



## On Generalized Soft Equality and Soft Lattice Structure

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**Abstract.** Molodtsov introduced soft sets as a mathematical tool to handle uncertainty associated with real world data based problems. In this paper we propose some new concepts which generalize existing comparable notions. We introduce the concept of generalized soft equality (denoted as  $g$ -soft equality) of two soft sets and prove that the so called lower and upper soft equality of two soft sets imply  $g$ -soft equality but the converse does not hold. Moreover we give tolerance or dependence relation on the collection of soft sets and soft lattice structures. Examples are provided to illustrate the concepts and results obtained herein.

### 1. Introduction

Mathematical models have been used extensively in real world problems related to engineering, computer sciences, economics, social, natural and medical sciences etc. It has become very common to use mathematical tools to solve, study the behavior and different aspects of a system and its different sub-systems. So it is very natural to deal with uncertainties and imprecise data in various situations. Fuzzy set theory has been evolved in mathematics as an important tool (initiated by Zadeh [28]) to resolve the issues of uncertainty and ambiguity. But there are certain limitations and deficiencies pertaining to the parametrization in fuzzy set theory (see [18]). Soft set theory aims to provide enough tools to deal with uncertainty in a data and to represent it in a useful way. The distinguishing attribute of soft set theory is that unlike probability theory and fuzzy set theory, it does not uphold a precise quantity. This attribute has facilitated applications in decision making, demand analysis, forecasting, information science, mathematics and other disciplines (see for detailed survey [3–7, 15, 19, 22, 29, 31]).

A lot of activity has been shown in soft set theory (see e.g. [1, 2, 8–13, 17, 18, 20, 23–27]) since Molodtsov [18] initiated the concept of soft sets. Maji et al. [14] introduced some basic algebraic operations on soft sets. They defined equality of two soft sets, subset and super set of soft sets, complement of soft sets, null soft set and absolute soft set with examples. Unfortunately, several basic properties in [14] do not hold true in general, these have been pointed out and improved by Yang [26], Ali et al. [2], and Li [16]. Ali et al. [2] defined some restricted intersection and union, the restricted difference and complement of a soft set. Zhu

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et al. [30] redefined the intersection, complement, and difference of soft sets and investigated the algebraic properties of these operations along with a known union operation. Their operations on soft sets inherit basic properties of operations on classical sets. With the newly defined operations the union of a soft set and its complement is exactly the whole universal soft set which is not true in general with the previously defined operations. In a recent and interesting paper [21], Qin and Hong defined soft equality relations (lower soft equality  $\approx_s$  and upper soft equality  $\approx^s$ ) and proved results by using already defined operations (see Ali et al. [2]) on union and intersection of soft sets.

In this article we redefine the concepts of null soft set and soft subset of a soft set reconsidered in [30]. We introduce the concept of  $g$ -null soft set and  $g$ -soft subset of a soft set and this lead us to give a new and generalized soft equality ( $g$ -soft equality) relation  $\approx_g$ . It is shown that  $g$ -soft equality relation  $\approx_g$  is more general than soft equality relations  $\approx_s$  and  $\approx^s$  on soft sets given in [21]. We provide examples to show that class of  $g$ -soft equal sets with respect to  $\approx_g$  is a more general class. We give tolerance (or dependence) relation and lattice structure on the class of soft sets. Examples are provided to illustrate the definitions and results obtained in this article.

## 2. Preliminaries

In this section we begin with some basic definitions and concepts related to soft sets needed in the sequel.

Let  $U$  be a given universe and  $E$  a set of parameters. Throughout this paper,  $P(U)$  and  $P^*(U)$  denote the family of all subsets of  $U$ , and the family of all nonempty subsets of  $U$ , respectively.

**Definition 2.1.** [18] *If  $F$  is a set valued mapping on  $A \subseteq E$  taking values in  $P(U)$ , then a pair  $(F, A)$  is called a soft set over  $U$ .*

A soft set  $(F, A)$  can be seen as a parametrized family of subsets of the set  $U$ . For each  $e$  in  $A$ , the set  $F(e)$  in  $U$  is called  $e$ - approximate element of the soft set  $(F, A)$ .

Moreover, in several places of this paper a soft set  $(F, A)$  will be identified with the set  $\{(e, F(e)) : e \in A\}$ .

**Definition 2.2.** [14] *A soft set  $(F, A)$  over  $U$  is said to be a null soft set over  $U$  if  $F(e) = \emptyset$  for all  $e \in A$ .*

**Definition 2.3.** [14] *Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$ , we say that  $(F, A)$  is a soft subset of  $(G, B)$  or  $(G, B)$  is super soft set of  $(F, A)$ , if  $A \subseteq B$ , and for all  $e \in A$ ,  $F(e) = G(e)$ . We write it as  $(F, A) \check{\subseteq} (G, B)$ .*

Zhu and Wen [30] gave a the following slight modification of Definitions 2.1 and 2.2 to inherit basic classical set operations in soft set theory.

**Definition 2.4.** [30] *If  $F$  is a set valued mapping on  $A \subseteq E$  taking values in  $P^*(U)$ , then a pair  $(F, A)$  is called a soft set over  $U$ .*

**Definition 2.5.** [30] *A soft set  $(F, A)$  over  $U$  (in the sense of Definition 2.4) is said to be a null soft set denoted by  $(\emptyset, \emptyset)$  whenever  $A = \emptyset$ .*

Maji et al. [14] gave Definitions 2.2 and 2.3 which do not inherit the property which reads as follows: "null set is a subset of any other set" in soft set theory (see Example 2.6 below). Zhu and Wen [30] presented Definitions 2.4 and 2.5 to incorporate this property.

In the following example we show that a null soft set in the sense of [14] need not be a soft subset of any other soft set. It also shows that Definitions 2.4 and 2.5 do not cover certain situations arising in soft set theory.

**Example 2.6.** Suppose that  $U$  is the set of persons given by

$$U = \{p_1, p_2, p_3, p_4, p_5, p_6\}$$

and

$$E = \{i, r, s\}, A = \{s, i\}, B = \{i, r\}$$

where  $s, i, r$  stand for susceptible, infectious and recovered persons. The soft set  $(F, A)$  describes the specific classes of people with respect to the set  $A$ , dependent upon their experience with respect to the disease and the corresponding approximations  $F(s)$  and  $F(i)$  with respect to parameter set  $A$ , are the the sets of susceptible and infected people respectively, given as:

$$F(s) = \emptyset = F(i).$$

We denote  $(F, A)$  as

$$(F, A) = \{(s, \emptyset), (i, \emptyset)\}.$$

The soft set  $(G, B)$  describes the specific classes of people with respect to the set  $B$ , dependent upon their experience with respect to the disease and the corresponding approximations  $G(i)$  and  $G(r)$  with respect to parameter set  $B$ , are the the sets of infected and recovered people respectively, given as:

$$G(i) = \emptyset, G(r) = \{p_1, p_2, p_3\}.$$

We denote  $(G, B)$  as

$$(G, B) = \{(i, \emptyset), (r, \{p_1, p_2, p_3\})\}.$$

Here according to Maji et al. [14],  $(F, A)$  is a null soft set but clearly according to them  $(F, A)$  is not a null soft subset of  $(G, B)$  because  $A \not\subseteq B$ , that is null soft set  $(F, A)$  is not a soft subset of  $(G, B)$ . As we mentioned before that Zhu and Wen [30] presented Definitions 2.4 and 2.5 to remove this shortcoming but according to them  $(F, A)$  cannot be regarded as a soft set as  $F(s) = F(i) = \emptyset$ .

**Definition 2.7.** [14] The union of two soft sets  $(F, A)$  and  $(G, B)$  over a common universe  $U$ , denoted by  $(F, A) \widetilde{\cup} (G, B)$ , is the soft set  $(H, C)$ , where  $C = A \cup B$  and  $H$  is defined on  $C$  as

$$H(e) = \begin{cases} F(e), & \text{if } e \in A \setminus B \\ G(e), & \text{if } e \in B \setminus A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases}.$$

**Definition 2.8.** [2] The restricted union of two soft sets  $(F, A)$  and  $(G, B)$  over a common universe  $U$ , denoted by  $(F, A) \cup_{\mathfrak{R}} (G, B)$ , is the soft set  $(H, C)$ , where  $C = A \cap B$  and for all  $e \in C$ ,  $H(e) = F(e) \cup G(e)$ .

**Definition 2.9.** [2] The extended intersection of two soft sets  $(F, A)$  and  $(G, B)$  over a common universe  $U$ , denoted by  $(F, A) \sqcap_{\epsilon} (G, B)$ , is the soft set  $(H, C)$ , where  $C = A \cup B$  and for all  $e \in C$ ,  $H(e)$  is defined as

$$H(e) = \begin{cases} F(e), & \text{if } e \in A \setminus B \\ G(e), & \text{if } e \in B \setminus A \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases}.$$

**Definition 2.10.** [2] The restricted intersection of two soft sets  $(F, A)$  and  $(G, B)$  over a common universe  $U$ , denoted by  $(F, A) \cap (G, B)$ , is the soft set  $(H, C)$ , where  $C = A \cap B$  and for all  $e \in C$ ,  $H(e) = F(e) \cap G(e)$ .

**Definition 2.11.** [2] The relative complement of a soft set  $(F, A)$  over a universe  $U$ , denoted by  $(F, A)^r$ , is defined as  $(F, A)^r = (F^r, A)$ , where  $F^r(e) = U \setminus F(e)$  for each  $e \in A$ .

Ali et al. [2] gave the following De Morgan's laws with respect to the relative complement of a soft set in soft set theory.

**Theorem 2.12.** [2] Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$  such that  $A \cap B \neq \emptyset$ , then

$$\begin{aligned}(F, A) \cup_{\mathfrak{R}} (G, B)^r &= (F, A)^r \cap (G, B)^r, \\ (F, A) \cap (G, B)^r &= (F, A)^r \cup_{\mathfrak{R}} (G, B)^r,\end{aligned}$$

hold true.

Recently, Qin and Hong [21] defined soft equalities  $\approx_s$  and  $\approx^s$ . We call these as lower soft equality and upper soft equality relations, respectively.

**Definition 2.13.** [21] Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$ . The soft set  $(F, A)$  is called lower soft equal to  $(G, B)$ , and we write  $(F, A) \approx_s (G, B)$ , if  $F(e) = G(e)$  whenever  $e \in A \cap B$ ,  $F(e) = \emptyset$  whenever  $e \in A \setminus B$  and  $G(e) = \emptyset$  whenever  $e \in B \setminus A$ .

**Definition 2.14.** [21] Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$ . The soft set  $(F, A)$  is called upper soft equal to  $(G, B)$ , and we write  $(F, A) \approx^s (G, B)$ , if  $F(e) = G(e)$ , whenever  $e \in A \cap B$ ,  $F(e) = U$  whenever  $e \in A \setminus B$  and  $G(e) = U$  whenever  $e \in B \setminus A$ .

For more on soft equal relations  $\approx_s$  and  $\approx^s$ , we refer to [21].

### 3. Generalized Soft Equality (or $g$ -soft Equality) Relation $\approx_g$

In this section we present a definition of  $g$ -soft equality of two soft sets which generalizes Definitions 2.13 and 2.14. We give the concept of a generalized null soft denoted as  $g$ -null soft set, which unifies Definitions 2.2 and 2.5. A notion of a " $g$ -soft subset of a soft set" is also introduced. It is worth mentioning that the definitions presented herein preserve the classical property of a crisp set theory that an empty set is a subset of every set. These notions not only generalize existing comparable concepts but also fit in the bigger set of situations.

**Definition 3.1.** A soft set  $(F, A)$  over  $U$  is said to be a  $g$ -null soft set if either (i)  $A = \emptyset$ , or (ii)  $F(e) = \emptyset$  for each  $e \in A$  whenever  $A \neq \emptyset$ . A  $g$ -null soft set over  $U$  is denoted by  $(F_\emptyset, A)$ .

**Definition 3.2.** A soft set  $(F, A)$  over  $U$  is called a  $g$ -universal soft set if  $A = E \neq \emptyset$  and  $F(e) = U$  for each  $e \in E$ . We denote  $g$ -universal soft set by  $(F_U, E)$ .

**Definition 3.3.** Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$ . We say that  $(F, A)$  is a  $g$ -soft subset of  $(G, B)$  if (i)  $A = \emptyset$ , or (ii) for each  $e \in A$ , there exists an  $e' \in B$  such that  $F(e) \subseteq G(e')$ . We denote it as  $(F, A) \sqsubseteq_g (G, B)$ .

According to Example 2.6,  $(F, A)$  is a  $g$ -null soft set and clearly  $(F, A) \sqsubseteq_g (G, B)$ . Hence Definitions 3.1 and 3.3 inherit the property from classical set theory which says that null set is subset of every other non-empty set.

**Example 3.4.** Suppose that  $U = \{s_1, s_2, s_3, s_4, s_5, s_6\}$  is a set of six students under consideration. Let  $s, g$ , and  $p$  stands for scholarship, good CGPA (we denote good CGPA if CGPA is greater or equal to 3.00 out of 4.00) and publications, respectively. If the sets of parameters are given as

$$A = \{s, g\} \quad \text{and} \quad B = \{s, g, p\},$$

and suppose that soft set  $(F, A)$  describes the choice of a person P-1 (say) with respect to the parameter set  $A$  and soft set  $(G, B)$  describes the choice of a person P-2 (say) with respect to the parameter set  $B$ , then corresponding approximations are given as:

$$\begin{aligned}(F, A) &= \{(s, \{s_1, s_2\}), (g, \{s_2, s_5\})\} \text{ and} \\(G, B) &= \{(s, \{s_2, s_5, s_4\}), (g, \{s_3, s_4\}), (p, \{s_1, s_2, s_6\})\},\end{aligned}$$

where

$$\begin{aligned}F(s) &= \{s_1, s_2\}, \text{ (set of students holding scholarship)} \\F(g) &= \{s_2, s_5\}, \text{ (set of students with good CGPA)} \\G(s) &= \{s_2, s_5, s_4\}, \text{ (set of students holding scholarship)} \\G(g) &= \{s_3, s_4\}, \text{ (set of students with good CGPA)} \\G(p) &= \{s_1, s_2, s_6\}, \text{ (set of students with publications)}.\end{aligned}$$

Clearly,  $(F, A) \sqsubseteq_g (G, B)$ . That means if according to P-1, a particular student has a certain attribute then that student also exists in the set of P-2's opinion with some attribute (same or different) because  $(F, A) \sqsubseteq_g (G, B)$ . Here according to P-1, students  $s_2$  and  $s_5$  have good CGPA and in P-2's opinion, these students hold scholarship as well.

**Definition 3.5.** Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$ . Then soft sets  $(F, A)$  and  $(G, B)$  are called  $g$ -soft equal if  $(F, A) \sqsubseteq_g (G, B)$  and  $(G, A) \sqsubseteq_g (F, B)$ . We denote it by  $(F, A) \approx_g (G, B)$ .

In above definition if we take  $A \subseteq B$  and  $e' = e$ , then Definition 3.5 reduces to Definition 2.3.

**Proposition 3.6.** Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$ . If  $(F, A) \approx_s (G, B)$  then  $(F, A) \approx_g (G, B)$ , that is, lower soft equality implies  $g$ -soft equality.

**Proof.** Suppose that  $(F, A) \approx_s (G, B)$ . We first show that  $(F, A) \sqsubseteq_g (G, B)$ . Since the conclusion is obvious when  $A = \emptyset$ , we shall assume  $A \neq \emptyset$ . Let  $e$  be an arbitrary parameter in  $A$ . If  $e \in A \cap B$ , then  $F(e) = G(e)$ , so  $F(e) \subseteq G(e')$  for  $e' = e \in B$ . If  $e \in A \setminus B$ , then  $F(e) = \emptyset$  and hence  $F(e) \subseteq G(e')$  for all  $e' \in B$ . Consequently for every  $e \in A$ , one may find an  $e' \in B$  such that  $F(e) \subseteq G(e')$ , that is  $(F, A) \sqsubseteq_g (G, B)$ . Note that the above argument also shows that  $(G, B) \sqsubseteq_g (F, A)$ , because  $(F, A) \approx_s (G, B)$  if and only if  $(G, B) \approx_s (F, A)$ . Hence we conclude that  $(F, A) \approx_g (G, B)$ .

**Proposition 3.7.** Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$ . If  $(F, A) \approx^s (G, B)$ , then  $(F, A) \approx_g (G, B)$ .

**Proof.** Following similar arguments to those given in Proposition 3.6, the result holds.

Now we give an example to show that if  $(F, A) \approx_g (G, B)$ , then the soft sets  $(F, A)$  and  $(G, B)$  are not necessarily lower soft equal or upper soft equal. Moreover this example shows that Definition 3.5 gives rise to the bigger class of soft subsets of a soft set.

**Example 3.8.** Suppose that  $U = \{h_1, h_2, h_3, h_4\}$  is a given universe and  $E = \{e_1, e_2, e_3, e_4\}$  a set of parameters. Put  $A = \{e_1, e_2, e_3\}$ , and  $B = \{e_1, e_2, e_4\}$ . Soft sets  $(F, A)$  and  $(G, A)$  are given as:

$$\begin{aligned}(F, A) &= \{(e_1, \{h_1, h_2\}), (e_2, \{h_3, h_4\}), (e_3, \emptyset)\} \text{ and} \\(G, B) &= \{(e_1, \{h_3, h_4\}), (e_2, \{h_1, h_2\}), (e_4, \emptyset)\},\end{aligned}$$

where

$$\begin{aligned}F(e_1) &= \{h_1, h_2\}, F(e_2) = \{h_3, h_4\}, F(e_3) = \emptyset, \text{ and} \\G(e_1) &= \{h_3, h_4\}, G(e_2) = \{h_1, h_2\} \text{ and } G(e_4) = \emptyset.\end{aligned}$$

As  $A \not\subseteq B$  and  $F(e_1) \not\subseteq G(e_1)$ , by Definition 2.3,  $(F, A)$  is not a soft subset of  $(G, A)$ . Note that  $F(e_1) = G(e_2)$  and  $F(e_2) = G(e_1)$ . Therefore we have  $(F, A) \sqsubseteq_g (G, B)$  and  $(G, A) \sqsubseteq_g (F, B)$  which implies that  $(F, A) \approx_g (G, B)$ . Also,  $F(e_1) \neq G(e_1)$  and  $F(e_2) \neq G(e_2)$ . Therefore, neither  $(F, A) \approx_s (G, B)$  nor  $(F, A) \approx^s (G, B)$  hold true. That is, generalized soft equality does not imply lower and upper soft equality.

**Proposition 3.9.** For any soft set  $(F, B)$  over  $U$  we have

$$(F_\emptyset, A) \sqsubseteq_g (F, B) \sqsubseteq_g (F_U, E).$$

**Proof.** If  $A = \emptyset$ , then assertion holds trivially. Let  $A \neq \emptyset$ , then for all  $e \in A$

$$F_\emptyset(e) = \emptyset \subseteq F(e') \text{ for all } e' \in B$$

and for any  $e$  in  $B$ , we have

$$F(e) \subseteq U = F_U(e') \text{ for all } e' \in E.$$

Hence

$$(F_\emptyset, A) \sqsubseteq_g (F, B) \sqsubseteq_g (F_U, E).$$

#### 4. Lattice Structure on the Soft Sets

In this section, we study soft algebraic operations  $\cap, \cup_{\mathfrak{R}}, \widetilde{\cup}, \Pi_\varepsilon$  with reference to  $g$ -soft equality relation  $\cong_g$ . We also give a lattice structure on a class of soft sets.

**Proposition 4.1.** Let  $(F, A)$  be any soft set over  $U$ , then

- (a)  $(F, A) \cap (F_U, E) \cong_g (F, A)$
- (b)  $(F, A) \cup_{\mathfrak{R}} (F_\emptyset, A) \cong_g (F, A)$ .

**Proof.** If  $A = \emptyset$ , then  $(F, A) = (F_\emptyset, A)$  and (a) and (b) hold true. If  $A \neq \emptyset$ , then  $(F, A) \cap (F_U, E) = (H, A \cap E)$ , with  $H$  defined as in Definition 2.10. Since for each  $e$  in  $A \cap E (= A)$ , we obtain

$$H(e) = F(e) \cap F_U(e) = F(e) \cap U = F(e),$$

then (a) follows. Similarly if  $(F, A) \cup_{\mathfrak{R}} (F_\emptyset, A) = (K, A)$ , then for each  $e$  in  $A$ , we have

$$K(e) = F(e) \cup F_\emptyset(e) = F(e) \cup \emptyset = F(e),$$

and (b) follows.

Then following theorem shows that the operation  $\widetilde{\cup}$  is idempotent, associative and commutative with respect to the  $g$ -soft equality relation  $\cong_g$ .

**Theorem 4.2.** If  $(F, A)$ ,  $(G, B)$  and  $(H, C)$  are soft sets over a common universe  $U$ , then

- (c)  $(F, A) \widetilde{\cup} (F, A) \cong_g (F, A)$ ,
- (d)  $(F, A) \widetilde{\cup} (G, B) \cong_g (G, B) \widetilde{\cup} (F, A)$ ,
- (e)  $[(F, A) \widetilde{\cup} (G, B)] \widetilde{\cup} (H, C) \cong_g (F, A) \widetilde{\cup} [(G, B) \widetilde{\cup} (H, C)]$ .

**Proof.** (c) Let  $(F, A) \widetilde{\cup} (F, A) = (K, A)$ , then  $K(e) = F(e)$ , hence (c) follows. It is straightforward to check (d). To prove (e), let

$$\begin{aligned} [(F, A) \widetilde{\cup} (G, B)] \widetilde{\cup} (H, C) &= (K_1, A \cup B) \widetilde{\cup} (H, C) = (K, D) \text{ and} \\ (F, A) \widetilde{\cup} [(G, B) \widetilde{\cup} (H, C)] &= (F, A) \widetilde{\cup} (L_1, B \cup C) = (L, D), \end{aligned}$$

where  $D = (A \cup B) \cup C = A \cup (B \cup C)$ . Let  $e \in D$ . Obviously  $e \in A$ , or  $e \in B$  or  $e \in C$ . First suppose that  $e \in C$ , then the following cases arise:

(e-i) If  $e \notin A$  and  $e \notin B$ , that is  $e \in C \setminus (A \cup B)$ , then  $K(e) = H(e)$ . Moreover  $e \in (B \cup C) \setminus A$  implies  $L(e) = L_1(e)$ . As  $e \in C \setminus B$ , so  $L_1(e) = H(e)$ . Consequently  $K(e) = L(e)$ .

(e-ii) If  $e \in A$  and  $e \notin B$ , then  $K_1(e) = F(e)$  and  $e \in A \cup B$  and  $e \in C$  implies  $K(e) = K_1(e) \cup H(e) = F(e) \cup H(e)$ . Moreover, since  $e \in C \setminus B$  we deduce that  $L_1(e) = H(e)$ . Since  $e \in A$  and  $e \in B \cup C$ , we obtain  $L(e) = F(e) \cup L_1(e) = F(e) \cup H(e)$ . Consequently  $K(e) = L(e)$ .

(e-iii) If  $e \notin A$  and  $e \in B$ , then  $K_1(e) = G(e)$ . If  $e \in A \cup B$  and  $e \in C$ , then we have  $K(e) = K_1(e) \cup H(e) = G(e) \cup H(e)$ . Since  $e \in B \cap C$ , this gives  $L_1(e) = G(e) \cup H(e)$ . Further  $e \notin A$  and  $e \in B \cup C$ , implies  $L(e) = L_1(e) = G(e) \cup H(e)$ . Consequently  $K(e) = L(e)$ .

(e-iv) If  $e \in A$  and  $e \in B$ , then  $K_1(e) = F(e) \cup G(e)$ . Also  $e \in A \cup B$  and  $e \in C$  implies  $K(e) = K_1(e) \cup H(e) = F(e) \cup G(e) \cup H(e)$ . As  $e \in B$  and  $e \in C$ , so  $e \in B \cap C$ , this implies  $L_1(e) = G(e) \cup H(e)$ . Since  $e \in A$  and  $e \in B \cup C$ , we obtain  $L(e) = F(e) \cup L_1(e) = F(e) \cup G(e) \cup H(e)$ . Consequently  $K(e) = L(e)$ .

Following arguments similar to those given in (e-i) to (e-iv), the result follows in each of the case when  $e \in B$  and  $e \in A$ . Since for all  $e \in D$ ,  $K$  and  $L$  are the same approximations, so we conclude that

$$[(F, A) \widetilde{\cup} (G, B)] \widetilde{\cup} (H, C) \cong_g (F, A) \widetilde{\cup} [(G, B) \widetilde{\cup} (H, C)].$$

**Theorem 4.3.** Let  $(F, A)$ ,  $(G, B)$  and  $(H, C)$  be soft sets over a common universe  $U$ . Then

- (f)  $(F, A) \sqcap_\epsilon (F, A) \cong_g (F, A)$ ,
- (g)  $(F, A) \sqcap_\epsilon (G, B) \cong_g (G, B) \sqcap_\epsilon (F, A)$ ,
- (h)  $[(F, A) \sqcap_\epsilon (G, B)] \sqcap_\epsilon (H, C) \cong_g (F, A) \sqcap_\epsilon [(G, B) \sqcap_\epsilon (H, C)]$ .

**Proof.** (f) Let  $(F, A) \sqcap_\epsilon (F, A) = (K, A)$ , then  $K(e) = F(e)$ , hence (f) follows. (g) is straightforward to check. To prove (h), let

$$\begin{aligned} [(F, A) \sqcap_\epsilon (G, B)] \sqcap_\epsilon (H, C) &= (K_1, A \cup B) \sqcap_\epsilon (H, C) = (K, D) \text{ and} \\ (F, A) \sqcap_\epsilon [(G, B) \sqcap_\epsilon (H, C)] &= (F, A) \sqcap_\epsilon (L_1, B \cup C) = (L, D), \end{aligned}$$

where  $D = A \cup B \cup C$ . Let  $e \in D$ . Obviously  $e \in A$ , or  $e \in B$  or  $e \in C$ . First suppose that  $e \in C$ , then there arise following cases:

(h-i) If  $e \notin A$  and  $e \notin B$ , that is  $e \in C \setminus (A \cup B)$ , then  $K(e) = H(e)$ . Moreover  $e \in (B \cup C) \setminus A$  implies  $L(e) = L_1(e)$ . As  $e \notin B$  and  $e \in C$ , that is  $e \in C \setminus B$ , so  $L_1(e) = H(e)$ . Consequently  $K(e) = L(e)$ .

(h-ii) If  $e \in A$  and  $e \notin B$ , then  $K_1(e) = F(e)$ , and  $e \in A \cup B$  and  $e \in C$  implies  $K(e) = K_1(e) \cap H(e) = F(e) \cap H(e)$ . Moreover, if  $e \in A$  and  $e \notin B$ , then  $L_1(e) = H(e)$ . Since  $e \in A$  and  $e \in B \cup C$ , we obtain  $L(e) = F(e) \cap L_1(e) = F(e) \cap H(e)$ . Consequently  $K(e) = L(e)$ .

(h-iii) If  $e \notin A$  and  $e \in B$ , then  $K_1(e) = G(e)$ , and  $e \in A \cup B$  and  $e \in C$  implies  $K(e) = K_1(e) \cap H(e) = G(e) \cap H(e)$ . Since  $e \in B \cap C$ , this gives  $L_1(e) = G(e) \cap H(e)$ . Further  $e \notin A$  and  $e \in B \cup C$ , implies  $L(e) = L_1(e) = G(e) \cap H(e)$ . Consequently  $K(e) = L(e)$ .

If  $e \in A$  and  $e \in B$ , then  $K_1(e) = G(e)$ , and  $e \in A \cup B$  and  $e \in C$  implies  $K(e) = K_1(e) \cup H(e) = G(e) \cup H(e)$ . Since  $e \in B \cap C$ , this gives  $L_1(e) = G(e) \cup H(e)$ . Further  $e \in A$  and  $e \in B \cup C$ , implies  $L(e) = L_1(e) = G(e) \cup H(e)$ . Consequently  $K(e) = L(e)$ .

(h-iv) If  $e \in A$  and  $e \in B$ , then  $K_1(e) = F(e) \cap G(e)$ , and  $e \in A \cup B$  and  $e \in C$  implies  $K(e) = K_1(e) \cap H(e) = F(e) \cap G(e) \cap H(e)$ . As  $e \in B$  and  $e \in C$ , this implies  $L_1(e) = G(e) \cap H(e)$ . Since  $e \in A$  and  $e \in B \cup C$ , we obtain  $L(e) = F(e) \cap L_1(e) = F(e) \cap G(e) \cap H(e)$ . Consequently  $K(e) = L(e)$ .

Following arguments similar to those given in (h-i) to (h-iv), the result follows in each of the case when  $e \in B$  and  $e \in A$ . Since for all  $e \in D$ ,  $K$  and  $L$  are the same approximations, so we conclude that

$$[(F, A) \sqcap_\epsilon (G, B)] \sqcap_\epsilon (H, C) \cong_g (F, A) \sqcap_\epsilon [(G, B) \sqcap_\epsilon (H, C)].$$

Note that Theorem 4.2 and Theorem 4.3 hold for the operations  $\cup_{\mathfrak{R}}$ , and  $\cap$  as well.

Following is an absorption law involving operations  $\cup$ ,  $\cap$  and  $g$ -soft equality.

**Theorem 4.4.** Let  $(F, A)$ ,  $(G, B)$  and  $(H, C)$  be soft sets over a common universe  $U$ . Then

- (i)  $[(F, A)\widetilde{\cup}(G, B)] \cap (F, A) \approx_g (F, A)$ .
- (j)  $[(F, A) \cap (G, B)]\widetilde{\cup}(F, A) \approx_g (F, A)$ .

**Proof.** Suppose

$$[(F, A)\widetilde{\cup}(G, B)] \cap (F, A) = (H, A \cup B) \cap (F, A) = (K, (A \cup B) \cap A) = (K, A).$$

So soft sets on both sides of (i) have the same parameter set  $A$ . Let  $e$  be an arbitrary element of  $A$ . If  $e \notin B$ , then  $H(e) = F(e) = H(e) \cap F(e) = K(e)$ . If  $e \in B$ , then  $H(e) = F(e) \cup G(e)$  and  $F(e) \subseteq H(e)$  which further implies that  $F(e) \subseteq H(e) \cap F(e) = K(e)$ , that is,  $F(e) \subseteq K(e)$ . Hence  $(F, A) \sqsubseteq_g (K, A)$ . On the other hand

$$K(e) = H(e) \cap F(e) = [F(e) \cup G(e)] \cap F(e) = F(e) \subseteq F(e),$$

implies that  $(K, A) \sqsubseteq_g (F, A)$ . Consequently  $(K, A) \approx_g (F, A)$ . Similarly, it can be shown that (j) holds true.

In the following theorem, we show that  $\widetilde{\cup}$  is distributive over  $\cap$ .

**Theorem 4.5.** Let  $(F, A)$ ,  $(G, B)$  and  $(H, C)$  be soft sets over a common universe  $U$ . Then

$$(k) (F, A)\widetilde{\cup}[(G, B) \cap (H, C)] \approx_g [(F, A)\widetilde{\cup}((G, B) \cap (H, C))].$$

**Proof.** Suppose

$$(F, A)\widetilde{\cup}[(G, B) \cap (H, C)] = (F, A)\widetilde{\cup}(K_1, (B \cap C)) = (K, A \cup (B \cap C)) = (K, D),$$

and

$$[(F, A)\widetilde{\cup}((G, B) \cap (H, C))] \cap (F, A) = (L_1, (A \cup B)) \cap (L_2, (A \cup C)) = (L, D),$$

where  $D = A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ . Now for all  $e \in D$ , it follows that  $e \in A$  or  $e \in B$ , and  $e \in A$  or  $e \in C$ . First suppose that  $e \in C$ , then there arise following cases:

(k-i) If  $e \in A$  and  $e \notin B$ , that is,  $e \in A \setminus (B \cap C)$ , then  $K(e) = F(e)$ . Since  $e \in A \setminus B$  and  $e \in A \cap C$ , so  $L_1(e) = F(e)$  and  $L_2(e) = F(e) \cup H(e)$ . Hence  $L(e) = L_1(e) \cap L_2(e) = F(e)$ . Consequently  $K(e) = L(e)$ .

(k-ii) If  $e \notin A$  and  $e \in B$ , that is,  $e \in (B \cap C) \setminus A$ , then  $K(e) = K_1(e) = G(e) \cap H(e)$ . Moreover,  $e \in B \setminus A$  and  $e \in C \setminus A$ , implies that  $L_1(e) = G(e)$  and  $L_2(e) = H(e)$ . Hence  $L(e) = L_1(e) \cap L_2(e) = G(e) \cap H(e)$ . Consequently  $K(e) = L(e)$ .

(k-iii) If  $e \in A$  and  $e \in B$ , then  $e \in A$  and  $e \in (B \cap C)$  implies that

$$K(e) = F(e) \cup K_1(e) = F(e) \cup [G(e) \cap H(e)].$$

Since  $e \in A \cap B$  and  $e \in A \cap C$ , so  $L_1(e) = F(e) \cup G(e)$  and  $L_2(e) = F(e) \cup H(e)$ . Hence

$$L(e) = L_1(e) \cap L_2(e) = F(e) \cup [G(e) \cap H(e)].$$

Consequently  $K(e) = L(e)$ .

The cases (k-i) to (k-iii) can be discussed for  $e \in B$  and  $e \in A$ . Since for all  $e \in A \cup (B \cap C)$ ,  $K$  and  $L$  are the same approximations, so we conclude that

$$(F, A)\widetilde{\cup}[(G, B) \cap (H, C)] \approx_g [(F, A)\widetilde{\cup}((G, B) \cap (H, C))].$$

Suppose that  $S(U, E)$  denotes the set of all soft sets over the common universe  $U$  and the parameter set  $E$ , that is,

$$S(U, E) = \{(F, A) : A \subseteq E \text{ and } F : A \rightarrow P(U)\}.$$

**Remark 4.6.** Let  $(F, A), (G, B) \in S(U, E)$ . If  $(F, A) \cong_g (G, B)$ , then  $(F, A) \cap (G, B) \cong_g (F, A)$  and  $(F, A) \widetilde{\cup} (G, B) \cong_g (G, B)$  do not hold true necessarily. See the following example.

**Example 4.7.** Let  $U = \{h_1, h_2\}$  be a universe under consideration and  $A = B = \{e_1, e_2\}$  is the set of parameters. Soft sets  $(F, A), (G, B)$  are given as:

$$\begin{aligned} (F, A) &= \{(e_1, \{h_1\}), (e_2, \{h_2\})\} \text{ and} \\ (G, B) &= \{(e_1, \{h_2\}), (e_2, \{h_1\})\} \end{aligned}$$

Clearly  $(F, A) \cong_g (G, B)$ . If  $(F, A) \cap (G, B) = (H, A \cap B)$  and  $(F, A) \widetilde{\cup} (G, B) = (K, A \cup B)$ , then

$$\begin{aligned} (H, A \cap B) &= \{(e_1, \emptyset), (e_2, \emptyset)\} \text{ and} \\ (K, A \cup B) &= \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_2\})\} \end{aligned}$$

Clearly  $(F, A) \cap (G, B) \not\cong_g (F, A)$  and  $(F, A) \widetilde{\cup} (G, B) \not\cong_g (G, B)$ .

Now we define soft ordering relation, denoted by  $\leq_s$  on  $S(U, E)$ . We say that  $(F, A) \leq_s (G, B)$  if and only if

$$(F, A) \cap (G, B) \cong_g (F, A) \text{ and } (F, A) \widetilde{\cup} (G, B) \cong_g (G, B).$$

The following example illustrates the fact that  $(F, A) \cap (G, B) \cong_g (F, A)$  does not always imply  $(F, A) \widetilde{\cup} (G, B) \cong_g (G, B)$ .

**Example 4.8.** Suppose that  $U = \{h_1, h_2, h_3, h_4\}$  and  $A = B = \{e_1, e_2\}$ . Soft sets  $(F, A)$  and  $(G, A)$  are given as:

$$\begin{aligned} (F, A) &= \{(e_1, \{h_3\}), (e_2, \{h_2, h_3\})\} \text{ and} \\ (G, B) &= \{(e_1, \{h_1, h_2\}), (e_2, \{h_2, h_3\})\} \end{aligned}$$

Suppose that  $(F, A) \cap (G, B) = (H, A \cap B)$  and  $(F, A) \widetilde{\cup} (G, B) = (K, A \cup B)$ . Note that

$$\begin{aligned} (H, A \cap B) &= \{(e_1, \emptyset), (e_2, \{h_2, h_3\})\} \text{ and} \\ (K, A \cup B) &= \{(e_1, \{h_1, h_2, h_3\}), (e_2, \{h_2, h_3\})\}. \end{aligned}$$

Clearly  $(F, A) \cap (G, B) \cong_g (F, A)$  but  $(F, A) \widetilde{\cup} (G, B) \not\cong_g (G, B)$  because  $K(e_1) \not\subseteq G(e)$  for any  $e \in B$ .

**Theorem 4.9.**  $(S(U, E), \widetilde{\cup}, \cap, \cong_g)$  is a distributive bounded lattice.

**Proof.** From Theorems 4.2, 4.3, 4.4 and 4.5, it follows that  $(S(U, E), \widetilde{\cup}, \cap, \cong_g)$  is a distributive lattice. As  $(F_\emptyset, \emptyset)$  and  $(F_U, E)$  are lower and upper bounds of  $S(U, E)$ , respectively so  $S(U, E)$  is a bounded lattice.

Note that  $(S(U, E), \cup_{\mathfrak{R}}, \cap_e, \cong_g)$  is also a distributive bounded lattice. Let  $A \subseteq E$  and

$$S_A(U, E) = \{(F, A) : F : A \rightarrow P(U)\}$$

be the set of all soft sets with parameter set  $A$  over a universe  $U$ . Then we have the following corollary.

**Corollary 4.10.**  $(S_A(U, E), \widetilde{\cup}, \cap, \cong_g)$  is a sublattice of  $(S(U, E), \widetilde{\cup}, \cap, \cong_g)$ .

**Proposition 4.11.** The soft ordering relation  $\leq_s$  is a tolerance relation on  $S(U, E)$  (i.e.,  $\leq_s$  is reflexive and symmetric).

**Proof.** Let  $(F, A), (G, B)$  and  $(H, C)$  be arbitrary elements of  $S(U, E)$ . Note that  $(F, A) \leq_s (F, A)$ , that is,  $\leq_s$  is reflexive. Now  $(F, A) \leq_s (G, B)$  implies that

$$(F, A) \cap (G, B) \cong_g (F, A) \text{ and } (F, A) \widetilde{\cup} (G, B) \cong_g (G, B)$$

and  $(G, B) \leq_s (F, A)$  implies that

$$(G, B) \cap (F, A) \cong_g (G, B) \text{ and } (G, B) \widetilde{\cup} (F, A) \cong_g (F, A).$$

As  $\cap$  and  $\widetilde{\cup}$  are commutative so  $(F, A) \cong_g (G, B)$ , that is  $\leq_s$  is symmetric.

The following example shows that  $\leq_s$  is not a transitive relation.

**Example 4.12.** Suppose that  $U = \{h_1, h_2, h_3, h_4\}$  and  $A = B = \{e_1, e_2\}$ . Let soft sets  $(F, A)$ ,  $(G, A)$ , and  $(H, A)$  be given by:

$$\begin{aligned} (F, A) &= \{(e_1, \{h_1\}), (e_2, \{h_2\})\} \\ (G, A) &= \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_2\})\} \text{ and} \\ (H, A) &= \{(e_1, \{h_3\}), (e_2, \{h_1, h_2, h_3\})\}. \end{aligned}$$

Let  $(F, A) \cap (G, A) = (J, A)$ ,  $(F, A) \cup (G, A) = (K, A)$ ,  $(G, A) \cap (H, A) = (L, A)$  and  $(G, A) \cup (H, A) = (M, A)$ . Note that

$$\begin{aligned} (J, A) &= \{(e_1, \{h_1\}), (e_2, \{h_2\})\} \\ (K, A) &= \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_2\})\} \\ (L, A) &= \{(e_1, \emptyset), (e_2, \{h_1, h_2\})\} \text{ and} \\ (M, A) &= \{(e_1, \{h_1, h_2, h_3\}), (e_2, \{h_1, h_2, h_3\})\}. \end{aligned}$$

Clearly  $(J, A) \cong_g (F, A)$  and  $(K, A) \cong_g (G, A)$  imply that  $(F, A) \leq_s (G, A)$ . Moreover  $(L, A) \cong_g (G, A)$  and  $(M, A) \cong_g (H, A)$  implies that  $(G, A) \leq_s (H, A)$ . Suppose that  $(F, A) \cap (H, A) = (N_1, A)$  and  $(F, A) \cup (H, A) = (N_2, A)$ . Note that

$$\begin{aligned} (N_1, A) &= \{(e_1, \emptyset), (e_2, \{h_2\})\} \text{ and} \\ (N_2, A) &= \{(e_1, \{h_1, h_3\}), (e_2, \{h_1, h_2, h_3\})\}. \end{aligned}$$

Since  $F(e_1) \not\subseteq N_1(e)$  for any  $e \in A$ , therefore  $(F, A) \not\subseteq_g (F, A) \cap (H, A)$ . This implies that  $(F, A) \not\cong_g (F, A) \cap (H, A)$ . Hence  $(F, A) \leq_s (G, A)$  and  $(G, A) \leq_s (H, A)$  but  $(F, A) \not\leq_s (H, A)$ .

**Proposition 4.13.**  $\cong_g$  is an equivalence relation on  $S(U, E)$ .

**Proof.** Let  $(F, A)$ ,  $(G, B)$  and  $(H, C)$  be arbitrary elements of  $S(U, E)$ . Then by definition  $(F, A) \cong_g (F, A)$ , hence  $\cong_g$  is reflexive. Also,  $(F, A) \cong_g (G, B)$  implies that  $(G, B) \cong_g (F, A)$ , that is  $\cong_g$  is symmetric. Suppose that  $(F, A) \cong_g (G, B)$  and  $(G, B) \cong_g (H, C)$ . Note that for any  $e \in A$  there exists an  $e' \in B$  such that  $F(e) \subseteq G(e')$  and for  $e' \in B$  there exists  $e'' \in C$  such that  $G(e') \subseteq H(e'')$ . Hence for every  $e \in A$  there is  $e'' \in C$  such that  $F(e) \subseteq H(e'')$ , thus  $(F, A) \subseteq_g (H, C)$ . Following similar arguments, we have  $(H, C) \subseteq_g (F, A)$ . Hence  $(F, A) \cong_g (H, C)$ .

From Definition 2.11, it follows that for any soft set  $(F, A)$ ,  $((F, A)^r)^r = (F, A)$  holds. Also, De Morgan's laws hold in soft set theory employing the concept of a  $g$ -soft equality relation  $\cong_g$ .

**Theorem 4.14.** Let  $(F, A)$ , and  $(G, B)$  be soft sets over a common universe  $U$  such that  $A \cap B \neq \emptyset$ , then

- (1)  $((F, A) \cup_{\mathfrak{R}} (G, B))^r \cong_g (F, A)^r \cap (G, B)^r$ .
- (2)  $((F, A) \cap (G, B))^r \cong_g (F, A)^r \cup_{\mathfrak{R}} (G, B)^r$ .

**Proof.** Suppose that

$$\begin{aligned} (F, A) \cup_{\mathfrak{R}} (G, B) &= (H, A \cap B), \\ ((F, A) \cup_{\mathfrak{R}} (G, B))^r &= (H^r, A \cap B), \\ (F, A)^r \cap (G, B)^r &= (K, A \cap B). \end{aligned}$$

Now for  $e \in A \cap B$ , we have

$$H^r(e) = U \setminus H(e) = U \setminus [F(e) \cup G(e)] = (U \setminus F(e)) \cap (U \setminus G(e)) = K(e).$$

Since for all  $e \in A \cap B$ ,  $H^r$  and  $K$  are same approximations, so we conclude that

$$((F, A) \cup_{\mathfrak{R}} (G, B))^r \cong_g (F, A)^r \cap (G, B)^r.$$

Now from (1), we obtain that

$$((F, A)^r \cup_{\mathfrak{R}} (G, B)^r)^r \cong_g ((F, A)^r) \cap ((G, B)^r)^r \cong_g (F, A) \cap (G, B).$$

Hence

$$(F, A)^r \cup_{\mathfrak{R}} (G, B)^r \cong_g ((F, A) \cap (G, B))^r.$$

**Theorem 4.15.** Let  $(F, A)$ , and  $(G, B)$  be soft sets over a common universe  $U$  such that  $A \cap B \neq \emptyset$ , then

- (1)  $((F, A) \widetilde{\cup} (G, B))^r \cong_g (F, A)^r \sqcap_{\epsilon} (G, B)^r$ .
- (2)  $((F, A) \sqcap_{\epsilon} (G, B))^r \cong_g (F, A)^r \widetilde{\cup} (G, B)^r$ .

**Proof.** Suppose that

$$\begin{aligned} (F, A) \widetilde{\cup} (G, B) &= (H, A \cup B), \\ ((F, A) \widetilde{\cup} (G, B))^r &= (H^r, A \cup B), \text{ and} \\ (F, A)^r \sqcap_{\epsilon} (G, B)^r &= (K, A \cup B). \end{aligned}$$

Now for  $e \in A \cup B$ , if  $e \in A$  and  $e \notin B$ , then  $H^r(e) = U \setminus H(e) = U \setminus F(e)$  and  $K(e) = F^r(e) = U \setminus F(e)$ . If  $e \notin A$  and  $e \in B$ , then  $H^r(e) = U \setminus H(e) = U \setminus G(e)$  and  $K(e) = G^r(e) = U \setminus G(e)$ . If  $e \in A$  and  $e \in B$ , then

$$H^r(e) = U \setminus H(e) = U \setminus [F(e) \cup G(e)] = (U \setminus F(e)) \cap (U \setminus G(e)),$$

and

$$K(e) = F^r(e) \cap G^r(e) = (U \setminus F(e)) \cap (U \setminus G(e)).$$

Hence

$$(H^r, A \cup B) = (K, A \cup B).$$

As for all  $e \in A \cup B$ ,  $H^r$  and  $K$  are the same approximations, so we conclude that

$$((F, A) \widetilde{\cup} (G, B))^r \cong_g (F, A)^r \sqcap_{\epsilon} (G, B)^r.$$

Now from (1) we obtain

$$((F, A)^r \widetilde{\cup} (G, B)^r)^r \cong_g ((F, A)^r \sqcap_{\epsilon} (G, B)^r)^r \cong_g (F, A) \sqcap_{\epsilon} (G, B).$$

Hence

$$(F, A)^r \widetilde{\cup} (G, B)^r \cong_g ((F, A) \sqcap_{\epsilon} (G, B))^r.$$

**Theorem 4.16.** [21, Theorem 24-26] Let  $(F, A)$  and  $(G, B)$  be two soft sets over a common universe  $U$ . Then  $(F, A) \approx_s (G, B)$  if and only if

- (a)  $(F, A) \widetilde{\cup} (G, B) \approx_s (F, A) \cap (G, B)$ ,
- (b)  $(F, A) \widetilde{\cup} (G, B) \approx_s (F, A) \sqcap_{\epsilon} (G, B)$ ,
- (c)  $(F, A) \cup_{\mathfrak{R}} (G, B) \approx_s (F, A) \cap (G, B)$ ,
- (d)  $(F, A) \cup_{\mathfrak{R}} (G, B) \approx_s (F, A) \sqcap_{\epsilon} (G, B)$ .

Theorem 4.16 does not hold if we replace  $\approx_s$  with soft equality  $\cong_g$ . Following example illustrates the fact.

**Example 4.17.** Suppose that  $U = \{h_1, h_2, h_3, h_4\}$  and  $A = \{e_1, e_2\}$ . Soft sets  $(F, A)$  and  $(G, A)$  are given as:

$$\begin{aligned}(F, A) &= \{(e_1, \{h_1, h_2\}), (e_2, \{h_3, h_4\})\} \text{ and} \\ (G, A) &= \{(e_1, \{h_3, h_4\}), (e_2, \{h_1, h_2\})\}.\end{aligned}$$

Clearly  $F(e_1) \cup G(e_1) = F(e_2) \cup G(e_2) = U$  and  $F(e_1) \cap G(e_1) = F(e_2) \cap G(e_2) = \emptyset$ . Hence

$$\begin{aligned}(F, A) \widetilde{\cup} (G, A) &\not\cong_g (F, A) \cap (G, A), \\ (F, A) \widetilde{\cup} (G, A) &\not\cong_g (F, A) \sqcap_\varepsilon (G, A), \\ (F, A) \cup_{\mathfrak{R}} (G, A) &\not\cong_g (F, A) \cap (G, A), \\ (F, A) \cup_{\mathfrak{R}} (G, A) &\not\cong_g (F, A) \sqcap_\varepsilon (G, A).\end{aligned}$$

**Remark 4.18.** The lower soft equality relation  $\approx_s$  is a congruence relation ([21, Theorem 28]) that is  $(F, A) \approx_s (G, A)$  and  $(H, A) \approx_s (I, A)$  imply that  $(F, A) \cap (H, A) \approx_s (G, A) \cap (I, A)$  and  $(F, A) \widetilde{\cup} (H, A) \approx_s (G, A) \widetilde{\cup} (I, A)$ , while  $\cong_g$  is not a congruence relation. To see it consider the following example.

**Example 4.19.** Suppose that  $U = \{h_1, h_2, h_3, h_4\}$  and  $A = \{e_1, e_2\}$ . Soft sets  $(F, A)$ ,  $(G, A)$ ,  $(H, A)$  and  $(I, A)$  are given as:

$$\begin{aligned}(F, A) &= \{(e_1, U), (e_2, \{h_1, h_3\})\}, \\ (G, A) &= \{(e_1, \{h_2, h_4\}), (e_2, U)\}, \\ (H, A) &= \{(e_1, \{h_1, h_2\}), (e_2, \{h_3, h_4\})\}, \text{ and} \\ (I, A) &= \{(e_1, \{h_3, h_4\}), (e_2, \{h_1, h_2\})\}.\end{aligned}$$

Clearly  $(F, A) \cong_g (G, A)$  and  $(H, A) \cong_g (I, A)$ . Now let  $(F, A) \cap (H, A) = (J, A)$ ,  $(G, A) \cap (I, A) = (K, A)$ ,  $(F, A) \widetilde{\cup} (H, A) = (L, A)$  and  $(G, A) \widetilde{\cup} (I, A) = (M, A)$ . Now

$$\begin{aligned}(J, A) &= \{(e_1, \{h_1, h_2\}), (e_2, \{h_3\})\}, \\ (K, A) &= \{(e_1, \{h_4\}), (e_2, \{h_1, h_2\})\}, \\ (L, A) &= \{(e_1, U), (e_2, \{h_1, h_3, h_4\})\}, \text{ and} \\ (M, A) &= \{(e_1, \{h_2, h_3, h_4\}), (e_2, U)\}.\end{aligned}$$

Note that  $(J, A) \not\cong_g (K, A)$  and  $(L, A) \not\cong_g (M, A)$ . Hence  $(F, A) \cap (H, A) \not\cong_g (G, A) \cap (I, A)$  and  $(F, A) \widetilde{\cup} (H, A) \not\cong_g (G, A) \widetilde{\cup} (I, A)$ . Consequently  $\cong_g$  is not a congruence relation.

## 5. Conclusion

In this manuscript, we introduced the concepts of  $g$ -null soft set and  $g$ -soft subset of a soft set along with the notion of  $g$ -soft equality relation  $\cong_g$  between the soft sets. It is shown that  $g$ -soft equality relation  $\cong_g$  generalizes existing comparable concepts about equality of soft sets. Moreover we gave example to show that  $\cong_g$  gives rise to the bigger class of soft subsets which ultimately will refine the bases for soft topological spaces. Furthermore, we give algebraic structure (lattice structure with order  $\leq_s$  on the class of soft sets) with  $g$ -soft equality relation  $\cong_g$  and already existing operations on union and intersection of soft sets. It is proved that order relation  $\leq_s$  is dependence relation and  $g$ -soft equality relation is an equivalence relation on the collection of soft sets. Some examples have been provided as illustrations and for comparisons. The new operations on soft sets will be important basis for the further developments on soft set theory. In our future research, we intend to establish more properties on soft sets and topological concepts.

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