FIXED POINTS RESULTS OF DOMINATED MAPPINGS ON A CLOSED BALL IN ORDERED PARTIAL METRIC SPACES WITHOUT CONTINUITY

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Fixed point results for mappings satisfying locally contractive conditions on a closed ball in a complete ordered partial metric space have been established without the assumption of continuity. Instead of monotone mapping, the notion of dominated mappings of Economics, Finance, Trade and Industry is also been applied to approximate the unique solution of non linear functional equations. We have used weaker contractive conditions and weaker restrictions to obtain unique fixed points. An example is given which shows that how this result can be used when the corresponding results can not. Our results improve some well-known,primary and conventional results.

Keywords: Fixed point; Banach Mapping; Kannan mapping; Chatterjea mapping; Closed ball; Continuous mapping; dominated mappings; partial metric spaces.

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1. Introduction

In most of the fixed point results, contractive condition holds on a whole space X. From the application point of view the situation is not yet completely satisfactory because it frequently happens that a mapping T is a contraction not on the entire space X but merely on a subset Y of X. However, if Y is closed then by imposing a subtle restriction, one can establish the existence of a fixed point of T. Arshad et. al. [3] proved some results concerning the existence of fixed points of a mapping satisfying a contractive conditions on closed ball in a complete dislocated metric space. Other results on closed ball can be seen in [5, 4, 7]. These results are very useful in the sense that they require the contraction of the mapping only on the closed ball instead on the whole space.

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Ran and Reurings [10] proved an analogue of Banach's fixed point theorem in metric space endowed with a partial order and gave applications to matrix equations. Subsequently, Nieto et. al. [9] extended the result in [10] for nondecreasing mappings and applied it to obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions.

Partial metric spaces have applications in theoretical computer science (see [8]). [2] used the idea of partial metric space and partial order and gave some fixed point theorems for contractive condition on ordered partial metric spaces. Consistent with [2] and [8], the following definitions and results will be needed in the sequel.

Definition 1.1. Let $p : X \times X \to R^+$, where X is a nonempty set, is said to be a partial metric on X if for any $x, y, z \in X$:

(P_1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$,
(P_2) $p(x,x) \leq p(x,y)$,
(P_3) $p(x,y) = p(y,x)$,
(P_4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is then called a partial metric space. Each partial metric p on X induces a T_0 topology p on X which has as a base the family of open balls $\{B_p(x,r) : x \in X,\ r > 0\}$, where $B_p(x,r) = \{y \in X : p(x,y) < p(x,x) + r\}$ for all $x \in X$ and $r > 0$.

It is clear that if $p(x,y) = 0$, then from P_1 and P_2, $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair (R^+, p) , where $p(x,y) = \max\{x,y\}$ for all $x, y \in R^+$. If (X,p) is a partial metric space, then $p_s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, $x, y \in X$, is a metric on X.

Lemma 1.1. [8] Let (X,p) be a partial metric space.

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p_s).
(b) A partial metric space (X,p) is complete if and only if the metric space (X,p_s) is complete. Furthermore, $\lim\limits_{n\to\infty} p_s(x_n,z) = 0$ if and only if $p(z,z) = \lim\limits_{n\to\infty} p(x_n,z) = \lim\limits_{n,m\to\infty} p(x_n,x_m)$.

Definition 1.2. Let X be a nonempty set. Then $(X,\preceq,p)$ is called an ordered partial metric space if: (i) p is a partial metric on X and (ii) $\preceq$ is a partial order on X.

Definition 1.3. [2] Let $(X,\preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

2. Fixed Points of Banach Mappings

Theorem 2.1. [6] *Let (X,d) be a complete metric space, $S : X \to X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exists $k \in [0,1)$ with*

$$d(Sx,Sy) \leq kd(x,y),\ \textit{for all } x,y \in Y = \overline{B(x_0,r)}$$

and $d(x_0, Sx_0) < (1-k)r$ *then there exists a unique point* x^* *in* $\overline{B(x_0, r)}$ *such that* $x^* = Sx^*$.

In the proof [6], the author considers an iterative sequence $x_n = Sx_{n-1}, n \geq 0$ and exploits the contraction condition on the points x_m's to see

$$d(x_m, x_n) \leq \frac{k^m}{1-k} d(x_0, x_1),$$

by using techniques of [6, Theorem 5.1.2] before proving that x_m's lie in the closed ball.

Following theorem not only extend above theorem to ordered partial metric spaces but also rectifies this mistake specially for those researchers who are utilizing the style of the proof of [6, Theorem 5.1.4] to study more general results.

Theorem 2.2. *Let* $(X, \preceq, p)$ *be a complete ordered partial metric space,* $S : X \to X$ *be a dominated map and* x_0 *be an arbitrary point in* X. *Suppose there exists* $k \in [0, 1)$ *with*

$$p(Sx, Sy) \leq kp(x, y), \tag{2.1}$$

for all comparable elements x, y *in* $\overline{B(x_0, r)}$ *and*

$$p(x_0, Sx_0) \leq (1-k)[r + p(x_0, x_0)]. \tag{2.2}$$

If, for a nonincreasing sequence $\{x_n\}$*in* $\overline{B(x_0, r)}$, $\{x_n\} \to u$ *implies that* $u \preceq x_n$, *then there exists a point* x^* *in* $\overline{B(x_0, r)}$ *such that* $x^* = Sx^*$. *Further* $p(x^*, x^*) = 0$.

Proof. Consider a Picard sequence $x_{n+1} = Sx_n$ with initial guess x_0. As $x_{n+1} = Sx_n \preceq x_n$ for all $n \in \{0\} \cup N$. Now by inequality (2.2)

$$p(x_0, Sx_0) \leq r + p(x_0, x_0).$$

$\Rightarrow x_1 \in \overline{B(x_0, r)}$. It follows that

$$p(x_1, x_2) = p(Sx_0, Sx_1) \leq kp(x_0, x_1) \leq k(1-k)r + k(1-k)p(x_0, x_0).$$

Now,

$$\begin{aligned} p(x_0, x_2) &\leq p(x_0, x_1) + p(x_1, x_2) - p(x_1, x_1) \\ &\leq (1-k)r + (1-k)p(x_0, x_0) + k(1-k)r + k(1-k)p(x_0, x_0) \\ &\leq r + p(x_0, x_0) \end{aligned}$$

$\Rightarrow x_2 \in \overline{B(x_0, r)}$ and hence all points of a sequence $\{x_n\}$ are in the closed ball $\overline{B(x_0, r)}$. Now by inequality (2.1), we have

$$p(x_n, x_n) \leq kp(x_{n-1}, x_{n-1}) \leq \cdots \leq k^n p(x_0, x_0) \longrightarrow 0 \text{ as } n \to \infty \tag{2.3}$$

Also

$$p(x_n, x_{n+1}) \leq k^n p(x_0, x_1). \tag{2.4}$$

Therefore by inequality (2.4) and by using definition of p_s,

$$p_s(x_n, x_{n+1}) \leq 2k^n p(x_0, x_1). \tag{2.5}$$

It follows that

$$\begin{aligned} p_s(x_n, x_{n+i}) &\leq p_s(x_n, x_{n+1}) + ... + p_s(x_{n+i-1}, x_{n+i}) \\ &\leq 2k^n p(x_0, x_1) + ... + 2k^{n+i-1} p(x_0, x_1), \text{ by (2.5)} \\ p_s(x_n, x_{n+i}) &\leq \frac{2k^n(1-k^i)}{1-k} p(x_0, x_1) \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Notice that the sequence $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0, r)}, p_s)$. By Lemma 1.1, $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0, r)}, p)$. Therefore there exists a point $x^* \in \overline{B(x_0, r)}$ with $\lim_{n\to\infty} p_s(x_n, x^*) = 0$. Then by Lemma 1.1 and inequality (2.3), we have

$$p(x^*, x^*) = \lim_{n\to\infty} p(x_n, x^*) = \lim_{n\to\infty} p(x_n, x_n) = 0. \tag{2.6}$$

Moreover by assumptions $x^* \preceq x_n \preceq x_{n-1}$, therefore

$$\begin{aligned} p(x^*, Sx^*) &\leq p(x^*, x_n) + p(x_n, Sx^*) - p(x_n, x_n) \\ &\leq p(x^*, x_n) + kp(x_{n-1}, x^*). \end{aligned}$$

On taking limit as $n \to \infty$ and by inequality (2.6), we obtain

$$p(x^*, Sx^*) \leq 0$$

and hence $x^* = Sx^*$. □

In the above result the fixed point of S may not be unique, whereas with some more restriction we can have unique fixed point of S which is proved now.

Theorem 2.3. *Let $(X, \preceq, p)$ be a complete ordered partial metric space, and $S : X \to X$ be a mapping with all conditions of Theorem 2.2. Also if for any two points x, y in $\overline{B(x_0, r)}$ there exists a point $z \in \overline{B(x_0, r)}$ such that $z \preceq x$ and $z \preceq y$ that is every pair of elements in $\overline{B(x_0, r)}$ has a lower bound, then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^*$. Also $p(x^*, x^*) = 0$.*

Proof. We prove uniqueness only. Let y be another point in $\overline{B(x_0, r)}$ such that $y = Sy$. If x^* and y are comparable then

$$p(x^*, y) = p(Sx^*, Sy) \leq kp(x^*, y).$$

This shows that $x^* = y$. Now if x^* and y are not comparable then there exists a point $z \in \overline{B(x_0, r)}$ which is lower bound of both x^* and y that is $z \preceq x^*$ and $z \preceq y$. Moreover by assumptions $z \preceq x^* \preceq x_n ... \preceq x_0$. Now we will prove that $S^n z \in \overline{B(x_0, r)}$.

$$\begin{aligned} p(x_0, Sz) &\leq p(x_0, x_1) + p(x_1, Sz) - p(x_1, x_1) \\ &\leq (1-k)[r + p(x_0, x_0)] + kp(x_0, z), \\ &\leq (1-k)[r + p(x_0, x_0)] + k[r + p(x_0, x_0)] \\ &= r + p(x_0, x_0). \end{aligned}$$

It follows that $Sz \in \overline{B(x_0, r)}$. Now

$$\begin{aligned} p(x_0, S^2z) &\leq p(x_0, x_1) + p(x_1, S^2z) - p(x_1, x_1) \\ &\leq (1-k)[r + p(x_0, x_0)] + kp(x_0, Sz), \end{aligned}$$

It follows that $S^2z \in \overline{B(x_0, r)}$. Hence $S^nz \in \overline{B(x_0, r)}$ for all $n \in N$. Now as S is dominated, it follows that $S^{n-1}z \preceq S^{n-2}z \preceq ... \preceq z \preceq x^*$ and $S^{n-1}z \preceq y$ for all $n \in N$. Which further implies $S^{n-1}z \preceq S^nx^*$ and $S^{n-1}z \preceq S^ny$ for all $n \in N$ as $S^nx^* = x^*$ and $S^ny = y$ for all $n \in N$.

$$\begin{aligned} p(x^*, y) &= p(S^nx^*, S^ny) \\ &\leq p(S^nx^*, S^{n-1}z) + p(S^{n-1}z, S^ny) - p(S^{n-1}z, S^{n-1}z) \\ &\leq kp(S^{n-1}x^*, S^{n-2}z) + kp(S^{n-2}z, S^{n-1}y) \\ &\quad \vdots \\ &\leq k^{n-2}p(x^*, Sz) + k^{n-2}p(Sz, y) \longrightarrow 0 \mathrm{as} n \to \infty \end{aligned}$$

Hence $x^* = y$. □

3. **Fixed Points of Kannan Mappings**

In 1969 Kannan established the following fixed point theorem:

Theorem 3.1. *Let (X, d) be a complete metric space. If a mapping $S : X \longrightarrow X$ satisfies,*

$$d(Sx, Sy) \leqslant \alpha[d(x, Sx) + d(y, Sy)],$$

for all $\alpha \in [0, \frac{1}{2})$. Then S has a unique fixed point in X.

Theorem 3.2. *Let $(X, \preceq, p)$ be a complete ordered partial metric space, $S : X \to X$ be a dominated map and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, \frac{1}{2})$ with*

$$p(Sx, Sy) \leq k[p(x, Sx) + p(y, Sy)], \tag{3.1}$$

for all comparable elements x, y in $\overline{B(x_0, r)}$ and

$$p(x_0, Sx_0) \leq (1-\theta)[r + p(x_0, x_0)], \tag{3.2}$$

where $\theta = \frac{k}{1-k}$. If for a nonincreasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$, then there exists a point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^*$ and $p(x^*, x^*) = 0$.

Proof. Consider a Picard sequence $x_{n+1} = Sx_n$ with initial guess x_0. Then $x_{n+1} = Sx_n \preceq x_n$ for all $n \in \{0\} \cup N$ and by using inequality (3.2), we have

$$p(x_0, Sx_0) \leq r + p(x_0, x_0).$$

Therefore, $x_1 \in \overline{B(x_0, r)}$. Thus, by using inequality (3.1), we have

$$\begin{aligned} p(x_1, x_2) &= p(Sx_0, Sx_1) \leq k[p(x_0, x_1) + p(x_1, x_2)] \\ &\leq \theta p(x_0, x_1) \leq \theta[r + p(x_0, x_0)] \end{aligned}$$

Now

$$\begin{aligned} p(x_0, x_2) &\leq p(x_0, x_1) + p(x_1, x_2) - p(x_1, x_1) \\ &\leq (1-\theta)[r + p(x_0, x_0)] + \theta[r + p(x_0, x_0)] \end{aligned}$$

It implies that $x_2 \in \overline{B(x_0, r)}$ and hence all points of a sequence $\{x_n\}$ are in the closed ball $\overline{B(x_0, r)}$. Now by using inequality (3.1), we have

$$\begin{aligned} p(Sx_n, Sx_{n+1}) &\leq k[p(x_n, Sx_n) + p(x_{n+1}, Sx_{n+1})] \\ &\leq \frac{k}{1-k} p(x_n, x_{n+1}) = \theta p(x_n, x_{n+1}) \\ p(x_{n+1}, x_{n+2}) &\leq \theta^2 p(x_{n-1}, x_n) \leq \ldots \leq \theta^{n+1} p(x_0, x_1). \end{aligned}$$

Again by using inequality (3.1), we have

$$p(x_n, x_n) \leq k[p(x_{n-1}, x_n) + p(x_{n-1}, x_n)]$$

which implies that

$$p(x_n, x_n) \leq \theta^{n-1} p(x_0, x_1) \longrightarrow 0 \text{ as } n \to \infty. \tag{3.3}$$

And by using definition of p_s,

$$p_s(x_{n+1}, x_{n+2}) \leq 2p(x_{n+1}, x_{n+2}) \leq 2\theta^{n+1} p(x_0, x_1), \tag{3.4}$$

Now, we have

$$p_s(x_n, x_{n+i}) \leq p_s(x_n, x_{n+1}) + \ldots + p_s(x_{n+i-1}, x_{n+i}),$$

by using inequality (3.4), we have

$$\begin{aligned} p_s(x_n, x_{n+i}) &\leq 2\theta^n p(x_0, x_1) + \ldots + 2\theta^{n+i-1} p(x_0, x_1) \\ &\leq 2\theta^n p(x_0, x_1)[1 + \ldots + \theta^{i-2} + \theta^{i-1}] \\ &\leq \frac{2\theta^n(1-\theta^i)}{1-\theta} p(x_0, x_1) \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

It follows that the sequence $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0, r)}, p_s)$. By Lemma 1.2, $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0, r)}, p)$. Therefore there exists a point $x^* \in \overline{B(x_0, r)}$ with $\lim\limits_{n\to\infty} p_s(x_n, x^*) = 0$. Then by using Lemma 1.1 and inequality (3.3), we have

$$p(x^*, x^*) = \lim_{n\to\infty} p(x_n, x^*) = \lim_{n\to\infty} p(x_n, x_n) = 0. \tag{3.5}$$

Moreover, by assumptions $x^* \preceq x_n \preceq x_{n-1}$, therefore

$$\begin{aligned} p(x^*, Sx^*) &\leq p(x^*, x_n) + p(x_n, Sx^*) - p(x_n, x_n) \\ &\leq p(x^*, x_n) + k[p(x_{n-1}, Sx_{n-1}) + p(x^*, Sx^*)] \\ (1-k)p(x^*, Sx^*) &\leq p(x^*, x_n) + kp(x_{n-1}, x_n). \end{aligned}$$

On taking limit as $n \to \infty$ and using inequality (3.5) we obtain

$$(1-k)p(x^*, Sx^*) \leq 0$$

and $x^* = Sx^*$. □

Theorem 3.3. *Let $(X, \preceq, p)$ be a complete ordered partial metric space, and $S : X \to X$ be a mapping with all conditions of Theorem (3.2). Also for every pair of elements x, y in $\overline{B(x_0, r)}$ there exists a point $z \in \overline{B(x_0, r)}$ such that $z \preceq x$ and $z \preceq y$ and*

$$p(x_0, Sx_0) + p(z, Sz) \leq p(x_0, z) + p(Sx_0, Sz), \tag{3.6}$$

then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^*$ and $p(x^*, x^*) = 0$.

Proof. It is sufficient to prove x^* is unique. Let y be another point in $\overline{B(x_0, r)}$ such that $y = Sy$. If x^* and y are comparable then

$$p(x^*, y) \leq k[p(x^*, x^*) + p(y, y)] = kp(y, y) \leq p(y, y).$$

Using the fact that $p(y, y) \leq p(x^*, y)$, we have $x^* = y$. Now if x^* and y are not comparable then there exists a point $z \in \overline{B(x_0, r)}$ which is a lower bound of both x^* and y. Now we will prove that $S^n z \in \overline{B(x_0, r)}$. Moreover by assumptions $z \preceq x^* \preceq x_n ... \preceq x_0$. Now by using inequality (3.1) , we have

$$\begin{aligned} p(Sx_0, Sz) &\leq k[p(x_0, x_1) + p(z, Sz)] \\ &\leq k[p(x_0, z) + p(x_1, Sz)], \quad \text{by using (3.6)} \end{aligned}$$

which further implies that

$$p(x_1, Sz) \leq \theta p(x_0, z). \tag{3.7}$$

Now,

$$\begin{aligned} p(x_0, Sz) &\leq p(x_0, x_1) + p(x_1, Sz) - p(x_1, x_1) \\ &\leq p(x_0, x_1) + \theta p(x_0, z), \quad \text{by using (3.7)} \\ p(x_0, Sz) &\leq (1-\theta)[r + p(x_0, x_0)] + \theta[r + p(x_0, x_0)] \\ &= r + p(x_0, x_0). \end{aligned}$$

It follows that $Sz \in \overline{B(x_0, r)}$. Now,

$$p(Sz, S^2 z) \leq k[p(z, Sz) + p(Sz, S^2 z)]$$

which implies that

$$p(Sz, S^2 z) \leq \theta p(z, Sz). \tag{3.8}$$

Also by using inequality (3.1), we have,

$$\begin{aligned} p(x_2, S^2 z) &\leq k[p(x_1, x_2) + p(Sz, S^2 z)] \\ &\leq k[\theta p(x_0, x_1) + \theta p(z, Sz)], \text{ by using (3.8)} \\ p(x_2, S^2 z) &\leq k\theta[p(x_0, z) + p(x_1, Sz)], \text{ by using (3.6)} \\ p(x_2, S^2 z) &\leq k\theta[p(x_0, z) + \theta p(x_0, z)], \text{ by using (3.7)} \\ p(x_2, S^2 z) &\leq k(\frac{k}{1-k})(\frac{1}{1-k})p(x_0, z) \end{aligned}$$

which imples that

$$p(x_2, S^2 z) \leq \theta^2 p(x_0, z). \tag{3.9}$$

Now,

$$\begin{aligned}
p(x_0, S^2z) &\leq p(x_0, x_1) + p(x_1, x_2) + p(x_2, S^2z) - p(x_1, x_1) - p(x_2, x_2) \\
&\leq p(x_0, x_1) + \theta p(x_0, x_1) + \theta^2 p(x_0, z), \text{ by using (3.9)} \\
p(x_0, S^2z) &\leq (1-\theta)[r + p(x_0, x_0)][1+\theta] + \theta^2[r + p(x_0, x_0)] \\
&= r + p(x_0, x_0).
\end{aligned}$$

It follows that $S^2z \in \overline{B(x_0, r)}$. Hence $S^nz \in \overline{B(x_0, r)}$. As $z \preceq x^*$ and $z \preceq y$ then $S^nz \preceq x^*$ and $S^nz \preceq y$ for all $n \in \{0\} \cup \mathbb{N}$. As $S^{n+1}z \preceq S^nz$ for all $n \in \{0\} \cup N$, we have

$$\begin{aligned}
p(S^{n-1}z, S^nz) &\leq k[p(S^{n-2}z, S^{n-1}z) + p(S^{n-1}z, S^nz)] \\
&\leq \theta p(S^{n-2}z, S^{n-1}z) \leq ...
\end{aligned}$$

$$p(S^{n-1}z, S^nz) \leq \theta^{n-1} p(z, Sz) \longrightarrow 0 \text{ as } n \to \infty. \tag{3.10}$$

Now,

$$\begin{aligned}
p(x^*, y) &= p(Sx^*, Sy) \\
&\leq p(Sx^*, S^nz) + p(S^nz, Sy) - p(S^nz, S^nz) \\
&\leq k[p(x^*, Sx^*) + p(S^{n-1}z, S^nz)] + k[p(S^{n-1}z, S^nz) + p(y, Sy)] \\
&\leq kp(x^*, x^*) + 2kp(S^{n-1}z, S^nz) + kp(y, y).
\end{aligned}$$

Hence by using inequality (3.10), $p(x^*, y) \leq p(y, y)$ as $n \to \infty$. A contradiction, so $x^* = y$. □

Theorem 3.4. *Let $(X, \preceq, p)$ be a complete ordered partial metric space, $S : X \to X$ be a dominated map and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, \frac{1}{2})$ with*

$$p(Sx, Sy) \leq k[p(x, Sx) + p(y, Sy)],$$

for all comparable elements x, y in X. If, for a nonincreasing sequence $\{x_n\}$in X, $\{x_n\} \to u$ implies that $u \preceq x_n$ and every pair of elements in X has a lower bound, then there exists a unique point x^* in X such that $x^* = Sx^*$ and $p(x^*, x^*) = 0$.

Theorem 3.5. *Let (X, p) be a complete partial metric space, $S : X \to X$ be a map and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, \frac{1}{2})$ with*

$$p(Sx, Sy) \leq k[p(x, Sx) + p(y, Sy)],$$

for all elements x, y in $\overline{B(x_0, r)}$ and

$$p(x_0, Sx_0) \leq (1-\theta)[r + p(x_0, x_0)],$$

where $\theta = \frac{k}{1-k}$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^*$ and $p(x^*, x^*) = 0$

Example 3.1. *Let* $X = R^+ \cup \{0\}$ *and* $\overline{B(x_0, r)} = [0, 1]$ *be endowed with the usual ordering and let* p *be the complete partial metric on* X *defined by* $p(x, y) = \max\{x, y\}$ *for all* $x, y \in X$. *Let* $S : X \to X$ *be defined by*

$$Sx = \begin{cases} \dfrac{3x}{70} \text{ if } x \in [0, \frac{1}{2}), \\ \dfrac{2x}{70} \text{ if } x \in [\frac{1}{2}, 1] \\ x - \frac{1}{2} \text{ if } x \in (1, \infty) \end{cases}$$

Clearly, $Sx \leq x$ *for all* $x \in X$ *that is,* S *is dominating map. For all comparable elements with* $k = \frac{1}{5} \in [0, \frac{1}{2})$, $x_0 = \frac{1}{2}$, $r = \frac{1}{2}$, $p(x_0, x_0) = \max\{\frac{1}{2}, \frac{1}{2}\} = \frac{1}{2}, \theta = \frac{k}{1-k} = \frac{1}{4}$

$$(1 - \theta)[r + p(x_0, x_0)] = (1 - \frac{1}{4})[\frac{1}{2} + \frac{1}{2}] = \frac{3}{4}$$

$$p(x_0, Sx_0) = p(\frac{1}{2}, S\frac{1}{2}) = p(\frac{1}{2}, \frac{1}{70}) = \max\{\frac{1}{2}, \frac{1}{70}\} = \frac{1}{2} < \frac{3}{4}$$

$$\begin{aligned} \textit{Also if } x, y &\in (1, \infty),\ p(Sx, Sy) = \max\{x - \frac{1}{2}, y - \frac{1}{2}\} \\ &\geq \frac{1}{5}[x + y] \\ &= \frac{1}{5}[\max\{x, x - \frac{1}{2}\} + \max\{y, y - \frac{1}{2}\}] \\ p(Sx, Sy) &\geq k[p(x, Sx) + p(y, Sy)]. \end{aligned}$$

So the contractive condition does not hold on $(1, \infty)$. *For the closed ball* $[0, 1]$ *the following cases arrises:*
(i) If $x, y \in [0, \frac{1}{2})$, *we have*

$$\begin{aligned} p(Sx, Sy) &= \max\{\frac{3x}{70}, \frac{3y}{70}\} = \frac{3}{70}\max\{x, y\} \\ &\leq \frac{1}{5}[x + y] = \frac{1}{5}[\max\{x, \frac{3x}{70}\} + \max\{y, \frac{3y}{70}\}] \\ &= k[p(x, Sx) + p(y, Sy)]. \end{aligned}$$

(ii) For $x \in [0, \frac{1}{2})$, $y \in [\frac{1}{2}, 1]$, *we have*

$$\begin{aligned} p(Sx, Sy) &= \max\{\frac{3x}{70}, \frac{2y}{70}\} = \frac{1}{70}\max\{3x, 2y\} \\ &\leq \frac{1}{5}[x + y] = \frac{1}{5}[\max\{x, \frac{3x}{70}\} + \max\{y, \frac{2y}{70}\}] \\ &= k[p(x, Sx) + p(y, Sy)]. \end{aligned}$$

(iii) When $y \in [0, \frac{1}{2})$, $x \in [\frac{1}{2}, 1]$, *we have*

$$\begin{aligned} p(Sx, Sy) &= \max\{\frac{2x}{70}, \frac{3y}{70}\} = \frac{1}{70}\max\{2x, 3y\} \\ &\leq \frac{1}{5}[x + y] = \frac{1}{5}[\max\{x, \frac{2x}{70}\} + \max\{y, \frac{3y}{70}\}] \\ &= k[p(x, Sx) + p(y, Sy)]. \end{aligned}$$

(iv) And if $x, y \in [\frac{1}{2}, 1]$, we obtain

$$\begin{aligned} p(Sx, Sy) &= \max\{\frac{2x}{70}, \frac{2y}{70}\} = \frac{2}{70}\max\{x, y\} \\ &\leq \frac{1}{5}[x+y] = \frac{1}{5}[\max\{x, \frac{2x}{70}\} + \max\{y, \frac{2y}{70}\}] \\ &= k[p(x, Sx) + p(y, Sy)]. \end{aligned}$$

Therefore, all the conditions of above theorem satisfied to obtain unique fixed point 0 *of S.*

4. **Fixed Points of Chatterjea Mappings**

Chatterjea established the following fixed point theorem:

Theorem 4.1. *Let (X, d) a complete metric space X. Let $S : X \longrightarrow X$ be a mapping satisfying contractive condition*

$$d(Sx, Sy) \leqslant \alpha[d(x, Sy) + d(y, Sx)],$$

for all $\alpha \in [0, \frac{1}{2})$. Then S has a fixed point.

Our extension of this theorem is as follows:

Theorem 4.2. *Let $(X, \preceq, p)$ be a complete ordered partial metric space, $S : X \to X$ be a dominated map and x_0 be an arbitrary point in X. Suppose there exists $k \in [0, \frac{1}{2})$ with*

$$p(Sx, Sy) \leq k[p(x, Sy) + p(y, Sx)], \tag{4.1}$$

for all comparable elements x, y in $\overline{B(x_0, r)}$ and

$$p(x_0, Sx_0) \leq (1-\theta)[r + p(x_0, x_0)], \tag{4.2}$$

where $\theta = \frac{k}{1-k}$. If, for a nonincreasing sequence $\{x_n\}$in $\overline{B(x_0, r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then there exists a point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^*$ and $p(x^*, x^*) = 0$.

Proof. Choose $x_0 \in X$ and consider a sequence of point such that

$$x_{n+1} = Sx_n, \ n \geq 0$$

Then, $x_{n+1} \preceq x_n$ for all $n \in \{0\} \cup N$ and by using inequality (4.2), we have

$$p(x_0, Sx_0) \leq r + p(x_0, x_0).$$

This implies that $x_1 \in \overline{B(x_0, r)}$, using inequality (4.1), we have

$$\begin{aligned} p(Sx_0, Sx_1) &\leq k[p(x_0, Sx_1) + p(x_1, Sx_0)] \\ &\leq k[p(x_0, x_1) + p(x_1, x_2) + p(x_1, x_1) - p(x_1, x_1)] \\ &\leq k(1-\theta)r + k(1-\theta)p(x_0, x_0) + kp(x_1, x_2) \\ &\leq \theta(1-\theta)r + \theta(1-\theta)p(x_0, x_0). \end{aligned}$$

It implies that

$$\begin{aligned} p(x_0,x_2) &\leq p(x_0,x_1)+p(x_1,x_2)-p(x_1,x_1) \\ &\leq (1-\theta)r+(1-\theta)p(x_0,x_0)+\theta(1-\theta)r+\theta(1-\theta)p(x_0,x_0) \\ &\leq r+p(x_0,x_0). \end{aligned}$$

That is $x_2 \in \overline{B(x_0,r)}$. Hence all points of a sequence $\{x_n\}$ are in the closed ball $\overline{B(x_0,r)}$, Now, by inequality (4.1), we have

$$\begin{aligned} p(x_{n+1},x_{n+2}) &= p(Sx_n,Sx_{n+1}) \\ &\leq k[p(x_n,x_{n+1})+p(x_{n+1},x_{n+2})], \end{aligned}$$

which further implies,

$$p(x_{n+1},x_{n+2}) \leq \theta p(x_n,x_{n+1})... \leq \theta^{n+1}p(x_0,x_1)$$

Moreover,

$$\begin{aligned} p(x_n,x_n) &\leq k[p(x_{n-1},x_n)+p(x_{n-1},x_n)] \\ &\leq 2k\theta^{n-1}p(x_0,x_1) \longrightarrow 0 \text{ as } n\to\infty \end{aligned}$$

Following similar arguments of Theorem (3.2), it can easily be seen that sequence $\{x_n\}$ is a Cauchy sequence in $\overline{B(x_0,r)}$ and there exists a point $x^* \in \overline{B(x_0,r)}$ with

$$p(x^*,x^*) = \lim_{n\to\infty} p(x_n,x^*) = \lim_{n\to\infty} p(x_n,x_n) = 0.$$

Moreover by assumptions $x^* \preceq x_n \preceq x_{n-1}$, therefore

$$\begin{aligned} p(x^*,Sx^*) &\leq p(x^*,x_n)+p(x_n,Sx^*)-p(x_n,x_n) \\ &\leq p(x^*,x_n)+p(Sx_{n-1},Sx^*)-p(x_n,x_n) \\ &\leq p(x^*,x_n)+k[p(x_{n-1},Sx^*)+p(x^*,x_n)]. \end{aligned}$$

On taking limit as $n\to\infty$, we have

$$p(x^*,Sx^*) \leq kp(x^*,Sx^*)$$

and therefore $x^* = Sx^*$. □

Remark 4.1. Since any metric is a partial metric, so above theorems holds in a metric space.

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