

> **Brave:** So in this case the cosine rule would actually state that: the longest side of the rectangle, assuming that it is AB.
> **Int:** Of a rectangle?
> **Brave:** Sorry, of a triangle would be 'a' squared is equal to uh mmm then the other two sides, I square the other two sides: 'b' squared plus 'c' squared then minus two 'bc' then cos, sorry sine, now the angle which is opposite to (pauses) 'a' in this case which is C.

Figure 5.26 and the foregoing excerpt show that Brave declared the rule $a^2 = b^2 + c^2 - 2bc\sin C$ as an interpretation of the cosine rule. He initially used the cosine of angle C (angle opposite the side denoted 'a') in the rule, but afterwards emphasised that it is the sine of angle C that should be used. This change suggests that he did not have the correct understanding of the cosine rule. Furthermore, Brave revealed a misconception when he insinuated that it is necessary to start with what he called ‘the longest side’ when stating the cosine rule. Consequently, a clarification was sought:

> **Int:** Now I note you mentioned that you start with the ‘longest side’ when stating the cosine rule. Why would you start with the ‘longest side’?
>
> **Brave:** Aah starting with the longest side actually will (pauses).
>
> **Int:** Ooh! Sorry repeat.
>
> **Brave:** So if I start with the longest side, starting with the longest side is the only way that will make this equation equal.

Brave claimed that if the rule did not start with ‘the longest side’ then the left hand side of the rule would not be equal to the right hand side.He explained that the longest side in Sketch K3 is AB, which he apparently drew as such. There is a likelihood that Brave was thinking in terms of Pythagoras’ theorem, which usually starts with the square of the hypotenuse.

Similar to the observation made concerning the student teachers from the high content knowledge category, Sara, Doreen, and Brave could not prove the cosine rule. While Brave posited that he had no idea how the rule is derived, Sara and Doreen had forgotten how to derive it.

Student teachers’ understanding of formulas for the computation of the areas of triangles

Mwila, Mubita, and John explained that the trigonometric formula $\frac{1}{2}ac\sin B$ is utilised to compute the area of non-right angled triangle ABC when *a*, *c*, and angle B are known. The student teachers explained that for a right angled triangle, the area is given by the formula $\frac{1}{2}\times\text{base length}\times\text{height}$. It was also clarified that this formula is applicable to non-right angled triangles as long as the perpendicular height is known. The three student teachers demonstrated similar conceptions concerning the basis of these formulas, therefore only Mwila’s understanding is reported on.

With regard to the derivation of the trigonometric formula $\frac{1}{2}ac\sin B$, Mwila drew the triangle ABC represented below as Figure 5.27 and used it to make an illustration.

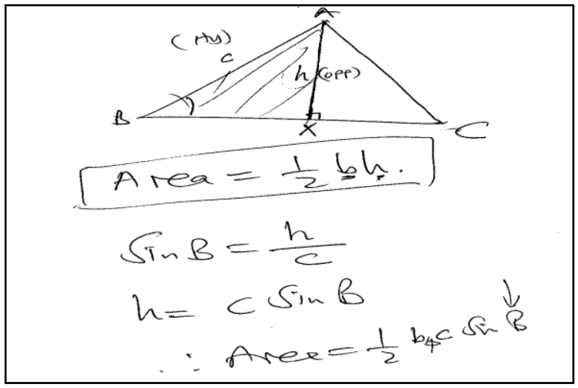


Figure 5.27: Sketch M1

Following his description that AX is the perpendicular height, while BC was assumed to be the base of triangle ABC, Mwila declared that the area of triangle ABC is calculated using the formula $\frac{1}{2}\times\text{base length}\times\text{height}$. He then clarified why the area of a triangle is normally obtained through this formula. Mwila posited that dividing any rectangle with a diagonal results in two equal right angled triangles. Since the area of a rectangle is the product of the length and the breadth, he then argued that the area of either of the two resultant triangles would be $\frac{1}{2}lb$. The letter *l* represented the perpendicular height and *b* represented the base length. However, Mwila could not explain why the formula $\frac{1}{2}\times\text{base length}\times\text{height}$ is applicable for non right-angled triangles.

Thereafter, Mwila used Sketch M1 to explain the basis of the trigonometric formula $\frac{1}{2}ac\sin B$. He assumed that the sides $AB = c$, and $BC = a$, together with the included angle B were known. Mwila then applied the sine ratio to triangle ABX and effectively showed that the perpendicular height (AX) of triangle ABC is the same as $c\sin B$. It is this expression that he substituted for the height in the formula $\frac{1}{2}\times\text{base length}\times\text{height}$, which resulted in $\frac{1}{2}ac\sin B$, which is obviously the trigonometric formula for the area of a triangle. In this regard, Mwila satisfactorily illustrated the basis for the trigonometric formula that is utilised when calculating the area of a triangle. An aspect which he did not mention is that the formula $\frac{1}{2}ac\sin B$ relates to both acute angled triangles and those triangles in which one of the angles is obtuse.

Mwila's understanding of the concept of area was that it relates to the surface covered by the shape or an object. He attempted to distinguish the concept of area from that of volume by suggesting that whereas area relates to 'covering', volume involves 'occupying'. Mwila's explanation of the concept of volume suggests that he understood the difference between the idea of area and that of volume. Mubita said much the same in suggesting that area is the surface that is covered by a shape. Furthermore, he explained that area is quoted in square units because it is a two dimensional quantity that involves multiplying two measurements that are normally cited in the same units.

Brave, Sara, and Doreen recited the formulas that are normally used to calculate the areas of triangles. They explained that when the perpendicular height of a triangle and the base length are known, the area is given by the formula $\frac{1}{2}\times \text{base length}\times \text{height}$. They added that when two sides and an included angle are provided, the area of a non-right angled triangle is calculated using $\frac{1}{2}ac\sin\theta$. Regardless of this capacity to state the formulas and the conditions that necessitate their use, none of the three student teachers could derive the formulas. The following extract, based on a discussion with Brave, testifies to this conclusion:

> **Int:** Okay, now how do you get this formula; how did you get this $\frac{1}{2}ac\sin\theta$?
>
> **Brave:** Silent.
>
> **Int:** How do you come up with that trigonometric formula?
>
> **Brave:** I think I'm not in a position to derive it.

The low content knowledge student teachers exhibited an inability to comprehensively explain the concept of area. Again, the discussion with Brave exemplifies the student teachers' superficial understanding of what area is all about:

Int: Are you able to explain to me the concept of area?

Brave: So like in this case I will probably say calculating the area entails uh mmm the (pauses).

Int: It entails the what?

Brave: It gives an idea of like how big or small this surface is.

While Brave, Sara, and Doreen were able to calculate the areas of triangles involving perpendicular heights as well as areas of non right-angled triangles, they could not competently explain the concept of area. Brave said that area is quoted in square units because it involves multiplying a unit of measurement by itself twice. The analysis of the student teachers' explanations of the methods that they employed to solve a basic trigonometric equation is presented next.

Student teachers' explanations based on a simple trigonometric equation

This section presents an analysis of the student teachers' understanding of what was assessed by an item that required the solving of the equation $\cos\beta = -\cos 64^\circ$ for $90^\circ < \beta < 180^\circ$. At the same time, this section includes an analysis of the student teachers' explanations of the methods that they applied to solve the equation, as well as their understanding of alternative methods. Sara was shown the equation $\cos\beta = -\cos 64^\circ$ for $90^\circ < \beta < 180^\circ$ and asked to explain what the item assessed. Her view was that the item involved 'the property of angles' as well as the association of trigonometric ratios to quadrants.

She explained that what she referred to as 'property of angles' was the sign of the cosine ratio in the second quadrant. She explained that the item involved knowledge of the sign of the cosine ratio in the second quadrant. However, she confirmed a misconception by indicating that the cosine ratio is positive in the second quadrant. Sara was shown a copy of her solution in the test and asked to explain the method applied. Her calculation is replicated as Figure 5.28.

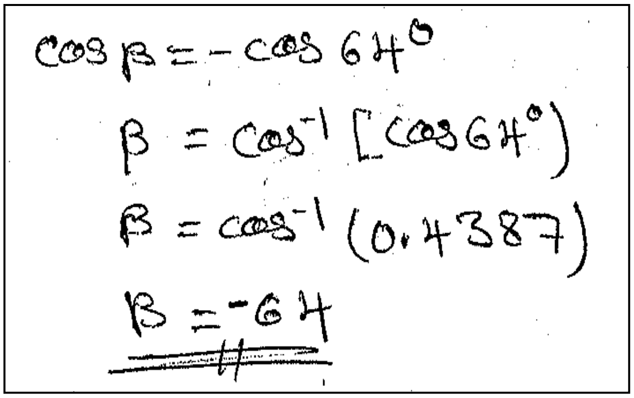


Figure 5.28: Sketch N1

Int: Maybe are you able to share with me or are you able to explain to me how you calculated for the value of β. Uh when you look at the answer that you gave me, are you able to carry me through your calculation, how you got the value of β?

Sara: Mmm okay here I just looked at the cos because cos, cos, cos; this side were common so it meant that since here there was cos 64° it meant that even β was 64°.

Int: β was equal to 64°?

Sara: But because of the coefficient negative, my angle was $\beta = -64$.

Sketch N1 reveals Sara's inability to correctly work on the interval $90^{\circ} < \beta < 180^{\circ}$ in which β lies. Also, the sample suggests her inability to distinguish between the clockwise and anti-clockwise rotations of angles. The foregoing excerpt shows that Sara explained a method that is consistent with how linear equations are solved. This is evident in her assertion that the value of β is equal to -64° since the word 'cos' is common on both sides of the equation $\cos\beta = -\cos 64^{\circ}$. She treated 'cos' as a constant and the negative sign on the right hand side of the equation as a coefficient. While Sketch N1 suggests that a calculator was utilised, Sara explained a different method. Interestingly though, when requested to explain an alternative method, she confidently declared that a calculator is the only option.

Brave's explanation regarding what the item assessed suggests that he was aware that β lies in the second quadrant. He mentioned that the trigonometric equation $\cos\beta = -\cos 64^{\circ}$ involved the determination of the sign of the cosine ratio in the second quadrant. However, he did not seem to know how to use this knowledge to solve the item. Additionally, he lacked the understanding that β is the supplementary angle to 64°.

Unlike Sara, Brave pointed out that he used a calculator to find the value of β. However, he could not account for the operations that were performed and claimed that 'this is the procedure'. His use of a calculator, therefore, showed a memorised process that was devoid of in-depth understanding of the reasons why particular operations are performed. Concerning an alternative method that can be used to solve $\cos\beta = -\cos 64^\circ$, Brave proposed trigonometric identities. After realising that 64° is not a special angle, he asserted that identities cannot be utilised to find the value of β.

John used a calculator to solve the equation $\cos\beta = -\cos 64^\circ$. He explained that computation of β requires knowledge of the trigonometric ratios, right-angled triangles, and Pythagoras' theorem. He could not elaborate how these concepts are used to solve the equation. Afterwards, he claimed that the equation $\cos\beta = -\cos 64^\circ$ can only be solved with a calculator because 64° is not a 'standard angle'. This suggests that he was not cognisant of the possibility of applying his knowledge of the sign of the cosine ratio in the second quadrant. John did not realise that the sought angle was a supplement of 64°.

Doreen contended that the equation $\cos\beta = -\cos 64^\circ$ for $90^\circ < \beta < 180^\circ$ gauged how the students calculate 'trigonometric' angles. She confessed that she did not know how to solve the equation. Nevertheless, she claimed that the calculation of β requires knowledge of the sine and cosine ratios, and an understanding of the signs of these ratios in the quadrants. She argued that it was difficult to solve the equation $\cos\beta = -\cos 64^\circ$ in the form it had been presented in. She seemingly had a particular format in mind in which the equation can be easily resolved. As a result, Doreen was encouraged to provide the trigonometric equation that she had in mind. She then suggested that –cos 64° be replaced by sin 64°, which resulted in $\cos\beta = \sin 64^\circ$. The discussion progressed as follows:

Int: Ehe, how would you have solved the equation $\cos\beta = \sin 64^\circ$?

Doreen: Because (pauses), and I'm asked to find β ?

Int: Aha.

Doreen: I would have put cos there, cos there, then I would remain with β .

Int: What do you mean 'I would have put cos'?

Doreen: I would divide by the inverse of (pauses), multiply by the inverse of cos.

Int: What is the inverse of cos there?

Doreen: I more like divide by cos here even here by cos it will give me this sin over cos, which is $\tan 64^\circ$.

This student teacher posited that she would have to multiply each term of the equation by the inverse of cosine. This revealed that Doreen had a misconception that $\cos^{-1}\theta = \dfrac{1}{\cos\theta}$. This became apparent when she clarified that she would have to divide each term of the equation $\cos\beta = \sin 64^\circ$ by 'cos' to obtain $\beta = \tan 64^\circ$. She pointed out that a calculator would then have to be used to evaluate tan 64°. While Doreen seemed to know that dividing the sine ratio by the cosine ratio gives the tangent ratio, her explanation typified the superficial understanding of trigonometric ratios. Whereas 'cos' is merely a word, '$\cos\beta$' is a trigonometric ratio, and therefore it was mathematically incorrect to say that $\beta = \dfrac{\cos\beta}{\cos}$, and that $\dfrac{\sin 64^\circ}{\cos} = \tan 64^\circ$. Evidently, she incorrectly conceived 'cos' to be a constant and as a coefficient of β .

Doreen confessed that, without the use of a calculator, she could only solve trigonometric equations that involve 'special' angles 60° and 45°. She asserted that in order to solve an equation like $\cos\beta = \sin 60^\circ$ it is necessary to use a special triangle that includes the angles 30° and 60°. However, she did not comprehensively explain how the special triangle can be used to solve the equation $\cos\beta = \sin 60^\circ$. Her explanation merely intimated that the special angle 60° and 30° can be applied to simplify trigonometric expressions.

Mubita and Mwila solved the trigonometric equation $\cos\beta = -\cos 64^\circ$ for $90^\circ < \beta < 180^\circ$ without utilising a calculator. These two student teachers explained that the equation assessed the relationship between the cosine of an obtuse angle and the cosine of the supplementary angle of that obtuse angle. Mwila and Mubita correctly interpreted the interval $90^\circ < \beta < 180^\circ$ and mentioned that β was the difference between 180° and the acute angle 64°. They explained that the cosine of an obtuse angle is equal to negative cosine of a supplementary angle to that obtuse angle. While Mubita felt that a calculator could also be used to solve the equation, Mwila showed a lack of knowledge regarding an alternative method:

> **Int:** Is there an alternative method you could have used other than this one you used?
>
> **Mwila:** Aah for me, no.
>
> **Int:** Okay, why do you think there is no other method (laughs)?
>
> **Mwila:** The other method could be there, but if it is there it is just that I'm not aware about that, but for now the tool that I have is this one.

Mwila declared that he only knew a single way of solving the equation. After being probed to provide a reason, he pointed out that other methods could exist, however, he was only aware of one method.

Explaining methods used to compute an angle of a non right-angled triangle

The analysis presented in this section is based on a test item that required the student teachers to find the smallest of the angles of a triangle whose sides are in the ratio $2:4:5$. Of the six interviewees, only Brave and John correctly solved the item in the test. Mwila and Mubita did not provide any solution, while Sara and Doreen presented answers that were incorrect. Sara was shown a copy of the test item together with Figure 5.29 below, which is the answer she provided in the test. She was then asked to explain what the item assessed:

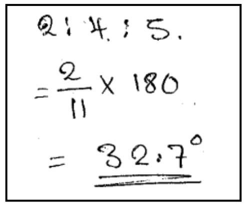


Figure 5.29: Sketch O1

Int: Ehe so what did you understand of that question?

Sara: Laughs and says: I think afterwards that is when I thought that I was supposed to draw a triangle.

Int: You were supposed to draw a triangle?

Sara: A triangle yes.

Int: Aha.

Sara: Then I say one side it will be two, then the other one it will be four and five and using the cosine rule I find the angle for the smallest which is two.

This excerpt shows that Sara realised that the components of the ratio $2:4:5$ represent the dimensions of the sides of the triangle in question. She mentioned that the cosine rule was applicable for the calculation of the required angle. To ascertain the implication of her explanation she was directly queried:

Int: So in other words are you saying the components of the ratio are actually the lengths of the sides of the triangle?

Sara: Yes, umm but I think on that day I did this (refers to Sketch O1) I added this (pauses), then I got the two then times 180° it gave me this [32.7°]. But I would have tested even for these other numbers.

Sara reaffirmed that the parts of the ratio were the lengths of the triangle, but suggested that the method used in Sketch O1 was equally appropriate. She was then reminded of her earlier idea regarding the suitability of applying the cosine rule to compute the required smallest angle:

Int: But today you are talking of the cosine rule?

Sara: Yah, I think even this one [method] can work.

Int: So the cosine rule or the same method you used in the test would still work?

Sara: Yes, aha.

Sara held on to the incorrect method while articulating an alternative, but appropriate method. On one hand, she considered the components of the ratio $2:4:5$ to be measurements of the sides of the triangle. On the other hand, she understood the components of the ratio to be fractions of the sum of the angles in a triangle. This scenario suggests that Sara's understanding was conflicted.

Doreen's answer was similar to Sketch O1, she did not consider the components of the ratio $2:4:5$ to be the lengths of the sides of the triangle. When requested to provide a reason for her choice of method, she claimed that it was the 'direct way of finding ratios'. In the determination of the smallest angle, Doreen was guided by the fact that the component 2 is the smallest in the ratio $2:4:5$. She could not see that the item involved the cosine rule.

Similarly, Mwila and Mubita did not understand that the components of the ratio of the sides are the lengths of the sides of the triangle. An extract from the dialogue held with Mubita characterises the two student teachers' understanding:

> **Int:** What did you understand of this question where you were asked to calculate the smallest of the angles of a triangle whose sides are in the ratio $2:4:5$?
>
> **Mubita:** In this particular case, first of all we know from geometry that the sum of the angles in a triangle is 180°. So here all I needed to do was to (stammers), to determine using my knowledge of ratios. It is possible to divide that 180° into these ratios. Using this ratio I can divide a quantity in a given ratio and that quantity is 180°. So it is possible for me to find the smallest angle.

Mubita failed to recognise that this item required the use of the cosine rule to calculate the smallest angle. However, John and Brave understood that the components of the ratio were the lengths of the triangle that had merely been 'reduced to the lowest terms'. Additionally, John and Brave explained that the cosine rule was applicable since the lengths of the three sides are known. While John knew that the smallest angle of a triangle is usually the one opposite to the shortest side, Brave had misconceptions in this area. Brave claimed that the smallest angle of a triangle is the one opposite to its longest side. He lacked the understanding that as the length of the opposite side increases, the opposite angle also increases. Moreover, he did not know that as the opposite side reduces in length, so the opposite angle reduces in size. Brave justified his viewpoint in the following extract:

Int: Why is the smallest angle opposite to the longest side?

Brave: Because the other two sides (pauses), because the other two sides if I consider them, if I consider the longest side, then the other two sides will be shorter. So joining them will actually give me the smallest angle.

This excerpt confirms that Brave did not know that the size of an angle of a triangle is not reliant on the lengths of the 'arms' that create it, but on the length of the opposite side. Lack of knowledge in this regard exposes his superficial understanding despite the correct answer he presented in the test.

Student teachers' understanding of the periods of the sine, cosine, and tangent functions

Mwila, Mubita, John, Brave, and Doreen were able to correctly state the periods of the sine, cosine, and tangent functions. They indicated, for instance, that the sine and the cosine functions have periods of 360°. Furthermore, they disclosed that the tangent function has a period of 180°. Sara initially mentioned that the period of the sine function is 360°, but afterwards declared that she was unsure. Similarly, Sara showed a lack of understanding concerning the period of the tangent function.

A common finding was that the six student teachers had no ability to comprehensively explain the idea of 'period'. Sara explained the period of a function as a graph that has a repetitive pattern. She could not distinguish the graph of a periodic function from the period of the function. She pointed out that the 'thing' that repeats itself does so 'after some time'. She did not relate the phrase 'after some time' to the number of degrees or radians that are equal to the width of the repeating pattern. Doreen claimed that the period of a function is 'the distance the curve of the function takes'. This description was open-ended as it did not suggest what the curve of the function does in the 'distance'. The student teacher did not indicate that a period is the width of a repeating pattern of a function. Brave explained the concept of period as follows:

Int: Okay, would you explain to me what the term 'period of a function' means?

Brave: Aah period of a function is actually the (pauses), it is the period that a particular function makes, a complete (stammers), a complete turn.

Brave's description of a period as being 'a complete turn' required unpacking. Consequently, he was asked to clarify:

> **Int:** Maybe what do you mean by making 'a complete turn'?
>
> **Brave:** Okay for example, if I was to draw the function of sine, from zero it (stammers), it actually, it will appear like this, so this is actually the period.
>
> **Int:** The question is: what do you mean by 'complete turn'?
>
> **Brave:** Long silence.

Evidently, Brave was not able to clarify what he meant by the phrase 'complete turn'. Instead, he resorted to sketching an incomplete graph of the sine function, although he said that the sine function has a period of 360°. This scenario pointed to his inability to explain the concept of period. Mwila also had difficulty explaining the idea of a period. He asserted that the period of a function is 'simply the period at which the given function repeats itself'. Mwila further clarified this description as depicted in the subsequent extract:

> **Int:** What do you mean by 'the function repeating itself'?
>
> **Mwila:** Okay, so if we have got let's say I will use the sine function. Let's say we know that from zero to 360° the sine graph behaves like that where this is 180°, the maximum there at 90°, here 270°, and this is 360°. So meaning that after 360° if we continue drawing this graph; we are going to obtain the same (stammers), the same graph that we have.

Mwila illustrated his understanding of the concept of period using the sine function. While he indicated that after 360°, the 'same graph' of the sine function is obtained, Mwila lacked the ability to thoroughly explain the concept of a period. Alternatively, John explained that a period relates to the number of degrees [or radians] that a function takes before it begins to repeat itself. He related the idea of 'repeating' to the aspect of having the same shape of the curve of the function after a particular number of degrees. John's conception of the period of a function was similar to the understanding that was demonstrated by Mubita.

Student teachers' understanding of the differences between the sine and tangent functions

While Doreen declared that she did not know any difference between the sine and the tangent functions, Mwila, John, Mubita, Brave, and Sara explained that the two functions have different periods. Surprisingly, Sara, who was uncertain of the periods of the sine and tangent functions, alluded to this difference. The five student teachers also explained that the sine and tangent functions have different ranges. However, only Mwila, John, and Mubita comprehensively explained that the sine function has a range of $-1 \leq \sin\theta \leq 1$ while the tangent function has the interval $-\infty \leq \tan\theta \leq \infty$ as its range. Brave and Sara exposed their misconceptions as they explained the differences. In view of this, it is their understanding that is reported henceforth. Brave expressed his ideas as follows:

> **Int:** Tell me any two differences that you know between $y = \sin\theta$ and $y = \tan\theta$?
>
> **Brave:** Of course one of the differences is that if I pick maybe the value of tan; if I pick on one value of theta then substitute it in those two functions I will be able to get two different values. So meaning that *y* equals sine theta, *y* equals tan theta will actually give me two different values.

This excerpt suggests that Brave had the misconception that for every angle, the sine and tangent functions have different values. He clearly lacked the understanding that for 0°, 180°, and 360°, the sine and tangent functions give the same value of zero. Another of Brave's misconceptions was revealed in the ensuing extract:

> **Int:** What is the range of the graph of $f(\theta) = \sin\theta$?
>
> **Brave:** $\sin\theta$ will actually range from zero to 360° then from 360° to uh should be 720°.

Brave claimed that the range of the sine function consisted of angles. This demonstrated that his understanding of the range of the sine function was incorrect. Moreover, he could not provide the range of the tangent function despite several promptings. These findings suggested that although he had stated that the sine and tangent functions are different in terms of their ranges, his understanding in that regard was superficial. Sara's perspective on the range of the sine function is shown below:

> **Int:** Okay, what is the range of $f(\theta) = \sin\theta$?
>
> **Sara:** The range is 1; -1 and 1.
>
> **Int:** -1 and 1?
>
> **Sara:** Yes.

This extract suggests that Sara regarded the maximum and minimum values of the sine function as the range. She did not seem to understand that the range of the sine function is the interval $-1 \leq \sin\theta \leq 1$ and not the values -1 and 1. Similarly, instead of presenting $-\infty \leq \tan\theta \leq \infty$ as the range of the tangent function, she indicated that the range is 'negative infinite, positive infinite'. These findings suggested that her declaration that the sine and tangent functions differed in respect to their ranges was not based on comprehensive understanding.

What justifications do the student teachers provide for their reasoning in trigonometry?

Why trigonometric ratios are positive or negative in the four quadrants

Doreen was among the five student teachers who correctly declared that the sine, cosine, and tangent ratios are positive in the first quadrant. When asked to justify her viewpoint in this regard, she responded as follows:

> **Int:** Would you give me justifications why all the three trigonometric ratios are positive in the first quadrant?
>
> **Doreen:** Mmm (laughs), no the justification I don't know.
>
> **Int:** You don't know why, you just know that they are all positive?
>
> **Doreen:** Yes.

In spite of her knowledge that the three trigonometric ratios were positive in the first quadrant, Doreen did not understand why this was the case. She did not have a justification for contending that only the sine ratio is positive in the second quadrant, and that only the tangent ratio is positive in the third quadrant. She admitted that she did not know the reasons or that she was not sure. Nevertheless, she used the characteristics of the sine curve to justify her view that the sine ratio is negative in the fourth quadrant:

> **Int:** Ehe, probably may you clarify what you mean that the sine curve gives an idea that the sine ratio is not positive in the fourth quadrant; how does it do so?
>
> **Doreen:** Because when you look at it, when starting from (stammers), from zero to π you see that it is in (pauses), above the x-axis, and then 270° it is below the x-axis. Maybe somehow that can determine its sign.
>
> **Int:** Okay, okay, so because of the sine curve you are saying that the sine ratio is negative in the fourth quadrant?
>
> **Doreen:** It is negative.

Doreen correctly observed that in the interval $0^{\circ} < \theta < 180^{\circ}$, the sine curve has positive values. She stated that in the interval $270^{\circ} < \theta < 360^{\circ}$, the sine curves have negative values. According to Doreen, this is what justified the assertion that the sine ratio is negative in the fourth quadrant. However, she did not explain why the sine curves have negative values in the interval $270^{\circ} < \theta < 360^{\circ}$. As a result, the purported justification was a mere declaration of fact. Similarly, Brave's justification was based on the characteristics of sine and cosine curves for specific intervals of angles. He equally did not highlight the reasons why such curves have positive or negative values for particular intervals of angles. To this extent, his supposed justifications were as inadequate as Doreen's.

Sara used comparable reasoning to justify her views on the four quadrants. Her justifications were consistent with those provided by John. To avoid repetition, the analysis of the explanations that she articulated regarding the first quadrant are reported. The discussion with Sara developed in the following manner:

> **Int:** Okay, why do you think that the sine ratio is positive in the first quadrant?
>
> **Sara:** Mmm because whatever number we pick from this quadrant and substitute in any sine uh equation given; sine, yes anything involving the sine it will give us a positive answer.
>
> **Int:** So what do you mean 'a number we substitute', we substitute where?
>
> **Sara:** We substitute in the, let us say for instance if I picked on sine (pauses), sine uh what is this (laughs), between here, I can say 45°, I can say $\sin 45^{\circ}$ is $\frac{1}{\sqrt{2}}$; that is positive.

Sara used one special angle to justify her view that the sine ratio is positive in the first quadrant. An example involving one angle (45°) was insufficient to conclude that for every other angle, the sine ratio is positive in the first quadrant. Moreover, it was not clear how many angles she needed to substitute in order to prove her point of view, especially as between 0° and 90° there are infinite real numbers. When 55° was suggested, she realised that it is difficult to show that sin 55° is positive in the absence of a calculator or a mathematical table.

Sara tried to use the isosceles right-angled triangle whose two base angles are 45° each. She expressed 55° as a sum of 45° and 10°. With this sum, she hoped to use trigonometric identities to show that the value of sin 55° is positive. A complication ensued in that she still needed to deal with 10°, which is not a special angle. At last, she exclaimed 'I think I need help'. Her failure to evaluate sin 55° confirmed that she lacked the competence and confidence to provide a justification for her reasoning.

Mwila's justification was similar to that given by Mubita. He used a diagram to provide his reasoning for asserting that the sine, cosine, and tangent ratios are positive in the first quadrant. In order to understand his explanation, the diagram is reproduced below as Figure 5.30.

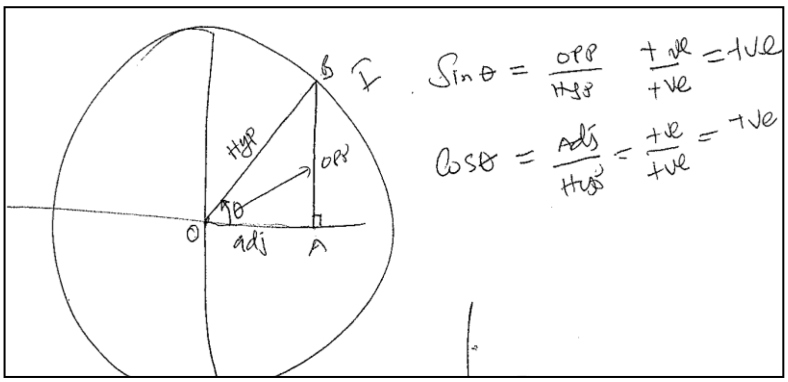


Figure 5.30: Sketch P1

Mwila drew a right angled triangle labelled OAB with an arm of rotation labelled OB in the first quadrant. The acute angle that the arm OB formed with the positive x-axis was denoted as θ. A perpendicular line from the point B (on the circumference) intersected the implied x-axis at point A. Mwila understood that $\sin\theta = \frac{opp}{hyp}, \cos\theta = \frac{adj}{hyp}$, and $\tan = \frac{opp}{adj}$. He explained that in right angled triangle OAB, the opposite side to the angle θ was AB, which was equal to positive *y*. In addition to this, he pointed out that the hypotenuse OB was equal to the radius of the circle, which is always positive. Furthermore, he indicated that the adjacent side OA was equal to positive *x*.

Mwila then explained that in the first quadrant, the sine ratio is positive because it is equal to $\frac{AB}{OB}$, which is a division of two positive numbers. Similarly, he argued that in the first quadrant the cosine and tangent, ratios are obtained by dividing positive numbers, thereby making them positive.

Mwila used another diagram to discuss the second quadrant, represented below as Figure 5.31. Prior to drawing the diagram, he posited that the same arguments as advanced for the first quadrant were applicable for the second quadrant.

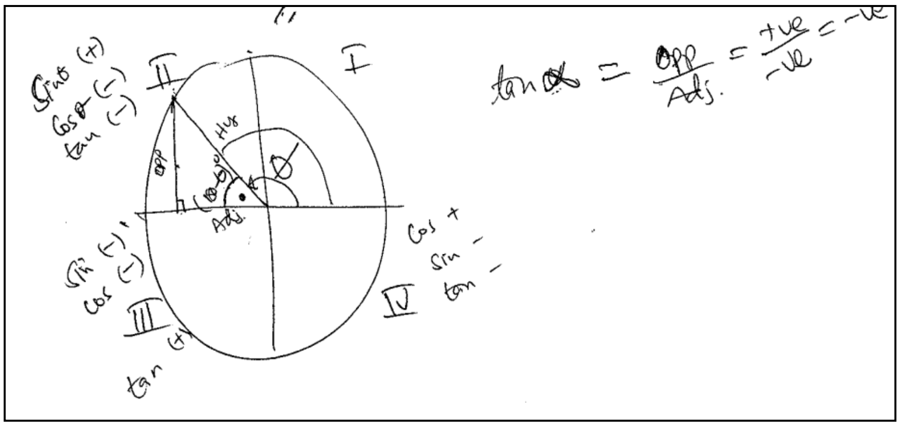


Figure 5.31: Sketch P2

Mwila indicated that the arm of rotation is in the second quadrant, and denoted the resulting obtuse angle θ. He explained that instead of using the obtuse angle θ, it was necessary to use the associated acute angle $180^{\circ} - \theta$. According to Mwila, such a move permits the application of the sine, cosine, and tangent ratios, which are defined in right angled triangles. He then posited that the side opposite to the acute angle $180^{\circ} - \theta$ is positive *y*, while the adjacent side is negative *x*. It was with this interpretation that he declared that $\sin(180^{\circ} - \theta) = \sin\theta$ is positive in the second quadrant. His argument was that the sine ratio comprises a division of two positive components: *y*, and the length of the hypotenuse.

Mwila considered the cosine ratio as negative in the second quadrant since it is a result of dividing negative *x* by the positive length of the hypotenuse. He clarified that the tangent ratio is negative in the second quadrant as it involves dividing positive *y* by negative *x*. He applied similar reasoning to justify his views on the signs of the sine, cosine, and tangent ratios in both the third and fourth quadrants.

Justifications relating to characteristics of the tangent function

The analysis reported on in this section is based on the student teachers' justification of their understanding of the specific features of the tangent function. Doreen insinuated that the value of tan 270° is zero. When asked to provide her reasoning she responded as follows:

> **Int:** Why do you think the value of tan 270° is zero?
>
> **Doreen:** Because when I look at the tan [tangent] curve, it goes; cuts at $\frac{\pi}{2}$. It goes 'asymptotically' at 270°. It just goes if I'm not mistaken, it goes straight it doesn't cut the x-axis at 270°.

Doreen claimed that the value of tan 270° is equal to zero because the tangent curve does not cross the x-axis at 270°. This explanation was contradictory because tan 270° can only be equal to zero if the curve intersects the x-axis at 270°. Moreover, Doreen's assertion that the tangent curve crosses the 90° axis was incorrect. She was shown a copy of the tangent graph that she had presented in the test. That graph is replicated below as Figure 5.32 and it suggests that she thought that the tangent curve intersects the $\frac{3\pi}{2}$ axis.

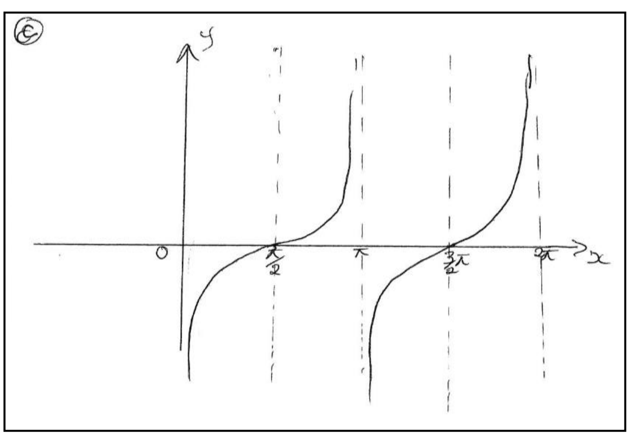


Figure 5.32: Sketch Q1

There was, therefore, a conflict between her interview declaration and Sketch Q1. Either Doreen did not know that $270°$ is equivalent to $\frac{3\pi}{2}$ or she changed her mind during the interview. The former was unlikely, as the following discussion attests:

> **Int:** You are saying 2π is the same as 360°; are you suggesting that there is a relationship between 2π and 360°?
>
> **Doreen:** Yes.
>
> **Int:** Okay, what is the relationship?
>
> **Doreen:** It is uh the relationship is that π is equivalent to 180°. So if you multiply 180° by two it will give you 360°.
>
> **Int:** So π is equal to 180°; why is that the case?
>
> **Doreen:** It is equivalent; that is how it is approximated to (laughs).

Although Doreen was unable to explain why π is equal to 180°, she appeared skilfull at converting angles presented in degrees to multiples of π. Sketch Q1 suggests that her conception of the domains of the tangent function was characterised by the concept of π. All the angles on the horizontal axis of this sample were quoted as multiples of π and yet the test item presented the domain in degrees. In spite of this, it seemed that Doreen had to firstly reason in terms of degrees.

After seeing her tangent curve, Doreen argued that tan 270° is zero because 270° is on the x-axis. Apart from exposing her superficial understanding regarding angles for which the tangent function is undefined, this view suggests that she had no reason, this is because every other angle of the curve lay on the x-axis. The following excerpt reveals another of Doreen's misconceptions:

> **Int:** Okay, okay. Maybe let me ask this last question: in your view what do you think is the value of tan 90°?
>
> **Doreen:** It is 1 (laughs).
>
> **Int:** It is 1?
>
> **Doreen:** Yes.

Doreen claimed that the value of $\tan 90^{\circ}$ is 1, which contradicted what she drew in Sketch Q1. In this sample, she showed that tan 90°is zero. Doreen clearly demonstrated a lack of understanding that the tangent function is undefined at 90° and 270°. Apart from this, she did not understand that $\tan \pi = 0$, $\tan 0 = 0$ and that $\tan 2\pi = 0$.

Sara declared that tan 270° is $\dfrac{1}{\sqrt{3}}$ and failed to provide a reason for this flawed point of view. This perspective suggests that she perceived tan 270° to be equal to tan 30°. She most likely used dimensions from the special triangle containing the special angle 30° to arrive at the incorrect value of tan 270°. Sara revealed another misconception regarding the value of tan 90°:

> **Int:** What is the value of $\tan 90^{\circ}$?
>
> **Sara:** $\tan 90^{\circ}$ is zero.
>
> **Int:** It is zero, why is it zero?
>
> **Sara:** Because the tangent curve doesn't pass through (pauses).
>
> **Int:** It doesn't pass through what?
>
> **Sara:** Through zero.

Sara exhibited a lack of understanding that between 0° and 360°, the tangent function is also undefined at 90°. She argued that tan 90° is zero because the tangent curve does not cross the point $(0,0)$. This confirmed that she did not know that tan 0° = 0. Sara declared that the tangent function is not defined at 180°, which suggests her lack of understanding that tan 180° = 0. She claimed that the tangent curve is what could help her to explain why tan 180° is undefined. Despite looking at her graph, she could not justify her statements. A copy of her tangent curve is replicated below as Figure 5.33.

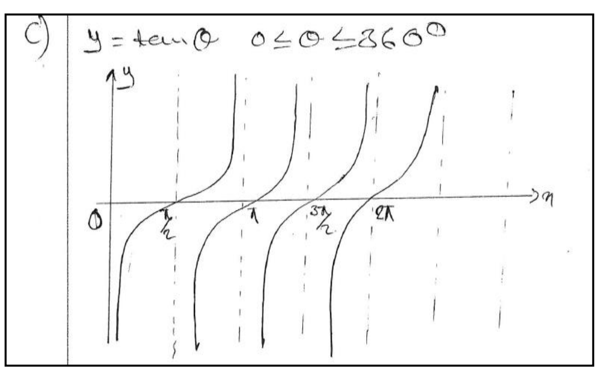


Figure 5.33: Sketch Q2

Figure 5.33 portrays that tan 180° is defined, and that it is equal to zero. The sample also shows that tan 270° is equal to zero. These observations suggest that Sara's understanding was conflicted regarding the values of tan 180° and tan 270°. Sketch Q2, and her explanation, showed that Sara had the misconception that tan 90° is equal to zero. She was consistently incorrect in saying that the tangent curve does not pass through the point $(0,0)$. Furthermore, Sara showed an inability to comprehensively interpret the domain $0^{\circ} \leq \theta \leq 360^{\circ}$ that had been provided, especially because her graph went beyond 360°. It appeared that she had merely memorised the appearance of the tangent curve without thoroughly understanding it.

Mwila understood that at 90° and 270°, the tangent function is undefined. While he did not justify why the function is undefined at 270°, he did so for the case of 90°alongside Figure 5.34.

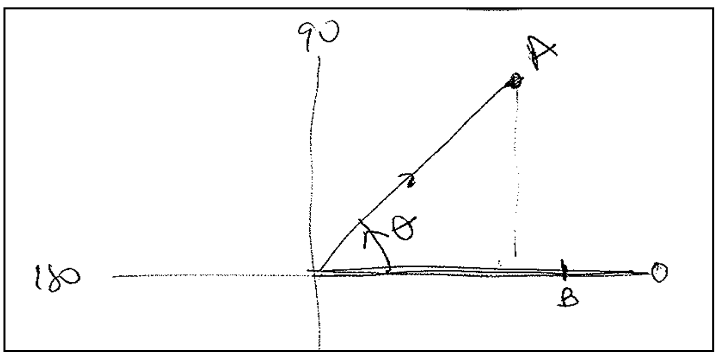


Figure 5.34: Sketch Q3

In Figure 5.34, Mwila implied that the point of intersection of the vertical and the horizontal axes was the origin. He then considered the slant line from the supposed origin to an arbitrary point A as the arm of rotation. After clarifying that $\tan\theta = \frac{opp}{adj}$, he explained that as the arm [OA] rotated in the anticlockwise direction; it would eventually reach the length equal to the hypotenuse (radius) at $\theta = 90^{\circ}$. He indicated that at $\theta = 90^{\circ}$, the opposite side would be of a length equal to the radius (hypotenuse), and that the adjacent side would be equal to zero.

Mwila then argued that at $\theta = 90^{\circ}, \tan\theta = \frac{opp}{0}$, which depicts a division by zero. It is this division by zero which he argued makes the tangent function undefined at 90°. Mwila used the relationship that exists between multiplication and division to explain why division by zero is undefined. He cited an example where the 6 was being divided by zero. He then argued that since there is no number whose product with zero is six, it follows that $\frac{6}{0}$ is undefined. Based on this example, he generalised that division by zero is undefined.

Brave correctly posited that tan 270° is undefined, but had a problem providing his justification. Instead, he kept on repeating that the tangent function is not defined at 270°. Insightfully, Brave claimed that tan 90° is equal to zero. He was asked to explain how he had arrived at this conclusion:

> **Int:** How have you found the zero for $\tan 90^{\circ}$?
>
> **Brave:** Uh by making use of the, I'm trying to visualise the graph of tan and how it comes out.
>
> **Int:** The graph, ehe?
>
> **Brave:** Yes, so uh at that point (pauses).
>
> **Int:** Ehe at which point?
>
> **Brave:** At 90° tan is zero.

Brave failed to explain why he thought tan 90° is equal to zero. His attempt to use the graph of the tangent function to justify this claim proved futile as he only restated that tan 90° is equal to zero. This confirmed that his understanding of the angles for which the tangent function is undefined was superficial. John and Mubita shared a common view that tan 270° is undefined, and were unable to comprehensively explain why this is the case, for example, Mubita claimed that the tangent function does not intersect the 270° axis because it has 'no maximum value' at that angle. This disclosure required a justification, which Mubita failed to provide.

Unlike Brave and Sara, who claimed that tan 90° = 0, John and Mubita declared that the tangent function is undefined at 90°. These two student teachers had the idea that tan 90° necessitates a division by zero. Additionally, they knew that it is the adjacent side of a right angled triangle that would have a value of zero. John had no ability to illustrate how a division by zero arises, whereas Mubita attempted to provide an illustration using a unit circle, which was unsuccessful as Mubita confessed that he could 'not recall the facts'.

Although Mubita knew that that tan 90° is equal to $\frac{\sin 90^{\circ}}{\cos 90^{\circ}} = \frac{1}{0}$, he did not seem to know the reasons why sin 90° = 1 and cos 90° = 0. Just like Mwila, Mubita used the relationship that exists between division and multiplication to explain why division by zero is undefined. He argued that $\frac{1}{0}$ is undefined because there is no number that can be multiplied by zero to give a result of 1.

Maximum and minimum values of trigonometric functions

Sara, Brave, John, Mwila, and Mubita correctly stated the maximum and minimum values of the sine, cosine, and tangent functions. Nonetheless, with the exception of Mwila, the other student teachers could not justify their reasoning. Doreen exhibited serious misconceptions concerning the aspect of maximum and minimum values of trigonometric functions. At the same time, she could not defend her position. An investigation of Doreen's understanding of the maximum value of the sine function was conducted in the following manner:

> **Int:** Okay, let me ask you this question: what's the maximum value of $f(\theta) = \sin\theta$?
>
> **Doreen:** 2π.
>
> **Int:** 2π; why 2π?
>
> **Doreen:** Because that is the complete revolution.

First and foremost, by asserting that 2π is the maximum value of $f(\theta) = \sin\theta$, Doreen showed an inability to differentiate between the values of a trigonometric function and the angles for which it is defined. Secondly, her answer typified her lack of understanding that the sine function has a maximum value of 1.

Furthermore, she had the misconception that the total number of degrees in a single revolution is synonymous with the maximum value of the sine function. With respect to the minimum value of the sine function, she explained that:

> **Int:** Okay, what is the minimum value of $f(\theta) = \sin\theta$?
>
> **Doreen:** It is π.
>
> **Int:** Okay do you have a justification why it is π?
>
> **Doreen:** No (laughs).
>
> **Int:** Not at all?
>
> **Doreen:** I can just tell from the graph of $f(\theta) = \sin\theta$.

Doreen had no understanding that the smallest value of the sine function is -1. Moreover, she did not know what the value of a trigonometric function entails. Doreen claimed that the sine curve is what informed her stance that the minimum value of the sine function is π. What was strange about this claim is that she had drawn the sine curve, which depicted - 1 as the minimum value. The sine curve that she drew is reproduced below as Figure 5.35.

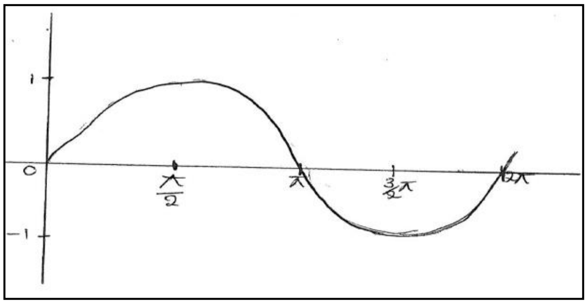


Figure 5.35: Sketch R1

An analysis of Doreen's views, together with Sketch R1, suggests that she associated the maximum and minimum values of the sine function with the angles at which the sine curve crosses the horizontal axis. It would appear that she thought that the values of the sine function can be computed in a way that is similar to how the solutions of quadratic equations are read from the horizontal axis $(y = 0)$. This conclusion is plausible because the two angles that she presented as maximum and minimum values $(2\pi, \text{and } \pi)$ give the sine function a value of zero. Doreen's views concerning the maximum and minimum values of $\cos\theta$ are revealed in the subsequent extract:

Int: Okay, what is the maximum value of $\cos\theta$?

Doreen: Maximum value?

Int: Ehe.

Doreen: It is also positive 360°.

Int: 360°?

Doreen: 360°.

In keeping with her notion of the values of trigonometric functions being angles, Doreen stated that the maximum value of $\cos\theta$ is 360°. In the case of $\sin\theta$, she cited the maximum value as a multiple of π, but used degrees when it came to $\cos\theta$. Doreen argued that $\cos\theta$ has a maximum value of 360° because 'the complete revolution ends at 360°'. She was requested to confirm her position in a hypothetical situation where a pupil provides -20 as one of the values of $\cos\theta$:

Int: Okay, supposing a pupil gave you -20 as one of the values for $\cos\theta$. How would you take that?

Doreen: Can you find a negative value? Unless you check if it is possible that the (stammers), the cosine inverse of the -20 can be found, because sometimes you find that those they don't exist.

Int: Okay, what does not exist?

Doreen: Certain values say negative something like that I don't know (laughs).

Doreen was reluctant to confirm whether $\cos\theta = -20$ is appropriate or not. At first she questioned if it is possible to find a negative value for $\cos\theta$. This reaction suggests that she was not sure that $\cos\theta$ can assume negative values. It appears that she did not have the understanding that the minimum value of the cosine function is -1. Moreover, her proposal of a requirement to first of all assess if $\cos^{-1}(-20)$ exists or not suggests that she was not certain that -20 cannot be a value for $\cos\theta$. These considerations confirmed that she did not understand the maximum and minimum values of $\cos\theta$.

When probed about the maximum value of $\tan\theta$, Doreen was hesitant to express her position. However, after being prompted, she pointed out that she did not know the maximum value of $\tan\theta$. Her reaction was different with respect to the minimum value of $\tan\theta$:

Int: What is the minimum value of $\tan\theta$?

Doreen: It is 90°.

Int: It is 90 what?

Doreen: 90°.

Doreen's notion that the minimum value of $\tan\theta$ is 90° indicated her lack of knowledge that the tangent function has unlimited values in specific intervals. Again, by using an angle as the minimum value of $\tan\theta$, she confirmed her lack of knowledge of the difference between elements of the domains and ranges of trigonometric functions.

Brave and Sara correctly identified the maximum and minimum values of the sine function. They both posited that the sine function has a maximum value of 1, and that it has a minimum value of -1. Brave's justification was elicited during the following conversation:

Int: Why is 1 the maximum value of $f(\theta) = \sin\theta$?

Brave: Anything above 1 would make it undefined.

Int: what do you mean it will be undefined?

Brave: Okay, meaning that uh mmm anything above 1 would (pauses), anything above 1 will make actually the function not to exist.

Brave seemed to lack meaningful reasons to support his stance. His claim that 'anything above 1' would make the sine function undefined was vague. He appeared to not be aware that a trigonometric function is defined in angles and not in its values. When asked to clarify the sense in which he used the word 'undefined', he provided an unclear explanation. He could not elucidate why between 0° and 360° the function $f(\theta)=\sin\theta$ reaches the maximum value of 1 only at $\theta=90^\circ$. In addition to this, he failed to explain why the function $f(\theta)=\sin\theta$ attains the lowest value of -1 at $\theta=270^\circ$ in a revolution.

The discussion with Sara developed in the following manner:

> **Int:** Why is it 1 the maximum value of $f(\theta)=\sin\theta$?
>
> **Sara:** I just know that it is 1 because it can't go beyond when it goes beyond.
>
> **Int:** You don't know why it can't go beyond?
>
> **Sara:** No, I don't.
>
> **Int:** Okay, what of the minimum value $f(\theta)=\sin\theta$?
>
> **Sara:** $\sin\theta$ it is -1.
>
> **Int:** That is the minimum value?
>
> **Sara:** Umm (yes).
>
> **Int:** Are you able to tell me the reason why?
>
> **Sara:** I'm not very sure, yes.

Evidently, Sara could not account for her assertion that the sine function has a maximum value of 1 and a minimum value of -1.

When Brave was asked to confirm whether a pupil who says $\cos\theta=-20$ would be correct or not, he indicated that such an equation is incorrect. Despite this standpoint being correct, the student did not seem to have a meaningful reasoning for this as he simply stated that the equation 'does not exist'. With regard to the minimum value for $\tan\theta$, Brave explained that it is 'infinite' due to the fact that the tangent curve does not touch the 270° axis. Firstly, positive infinity cannot be associated with the minimum value of the tangent function as it represents the aspect of values that increases endlessly. Secondly, although it is true that the tangent curve does not intersect the 270° axis, it is equally true that the tangent function does not intersect the 90° axis. Moreover, as the values of θ approach 270°, $\tan\theta$ approaches positive ∞.

Sara showed the understanding that particular values are inappropriate for the cosine function. She contended that a pupil who presents -20 and 40 as values of $\cos\theta$ would be marked incorrect. Her explanation was that -20 and 40 cannot be possible values of $\cos\theta$ because the maximum and minimum values that the cosine function can assume are 1 and -1. While this view is correct, Sara confessed that she did not know why this is the case.

With respect to the tangent function, Sara initially indicated that its maximum value is $-\infty$ to ∞. However, after being queried on this, she changed her mind and mentioned that the maximum value of $\tan\theta$ is infinity. She explained that the word 'infinite' is used to signify the unlimited nature of the values of $\tan\theta$. Notwithstanding, Sara showed that she did not have the capacity to comprehensively justify her statement. This deduction was arrived at when she stated that $\tan\theta$ has no maximum value 'because this one will keep going up and up'. Instead of presenting a reason, she described the features.

Mwila, Mubita, and John were of the correct opinion that the maximum and minimum values of both $\sin\theta$ and $\cos\theta$ are 1 and -1 respectively. Mwila presented judicious justifications for his point of view regarding $\sin\theta$, unlike Mubita and John. An analysis of John and Mubita's reasoning did not suggest any difference in understanding. As was the case with Brave and Sara, Mubita and John did not have a satisfactory justification for their assertions. Extracts based on the discussions that were conducted with John and Mubita confirm this conclusion:

> **Int:** Why do you say that the maximum value is 1 for $\sin\theta$?
>
> **John:** Because when we, if we put the values from 0° to 360° into $\sin\theta$ we will get the highest number as 1, which is its turning point.

John was right to indicate that for all the angles in a revolution the highest value that $\sin\theta$ can attain is 1. However, this was not a justification, but a mere declaration of facts. John's assertion that the maximum value of 1 is the same as a turning point suggested that he had limited knowledge of this topic. He failed to explain why the smallest value of $\sin\theta$ is -1, while Mubita kept on repeating that the sine function can not have a value greater than 1:

> **Int:** That is where the question is: why is it that we cannot have a value greater than 1 for $\sin\theta$?
>
> **Mubita:** Because this is a periodic function. It repeats itself after a certain (stammers), after a certain interval. So it doesn't exceed that same value because after a given interval it repeats the cycle. So you are always ending up with the same maximum value.

Instead of explaining why the sine function has a maximum value of 1, Mubita described the characteristic of the function. He claimed that the sine function has a maximum value of 1 because of its periodic nature. If this explanation were to hold true, it would imply that the tangent function, which is equally periodic, must have a maximum value of 1. Therefore, Mubita's claim was not only incorrect, but it exposed his inability to provide an explanation for his standpoint.

With regard to the cosine function, John stated that he would mark a pupil wrong if he/she gave -20 and 40 as possible values for $\cos\theta$. This was his explanation:

> **Int:** But why would you mark that pupil wrong if he/she gave you -20 and 40 as two possible values of $\cos\theta$?
>
> **John:** $\cos\theta$ is always between 0 and mmm between -1 and 1.
>
> **Int:** Why is that the case?
>
> **John:** Because those are the only values obtained when you 'punch' [press] the value of the degrees on the calculator.

John failed to provide a reason why, for example, the cosine function has a maximum value of 1. To suggest that 1 and -1 are 'the only values obtained' when a calculator is used showed a limited understanding. Similarly, Mubita demonstrated superficial knowledge when he failed to account for his stance that the cosine function has a minimum value of -1:

> **Int:** Why is -1 the minimum value of $f(\theta) = \cos\theta$?
>
> **Mubita:** Because similarly it repeats itself. It is also a periodic function and it does not (stammers), it does not drop further than -1 as it repeats itself. The lowest it can go is -1.

Mubita claimed that $f(\theta) = \cos\theta$ has a minimum value of -1 due to the characteristic of its graph being periodic. Again, this notion suggested that he did not understand the reasons why $\cos\theta$ has the lowest value of -1 at $\theta = 270^\circ$. He was asked to clarify:

> **Int:** What is the relationship between the period of a function, and the maximum and minimum values of functions; picking it from your answers?
>
> **Mubita:** Uh what is (stammers), what is the relationship between, yes, uh these are wave functions. These are wave functions and they can only go up to a certain height and they can drop up to a certain low. So since they repeat themselves after a certain; they are cyclic, they are cyclic they are not continuing in a (pauses), they are not continuing in a certain direction upwards, but they are 'wavey' along x [x-axis], so I

should think that is what explains (pauses again), that is the explanation I can give as to why they cannot have eeh an infinite; an infinite maximum or an infinite minimum.

Mubita could not establish a relationship between the maximum and minimum values of the sine and cosine functions, and the periods of these functions. He merely described some of the characteristics of the graphs of the functions. This confirmed that he had no basis for relating the periods of the functions to their maximum and minimum values.

Mwila provided appropriate reasons for his opinion that $\sin\theta$ has maximum and minimum values of 1 and -1 respectively. He used Sketch R2 in Figure 5.36 to explain that at $\theta = 90^\circ$ a scenario arises where the radius of the circle becomes equal in length to a side that is opposite to the angle formed by the radius and the positive x-axis. This sample is reproduced below for easy reference:

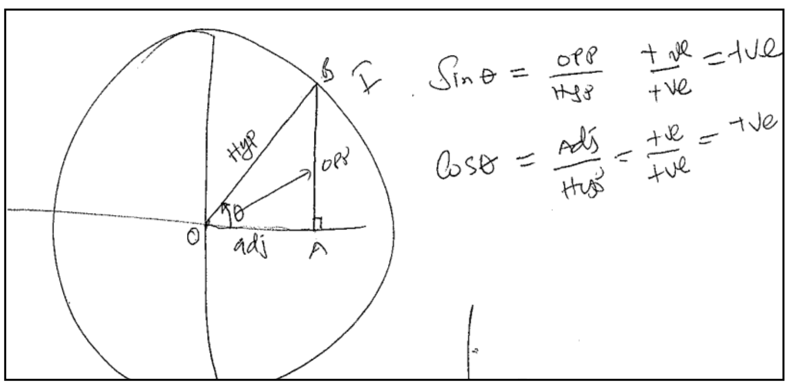


Figure 5.36: Sketch R2

He elaborated that $\sin\theta = \dfrac{opp}{hyp} = \dfrac{AB}{OB}$ and explained that it is only when the numerator and the denominator are equal that a quotient of 1 is attained. He emphasised that such a situation occurs when $\theta = 90^\circ$. Thus, as θ approaches 90°, *AB* approaches *OB*. He agreed that his views are equally applicable to a unit circle. In a unit circle $\sin\theta = y$, therefore, it follows that at $\theta = 90^\circ$, $\sin\theta = \dfrac{OB}{OB}$ which gives a value of 1. Similarly, Mwila provided a satisfactory explanation to support his notion that the minimum value of $\sin\theta$ is -1. He presented similar arguments as those given for the maximum value of $\sin\theta$. Mwila was, nonetheless, unable to give his reason regarding the maximum and minimum values of $\cos\theta$.

Mwila, Mubita, and John explained that for $\tan\theta$, there is no particular number that would represent its maximum value and that no number exists as its minimum value. While these student teachers expressed the facts, none of them showed the capacity to satisfactorily explain why the tangent function consists of unbounded ranges of values. They could not explain, for instance, why $\tan\theta$ has unlimited ranges of values in intervals such as $0^\circ < \theta < 90^\circ$, and $90^\circ < \theta < 270^\circ$. Moreover, they essentially neglected to mention that the tangent function is not continuous.

5.4. SUMMARY OF CHAPTER 5

Chapter 5 presented the results and analyses of the research data on trigonometry that was collected using a mathematics test, as well as the semi-structured interviews. Regarding the test, descriptive statistics were used to provide summaries of the data, followed by qualitative analysis techniques. The analysis of the test results was carried out through the CCK and SCK categories of the study's conceptual framework. Only the following component of SCK was utilised: the ability to use different representations. This is because it was considered that the interviews would sufficiently use the second component of SCK. Consequently, the results and analysis of the interviews were solely presented via the SCK component: the ability to explain and justify reasoning. The next chapter provides a synthesis of the study's findings.

6. FINDINGS AND DISCUSSION

6.1. INTRODUCTION

Chapter 6 provides a synthesis of the study's findings. This is done with a view to drawing a conclusion concerning the research's main question, which reads: *How can the University of Zambia's mathematics student teachers' content knowledge of functions and trigonometry at secondary school level be described?* The data was collected in two phases: Phase 1 involved a mathematics test that assessed the student teachers' Common Content Knowledge (CCK) and Specialised Content Knowledge (SCK) of functions and trigonometry at secondary school level. Phase 2 used the semi-structured interviews to explore the student teachers' SCK. In this respect, an exploration of the student teachers' ability to explain and justify their reasoning, as well as their ability to work with different representations, was conducted.

In terms of organisation, this chapter has two main sections. Section 6.2 provides an overview of the findings based on functions, followed by Section 6.3, which presents the results concerning trigonometry. Section 6.2 is split into Sections 6.2.1 and 6.2.2, while Section 6.3 is divided into Sections 6.3.1 and 6.3.2. Table 6.1 below provides a synopsis of the organisation of Chapter 6, after which a discussion regarding the findings on functions is presented.

Table 6.1: Summary of organisation of Chapter 6: synthesis of the research findings

Sections 6.2 and 6.3	Research questions	Features
Section 6.2.1	Research question 1: How proficient are the student teachers in the Common Content Knowledge of functions at secondary school level?	An overview of the findings on the proficiency of the student teachers in the Common Content Knowledge of functions.
Section 6.2.2	Research question 2: What Specialised Content Knowledge of functions at secondary school level is held by the student teachers?	Student teachers' explanations of their understanding of concepts; Student teachers' justification of their reasoning; Student teachers' understanding regarding the use of different representations.
Section 6.3.1	Research question 1: How proficient are the student teachers in the Common Content Knowledge of trigonometry at secondary school level?	Synthesis of findings on the proficiency of the student teachers in the Common Content Knowledge of trigonometry.
Section 6.3.2	Research question 2: What Specialised Content Knowledge of trigonometry at secondary school level is held by the student teachers?	Student teachers' explanation of concepts (including quadrants, rules, and formulas); Student teachers' conception of the periods of the sine, cosine, and tangent functions; Student teachers' perspectives on the differences between the sine and tangent functions; Student teachers' explanation of their methods in selected items; Student teachers' justification of their reasoning; Student teachers' ability to translate the sine and tangent functions between the algebraic and graphic representations; Student teachers' ability to draw two special triangles.

6.2. DISCUSSION OF THE FINDINGS ON FUNCTIONS

6.2.1. How proficient are the student teachers in the CCK of functions at secondary school level?

The research findings indicate that the student teachers achieved a mean performance score of approximately 52% in the CCK category as it related to functions. An item by item analysis of the data shows that 77% of the student teachers had difficulty defining a relation. Some of their definitions of a relation excluded examples that cannot be represented by standard formulas. There were student teachers who defined a relation as a one-to-one mapping, while others indicated that relations are ‘functions’ that map elements from the domain to the range. Definitions like this demonstrate that most of the student teachers had a limited understanding of relations. Similarly, it appeared that the majority of the students could not articulate the difference between an ordinary relation and a function.

70% of the student teachers demonstrated an inability to provide a valid definition of a function. A number of the student teachers’ definitions included relations such as many-to-many. At the same time, restrictive definitions were presented that categorised functions as solely one-to-one relations to the exclusion of the many-to-one types of relations. These findings show that the student teachers had limited knowledge of the univalence condition of functions. Interestingly, a significant number of the students were proficient in the identification of functions and non-functions. In this regard, 82% of the students recognised the many-to-one function, which was represented as an arrow diagram. In terms of the one-to-many relation, which was depicted as a graph, 59% of the student teachers identified it as a non-example of a function. In addition to this, 68% of the student teachers recognised a circle on a Cartesian plane as a non-example of a function (see interview discussions in Section 6.2.2).

Similarly, 95% of the student teachers did not present a valid definition of an inverse function. While some of the definitions suggest that they had the idea that an inverse function ‘reverses’ the operations of the function for which it is an inverse, this view was not comprehensively articulated, for example, one student teacher asserted that an inverse function ‘is the function that takes back the element in the range back to the domain’. This definition appears to be acceptable, except that it does not clarify whether or not an element of the range that is taken ‘back’ to the domain is primarily a member of the domain. It seems that this student teacher considered the characteristic of inverse functions operating on images. However, he was unable to clearly express this view. Nonetheless, approximately 64% of the sample provided appropriate calculations that lead to an expression for the algebraic inverse function of a quadratic function. Also, a significant number of the student teachers correctly calculated the inverse of a linear function. These results imply that the majority of the student teachers were satisfactory adept at calculating inverse functions, but lacked the ability to comprehensively define an inverse function. There seemed to be a disconnection between the student teachers' ability to correctly calculate algebraic inverse functions, and their ability to provide a valid definition of an inverse function.

A slightly different picture emerged when it came to the one-to-one functions. Whereas 96% of the sample failed to provide a valid definition for a one-to-one function, about 73% equally lacked the proficiency to identify Cartesian plane examples or non-examples of one-to-one functions. These statistics suggest that most of the student teachers did not have in-depth understanding of one-to-one functions. A classic example of the student teachers' flawed definitions reads: 'a one-to-one function is a function that maps each element in the domain to only one image in the range'. Basically, this definition depicts the feature that distinguishes ordinary relations from functions, but does not indicate what differentiates many-to-one from one-to-one functions. Moreover, two arrow diagrams were provided alongside this definition. One of these diagrams correctly depicts a one-to-one function, while the other represents a many-to-one function (Section 4.2.1). In fact, the arrow diagrams point to this student teacher's lack of capacity to distinctly define a one-to-one function.

There were other aspects of CCK in which most of the student teachers showed a lack of expertise. Particularly, the data suggests that a significant number of the student teachers (36%) had no idea how functions are composed. Additionally, 18% of the student teachers had no understanding that the order of composing functions is not always commutative. In this regard, they assumed that the expression $(z \circ g)(x)$ is generally equal to $(g \circ z)(x)$.

Similarly, the majority of the student teachers (73%) were unable to complete the square to determine the turning point of $f(x) = -2x^2 - x + 8$. Instead of completing the square, some of the student teachers solved the equation $f(x) = 0$. At the same time, most of the student teachers (64%) were unable to compute the maximum value of $f(x) = -2x^2 - x + 8$.

Most of the student teachers (64%) were unsuccessful in calculating the range of the function $f(x) = -2x^2 - x + 8$. Furthermore, the majority of the student teachers were unable to determine the domains in which the function $z: x \rightarrow x^2 - 2x$ has an inverse. While the sample of the present study was comprised of student teachers, the last two findings are consistent with the results of Hitt's (1998) study, which revealed that teachers have difficulty in identifying the domains and ranges of graphical functions. The current study used an algebraic quadratic function to assess the student teachers' ability to determine domains, whereas Hitt's study made use of graphical functions. In the present study, the concept of domain was examined in association with inverse functions.

6.2.2. What SCK of functions at secondary school level is held by the student teachers?

Student teachers' explanation of their understanding of concepts

While the student teachers were required to present the definitions of certain concepts in the test as part of the CCK category, they were also asked to provide an explanation during the interviews regarding their understanding of the definitions of these concepts as a part of the SCK. Researchers have suggested that any valid definition of a function should include the univalence and arbitrariness properties of functions (Even, 1990; Lloyd et al., 2010; Nyikahadzoyi, 2013). Another necessary condition is that a function should be defined on every object in the domain. These aspects are the basis upon which the student teachers' understanding of the definition of a function are discussed in this section.

The interview findings show that the student teachers demonstrated knowledge of the rule of correspondence (Markovits et al., 1986) between sets as they explained the function concept. Nonetheless, even though most of them shared a common understanding that a function relates members of two sets; their explanations lacked depth, and at times were restrictive. A typical example involves a student teacher named John, who confined his description of a function to the rule of one-to-one correspondence. He explained that for a relation to be a function, it only needs to fulfil the one-to-one correspondence property. This conception excluded the many-to-one function, and validated the test findings, which showed that some of the student teachers claimed that relations have to be one-to-one to qualify as functions. Similar results were obtained by other researchers, who report that a function is normally narrowly viewed as a relation that only satisfies the one-to-one correspondence property (Evangelidou et al., 2004; Spyrou & Zagorianakos, 2010).

Some student teachers could not accept that a many-to-one relation is also a function, while others showed a lack of ability to competently describe a many-to-one function, for example, the researcher cited a relation comprising three elements in the domain and two images in the range. Two objects were associated with the same image in the range, while the third object was linked to the second image. There was, of course, no regular or standard formula that was utilised to link the elements of the two sets. A student teacher posited that this example is not a function because ‘all’ objects are not linked to the same image. Generally, this was a case of a shallow understanding of functions and of many-to-one functions. This discovery complemented the results of prior research, which shows that students have difficulty perceiving a many-to-one relation as a type of functions (Spyrou & Zagorianakos, 2010).

The univalence property of functions that requires each element of the domain to be mapped onto a unique element in the range characterises the student teachers’ explanations of a function. Regardless, the findings suggest that this requirement was not well expounded by most of the student teachers. There was general ambiguity as the student teachers attempted to explain that for a function, every element of the domain should be associated with only one image in the range. In some cases, the univalence condition was applied as though it means that there should always be a single element in the range of a function, and that all elements of a domain must be connected to that single image. This was in addition to the instances when the condition was used as though it was synonymous with a one-to-one correspondence.

Another interesting finding was that the arbitrariness property of functions was disregarded in the student teachers' explanations of a function. Some student teachers emphasised the necessity of a rule, while others contended that functions involve 'patterns' that are used to generate images in the range. In the strictest sense, these views suggest that relations that are ordinarily functions should be disqualified in the absence of noticeable 'rules' or 'patterns'. In other words, the student teachers' explanations insinuated that functions are normally accompanied by algebraic formulas. Other investigators point out that students usually think that functions must include algebraic formulas (Bayazit, 2011; Clement, 2001). In the current study, one student teacher's understanding of functions included concept images involving machines that produce outputs when inputs are provided.

In order to 'understand' the function concept, students should learn not only the rule of correspondence, but also the sub-concepts of domain, and range (Markovits et al., 1986). On the one hand, the student teachers from the low content knowledge category did not have the aptitude to satisfactorily explain the concepts of domain and range. They showed a lack of capacity to distinguish between the elements in the domain, and the domain itself. Similarly, these student teachers had difficulties differentiating between images and range. Although one student delivered a meaningful definition of both the domain and range, she was unable to explain these definitions. On the other hand, the student teachers from the high content knowledge category explained the concepts of domain and range, but in a restrictive manner. These student teachers articulated ideas that were limited to the domain and range of functions that can be represented as algebraic formulas.

An additional finding that corroborated the test results is that the majority of the student teachers struggled to comprehensively explain their understanding of a one-to-one function. Only one out of the six student teachers demonstrated the ability to thoroughly explain the concept of a one-to-one function. Instead of explaining what a one-to-one function is, most of the students endeavoured to articulate the requirement that distinguishes relations from functions. To an extent, this result validates past research studies indicating that students have difficulty differentiating between the definition of one-to-one functions and the univalence property (Dubinsky & Harel, 1992; Leinhardt et al., 1990; Markovits et al., 1986).

Two student teachers did not know the difference between a one-to-one and a many-to-one function. One of these erroneously declared an example of a many-to-one function to be a one-to-one function. This exposed his limited knowledge of the one-to-one type of functions. Nevertheless, one student teacher appropriately used the heads of states and the flags of their countries to illustrate a one-to-one function.

In their explanations of the concept of an inverse function, most of the student teachers underscored the aspect of 'reversing' that which is done by another function. Interestingly, one student teacher explained an inverse function as 'a process of reverting the function to the original expression'. While the idea of a function being perceived as a 'process' is dealt with in the research literature (Breidenbach, Dubinsky, Hawks, & Nichols, 1992), the student teacher's explanation was inaccurate because an inverse function does not 'revert' to the initial expression, but rather relates images to their corresponding objects. Otherwise, the student teacher explained that he was referring to the procedure of calculating an algebraic inverse of a function.

Two student teachers included the notion of an inverse function 'un-doing' that which is done by another function. The characteristic of 'un-doing' is considered as a defining feature of the definition of an inverse function (Bayazit & Gray, 2004). However, they limited their interpretation to the procedure of calculating an algebraic inverse function. Overall, it appeared that the six student teachers confined their explanations of an inverse function to the formulaic representation. Instead of being holistic in their approach, their explanations were limited to the procedure that is performed when calculating an algebraic inverse function. This finding validated the test findings, which indicated that the student teachers were proficient at computing algebraic inverse functions even when they could not competently define an inverse function.

With respect to composite functions, the student teachers provided explanations that suggested that they had a general understanding thereof. Specifically, Brave and Doreen described composite functions as combinations of functions. Sara contended that composite functions depict a situation where one function is contained in another function. Similarly, John, Mubita, and Mwila insinuated that composite functions are a combination of functions. It therefore seems that the student teachers perceived composite functions as functions of other functions. A common finding from the test and the interviews was that the majority of the student teachers did not understand that, generally, the order in which functions are composed cannot be interchanged.

Student teachers' justification of their reasoning

Research has shown that there are instances when teachers may know how to solve a question or use a procedure, but may fail to explain why these procedures are used or why they work (Bryan, 1999; Even & Tirosh, 1995). It was, therefore, in the context of relational understanding (Skemp, 2006) that the student teachers were interviewed regarding their capacity to provide an explanation or justification. Although in this study, the ability to identify functions and non-functions is conceptualised as a feature of the CCK, the student teachers' capacities in this regard were equally investigated during the interviews. The idea was to create a foundation for exploring the student teachers' justifications.

In Section 6.2.1, the test data was discussed, which indicated that the majority of the student teachers were proficient in recognising the graphs of functions and non-functions. The interview data corroborates the test result in that the majority of the student teachers were able to identify the graphs of functions and non-functions. A significant number of the students posited that they used the vertical line test and definitions of a function to identify functions and non-functions. Remarkably, unlike in previous research where most of the university students considered a circle as a function (Tall & Bakar, 1992), the current study's findings suggest that most of the student teachers identified a circle as a non-function. Among those who identified functions and non-functions were two student teachers (Brave and Sara) who exhibited isolated challenges. Brave, for example, failed to recognise that a many-to-one arrow diagram is a function. At the same time, Sara could not recognise that a one-to-many Cartesian graph is not a function.

The test data indicates that the majority of the student teachers struggled to provide a sound explanation for their assertion that particular figures represent functions or non-functions (see Table 4.4). The interview findings suggest that the majority of the student teachers' justifications were based on their understanding of the definition of a function. In that context, some of the explanations legitimised the student teachers' superficial idea of the univalence condition of functions. Some of the student teachers could not explain why a circle is not a function. John, for instance, indicated that a circle is not a function because a vertical line crosses it at more than one point. While the vertical line test enabled the student teacher to make a correct determination, it is a mere strategy that is applied to determine the graphs of functions and non-functions, and therefore did not amount to an explanation.

Doreen was generally unable to identify functions and non-functions or to justify her viewpoints. This student teacher claimed that a many-to-one arrow diagram was not a function because there was no one-to-one association between the elements of the domain and the range. This explanation suggests that she could not distinguish between a one-to-one function and a function in general. She also contended that the one-to-many Cartesian graph was a function. A one-to-many Cartesian graph can be a function if the vertical axis is taken as the domain and the horizontal axis as the range. Doreen's reasoning was that for every point on the one-to-many Cartesian graph there was 'one' x-value. This perspective exposed her lack of in-depth understanding of the condition that distinguishes relations from functions. Doreen asserted that there is not 'much' difference between relations and functions. In fact, she could not provide a comprehensive justification, and declared that there is a possibility that the Cartesian graph of a circle is a function. Initially, she mentioned that a circle is not a function, but changed her point of view after being requested to provide a reason. Moreover, she did not provide any justification for contending that the graph of a circle can be a function.

Other areas concerning the student teachers' capacity to provide justifications related to the vertical and horizontal lines drawn on a Cartesian plane. Three student teachers explained that a vertical line drawn on a Cartesian plane is a function. Despite having this misconception, they could not justify this perspective. One of the students claimed that vertical lines are functions since they are the same as graphs of linear functions. What this student teacher did not know is that the graphs of linear functions are slanted straight lines. The other three student teachers appropriately concluded that vertical lines that are drawn on a Cartesian plane are not functions. They explained that on a vertical line, a single *x*-value is usually associated with more than one *y*-value. This suggests that the student teachers applied the univalence condition in deciding that a vertical line is not a function. Some of the students added that vertical lines have no gradient. The implication of this disclosure is that while the images on a vertical line vary, the object is constant.

Again, three student teachers indicated that a horizontal line drawn on a Cartesian plane is not a function. It would seem that they did not know that horizontal lines are special examples of many-to-one functions called constant functions. At the same time, these student teachers were unable to provide a reason to support their view that horizontal lines are not functions. The perspective of these three student teachers complements the results of another study in which the majority of the university students considered constant functions expressed graphically or algebraically to be non-functions (Tall & Bakar, 1992). Nevertheless, the other three student teachers correctly identified horizontal lines as examples of functions. Their uniform justification was that for a horizontal line there are several values from the domain that are associated with one image. They explained that as the *x*-values changed, the *y*-value would remain constant, giving a gradient of zero. These student teachers used the univalence condition implicitly to conclude that horizontal lines are functions.

In terms of explaining their reasoning based on examples and non-examples of graphs of one-to-one functions, the interview results complement the test results. The majority of the student teachers showed a lack of ability to provide a justification for their viewpoints. Those who correctly identified the graphs attempted to use their understanding of the definition of a function to justify their position. Even the two student teachers who incorrectly identified the graphs tried to use the univalence condition of functions to justify their stance. This suggested that the student teachers had no in-depth understanding of the difference between a condition that qualifies relations as functions and the definition of a one-to-one function.

One of the student teachers who had initially correctly identified a non-example of a one-to-one graph argued that the graph was not a one-to-one function because it did not satisfy the horizontal line test. What this student articulated could not be categorised as an explanation, but rather as a strategy that helps in the identification process. This scenario points to the student teacher's lack of ability to provide a justification for his position. The student teacher then later changed his mind and incorrectly claimed that the graph in question was one-to-one. This reveals his lack of in-depth understanding of the horizontal line test.

All of the student teachers adeptly explained the process involved in computing the inverse of an algebraic function. However, none of them could explain *why* they performed the specific procedure. None of the students could, for example, give a reason as to why the independent variable is normally expressed in terms of the dependent variable in the process of computing an inverse function. None of them were able to justify the necessity of replacing the y-variable with the x-variable in the final expression of the inverse function. A student teacher who successfully calculated the inverse of $y = 2x + 3$ declared that he had to make x the subject as it is a 'method' that is essential when calculating the inverse. When asked for an explanation concerning the replacement of y with x in the final expression $\frac{x-3}{2}$, he claimed that he did not want to 'distort'. This response confirms of a deficiency of substantive reasoning. As Even and Tirosh (1995) found, the student teachers' experience with the computation of inverse functions was a typical case of 'knowing how' without understanding 'why'.

Student teachers' understanding of different representations

On the one hand, the student teachers who were considered to have high content knowledge demonstrated an extensive awareness of the different forms in which functions can be represented. They mentioned the formula, arrow diagrams, Cartesian representations, and sets of ordered pairs as some of the ways in which functions can be represented. On the other hand, the student teachers in the low content knowledge group showed varied degrees of awareness with regard to the different forms in which functions can be characterised. Nonetheless, each one of these student teachers indicated formulas as a form of function representation. Sara and Brave included the graphical representations as another form of representing functions.

Brave alluded to what he described as the 'set notation', while Doreen explained the set builder notation. Doreen lacked the ability to comprehensively explain the set of ordered pairs representation, which she referred to as the set builder notation. Brave's description of the 'set notation' suggests that he had the arrow diagrams in mind. An insightful finding, however, was that Brave and Doreen believed that all functions can be represented as algebraic formulas. Although Sara did not directly advance this view, she implicitly insinuated it. A notion that all functions can be represented as formulas disregards the arbitrariness property of functions. In a way, the three student teachers' stance suggests that their understanding of the forms of function representations was superficial.

In terms of the representations of linear functions, the test data shows that the majority of the student teachers were able to sketch a Cartesian straight line graph. This was done in response to a requirement to draw the graph of a function that passes through two points on a Cartesian plane. Generally, the interview data complemented this finding. Five student teachers explained that a linear function can be represented graphically, and asserted that its graph is a straight line. One student teacher's explanation suggested that linear functions can only be expressed in the form of a formula. Nonetheless, most of the student teachers explained that, given an algebraic linear function, it is possible to draw its graph when at least two points are known. They emphasised that it is easier to use the *x* and *y* intercepts when sketching the graph of a linear function.

Furthermore, the student teachers explained how to translate from the graphical to the algebraic representation of a linear function. They recited the standard formula $y = mx + c$ and indicated that m represents the gradient, while c is the *y*-intercept. Additionally, they mentioned that it is necessary to use two points from the graph to calculate the gradient and the *y*-intercept. They explained that the values of the gradient and *y*-intercept are then supposed to be substituted into the the standard formula $y = mx + c$. Regardless of these explanations, none of the student teachers could explain the basis of the formula $y = mx + c$. This result is similar to what Bryan (1999) discovered where most of the students could not explain why m represents the gradient and c is the *y*-intercept in $y = mx + c$. This was in spite of the fact that the students easily identified the gradient and the *y*-intercept of an algebraic linear function.

Another sphere in which the majority of the student teachers could recite facts without relational understanding involved the shapes of the curves of quadratic functions. All of the student teachers explained that $f(x) = ax^2 + bx + c$ for $a \neq 0$ has a minimum turning point when $a > 0$ and that it has a maximum turning point when $a < 0$. However, none of the student teachers provided an appropriate explanation of why these relationships exist. Even the student teacher who attempted to use the 'completing of the square' method did not effectively clarify why the relationship exists between the sign of the leading coefficient of $f(x) = ax^2 + bx + c$ and the shape of the parabola. This finding agrees with the results of past research in which students have recounted the relationship between the coefficient of x^2 and the shape of a parabola, but were unable to explain why the relationship exists (Even & Tirosh, 1995).

A conflict was discovered in the understanding of three student teachers, namely, Doreen, Mubita, and Sara. This conflict related to the total number of characteristic shapes of the curves of quadratic functions. They asserted that the graphs of quadratic functions take four forms instead of merely opening upwards or downwards. They contended that parabolas open either towards the positive or towards the negative direction of the *x*-axis (horizontal axis). While this is possible in cases where the vertical axis represents the domain and the horizontal axis represents the range, the student teachers did not portray this understanding. The student teachers' conceptions were otherwise conflicted with respect to the types of turning points. Initially, each one of the students had intimated that there are two types of turning points for a quadratic function, namely, minimum and maximum turning points.

In the test, most of the student teachers could draw the graph of a symbolic quadratic function when a table of values is provided. The interview findings suggest that the student teachers from the low content knowledge category espoused the method of using a table of values when drawing parabolas. However, Doreen, Sara, and Brave had problems explaining how to decide if the generated points were sufficient to draw an appropriate curve.

One of the student teachers alluded to the need to calculate a turning point, but was quick to confess that it was difficult to remember the formula used. This indicated the absence of in-depth relational understanding. This finding complemented the test results, which showed that some student teachers did not know how to use the 'completing of the square method' to determine the turning point of the function $f(x) = -2x^2 - x + 8$.

In contrast, the student teachers from the high content knowledge classification calculated the turning point as well as the *x* and *y* intercepts. Mwila, Mubita, and John indicated that for the function $f(x) = ax^2 + bx + c$, the *x*-intercepts are calculated via the equation $f(x) = 0$. The students mentioned that the *y*-intercept is obtained when $x = 0$ in $f(x) = ax^2 + bx + c$. Additionally, they indicated that the use of the discriminant $b^2 - 4ac$ allows one to know in advance the position of the curve in relation to the *x*-axis. Mwila, Mubita, and John explained that when the discriminant is positive it, means that there are two different *x*-intercepts involved, whereas a negative discriminant implies that there are no *x*-intercepts. At the same time, the students explained that when the discriminant has a value of zero, the parabola touches the x-axis at one point. The student teachers indicated that through a process of completing the square of the quadratic function, the *x*-value at the turning point turns out to be $-\frac{b}{2a}$, while the *y*-value is $f\left(-\frac{b}{2a}\right)$.

While it may seem reasonable to assume that student teachers who have studied advanced university mathematics are familiar with the general form $f(x) = ax^2 + bx + c$, $a \neq 0$ of an algebraic quadratic function, this study's findings suggest otherwise. The student teachers in the low content knowledge group showed a deficiency of understanding regarding the formulaic representation. One of the student teachers asserted that $f(x) = x^2 + y^2 + 3x + 5$ is a quadratic function. Her rationale was that 'at least one variable' is raised to the power of two. Evidently, she did not know that only one term is supposed to be of degree two in an algebraic quadratic function.

Thus, in cases where more than one term is of degree two then such terms are expected to be like terms. Moreover, she presented $3x^2+5x+4=0$ as an example of a quadratic function. This suggests that she could not differentiate between a quadratic function and a quadratic equation. Another student teacher declared that $y=x^2$ is not a quadratic function. Some researchers have observed that students cultivate their own model examples of what they consider to be functions, and it is those model examples that come to mind when they are asked to identify an example or non-example of a function (Tall & Bakar, 1992). In the current study, it appears that the many-to-one quadratic function $y=x^2$ was not identifiable as an example of a quadratic function in the mind of this student teacher.

The third student teacher contended that $y=3x^2+4$ can only be a quadratic function if the y variable is replaced by $f(x)$. Interestingly, the student who argued that $y=x^2$ is not a quadratic function asserted that $f(x)=3x^2+4$ is a quadratic function. These findings point to the student teachers' limited conception of the notations that are used to denote functions. The students did not seem to realise that both y and $f(x)$ denote an image of a function. This finding confirms the views of two scholars who posit that mathematical notation is not easy to learn (Schoenfeld & Arcavi, 1988). Specifically, Schoenfeld and Arcavi (1988) point out that "mathematical notation is a wonderful and powerful tool. It is also subtle and difficult to learn. When we have mastered it, we often forget just how hard it was and just what went into the development of our understanding" (p. 426).

Another area that proved problematic for the student teachers was translating functions between representations. The test findings indicate that a significant number of the student teachers could not change the representation of a relation from ordered pairs to a Cartesian plane. The interview results confirm that some of the student teachers struggled to translate to the Cartesian plane. The common challenge was the student teachers' inability to distinguish between relations whose domains are continuous and those whose domains are discrete. In this respect, the relation was defined on a domain, which required that the plotted points on the Cartesian plane remain in their unconnected form. Some of the student teachers erroneously joined the plotted points. There were other students who, instead of presenting Cartesian graphs, provided inequalities and arrow diagrams that did not even depict the relation in question.

Similarly, the test and interview findings indicate that a significant number of the student teachers were incapable of translaing the algebraic linear function $g(x)=|x|$ for $-3 \leq x \leq 2$ and $x \in Z$ to the Cartesian plane. Most of the student teachers connected the plotted points in the Cartesian plane, even when the domain that was provided was discrete. When requested to explain why they had connected the points, most of these student teachers could not provide a reason, or the reasons provided were incoherent, for example, one student teacher indicated that it is a requirement for all graphs of functions to have the plotted points connected using straight lines or curves. This points to the student teacher's lack of understanding of the difference between the graphs of functions defined on continuous domains and those defined on discrete domains. In Section 6.3, a discussion of the research findings as they relate to trigonometry is provided.

6.3. DISCUSSION OF THE FINDINGS ON TRIGONOMETRY

6.3.1. How proficient are the student teachers in the CCK of trigonometry at secondary school level?

The CCK category included the assessment of the student teachers' ability to apply rules, trigonometric ratios, special angles, formulas, and theorems, correctly solve trigonometric equations, angles and lengths of triangles, area of triangles, and three-figure bearings. An examination of the student teachers' performance in specific concepts showed that the majority were generally proficient concerning the application of the sine rule to calculate the sides and angles of non right-angled triangles. 82% of the student teachers effectively utilised the sine rule to determine the length of a non right-angled triangle. 73% of the students successfully applied the sine rule to calculate one of the two possible angles of a non right-angled triangle. However, about 32% of the student teachers struggled to find the second possible value of the same angle (obtuse angle). This finding suggests that some students were not familiar with the ambiguous case that occurs when two conditions exist. Firstly, the dimensions of two sides and a non-included acute angle in a non right-angled triangle must be known. Secondly, the length of a side that is opposite to the given acute angle should be less than the length of the other given side.

Most of the student teachers (77%) could use the cosine rule to calculate the length of a side of a triangle. Notwithstanding, the majority of the students (about 82%) were unable to recognise that the cosine rule was supposed to be used to find an angle of a triangle. These students demonstrated a lack of understanding about the components of the ratio of the sides of a triangle representing the lengths of the sides. Instead of applying the cosine rule, the students used the ratio of the sides to divide the total sum of angles in a triangle. This result suggests that the student teachers had limited ability in applying the cosine rule.

With regard to the computation of the area of a non right-angled triangle, it was discovered that most of the student teachers (approximately 73%) had ability to apply the formula $\frac{1}{2}ab\sin C$. The successful application of this formula suggests that the student teachers understood that it is supposed to be applied when the dimensions of two sides and an included angle are known. Similarly, most of the student teachers (59%) exhibited an understanding of the application of Pythagoras' theorem. In this study, the student teachers were required to identify an appropriate right-angled triangle from the cuboid in order to utilise Pythagoras' theorem. Therefore, the successful use of the theorem by most of the students indicates that they had the added capacity to recognise a right angled-triangle from a solid. Nonetheless, 9% of the students struggled to separate the appropriate right-angled triangle from the cuboid even when they knew how to use the theorem. A considerable number of student teachers (32%) did not provide any answer. By implication, these students lacked the ability to identify the appropriate right-angled triangle in a three dimensional figure. At the same time, it appeared that they did not know how to apply Pythagoras' theorem. This result was surprising given that Pythagoras' theorem is one of the most common theorems taught in school mathematics.

One aspect of the CCK related to the evaluation of the trigonometric expression sin 315° without using a calculator. Although most of the student teachers were able to evaluate this expression, 41% were inept at doing so. This result indicates that some of the student teachers' understanding of the signs of trigonometric ratios in quadrants as well as special angles lacked depth. There were student teachers whose calculations suggested that trigonometric ratios can be subtracted the way whole numbers or algebraic-like terms are subtracted. For instance, a student teacher attempted to subtract sin 45° from sin 360° to obtain sin 315°. Such findings suggest that the student teacher had a limited understanding of the evaluation of trigonometric expressions.

This result confirms one of the results of prior research in which it was shown that most of the students in the control group had the misconception that $\sin 40 - \sin 10 = \sin 30$ (Tuna, 2013).

The student teachers were also assessed regarding their ability to solve a basic trigonometric equation $\cos\beta = -\cos 64^{\circ}$ for the domain $90^{\circ} < \beta < 180^{\circ}$. Again, although the majority of the student teachers managed to solve this item, those who had no aptitude (45%) constituted a large percentage of the sample. By implication, there were student teachers who lacked an understanding of the sign of the cosine ratio in the second quadrant. Additionally, such students showed an inability to relate the acute angle 64° to its corresponding obtuse angle in the second quadrant.

Another aspect of the CCK in which the majority of the student teachers showed ability involved locating bearings. At the same time, most of the students (59%) demonstrated that they had the capacity to compute a three-figure bearing of one point from another. In spite of these results, it was observed that some of the student teachers could not locate the position of bearings. Others presented calculations that suggested that they did not know that a three-figure bearing is calculated in the clockwise direction from the north to a line connecting two points.

It was discovered that the majority of the student teachers (64%) could calculate the shortest distance from a point to a straight line on a plane figure. Even though some of the students used the area of a triangle, others applied trigonometric ratios to calculate the distance. Another finding was that most of the student teachers applied their knowledge of the definitions of the sine, cosine, and tangent ratios to make evaluations. Specifically, in a situation where $\sin\theta = \frac{7}{25}$ was provided, half of the sample proficiently applied their understanding of trigonometric ratios, and Pythagoras' theorem to evaluate $\cos(180^{\circ} - \theta)$. These results suggest the following: (1) the students knew that the shortest distance referred to the perpendicular distance between the point and the line, (2) the student teachers were generally familiar with the definitions of the sine, cosine, and tangent ratios, and (3) the student teachers could apply trigonometric ratios on plane figures such as triangles.

The majority of the student teachers (91%) seemed unable to use trigonometric ratios to calculate the shortest distance when the figure involved was a cuboid. Furthermore, most of the student teachers (about 73%) were unable to successfully utilise the applicable trigonometric ratios to find an angle in a cuboid. It seems that while most of the student teachers could apply the sine, cosine, and tangent ratios on a plane figure, they struggled to do so on a three-dimensional figure. This observation is consistent with Haambokoma et al.'s (2002) report, which indicates that Zambian practising mathematics teachers had difficulty teaching three-dimensional trigonometry. A synthesis of the research findings, therefore, suggests that the majority of the student teachers were proficient in the CCK of trigonometry that is taught at secondary school level in Zambia.

6.3.2. What SCK of trigonometry at secondary school level is held by the student teachers?

Student teachers' explanation of concepts (quadrants, rules, and formulas)

The interview data showed that the majority of the student teachers were capable of describing the intervals of angles for each of the four quadrants. At the same time, most of the students could explain in which quadrants sine, cosine, and tangent ratios are positive or negative. However, two student teachers were unable to provide satisfactory explanations on this topic. One of the two students indicated that in the second quadrant, the cosine ratio is positive, while the sine and tangent ratios are negative. The tangent ratio is, of course, negative between 90° and 180°, but it was incorrect of the student to assert that the sine ratio is negative and that the cosine ratio is positive. The student teacher otherwise contended that the cosine ratio is negative in the fourth quadrant and that the sine ratio is positive in that quadrant.

The second student teacher claimed that the cosine ratio is negative in the fourth quadrant. These views suggest that the two student teachers had a superficial knowledge of the quadrants in which the trigonometric ratios are positive or negative. While the test results showed that most of the student teachers could apply the sine rule, the interview data indicates that the majority of the student teachers did not have the ability to comprehensively explain this rule. Only one student teacher indicated that the sine rule is a relationship between the sides of a triangle and the sines of the corresponding opposite angles. Furthermore, only two student teachers could comprehensively explain the conditions that warrant the use of the sine rule. At the same time, only two student teachers out of six could explain the process of deriving the sine rule. These findings suggest that most of the student teachers' understanding of the sine rule were instrumental and not relational.

There was a similar scenario concerning the student teachers' ability to explain the cosine rule. The students from the high content knowledge category showed a satisfactory understanding of what the cosine rule is, and when to apply it. These student teachers explained that the rule is used to calculate the angles and lengths in non right-angled triangles. They posited that the rule is relevant when the sine rule does not apply as long as specific information is available. The students asserted that, given two sides and an included angle in a non right-angled triangle, the cosine rule can be employed to find the length of the third side. Likewise, they explained that the cosine rule can be used to compute angles of non right-angled triangles in situations where the dimensions of the three sides are known.

Alternatively, the discussions with the student teachers from the low content knowledge bracket provided unique insight into their conceptions of the cosine rule. One student teacher knew the conditions that necessitate the use of the cosine rule, but said that in order for the rule to balance, it is necessary to always start with the longest side of the triangle when stating it. Similarly, the other two student teachers had misconceptions regarding the cosine rule, for example, one of the two claimed that when three angles are known in a non right-angled triangle, the cosine rule can be applied to find the lengths of sides.

This student then explained that the rule can be utilised to compute an unknown length when three angles and one side are known. These perspectives suggest that this student teacher had no understanding that the rule is utilised to compute lengths when two lengths and an included angle are known. The third student explained that two known sides and an included angle are sufficient for the use of the cosine rule to calculate both angles and sides on a non right-angled triangle. Evidently, this student did not know that the cosine rule is used to calculate angles when the measurements of the three sides of a non right-angled triangle are known.

A common finding between the understanding of the student teachers classified as possessing high content knowledge and the students in the low content knowledge group was the lack of ability to prove the cosine rule. Some mentioned that they were merely reproducing the rule as they had read it in a text book, while others simply stated that they did not know how to prove it. One student teacher claimed that the derivation of the cosine rule is rarely a feature at secondary school level. When challenged to derive the rule, he intimated that Pythagoras' theorem can be used to derive the rule, but was quick to indicate that 'it is not handy'. All these views point to the student teachers' inability to explain the process of deriving the cosine rule, and suggest that they had no relational understanding of it. If the student teachers had such understanding they could have used Pythagoras's theorem and trigonometric ratios to derive the cosine rule.

The analysis of the interview data suggests that the student teachers could recite the formulas that are used to calculate the areas of triangles. They were, for instance, able to state the formula $\frac{1}{2} \times \text{base length} \times \text{height}$ which is used to find area of a triangle when the perpendicular height is known. Each one of the interviewed students also presented an accurate trigonometric formula of the kind $\frac{1}{2}ab\sin C$.

The students explained that this formula is applicable when two sides and an included angle are provided in a non right-angled triangle. This finding complemented the test results, which indicated that the majority of the student teachers effectively used the trigonometric formula to calculate the area of a triangular region. Nevertheless, the student teachers from the low content knowledge bracket showed a lack of understanding of the concept of area. In addition to this, they could not explain the process of deriving the formula $\text{area} = \frac{1}{2}ab\sin C$. This discovery is consistent with the findings of another study in which the participants struggled to explain the basis of the trigonometric formula that is used to calculate the area of a triangle (Bryan, 1999).

In contrast, the three student teachers from the high content knowledge bracket applied the sine ratio on acute angled triangles, and effectively derived the trigonometric formula: $\text{area} = \frac{1}{2}ab\sin C$. While none of these students demonstrated how the formula is proved for an obtuse angled triangle, it was clear that their understanding thereof was relational. Additionally, the student teachers' explanations suggest that they understood that area is a two dimensional quantity. Moreover, one of them provided an explanation that confirmed an understanding of the reason why the area of a triangle is generally given by $\frac{1}{2} \times \text{base} \times \text{height}$.

Student teachers' conception of the periods of the sine, cosine, and tangent functions

The test data suggests that most of the student teachers could not correctly determine the period of the sine function, which was expressed as a table of values. A classic example involves a student teacher who indicated that the period of the sine function is 'from -1 to 1'. He clearly lacked the understanding that the period of a function relates to the number of degrees or radians.

The interview findings show that the majority of the student teachers were able to state the periods of the sine, cosine, and tangent functions. Most of the interviewees pointed out that the sine and cosine functions have periods of 360° and the tangent function 180°. Half of the student teachers who were interviewed intimated that a period relates to the width, measured in degrees or radians, of the pattern that repeats for a given function. The other half of the student teachers could not comprehensively explain the concept of the period of a function.

Student teachers' perspectives on the differences between the sine and tangent functions

An exploration of the student teachers' ability to explain the differences between the sine and the tangent functions suggests that the student teachers in the high content knowledge category provided thorough explanations. They could explain the difference in the period of the sine and tangent functions. Additionally, they explained that the tangent function has unbounded values $-\infty \leq \tan\theta \leq \infty$ while the sine function is continuous and has a range of $-1 \leq \sin\theta \leq 1$ for all the real angles θ. One of the student teachers from the low content knowledge group pointed out that she did not know the differences between the sine and tangent functions. Another student teacher explained that the range of the sine function is from 0° to 360° and from 360° to 720°. This explanation suggests that the range of the sine function is synonymous with the intervals of angles. In addition to this, the student teacher seemed to think that the sine function is defined on its maximum and minimum values and not on angles.

Furthermore, the student teacher did not know the range of the tangent function. He also did not know that there are angles such as 0°, 180°, and 360° for which the sine and tangent functions have the same value of zero. The second student teacher contended that the sine function has a range of -1 and 1. This perspective suggests that the student teacher lacked the understanding that the range of the sine function is a continuous interval $-1 \le \sin\theta \le 1$. A similar misconception was revealed when the student claimed that the range of the tangent function is 'negative infinite, positive infinite'. These findings point to the student teachers' inability to comprehensively explain concepts.

Student teachers' explanation of their methods in selected items

Although the test findings suggest that most of the student teachers could solve the equation $\cos\beta = -\cos 64^\circ$ for $90^\circ < \beta < 180^\circ$, the interview findings show a different picture. The majority of the student teachers did not have the competence to explain the methods they had applied to solve the item, and had difficulty elucidating what the item assessed. In other words, with the exception of Mwila and Mubita, the student teachers demonstrated a limited understanding regarding the procedures required to resolve the trigonometric equation $\cos\beta = -\cos 64^\circ$ for the interval $90^\circ < \beta < 180^\circ$.

John, Sara, and Brave used a calculator to solve the trigonometric equation. While John effectively used a calculator, and understood that β lay in the second quadrant, he too, like Sara and Brave, struggled to describe the operations carried out. John knew that the cosine ratio is negative in the second quadrant, but failed to relate this understanding to the computation of β. Brave and Sara showed a lack of in-depth understanding of the relationship between an acute angle and its corresponding obtuse angle. In their view, only a calculator can be used to solve the equation $\cos\beta = -\cos 64^\circ$ for $90^\circ < \beta < 180^\circ$.

Doreen mistakenly believed that a trigonometric equation can be solved like a linear equation. A typical example of this was her evaluation of the angle as follows: $\frac{\cos\beta}{\cos} = \beta$. Furthermore, she demonstrated a superficial understanding of the tangent ratio by indicating that $\frac{\sin 64^\circ}{\cos} = \tan 64^\circ$. Alternatively, Mubita and Mwila applied their knowledge of quadrants to resolve the equation

$\cos\beta = -\cos 64^{\circ}$ for the interval $90^{\circ} < \beta < 180^{\circ}$. These two student teachers demonstrated in-depth understanding of the relationship between an acute angle angle in the first quadrant and its corresponding obtuse angle in the second quadrant. They also related the sign of the cosine ratio in the second quadrant to the trigonometric equation.

There was corroboration between the test and interview findings regarding the student teachers' inability to recognise that the cosine rule is used to calculate an angle of a non right-angled triangle. In this context, the ratio $2:4:5$ of the sides of a triangle was provided, and the computation of the smallest angle of the triangle was required. Most of the student teachers were unable to relate the components of the ratio to the sides of a triangle. Instead of applying the cosine rule to calculate the smallest of the angles of the triangle, the majority of the student teachers used the ratio of the sides to divide the sum of the angles of a triangle.

Even the student teachers who had discussed the application of the cosine rule to calculate an angle did not use this knowledge. This result confirms that the majority of the student teachers' understanding of the cosine rule was limited. One of the two student teachers who successfully applied the cosine rule to determine the smallest angle exhibited a misunderstanding. He claimed that the smallest angle in a triangle is the one opposite the longest side of a triangle. It seems that he did not know that the smallest angle of a triangle is normally opposite the shortest side.

Student teachers' justification of their reasoning

One of the content areas explored regarding the student teachers' justifications was the signs of the trigonometric ratios in each of the four quadrants. This was a follow-up on the students' identification of the quadrants in which the sine, cosine, and tangent ratios are either positive or negative. An analysis of the data suggests that the majority of the student teachers were unable to provide a coherent explanation to support their viewpoints about the quadrants in which the trigonometric ratios are either positive or negative.

Doreen could not give any reason why the sine, cosine, and tangent ratios are either positive or negative in each of the first three quadrants. In terms of the fourth quadrant, she explained that the sine ratio is negative because the sine curve has negative values in the interval $270^{\circ} < \theta < 360^{\circ}$. Similarly, Brave used the characteristics of the curves of the trigonometric functions in particular intervals to justify his stance. These were not classified as explanations because the students never explained why the trigonometric curves have positive or negative values in specific intervals. Sara and John used a technique of evaluating the trigonometric ratios using angles from each of the quadrants to show that the ratios were either positive or negative. This strategy only worked for the students when special angles were involved since they could generate the values of the trigonometric expressions using special triangles. Alternatively, Mwila and Mubita provided satisfactory reasons for their perspectives. Both student teachers provided justifications alongside circles and right-angled triangles, which were drawn on Cartesian planes. Their explanations were similar to a situation where a unit circle is used, resulting in $\sin\theta = y$, and $\cos\theta = x$.

Maximum and minimum values of the sine, cosine, and tangent functions

Most of the student teachers were able to correctly point out the maximum and minimum values of the sine, and cosine functions. They indicated that the sine function has maximum and minimum values of 1 and -1 respectively. The students also mentioned that the cosine function has a maximum value of 1 and that its minimum value is -1. The students otherwise asserted that the tangent function is unbounded.

In spite of this, the majority of the student teachers struggled when it came to providing a justification for their views. Of the five student teachers who correctly indicated the minimum and maximum values of trigonometric functions only one provided a comprehensive justification for his point of view on the sine function. However, this student was incapable of justifying his views regarding the cosine and tangent functions. The other four student teachers were not able to comprehensively explain why the sine function has a minimum value of -1 and a maximum value of 1. Instead of providing an explanation, they only described the characteristics of the sine function.

Similarly, the four student teachers failed to satisfactorily account for their view that $\cos\theta = -20$ and $\cos\theta = 40$ are inappropriate. While one student indicated that the equation could not be solved, the other said that the cosine function has a minimum value of -1 and a maximum value of 1. Again, these students were merely restating facts and not providing justifications. They needed to explain, for instance, why the cosine function has a minimum value of -1 and a maximum value of 1. One of the student teachers revealed his superficial understanding when he explained that the sine and cosine functions have 1 for their maximum values and -1 for their minimum values due to their periodic nature.

Despite knowing the characteristics of the cosine function, the student teachers did not have the ability to justify why: (1) the values of $\cos\theta$ decrease from 1 to 0 as θ increases in the interval $0^\circ \leq \theta \leq 90^\circ$, (2) $\cos\theta$ decreases from 0 to -1 in the interval $90^\circ < \theta \leq 180^\circ$, (3) $\cos\theta$ increases from -1 to 0 in the interval $180^\circ < \theta \leq 270^\circ$, and (4) $\cos\theta$ increases from 0 to 1 in the Interval $270^\circ < \theta \leq 360^\circ$. There was a similar situation concerning the tangent function as none of the student teachers provided a justification for their views. In this regard, the student teachers demonstrated 'knowing that' without 'knowing why' (Even & Tirosh, 1995).

One student teacher revealed distinctive misconceptions in terms of specifying the maximum and minimum values of trigonometric functions. This student indicated that the maximum value of the sine function is 2π and that its minimum value is π. With regard to the cosine function, the student teacher explained that its maximum value is 360°. The student was quick to confess a lack of knowledge of the maximum value of the tangent function, but mentioned that its minimum value is 90°. These answers suggest that the student teacher had a serious lack of understanding that trigonometric functions are defined on angles, and that their values are not angles. Similarly, the student teacher did not have a reasonable justification for this stance. For instance, the student claimed that the maximum value of the cosine function is 360° because a complete revolution is 360°. This explanation clearly confirms that the student teacher's understanding was superficial.

Student teachers' justifications regarding some of the characteristics of the tangent function

The interview data revealed that three student teachers had misconceptions concerning the angles for which the tangent function is defined and undefined. One of these student teachers claimed that the value of tan 270° is 0, and another student contended that tan 270° has a value of $\frac{1}{\sqrt{3}}$. It was also mentioned by one of the students that tan 90° = 1, while the other two claimed that tan 90° = 0. Furthermore, one of these two student teachers posited that the tangent function is undefined at 180°. These findings suggest that the three student teachers had no understanding of the angles for which the tangent function is defined or undefined. To this extent, the interview discoveries corroborated the test findings, which suggest that there were student teachers who did not know the angles for which the tangent function is defined and undefined.

Apart from lacking an understanding of the angles for which the tangent function is undefined, the three student teachers could not account for their reasoning. For instance, one student explained that tan 90° = 0 because the tangent curve does not intersect the origin on a Cartesian plane. Another student teacher claimed that tan 270° = 0 because the tangent curve intersects the 90° axis. These explanations confirm the student teachers' lack of in-depth understanding of the tangent function.

The remaining three student teachers knew the angles for which the tangent function is defined. The students were aware that at angles such as -90°, 90°, and 270°, this function is undefined. Additionally, the student teachers' tangent graphs suggest that they knew that $\tan\theta$ increases from $-\infty$ to 0 in the interval $90^{\circ} < \theta \leq 180^{\circ}$, while for angles in the interval $180^{\circ} \leq \theta < 270^{\circ}$, it increases from $0 \text{ to } \infty$. Furthermore, it appeared that the three students knew that the tangent function increases from $-\infty \text{ to } 0$ in the interval $270^{\circ} < \theta \leq 360^{\circ}$. However, only one of these student teachers could provide a comprehensive explanation for tan 90° being undefined. This student teacher explained, for instance, that tan 90° is undefined because it involves a division by 0. He successfully illustrated how the division by 0 arises, and explained why such a division is undefined.

A second student teacher attempted to provide a justification as to why the tangent function is undefined at 90°. He showed that $\tan 90^{\circ} = \dfrac{\sin 90^{\circ}}{\cos 90^{\circ}} = \dfrac{1}{0}$, and explained why a division by 0 is undefined. Nevertheless, he could not justify why sin 90° = 1 and cos 90° = 0. He also could not explain why tan 270° is undefined. The experience of the third student teacher shows that he knew the facts, but could not explain why tan 90° and tan 270° are undefined.

Student teachers' ability to translate the sine and tangent functions

The test findings indicate that the vast majority of the student teachers could not accurately draw sine and tangent functions. While 32% of the student teachers managed to correctly draw the sine curve, only 5% drew accurate tangent curves. By implication, 68% of the student teachers could not work from the algebraic representation of the sine function to a graphical one. Additionally, 95% of the student teachers could not work from the algebraic to the graphical representation of the tangent function. Moreover, the student teachers did not know the following facts concerning the tangent function: (1) $y = \tan\theta$ increases from 0 to ∞ in the interval $0 \le \theta < \frac{\pi}{2}$, (2) $y = \tan\theta$ increases from $-\infty$ to 0 on the interval $\frac{\pi}{2} < \theta \le \pi$, (3) $y = \tan\theta$ increases from 0 to ∞, on the interval $\pi \le \theta < \frac{3\pi}{2}$, and (4) $y = \tan\theta$, increases from $-\infty$ to 0 on the interval $\frac{3\pi}{2} < \theta \le 2\pi$.

A prominent observation drawn from the trigonometric curves that were drawn by some of the student teachers was that the angles were given as multiples of π even when the domains in the test items were presented in degrees. Previous research has shown that student teachers view π either as an angle in radian form or as an irrational number (Akkoc, 2008). While Akkoc (2008) indicates that the students were more comfortable using degrees, the current study's findings suggest that some of the student teachers believed that graphs needed to reflect angles as multiples of π. It was established that the student teachers firstly reasoned in terms of degrees before presenting the angles in terms of π.

Student teachers' ability to accurately draw two special triangles

The research findings indicate that most of the student teachers had a general understanding of the two triangles that are used to evaluate trigonometric expressions. Notwithstanding, the majority of the student teachers did not completely label these triangles. Some of the triangles were drawn without the dimensions of the angles and sides such as 45°, 45°, and 90° and $1, 1$, and $\sqrt{2}$ as well as 30°, 30°, 60°, 60° and $2, 2, 1, 1, \sqrt{3}$. Other student teachers had the misconception that the equilateral triangle with dimensions 30°, 30°, 60°, 60° and $2, 2, 1, 1, \sqrt{3}$ is an isosceles triangle. Additionally, others did not know that the right-angled triangle with dimensions 30°, 60°, 90° and $2, 1, \sqrt{3}$ is also applicable.

Student teachers' ability to translate information on bearings onto a diagram

The majority of the student teachers demonstrated the ability to translate information on bearings onto a diagram. However, a significant number of the student teachers provided diagrams that did not accurately represent the information. In this regard, either the given points were interchanged or wrongly placed. A notable finding relates to the student teachers' inability to depict a bearing as an angle that is measured in the clockwise direction from the north to a line connecting two points.

6.4. SUMMARY OF CHAPTER 6

Chapter 6 provided a synthesis of the study's findings concerning the UNZA's mathematics student teachers' content knowledge of functions and trigonometry at secondary school level. The first section provided a discussion relating to the students' proficiency in the CCK of functions, followed by the SCK of functions held by the students. The second section presented a discussion of the findings regarding the student teachers' proficiency in the CCK of trigonometry. Thereafter, a synthesis of the study's findings was provided in relation to the SCK of trigonometry held by the participating UNZA mathematics student teachers.

7. CONCLUSIONS

7.1. HOW PROFICIENT ARE THE STUDENT TEACHERS IN CCK?

7.1.1. How proficient are the student teachers in CCK of functions for secondary schools?

For the topic of functions, the CCK category involved the assessment of the student teachers' ability to: (1) provide valid definitions of concepts, (2) identify examples and non-examples of concepts, and (3) correctly solve items on functions. Most of the student teachers managed to identify functions and non-functions, however, the majority of the students could not provide comprehensive definitions of concepts and correctly solve some of the items (see Chapter 6, Section 6.2.1 and Section 7.5.2 for specific details). While the mean achievement score was approximately 52%, an item by item analysis suggested that the majority of the student teachers were generally not proficient in the CCK of functions at secondary school level.

7.1.2. How proficient are the student teachers in the CCK of trigonometry at secondary school level?

An analysis of the data in the CCK category showed that the student teachers attained a mean score of approximately 53%. Furthermore, of the 22 student teachers who participated in the mathematics test, seven scored below 50%, while 15 achieved scores that were above 50%. These results indicate that the majority of the student teachers scored above 50% in items that involved the CCK of trigonometry at secondary school level in Zambia. Similarly, most of the student teachers were proficient in most of the concepts that were assessed in the CCK category. Therefore, it appears that most of the student teachers were generally proficient in the CCK of trigonometry at secondary school level (see Chapter 6, Section 6.3.1 and Section 7.5.4).

7.2. WHAT SCK IS HELD BY THE STUDENT TEACHERS?

7.2.1. What SCK of functions at secondary school level is held by the student teachers?

In terms of this question, an exploration was conducted through the two components of the SCK, as conceptualised in the study's conceptual framework: (1) the ability to explain and justify reasoning, and (2) the ability to use different representations. The majority of the student teachers could not coherently explain concepts. They also could not provide comprehensive justifications for their statements and perspectives. Furthermore, the students showed limited understanding of the different forms in which functions can be represented (see Chapter 6, Section 6.2.2 and Section 7.5.3).

7.2.2. What SCK of trigonometry at secondary school level is held by the student teachers?

Similar to the findings concerning functions, most of the student teachers' SCK of trigonometry was characterised by an inability to comprehensively explain trigonometric concepts. While the student teachers could apply mathematical rules and formulas, most of them could not account for their reasoning. It was also established that most of the student teachers could not change the representation of trigonometric functions from the algebraic form to a graphic representation. The majority of the students showed a limited understanding of trigonometric functions in graph form (see Chapter 6, Section 6.3.2 and Section 7.5.5).

7.3. DESCRIBING UNZA'S MATHEMATICS STUDENT TEACHERS' CONTENT KNOWLEDGE OF FUNCTIONS AND TRIGONOMETRY AT SECONDARY SCHOOL LEVEL

This study set out to answer the following main question: *How can the University of Zambia's mathematics student teachers' content knowledge of functions and trigonometry at secondary school level be described?* The conclusions that have been highlighted in Section 7.1.1 point to the fact that the student teachers generally lacked proficiency in the CCK of functions at secondary school level. Section 7.2.1 indicates that the student teachers could not comprehensively explain concepts. Similarly, they could not generally provide explanations for their perspectives. In this regard, they had instrumental, and not relational, understanding of concepts in functions at secondary school level. At the same time, it appeared that most of the student teachers lacked the ability to work with different representations of functions. Based on these considerations, it seems that the University of Zambia's mathematics student teachers' content knowledge of functions at secondary school level lacked depth.

In terms of trigonometry, the study's findings suggest that the student teachers were generally proficient in the CCK of trigonometry at secondary school level (Section 7.1.2). However, the student teachers lacked the competence to exhaustively explain concepts and justify their viewpoints. They could state mathematical facts without coherently explaining these 'facts'. Additionally, their understanding appeared instrumental in that they could not explain why these 'facts' were true. While the student teachers could translate information on bearings, they generally lacked the ability to translate algebraic trigonometric functions onto the Cartesian plane (Section 7.2.2). The findings suggest that, generally, there seems to be a disconnection between the student teachers' CCK and the SCK of trigonometry at secondary school level.

7.4. REFLECTIONS

7.4.1. Reflection on the conceptual framework

Generally, the conceptual framework that was developed served the objectives of this study. It allowed for the investigation of the student teachers' proficiency in the CCK of functions and trigonometry at secondary school level. Furthermore, it allowed for an exploration of the SCK of functions and trigonometry as understood by the student teachers. The descriptors for the CCK and SCK categories guided the development of the test items and formed the basis of the preparation of the interview schedules. At the same time, the descriptors of the CCK and SCK guided the analysis of the test and interview data.

Notwithstanding, there are aspects of the SCK, such as problem posing, which the conceptual framework did not include. While problem posing was initially included as a component of the SCK, it was removed later on as the student teachers did not have classroom opportunities. It was felt that this component could be effectively investigated in a situation where the student teachers are observed posing problems to the learners. In addition to this, the conceptual framework was limited in that it did not directly provide for the determination of the links between advanced university mathematics and secondary school mathematics.

7.4.2. Reflection on the methodology

The methodology that was employed for this study enabled the description of the UNZA's mathematics student teachers' content knowledge of functions and trigonometry at secondary school level. The convenience sampling techniques that were employed helped to select student teachers who were available and willing to participate in the study. The purposive sampling techniques enabled the selection of the student teachers who could provide the rich descriptive data that was sought. The mathematics test and semi-structured interviews were suited for the nature of the study and they effectively allowed for the gathering of the relevant data. Specifically, the mathematics test allowed for the collection of CCK and SCK at once. This was important because the student teachers had limited time as they were preparing for their university final examinations. At the same time, the test allowed the student teachers to present their calculations, which assisted during the data analysis. The semi-structured interviews permitted the exploration of the student teachers' understanding of concepts. In this regard, the student teachers were probed, and prompted to explain their understanding. The interviews allowed for the prompting of the student teachers' justification for their reasoning and also allowed for follow-ups concerning the methods and procedures that the student teachers had employed when solving particular test items.

However, the study could have been enriched if the student teachers had also been interviewed before the administration of the mathematics test. In this regard, the student teachers could have been asked to identify common concepts between advanced university mathematics and secondary school mathematics in the area of functions and trigonometry. Additionally, the student teachers could have been asked to explain if they thought advanced university mathematics courses had helped them to acquire an in-depth understanding of functions and trigonometry at secondary school level. The test data could then have confirmed or disproved the student teachers' views. The interviews conducted after the administration of the test could have been used to triangulate the data from the test. Furthermore, it could have been interesting, for instance, to observe the student teachers explaining and justifying the mathematics concepts in a classroom set-up.

7.5. CONCLUSIONS

7.5.1. Contribution and general result of the study

A unique contribution of this study lies in the fact that a mathematics test instrument was developed, in the Zambian context, using the descriptors that are aligned to the conceptual framework. The items of this test could be used by other researchers to assess mathematics student teachers' CCK and the SCK of functions and trigonometry at secondary school level. Similarly, the descriptors of the CCK and those of the components of the SCK can be adapted and utilised by researchers to describe mathematics student teachers' content knowledge of other secondary school mathematics topics. This study has also contributed by showing the nature of the SCK of functions and trigonometry at secondary school level as was held by the UNZA mathematics student teachers (see Chapter 6, Sections 6.2.2 and 6.3.2)

There are studies whose findings suggest that student teachers' study of higher mathematics courses in tertiary institutions does not guarantee in-depth understanding of the mathematics that is taught at school level (Bryan, 1999; Even, 1993; Fi, 2003, 2006; Wilburne & Long, 2010; Wilson, 1994; Wood, 1993). The current study corroborated this view. While the student teachers had studied advanced UNZA mathematics courses, and were in the final phase of their training, it was found out that they had not acquired an in-depth understanding of functions and trigonometry concepts at secondary school level. The student teachers did not have the ability to comprehensively explain concepts and justify their reasoning. Also, the students exhibited limited understanding with regard to the different representation of functions and trigonometric concepts.

7.5.2. Specific conclusions about the student teachers' CCK of functions

While the student teachers could identify the graphs of functions and non-functions, and correctly solve inverse functions, most of them could not identify the graphs of one-to-one functions and non-examples of one-to-one functions. The majority of the student teachers could not present comprehensive definitions of concepts such as relations, functions, one-to-one functions, and inverse functions. Some of the definitions of functions provided by the student teachers were restricted to the one-to-one correspondence property, while others revealed a flawed conception of the univalence condition of functions. Other student teachers presented definitions of a function which suggested that they were not familiar with the arbitrariness property of functions. Furthermore, the majority of the student teachers could not correctly find the maximum value, and range of a quadratic function. They could not complete the square of a quadratic function and were unable to determine the turning point of a quadratic function. Similarly, they could not successfully resolve composite functions and determine the domains for which a given quadratic function has no inverse.

7.5.3. Conclusions about the student teachers' SCK of functions

Some of the student teachers confined their explanations of a function to the one-to-one correspondence feature. While some students showed a lack of ability to competently explain a many-to-one function, others could not accept that a many-to-one relation is a function. This suggests a limited understanding of the univalence condition of functions. Moreover, this condition was not comprehensively explained and most of the students confessed that they did not know it even though they used it. Similarly, the arbitrariness condition of functions was not part of the student teachers' explanations of a function. Some student teachers' explanations seemed to suggest that functions are always supposed to be accompanied by algebraic formulas. A significant number of the student teachers could not provide a comprehensive explanation of what the domain and the range of a function are. The student teachers could not distinguish between a domain and the elements in a domain. Similarly, they could not differentiate the elements in a range from the range itself.

Most of the student teachers could otherwise not explain a one-to-one function. Some of the students saw functions as synonymous with one-to-one relations. Even when most of the student teachers had the idea that an inverse function 'reverses' or 'un-does' the operations of another function, they confined their explanations to the procedures that are performed when calculating the inverse of a function. Their understanding was generally characterised by making the independent variable the subject of the formula. This was a restrictive view of the concept of inverse functions as compared to functions that can be expressed as a formula. The student teachers appeared to generally understand the procedure of composing functions, but most of them could not explain that composite functions are not generally commutative. One student teacher thought that when composing two functions, an independent variable of one of the functions needed to be made the subject of the formula.

While the student teachers could state mathematical facts, they could not provide justifications for their views. Most of the student teachers who easily identified the figures that represented functions and non-functions could not effectively justify their positions. Some of them attempted to use the definition of a function, but due to their superficial understanding of the univalence condition, their explanations were flawed. The student teachers who held the misconception that a vertical line drawn in a Cartesian plane is a function could not account for their stance. Also, the student teachers who contended that a horizontal line drawn on Cartesian plane is not a function failed to provide their reasoning. Notwithstanding, those who correctly identified horizontal lines as functions and vertical lines as non-functions used the univalence condition of functions to account for their perspectives. They argued, for instance, that for a horizontal line every x-value from the horizontal axis has a unique y-value on the vertical axis.

The student teachers who correctly identified the graph of a one-to-one function, as well as those who did not do so attempted to use the univalence condition of functions to justify their perspective. This exposed their lack of in-depth understanding of the definition of a one-to-one function. One student teacher gave the vertical line test as a justification. Again, this demonstrated a lack of understanding that the vertical line test is a strategy used to identify graphs of functions and non-functions, and that it does not amount to a rationalisation.

The student teachers could not provide an explanation of the procedures that are performed when calculating the inverse of a function, for example, none of them explained why it is necessary to express the independent variable as a subject of the formula when calculating the inverse function. It was also found out that the student teachers could not provide an explanation for the relationships that exist between the leading coefficients of the algebraic quadratic functions and the nature of their curves.

The student teachers demonstrated awareness of the types of function representations. However, a significant number of students indicated that all functions can be represented as formulas. This perspective suggests that they lacked an understanding of the arbitrariness property of functions. Interestingly, most of the student teachers were able to explain the procedure of changing representation of linear function from the algebraic to the graphical representation. Additionally, most of the students could explain the process of translating from the graphic to the algebraic representation of a linear function. Nevertheless, none of them could explain why the standard formula of a linear function is $y = mx + c$.

While most of the student teachers successfully drew the graph of a quadratic function, the majority could not translate a relation that was defined on a discrete domain to the Cartesian plane. Most of them were also unable to translate an algebraic linear function which was defined on a discrete domain to the Cartesian plane. A classic example involves a student teacher who believed that graphs must always have their points connected either by straight lines or smooth curves. Furthermore, some of the student teachers only superficially understood the algebraic representation of quadratic functions. They cited inaccurate examples and were unable to recognise accurate representations of quadratic functions. Other student teachers did not know that the graphs of quadratic functions can only take two forms: opening upwards or downwards. In this regard, they contended that graphs of quadratic functions take four forms. To conclude, the research findings suggest that the majority of the student teachers could not explain concepts or justify their reasoning. Similarly, most of the student teachers seemed to have problems working with the different representations of functions.

7.5.4. Conclusions about the student teachers' CCK of trigonometry

The student teachers were proficient in the application of the following: sine rule, cosine rule as it relates to calculation of lengths, trigonometric formula for area of a triangle, and Pythagoras's theorem. The majority of the student teachers were further able to successfully do the following: calculate a three-figure bearing, evaluate a trigonometric expression without the use of a calculator, solve a simple trigonometric equation, calculate the shortest distance from a point to a straight line on a figure, and cite the definitions of the sine, cosine, and tangent ratios in relation to a right-angled triangle. However, most of the student teachers could not apply the cosine rule to find an angle. Also, the majority of the student teachers struggled to apply the trigonometric ratios in a three dimensional figure.

7.5.5. Conclusions about the student teachers' SCK of trigonometry

The majority of the student teachers could state the quadrants in which the sine, cosine, and tangent ratios are positive or negative. However, they could not provide an explanation as to why the trigonometric ratios are positive or negative in particular quadrants. Although most of the student teachers were able to apply the sine, and cosine rules, half of the interviewees was unable to comprehensively explain the conditions that warrant the use of these rules. The majority of the student teachers could not derive the sine rule, whereas none of them could prove the cosine rule. The trigonometric formula that is used to calculate the area of a non right-angled triangle was explained by most of the student teachers. Half of the student teachers could derive the formula, while the other half could not derive it which was complemented by their failure to clearly explain the concept of area. The student teachers memorised rules and formulas as they appear in textbooks without understanding why such rules and formulas hold.

It was established that the majority of the student teachers had a general understanding of the two special triangles containing special angles. Notwithstanding, most of the students could not completely label the triangles. While some of the students struggled, it seemed that most could translate information on bearings onto a diagram. In contrast, the majority of the student teachers could not translate the sine and tangent functions onto the Cartesian plane. The student teachers could explain the differences that exist between the sine and the tangent functions, but a significant number of them could not provide coherent explanations. In addition to this, most of the student teachers correctly determined the periods of the trigonometric functions, but could not comprehensively explain the concept of a period of a function.

Most of the student teachers who knew the maximum and minimum values of the sine and cosine functions could not justify their statements. Some of the student teachers showed that they had misconceptions concerning the angles at which the tangent function is defined or undefined. None of the student teachers could provide an explanation for their view on the angles at which the tangent function is either defined or undefined. The majority of the student teachers were unable to coherently explain the methods that they employed to solve a trigonometric equation. Most of the student teachers could not comprehensively explain what the trigonometric equation assessed. Furthermore, the student teachers were unable to recognise that the cosine rule rule was supposed to be applied to solve an angle. This was in spite of their capacity to recite when the rule is supposed to be used. It was interesting to note that the student teachers mistook the ratio of the sides of a triangle for a ratio of the angles of a triangle.

7.5.6. Specific findings of the study in relation to previous research

This study validated the findings of previous research, which showed that students struggle to articulate the definitions of a function, one-to-one functions, and many-to-one functions (Bayazit, 2011; Clement, 2001; Evangelidou et al., 2004; Spyrou & Zagorianakos, 2010). For example, some of the student teachers involved in the present study viewed functions as those relations that only satisfy the one-to-one correspondence property while refusing to believe that a many-to-one relation is a function. At the same time, this study confirmed the results of other studies indicating that students usually confuse the definition of a one-to-one function with the univalence condition of functions (Leinhardt et al., 1990; Markovits et al., 1986). There were students whose explanations suggested that the univalence condition means that the range of a function should only have a single element. Similar to the findings of Bayazit (2011), this study showed that student teachers think that functions must always be accompanied by formulas or rules. Some of the student teachers also claimed that all functions can be represented as formulas. While some of the students could not coherently explain the idea of a domain and range, others provided explanations that were restrictive. In this regard, they confined domains and ranges to functions that can be expressed as formulas (see Chapter 6 for detailed findings as related to previous research).

7.5.7. Student teachers' misconceptions

This study revealed unique misconceptions that were held by the University of Zambia mathematics student teachers regarding functions and trigonometry at secondary school level. Concerning functions, it was established that there were student teachers whose understanding of the characteristic shapes of graphs of quadratic functions was superficial. Some of the students understood that there are four shapes that parabolas can assume. Additionally, there were students who contended that the vertical lines drawn on Cartesian planes are functions. They mistakenly assumed vertical lines to be synonymous with the graphs of linear functions (see Chapter 6, Section 6.2.2).

Regarding trigonometry, the study showed that there were students who thought that trigonometric equations can be solved the same way as algebraic linear equations. The concepts of the range and period of trigonometric functions were also misunderstood. Some students mistook the periods of trigonometric functions for the ranges of the functions, and vice versa. One student teacher declared that the range of the sine function is from 0 to 360° and then to 720°. Another student teacher claimed that the maximum and minimum values of the sine function are 2π and π respectively. This student also indicated that the maximum value of the cosine function is 360°, while he claimed that the minimum value of the tangent function is 90°. Other striking misconceptions involved the values of tan 270° and tan 90°, the cosine rule, and smallest angle of a triangle. For instance, one student declared that the smallest angle of a triangle is usually opposite the longest side of a triangle.

There were students who believed that in order to balance the cosine rule, it is necessary to start with the longest side when stating it. Additionally, some of the students claimed that the cosine rule can be used to solve the length of a triangle when the three angles of a triangle are known (see Chapter 6, Section 6.3.2).

7.6. LIMITATIONS OF THE STUDY

This study was a qualitative case study that involved the mathematics student teachers from a single university in Zambia. In view of this, the findings cannot be generalised, although they can be related to other universities within the country. Moreover, the mathematics education programmes of the universities within Zambia are not uniform. Also, the content of the courses offered in the different Zambian universities that train secondary school teachers of mathematics are not the same.

While the specific student teachers were conveniently selected to write the Phase 1 test, some of these student teachers did not arrive. Similarly, some of the student teachers who had been purposefully chosen to participate in the interviews did not arrive. At the time of data collection, the University of Zambia was nearing the period of examinations. It is possible that this is what could have caused some of the student teachers to not participate.

The Zambian Curriculum Development Centre launched the new secondary school mathematics syllabus after the data collection instruments in this study had already been administered. This impacted the document analysis as there was the possibility of some of the concepts in the new syllabus not having been included in this study's assessments. Furthermore, during the research study, the University of Zambia was transitioning from the semester system to the three term calendar system. The mathematics and mathematics education course titles and course contents were in the process of being revised. As a result, there was the possibility of using titles that are no longer in use.

7.7. RECOMMENDATIONS FOR PRACTICE AND RESEARCH

In view of the findings of this study, it is hereby recommended that:

- During training, the UNZA mathematics student teachers should be provided opportunities to study all the secondary school mathematics topics that they are supposed to teach upon graduation.
- The MSE department at UNZA should develop mathematics education courses that are based on secondary school mathematics topics. In those courses, aspects of SCK such as explaining concepts and justifying reasoning, as well as use of different representations should be incorporated.
- The MSE department at UNZA should identify common concepts between advanced university mathematics courses and secondary school mathematics topics. Mathematics courses should then be developed comprising the identified common concepts.
- A quasi-experimental study should be conducted based on a mathematics education course on functions and trigonometry at secondary school level.
- A study should be conducted to determine the relationship between the mathematics student teachers' CCK and SCK of either functions or trigonometry at secondary school level.
- This study should be replicated using other topics in the secondary school curriculum or involving practising teachers of secondary school mathematics with the possibility of utilising observations as a data collection instrument.

This study provided a description of the University of Zambia's mathematics student teachers content knowledge of functions and trigonometry at secondary school level. Firstly, this aspiration was pursued through the assessment of the student teachers' proficiency in CCK. Secondly, an exploration was conducted regarding the SCK of the student teachers. The findings suggested that the student teachers' content knowledge of functions and trigonometry at secondary school level lacked depth. This raises concern since the student teachers had studied advanced university mathematics courses, and were in the final phase of their training. In a short while, they would be employed as graduate teachers of secondary school mathematics. Therefore, it is hoped that implementation of some of the recommendations indicated above will assist in addressing the scenario that the findings of this study has highlighted.

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APPENDIX 1 APPLICATION LETTER TO THE REGISTRAR OF THE UNIVERSITY OF ZAMBIA

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA
Faculty of Education

The Registrar,
University of Zambia,
Lusaka.

Dear Sir,

I respectfully request your office to allow me to involve students of the University of Zambia in my PhD research in Mathematics Education. My study is entitled: **Exploring Zambian Mathematics student teachers' content knowledge of functions and trigonometry for secondary schools.** My research study involves University of Zambia mathematics student teachers as participants. I am a lecturer of mathematics education at the University of Zambia, School of Education, in the department of Mathematics and Science Education. Currently, I'm enrolled for a PhD in Mathematics Education at the University of Pretoria, in South Africa.

My research requires a document analysis of the University of Zambia mathematics courses that are studied by the mathematics student teachers with a view to relating them to the Zambian secondary school mathematics curriculum. I propose to administer, within the University of Zambia, a paper and pencil mathematics test instrument to a conveniently selected sample of University of Zambia mathematics student teachers. The items in the mathematics test instrument are based on functions and trigonometry for secondary schools and are intended to enable me to gather content knowledge data for analysis. Subsequently, a purposive sub-sample of University of Zambia mathematics student teachers will be invited to participate in semi-structured interviews for elucidation of the test data and further exploration of their content knowledge. The interviews will be audio recorded at convenient places and times within the University.

I would like to assure your office, Sir, that a high level of confidentiality of the documentary analysis and participants' results during and after the test and interviews will be maintained. In this regard, only my supervisor and I will have access to the collected information. If you allow me to conduct this study at the University of Zambia, please do fill in the consent form attached to this letter. For any questions that you may have, you can either contact me or my supervisor using the contact details given below.

Yours Sincerely

Signature of student: Signature of supervisor:

Names of student: Names of supervisor:

Mobile number: E-mail address:

E-mail address:

Consent form

I,……………………………………(Your name), Registrar of ……………………………......agree/do not agree (delete what is not applicable) to allow…………………………………………to conduct PhD studies research in this University. The topic of his research is: **Exploring Zambian Mathematics student teachers' content knowledge of functions and trigonometry for secondary schools.**

I understand that the researcher will conduct document analysis of the mathematics courses studied by student teachers at the University of Zambia. I also understand that a conveniently chosen sample of the University of Zambia mathematics student teachers will write a paper and pencil mathematics test based on functions and trigonometry for secondary schools for a maximum of 3 hours. I equally understand that a purposive sub-sample will be selected from among those who will have written the mathematics test for participation in semi-structured interviews and that the said interviews will be audio taped and transcribed for analysis. All these activities will take place in venues within the University and at times convenient to the participants without interference of the University activities.

I understand that the researcher will explain the objectives of the study to the mathematics student teachers and that he will request them to voluntarily participate in the study. I further understand that the researcher subscribes to the principles of:

1. *Voluntary participation in research* which implies that participants might withdraw from research at any time.
2. *Informed consent* which entails that participants must be fully informed at all times in respect of the purpose of the study and must give consent to their participation in research.
3. *Safety in participation*, which means that human respondents should not be put at risk or harm of any kind.
4. *Privacy*, which implies that confidentiality and anonymity of the human respondents should be protected at all times.

5. *Trust*, meaning that human respondents should not be subjected to any act of deception or betrayal in the research process or its published outcomes.

Signature:……………………………………………………Date:...............................
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APPENDIX 2 LETTER REQUESTING STUDENT TEACHERS' PARTICIPATION IN THE STUDY

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA
Faculty of Education

Dear Student teacher,

I write to request your voluntary participation in my PhD study entitled: **Exploring Zambian Mathematics student teachers' content knowledge of functions and trigonometry for secondary schools.**

In this study, you are requested to participate by writing a paper and pencil mathematics test based on functions and trigonometry for secondary schools. This test will be administered at a mutually agreed venue and convenient time within the University of Zambia. The test is intended to enable me gather content knowledge data for my study. You may be requested to participate in an interview for purposes of elucidation of the test data and further exploration of content knowledge. The interviews, which will be audio recorded and transcribed for analysis, will be conducted at the time and place convenient to you, within the University of Zambia.

The data collected from the mathematics test instrument and subsequent interview will be analysed and used in my PhD thesis. I wish to assure you of confidentiality of your results during and after your participation in my study. Furthermore, be assured that data collected from the mathematics test instrument and interview will not have any effect on the courses you are enrolled for at the University of Zambia.

If you consent to participate in the mathematics test and a subsequent interview, please fill in and sign the consent form attached to this letter. For any questions, you are free to contact me using the contact details below.

Yours Sincerely,

Signature of student: Signature of supervisor:

Names of student: Names of supervisor:

Mobile number:

E-mail address:

Consent form

I,………………………………(your name), agree/do not agree (delete what is not applicable) to participate in a research entitled: **Exploring Zambian Mathematics student teachers' content knowledge of functions and trigonometry for secondary schools.** I understand that I will be writing a mathematics test based on functions and trigonometry for secondary schools for a maximum of 3 hours at a convenient venue and time within the University of Zambia. I understand that I may subsequently be asked to participate in an interview, which will be audio recorded, at a convenient place and time within the University. I also understand that my results in the said mathematics test and subsequent interview will only be used for purposes of this research and that they will in no way affect my performance in my academic courses at the University of Zambia. I further understand that the researcher will not provide any information, during and after data collection, which will directly identify the results with me.

I also understand that the researcher subscribes to the principles of:

1. *Voluntary participation in research* which implies that participants might withdraw from research at any time.
2. *Informed consent* which entails that participants must be fully informed at all times in respect of the purpose of the study and must give consent to their participation in research.
3. *Safety in participation,* which means that human respondents should not be put at risk or harm of any kind.
4. *Privacy,* which implies that confidentiality and anonymity of the human respondents should be protected at all times.
5. *Trust,* meaning that human respondents should not be subjected to any act of deception or betrayal in the research process or its published outcomes.

Signature:……………………………………………………Date:...............................
.

APPENDIX 3 FINAL VERSION OF THE MATHEMATICS TEST ADMINISTERED DURING PHASE 1

Instructions:

1. This test instrument is composed of **two** sections: A and B. Each of the sections has 6 questions.
2. The questions in the instrument are based on functions and trigonometry as prescribed in the Zambian secondary school mathematics curricular.
3. Answer **all** the questions in each of the sections on the answer sheets provided. You have a maximum of **3 hours** to answer the questions in the test instrument.
4. Show all the necessary calculations on the answer sheets and number your answers correctly in each section.
5. Indicate your names and mobile phone number on every answer sheet used. These details are only intended for the information of the researcher and shall remain strictly confidential.
6. You are allowed to use a calculator to answer questions on trigonometry except for question 7 (b).

SECTION A (FUNCTIONS)

1. (a) Define what a *relation* is as you would teach it to secondary school pupils.

   (b) The relation R on the set $X = \{3,4,5,6\}$ is defined by the rule "is greater than".

   Express R as a set of ordered pairs.

   (c) If R_1 is another relation on the set $X = \{3,4,5,6\}$ defined by the rule "is less than", express R_1 as a graph in the Cartesian plane.

   (d) Explain the relationships of the domains and ranges of R and R_1.

2. (a) Give a definition of what a function is. Is there a difference between a *relation* and a *function*? Explain your view.

   (b) In each of the cases below, state whether the figure represents a function or not. Justify your answers.

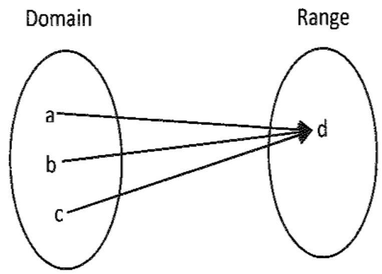


Figure 1

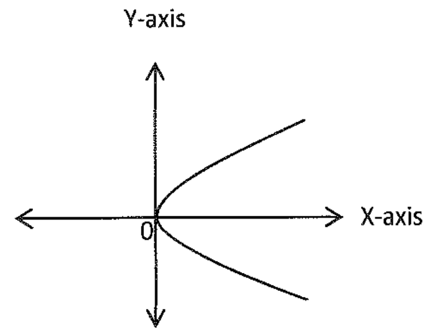


Figure 2

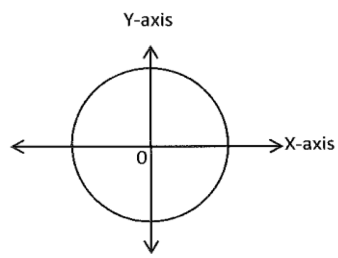


Figure 3

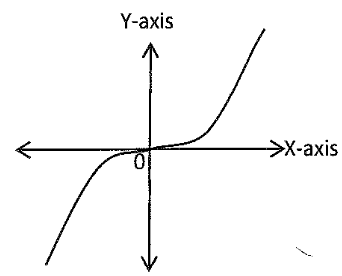
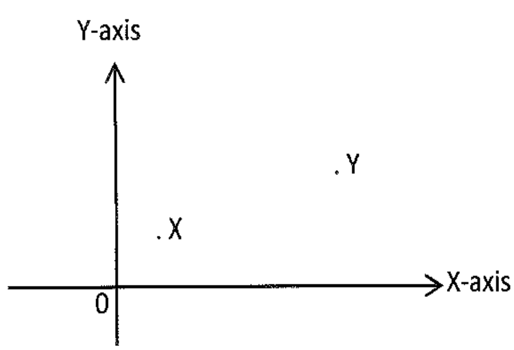


Figure 4

(c) Draw a graph of a function that passes through the points X and Y in the following figure:



Are there other functions whose graphs pass through the two points X and Y? If yes, draw the graph of such a function and if no such other function exists, explain why.

(d) A secondary school pupil gave $y^2 = x + 9$ with domain $\{x : 0 \leq x < 2 \text{ and } x \in Z\}$ as an example of a function. Is the pupil right or wrong? Explain.

3. (a) Represent $g : x \rightarrow |x|$ whose domain is $\{x : -3 \leq x \leq 2 \text{ and } x \in Z\}$ on a Cartesian plane.

(b) Answer this question on the sheet of graph paper provided. The table below shows corresponding values of the objects and images of the function $f(x) = -2x^2 - x + 8$.

x	-3	-2	-1	-0.5	0	0.5	1	2	3
$f(x)$	-7	2	7	8	8	7	5	-2	-13

Taking 2cm to represent 1 unit on the **x-axis** for $-3 \leq x \leq 3$ and 1cm to represent 2 units on the **y-axis**, draw the graph of $f(x) = -2x^2 - x + 8$.

(c) Use the graph of $f(x) = -2x^2 - x + 8$ to solve $f(x) = 2$.

(d) Complete the square for $f(x) = -2x^2 - x + 8$ and hence determine the turning point of f.

(e) State the maximum value of $f(x) = -2x^2 - x + 8$ and explain how that value relates to the range of the function f.

4. (a) Give a definition of a *one-to-one function*.

(b) A mathematics text book shows the following graphs as examples of one-to-one functions:

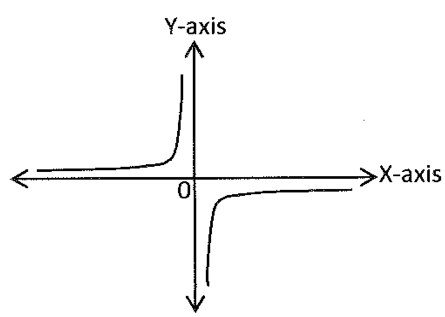


Figure 1

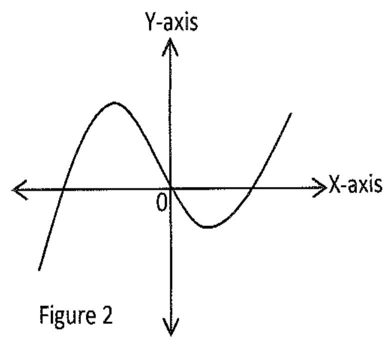


Figure 2

Is the textbook correct in this regard, or not? Explain.

5. Let $h(x) = x^2 + 1$ for $-2 \leq x \leq 2$ and $f(x) = x^2 + 1$ for $0 \leq x \leq 2$.

(a) State **two** differences between h and f.

(b) Define an inverse function.

(c) Find an expression for $f^{-1}(x)$ and specify the domain of f^{-1}.

(d) Determine the range of h.

(e) Describe the relationship between the range of h and the domain of f^{-1}.

6. Given that $g(x) = \frac{3}{2x+1}$ and $z : x \to x^2 - 2x$,

(a) Find the value of $g^{-1}(-5)$.

(b) State, with justification, two domains on which the function $z : x \to x^2 - 2x$ has an inverse.

(c) Find an expression for $(z \circ g)(x)$ where $z \circ g$ denotes the composite function of z and g.

(d) Evaluate $(g \circ g^{-1})(-5)$ where -5 belongs to the domain of g^{-1}.

SECTION B (TRIGONOMETRY)

7. (a) Calculate the value of β given that $\cos\beta = -\cos 64^\circ$ for $90^\circ < \beta < 180^\circ$.

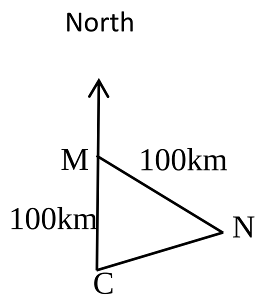
(b) Without use of a calculator, find the value of $\sin 315^\circ$.

(c) Draw two special triangles which are usually used in trigonometry to calculate values of trigonometric expressions.

(d) Given that θ is an acute angle and that $\sin\theta = \frac{7}{25}$, find the exact value of $\cos(180^\circ - \theta)$.

8. (a) The University of Zambia library denoted L, is 8 metres away on a bearing of 040° from a Mathematics Department office denoted M. Draw a diagram depicting this information and determine how far L is north of M.

(b) The diagram below shows the town of Choma denoted C, which is 100 km due south of the town of Monze denoted M.

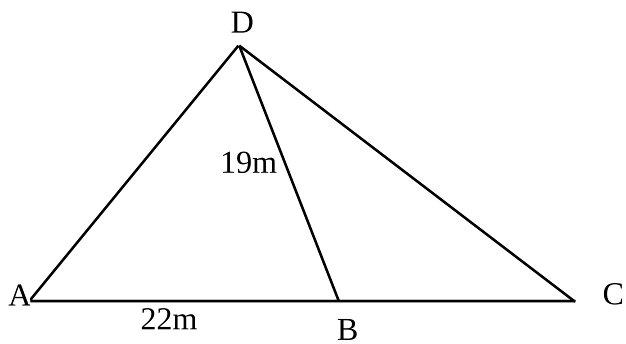


Given that Namwala District, denoted N, is 100km away from Monze on a bearing of 130°,

(i) Label, on the diagram, the angle that represents the bearing of 130° on which the District of Namwala lies using the letter **x**.

(ii) Find the 3-figure bearing of the town of Choma from the District of Namwala.

9. [For this question, give the answers to 3 significant figures where possible.]
The diagram below shows stopcocks connected to some water pipes at the University of Zambia, Great East Road Campus. The stopcocks are represented by points *A*, *B*, *C* and *D*, while the water pipes are represented by straight lines *AB*, *BC*, *AD*, *DC* and *BD*.



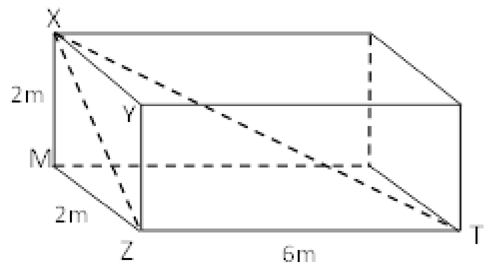
Given that $AB = 22m$, $BD = 19m$, $A\hat{B}D = 60^{\circ}$ and $B\hat{C}D = 34^{\circ}$, calculate:

(a) The length of the water pipe from stopcock *B* to stopcock *C*.
(b) The length of the water pipe from stopcock *A* to stopcock *D*.
(c) The area bounded by the water pipes *AB*, *BD* and *AD*.
(d) The shortest distance from stopcock *B* to the water pipe *AD*.

10. (a) If a triangle XYZ is such that $XY = 7cm$, $YZ = 6cm$, and angle $X = 44^\circ$, find, to one decimal place, two possible values for angle Z.

(b) Giving the answer to one decimal place, calculate the smallest of the angles of a triangle whose sides are in the ratio $2:4:5$.

11. The figure below shows a petrol tank which is made in the form of a cuboid.



Given that one end of the petrol tank is a square XYZM of side 2 meters, $ZT = 6$ metres and that XT is a diagonal of the tank, calculate:

(a) The length of XT giving the answer to 1 decimal place.
(b) The angle between XT and the plane XYZM to the nearest degree.
(c) The shortest distance between the plane XTM and YZ to 1 decimal place.

12. (a) Sketch the graph of the function $y = \sin\theta$ for $0^\circ \leq \theta \leq 360^\circ$.

(b) The table of values below represents the function *f* for $0^\circ \leq \theta \leq 360^\circ$:

θ	0°	90°	180°	270°	360°
$f(\theta)$	0	1	0	-1	0

State the *range* and *period* of the function *f*.

(c) Sketch the graph of $y = \tan\theta$ for $0^\circ \leq \theta \leq 360^\circ$.

APPENDIX 4 APPLICATION LETTER TO THE DIRECTOR OF THE ZAMBIA CURRICULUM DEVELOPMENT CENTRE

UNIVERSITEIT VAN PRETORIA
UNIVERSITY OF PRETORIA
YUNIBESITHI YA PRETORIA
Faculty of Education

The Director,
Zambia Curriculum Development Centre,
Lusaka,
Zambia.

Dear Sir/ Madam,

I write to request permission from your office for me to conduct a document analysis of the Zambian secondary school mathematics curriculum with a special focus on functions and trigonometry.

I am Zambian and a lecturer at the University of Zambia, and am currently enrolled as a PhD student of Mathematics Education at the University of Pretoria in South Africa. The title of my research study is: **Exploring Zambian Mathematics student teachers' content knowledge of functions and trigonometry for secondary schools.** One of the aspirations of my study is to determine the proficiency of University of Zambia's mathematics student teachers in Common Content Knowledge of functions and trigonometry for secondary schools by the end of their training. In order to meet this aspiration, I need to conduct an in-depth document analysis of the secondary school mathematics curriculum during my research study.

I wish to assure your office that the findings of the document analysis are purely for my PhD research study and shall not in any way compromise the work of the Zambia Curriculum Development Centre. If you accept my request please fill in the form of consent attached to this letter. For any questions, you are free to contact either me or my supervisor using the contact details below.

Yours Sincerely,

Signature of student: Signature of supervisor:

Names of student: Names of supervisor:

E-mail address: E-mail address:

Consent form

I,……………………………(Your name), Director of ……………………………….….. agree/do not agree (delete what is not applicable) to allow………………………………to conduct document analysis of the Zambian secondary school mathematics curriculum for his PhD studies at the University of Pretoria. His research topic is: **Exploring Zambian Mathematics student teachers' content knowledge of functions and trigonometry for secondary schools.**

I understand that the researcher will conduct document analysis during the course of the study with a view to determine the content to include in the mathematics test instrument and interview schedules which shall be administered to University of Zambia mathematics student teachers. I also understand that the said document analysis will be conducted with a special focus on functions and trigonometry.

I understand that the researcher will not use the data collected from the document analysis in any way that will be detrimental to the aspirations and welfare of the Curriculum Development Centre. I equally understand that the researcher subscribes to the principles of:

1. *Voluntary participation in research* which implies that participants might withdraw from research at any time.
2. *Informed consent* which entails that participants must be fully informed at all times in respect of the purpose of the study and must give consent to their participation in research.
3. *Safety in participation*, which means that human respondents should not be put at risk or harm of any kind.
4. *Privacy*, which implies that confidentiality and anonymity of the human respondents should be protected at all times.
5. *Trust*, meaning that human respondents should not be subjected to any act of deception or betrayal in the research process or its published outcomes.

Signature:……………………………………………………Date:...............................
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APPENDIX 5 INTERVIEW SCHEDULE ON FUNCTIONS

INTERVIEW SCHEDULE 1

Name of researcher	Mr P. Malambo
Names of interviewee	
Place of interview	University of Zambia
Topic	Functions
Date of interview	
Time of interview	

1. Which aspects of secondary school functions do you like most? Why?
   - Describe for me what a function is.
   - In the mathematics test, you defined a function as, "……………..." Briefly explain to me what you meant by this definition.
   - Can you share with me why you defined a function that/this way?
   - Tell me another way in which a function can be described.
   - What do you understand by the phrase: domain of a function? What of a range of a function?
   - In the test you stated that there is a difference/no difference between a relation and a function. Tell me a bit more about this difference/why you think there is no difference.
   - Give me an example to illustrate your position?
   - In your view, is a one-to-many relation a function? Why do you take that position? What of a many-to-one relation? Give a justification for your position.
   - In the test, you defined a one-to-one function as, "………………" Could you share with me what this definition means?
   - Tell me an alternative definition of one-to-one functions that you know of.
   - How would you define an inverse function for a pupil?
   - In the test you defined an inverse function as, "…………..." Why did you define an inverse function that way?
   - What conditions should be fulfilled for a function to have an inverse function? Why?
   - Define a linear function for me. Why do you define a linear function that way?
   - Give me the definition of a quadratic function.
   - What conditions should be satisfied for a function to qualify as quadratic? Why?
   - Let us now turn to the issue of turning points. In your view, what is a turning point of a function?
   - Which one between a linear function and a quadratic function has a turning point? Explain why.

- Tell me two types of turning points that are associated with quadratic functions? Describe each one of these turning points for me.
- Let us now shift to composite functions. Describe for me what a composite function is.
- What does the composition of a function by itself entail/mean?
- Have a look at Question 6 part 'd' of the test. Tell me what this question is assessing. Explain to me how you found the answer to this question.
- What is the meaning of the answer that you got?
- Why is it necessary that -5 should lie in the domain of g^{-1}?
- Other than the method you used in the test, how else could you have found the answer to this question?
- In your view, are composite functions important in the secondary school curriculum? Why?

2. Explain one of the forms in which functions can be represented.
   - Is it possible to represent every function using a formula? Why do you say so?
   - Look at Figure 1 in Question 2 (b). Explain to me an alternative way in which this relation can be represented.
   - Look at question 3 (b). How else can you represent the function *f*? Why?
   - Have a look at Question 1 (c). What are some of the factors that you considered when coming up with the graph?
   - Do you think the relation R_1 represents a function or not? Why do you say so?
   - Do you agree or not that every vertical line drawn in the Cartesian plane represents a function? Why?
   - Is every horizontal line drawn in the Cartesian plane an example or non-example of a function? Why do you take that view?
   - Have a look at the Figures in question 2 (b). In the test you indicated that Figure……is an example/non-example of a function. Explain to me why you thought so (I will ask respondents to justify their reasoning for each of the four figures).
   - Describe a specific method that you know of that can be used to determine examples and non-examples of functions?
   - Could you tell me what you know of the vertical line test? What do you think is the importance of the vertical line test?
   - Would you tell me what univalence of functions is? What of the arbitrariness of functions?
   - Share with me how you can sketch the graph of a linear function which is expressed in formula form? Why would you do what you have just told me?
   - Supposing you are given the graph of linear function in the Cartesian plane, explain to me how you can find the equation of that linear function.

- Explain how you can sketch the graph of a quadratic function.
- Why would you proceed in that manner?
- Let me take you back to one-to-one functions. In Question 4 (b) of the test, you suggested that Figure…..represents/does not represent a one-to-one function. Tell me why you suggested so (respondents will be asked on both figures).
- Share with me a specific method that can be used to determine whether a function is one-to-one or not.

3. Explain to me one difference that exists between a many-to-one function and a one-to-one function.
    - Let us get back to the idea of linear and quadratic functions. In your view, what differences exist between a linear and a quadratic function?
    - Given a quadratic function in symbolic form (formula), how can you determine the shape of its graph? Tell me why that is the case.
    - How else can you determine the shape of the graph?
    - What relationship exists between the leading coefficient of a quadratic function and the nature of its turning point? Why does this relationship exist?
    - Give me one characteristic of a graph of function that is defined on a discrete domain/ defined on continuous domain.
    - What difference exist between a graph of a function defined on a discrete domain and that of a function defined on a continuous domain?
    - Take a look at questions 3 (a). Explain to me how you sketched the graph of the function *g*. Why did you do what you did?
    - Consider 3 (b) of the mathematics test instrument. Is the function *f* continuous or discontinuous over the given domain? Why do you say so?
4. Explain how you can find the minimum value of a quadratic function. Why would you do what you have explained?
    - How can you find a maximum value of a function? Why?
    - Take a look at Question 3 (e) of the test. How did you determine the maximum value of the quadratic function *f*? Why did you use that method? How else could you have found the maximum value?
    - Given a quadratic function and its domain, how can you determine its range?
    - Explain how you can find the range of a quadratic function when the domain is not given?
    - What of the range of a linear function? Why?
    - Describe how you can determine the range of a quadratic function $h(x) = x^2 + 4x$. Why is your suggested method important?
    - In question 3 (d), how did you use the method of completing the square to determine the turning point of the function *f*? Why did you do what you did?

- Explain to me another method which can be used to find a turning point of a quadratic function.
- Take a look at Question 3 (c) of the mathematics test. How many real roots did you find for the equation $f(x) = 2$? Tell me why.
- Briefly share with me how you used the graph of the quadratic function *f* in Question 3 (c) to find the roots of the equation $f(x) = 2$? Why did you do what you did?
- Let me take you back to inverse functions. Refer to Question 5 (c) of the mathematics test instrument. How did you calculate the inverse function of the function *f*? Why did you use that method?
- In Question 5, is it possible to find the inverse of *h*? Why?
- Supposing the domain for *f* in question 5 (c) was not provided, would it have been possible for you to compute its inverse? If yes or no explain why.
- In 6 (b), you indicated that the function *z* has an inverse on domains……and ……Explain why. How did you derive these domains?
- Take a look at Question 2 (d) in the mathematics test instrument. What do you think is being asked for in question 2 (d)? What are some of the factors that should be taken into account when answering question 2 (d)?
- Suggest one example of a topic in the secondary school mathematics curriculum which is related to functions. How is this topic connected to functions?

APPENDIX 6 INTERVIEW SCHEDULE ON TRIGONOMETRY

INTERVIEW SCHEDULE 2

Name of researcher	Mr P. Malambo
Names of interviewee	
Place of interview	University of Zambia
Topic	Trigonometry
Date of interview	
Time of interview	

5. Do you think the topic trigonometry is important in the secondary school curriculum? Why do you take that view?

6. In your view, is the sine ratio positive or negative in the fourth quadrant? Could you explain your view? What of the tangent ratio in the third quadrant? Can you explain why?
   - Why is the sine ratio positive in the second quadrant?
   - Take a look at Question 7 (a) of the mathematics test. Briefly share with me what this question is assessing.
   - Explain the method which you used to calculate for the value of β in Question 7 (a). Why did you use this particular method?
   - What knowledge is necessary in order to calculate the value of β in Question 7 (a)?
   - Is the value of $\cos\beta$ positive or negative? How have you determined this sign? Why?
   - Explain the method you used to find the value of $\sin 315^0$ in Question 7 (b).
   - Can you explain an alternative method which can be used to find the value of $\sin 315^0$?
   - In your view, which one is greater between $\sin 315^0$ and $\sin 30^0$? Why do you take this view?
   - What does $\sin\theta = 0.28$ mean to you? In which quadrant does θ lie? How are you able to determine this quadrant?
   - In Question 7 (c), how did you come up with the two special triangles?
   - How did you determine the dimensions of the triangles e.g. the sides?
   - What would happen if the sides of the triangles are changed? (Would you still be able to get the same trigonometric values for sine, cosine and tangent ratios?)
7. Share with me what a three figure bearing is.
   - Consider question 8 (b) of the test instrument. Explain the method that you used to calculate the bearing of Choma from Namwala. Why did you calculate the bearing that way?

- What knowledge is necessary in order to calculate the bearing of Choma from Namwala [question 8 (b) part ii]?
- How can you find the distance from Choma to Namwala? Why would you use that method?

8. Refer to Question 9 (a) of the mathematics test instrument.
   - Tell me what you know of the sine rule. What is the sine rule used for?
   - State the sine rule on the answer sheet provided.
   - What conditions are necessary in order for the sine rule to be used?
   - Explain what the Cosine rule is.
   - State the cosine rule on the answer sheet provided.
   - When is it necessary to use the cosine rule?
   - For which questions in the test instrument did you use the sine rule? Why? What of the cosine rule? Why?
   - Explain the method which you used to calculate the length of the water pipe from stopcock B to stopcock C. Why did you use this particular method?
   - Is there another method that you could have used to obtain the same answer? If yes, explain, and if no, why?
   - In Question 9 (c) briefly explain the method you used to calculate the area. Why did you calculate the area the way you did?
   - Describe an alternative method that can be used to compute the area.
   - What did you understand of Question 9 (d)?
   - Explain to me the method you used to compute the shortest distance from B to AD. Why did you use this method?
   - Is there a difference between a perpendicular distance and a perpendicular bisector? Clarify.
   - How can you rephrase Question 9 (d) without changing what is being examined?
9. Take a look at Question 10 (a) of the test. What is the maximum number of possible values of angle *Z*? Why?
   - If a pupil gives 191° and 0° as two possible values for angle Z, would you consider these answers correct or wrong? Why would you?
   - What did you understand of Question 10 (b) of the test?
   - How can you identify the smallest of the three angles of a triangle? Why?
   - Describe the method which you used to calculate the smallest angle of the triangle in Question 10 (b).
   - Why did you use the method which you have described?
   - Take a look at Question 11 (a) of the mathematics test. Describe the method which you used to calculate the length of XT.
   - Why did you use that particular method?
   - Describe Pythagoras' theorem for me. When is it appropriate to use this method?

- Consider Question 11 (b). Identify the angle that lies between XT and the plane XYZM.
- How did you calculate this angle? Why?
- What does the phrase ‘shortest distance’ imply in Question 11 (c) of the test?
- Use the letters in the figure in Question 11 to draw a diagram that identifies the shortest distance between the plane XTM and YZ.

10. In your view, what is an angle?

- Describe an angle measure that you know of. Tell me something about the degree measure. What of the radian measure of angles?
- Give me a description of what $\sin\theta$ is. How else can you describe $\sin\theta$?
- In your view, is $\sin\theta$ a function or not? Explain why.
- Define a sine function? What of a cosine function? Tangent function?
- What is the maximum value of $\sin\theta$? Why is the value that you have given the maximum value of $\sin\theta$?
- What is the minimum value of $\sin\theta$? Why?
- Give me one characteristic of the graph of $y = \sin\theta$.
- A pupil obtained -20 and 40 as two possible values for $\cos\theta$, is the pupil correct or wrong? Motivate your answer.
- What is the range of $\cos\theta$? Justify your answer.
- What is the value of $\tan 270^\circ$? Can you explain your answer?
- Give me one value of an angle for which $\tan\theta$ is undefined. Why is $\tan\theta$ undefined at that value?
- What is the maximum value of $\tan\theta$? Can you justify your answer?
- How can you get the minimum value of $\tan\theta$? Can you motivate your answer?
- What does the term *period* of a function mean? In Question 12 (b), how did you determine the period of the function *f*?
- Share with me any two differences that you know of between $y = \sin\theta$ and $y = \tan\theta$.
- Is there anything else that you would like to tell me on secondary school trigonometry?

APPENDIX 7 MATHEMATICS EDUCATION COURSES STUDIED BY MATHEMATICS STUDENT TEACHERS

COURSE TITLE	CONTENT
MSE 131-Foundation Mathematics For Teachers	Rectangular Cartesian co-ordinates; distance between two points; the gradient and the equation of a straight line; condition for two lines to be parallel or perpendicular; functions; inverse of a one-one function; composition of functions; graphical illustration of the relationship between a function and its inverse; the quadratic function $f(x) = ax^2 + bx + c$; finding its maximum or minimum by any method; sketching its graph or determining its range for a given domain; the condition for the equation $ax^2 + bx + c = 0$ to have: (a) two real roots, (b) two equal roots, and (c) no real roots; solution of quadratic inequalities; simultaneous equations, at least one linear, in two unknowns; the use of the expansion of $(a+b)^n$ for positive integral *n*; circular measure: arc length, area of a sector of a circle; the six trigonometric functions of angles of any magnitude; the graphs of sine, cosine and tangent; knowledge of the relationships: $\frac{\sin A}{\cos A} = \tan A$, $\frac{\cos A}{\sin A} = \cot A$, $\sin^2 A + \cos^2 A = 1$, $\sec^2 A = \tan^2 A + 1$, and $1 + \cot^2 A = \csc^2 A$; solution of simple trigonometric equations involving any of the six trigonometric functions and the above relationships between them; simple identities; vectors in two directions: magnitude of a vector, addition and subtraction of vectors, multiplication by scalars; position vectors and unit vectors; scalar product and its use to determine the angle between two lines; the idea of a derived function; derivative of kx^n for *n* a positive or negative integer; derivative of a sum and of a composite function; applications of differentiation to gradients, tangents and normal, stationary points, velocity and acceleration, connected rates of change, small increments and approximations; practical problems involving maxima and minima; integration as the reverse process of differentiation; integration of sums of terms in integer powers of x; definite integrals; and applications of integration to plane areas, volumes, displacement, velocity and acceleration.
MSE 331-Mathematics Education I	Aims and objectives of teaching mathematics; domains of learning & behavioural objectives; sequencing instruction; teaching methods; use of basic teaching aids; organising for teaching (syllabuses, schemes of work, lesson plan &records of work); assessment; and peer teaching.
MSE 332-Mathematics Education II	Psychology of learning mathematics; strategies for teaching mathematics skills, concepts and processes; problem solving; classroom interaction analysis; gender issues in mathematics education; use of Audio-Visual Aids; use of textbooks; school mathematics; research in mathematics education; and practical classroom teaching.
MSE 431-Mathematics Education III	Analysis of teaching strategies; classroom organisation and management; teaching children with special needs; aspects of the history of mathematics; examinations & assessment; philosophy of mathematics education; curriculum development in mathematics; professional development of mathematics teachers; mathematics clubs and projects; and managing a department.

APPENDIX 8 MATHEMATICS COURSES STUDIED BY STUDENT TEACHERS AT UNZA

COURSE TITLE	CONTENT
M111 – Mathematical Methods I	Introduction set theory, preliminary algebra, elementary functions, elementary differential calculus.
M112 – Mathematical Methods II-A	Algebra, analytical geometry and vector analysis, matrices and determinants, complex numbers, differential calculus, integral calculus.
M211 - Mathematical Methods III.	Analytic geometry, differential calculus of functions of one variable, integral calculus of functions of one variable.
M221 – Linear Algebra I	Linear equations, matrices, determinants, vector spaces, linear transformations.
M231 – Real Analysis I	Set theory, order relations, the real line R, open and closed sets on the real line.
M212 - Mathematical Methods IV	Vector analysis, differential calculus of function of several variables, ordinary differential equations.
M222 – Linear Algebra II	Orthogonality, characteristic roots and vectors, quadratic forms.
M232 – Real Analysis II	Sequences in R, infinite series, the Cantor Set.
M325 – Introduction to Group & Ring theory	Permutations, introduction to group theory, further group theory, integers, ring theory, factorization in integral domains.
M331 – Real Analysis III	Elements of topology, continuous functions and functions of bounded variation.
M335 – Topology	Set theory, metric spaces, topological spaces, compactness, connectedness, approximation.
M911 - Mathematical Methods V	Functions of several variables, vector analysis, and further differential calculus.
M332 – Real Analysis IV	Differentiable functions, the Riemann integral.
M912 - Mathematical Methods VI	Further integral calculus, Fourier series and integral transforms, and differential equations.
M411 – Theory of Functions of a Complex Variable I	The complex number system, elementary theory of power series, elementary point set topology of C, analytic functions, analytic functions as mappings, complex integration.
M421 – Structure & Representation of Groups	Basic group theory, structure and theorems of groups, representations of groups, group rings and group algebras, introduction to group characters.
M431 – Real Analysis V	The ℓ^P Sets, Compact Metric Spaces.
M412 – Theory of Functions of a Complex Variable II	Further complex integration, calculus of residues, the maximum modulus theorem, analytic continuation.
M422 – Module & Field Theory	Elementary module theory, decomposition of finitely generated modules over PIDs, field extensions and Galois splitting fields, field automorphisms and the Galois Group, finite fields.
M432 – Real Analysis VI	Complete metric spaces, normed linear spaces, inner product spaces.

APPENDIX 9 DESCRIPTIVE STATISTICS PER ITEM: ASSESSING PROFICIENCY IN CCK OF FUNCTIONS

Items	Total marks	Range	Minimum score	Maximum score	Mean score	Standard deviation
1a	2	2	0	2	.91	.750
2a	2	2	0	2	.68	.839
2b	4	4	0	4	2.86	1.320
3c	3	3	0	3	2.45	1.057
3d	3	3	0	3	1.18	1.332
3e	1	1	0	1	.36	.492
4a	2	2	0	2	.23	.528
4b	2	2	0	2	.91	.811
5b	2	2	0	2	.27	.550
5c	3	3	0	3	1.77	.973
5d	1	1	0	1	.36	.492
6a	3	3	0	3	2.23	1.110
6c	3	3	0	3	1.68	1.393
6d	2	2	0	2	1.18	1.006

APPENDIX 10 STUDENTS' PERFORMANCE: ABILITY TO EXPLAIN AND JUSTIFY REASONING IN FUNCTIONS

Items	1(d)	2(a)	2(b)	2(c)	2(d)	3(e)	4(b)	5(a)	5(e)	6(b)	**Total score**	**Total %**
Possible marks	2	3	8	1	2	2	4	2	2	2	28	100
	Student teachers' scores per item											
Names	1(d)	2(a)	2(b)	2(c)	2(d)	3(e)	4(b)	5(a)	5(e)	6(b)	Students' total scores	Total %
Sara*	0	0	0	0	0	0	0	1	0	0	1	4
Dorika	0	0	8	0	2	1	0	2	0	0	13	46
Jolly	2	0	6	0	2	0	0	1	1	0	12	43
Doreen*	2	0	0	0	0	0	0	1	0	0	3	11
James	2	2	2	0	1	0	0	0	2	0	9	32
Soma	0	2	1	0	0	0	0	1	0	0	4	14
Sarafina	2	2	6	0	0	1	1	2	2	1	17	60
John*	2	3	8	1	2	0	0	1	2	0	19	67
Mubita*	2	2	8	1	0	2	0	2	2	1	20	71
Mwila*	2	3	8	0	2	1	1	1	2	0	20	71
Mubika	0	0	0	0	0	0	0	1	0	0	1	4
Mukela	0	2	0	1	0	0	0	0	0	0	3	11
Mwiitwa	0	1	0	0	0	1	0	1	0	0	3	11
Moonde	2	1	0	0	0	1	0	0	2	0	6	21
Sareka	0	0	0	0	0	0	0	0	0	0	0	0
Mwansa	0	1	2	0	0	0	1	1	0	0	5	18
Beverly	0	0	1	0	2	0	1	1	0	0	5	18
Johanes	0	0	0	0	0	0	0	1	0	0	1	4
Mundia	0	0	0	0	0	0	0	0	0	0	0	0
Brian	2	1	4	1	2	1	0	1	0	0	12	43
Mwiilula	0	1	4	0	2	0	0	1	0	0	8	29
Brave*	0	0	5	0	1	0	0	0	0	0	6	21
Mean											8	27

Key: * represents the interviewees in Phase 2 of the study

APPENDIX 11 STUDENTS' PERFORMANCE: THE USE OF DIFFERENT REPRESENTATIONS IN FUNCTIONS

Assessing competence in use of different representations								
Items	1(b)	1(c)	2(c)	2(d)	3(a)	3(b)	Total	Total %
Possible marks	3	3	2	1	3	3	15	100
	Student teachers' scores per item						Students' total scores	Student total %
Names	1(b)	1(c)	2(c)	2(d)	3(a)	3(b)		
Sara*	0	0	2	0	0	3	5	33
Dorika	0	0	2	1	0	0	3	20
Jolly	2	3	1	1	0	0	7	47
Doreen*	3	3	1	0	0	3	10	67
James	3	3	2	1	3	3	15	100
Soma	3	3	2	0	0	3	11	73
Sarafina	3	0	1	1	0	3	8	53
John*	3	3	2	1	0	0	9	60
Mubita*	3	0	2	0	0	3	8	53
Mwila*	3	3	1	1	3	3	14	93
Mubika	0	0	1	1	0	1	3	20
Mukela	0	0	2	0	0	3	5	33
Mwiitwa	0	0	1	1	0	3	5	33
Moonde	0	0	2	0	0	0	2	13
Sareka	0	0	1	1	0	0	2	13
Mwansa	0	0	0	0	1	1	2	13
Beverly	0	0	2	1	1	3	7	47
Johanes	0	0	1	0	0	3	4	27
Mundia	0	0	2	1	0	1	4	27
Brian	3	3	2	1	1	1	11	73
Mwiilula	0	0	1	1	0	3	5	33
Brave*	0	0	1	1	1	3	6	40
Mean							7	44

Key: * represents the interviewees in Phase 2 of the study

APPENDIX 12 STUDENTS' PERFORMANCE PER ITEM: PROFICIENCY IN CCK OF TRIGONOMETRY

Items	7a	7b	7d	8a	8b	9a	9b	9c	9d	10a	10b	11a	11b	11c	12b
Possible mark	3	3	3	3	4	3	3	3	3	3	3	3	3	3	2
Names of student teachers	Student teachers' scores per item														
	7a	7b	7d	8a	8b	9a	9b	9c	9d	10a	10b	11a	11b	11c	12b
Sara*	0	0	0	0	0	3	3	3	0	1	0	3	0	0	0
Dorika	0	3	0	0	4	3	3	3	3	3	0	3	0	0	2
Jolly	3	3	3	0	4	3	3	3	3	3	0	3	3	1	1
Doreen*	0	0	0	0	4	3	3	3	0	2	0	0	0	0	1
James	3	3	3	0	4	2	2	2	0	0	0	3	3	0	2
Soma	3	0	0	0	4	3	3	3	3	3	0	0	0	0	2
Sarafina	3	3	3	0	0	3	3	3	3	3	3	3	0	0	0
John*	3	3	3	1	4	3	3	2	3	2	3	3	3	0	2
Mubita*	3	2	3	3	4	3	3	3	3	3	0	3	0	0	1
Mwila*	3	3	3	3	4	3	3	3	3	3	0	3	3	2	2
Mubika	3	3	3	0	1	3	3	2	3	2	0	2	0	0	1
Mukela	3	3	0	0	1	3	3	3	3	3	3	3	3	0	2
Mwiitwa	0	0	3	0	4	3	2	3	3	3	0	3	0	0	1
Moonde	3	0	3	3	4	3	3	3	0	2	0	0	0	0	0
Sareka	3	3	1	1	4	3	3	3	3	2	0	2	3	0	0
Mwansa	0	3	3	0	3	3	3	3	3	2	0	3	0	0	1
Beverly	3	3	3	3	4	3	3	3	3	2	0	3	0	0	0
Johanes	0	0	1	0	4	3	3	0	0	1	0	3	0	0	1
Mundia	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Brian	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Mwiilula	0	0	0	0	1	3	3	3	3	3	0	0	0	0	0
Brave*	0	3	0	0	0	0	1	3	0	0	3	0	0	0	0

Key: * represents the interviewees in Phase 2 of the study.

APPENDIX 13 STUDENT TEACHERS' TOTAL SCORES IN THE CCK CATEGORY OF TRIGONOMETRY

Names of student teachers	Total score	Total percentage
Sara*	13	29
Dorika	27	60
Jolly	36	80
Doreen*	16	36
James	27	60
Soma	24	53
Sarafina	30	67
John*	38	84
Mubita*	34	76
Mwila*	41	91
Mubika	26	58
Mukela	33	73
Mwiitwa	25	56
Moonde	24	53
Sareka	31	69
Mwansa	27	60
Beverly	33	73
Johanes	16	36
Mundia	0	0
Brian	0	0
Mwiilula	16	36
Brave*	10	22

Key: * represents the interviewees in Phase 2 of the study.

APPENDIX 14 DESCRIPTIVE STATISTICS PER ITEM: ASSESSING PROFICIENCY IN CCK OF TRIGONOMETRY

Items	Range	Minimum	Maximum	Mean	Standard deviation
7(a)	3	0	3	1.64	1.529
7(b)	3	0	3	1.73	1.486
7(d)	3	0	3	1.59	1.469
8(a)	3	0	3	.64	1.177
8(b)	4	0	4	2.64	1.787
9(a)	3	0	3	2.55	1.057
9(b)	3	0	3	2.55	.963
9(c)	3	0	3	2.45	1.057
9(d)	3	0	3	1.91	1.477
10(a)	3	0	3	1.95	1.133
10(b)	3	0	3	.55	1.184
11(a)	3	0	3	1.95	1.397
11(b)	3	0	3	.82	1.368
11(c)	2	0	2	.14	.468
12(b)	2	0	2	.86	.834

APPENDIX 15 STUDENTS' PERFORMANCE IN ITEMS ASSESSING USE OF DIFFERENT REPRESENTATIONS IN TRIGONOMETRY

Items	7(c)	8(a)	12(a)	12(c)	Total score	Total %
Possible mark	6	3	3	3	15	100
	Student teachers' scores per item					
Names	7(c)	8(a)	12(a)	12(c)	Students' total scores	Students' total %
Sara*	0	0	0	0	0	0
Dorika	4	3	2	2	11	73.33333
Jolly	6	3	2	0	11	73.33333
Doreen*	5	3	1	1	10	66.66667
James	6	0	3	2	11	73.33333
Soma	5	0	1	1	7	46.66667
Sarafina	5	0	3	2	10	66.66667
John*	6	3	2	1	12	80
Mubita*	6	3	1	0	10	66.66667
Mwila*	6	3	3	3	15	100
Mubika	4	0	2	1	7	46.66667
Mukela	4	3	3	2	12	80
Mwiitwa	5	0	3	2	10	66.66667
Moonde	0	3	1	0	4	26.66667
Sareka	6	3	1	0	10	66.66667
Mwansa	6	3	3	0	12	80
Beverly	5	3	0	0	8	53.33333
Johanes	3	2	1	0	6	40
Mundia	0	0	0	0	0	0
Brian	0	0	0	0	0	0
Mwiilula	6	3	0	0	9	60
Brave*	1	0	3	1	5	33.33333
Mean					8.181818	54.54545

Key: * represents the interviewees in Phase 2 of the study.