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# Optimal coincidence point results in partially ordered non-Archimedean fuzzy metric spaces

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**Abstract**
In this paper, we introduce best proximal contractions in complete ordered non-Archimedean fuzzy metric space and obtain some proximal results. The obtained results unify, extend, and generalize some comparable results in the existing literature.

**MSC:** 47H10; 47H04; 47H07

**Keywords:** fuzzy metric space; monotone fuzzy proximal contraction; fuzzy ordered proximal contractions; fuzzy expansive; fuzzy nonexpansive; *s*-increasing sequence; *t*-norm

## 1 Introduction and preliminaries

In 1969, Fan [1], introduced the concept of a best approximation in Hausdorff locally convex topological vector spaces as follows.

**Theorem 1.1** *Let $X$ be a nonempty compact convex set in a Hausdorff locally convex topological vector space $E$ and $T : X \to E$ a continuous mapping, then there exists a fixed point $x$ in $X$, or there exist a point $x_0 \in X$ and a continuous semi-norm $p$ on $E$ satisfying* $\min_{y\in X} p(y - Tx_0) = p(x_0 - T(x_0)) > 0$.

A fixed point problem is to find a point $x$ in $A$ such that $Tx = x$. There are certain situations where solving an equation $d(x, Tx) = 0$ for $x$ in $A$ is not possible, then a compromise is made on the point $x$ in $A$ where $\inf\{d(y, Tx) : y \in A\}$ is attained, that is, $d(x, Tx) = \inf\{d(y, Tx) : y \in A\}$ holds. Such a point is called an approximate fixed point of $T$ or an approximate solution of an equation $Tx = x$. It is significant to study the conditions that ensure the existence and uniqueness of an approximate fixed point of the mapping $T$.

Let $A$ and $B$ be two nonempty subsets of $X$ and $T : A \to B$. Suppose that $d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\}$ is the distance between two sets $A$ and $B$ where $A \cap B = \phi$. A point $x^*$ is called a best proximity point of $T$ if $d(x^*, Tx^*) = d(A, B)$. Indeed, if $T$ is a multifunction from $A$ to $B$ then

$$d(x, Tx) \geq d(A, B),$$

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for all $x \in A$, always. Note that if $A = B$, then the best proximity point will reduce to a fixed point of the mapping $T$. Hence the results dealing with the best proximity point problem extend fixed point theory in a natural way.

For more results in this direction, we refer to [2–7] and references therein.

On the other hand, Zadeh [8] introduced the concept of fuzzy sets. Meanwhile Kramosil and Michalek [9] defined fuzzy metric spaces. Later, George and Veeramani [10, 11] further modified the notion of fuzzy metric spaces with the help of a continuous *t-norm* and generalized the concept of a probabilistic metric space to the fuzzy situation. In this direction, Vetro and Salimi [12] obtained best proximity theorems in non-Archimedean fuzzy metric spaces.

The aim of this paper is to obtain a coincidence best proximity point solution of $M(gx, Tx, t) = M(A, B, t)$ over a nonempty subset $A$ of a partially ordered non-Archimedean fuzzy metric space $X$, where $T$ is a nonself mapping and $g$ is a self mapping on $A$. Our results unify, extend, and strengthen various results in [13].

Let us recall some definitions.

**Definition 1.2** ([14]) A binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a continuous *t-norm* if

(1) $*$ is associative, commutative and continuous;
(2) $a * 1 = a$ for all $a \in [0,1]$;
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Typical examples of continuous $t$-norm are $\wedge$, $\cdot$, and $*_L$, where, for all $a, b \in [0,1]$, $a \wedge b = \min\{a, b\}$, $a \cdot b = ab$, and $*_L$ is the Lukasiewicz $t$-norm defined by $a *_L b = \max\{a + b - 1, 0\}$.

It is easy to check that $*_L \leq \cdot \leq \wedge$. In fact $* \leq \wedge$ for all continuous $t$-norms $*$.

**Definition 1.3** ([11]) Let $X$ be a nonempty set, and $*$ be a continuous *t-norm*. A fuzzy set $M$ on $X \times X \times [0, +\infty)$ is said to be a fuzzy metric if, for any $x, y, z \in X$, the following conditions hold:

(i) $M(x, y, t) > 0$,
(ii) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$,
(v) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

The triplet $(X, M, *)$ is called a fuzzy metric space.

Since $M$ is a fuzzy set on $X \times X \times [0, \infty)$, the value $M(x, y, t)$ is regarded as the degree of closeness of $x$ and $y$ with respect to $t$.

It is well known that for each $x, y \in X$, $M(x, y, \cdot)$ is a nondecreasing function on $(0, +\infty)$ [15].

If we replace (iv) with

(vi) $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$ for all $t, s > 0$,

then the triplet $(X, M, *)$ is said to be a non-Archimedean fuzzy metric space.

As (vi) implies (iv), every non-Archimedean fuzzy metric space is a fuzzy metric space. Also, if we take $s = t$, then (vi) reduces to $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$ for all $t > 0$. And $M$ in this case is said to be a strong fuzzy metric on $X$.

Each fuzzy metric $M$ on $X$ generates a Hausdorff topology $\tau_M$ whose base is the family of open $M$-balls $\{B_M(x,\varepsilon,t) : x \in X, \varepsilon \in (0,1), t > 0\}$, where

$$B_M(x,\varepsilon,t) = \{y \in X : M(x,y,t) > 1 - \varepsilon\}.$$

Note that a sequence $\{x_n\}$ converges to $x \in X$ (with respect to $\tau_M$) if and only if $\lim_{n\to\infty} M(x_n,x,t) = 1$ for all $t > 0$.

Let $(X,d)$ be a metric space. Define $M_d : X \times X \times [0,\infty) \to [0,1]$ by

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}.$$

Then $(X,M_d,\cdot)$ is a fuzzy metric space and is called the standard fuzzy metric space induced by a metric $d$ [10]. The topologies $\tau_{M_d}$ and $\tau_d$ (the topology induced by the metric $d$) on $X$ are the same. Note that if $d$ is a metric on a set $X$, then the fuzzy metric space $(X,M_d,*)$ is strong for every continuous $t$-norm '$*$' such that for all $* \leq \cdot$, where $M_d$ is the standard fuzzy metric (see [16]).

A sequence $\{x_n\}$ in a fuzzy metric space $X$ is said to be a Cauchy sequence if for each $t > 0$ and $\varepsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1 - \varepsilon$ for all $n,m \geq n_0$. A fuzzy metric space $X$ is complete [11] if every Cauchy sequence converges in $X$. A subset $A$ of $X$ is closed if for each convergent sequence $\{x_n\}$ in $A$ with $x_n \longrightarrow x$, we have $x \in A$. A subset $A$ of $X$ is compact if each sequence in $A$ has a convergent subsequence.

**Lemma 1.4** ([15]) *$M$ is a continuous function on $X^2 \times (0,\infty)$.*

**Definition 1.5** ([7]) Let $A$ and $B$ be two nonempty subsets of a fuzzy metric space $(X,M,*)$. We define $A_0(t)$ and $B_0(t)$ as follows:

$$A_0(t) = \{x \in A : M(x,y,t) = M(A,B,t) \text{ for some } y \in B\},$$
$$B_0(t) = \{y \in B : M(x,y,t) = M(A,B,t) \text{ for some } x \in A\}.$$

The distance of a point $x \in X$ from a nonempty set $A$ for $t > 0$ is defined as

$$M(x,A,t) = \sup_{a\in A} M(x,a,t),$$

and the distance between two nonempty sets $A$ and $B$ for $t > 0$ is defined as

$$M(A,B,t) = \sup\{M(a,b,t) : a \in A, b \in B\}.$$

**Definition 1.6** ([4]) Let $\Psi$ be the set of all mappings $\psi : [0,1] \to [0,1]$ satisfying the following properties:

(i) $\psi$ is continuous and nondecreasing on $(0,1)$ and $\psi(t) > t$ also $\psi(0) = 0$ and $\psi(1) = 1$.

(ii) $\lim_{n\to\infty} \psi^n(t) = 1$ if and only if $t = 1$.

Let $\Lambda$ be the set of all mappings $\eta : [0,1] \to [0,1]$ which satisfy the following properties:

(i) $\eta$ is continuous and strictly decreasing on $(0,1)$ and $\eta(t) < t$ for all $t \in (0,1)$,

(ii) $\eta(1) = 1$ and $\eta(0) = 0$.

If we take $\eta(t) = 2t - t^2$, then $\eta \in \Lambda$ and hence $\Lambda \neq \phi$.

## 2 Best proximity point in partially ordered non-Archimedean fuzzy metric space

**Definition 2.1** Let $A$ be a nonempty subset of a non-Archimedean fuzzy metric space $(X, M, *)$. A self mapping $f$ on $A$ is said to be (a) fuzzy isometry if $M(fx, fy, t) = M(x, y, t)$ for all $x, y \in A$ and $t > 0$ (b) fuzzy expansive if, for any $x, y \in A$ and $t > 0$, we have $M(fx, fy, t) \leq M(x, y, t)$, (c) fuzzy nonexpansive if, for any $x, y \in A$ and $t > 0$, we have $M(fx, fy, t) \geq M(x, y, t)$.

**Example 2.2** Let $X = [0,1] \times \mathbb{R}$ and $d : X \times X \to \mathbb{R}$ be a usual metric on $X$. Let $A = \{(0, x) : x \in \mathbb{R}\}$. Note that $(X, M_d, \cdot)$ is non-Archimedean fuzzy metric space, where $M_d$ is standard fuzzy metric induced by $d$. Define the mapping $f : A \to A$ by $f(0, x) = (0, -x)$. Note that $M_d(w, u, t) = \frac{t}{t+|x-y|} = M(fw, fu, t)$, where $w = (0, x)$, $u = (0, y) \in A$.

Note that every fuzzy isometry is fuzzy expansive but the converse does not hold in general.

**Example 2.3** Let $X = [0,4] \times \mathbb{R}$ and $d : X \times X \to \mathbb{R}$ be a usual metric on $X$. Let $A = \{(0, x) : x \in \mathbb{R}\}$. Note that $(X, M_d, \cdot)$ is a non-Archimedean fuzzy metric space, where $M_d$ is the standard fuzzy metric induced by $d$. Define the mapping $f : A \to A$ by

$$f(0, x) = 100(0, x).$$

If $x = (0,0)$ and $y = (0,4)$ then $M(x, y, t) = \frac{t}{t+4}$ and $M(fx, fy, t) = \frac{t}{t+400}$. This shows that $f$ is fuzzy expansive but not a fuzzy isometry.

**Example 2.4** Let $X = [0,1] \times \mathbb{R}$, $d : X \times X \to \mathbb{R}$ a usual metric on $X$ and $A = \{(0, x) : x \in \mathbb{R}\}$. Define a mapping $f : A \to A$ by

$$f(0, x) = \left(0, \frac{x}{10}\right).$$

If $x = (0,0)$ and $y = (0,1)$ then $M(x, y, t) = \frac{t}{t+1}$ and $M(fx, fy, t) = \frac{t}{t+\frac{1}{10}} \geq \frac{t}{t+1} = M(x, y, t)$. Thus $f$ is fuzzy nonexpansive but not a fuzzy isometry.

Note that the fuzzy expansive and nonexpansive mapping are fuzzy isometries. However, the converse is not true in general.

**Definition 2.5** Let $A$, $B$ be nonempty subsets of a non-Archimedean fuzzy metric space $(X, M, *)$. A set $B$ is said to be fuzzy approximatively compact with respect to $A$ if for every sequence $\{y_n\}$ in $B$ and for some $x \in A$, $M(x, y_n, t) \longrightarrow M(x, B, t)$ implies that $x \in A_0(t)$.

**Definition 2.6** ([17]) A sequence $\{t_n\}$ of positive real numbers is said to be $s$-increasing if there exists $n_0 \in \mathbb{N}$ such that $t_{n+1} \geq t_n + 1$ for all $n \geq n_0$.

**Definition 2.7** (compare [18]) A fuzzy metric space $(X, M, *)$ is said to satisfy *property T* if, for any $s$-increasing sequence, there exists $n_0 \in \mathbb{N}$ such that $\prod_{n \geq n_0}^{\infty} M(x, y, t_n) \geq 1 - \varepsilon$ for all $n \geq n_0$.

A 4-tuple $(X, M, *, \preceq)$ is called a partially ordered fuzzy metric space if $(X, \preceq)$ is a partially ordered set and $(X, M, *)$ is a non-Archimedean fuzzy metric space. Unless otherwise stated, it is assumed that $A$, $B$ are nonempty closed subsets of partially ordered fuzzy metric space $(X, M, *, \preceq)$.

**Definition 2.8** ([13]) A mapping $T : A \longrightarrow B$ is called (a) nondecreasing or order preserving if, for any $x$, $y$ in $A$ with $x \preceq y$, we have $Tx \preceq Ty$; (b) an ordered reversing if, for any $x$, $y$ in $A$ with $x \preceq y$, we have $Tx \succeq Ty$; (c) monotone if it is order preserving or order reversing.

**Definition 2.9** ([19]) Let $A$, $B$ be nonempty subsets of partially ordered fuzzy metric space $(X, M, *, \preceq)$ and $\psi : [0,1] \longrightarrow [0,1]$ be a continuous mapping. A mapping $T : A \longrightarrow B$ is said to be a fuzzy ordered $\psi$-contraction if, for any $x, y \in A$ with $x \preceq y$, we have $M(Tx, Ty, t) \geq \psi[M(x, y, t)]$ for all $t > 0$.

**Definition 2.10** A mapping $T : A \longrightarrow B$ is called a fuzzy ordered proximal $\psi$-contraction of type-$I$ if, for any $u$, $v$, $x$, and $y$ in $A$, the following condition holds:

$$\left.\begin{aligned} x \preceq y \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{aligned}\right\} \quad \Longrightarrow \quad M(u, v, t) \geq \psi\bigl[M(x, y, t)\bigr], \quad \text{where } \psi \in \Psi.$$

**Definition 2.11** A mapping $T : A \longrightarrow B$ is said to be a fuzzy ordered proximal $\psi$-contraction of type-$II$ if, for any $u$, $v$, $x$, and $y$ in $A$, and for some $\alpha \in (0,1)$, the following condition holds:

$$\left.\begin{aligned} x \preceq y \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{aligned}\right\} \quad \Longrightarrow \quad M(u, v, t) \geq \psi\left[M\left(x, y, \frac{t}{\alpha}\right)\right], \quad \text{where } \psi \in \Psi.$$

**Definition 2.12** A mapping $T : A \longrightarrow B$ is called a fuzzy ordered $\eta$-proximal contraction if, for any $u$, $v$, $x$, and $y$ in $A$, the following condition holds:

$$\left.\begin{aligned} x \preceq y \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{aligned}\right\} \quad \Longrightarrow \quad M(x, y, t) \leq \eta\bigl[M(u, v, t)\bigr], \quad \text{where } \eta \in \Lambda.$$

**Definition 2.13** A mapping $T : A \longrightarrow B$ is said to be a proximal fuzzy order preserving if, for any $u$, $v$, $x$, and $y$ in $A$, the following implication holds:

$$\left.\begin{aligned} x \preceq y \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{aligned}\right\} \quad \Longrightarrow \quad u \preceq v.$$

If $A = B$, then a proximal fuzzy order preserving mapping will become fuzzy order preserving.

**Definition 2.14** A mapping $T : A \longrightarrow B$ is said to be a proximal fuzzy order reversing if for any $u$, $v$, $x$, and $y$ in $A$, the following implication holds:

$$\left.\begin{aligned} x \preceq y \\ M(u, Tx, t) = M(A, B, t) \\ M(v, Ty, t) = M(A, B, t) \end{aligned}\right\} \quad \Longrightarrow \quad u \succeq v.$$

If $A = B$, then proximal fuzzy order reversing mapping will become fuzzy order reversing.

**Definition 2.15** A point $x$ in $A$ is said to be an optimal coincidence point of the pair of mappings $(g, T)$, where $T : A \longrightarrow B$ is a nonself mapping and $g : A \longrightarrow A$ is a self mapping if

$$M(gx, Tx, t) = M(A, B, t)$$

holds.

From now on, we use the notation $\Delta_{(t)}$ for a set $\{(x, y) \in A_0(t) \times A_0(t) : \text{either } x \preceq y \text{ or } y \preceq x\}$.

We start with the following result.

**Theorem 2.16** *Let $T : A \to B$ be continuous, proximally monotone, and proximal fuzzy ordered $\psi$-contraction of type-I, $g : A \to A$ surjective, fuzzy expansive and inverse monotone mapping. Suppose that each pair of elements in $X$ has a lower and upper bound and for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty such that $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$. If there exist some elements $x_0$ and $x_1$ in $A_0(t)$ such that*

$$M(gx_1, Tx_0, t) = M(A, B, t) \quad \text{with } (x_0, x_1) \in \Delta_{(t)},$$

*then there exists a unique element $x^* \in A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$, that is, $x^*$ is an optimal coincidence point of the pair $(g, T)$. Further, for any fixed element $\overline{x}_0 \in A_0(t)$, the sequence $\{\overline{x}_n\}$ defined by $M(g\overline{x}_{n+1}, T\overline{x}_n, t) = M(A, B, t)$ converges to $x^*$.*

*Proof* Let $x_0$ and $x_1$ be given points in $A_0(t)$ such that

$$M(gx_1, Tx_0, t) = M(A, B, t) \quad \text{with } (x_0, x_1) \in \Delta_{(t)}. \tag{1}$$

Since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, and $A_0(t) \subseteq g(A_0(t))$, we can choose an element $x_2 \in A_0(t)$ such that

$$M(gx_2, Tx_1, t) = M(A, B, t). \tag{2}$$

As $T$ is proximally monotone, we have $(gx_1, gx_2) \in \Delta_{(t)}$ which further implies that $(x_1, x_2) \in \Delta_{(t)}$. Continuing this way, we obtain a sequence $\{x_n\}$ in $A_0(t)$, such that it satisfies

$$M(gx_n, Tx_{n-1}, t) = M(A, B, t) \quad \text{with } (x_{n-1}, x_n) \in \Delta_{(t)} \tag{3}$$

for each positive integer $n$. Having chosen $x_n$, one can find a point $x_{n+1}$ in $A_0(t)$ such that

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t). \tag{4}$$

Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, $T$ is proximally monotone mapping, so from (3) and (4) it follows that $(gx_n, gx_{n+1}) \in \Delta_{(t)}$ and $(x_n, x_{n+1}) \in \Delta_{(t)}$. Note that

$$M(x_n, x_{n+1}, t) \geq M(gx_n, gx_{n+1}, t) \geq \psi\big[M(x_{n-1}, x_n, t)\big]. \tag{5}$$

Denote $M(x_n, x_{n+1}, t) = \tau_n(t)$ for all $t > 0$, $n \in \mathbb{N} \cup \{0\}$. The above inequality becomes

$$\tau_n(t) \geq \psi\big(\tau_{n-1}(t)\big) > \tau_{n-1}(t) \tag{6}$$

and

$$\tau_n(t) > \tau_{n-1}(t).$$

Thus $\{\tau_n(t)\}$ is an increasing sequence for all $t > 0$. Consequently, there exists $\tau(t) \leq 1$ such that $\lim_{n\to+\infty} \tau_n(t) = \tau(t)$. Note that $\tau(t) = 1$. If not, there exists some $t_0 > 0$ such that $\tau(t_0) < 1$. Also, $\tau_n(t_0) \leq \tau(t_0)$. By taking limit as $n \to \infty$ on both sides of (6), we have

$$\tau(t_0) \geq \psi\big(\tau(t_0)\big) > \tau(t_0),$$

a contradiction. Hence $\tau(t) = 1$. Now we show that $\{x_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon \in (0, 1)$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$, with $m_k > n_k \geq k$ such that

$$M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \varepsilon. \tag{7}$$

Assume that $m_k$ is the least integer exceeding $n_k$ and satisfying the above inequality, then we have

$$M(x_{m_k-1}, x_{n_k}, t_0) > 1 - \varepsilon. \tag{8}$$

So, for all $k$,

$$\begin{aligned} 1 - \varepsilon &\geq M(x_{m_k}, x_{n_k}, t_0) \\ &\geq M(x_{m_k}, x_{m_k-1}, t_0) * M(x_{m_k-1}, x_{n_k}, t_0) \\ &> \tau_{m_k}(t_0) * (1 - \varepsilon). \end{aligned} \tag{9}$$

On taking the limit as $k \to \infty$ on both sides of the above inequality, we obtain $\lim_{k\to+\infty} M(x_{m_k}, x_{n_k}, t_0) = 1 - \varepsilon$. Note that

$$M(x_{m_k+1}, x_{n_k+1}, t_0) \geq M(x_{m_k+1}, x_{m_k}, t_0) * M(x_{m_k}, x_{n_k}, t_0) * M(x_{n_k}, x_{n_k+1}, t_0)$$

and

$$M(x_{m_k},x_{n_k},t_0) \geq M(x_{m_k},x_{m_k+1},t_0) * M(x_{m_k+1},x_{n_k+1},t_0) * M(x_{n_k+1},x_{n_k},t_0),$$

imply that

$$\lim_{k\to+\infty} M(x_{m_k+1},x_{n_k+1},t_0) = 1 - \varepsilon.$$

From (4), we have

$$M(gx_{m_k+1}, Tx_{m_k}, t_0) = M(A,B,t_0) \quad \text{and} \quad M(gx_{n_k+1}, Tx_{n_k}, t_0) = M(A,B,t_0).$$

Thus

$$M(x_{m_k+1},x_{n_k+1},t_0) \geq M(gx_{m_k+1},gx_{n_k+1},t_0) \geq \psi\big[M(x_{m_k},x_{n_k},t_0)\big].$$

On taking the limit as $k \to \infty$ in the above inequality, we get $1-\varepsilon \geq \psi(1-\varepsilon) > 1-\varepsilon$, a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in the closed subset $A(t)$ of complete partially ordered fuzzy metric space $(X,M,*,\preceq)$. There exists $x^* \in A(t)$ such that $\lim_{n\to\infty} M(x_n,x^*,t) = 1$, for all $t > 0$. This further implies that

$$M\big(gx^*, Tx^*, t\big) = \lim_{n\longrightarrow\infty} M(gx_{n+1}, Tx_n, t) = M(A,B,t).$$

Hence $x^* \in A_0(t)$ is the optimal coincidence point of a pair $\{g,T\}$. To prove the uniqueness of $x^*$; We show that, for any fixed element $\overline{x}_0 \in A_0(t)$, the sequence $\{\overline{x}_n\} \in A_0(t)$ defined by $M(g\overline{x}_{n+1}, T\overline{x}_n, t) = M(A,B,t)$ converges to $x^*$. Suppose that there is another element $\overline{x}_0 \in A(t)$ such that $0 < M(x_0,\overline{x}_0,t) < 1$ for all $t > 0$ satisfying

$$M(g\overline{x}_0, T\overline{x}_0, t) = M(A,B,t). \tag{10}$$

Suppose that $(\overline{x}_0,x_0) \in \Delta_{(t)}$, that is, $\overline{x}_0 \preceq x_0$ or $\overline{x}_0 \succeq x_0$. Then by the given assumption, we have

$$M(\overline{x}_0,x_0,t) \geq M(g\overline{x}_0,gx_0,t) \geq \psi\big(M(\overline{x}_0,x_0,t)\big) > M(\overline{x}_0,x_0,t)$$

a contradiction. So $x^*$ is unique. If $(\overline{x}_0,x_0) \notin \Delta_{(t)}$, then by assumption, suppose that $u_0$ be a lower bound of $x_0$ and $\overline{x}_0$, also assume that $\overline{u}_0$ is an upper bound of $x_0$ and $\overline{x}_0$. That is,

$$\overline{u}_0 \succeq x_0 \succeq u_0 \quad \text{or} \quad \overline{u}_0 \succeq \overline{x}_0 \succeq u_0.$$

Recursively, construct the sequences $\{u_n\}$ and $\{\overline{u}_n\}$, such that

$$M(gu_{n+1}, Tu_n, t) = M(A,B,t) \quad \text{and} \quad M(g\overline{u}_{n+1}, T\overline{u}_n, t) = M(A,B,t).$$

The proximal monotonicity of the mapping $T$ and the monotonicity of the inverse of $g$ imply that

$$\overline{u}_n \succeq \overline{x}_n \succeq u_n \quad \text{or} \quad \overline{u}_n \preceq \overline{x}_n \preceq u_n.$$

Since $(x_0, u_0) \in \Delta_{(t)}$, also $(x_0, \overline{u}_0) \in \Delta_{(t)}$, similarly we have $(x_n, u_n) \in \Delta_{(t)}$ and $(x_n, \overline{u}_n) \in \Delta_{(t)}$, therefore

$$\lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} u_n = x^*.$$

Hence

$$\lim_{n\to\infty} \overline{x}_n = x^*.$$

This completes the proof. □

**Example 2.17** Let $X = [0,1] \times \mathbb{R}$ and $\preceq$ be the usual order on $\mathbb{R}^2$, that is, $(x,y) \preceq (z,w)$ if and only if $x \leq z$ and $y \leq w$. Suppose that $A = \{(-1,x) : \text{for all } x \in \mathbb{R}\}$ and $B = \{(1,y) : \text{for all } y \in \mathbb{R}\}$. $(X, M, *, \preceq)$ is a complete ordered metric space under $M(x,y,t) = \frac{t}{t+d(x,y)}$ for all $t > 0$, where $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$ for all $x = (x_1, y_1)$, $y = (x_2, y_2)$. Note that $M(A,B,t) = \frac{t}{t+2}$, $A_0(t) = A$, and $B_0(t) = B$. Define $T : A \to B$ by

$$T(-1,x) = \left(1, \frac{x}{2}\right).$$

Let $g : A \to A$ be defined by $g(-1,x) = (-1,2x)$. Note that $g$ is fuzzy expansive and its inverse is monotone. Obviously, $T(A_0(t)) = B_0(t)$, and $A_0(t) = g(A_0(t))$. Note that $u = (-1, \frac{y_1}{4})$, $v = (-1, \frac{y_2}{4})$, $x = (-1, y_1)$, and $y = (-1, y_2) \in A$ satisfy

$$M(gu, Tx, t) = M(A,B,t), \tag{11}$$

$$M(gv, Ty, t) = M(A,B,t). \tag{12}$$

Also, note that

$$M(gu, gv, t) = M\left(\left(-1, \frac{y_1}{2}\right), \left(-1, \frac{y_2}{2}\right), t\right) \geq \psi\big(M\big((-1, y_1), (-1, y_2), t\big) = \psi\big(M(x,y,t)\big),$$

where $\psi(t) = \sqrt{t}$. Thus all conditions of Theorem 2.16 are satisfied. However, $(-1,0)$ is the optimal coincidence point of $g$ and $T$, satisfying the conclusion of the theorem.

The above example shows that our result is a potential generalization of Theorem 3.1 in [13].

**Corollary 2.18** *Let $T : A \to B$ is continuous, proximally monotone, and proximal fuzzy ordered $\psi$-contraction of type-I, $g : A \to A$ surjective, a fuzzy isometry, and an inverse monotone mapping. Suppose that each pair of elements in $X$ has a lower and upper bound, for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty such that $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$. If there exist some elements $x_0$ and $x_1$ in $A_0(t)$ such that*

$$M(gx_1, Tx_0, t) = M(A,B,t) \quad \textit{with } (x_0, x_1) \in \Delta_{(t)},$$

*then there exists a unique element $x^* \in A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A,B,t)$. Further, for any fixed element $\overline{x}_0 \in A_0(t)$, the sequence $\{\overline{x}_n\}$ defined by $M(g\overline{x}_{n+1}, T\overline{x}_n, t) = M(A,B,t)$ converges to $x^*$.*

*Proof* Every fuzzy isometry is fuzzy expansive, and this corollary satisfies all the conditions of Theorem 2.16. □

**Example 2.19** Let $X = [-1,1] \times \mathbb{R}$ and $\preceq$ a usual order on $\mathbb{R}^2$. Let $A = \{(-1,x) : \text{for all } x \in \mathbb{R}\}$, $B = \{(1,y) : \text{for all } y \in \mathbb{R}\}$, and $(X,M,*,\preceq)$ a complete fuzzy ordered metric space as given in Example 2.17. Note that $M(A,B,t) = \frac{t}{t+2}$, $A_0(t) = A$ and $B_0(t) = B$. Define $T : A \to B$ by

$$T(-1,x) = \left(1, \frac{x}{5}\right).$$

Let $g : A \to A$ be defined by $g(-1,x) = (-1,-x)$. Note that $g$ is a fuzzy isometry and its inverse is monotone. Obviously, $T(A_0(t)) = B_0(t)$, and $A_0(t) = g(A_0(t))$. Note that $u = (-1,-\frac{y_1}{5})$, $v = (-1,-\frac{y_2}{5})$, $x = (-1,y_1)$, and $y = (-1,y_2) \in A_0(t)$ satisfy

$$M(gu,Tx,t) = M(A,B,t),$$
$$M(gv,Ty,t) = M(A,B,t).$$

Also, note that

$$M(gu,gv,t) = M\left(\left(-1,\frac{y_1}{5}\right),\left(-1,\frac{y_2}{5}\right),t\right) \geq \psi\left(M\left((-1,y_1),(-1,y_2),t\right)\right) = \psi\left(M(A,B,t)\right),$$

where $\psi(t) = \sqrt{t}$. All conditions of Corollary 2.18 are satisfied. Moreover, $(-1,0)$ is an optimal coincidence point of $g$ and $T$.

**Corollary 2.20** *Let $T : A \to B$ be a continuous, proximally monotone, and proximal fuzzy ordered $\psi$-contraction of type-I. Suppose that each pair of elements in $X$ has a lower and upper bound for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty such that $T(A_0(t)) \subseteq B_0(t)$. If there exist some elements $x_0$ and $x_1$ in $A_0(t)$ such that*

$$M(x_1,Tx_0,t) = M(A,B,t) \quad \text{with } (x_0,x_1) \in \Delta_{(t)},$$

*then there exists a unique element $x^* \in A_0(t)$ such that $M(x^*,Tx^*,t) = M(A,B,t)$. Further, for any fixed element $\overline{x}_0 \in A_0(t)$, the sequence $\{\overline{x}_n\}$ defined by $M(\overline{x}_{n+1},T\overline{x}_n,t) = M(A,B,t)$ converges to $x^*$.*

*Proof* This corollary satisfies all the conditions of Theorem 2.16 by taking $gx = I_A$ (an identity mapping on $A$). □

## 3 Best proximity point in partially ordered non-Archimedean fuzzy metric spaces for proximal $\psi$-contractions of type-*II*

**Theorem 3.1** *Let $T : A \to B$ is continuous, proximally monotone, and proximal ordered fuzzy $\psi$-contraction of type-II, $g : A \to A$ surjective, fuzzy expansive, and inverse monotone mapping. Suppose that each pair of elements in $X$ has a lower and upper bound, and an s-increasing sequence $\{t_n\}$ satisfying property $T$, for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty* *such that $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$. If there exist some elements $x_0$ and $x_1$ in $A_0(t)$ such that*

$$M(gx_1, Tx_0, t) = M(A, B, t) \quad \textit{with } (x_0, x_1) \in \Delta_{(t)},$$

*then there exists a unique element $x \in A_0(t)$ such that $M(gx, Tx, t) = M(A, B, t)$. Further, for any fixed element $\overline{x}_0 \in A_0(t)$, the sequence $\{\overline{x}_n\} \in A_0(t)$, defined by $M(g\overline{x}_{n+1}, T\overline{x}_n, t) = M(A, B, t)$, converges to $x$.*

*Proof* Let $x_0$ and $x_1$ be given elements in $A_0(t)$. such that

$$M(gx_1, Tx_0, t) = M(A, B, t) \quad \text{with } (x_0, x_1) \in \Delta_{(t)}. \tag{13}$$

Since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$, $A_0(t) \subseteq T(A_0(t)) \subseteq B_0(t)$, and $A_0(t) \subseteq g(A_0(t))$, it follows that there exists an element $x_2 \in A_0(t)$ such that it satisfies

$$M(gx_2, Tx_1, t) = M(A, B, t). \tag{14}$$

As $T$ is proximal monotone, we have $(gx_1, gx_2) \in \Delta_{(t)}$, which further implies that $(x_1, x_2) \in \Delta_{(t)}$. Continuing this way, we obtain a sequence $\{x_n\}$ in $A_0(t)$ such that

$$M(gx_n, Tx_{n-1}, t) = M(A, B, t) \quad \text{with } (x_{n-1}, x_n) \in \Delta_{(t)} \tag{15}$$

for each positive integer $n$. Hence after finding $x_n$, we can find an element $x_{n+1}$ in $A_0(t)$ such that

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t). \tag{16}$$

Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, $T$ is proximally monotone mapping, so from (15) and (16), it follows that $(gx_n, gx_{n+1}) \in \Delta_{(t)}$ and $(x_n, x_{n+1}) \in \Delta_{(t)}$. Note that

$$M(x_{n+1}, x_n, t) \geq M(gx_{n+1}, gx_n, t) \geq \psi\left(M\left(x_n, x_{n-1}, \frac{t}{\alpha}\right)\right). \tag{17}$$

for all $n \geq 0$. Recursively,

$$\begin{aligned} M(x_{n+1}, x_n, t) &\geq \psi\left(M\left(x_{n+1}, x_n, \frac{t}{\alpha}\right)\right) \geq \psi^2\left(M\left(x_n, x_{n-1}, \frac{t}{\alpha^2}\right)\right) \geq \cdots \\ &\geq \psi^n\left(M\left(x_1, x_0, \frac{t}{\alpha^n}\right)\right) > M\left(x_1, x_0, \frac{t}{\alpha^n}\right), \end{aligned} \tag{18}$$

for all $t > 0$ and $m, n \in \mathbb{N}$, where $m \geq n$, so we have

$$\begin{aligned} M(x_n, x_m, t) &\geq M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t) * M(x_{n+2}, x_{n+3}, t) * \cdots * M(x_{m-1}, x_m, t) \\ &> M\left(x_0, x_1, \frac{t}{\alpha^n}\right) * M\left(x_0, x_1, \frac{t}{\alpha^{n+1}}\right) * \cdots * M\left(x_0, x_1, \frac{t}{\alpha^{m-1}}\right) \\ &> \prod_{i=n}^{\infty} M\left(x_0, x_1, \frac{t}{\alpha^i}\right), \end{aligned}$$

where $t_i = \frac{t}{\alpha^i}$. As $\lim_{n\to\infty}(t_{n+1} - t_n) = \infty$, $\{t_n\}$ is an $s$-increasing sequence satisfying the property $T$. Consequently for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, so we have $\prod_{n=1}^{\infty} M(x_0, x_1, t_n) \geq 1 - \varepsilon$ for all $n \geq n_0$. Hence we obtain $M(x_n, x_m, t) \geq 1 - \varepsilon$ for all $n, m \geq n_0$ and $\{x_n\}$ is a Cauchy sequence in $A(t)$. By the completeness of $X$, there exists $x$ in $A(t)$ such that $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. This further implies that

$$\begin{aligned} M(gx, B, t) &\geq M(gx, Tx_n, t) \\ &\geq M(gx, gx_{n+1}, t) * M(gx_{n+1}, Tx_n, t) \\ &= M(gx, gx_{n+1}, t) * M(A, B, t) \\ &\geq M(gx, gx_{n+1}, t) * M(gx, B, t). \end{aligned}$$

Since $g$ is continuous, the sequence $\{gx_n\}$ converges to $gx$. Therefore, $M(gx, Tx_n, t) \to M(gx, B, t)$. Since $B(t)$ is fuzzy approximately compact with respect to $A(t)$, $\{Tx_n\}$ has a subsequence which converges to $y$ in $B(t)$ such that

$$M(gx, y, t) = M(A, B, t),$$

for some $y \in B(t)$, hence $gx \in A_0(t)$ implies $gx = gu$ for some $u \in A_0(t)$. Hence $M(x, u, t) \geq M(gx, gu, t) = 1$, which implies that $M(x, u, t) = 1$. Thus $x$ and $u$ are identical, and hence $x \in A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$,

$$M(z, Tx, t) = M(A, B, t) \tag{19}$$

for some $z$ in $A(t)$. From (16) and (19) we obtain

$$M(gx_{n+1}, z, t) \geq \psi\left(M\left(x, x_n, \frac{t}{\alpha}\right)\right). \tag{20}$$

Taking the limit as $n \to \infty$, the above inequality becomes

$$\lim_{n\to\infty} M(gx_{n+1}, z, t) \geq \lim_{n\to\infty} \psi\left(M\left(x, x_n, \frac{t}{\alpha}\right)\right) = 1,$$

which shows that $\{gx_n\}$ converges to $z$

$$M(gx_n, z, t) = 1. \tag{21}$$

Since $g$ is continuous, the sequence $\{gx_n\}$ converges to $gx$ such that

$$M(gx_n, gx, t) = 1. \tag{22}$$

Hence we have $gx = z$,

$$M(gx, Tx, t) = M(A, B, t) = M(z, Tx, t). \tag{23}$$

Suppose that there is another element $x^*$ such that

$$M\left(gx^*, Tx^*, t\right) = M(A, B, t). \tag{24}$$

First suppose that $(x,x^*) \in \Delta_{(t)}$. From (23) and (24), it follows that

$$M(x,x^*,t) \geq M(gx,gx^*,t) \geq \psi\left(M\left(x,x^*,\frac{t}{\alpha}\right)\right),$$

which further implies that

$$M(x,x^*,t) > M\left(x,x^*,\frac{t}{\alpha}\right),$$

a contradiction. Hence $x$ is unique.

Now, suppose that $(x,x^*) \notin \Delta_{(t)}$. Let $\overline{x}_0$ be any element in $A_0(t)$, $u_0$ and $\overline{u}_0$ be lower and upper bounds of $x_0$ and $\overline{x}_0$, respectively such that

$$\overline{u}_0 \succeq x_0 \succeq u_0 \quad \text{or} \quad \overline{u}_0 \succeq \overline{x}_0 \succeq u_0.$$

Recursively, we can find sequences $\{u_n\}$ and $\{\overline{u}_n\}$ such that

$$M(gu_{n+1}, Tu_n, t) = M(A,B,t) \quad \text{and} \quad M(g\overline{u}_{n+1}, T\overline{u}_n, t) = M(A,B,t).$$

The proximal monotonicity of the mapping $T$ and the monotonicity of the inverse of $g$ implies that

$$\overline{u}_n \succeq \overline{x}_n \succeq u_n \quad \text{or} \quad \overline{u}_n \preceq \overline{x}_n \preceq u_n.$$

Since $(x_0,u_0) \in \Delta_{(t)}$, also $(x_0,\overline{u}_0) \in \Delta_{(t)}$. It follows that

$$\lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} u_n = x^*.$$

Hence

$$\lim_{n\to\infty} x_n = x^*.$$

This completes the proof. □

**Example 3.2** Let $X = [0,2] \times \mathbb{R}$ and $\preceq$ a usual order on $\mathbb{R}^2$. Let $A = \{(0,x) : x \geq 0 \text{ and } x \in \mathbb{R}\}$, $B = \{(2,y) : \text{for all } y \in \mathbb{R}\}$, and $(X,M,*,\preceq)$ a complete fuzzy ordered metric space as given in Example 2.17. Note that $A_0(t) = A$, $B_0(t) = \{(2,y) : y \geq 0 \text{ and } y \in \mathbb{R}\}$. Define $T : A \to B$ by

$$T(0,x) = \left(2,\frac{x}{10}\right).$$

Let $g : A \to A$ be defined by $g(0,x) = (0,10x)$. Note that $g$ is a fuzzy expansive and its inverse is monotone. Obviously, $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$. Note that $u = (0,\frac{y_1}{100})$, $v = (0,\frac{y_2}{100})$, $x = (0,y_1)$, and $y = (0,y_2) \in A_0(t)$ satisfy

$$M(gu, Tx, t) = M(A,B,t),$$

$$M(gv, Ty, t) = M(A,B,t).$$

Also, note that

$$M(gu,gv,t) = M\left(\left(0,\frac{y_1}{10}\right),\left(0,\frac{y_2}{10}\right),t\right) \geq \psi\left(M\left((0,y_1),(0,y_2),\frac{t}{\alpha}\right)\right)$$
$$= \psi\left(M\left(x,y,\frac{t}{\alpha}\right)\right),$$

where $\psi(t) = \sqrt{t}$ and for all $\alpha \in [\frac{1}{10},1]$. All conditions of Theorem 3.1 are satisfied. Moreover, $(0,0)$ is optimal coincidence point of $g$ and $T$.

**Corollary 3.3** *Let $T : A \to B$ is continuous, proximally monotone, and proximal ordered fuzzy $\psi$-contraction of type-II, $g : A \to A$ surjective, fuzzy isometry and inverse monotone mapping. Suppose that each pair of elements in $X$ has a lower and upper bound, and an s-increasing sequence $\{t_n\}$ satisfying property $T$, for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty such that $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$. If there exist some elements $x_0$ and $x_1$ in $A_0(t)$ such that*

$$M(gx_1, Tx_0, t) = M(A,B,t) \quad \textit{and} \quad (x_0,x_1) \in \Delta_{(t)},$$

*then there exists a unique element $x^* \in A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A,B,t)$. Further, for any fixed element $\overline{x}_0 \in A_0(t)$, the sequence $\{\overline{x}_n\} \in A_0(t)$, defined by $M(g\overline{x}_{n+1}, T\overline{x}_n, t) = M(A,B,t)$, converges to $x^*$.*

*Proof* Here the $T$ satisfy all the conditions of Theorem 3.1 if we consider $g$ as fuzzy isometry mapping. □

**Corollary 3.4** *Let $T : A \to B$ is continuous, proximally monotone, and proximal ordered fuzzy $\psi$-contraction of type-II. Suppose that each pair of elements in $X$ has a lower and upper bound, and an s-increasing sequence $\{t_n\}$ satisfying property $T$, for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty such that $T(A_0(t)) \subseteq B_0(t)$.*

*Then there exists a unique element $x^* \in A$ such that $M(x^*, Tx^*, t) = M(A,B,t)$. Further, for any fixed element $x_0 \in A_0(t)$, the sequence $\{x_n\} \in A_0(t)$, defined by $M(x_{n+1}, Tx_n, t) = M(A,B,t)$, converges to $x^*$.*

*Proof* Here the $T$ satisfy all the conditions of Theorem 3.1 if $g(x) = I_A$ (an identity mapping on $A$). □

## 4 Best proximity point in partially ordered non-Archimedean fuzzy metric spaces for proximal $\eta$-contractions

**Theorem 4.1** *Let $T : A \to B$ be continuous, proximally monotone, and proximal fuzzy ordered $\eta$-contraction such that, for any $t > 0$, $A_0(t)$ and $B_0(t)$ are nonempty with $T(A_0(t)) \subseteq B_0(t)$, $g : A \to A$ surjective, fuzzy nonexpansive and inverse monotone mapping with $A_0(t) \subseteq g(A_0(t))$ for any $t > 0$. If there exist some elements $x_0$ and $x_1$ in $A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A,B,t)$ with $(x_0,x_1) \in \Delta_{(t)}$, then there exists a unique element $x^* \in A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A,B,t)$ provided that each pair of elements in $X$ has a lower and upper bound. Further, for any fixed element $\overline{x}_0 \in A_0(t)$, the sequence $\{\overline{x}_n\}$ defined by $M(g\overline{x}_{n+1}, T\overline{x}_n, t) = M(A,B,t)$ converges to $x^*$.*

*Proof* Let $x_0$ and $x_1$ be given points in $A_0(t)$ such that

$$M(gx_1, Tx_0, t) = M(A,B,t) \quad \text{with } (x_0, x_1) \in \Delta_{(t)}. \tag{25}$$

Since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can choose an element $x_2 \in A_0(t)$ such that

$$M(gx_2, Tx_1, t) = M(A,B,t). \tag{26}$$

As $T$ is proximally monotone, we have $(gx_1, gx_2) \in \Delta_{(t)}$, which further implies that $(x_1, x_2) \in \Delta_{(t)}$. Continuing this way, we can obtain a sequence $\{x_n\}$ in $A_0(t)$, such that it satisfies

$$M(gx_n, Tx_{n-1}, t) = M(A,B,t) \quad \text{with } (x_{n-1}, x_n) \in \Delta_{(t)}. \tag{27}$$

for each positive integer $n$. Having chosen $x_n$, one can find a point $x_{n+1}$ in $A_0(t)$ such that

$$M(gx_{n+1}, Tx_n, t) = M(A,B,t). \tag{28}$$

Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, $T$ is proximally monotone mapping, so from (27) and (28) it follows that $(gx_n, gx_{n+1}) \in \Delta_{(t)}$ and $(x_n, x_{n+1}) \in \Delta_{(t)}$. Note that

$$M(x_n, x_{n-1}, t) \leq \eta\bigl[M(gx_{n+1}, gx_n, t)\bigr] \leq \eta\bigl[M(x_{n+1}, x_n, t)\bigr] < M(x_{n+1}, x_n, t). \tag{29}$$

Denote $M(x_n, x_{n+1}, t) = \tau_n(t)$ for all $t > 0$, $n \in \mathbb{N} \cup \{0\}$. The above inequality becomes

$$\tau_{n-1}(t) \leq \eta\bigl(\tau_n(t)\bigr) < \tau_n(t). \tag{30}$$

Thus $\{\tau_n(t)\}$ is an increasing sequence for each $t > 0$. Consequently, $\lim_{n\to+\infty} \tau_n(t) = \tau(t)$. We claim that $\tau(t) = 1$ for each $t > 0$. If not, there exist some $t_0 > 0$ such that $\tau(t_0) < 1$. Also, $\tau_n(t_0) \leq \tau(t_0)$. On taking limit as $n \to \infty$ on both sides of (30), we have $\tau(t_0) \leq \eta(\tau(t_0)) < \tau(t_0)$, a contradiction. Hence $\tau(t) = 1$ for each $t > 0$. Now we show that $\{x_n\}$ is a Cauchy sequence. If not, then there exist some $\varepsilon \in (0,1)$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$, with $m_k > n_k \geq k$ such that

$$M(x_{m_k}, x_{n_k}, t_0) \leq 1 - \varepsilon. \tag{31}$$

If $m_k$ is the least integer exceeding $n_k$ and satisfying the above inequality, then

$$M(x_{m_k-1}, x_{n_k}, t_0) > 1 - \varepsilon. \tag{32}$$

So, for all $k$,

$$\begin{aligned} 1 - \varepsilon &\geq M(x_{m_k}, x_{n_k}, t_0) \\ &\geq M(x_{m_k}, x_{m_k-1}, t_0) * M(x_{m_k-1}, x_{n_k}, t_0) \\ &> \tau_{m_k}(t_0) * (1 - \varepsilon). \end{aligned} \tag{33}$$

On taking the limit as $k \to \infty$ on both sides of above inequality, we obtain $\lim_{k\to+\infty} M(x_{m_k}, x_{n_k}, t_0) = 1 - \varepsilon$. Note that

$$M(x_{m_k+1}, x_{n_k+1}, t_0) \geq M(x_{m_k+1}, x_{m_k}, t_0) * M(x_{m_k}, x_{n_k}, t_0) * M(x_{n_k}, x_{n_k+1}, t_0)$$

and

$$M(x_{m_k}, x_{n_k}, t_0) \geq M(x_{m_k}, x_{m_k+1}, t_0) * M(x_{m_k+1}, x_{n_k+1}, t_0) * M(x_{n_k+1}, x_{n_k}, t_0),$$

imply that

$$\lim_{k\to+\infty} M(x_{m_k+1}, x_{n_k+1}, t_0) = 1 - \varepsilon.$$

From (28), we have

$$M(gx_{m_k+1}, Tx_{m_k}, t_0) = M(A, B, t_0) \quad \text{and} \quad M(gx_{n_k+1}, Tx_{n_k}, t_0) = M(A, B, t_0).$$

Thus

$$M(x_{m_k}, x_{n_k}, t_0) \leq \eta\big[M(gx_{m_k+1}, gx_{n_k+1}, t_0)\big] \leq \eta\big[M(x_{m_k+1}, x_{n_k+1}, t_0)\big] < M(x_{m_k+1}, x_{n_k+1}, t_0).$$

On taking the limit as $k \to \infty$ in the above inequality, we get $1 - \varepsilon \leq \eta(1 - \varepsilon) < 1 - \varepsilon$, a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in the closed subset $A(t)$ of complete partially ordered fuzzy metric space $(X, M, *, \preceq)$. Thus there exists $x^* \in A(t)$ such that $\lim_{n\to\infty} M(x_n, x^*, t) = 1$, for all $t > 0$. This further implies that $M(gx^*, Tx^*, t) = \lim_{n\longrightarrow\infty} M(gx_{n+1}, Tx_n, t) = M(A, B, t)$ and hence $x^* \in A_0(t)$ is the optimal coincidence point of a pair $\{g, T\}$. To prove the uniqueness of $x^*$, we show that, for any fixed element $\overline{x}_0 \in A_0(t)$, the sequence $\{\overline{x}_n\} \in A_0(t)$ defined by $M(g\overline{x}_{n+1}, T\overline{x}_n, t) = M(A, B, t)$ converges to $x^*$. Suppose that there is another element $\overline{x}_0 \in A(t)$ such that $0 < M(x_0, \overline{x}_0, t) < 1$ for all $t > 0$ satisfying

$$M(g\overline{x}_0, T\overline{x}_0, t) = M(A, B, t). \tag{34}$$

Suppose that $(\overline{x}_0, x_0) \in \Delta_{(t)}$. Then, by the given assumption, we have

$$M(\overline{x}_0, x_0, t) \leq \eta\big(M(g\overline{x}_0, gx_0, t)\big) \leq \eta\big(M(\overline{x}_0, x_0, t)\big) < M(\overline{x}_0, x_0, t),$$

a contradiction and hence the result follows. If $(\overline{x}_0, x_0) \notin \Delta_{(t)}$, then let $u_0$ be a lower bound of $x_0$ and $\overline{x}_0$, and $\overline{u}_0$ an upper bound of $x_0$ and $\overline{x}_0$. That is,

$$\overline{u}_0 \succeq x_0 \succeq u_0 \quad \text{or} \quad \overline{u}_0 \succeq \overline{x}_0 \succeq u_0.$$

Recursively, construct sequences $\{u_n\}$ and $\{\overline{u}_n\}$, such that

$$M(gu_{n+1}, Tu_n, t) = M(A, B, t) \quad \text{and} \quad M(g\overline{u}_{n+1}, T\overline{u}_n, t) = M(A, B, t).$$

The proximal monotonicity of the mapping $T$ and the monotonicity of the inverse of $g$ imply that

$$\overline{u}_n \succeq \overline{x}_n \succeq u_n \quad \text{or} \quad \overline{u}_n \preceq \overline{x}_n \preceq u_n.$$

From $(x_n, u_n) \in \Delta_{(t)}$ and $(x_n, \overline{u}_n) \in \Delta_{(t)}$, it follows that

$$\lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} u_n = x^*.$$

Hence $\lim_{n\to\infty} \overline{x}_n = x^*$. □

**Example 4.2** Let $X = [-1,1] \times \mathbb{R}$ and $\preceq$ a usual order on $\mathbb{R}^2$. Let $A = \{(-1,x) : \text{for all } x \in \mathbb{R}\}$, $B = \{(1,y) : \text{for all } y \in \mathbb{R}\}$, and $(X, M, *, \preceq)$ a complete fuzzy ordered metric space as given in Example 2.17. Note that $M(A,B,t) = \frac{t}{t+2}$, $A_0(t) = A$ and $B_0(t) = B$. Define $T : A \to B$ by

$$T(-1,x) = \left(1, \frac{x}{5}\right).$$

Let $g : A \to A$ be defined by $g(-1,x) = (-1, \frac{x}{2})$. Note that $g$ is fuzzy nonexpansive and its inverse is monotone. Obviously, $T(A_0(t)) \subseteq B_0(t)$, and $A_0(t) \subseteq g(A_0(t))$. Note that $u = (-1, \frac{2}{5}y_1)$, $v = (-1, \frac{2}{5}y_2)$, $x = (-1, y_1)$, and $y = (-1, y_2) \in A$. Also, note that

$$M\bigl((-1,y_1),(-1,y_2),t\bigr) \leq \eta\left(M\left(\left(-1,\frac{y_1}{5}\right),\left(-1,\frac{y_2}{5}\right),t\right)\right).$$

Here $\eta(t) = 2t - t^2$. Thus all conditions of Theorem 4.1 are satisfied. Moreover, $(-1,0)$ is the optimal coincidence point of $g$ and $T$.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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