Dynamics of Discrete-Time Sliding-Mode-Control Uncertain Systems With a Disturbance Compensator

Shaocheng Qu, *Member, IEEE*, Xiaohua Xia, *Fellow, IEEE*, and Jiangfeng Zhang

Abstract—In this paper, dynamical behaviors of the discrete-time sliding-mode-control (DSMC) uncertain systems are studied. First, a discrete reaching law with a disturbance compensator is presented, and a sliding-mode controller is designed by using the proposed reaching law. Second, the time steps for the system trajectories to converge to the switching manifold are found. Then, a quasi-sliding-mode domain (QSMD) of the DSMC uncertain systems is obtained, and the system dynamics of the DSMC systems in QSMD are described. Finally, numerical simulations are given to demonstrate the effectiveness of the proposed strategies.

Index Terms—Discrete-time uncertain systems, disturbance compensator (DSMC), dynamics, reaching law, sliding-mode control (SMC).

I. INTRODUCTION

AS a general design approach for robust control systems, sliding-mode-control (SMC) strategies have received wide attention, and the relevant results can be found in various publications [1]–[4], [22], [29]–[31]. The need for research in discrete-time SMC (DSMC) systems is evident. A primary reason is that most controllers nowadays are implemented in discrete time. Usually, there are two types of DSMC systems: one associated with the SMC of discrete-time systems [5]–[7], [9]–[15] and the other resulting from discretization of the SMC of continuous-time systems [16]–[25].

DSMC systems cannot be directly transformed from their continuous counterparts by means of simple equivalence [9]. Recently, some intrinsic dynamic properties within the discretized SMC systems have been found, such as periodic orbits [16]–[21] and delta modulation effects in the sliding direction [18]. The SMC of discrete-time systems is important, particularly when the implementation of the controller is realized by using computers with relatively low sampling frequencies [5]–[7], [9]–[15].

Some of the work on the DSMC systems focuses on the reaching conditions of the DSMC systems. There are normally two types of reaching conditions: the inequality conditions [5]–[7] and the equality conditions [8]–[12], [14]. Gao and Hung [8] propose a reaching law which is based on the equality type of the reaching conditions; then, an SMC controller is designed by using the reaching law. Gao *et al.* [9] further develop a discrete reaching law, which can steer the switching function to converge to the switching manifold in finite time and, then, to undergo a zigzag motion in the vicinity of the switching manifold. A notion of the quasi-sliding mode (QSM) in the vicinity of the switching manifold is developed in [9].

Usually, the system trajectories of the DSMC systems cannot reach the origin but only tend to reach a chattering surrounding the origin [9], [10]. There are some methods developed to alleviate the chattering. A reaching law is developed to overcome the chattering in [11]. A strategy combining the reaching law and the fast-output-sampling method with output feedback is developed in [12]. A two-scale reaching law with Euler velocity estimation is proposed in [14]. A no-switching type of DSMC is discussed, and an $o(T^2)$ boundary layer in the vicinity of the switching manifold is found in [15]; however, such a layer is inevitable for any sampled-data systems by nature of the zero-order-holder (ZOH) sampling method. By redefining a multivalued sign function, a chattering-free digital SMC with state observer and disturbance rejection is presented in [23]–[25]. There have been other efforts to reduce the chattering, such as state observers, disturbance compensators, and adaptive techniques [13], [14].

The objectives of this paper are to develop a discrete reaching law with a disturbance compensator and to describe the dynamics of the DSMC uncertain systems designed by using the proposed reaching law. First, a measure of the uncertain parameters and external disturbances is established by the deviations between the switching function and the reaching law, and a discrete reaching law with a disturbance compensator is presented. Second, the time steps for the system trajectories to converge to the switching manifold are found, and a QSM domain (QSMD) of the closed-loop SMC system is obtained. Third, the system dynamics of the closed-loop system in the vicinity of the switching manifold are described. Finally, simulation examples are given to verify the theoretical results.

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II. System Description

Consider a general description of an uncertain single-input system

$$\dot{x} = Ax + Bu + Df \tag{1}$$

where $x \in R^n$ is the state, $u \in R^1$ is the control, the generalized uncertainty $f \in R^l$ includes the variations of system parameters, control uncertainties, and external disturbances, and A, B, and D are constant matrices of appropriate dimensions. The following assumptions are needed throughout this paper.

Assumption I: The pair (A, B) is completely controllable.

Assumption II: The generalized uncertainty is smooth and bounded.

Assumption III: The generalized uncertainty satisfies the matching condition [1], i.e.,

$$\text{rank}[B, D] = \text{rank}[B]. \tag{2}$$

To alleviate chattering, a discrete-time approach which takes the sample-hold effects into account is necessary. By applying control through a ZOH sampling, i.e., $u(t) = u(k)$ for time $t \in [kT, (k+1)T]$, where T is the sampling period, the discrete-time representation of the system (1) with the ZOH is written as

$$x_{k+1} = \Phi x_k + \Gamma u_k + d_k \tag{3}$$

where $x_k = x(kT)$, $u_k = u(kT)$, $d_k = d(kT)$, $\Phi = e^{AT}$, $\Gamma = \int_0^T e^{A\tau} d\tau B$, and $d_k = \int_0^T e^{A\tau} Df((k+1)T - \tau) d\tau$.

It should be pointed out that d_k in (3) is obtained under the assumption that Df in (1) is considered constant over the sampling period, i.e., the matched condition after discretization only holds in the sense of approximation.

The switching function is defined as

$$s_k = Cx_k \tag{4}$$

where the vector $C \in R^{1 \times n}$ is to be designed such that the system will achieve the desired dynamics when traveling along the switching manifold

$$S = \{x_k | Cx_k = 0\}. \tag{5}$$

Similar to the case of continuous-time systems, once system (3) reaches the switching manifold (5) and is maintained on it, the order of the closed-loop system (3) will be reduced to $(n-1)$. The desired dynamics of the sliding mode, governed by this $(n-1)$ order system, can be designed by an appropriate choice of vector C. Furthermore, $C\Gamma \neq 0$ is assumed.

The control objective is to steer the system trajectories to converge toward the switching manifold S defined in (5) and also to travel to the origin along the switching manifold S.

Now, let us specify how the QSM and the reaching condition are understood in this paper.

Definition 1: System (3) is said to be in a QSM in the Δ vicinity of the switching manifold (5) if a motion of the system (3) satisfies $\{x_k | |Cx_k| \leq \Delta\}$ for all $k \geq k^*$ (k^* is a constant integer). The specified space domain where the QSM occurs is called the QSMD, and the positive constant Δ is called the width of the QSMD.

Definition 2: System (3) is said to satisfy the reaching condition of the QSM in the Δ vicinity of the switching manifold (5) if the following conditions hold: When $s_k > \Delta$, then $-\Delta \leq s_{k+1} < s_k$; when $s_k < -\Delta$, then $s_k < s_{k+1} \leq \Delta$; and when $|s_k| \leq \Delta$, then $|s_{k+1}| \leq \Delta$.

The following result is needed in the proof of Lemma 2.

Lemma 1 [18], [28]: For a 1-D delta-modulated feedback system in the form of a discrete-time dynamics, $x^+ = f(x) \triangleq ax - \Delta\text{sgn}(ax)$, where x^+ denotes the system state at the next discrete-time step, a is a real number, and $\text{sgn}(x)$ is defined as

$$\text{sgn}(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0. \end{cases}$$

The following results are valid.

1) If $|a| = 1$, then $\Omega = [-\Delta, \Delta]$ is a global attractor on $(-\infty, \infty)$.
2) If $|a| < 1$, then the global attractor Ω is a set of two points: $\{-\Delta/(1 + |a|), \Delta/(1 + |a|)\}$.
3) If $0 \leq a < 1$, then two points $\{-\Delta/(1 + |a|), \Delta/(1 + |a|)\}$ are two-periodic points; if $-1 < a < 0$, then two points $\{-\Delta/(1 + |a|), \Delta/(1 + |a|)\}$ are (one-periodic) fixed points.

III. Design of the DSMC Systems With the Proposed Reaching Law

The design of the DSMC systems includes the following two steps.

1) Determine a sliding-mode controller u_k and a switching function (4) such that the closed-loop system (3) satisfies the reaching condition in Definition 2. It shows that the closed-loop system will be forced to the QSMD defined in Definition 1. Furthermore, once entering into the QSMD, the system trajectories of the closed-loop system will stay within QSMD and never escape it.
2) Further design a switching function (4) such that the closed-loop system (3) restricted to the switching manifold (5) is asymptotically stable.

Gao *et al.* developed a discrete reaching law in [9]. On the one hand, a sliding-mode controller can be synthesized by this reaching law. On the other hand, it not only can guarantee that the closed-loop system satisfies the reaching condition but also can be directly used to dictate the dynamics of the switching function of the closed-loop SMC system.

A. New Reaching Law With a Disturbance Compensator

The following new reaching law with a disturbance compensator is proposed:

$$s_{k+1} = (1 - qT)s_k - \varepsilon T\text{sgn}(s_k) + Cd_k$$
$$- \sum_{i=2}^{k} \{s_i - [(1 - qT)s_{i-1} - \varepsilon T\text{sgn}(s_{i-1})]\} \tag{6}$$

where T is the sampling period, $\varepsilon > 0$ is a switching parameter, $q > 0$ is a converging parameter, and $0 < 1 - qT < 1$ is required. It is assumed that the change rate of the generalized uncertainty d_k is bounded as follows:

$$\delta_k \triangleq C(d_k - d_{k-1}) \tag{7}$$

$$|\delta_k| \leq \delta \leq \varepsilon T, \qquad k = 0, 1, 2, \ldots. \tag{8}$$

It implies that the uncertainties are assumed to be slowly time varying.

It follows from (6) that

$$s_k = (1 - qT)s_{k-1} - \varepsilon T \mathrm{sgn}(s_{k-1}) + Cd_{k-1}$$
$$- \sum_{i=2}^{k-1} \{s_i - [(1 - qT)s_{i-1} - \varepsilon T \mathrm{sgn}(s_{i-1})]\} \tag{9}$$

$$\sum_{i=2}^{k} \{s_i - [(1 - qT)s_{i-1} - \varepsilon T \mathrm{sgn}(s_{i-1})]$$
$$= s_k - [(1 - qT)s_{k-1} - \varepsilon T \mathrm{sgn}(s_{k-1})]$$
$$+ \sum_{i=2}^{k-1} \{s_i - [(1 - qT)s_{i-1} - \varepsilon T \mathrm{sgn}(s_{i-1})]\}. \tag{10}$$

Substituting (9) into (10) yields

$$\sum_{i=2}^{k} \{s_i - [(1 - qT)s_{i-1} - \varepsilon T \mathrm{sgn}(s_{i-1})] = Cd_{k-1}. \tag{11}$$

Equation (11) shows that the accumulated deviations between the switching function s_i and $(1 - qT)s_i - \varepsilon T \mathrm{sgn}(s_i)$ reveal the generalized uncertainty at the $k - 1$ steps. Su *et al.* [15] and Morgan and Özgüner [26] predict and replace d_{k-1} by $x_k - \Phi x_{k-1} - \Gamma u_{k-1}$ in (11), and it is successfully implemented for robotics manipulators in [27].

Substituting (11) into (6), we obtain an explicit expression of the proposed reaching law (6)

$$s_{k+1} = (1 - qT)s_k - \varepsilon T \mathrm{sgn}(s_k) + C(d_k - d_{k-1}) \tag{12}$$

which will be discussed later.

B. Design of a Sliding-Mode Controller

In the following, a sliding-mode controller for DSMC uncertain systems is designed by using the proposed reaching law (6).

Substituting the system dynamics (3) into the switching function (4) yields

$$s_{k+1} = Cx_{k+1}$$
$$= C\Phi x_k + C\Gamma u_k + Cd_k. \tag{13}$$

Comparing (13) to the improved reaching law (6) gives

$$s_{k+1} = (1 - qT)s_k - \varepsilon T \mathrm{sgn}(s_k) + Cd_k$$
$$- \sum_{i=2}^{k} [s_i - (1 - qT)s_{i-1} + \varepsilon T \mathrm{sgn}(s_{i-1})]$$
$$= C\Phi x_k + C\Gamma u_k + Cd_k.$$

Note that $C\Gamma \neq 0$; the controller u_k can be solved from the previous equation

$$u_k = -(C\Gamma)^{-1} \Bigg\{ C\Phi x_k - (1 - qT)s_k + \varepsilon T \mathrm{sgn}(s_k)$$
$$+ \sum_{i=2}^{k} [s_i - (1 - qT)s_{i-1} + \varepsilon T \mathrm{sgn}(s_{i-1})] \Bigg\} \tag{14}$$

which is the sliding-mode controller designed by using the proposed reaching law (6). Note that the controller (14) does not depend on the uncertainties and therefore can be directly implemented.

Under the assumption

$$|C(d_k - d_{k-1})| = 0 \tag{15}$$

the proposed reaching law (6) is simplified to the following reaching law [9, eq. (11)]:

$$s_{k+1} = (1 - qT)s_k - \varepsilon T \mathrm{sgn}(s_k). \tag{16}$$

Moreover, the sliding-mode controller (14) is reduced to the following sliding-mode controller [9, eq. (20)]:

$$u_k = -(C\Gamma)^{-1} [C\Phi x_k - (1 - qT)s_k + \varepsilon T \mathrm{sgn}(s_k)]. \tag{17}$$

Remark 1: Comparing (6) and (16), it is noted that only iterative terms are added in the proposed reaching law (6). Furthermore, the iterative terms only consist of the historical information. It shows that all attributes of the reaching law in [9] are preserved. Compared to (17), only accumulated deviations are added in controller (14). This is very convenient when the proposed controller (14) is implemented by microprocessors.

Remark 2: Some other novel techniques on the reaching law can be incorporated into the proposed reaching law (6). For example, the two-scale reaching law in [14] can be applied to the proposed reaching law (6) by adding an adjusting parameter to εT in (6) (see [14, eq. (22)]). Nevertheless, it should be pointed out that the discrete dynamics (12) and (16) cannot reach the origin in view of the traditional definition of the sgn function in Lemma 1.

IV. SYSTEM DYNAMICS OF THE DSMC SYSTEM WITH THE PROPOSED REACHING LAW

The DSMC systems designed by using the reaching law are not asymptotically stable, i.e., the system trajectories do not converge to zero asymptotically due to the switching with a finite frequency and system uncertainties. It will be shown hereinafter that, under certain conditions, system (3) with the switching manifold (5) and controller (14) is stable and the system trajectories are bounded.

If the generalized uncertainty d_k satisfies (15), i.e., d_k is zero or constant, the following Lemma 2 can be obtained.

Lemma 2: Consider the system (3) with the assumption (15); if the sliding-mode controller (14) is implemented, then the

system trajectories of the closed-loop system (3) from any initial state will converge to the QSMD defined by

$$s_{\Delta 1} = \left\{ x_k \,\middle|\, |Cx_k| \leq \frac{\varepsilon T}{2 - qT} \right\} \qquad (18)$$

and the following two points at the two ends of the QSMD

$$\left\{ \frac{-\varepsilon T}{2 - qT}, \frac{\varepsilon T}{2 - qT} \right\} \qquad (19)$$

are a global two-periodic attractor.

Proof: If the controller (14) is implemented for the system (3) with assumption (15), the dynamics of the switching function (4) of the closed-loop system (3) will be completely described by the reaching law (16). According to the aforementioned Lemma 1(b) and Lemma 1(c), and noting the assumption $0 < 1 - qT < 1$, the proof is straightforward.

Remark 3: If the sampling period T is sufficiently small, then the QSMD (18) will be very close to the switching manifold (5). Moreover, the system trajectories in steady state will approach the origin. It should be indicated that the same conclusion (18) is found with a different method by Bartoszewicz in [10].

In the following, we divide the discussions on the dynamics of the DSMC uncertain system into two parts: One is how many steps that the system trajectories need to converge to the QSMD, and the other is system dynamics in the vicinity of the switching manifold.

A. Convergence of System Trajectories Toward QSMD

Theorem 1: Consider system (3) with assumptions (7) and (8); if the controller (14) is implemented, then the trajectories of the switching function (4) from any initial state will first cross the switching manifold (5) within $k^* + 1$ steps at most, where $k^* = \lfloor m_{1st}^* \rfloor$

$$m_{1st}^* = \log_{(1-qT)} \left(\frac{\varepsilon T - \delta}{\varepsilon T - \delta + qT|s_0|} \right). \qquad (20)$$

s_0 is an initial point of the switching function, and $\lfloor m_{1st}^* \rfloor$ denotes the floor operation, i.e., the maximal integer bounded below the real number m_{1st}^*.

Proof: When the controller (14) is implemented for the system (3) with assumptions (7) and (8), the dynamics of the switching function of the closed-loop system will be described by (12). From (12) and (7), it yields that

$$s_{k+1} = (1 - qT)s_k - \varepsilon T \operatorname{sgn}(s_k) + \delta_k. \qquad (21)$$

In the following, we prove the theorem by contradiction according to (21).

Assume that there is an initial state, resulting in the initial point s_0 of the switching function (4), such that $s_0, s_1, \ldots, s_{k^*+1}$ do not change signs. There are two cases: $s_0 \geq 0$ and $s_0 < 0$.

Case I: Assume that $s_0 \geq 0$, and the system trajectories from the initial state do not cross the switching manifold in

$k^* + 1$ number of steps, i.e., s_m are nonnegative for all $m \leq k^* + 1$. It follows from (21) and (8) that

$$s_1 = (1 - qT)s_0 - (\varepsilon T - \delta_0)$$

$$s_2 = (1 - qT)s_1 - \varepsilon T + \delta_1$$
$$= (1 - qT)^2 s_0 - (1 - qT)(\varepsilon T - \delta_0) - (\varepsilon T - \delta_1)$$

$$\vdots$$

$$s_m = (1 - qT)s_{m-1} - \varepsilon T + \delta_{m-1}$$
$$= (1 - qT)^m s_0 - (1 - qT)^{m-1}(\varepsilon T - \delta_0) - \cdots$$
$$\quad - (1 - qT)(\varepsilon T - \delta_{m-2}) - (1 - qT)^0 (\varepsilon T - \delta_{m-1})$$
$$= (1 - qT)^m s_0 - \sum_{i=0}^{m-1} (1 - qT)^{m-1-i}(\varepsilon T - \delta_i)$$
$$\leq (1 - qT)^m s_0 - \sum_{i=0}^{m-1} (1 - qT)^{m-1-i}(\varepsilon T - \delta)$$
$$= (1 - qT)^m |s_0| - (\varepsilon T - \delta)\frac{1 - (1 - qT)^m}{qT}.$$

Noting assumption (8) and that $0 < 1 - qT < 1$, it is straightforward to verify that the real number m_{1st}^* in (20) satisfies

$$(1 - qT)^{m_{1st}^*} |s_0| - (\varepsilon T - \delta)c\frac{1 - (1 - qT)^{m_{1st}^*}}{qT} = 0.$$

It follows from $k^* = \lfloor m_{1st}^* \rfloor$ that

$$s_{k^*+1} \leq (1 - qT)^{k^*+1}|s_0| - (\varepsilon T - \delta)\frac{1 - (1 - qT)^{k^*+1}}{qT}$$
$$< (1 - qT)^{m_{1st}^*}|s_0| - (\varepsilon T - \delta)\frac{1 - (1 - qT)^{m_{1st}^*}}{qT}$$
$$= 0$$

which contradicts the assumption that $s_m \geq 0$ is nonnegative for all $m \leq k^* + 1$.

Case II: Assume that $s_0 < 0$, and s_m is negative for all $m \leq k^* + 1$; similarly, it follows that

$$s_m = (1 - qT)^m s_0 + \sum_{i=0}^{m-1} (1 - qT)^{m-1-i}(\varepsilon T + \delta_i)$$
$$\geq (1 - qT)^m s_0 + \sum_{i=0}^{m-1} (1 - qT)^{m-1-i}(\varepsilon T - \delta)$$
$$= - \left[(1 - qT)^m |s_0| - (\varepsilon T - \delta)\frac{1 - (1 - qT)^m}{qT} \right].$$

It can also be verified that the real number m_{1st}^* in (20) satisfies

$$(1 - qT)^{m_{1st}^*} |s_0| - (\varepsilon T - \delta)\frac{1 - (1 - qT)^{m_{1st}^*}}{qT} = 0.$$

It follows from $k^* = \lfloor m^*_{1\text{st}} \rfloor$ that

$$s_{k^*+1} \geq -(1-qT)^{k^*+1}|s_0| + (\varepsilon T - \delta)\frac{1-(1-qT)^{k^*+1}}{qT}$$

$$> -(1-qT)^{m^*_{1\text{st}}}|s_0| + (\varepsilon T - \delta)\frac{1-(1-qT)^{m^*_{1\text{st}}}}{qT}$$

$$= 0$$

which is again a contradiction to the assumption that s_m is all negative for all $m \leq k^* + 1$.

The aforementioned two cases show that $k^* = \lfloor m^*_{1\text{st}} \rfloor$ is the smallest integer such that the system trajectories are guaranteed to first cross the switching manifold (5) within $k^* + 1$ steps. The proof of Theorem 1 is completed.

If $\delta = 0$, (20) is reduced to

$$m^{**}_{1\text{st}} = \log_{(1-qT)}\left(\frac{\varepsilon}{\varepsilon + q|s_0|}\right) \qquad (22)$$

which shows that the system trajectories in Lemma 2 will first cross the switching manifold (5) in $\lfloor m^{**}_{1\text{st}} \rfloor + 1$ number of steps and eventually enter into the global two-periodic orbit (19) within the QSMD defined by (18).

B. System Dynamics in the Vicinity of the Switching Manifold

According to Theorem 1, the system trajectories from any initial state will cross the switching manifold in a finite number of steps. The following results characterize the system dynamics of the closed-loop system (3) in the vicinity of the switching manifold.

Theorem 2: The following results hold for the system (3) with assumptions (8) and controller (14).

1) The system trajectories from any initial state will enter into a QSMD defined by

$$s_{\Delta 2} = \{x_k | |Cx_k| \leq \varepsilon T + \delta\} \qquad (23)$$

   i.e., the QSMD is globally attractive for the switching function of the closed-loop system.
2) Once the system trajectories enter into the QSMD, they cannot escape it.
3) Any system trajectories in QSMD will cross the switching manifold repeatedly within each $k^{**} + 1$ new steps, where $k^{**} = \lfloor m^*_2 \rfloor$ and

$$m^*_2 = \log_{(1-qT)}\left(\frac{\varepsilon T - \delta}{\varepsilon T - \delta + qT(\varepsilon T + \delta)}\right). \qquad (24)$$

Proof: In a similar way as in the proof of Theorem 1, we can prove that the dynamics of the switching function of the closed-loop system (3) are described by (12).

1) If $s_k > 0$, it follows from (8) that

$$s_{k+1} = (1 - qT)s_k - \varepsilon T + \delta_k$$
$$< s_k - \varepsilon T + \delta_k < s_k$$

   namely, the sequence $\{s_k\}$ is decreasing as long as $s_k > 0$. Similarly, it can be proved that the sequence $\{s_k\}$

is increasing as long as $s_k < 0$. According to Theorem 1, the sequence $\{s_k\}$ will cross the switching manifold in a finite number of steps. Assuming that the sequence $\{s_k\}$ does not enter into the QSMD, then there are only two situations, i.e., there exists a positive integer l such that
Case I: $s_0 > s_1 > \ldots > s_l > 0$, but $s_{l+1} < -(\varepsilon T + \delta)$
   or
Case II: $s_0 < s_1 < \ldots < s_l < 0$, but $s_{l+1} > (\varepsilon T + \delta)$.

In Case I, it follows that

$$s_{l+1} = (1 - qT)s_l - \varepsilon T + \delta_l$$

but

$$s_{l+1} < -\varepsilon T - \delta.$$

This is impossible, since $s_l > 0$ and $|\delta_l| \leq \delta \leq \varepsilon T$. Similarly, Case II is also impossible. It follows that the sequence $\{s_k\}$ will enter into the QSMD, i.e., the system trajectories from any initial state will enter into the QSMD defined by (23). The proof of Theorem 2(a) is completed.

2) By Theorem 2(a), the system trajectories from any initial state will enter into a QSMD. When $0 \leq s_k \leq \varepsilon T + \delta$, then

$$s_{k+1} = (1 - qT)s_k - \varepsilon T + \delta_k \geq -\varepsilon T - \delta$$

also

$$s_{k+1} = (1 - qT)s_k - \varepsilon T + \delta_k$$
$$\leq \varepsilon T + \delta - \varepsilon T + \delta_k \leq \varepsilon T + \delta.$$

When $-(\varepsilon T + \delta) \leq s_k < 0$, then

$$s_{k+1} = (1 - qT)s_k + \varepsilon T + \delta_k \leq \varepsilon T + \delta$$

also

$$s_{k+1} = (1 - qT)s_k + \varepsilon T + \delta_k$$
$$\geq -(\varepsilon T + \delta) + \varepsilon T + \delta_k \geq -(\varepsilon T + \delta).$$

It can be concluded from the aforementioned two cases that, once the system trajectories enter into the QSMD, they never escape it.

3) According to Theorem 2(a) and Theorem 2(b), the system trajectories will enter into the QSMD defined in (23) and never escape it. Assuming the worst case when $s_0 = \pm(\varepsilon T + \delta)$ in (20), it is then straightforward to obtain the conclusion (24).

This completes the proof of Theorem 2.

In the following, we consider a special case: once the system trajectories have crossed the switching manifold, they will also cross the switching manifold in each successive step. This motion can adequately exhibit robust control of the DSMC systems [9], [10]. It is interesting to further characterize this behavior within the QSMD.

The aforementioned description can be characterized as follows:

$$\text{sgn}(s_{k+2}) = -\text{sgn}(s_{k+1}) = \text{sgn}(s_k). \tag{25}$$

It follows from (12) that

$$
\begin{aligned}
s_{k+2} &= (1-qT)s_{k+1} - \varepsilon T\text{sgn}(s_{k+1}) + C(d_{k+1} - d_k) \\
&= (1-qT)\left[(1-qT)s_k - \varepsilon T\text{sgn}(s_k) + C(d_k - d_{k-1})\right] \\
&\quad - \varepsilon T\text{sgn}(s_{k+1}) + C(d_{k+1} - d_k) \\
&= (1-qT)^2 s_k + qT\varepsilon T\text{sgn}(s_k) \\
&\quad + (1-qT)C(d_k - d_{k-1}) + C(d_{k+1} - d_k).
\end{aligned} \tag{26}
$$

It can be concluded from (26) that the s_{k+2} and s_k will have the same signs if the following condition holds:

$$|(1-qT)C(d_k - d_{k-1}) + C(d_{k+1} - d_k)| \le qT\varepsilon T. \tag{27}$$

Solving (27) yields

$$\delta \le \frac{qT\varepsilon T}{2 - qT}. \tag{28}$$

Remark 4: Assume that the uncertainty boundary δ in (8) satisfies (28), once s_k has crossed the switching manifold (5), then it also will cross the switching manifold (5) in all successive steps within the QSMD defined by (23). If the uncertainty boundary does not satisfy (28), it cannot maintain to cross the switching manifold in all successive steps, but the system trajectories still stay within QSMD defined by (23).

Finally, let us consider the system states of the closed-loop system.

Theorem 3: If the vector C in (4) is chosen such that $\|M\| < 1$, where $M = \Phi - \Gamma(C\Gamma)^{-1}C\Phi$, and the uncertainty d_k is bounded as $\|d_k\| \le \zeta$, then the system states of the SMC system (3) with control (14) are bounded by

$$\|x_\infty\| \le \Psi(1 - \|M\|)^{-1} \tag{29}$$

where $\Psi = \|\Gamma(C\Gamma)^{-1}\|[(1-qT)(\delta + \varepsilon T) + \varepsilon T] + \zeta(1 + \|\Gamma(C\Gamma)^{-1}C\|)$.

Proof: Substituting control (14) into the system (3) yields

$$x_{k+1} = Mx_k + W_k + V_k \tag{30}$$

where

$$
\begin{aligned}
W_k &= \Gamma(C\Gamma)^{-1}\left[(1-qT)s_k - \varepsilon T\text{sgn}(s_k)\right] \\
V_k &= d_k - \Gamma(C\Gamma)^{-1}Cd_{k-1}.
\end{aligned}
$$

Since d_k is bounded by $\|d_k\| \le \zeta$, then

$$\|V_k\| \le \zeta\left(1 + \|\Gamma(C\Gamma)^{-1}C\|\right). \tag{31}$$

Furthermore, once the system trajectories have entered into the QSMD, then

$$
\begin{aligned}
\|W_k\| &\le \|\Gamma(C\Gamma)^{-1}\|[(1-qT)\|s_k\| + \varepsilon T] \\
&\le \|\Gamma(C\Gamma)^{-1}\|[(1-qT)(\delta + \varepsilon T) + \varepsilon T].
\end{aligned} \tag{32}
$$

Noting definition Ψ, it follows from (30) that

$$
\begin{aligned}
\|x_{k+1}\| &= \|Mx_k + W_k + V_k\| \\
&\le \|M\|\|x_k\| + \|W_k\| + \|V_k\| \\
&\le \|M\|\|x_k\| + \Psi
\end{aligned}
$$

iterating it for n times from the kth step, we have

$$\|x_{k+n}\| \le \|M\|^n\|x_k\| + \Psi\sum_{i=0}^{n-1}\|M\|^{n-1-i}. \tag{33}$$

Therefore, if $\|M\| < 1$, it follows from (31)–(33) that the state trajectories of the SMC system in steady state will be bounded by (29). This completes the proof of Theorem 3.

V. Numerical Examples

Consider the second-order system with time-varying uncertainties in the form of (1), where

$$
A = \begin{bmatrix} 0 & 1 \\ 5 & -2 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

$$
Df = \begin{bmatrix} 0 \\ a\,(1 + 2.2\cos(0.5\pi t)) \end{bmatrix}
$$

and a is an adjustable parameter. The initial state is assumed as $x(0) = [2.1\ 1]^T$.

The sampling period is chosen as $T = 0.1$. Then, the discrete-time representation of the system with ZOH sampling is obtained as

$$x_{k+1} = \Phi x_k + \Gamma u_k + d_k \tag{34}$$

where

$$
\Phi = \begin{bmatrix} 1.02351 & 0.09139 \\ 0.45696 & 0.84073 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} 0.00470 \\ 0.09139 \end{bmatrix}
$$

$$
d_k = a \times \begin{bmatrix} 0.10080 & 0.00470 \\ 0.02351 & 0.09139 \end{bmatrix} \begin{bmatrix} 0 \\ 1 + 2.2\cos(0.5\pi KT) \end{bmatrix}
$$

and $C = [1\ 1]$ in (4).

In the following, we discuss the system dynamics in three cases.

Case I: Consider the system (34) with an adjustable parameter $a = 0$ (i.e., $d_k = 0$); when the control (14) with parameters $q = 5$ and $\varepsilon = 3$ is implemented, then the switching function of the closed-loop SMC system is shown in Fig. 1. Here, the big switching parameter $\varepsilon = 3$ is chosen so as to show the dynamics depicted by Lemma 2. As stated by Lemma 2, the system trajectories starting from the initial state enter into the QSMD $s_{\Delta 1} = \{x_k | |Cx_k| \le 0.2\}$ as defined in (18), and the two points $\{-0.2, 0.2\}$ at the two ends of the QSMD are a global two-periodic attractor. Moreover, it can be observed from Fig. 1 that the system trajectories first cross the switching manifold in $k^{**} + 1$ number of steps, where $k^{**} = 2$ is the maximal integer bounded below the real number 2.7370.
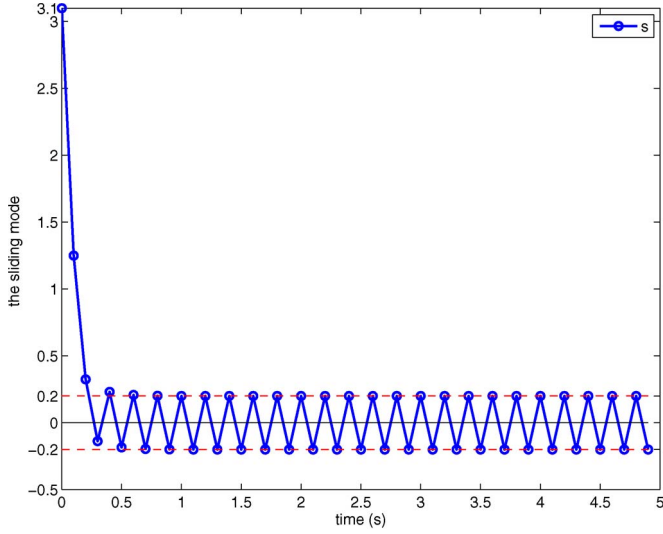
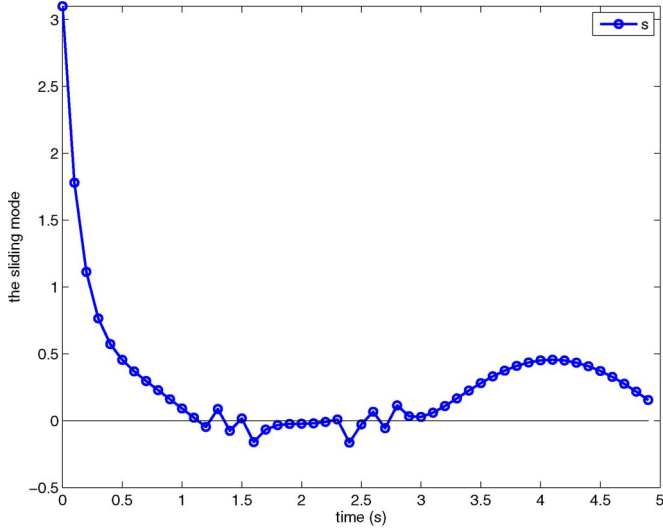Fig. 1.   Switching function with the reaching law in case I.

Fig. 2.   Switching function with the reaching law in case II.
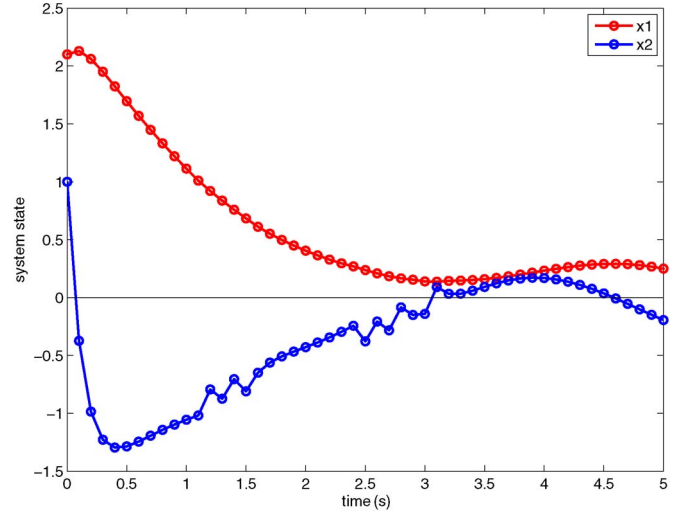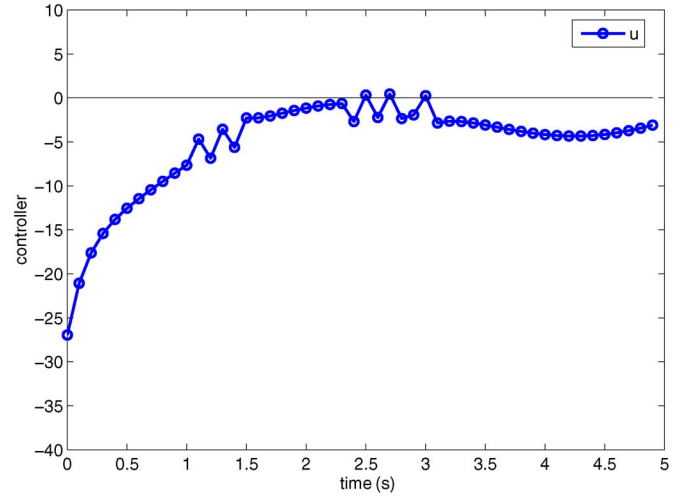

Fig. 3.   System states with the reaching law in case II.

Fig. 4.   Controller with the reaching law in case II.

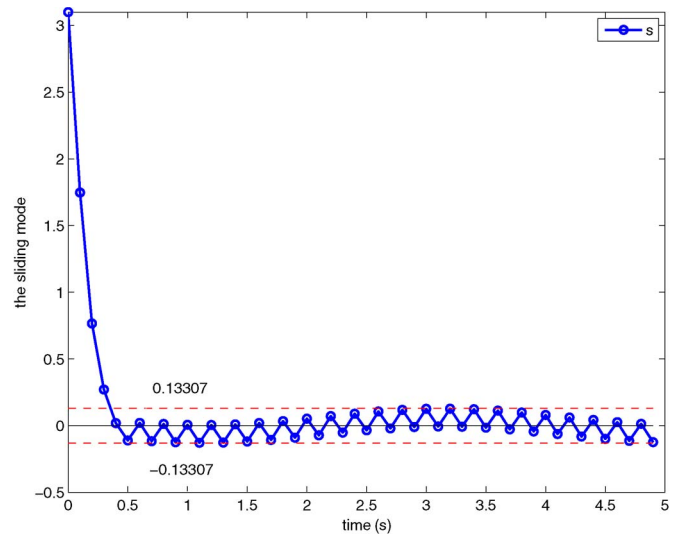Fig. 5.   Switching function with the improved reaching law in case II.

Case II: When the control (17) with parameters $q = 5$ and $\varepsilon = 1$ is implemented for the system (34) with an adjustable parameter $a = 1$, then the switching function, system states, and controller are shown in Figs. 2–4, respectively. When the proposed SMC (14) with the same parameters is implemented, then the switching function, system states, and controller are shown in Figs. 5–7, respectively. By comparison, we can observe that the dynamics of the closed-loop system designed by using the proposed reaching law are effectively improved in the presence of uncertainties.

The uncertainty boundary $\delta = 0.033070$ in (8) satisfies the condition (28), i.e., $\delta \leq (qT\varepsilon T/2 - qT) = 0.033333$; Fig. 5 confirms Remark 4 that, once s_k has crossed the switching manifold, it maintains to cross the switching manifold in all subsequent steps. The QSMD is defined as $s_{\Delta 2} = \{x_k\|Cx_k| \leq \delta + \varepsilon T = 0.13307\}$ by (23) in Theorem 2.

It should be indicated that the long sampling period $T = 0.1$ is chosen in order to clearly exhibit the system

dynamics of the closed-loop system. If the sampling period $T = 0.01$ and the same control parameters $q = 5$ and $\varepsilon = 1$ are implemented, then the magnitude of the controller will be alleviated as Fig. 8.
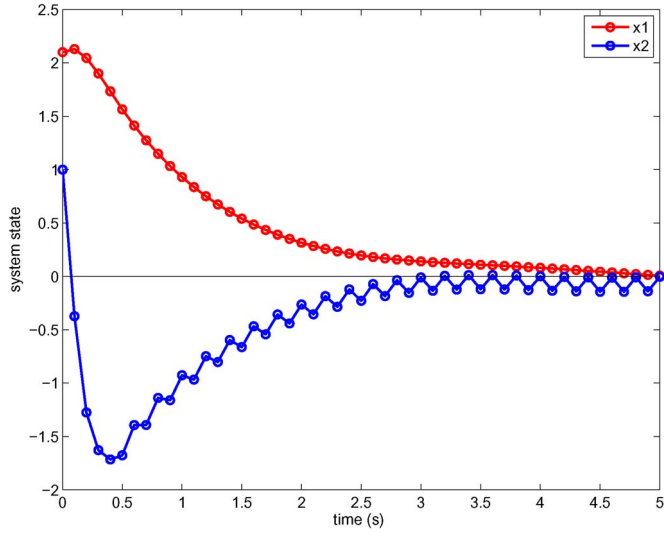
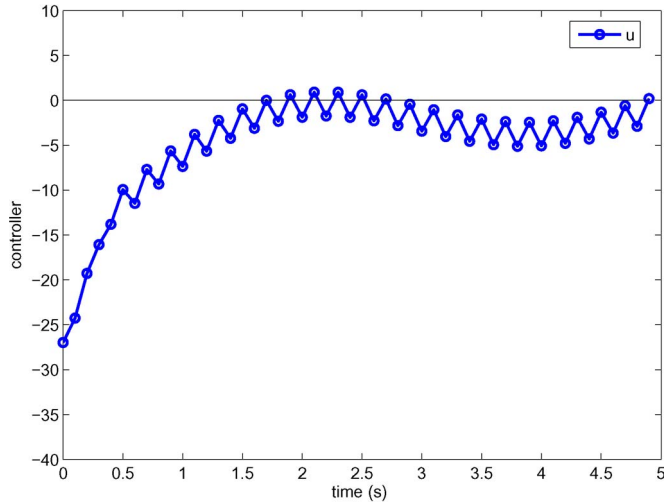Fig. 6. System states with the improved reaching law in case II.

Fig. 8. Controller with the improved reaching law in the case of the sampling period $T = 0.01$ in case II.

Fig. 7. Controller with the improved reaching law in case II.

Fig. 9. Switching function with the improved reaching law in case III.

Case III: Consider the system (34) with an adjustable parameter $a = 1.05$; when the proposed SMC (14) with parameters $q = 5$ and $\varepsilon = 1$ is implemented, the switching function of the closed-loop system is shown in Fig. 9. Because the uncertainty boundary $\delta = 0.034723$ in (8) and it does not satisfy the condition (28), i.e., $\delta > (qT\varepsilon T/2 - qT) = 0.033333$, it is observed from Fig. 9 that the sequence $\{s_k\}$ does not maintain to cross the switching manifold in all subsequent steps, but it still lies in the QSMD defined by $s_{\Delta 2} = \{x_k | |Cx_k| \le \delta + \varepsilon T = 0.13472\}$ by (23) in Theorem 2; moreover, it is still attracting and invariant.

VI. CONCLUSION

In this paper, a new discrete reaching law with a disturbance compensator has been presented. A discrete sliding-mode controller is designed by using the proposed reaching law. The time steps for system trajectories to converge to the switching manifold are found, and the QSMD of the closed-loop system is obtained. Furthermore, the system trajectories of the closed-
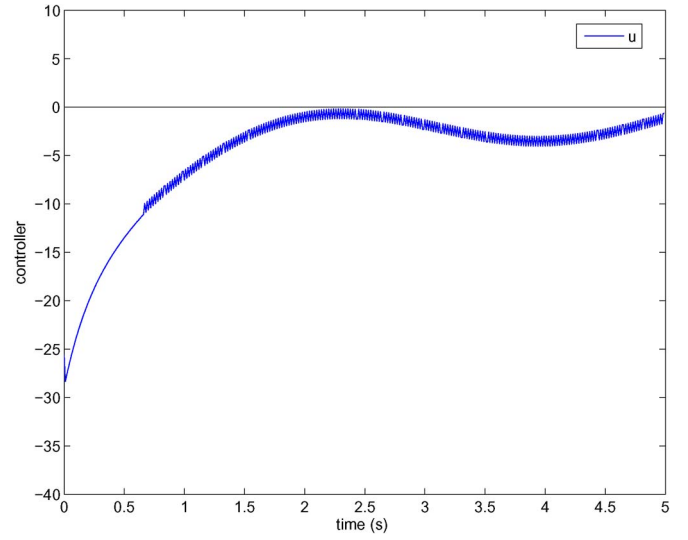
loop system in QSMD are described. Future work will focus on the nonlinear discrete systems with DSMC and industry applications.

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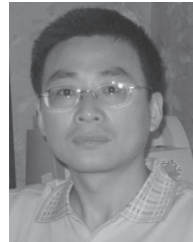
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