

DISJOINTNESS OF C*-DYNAMICAL SYSTEMS

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ABSTRACT. We study ergodic theorems for disjoint C*- and W*-dynamical systems, where disjointness here is a noncommutative version of the concept introduced by Furstenberg for classical dynamical systems. We also consider specific examples of disjoint W*-dynamical systems. Lastly we use unique ergodicity and unique ergodicity relative to the fixed point algebra to give examples of disjoint C*-dynamical systems.

1. INTRODUCTION

In [11, 13, 14] we studied joinings of W*-dynamical systems, generalizing aspects of the classical case [20, 21]. In particular disjointness and some of its implications were studied. This included the relative case of disjointness over a common subsystem of the W*-dynamical systems. Here we continue studying disjointness and its consequences, in the context of C*-dynamical systems possessing an invariant state. In the process we illustrate how joining techniques can be applied to C*- and W*-dynamical systems.

In particular we obtain an ergodic theorem for two disjoint C*-dynamical systems, and apply it to W*-dynamical systems. A number of results on ergodic theorems for C*- and W*-dynamical systems have appeared recently (see for example [23], [12], [6], [3] and [15, Section 4]), and the ergodic theorems we obtain here for pairs of disjoint C*- and W*-dynamical systems, fit into this broader research effort, providing evidence that joinings are useful in this field.

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A number of examples of disjoint systems illustrating these results, including for the relative case, are constructed using \mathbb{Z} -actions on group von Neumann algebras, as well as \mathbb{R} and \mathbb{R}^2 actions on quantum tori and tensor products of quantum tori.

A concept closely related to disjointness is unique ergodicity relative to the fixed point algebra of a C^* -dynamical system. Abadie and Dykema introduced the latter notion for \mathbb{Z} -actions in [1] as a generalization of unique ergodicity. Aspects of unique ergodicity of C^* -dynamical systems have already been explored by Avitzour [4, 5] and Longo and Peligrad [22] for general group actions. Here we use unique ergodicity relative to the fixed point algebra for actions of σ -compact locally compact amenable groups to obtain examples of disjoint C^* -dynamical systems.

The main discussion of disjointness and its implications appear in Section 3. Unique ergodicity and unique ergodicity relative to the fixed point algebra, and their connection to disjointness, are studied in Sections 4 and 5 respectively.

2. BASIC DEFINITIONS AND TOOLS

Here we discuss some of the definitions and an example that we will use in the rest of the paper.

First we define the dynamical systems that we will be working with.

Definition 2.1. A C^* -dynamical system (A, α) consists of a unital C^* -algebra A and an action α of a group G on A as $*$ -automorphisms. We use the notation α_g for an element of the group action. If B is an α -invariant C^* -subalgebra of A containing the unit of A , in other words $\alpha_g(B) = B$ for all $g \in G$, we can define $\beta_g := \alpha_g|_B$ to obtain a C^* -dynamical system (B, β) called a *subsystem* of (A, α) . The *fixed point algebra* of a C^* -dynamical system (A, α) is defined as

$$A^\alpha := \{a \in A : \alpha_g(a) = a \text{ for all } g \in G\}.$$

Note that this gives an example of a subsystem of (A, α) , namely (A^α, id) where id is the identity map.

Subsystems here correspond to factors in classical topological dynamics, but we use the term subsystem in the noncommutative case to avoid confusion.

In any situation in the rest of the paper involving more than one C^* -dynamical system, all the systems will be assumed to make use of actions of the same group G . A simple but important construction in our work is the tensor product of two C^* -dynamical systems (A, α) and (B, β) , namely the C^* -dynamical system

$(A \otimes_m B, \alpha \otimes_m \beta)$ where $A \otimes_m B$ is the maximal C*-algebraic tensor product of A and B , and $(\alpha \otimes_m \beta)_g := \alpha_g \otimes_m \beta_g$ for all $g \in G$.

We will in fact focus on a specific type of C*-dynamical system defined as follows:

Definition 2.2. A C*-dynamical system (A, α) is called *amenable* if G is an amenable σ -compact locally compact group and the function $G \rightarrow A : g \mapsto \alpha_g(a)$ is continuous for every $a \in A$.

Remember that an amenable σ -compact locally compact group G has a right Følner sequence (Λ_n) consisting of compact subsets of G with strictly positive right Haar measure. This means that

$$(2.3) \quad \lim_{n \rightarrow \infty} \frac{|\Lambda_n \triangle (\Lambda_n g)|}{|\Lambda_n|} = 0$$

for every $g \in G$, where $|\cdot|$ denotes the right Haar measure on G . If in (2.3) we were to replace $\Lambda_n g$ by $g\Lambda_n$ and we work in terms of the left Haar measure, then (Λ_n) is rather called a *left* Følner sequence. One can even choose each Λ_n to be symmetric, i.e. $\Lambda_n^{-1} = \Lambda_n$. Note that this implies that when the group is unimodular (i.e. when the right Haar measure is also a left Haar measure) we always have a right Følner sequence which is also a left Følner sequence, namely any Følner sequence consisting of symmetric sets. Of course a Følner sequence need not be symmetric in order to be both a right and left Følner sequence, for example the sequence $\Lambda_n = \{1, \dots, n\}$ for the group \mathbb{Z} . All these facts (and more) regarding Følner sequences can be found in [17, Theorems 1 and 2] and [16, Theorem 4], and will be used in this paper.

In the rest of the paper the notation (Λ_n) will refer to a right Følner sequence consisting of compact subsets, each with strictly positive right Haar measure, of the relevant group G . Furthermore, we will write dg when integrating with respect to the right Haar measure.

A central notion in our work will be that of an invariant state:

Definition 2.3. Given a C*-dynamical system (A, α) , a state μ on A is called an *invariant* state of (A, α) , or alternatively an α -*invariant* state, if $\mu \circ \alpha_g = \mu$ for all $g \in G$. In this case we say that $\mathbf{A} = (A, \alpha, \mu)$ is a *state preserving C*-dynamical system*.

We end this section with a basic and standard example that we will use a number of times in the paper, namely the quantum torus.

Example 2.4. Our discussion is based on the exposition of the quantum torus in [27, Section 5.5], although our notation and conventions are somewhat different.

We consider the classical torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the Hilbert space $H := L^2(\mathbb{T}^2)$ with respect to the normalized Haar measure on \mathbb{T}^2 . With every $\theta \in \mathbb{R}$ we associate a C*-algebraic quantum torus A_θ which is the C*-subalgebra of $B(H)$ generated by the set of operators $\{u, v\} \subset B(H)$ defined by

$$(uf)(x, y) := e^{2\pi i x} f(x, y + \theta/2)$$

and

$$(vf)(x, y) := e^{2\pi i y} f(x - \theta/2, y)$$

for all $f \in H$ and $(x, y) \in \mathbb{T}^2$. We will also consider the von Neumann algebraic quantum torus $M_\theta = A_\theta''$. We note that $uv = e^{2\pi i \theta} vu$ so in general A_θ and M_θ are not abelian. We can set $\Omega := 1 \in H$ which is then a cyclic vector for A_θ and M_θ , and it can be verified using harmonic analysis on \mathbb{T}^2 that it is also separating for A_θ and M_θ . Hence we can define a faithful trace Tr on both A_θ and M_θ by

$$\text{Tr}(a) := \langle \Omega, a\Omega \rangle$$

for all $a \in M_\theta$. Of course, Tr is normal on M_θ . We call Tr the *canonical trace* of the quantum torus.

A very simple \mathbb{R}^2 action can be defined on both A_θ and M_θ as follows: Set

$$T_{s,t}(x, y) := (x + s, y + t) \in \mathbb{T}^2$$

for all $(x, y) \in \mathbb{T}^2$ and then define

$$U_{s,t} : H \rightarrow H : f \mapsto f \circ T_{s,t}$$

for all $(s, t) \in \mathbb{R}^2$. One can then check that this leads to a well defined action τ of \mathbb{R}^2 as *-automorphisms on both A_θ and M_θ defined by

$$\tau_{s,t}(a) := U_{s,t} a U_{s,t}^*$$

for all a in A_θ or M_θ , and all $(s, t) \in \mathbb{R}^2$. Furthermore, Tr is an invariant state of this dynamics in both the C*-algebraic and von Neumann algebraic cases. It can be shown that

$$\mathbb{R}^2 \rightarrow A_\theta : (s, t) \mapsto \tau_{s,t}(a)$$

is norm continuous for every $a \in A_\theta$. So $(A_\theta, \tau, \text{Tr})$ is an amenable state preserving C*-dynamical system. A useful fact is that

$$(2.4) \quad \tau_{s,t}(u) = e^{2\pi i s} u \quad \text{and} \quad \tau_{s,t}(v) = e^{2\pi i t} v$$

for all $(s, t) \in \mathbb{R}^2$. We will also consider variations on this dynamics in the rest of the paper. The C*-algebraic case will appear in Sections 4 and 5, while the von Neumann algebraic case will appear in Section 3.

We note that at least in the case where θ is irrational, A_θ is nuclear. This follows for example from the fact that A_θ can be written as a crossed product of the abelian (and therefore nuclear) C*-algebra $C(\mathbb{T})$ by an action of the amenable group \mathbb{Z} as explained in [8, Theorem VI.1.4 and Example VIII.1.1], combined with the fact that such crossed products are nuclear [24, Proposition 2.1.2]. Nuclearity of A_θ will be useful for us when discussing certain examples in Section 5.

3. DISJOINTNESS

Disjointness of C*-dynamical systems was first considered by Avitzour in [5], extending the concept of disjointness in topological dynamical systems [20, Part II]. In this section however our approach is from the ergodic theory point of view, in that we define disjointness of a pair of state preserving C*-dynamical systems in terms of invariant states on a tensor product of the pair, similar to [11, 13, 14]. It is based on the idea of disjointness in classical ergodic theory, also originating in Furstenberg's paper [20], and treated extensively in [21].

Definition 3.1. Let $\mathbf{A} = (A, \alpha, \mu)$ and $\mathbf{B} = (B, \beta, \nu)$ be state preserving C*-dynamical systems. A *joining* of \mathbf{A} and \mathbf{B} is an invariant state ω of $(A \otimes_m B, \alpha \otimes_m \beta)$ such that $\omega(a \otimes 1) = \mu(a)$ and $\omega(1 \otimes b) = \nu(b)$ for all $a \in A$ and $b \in B$. The set of all joinings of \mathbf{A} and \mathbf{B} is denoted by $J(\mathbf{A}, \mathbf{B})$. If $J(\mathbf{A}, \mathbf{B}) = \{\mu \otimes_m \nu\}$, then \mathbf{A} and \mathbf{B} are called *disjoint*. More generally we can consider a subsystem (R, ρ) of $(A \otimes_m B, \alpha \otimes_m \beta)$, and a ρ -invariant state ψ on R which has at least one extension to a joining of \mathbf{A} and \mathbf{B} . So we obtain a state preserving C*-dynamical system $\mathbf{R} = (R, \rho, \psi)$. Denote by $J_{\mathbf{R}}(\mathbf{A}, \mathbf{B})$ the subset of elements ω of $J(\mathbf{A}, \mathbf{B})$ such that $\omega|_R = \psi$. If $J_{\mathbf{R}}(\mathbf{A}, \mathbf{B})$ contains exactly one element, then we say that \mathbf{A} and \mathbf{B} are *disjoint with respect to \mathbf{R}* .

In this definition one can equivalently define a joining of \mathbf{A} and \mathbf{B} in terms of the algebraic tensor product $(A \odot B, \alpha \odot \beta)$ instead of $(A \otimes_m B, \alpha \otimes_m \beta)$, since a state on the algebraic tensor product can be extended to a state on the maximal C*-algebraic tensor product; see for example [13, Proposition 4.1]. This fact is used later on. In the rest of this section, the notation \mathbf{A} , \mathbf{B} and \mathbf{R} will refer to triples as in Definition 3.1, and we also use the notation \mathbf{F} for the triple (F, φ, λ) , with the symbol G for the group always implied.

We also need the following weaker concept:

Definition 3.2. Let A and B be unital C*-algebras with states μ and ν respectively. A *coupling* of the pairs (A, μ) and (B, ν) is a state κ on $A \otimes_m B$ such that $\kappa(a \otimes 1) = \mu(a)$ and $\kappa(1 \otimes b) = \nu(b)$ for all $a \in A$ and $b \in B$. If furthermore ψ

is a state on a C^* -subalgebra R of $A \otimes_m B$ such that $\kappa|_R = \psi$, then we call κ a *coupling* of (A, μ) and (B, ν) with respect to (R, ψ) .

Our first result is the following simple ergodic theorem, which can be viewed as a generalization of [13, Proposition 2.3], and indeed its proof is a straightforward adaptation of that in [13].

Theorem 3.3. *Let \mathbf{A} and \mathbf{B} be state preserving amenable C^* -dynamical systems which are disjoint with respect to \mathbf{R} . Let (κ_n) be a sequence of couplings of (A, μ) and (B, ν) with respect to (R, ψ) . Then*

$$(3.1) \quad \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \kappa_n \circ (\alpha_g \otimes_m \beta_g)(c) dg = \omega(c)$$

for all $c \in A \otimes_m B$, where ω is the unique element of $J_{\mathbf{R}}(\mathbf{A}, \mathbf{B})$. Alternatively, instead of assuming that $g \mapsto \alpha_g(a)$ and $g \mapsto \beta_g(b)$ are continuous for all $a \in A$ and $b \in B$, we can simply assume that for every n the function $g \mapsto \kappa_n \circ (\alpha_g \otimes_m \beta_g)(a \otimes b)$ is Borel measurable on G for all $a \in A$ and $b \in B$, and then the result still holds.

PROOF. Since $g \mapsto \alpha_g(a)$ and $g \mapsto \beta_g(b)$ are both continuous, so is $g \mapsto \alpha_g(a) \otimes \beta_g(b)$. Hence $g \mapsto \kappa_n(\alpha_g(a) \otimes \beta_g(b))$ is continuous on G , or more generally measurable if we use the alternative assumption in the theorem. Either way, from Lebesgue's dominated convergence theorem it then follows that $g \mapsto \kappa_n \circ (\alpha_g \otimes_m \beta_g)(c)$ is measurable on Λ_n for all $c \in A \otimes_m B$, which means that the integrals in (3.1) indeed exist.

We define a sequence of states (ω_n) on $A \otimes_m B$ by

$$\omega_n(c) := \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \kappa_n \circ (\alpha_g \otimes_m \beta_g)(c) dg$$

which then has a cluster point ω' in the weak* topology in the set of all states on $A \otimes_m B$. By an argument similar to that in the proof of [13, Proposition 2.3], it follows from eq. (2.3) that $\omega' \in J(\mathbf{A}, \mathbf{B})$. It is also easy to check, using the ρ -invariance of ψ , that $\omega_n|_R = \psi$, hence $\omega'|_R = \psi$. So by the assumed disjointness, $\omega' = \omega$. Therefore ω is the unique cluster point of (ω_n) in the weak* topology, from which we conclude that (ω_n) converges to ω in the weak* topology, proving the theorem. \square

This theorem is quite abstract, but has a number of interesting consequences as we will see. To begin to understand its meaning, and for later reference, we state the following special case explicitly.

Corollary 3.4. *Let \mathbf{A} and \mathbf{B} be disjoint state preserving amenable C*-dynamical systems, and (κ_n) a sequence of couplings of (A, μ) and (B, ν) . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \kappa_n \circ (\alpha_g \otimes \beta_g)(c) dg = \mu \otimes_m \nu(c)$$

for all $c \in A \otimes_m B$. Again we can also use the alternative measurability assumptions given in Theorem 3.3

Remark 3.5. This theorem and its corollary in fact continues to hold if we use a Følner net instead of a Følner sequence, and a net of couplings with the same directed set as the Følner net. This allows one to use for example the form $\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T}^T$ in the case where $T \in G = \mathbb{R}$.

The corollary is the one extreme of Theorem 3.3, namely when $R = \mathbb{C}1$. The other extreme is when $R = A \otimes_m B$, in which case Theorem 3.3 becomes trivial, since $\kappa_n = \psi = \omega$ is then $\alpha \otimes_m \beta$ -invariant. Broadly the idea is therefore to apply Theorem 3.3 to situations where R is a proper subalgebra of $A \otimes_m B$.

In the remainder of this section we apply Theorem 3.3 to W*-dynamical systems, using results and ideas on joinings from [11, 13, 14]. In subsequent sections we will illustrate the C*-algebraic case itself, by way of examples.

Definition 3.6. A *W*-dynamical system* is a state preserving C*-dynamical system $\mathbf{A} = (A, \alpha, \mu)$ where μ is a faithful normal state on a (necessarily σ -finite) von Neumann algebra A .

We now also use Definitions 3.1 and 3.2 for W*-dynamical systems, since W*-dynamical systems are state preserving C*-dynamical systems. However, in [11, 13, 14] joinings were expressed in terms of algebraic tensor products, so as mentioned above, it is important to keep in mind that any state on the algebraic tensor product of two C*-algebras, and in particular two von Neumann algebras, can be extended to a state on the maximal C*-algebraic tensor product of the two algebras, as explained in [13, Section 4]. So even though we are working on the maximal C*-algebraic tensor product, we will still be able to apply results from [11, 13, 14].

We need some additional background and notation regarding a W*-dynamical system \mathbf{A} before we proceed (a more general setup and more details can be found in [11, 13, 14]): The cyclic representation of A obtained from μ by the GNS construction will be denoted by (H, π, Ω) . The associated modular conjugation will be denoted by J and we set

$$j : B(H) \rightarrow B(H) : a \mapsto Ja^*J.$$

Note that $j^{-1} = j$. Then α can be represented by a unitary group U on H defined by extending

$$U_g \pi(a) \Omega := \pi(\alpha_g(a)) \Omega.$$

It satisfies

$$U_g \pi(a) U_g^* = \pi(\alpha_g(a))$$

for all $a \in A$ and $g \in G$; also see [7, Corollary 2.3.17]. We are particularly interested in the case where we have a second W^* -dynamical system \mathbf{B} with $(B, \nu) = (A, \mu)$, in other words we want to consider two dynamics on the same von Neumann algebra and state. Exactly as for U above, we obtain a unitary representation V of β .

Following the plan in [11, Construction 3.4] the “commutant” $\tilde{\mathbf{B}}$ of \mathbf{B} can be defined as follows: Set $\tilde{B} := \pi(B)'$ and then carry the state and dynamics of \mathbf{B} over to \tilde{B} in a natural way using j , by defining a state $\tilde{\nu}$ and $*$ -automorphism $\tilde{\beta}_g$ on \tilde{B} by $\tilde{\nu}(b) := \nu \circ \pi^{-1} \circ j(b)$ and $\tilde{\beta}_g(b) := j \circ \pi \circ \beta_g \circ \pi^{-1} \circ j(b)$ for all $g \in G$. From Tomita-Takesaki theory one has that $V_g J = J V_g$ (see [11, Construction 3.4]). Then

$$\tilde{\nu}(b) = \langle \Omega, b \Omega \rangle$$

and

$$\tilde{\beta}_g(b) = V_g b V_g^*$$

for all $b \in \tilde{B}$ and $g \in G$. This tells us that the unitary representation of $\tilde{\beta}$ is the same as that of β , namely V .

A *subsystem* of \mathbf{A} is a W^* -dynamical system $\mathbf{F} = (F, \varphi, \lambda)$ such that (F, φ) is a subsystem of (A, α) as in Definition 2.1, but with F now a von Neumann subalgebra of A , and $\lambda = \mu|_F$. In [11, 13, 14] a more general situation was considered, but this definition of a subsystem will do in this paper. We say that \mathbf{F} is a *modular* subsystem of \mathbf{A} if F is invariant under the modular group σ associated to (A, μ) , i.e. $\sigma_t(F) = F$ for all $t \in \mathbb{R}$. Assuming that \mathbf{F} is also a modular subsystem of \mathbf{B} above, and writing

$$\tilde{F} := j \circ \pi(F)$$

we obtain a modular subsystem $\tilde{\mathbf{F}} = (\tilde{F}, \tilde{\varphi}, \tilde{\lambda})$ of $\tilde{\mathbf{B}}$. One can then define a *diagonal* state Δ_λ on the algebraic tensor product $F \odot \tilde{F}$ by extending

$$\delta(a \otimes b) := \pi(a)b$$

to a unital $*$ -homomorphism $\delta : F \odot \tilde{F} \rightarrow B(H)$ and setting

$$\Delta_\lambda(c) := \langle \Omega, \delta(c) \Omega \rangle$$

for all $c \in F \odot \tilde{F}$. Since \mathbf{F} is a modular subsystem, it turns out that there is at least one joining of \mathbf{A} and $\tilde{\mathbf{B}}$ extending the state Δ_λ , as explained in [14, Section 3].

Definition 3.7. Let \mathbf{A} and \mathbf{B} be W^* -dynamical systems with $(A, \mu) = (B, \nu)$, and assume both have \mathbf{F} as a modular subsystem. If Δ_λ has a unique extension to a joining of \mathbf{A} and $\tilde{\mathbf{B}}$, then we call \mathbf{A} and $\tilde{\mathbf{B}}$ *disjoint over \mathbf{F}* .

Clearly this definition is closely related to Definition 3.1, and the connection will be further clarified in the proof of Theorem 3.8 below.

Using Theorem 3.3 we can now prove the main result of this section, where the notation id_F refers to the identity map $F \rightarrow F$.

Theorem 3.8. *Let \mathbf{A} and \mathbf{B} be W^* -dynamical systems with $(A, \mu) = (B, \nu)$ and with G amenable, σ -compact and locally compact. Let \mathbf{F} be a modular subsystem of both \mathbf{A} and \mathbf{B} , and assume that \mathbf{A} and $\tilde{\mathbf{B}}$ are disjoint over \mathbf{F} . Let $\eta_n : A \rightarrow A$ be any sequence of $*$ -automorphisms with $\eta_n|_F = \text{id}_F$ and $\mu \circ \eta_n = \mu$, and assume that $G \rightarrow \mathbb{C} : g \mapsto \mu(\eta_n(\alpha_g(a))\beta_g(b))$ is Borel measurable for all $a, b \in A$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \mu(\eta_n(\alpha_g(a))\beta_g(b)) dg = \mu(D(a)D(b))$$

for all $a, b \in A$, where $D : A \rightarrow F$ is the unique conditional expectation such that $\mu \circ D = \mu$.

In particular, if \mathbf{A} and $\tilde{\mathbf{B}}$ are disjoint (that is, $F = \mathbb{C}$), then

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \mu(\eta_n(\alpha_g(a))\beta_g(b)) dg = \mu(a)\mu(b)$$

for all $a, b \in A$.

PROOF. The existence and uniqueness of D follow from a result of Takesaki [26] and the fact that \mathbf{F} is a modular subsystem of \mathbf{A} . Similarly we have a unique conditional expectation $\tilde{D} : \tilde{A} \rightarrow \tilde{F}$ such that $\tilde{\mu} \circ \tilde{D} = \tilde{\mu}$. Note that if P is the projection of H onto $\overline{\pi(F)\Omega}$, in terms of the notation above, then $\pi(D(a)) = P\pi(a)P$ and $\tilde{D}(b) = PbP$ for all $a \in A$ and $b \in \tilde{B}$, as discussed for example in [25, Subsection 10.2].

Let R be the closure of $F \odot \tilde{F}$ in $A \otimes_m B$. Note that in terms of the notation above, Δ_λ has at least one extension to a state on $A \odot \tilde{B}$, namely the relatively independent joining $\mu \odot_\lambda \tilde{\nu} = \Delta_\lambda \circ (D \odot \tilde{D})$ [14, Section 3], which can in turn be uniquely extended to a state, say $\mu \otimes_\lambda \tilde{\nu}$, on $A \otimes_m \tilde{B}$ [13, Proposition 4.1], and therefore in particular Δ_λ can be extended to a (necessarily unique) state on R . Since \mathbf{A} and $\tilde{\mathbf{B}}$ are in fact disjoint over \mathbf{F} , $\mu \otimes_\lambda \tilde{\nu}$ is the unique joining extending

Δ_λ . In the sense of Definition 3.1, \mathbf{A} and $\tilde{\mathbf{B}}$ are therefore disjoint with respect to $\mathbf{R} = (R, \rho, \Delta_\lambda)$ where $\rho_g := \alpha_g \otimes_m \tilde{\beta}_g|_R$.

We define a coupling μ_Δ of (A, μ) and $(\tilde{B}, \tilde{\nu})$ by $\mu_\Delta(c) := \langle \Omega, \delta(c)\Omega \rangle$ where the unital $*$ -homomorphism $\delta : A \otimes_m \tilde{B} \rightarrow B(H)$ is obtained from $\delta(a \otimes b) := \pi(a)b$ for all $a \in A$ and $b \in \tilde{B}$, extending δ appearing above. This type of coupling has also been used in [18]. From this we then define the sequence of couplings

$$\kappa_n := \mu_\Delta \circ (\eta_n \otimes_m \text{id}_{\tilde{B}}).$$

It is easily verified using elementary tensors, that each κ_n is a coupling with respect to (R, Δ_λ) in the sense of Definition 3.2. Let W_n be the unitary representation of η_n in the cyclic representation obtained from μ , and note that $\mu(\eta_n(\alpha_g(a))\beta_g(b)) = \langle \pi(a^*)\Omega, U_g^*W_n^*V_g\pi(b)\Omega \rangle$ for all $a, b \in A$, so from Lebesgue's dominated convergence theorem and the fact that $\pi(A)\Omega$ is dense in H it follows that $g \mapsto \langle x, U_g^*W_n^*V_gy \rangle$ is measurable on compact subsets of G for all $x, y \in H$. This in turn implies that $g \mapsto \kappa_n \circ (\alpha_g \otimes_m \tilde{\beta}_g)(a \otimes b) = \langle \pi(a^*)\Omega, U_g^*W_n^*V_gb\Omega \rangle$ is measurable on compact subsets of G for all $a \in A$ and $b \in \tilde{B}$. Applying Theorem 3.3, it follows that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \langle \pi(a)\Omega, U_g^*W_n^*V_gb\Omega \rangle dg \\ &= \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \kappa_n \circ (\alpha_g \otimes_m \tilde{\beta}_g)(a^* \otimes b) dg \\ &= \mu \otimes_\lambda \tilde{\nu}(a^* \otimes b) \\ &= \langle \pi(D(a))\Omega, \tilde{D}(b)\Omega \rangle \\ &= \langle P\pi(a)\Omega, Pb\Omega \rangle \end{aligned}$$

and therefore, since $\tilde{B}\Omega$ is also dense in H ,

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \langle x, U_g^*W_n^*V_gy \rangle = \langle Px, Py \rangle$$

for all $x, y \in H$. Finally we can apply this to our current problem, namely for all $a, b \in A$ we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \mu(\eta_n(\alpha_g(a))\beta_g(b))dg \\ &= \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \langle \pi(a^*)\Omega, U_g^* W_n^* V_g \pi(b)\Omega \rangle dg \\ &= \langle P\pi(a^*)\Omega, P\pi(b)\Omega \rangle \\ &= \langle \pi(D(a^*))\Omega, \pi(D(b))\Omega \rangle \\ &= \langle \Omega, \pi(D(a)D(b))\Omega \rangle \\ &= \mu(D(a)D(b)) \end{aligned}$$

as required.

In particular, when \mathbf{A} and $\tilde{\mathbf{B}}$ are disjoint, we have $D = \mu$, so $\mu(a)\mu(b)$. Alternatively we can follow the above proof, but using Corollary 3.4 instead of Theorem 3.3, to obtain this special case. \square

We are therefore interested in pairs of disjoint W^* -dynamical systems. An archetypal example is weak mixing versus compactness; see [23, 13, 14] for an extended discussion and definitions. In particular we recall from [23, Section 4] and its extension to more general groups in [14, Section 5], that a W^* -dynamical system with G abelian, has a biggest compact subsystem, which we denote by $\mathbf{A}^K = (A^K, \alpha^K, \mu^K)$, and this subsystem is necessarily modular.

Corollary 3.9. *Let \mathbf{A} and \mathbf{B} be W^* -dynamical systems with $(A, \mu) = (B, \nu)$. Assume that G is abelian, amenable, σ -compact and locally compact, and that $G \rightarrow \mathbb{C} : g \mapsto \mu(\alpha_g(a)\beta_g(b))$ is Borel measurable for all $a, b \in A$. Assume furthermore that \mathbf{B} is compact and that \mathbf{A}^K is a subsystem of \mathbf{B} . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \mu(\alpha_g(a)\beta_g(b))dg = \mu(D(a)D(b))$$

for all $a, b \in A$, where $D : A \rightarrow A^K$ is the unique conditional expectation such that $\mu \circ D = \mu$.

In particular, if \mathbf{A} is weakly mixing (in which case \mathbf{A}^K is trivial, that is, $A^K = \mathbb{C}1$), then

$$(3.2) \quad \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \mu(\alpha_g(a)\beta_g(b))dg = \mu(a)\mu(b)$$

for all $a, b \in A$.

PROOF. Since $\nu = \mu$, while \mathbf{A}^K is a modular subsystem of \mathbf{A} , it follows that \mathbf{A}^K is also a modular subsystem of \mathbf{B} . Therefore \mathbf{A} and $\tilde{\mathbf{B}}$ are disjoint over \mathbf{A}^K by [14, Theorem 5.6], and the result follows from Theorem 3.8. When \mathbf{A} is weakly mixing, then $\mathbf{A}^K = \mathbb{C}1$ by [14, Corollary 5.5], therefore $D = \mu$, and the special case follows. \square

Remark 3.10. Using [13, Theorem 2.8], we can obtain a variation on the special case in the result above, namely assuming that \mathbf{B} is ergodic with discrete spectrum, instead of compact, but relaxing the assumption that G be abelian, eq. (3.2) still holds.

Remark 3.11. Following a similar proof, using [14, Theorem 4.3] instead of [14, Theorem 5.6], one finds that Corollary 3.9 also holds when \mathbf{A} is any W^* -dynamical system, \mathbf{B} is an identity system (i.e. $\beta_g = \text{id}_B$), and \mathbf{A}^K is replaced by the fixed point subsystem $\mathbf{A}^\alpha = (A^\alpha, \text{id}, \mu|_{A^\alpha})$ of \mathbf{A} , without G having to be abelian. Note that the fixed point algebra indeed gives a modular subsystem \mathbf{A}^α of \mathbf{A} [14, Proposition 4.2]. The special case eq. (3.2) now holds for ergodic \mathbf{A} , which means that $\mathbf{A}^\alpha = \mathbb{C}1$, and is then a standard characterization of ergodicity (see also Section 4). This version of Corollary 3.9 however also follows directly from the mean ergodic theorem and the representation of the conditional expectations in terms of Hilbert space projections, so does not require joining techniques for its proof.

We end this section with a discussion of examples to illustrate the W^* -dynamical results above. Simple examples of weakly mixing (in fact strongly mixing) and compact W^* -dynamical systems on the same algebra and state, for \mathbb{Z} -actions, are provided by group von Neumann algebras; see [13, Theorems 3.4 and 3.6]. We now firstly look at the relative case of this example as well, namely where \mathbf{A}^K is not trivial.

We use the same basic setting as in [13, Section 3], namely we consider an automorphism T of an arbitrary group Γ to which we assign the discrete topology. This leads to a *dual system* $\mathbf{A} = (A, \alpha, \mu)$ with $G = \mathbb{Z}$, where A is the group von Neumann algebra of Γ , μ is its canonical trace, and $\alpha(a) := UaU^*$ for all $a \in A$, with the unitary operator $U : H \rightarrow H$ defined by

$$Uf := f \circ T^{-1}$$

where $H := \ell^2(\Gamma)$. For any $g \in \Gamma$ we define $\delta_g \in H$ by $\delta_g(g) = 1$ and $\delta_g(h) = 0$ for $h \neq g$. Note that $U\delta_g = \delta_{Tg}$. Setting $\Omega := \delta_1$ where 1 here refers to the identity element of Γ , it is then easily seen that (H, id_A, Ω) is the cyclic representation of (A, μ) and that $U\Omega = \Omega$, so U is the unitary representation of α in the cyclic

representation. We can define the *finite orbit subsystem* \mathbf{F} of \mathbf{A} by defining F to be the von Neumann subalgebra of A generated by the subgroup of elements of Γ which have finite orbits under T . Then we have the following fact.

Proposition 3.12. *For a dual system \mathbf{A} as described above, the finite orbit subsystem is exactly \mathbf{A}^K .*

PROOF. Let \mathbf{F} denote the finite orbit subsystem of \mathbf{A} , then we simply have to prove that $F = A^K$. Since \mathbf{F} is compact [13, Theorem 3.5], it follows that $F \subset A^K$. So let's look at the converse. The proof uses a similar idea as in the proof of [13, Theorem 3.4].

Let l be the left regular representation of Γ on H , i.e. $[l(g)f](h) = f(g^{-1}h)$ for all $g, h \in \Gamma$ and $f \in H$. So A is generated by $\{l(g) : g \in \Gamma\}$ and F is generated by $\{l(g) : g \in E\}$, where $E := \{g \in \Gamma : T^{\mathbb{Z}}(g) \text{ is finite}\}$. Setting $H_F := \overline{F\Omega}$, we see that since $l(g)\Omega = \delta_g$, the space H_F^\perp is spanned by the orthonormal set of vectors $\{\delta_g : g \in G \setminus E\}$. Since the elements of $G \setminus E$ have infinite orbits under T , we have $\langle U^n \delta_g, \delta_h \rangle = 0$ for n large enough, for all $g, h \in G \setminus E$. It follows that

$$\lim_{n \rightarrow \infty} \langle U^n x, y \rangle = 0$$

for all $x, y \in H_F^\perp$. However, H_F is invariant under U , so it follows that

$$\lim_{n \rightarrow \infty} \langle U^n x, y \rangle = 0$$

for any $x \in H$ and $y \in H_F^\perp$.

Now, let H_0 be the subspace of H spanned by eigenvectors of U . Since $F \subset A^K$, it follows from [14, Theorem 5.4] that $H_F \subset \overline{A^K\Omega} = H_0$. Suppose there is an eigenvector v of U which is not in H_F , and let the corresponding eigenvalue be denoted by c . (Since U is unitary, $|c| = 1$.) Then there is a $y \in H_F^\perp$ such that $\langle v, y \rangle \neq 0$, hence the limit $\lim_{n \rightarrow \infty} \langle U^n v, y \rangle = (\lim_{n \rightarrow \infty} c^n) \langle v, y \rangle$ is not zero (and does not even necessarily exist), contradicting what we found above. It follows that such a v does not exist, therefore $H_F = H_0$.

Next we need to carry this result over to the algebras themselves. What we have shown is that $\overline{F\Omega} = \overline{A^K\Omega}$, but F is invariant under the modular group of (A^K, μ^K) , since $\mu^K = \mu|_{A^K}$ is tracial and its modular group therefore trivial. We have a resulting conditional expectation $D : A^K \rightarrow F$ which is implemented by the projection of H_0 onto H_F , which means that D is the identity mapping. So $F = A^K$. \square

Example 3.13. Consider in particular the case where Γ is the free group on any infinite set of symbols S . Partition S into two sets S_1 and S_2 , with at least

S_2 infinite. Define the automorphism $T : \Gamma \rightarrow \Gamma$ by extending any bijection $T : S \rightarrow S$ satisfying $T(S_1) = S_1$ and $T(S_2) = S_2$, and with the orbits of T on all elements of S_1 being finite, but infinite on all elements of S_2 . In this way we obtain a dual system \mathbf{A} as discussed above, and because of Proposition 3.12, \mathbf{A}^K is given by the von Neumann subalgebra of A generated by the subgroup of Γ generated by S_1 . In the same way we also consider another dual system \mathbf{B} with $(B, \nu) = (A, \mu)$, given by a bijection $E : S \rightarrow S$, again satisfying $E(S_1) = S_1$ and $E(S_2) = S_2$, and such that $E|_{S_1} = T|_{S_1}$ while all the orbits of E on S_2 are finite. Note that \mathbf{B} is compact [13, Theorem 3.5]. Furthermore, \mathbf{A}^K is a subsystem of the compact system \mathbf{B} , and it is necessarily a modular subsystem, since μ is tracial and its modular group therefore trivial. This example satisfies all the requirements of Corollary 3.9. In particular it provides examples where \mathbf{A}^K is not trivial, and not even an identity system, namely when S_1 has more than one element and $T|_{S_1}$ is not the identity map. Note that the dynamics of \mathbf{A}^K can in fact be fairly complicated. As an illustration, consider the countable set of symbols $S_1 = \{s_1, s_2, s_3, \dots\}$, and let $T|_{S_1}$ be given by the cycles of increasing length (s_1, s_2) , (s_3, s_4, s_5) , (s_6, s_7, s_8, s_9) , and so on. So, for example, $T(s_6) = s_7$ and $T(s_9) = s_6$. Then there are elements of $\overline{A^K\Omega}$ that do not have finite orbits under U , for example $\sum_{n=1}^\infty \delta_{s_n}/n$, so we don't have periodic dynamics.

Next we exhibit a pair of disjoint compact systems by considering $G = \mathbb{R}^2$ actions on quantum tori.

Example 3.14. Refer to Example 2.4, however consider $\theta_1, \theta_2 \in \mathbb{R}$. Let $A := M_{\theta_1}$ and $B := M_{\theta_2}$, and let μ and ν be their respective canonical traces. Furthermore, set

$$\alpha_{s,t} := \tau_{ps,qt} \quad \text{and} \quad \beta_{s,t} := \tau_{cs,dt}$$

for all $(s, t) \in \mathbb{R}^2$, where $p, q, c, d \in \mathbb{R} \setminus \{0\}$ are fixed. Then $\mathbf{A} = (A, \alpha, \mu)$ and $\mathbf{B} = (B, \beta, \nu)$ are W^* -dynamical systems.

Setting $\chi_{g,h}(s, t) := e^{2\pi i(gs+ht)}$ for all $(g, h), (s, t) \in \mathbb{R}^2$, one can use harmonic analysis on \mathbb{T}^2 to show that H is spanned by the eigenvectors of the unitary group U and of the unitary group V , namely

$$U_{s,t}\chi_{m,n} = \chi_{mp,nq}(s, t)\chi_{m,n}$$

and

$$V_{s,t}\chi_{m,n} = \chi_{mc,nd}(s, t)\chi_{m,n}$$

for all $(s, t) \in \mathbb{R}^2$ and $(m, n) \in \mathbb{Z}^2$. Therefore \mathbf{A} and $\tilde{\mathbf{B}}$ are compact W^* -dynamical systems [13, Definition 2.5 and Proposition 2.6], and their point spectra are

$$\sigma_{\mathbf{A}} = \{\chi_{mp,nq} : (m, n) \in \mathbb{Z}^2\}$$

and

$$\sigma_{\tilde{\mathbf{B}}} = \{\chi_{mc,nd} : (m, n) \in \mathbb{Z}^2\}$$

respectively. Since $p, q, c, d \neq 0$, one simultaneously sees that \mathbf{A} and $\tilde{\mathbf{B}}$ are also ergodic, since the fixed point spaces of U and V are one dimensional (they are spanned by the eigenvector $\chi_{0,0} = 1 = \Omega$). Now, if either p/c or q/d , or both, are irrational, we see that

$$\sigma_{\mathbf{A}} \cap \sigma_{\tilde{\mathbf{B}}} = \{1\}$$

and therefore by [13, Theorem 2.8] \mathbf{A} and $\tilde{\mathbf{B}}$ are disjoint. A C*-algebraic version of this disjointness (though requiring both p/c and q/d to be irrational) will be considered in Section 4.

When $\theta_1 = \theta_2$, so $(B, \nu) = (A, \mu)$, one can use harmonic analysis on \mathbb{T}^2 and the Lebesgue dominated convergence theorem, to show that for any *-automorphism $\eta : A \rightarrow A$ with $\mu \circ \eta = \mu$ the function

$$\mathbb{R}^2 \rightarrow \mathbb{C} : (s, t) \mapsto \mu(\eta(\alpha_{s,t}(a))\beta_{s,t}(b))$$

is continuous for all $a, b \in A$, and therefore Theorem 3.8 applies.

We are not aware of pairs of disjoint weakly mixing W*- (or state preserving C*-) dynamical systems on the same algebra and state.

4. UNIQUE ERGODICITY

We consider a simple connection between disjointness and unique ergodicity, and discuss examples of uniquely ergodic C*-dynamical systems. This is used to obtain a C*-algebraic (rather than W*-algebraic) example satisfying the assumptions of Corollary 3.4. We start with the definition of unique ergodicity (also see [4, Definition 4.5]).

Definition 4.1. A C*-dynamical system is *uniquely ergodic* if it has a unique invariant state.

Using an argument similar to the one in the proof of Theorem 3.3, one can show that an amenable C*-dynamical system has at least one invariant state by starting with an arbitrary state and considering its sequence of averages over (Λ_n) , and furthermore that an amenable C*-dynamical system is uniquely ergodic if and only if the limit

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \alpha_g(a) dg$$

exists in the norm topology and is a scalar multiple of the identity for every $a \in A$, where the integral used here is the Bochner integral (see for example [10, Chapter II] for an exposition).

From properties of the Bochner integral (see [10, Theorem II.2.6]) it also follows that unique ergodicity implies ergodicity in the sense that in the GNS representation obtained from the unique invariant state, the unitary representation of the dynamics has a one dimensional fixed point space, or equivalently

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \mu(\alpha_g(a)b)dg = \mu(a)\mu(b)$$

for all $a, b \in A$. See for example [9] for the relevant general background regarding ergodicity.

The converse is not true however. For example, consider the shift $*$ -automorphism on a countably infinite tensor product A of a C^* -algebra C with itself. Any state on C then gives a shift invariant state on A , leading to an ergodic (and in fact strongly mixing) state preserving dynamical system. Since any state on C will do, the shift is not uniquely ergodic though.

An example of unique ergodicity is provided by the quantum torus which illustrates that one should indeed consider unique ergodicity for group actions other than \mathbb{Z} :

Proposition 4.2. *The C^* -dynamical system (A_θ, τ) given by Example 2.4 is uniquely ergodic.*

PROOF. We know that the canonical trace Tr is invariant. Let μ be any invariant state of (A_θ, τ) . Let B be the $*$ -subalgebra of A_θ generated by $\{u, v\}$, so B consists of finite linear combinations of elements of the form $u^m v^n$, with $m, n \in \mathbb{Z}$. By (2.4) we have

$$\mu(u^m v^n) = \mu(\tau_{s,t}(u^m v^n)) = e^{2\pi i(ms+nt)}\mu(u^m v^n)$$

which means that $m = n = 0$ or $\mu(u^m v^n) = 0$. However Tr satisfies these conditions as well, so $\mu|_B = \text{Tr}|_B$ and since B is dense in A_θ , we have $\mu = \text{Tr}$. \square

This example is of course very simple in the sense that it is periodic in each of the two real parameters of its action; in effect \mathbb{R}^2 acts on \mathbb{T}^2 which in turn acts on A_θ . A slightly more complicated example is given by the following proposition.

Proposition 4.3. *Let $\theta_1, \theta_2 \in \mathbb{R}$ and in terms of Example 2.4 set $A := A_{\theta_1}$ and $B := A_{\theta_2}$. Furthermore, set $\alpha_{s,t} := \tau_{ps,qt}$ and $\beta_{s,t} := \tau_{cs,dt}$ for all $(s, t) \in \mathbb{R}^2$, where $p, q, c, d \in \mathbb{R} \setminus \{0\}$ are fixed. Also assume that p/c and q/d are both irrational. Then the C^* -dynamical system $(A \otimes_m B, \alpha \otimes_m \beta)$ is uniquely ergodic.*

PROOF. Let u and v be the generators of A as in Example 2.4, and let w and z correspondingly be the generators of B . Using the notation $\dot{u} = u \otimes 1$, $\dot{v} = v \otimes 1$, $\dot{w} = 1 \otimes w$ and $\dot{z} = 1 \otimes z$, we see that the C*-algebra $A \otimes_m B$ is generated by $\{\dot{u}, \dot{v}, \dot{w}, \dot{z}\}$. Therefore the *-algebra C consisting of all finite linear combinations of elements of the form $\dot{u}^j \dot{v}^k \dot{w}^l \dot{z}^m$, where $j, k, l, m \in \mathbb{Z}$, is dense in $A \otimes_m B$. If ω is any invariant state of $(A \otimes_m B, \alpha \otimes_m \beta)$, then we as in Proposition 4.2's proof

$$\omega(\dot{u}^j \dot{v}^k \dot{w}^l \dot{z}^m) = e^{2\pi i[(jp+lc)s+(kq+md)t]} \omega(\dot{u}^j \dot{v}^k \dot{w}^l \dot{z}^m)$$

for all $(s, t) \in \mathbb{R}^2$, therefore $\omega(\dot{u}^j \dot{v}^k \dot{w}^l \dot{z}^m) = 0$ or $jp + lc = kq + md = 0$. Since p/c and q/d are irrational, the latter implies $j = k = l = m = 0$, and in this case $\omega(\dot{u}^j \dot{v}^k \dot{w}^l \dot{z}^m) = \omega(1 \otimes 1) = 1 = \text{Tr}_{\otimes_m} \text{Tr}(\dot{u}^j \dot{v}^k \dot{w}^l \dot{z}^m)$. When at least one of j, k, l or m is not zero, $\text{Tr}_{\otimes_m} \text{Tr}(\dot{u}^j \dot{v}^k \dot{w}^l \dot{z}^m) = \text{Tr}(u^j v^k) \text{Tr}(w^l z^m) = 0$. Therefore $\omega = \text{Tr}_{\otimes_m} \text{Tr}$. \square

Note that if $ps, cs \in \mathbb{Z} \setminus \{0\}$ then p/c is rational, and similarly for q/d . So the product system in Proposition 4.3 is not periodic in either of the two real parameters; consider for example the orbit of $\dot{u} + \dot{w}$.

Remark 4.4. In general, for two C*-dynamical systems (A, μ) and (B, ν) with G a topological group and $g \mapsto \alpha_g(a)$ and $g \mapsto \beta_g(b)$ both continuous, it is fairly straightforward to show that $g \mapsto \alpha_g \otimes_m \beta_g(c)$ is continuous for all $c \in A \otimes_m B$. Hence, if furthermore (A, μ) and (B, ν) are also amenable, so is their product system. In particular the dynamics of the product system in Proposition 4.3 is continuous in the sense just described, and the product system is therefore amenable.

Clearly Proposition 4.3 is similar to the disjointness result in Example 3.14, although now we are in the C*-algebra context. Let us formalize this connection: Two uniquely ergodic C*-dynamical systems (A, α) and (B, β) are called *disjoint* if the corresponding state preserving C*-dynamical systems are disjoint. Clearly this is the case if and only if the product system $(A \otimes_m B, \alpha \otimes_m \beta)$ is uniquely ergodic, since any invariant state of this product system is a joining because it restricts to invariant states of (A, α) and (B, β) , which are unique. In particular we have the following corollary which tells us that examples of disjointness are not restricted to W*-dynamical systems:

Corollary 4.5. *The C*-dynamical systems (A, α) and (B, β) described in Proposition 4.3 are disjoint.*

In Section 3 we considered consequences of Theorem 3.3 for W^* -dynamical systems. We now consider an example satisfying all the requirements of Theorem 3.3 (specifically those of Corollary 3.4) in the C^* -algebraic context.

Example 4.6. Still consider the situation in Proposition 4.3, and let μ and ν be the canonical traces of A and B respectively. Let u, v and w, z be the generators of A and B respectively, as in the proof of Proposition 4.3. It is easily verified that $vz = zv$, so the C^* -subalgebras $C^*(v)$ and $C^*(z)$ of $B(H)$ generated by v and z respectively, are mutually commuting. We can therefore use a similar idea as in Section 3 to construct a coupling of (A, μ) and (B, ν) . Let

$$\delta : C^*(v) \odot C^*(z) \rightarrow B(H)$$

be the $*$ -homomorphism obtained by extending $\delta(a \otimes b) = ab$. We can construct a conditional expectation $D_1 : A \rightarrow C^*(v)$ such that $D_1(u^m v^n) = 0$ for nonzero $m \in \mathbb{Z}$, and $D_1(v^n) = v^n$ for all $n \in \mathbb{Z}$. Since $\mu(u^m v^n)$ is 1 when $m = n = 0$, and 0 otherwise, it follows that $\mu \circ D_1 = \mu$. Similarly we obtain $D_2 : B \rightarrow C^*(z)$ with $\nu \circ D_2 = \nu$. We then define a state κ on $A \otimes_m B$ by continuously extending the following:

$$\kappa(c) := \langle \Omega, \delta \circ (D_1 \odot D_2)(c) \Omega \rangle$$

for all $c \in A \odot B$. It is now easy to verify that κ is a coupling of (A, μ) and (B, ν) . It is indeed a nontrivial coupling, that is, $\kappa \neq \mu \otimes_m \nu$, since $\kappa(v \otimes z^{-1}) = 1$ while $\mu \otimes_m \nu(v \otimes z^{-1}) = 0$. We therefore have all the ingredients required in Theorem 3.3, or more specifically, Corollary 3.4. Note that as opposed to the W^* -dynamical applications considered in Section 3, this gives a case of Theorem 3.3 for two different algebras. Explicitly, Corollary 3.4 and Corollary 4.5 say that

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \kappa(\alpha_g(a) \otimes \beta_g(b)) dg = \mu(a)\nu(b)$$

for all $a \in A$ and $b \in B$, where (Λ_n) is any Følner sequence in $G = \mathbb{R}^2$.

In particular, when $\theta_1 = \theta_2$ (so $A = B$), and if we write $D = D_1 = D_2$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \mu(D(\alpha_g(a))D(\beta_g(b))) dg = \mu(a)\mu(b)$$

for all $a, b \in A$.

5. RELATIVE UNIQUE ERGODICITY

In this section we consider a relative version of unique ergodicity and its connections with disjointness. In particular we use these results to obtain an example

of (relative) disjointness in the C*-algebraic context, and use it to illustrate Theorem 3.3. To do this, we first formulate [1, Theorem 3.2] for actions of more general groups than \mathbb{Z} , followed by examples for actions of the group \mathbb{R} using quantum tori. Note that [1, Theorem 3.2] has already lead to further work, in particular [19] and [2], but only for actions of the group \mathbb{Z} .

Definition 5.1. We call the C*-dynamical system (A, α) *uniquely ergodic relative to its fixed point algebra* if every state on A^α has a unique extension to an invariant state of (A, α) .

From the remarks following Definition 4.1, the fixed point algebra of a uniquely ergodic amenable C*-dynamical system is $A^\alpha = \mathbb{C}1$.

The proof of the next result follows the basic plan of [1]'s proof, with fairly straightforward modifications and refinements. We therefore omit the proof here. Note that if we say that a conditional expectation $E : A \rightarrow A^\alpha$ is α -invariant, we mean that $E \circ \alpha_g = E$ for all $g \in G$, and similarly for linear functionals. Existence of limits, closures etc. are all in terms of the norm topology on A . The integrals are Bochner integrals.

Theorem 5.2. *Let (A, α) be an amenable C*-dynamical system with G unimodular, and let (Λ_n) be both a right and left Følner sequence. Then statements (i) to (v) below are equivalent.*

- (i) *The system (A, α) is uniquely ergodic relative to its fixed point algebra.*
- (ii) *The limit*

$$\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \alpha_g(a) dg$$

exists for every $a \in A$.

- (iii) *The subspace $A^\alpha + \overline{\text{span}\{a - \alpha_g(a) : g \in G, a \in A\}}$ is dense in A .*
- (iv) *The equality $A = A^\alpha + \overline{\text{span}\{a - \alpha_g(a) : g \in G, a \in A\}}$ holds.*
- (v) *Every bounded linear functional on A^α has a unique bounded α -invariant extension to A with the same norm.*

Furthermore, statements (i) to (v) imply the following statements:

- (vi) *There exists a unique α -invariant conditional expectation E from A onto A^α .*
- (vii) *The conditional expectation E in (vi) is given by*

$$E(a) = \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \alpha_g(a) dg$$

for all $a \in A$.

Remark 5.3. Note that if G is abelian, or if more generally we restrict ourselves to Følner sequences that are both right and left Følner sequences, then the value of the ergodic average in (ii) is independent of the Følner sequence because of (vi) and (vii).

The equivalence of conditions (i) and (ii) is what we are most interested in now, since it is useful in showing relative unique ergodicity in the examples below. These examples are simple variations on Propositions 4.2 and 4.3.

Proposition 5.4. *Consider the situation in Example 2.4, write $A = A_\theta$ and set $\alpha_s := \tau_{s,0}$ for all $s \in \mathbb{R}$. Then (A, α) is an example of a C^* -dynamical system for an action of \mathbb{R} which is uniquely ergodic relative to its fixed point algebra.*

PROOF. Take any Følner sequence (Λ_n) in \mathbb{R} , for example $\Lambda_n = [0, n]$. Let $a = u^j v^k$ for any $j, k \in \mathbb{Z}$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \alpha_s(a) ds &= \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} e^{2\pi i j s} a ds \\ &= a \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} e^{2\pi i j s} ds \\ &= \begin{cases} a & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} \alpha_s(a) ds$ exists for all a in the dense $*$ -subalgebra of A generated by $\{u, v\}$. Then it is easily verified that it exists for all $a \in A$. According to Theorem 5.2(i) and (ii) this means that (A, α) is indeed uniquely ergodic relative to its fixed point algebra. \square

Remark 5.5. With some more work one can show that the fixed point algebra of the C^* -dynamical system in Proposition 5.4 is the C^* -subalgebra of A generated by v , as one would expect. One way of doing this is to use the conditional expectation D_1 (for $\theta_1 = \theta$) mentioned in Example 4.6; this is exactly the conditional expectation also given by Theorem 5.2(vi) for the system in Proposition 5.4.

Proposition 5.6. *Consider the situation in Proposition 4.3 except that p/c and q/d need not be irrational. Write $\alpha_s \equiv \alpha_{s,0}$ and $\beta_s \equiv \beta_{s,0}$ for all $s \in \mathbb{R}$, to define actions α and β of \mathbb{R} on A and B respectively. Then $(A \otimes_m B, \alpha \otimes_m \beta)$ is an example of a C^* -dynamical system for an action of \mathbb{R} which is uniquely ergodic relative to its fixed point algebra.*

PROOF. Using the notation in the proof of Proposition 4.3 and setting $a = \dot{u}^j \dot{v}^k \dot{w}^l \dot{z}^m$ for any $j, k, l, m \in \mathbb{Z}$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} (\alpha \otimes_m \beta)_s(a) ds &= \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \int_{\Lambda_n} e^{2\pi i(jp+lc)s} a ds \\ &= \begin{cases} a & \text{if } jp + lc = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and proceeding as in Proposition 5.4's proof, the result follows. \square

As in Section 4, in the case where p/c is irrational, the C*-dynamical system $(A \otimes_m B, \alpha \otimes_m \beta)$ in Proposition 5.6 is not periodic.

Using higher dimensional quantum tori one should similarly be able to construct C*-dynamical systems for actions of \mathbb{R}^n which are uniquely ergodic relative to their fixed point algebras.

Finally we return to disjointness. In Section 3 we focussed on state preserving C*-dynamical systems, that is to say, we considered a specific invariant state. Now however we in principle allow many states, so let us consider an appropriate version of disjointness for relatively uniquely ergodic systems. In line with relative unique ergodicity, the focus here will be on fixed point algebras, rather than more general subsystems as in the W*-dynamical case in Section 3.

Note that if (A, α) and (B, β) are C*-dynamical systems which are uniquely ergodic relative to their respective fixed point algebras, then any state ψ on the algebraic tensor product $A^\alpha \odot B^\beta$ induces a unique pair of invariant states μ and ν of (A, α) and (B, β) respectively through the following process: $\psi|_{A^\alpha \odot 1}$ gives a state on A^α , which has a unique extension to an invariant state μ of (A, α) , and similarly for ν .

Definition 5.7. Let the C*-dynamical systems (A, α) and (B, β) be uniquely ergodic relative to their respective fixed point algebras. Let R be the closure of $A^\alpha \odot B^\beta$ in $A \otimes_m B$. We call (A, α) and (B, β) *disjoint relative to their fixed point algebras* if every state ψ on R has a unique extension to a joining of (A, α, μ) and (B, β, ν) , where μ and ν are the states induced by ψ on A and B respectively.

The disjointness defined in Definition 5.7 implies that for any state ψ on R , the induced state preserving C*-dynamical systems (A, α, μ) and (B, β, ν) are disjoint with respect to (R, id, ψ) as in Definition 3.1.

As in Section 4, the connection with relative unique ergodicity we want to exploit is quite simple: Let the C*-dynamical systems (A, α) and (B, β) be uniquely ergodic relative to their respective fixed point algebras. Since $A^\alpha \odot B^\beta$ is in the fixed point algebra of $(A \otimes_m B, \alpha \otimes_m \beta)$, so is R . If $(A \otimes_m B, \alpha \otimes_m \beta)$ is uniquely

ergodic relative to R (so in particular R is equal to the fixed point algebra of $(A \otimes_m B, \alpha \otimes_m \beta)$), then (A, α) and (B, β) are disjoint relative to their fixed point algebras. This is the case, since any state ψ on R has a unique extension to an invariant state of $(A \otimes_m B, \alpha \otimes_m \beta)$, which by the relative unique ergodicity of (A, α) and (B, β) is then necessarily the unique joining of (A, α, μ) and (B, β, ν) as in Definition 5.7. In general though, R will not be the fixed point algebra, so this is something we need to check.

Corollary 5.8. *Consider the situation in Proposition 5.6, and assume furthermore that p/c is irrational, and that θ_1 or θ_2 are irrational. Then (A, α) and (B, β) are disjoint relative to their fixed point algebras.*

PROOF. The irrationality of θ_1 or θ_2 ensures that either A or B are nuclear, so $A \otimes_m B = A \otimes B$ is the spatial tensor product, which allows us to use the spatial representation as will be seen below. The key technical point is to show that the fixed point algebra of $(A \otimes B, \alpha \otimes \beta)$, is indeed $A^\alpha \otimes B^\beta$. We again use the notation in the proof of Proposition 4.3, and $H = L^2(\mathbb{T}^2)$.

Denote the unitary group representing $\alpha \otimes \beta$ on $H \otimes H$ by V . So $V_s f = f \circ (T_{ps,0} \times T_{cs,0})$ in terms of Example 2.4. Using harmonic analysis on \mathbb{T}^2 and the fact that p/c is irrational, one can show that the fixed point space $(H \otimes H)^V$ of V is the Hilbert subspace of $H \otimes H$ spanned by $\{e_{0,k} \otimes e_{0,m} : k, m \in \mathbb{Z}\}$, where $e_{j,k}$ denote the functions in the total orthonormal set of H given by $e_{j,k}(s, t) := e^{2\pi i j s} e^{2\pi i k t}$.

Furthermore, taking the spatial tensor product of the conditional expectations D_1 and D_2 described in Example 4.6, we obtain the conditional expectation $E : A \otimes B \rightarrow C^*(v) \otimes C^*(z)$. One can then show that $[(1 - E)(A \otimes B)](\Omega \otimes \Omega)$ is orthogonal to $(H \otimes H)^V$. Since $\Omega \otimes \Omega$ is separating for $A \otimes B$ on $H \otimes H$, it follows that the only fixed point of $(1 - E)(A \otimes B)$ under $\alpha \otimes \beta$ is 0. For any $a \in (A \otimes B)^{\alpha \otimes \beta}$ we then have $a = (\alpha \otimes \beta)_g(a) = (\alpha \otimes \beta)_g(Ea + (1 - E)a) = Ea + (\alpha \otimes \beta)_g((1 - E)a)$, since $C^*(v) \otimes C^*(z) \subset (A \otimes B)^{\alpha \otimes \beta}$. So $(1 - E)a$ is a fixed point of $\alpha \otimes \beta$, which means that $(1 - E)a = 0$. Therefore $a = Ea \in C^*(v) \otimes C^*(z)$, so $(A \otimes B)^{\alpha \otimes \beta} = C^*(v) \otimes C^*(z)$.

As mentioned above, $C^*(v)$ and $C^*(z)$ are the respective fixed point algebras of (A, α) and (B, β) . By Proposition 5.6 we now know that (A, α) and (B, β) are disjoint relative to their fixed point algebras. \square

In order to illustrate Theorem 3.3 by an example satisfying all its assumptions, we however also need a nontrivial coupling. In the next example we consider a case of Corollary 5.8 where we are able to construct such a coupling.

Example 5.9. Again we use ideas from Section 3 and the notation in Example 2.4. Write $A = A_\theta$, but with θ irrational to ensure A is nuclear, and let μ be its canonical trace. From the modular conjugation J of (A'', μ) , we obtain an idempotent map j as in Section 3. Setting $\tilde{u} := j(u)$ and $\tilde{v} := j(v)$, we define \tilde{A} to be the C*-subalgebra of $B(H)$ generated by $\{\tilde{u}, \tilde{v}\}$, and we naturally carry μ to the state $\tilde{\mu}$ on \tilde{A} given by $\tilde{\mu} := \mu \circ j$, which turns out to be exactly $\tilde{\mu} = \langle \Omega, (\cdot)\Omega \rangle$. So A and \tilde{A} are mutually commuting, allowing us as in Section 3 to define a unital *-homomorphism $\delta : A \otimes \tilde{A} \rightarrow B(H)$ by extending $\delta(a \otimes b) := ab$, which gives the coupling κ of (A, μ) and $(\tilde{A}, \tilde{\mu})$ defined by

$$\kappa(a) := \langle \Omega, \delta(a)\Omega \rangle$$

for all $a \in A \otimes \tilde{A}$.

This abstract approach actually has a simple concrete meaning. It is easily verified that J is just complex conjugation on $H = L^2(\mathbb{T}^2)$, and that \tilde{u} and \tilde{v} are the generators of $A_{-\theta}$, so they are given by precisely the same formulas as u and v respectively in Example 2.4, but with θ replaced by $-\theta$.

So we simply apply Corollary 5.8 to the case $\theta_1 = \theta = -\theta_2$ to obtain disjoint (A, α) and (B, β) , assuming p/c is irrational. In terms of the notation in Corollary 5.8's proof, we can define a state ω on $A \otimes B$ by

$$\omega(a) := \langle \Omega, \delta \circ E(a)\Omega \rangle$$

for all $a \in A \otimes B$. Setting $R := C^*(v) \otimes C^*(z)$ and $\psi := \omega|_R$, we have $\kappa|_R = \psi$, and by the above mentioned disjointness, ω is easily checked to be the unique joining of (A, α, μ) and (B, β, ν) extending ψ , where μ and ν are the canonical traces of A and B as usual.

Keeping in mind Remark 4.4, we therefore satisfy all the assumptions in Theorem 3.3 in a nontrivial way, since $\kappa \neq \omega$, for example $\kappa(u \otimes \tilde{u}^{-1}) = 1$ while $\omega(u \otimes \tilde{u}^{-1}) = 0$.

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