DSGE Model-Based Forecasting of Modeled and Non-Modeled Inflation Variables in South Africa [*]

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Abstract

Inflation forecasts are a key ingredient for monetary policymaking - especially in an inflation targeting country such as South Africa. Generally, a typical Dynamic Stochastic General Equilibrium (DSGE) only includes a core set of variables. As such, other variables, *e.g.* such as alternative measures of inflation that might be of interest to policymakers, do not feature in the model. Given this, we implement a closed-economy New Keynesian DSGE model-based procedure which includes variables that do not explicitly appear in the model. We estimate such a model using an in-sample covering 1971Q2 to 1999Q4, and generate recursive forecasts over 2000Q1-2011Q4. The hybrid DSGE performs extremely well in forecasting inflation variables (both core and non-modeled) in comparison with forecasts reported by other models such as AR(1).

Keywords: DSGE model, inflation, core variables, non-core variables
JEL Codes: C11, C32, C53, E27, E47

1   Introduction

Forecasting inflation is a key component of an inflation targeting central bank such as the case of the South African Reserve Bank, which adopted an inflation targeting framework since the February of 2000. Essentially, a typical inflation targeting central bank uses its monetary policy instruments (for example, the Repurchase (Repo) rate in South Africa) to bring inflation forecasts close to the inflation target (Croce and Khan, 2000), which happens to be between 3 percent and 6 percent for South Africa. Given this backdrop, accurate forecasting of inflation is important in the conduct of monetary policy.

As tools for forecasting and policy analysis, Central Banks propose different Dynamic Stochastic General Equilibrium (DSGE) models[1] Schorfheide *et al.* (2010) argue that, unlike traditional

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system of equations models, the DSGE paradigm provides empirical models with a strong degree of theoretical coherence. However, this comes at a cost: First, tight cross-equation restrictions result in inferior fit compared to less restrictive time series models. Second, DSGE models only feature a core set of macroeconomic variables *e.g.* real GDP, consumption, investment as well as wages, hours worked, inflation and interest rate. Also, it is relatively easier to incorporate additional variables of interest in traditional econometric models compared to DSGE models.

That being said, in practice, forecasts for macroeconomic variables that do not feature in the DSGE model could be of interest to a researcher (Schorfheide *et al.*, 2010). For instance, core inflation, which excludes goods with volatile prices *e.g.* food and energy, is a gauge of the underlying long-term trend of inflation. However, a typical DSGE specification does not include this variable. In the same way, although unemployment is not included in a typical DSGE model framework, it remains a key macroeconomic variable. For instance, unemployment is a persistent concern in the South African economy (Amusa *et al.,* 2013). As such, unemployment forecasts will provide a framework for assessing the effectiveness of policy geared at job creation. Further, neither is a variable, such as building plans passed, is included in a typical DSGE model. However as shown by Aye *et al.,* (2013), such a variable have historically played an important role in predicting South African business cycles.

To address this issue, we implement the Schorfheide *et al.* (2010) DSGE model-based method for forecasting variables that do not explicitly feature in the model. Essentially, Schorfheide *et al.* (2010) use auxiliary equations to link the so-called non-core variables to the state variables of the DSGE model. To apply this method, we proceed in three steps: (1) Bayesian estimation of the DSGE model using the core variables as measurements; (2) based on the DSGE model parameter estimates, we apply the Kalman filter to obtain estimates of the latent state variables given the most recent information set; (3) we then use the filtered state variables as regressors to estimate simple linear measurement equations with serially correlated idiosyncratic errors.

Our paper contributes to the existing and growing literature of forecasting macroeconomic variables for South Africa using DSGE models, by being the first study to forecast both core and non-core variables based on a DSGE model. According to Liu and Gupta (2007); Liu *et al.* (2009); Liu *et al.* (2010) and Gupta and Kabundi (2011), Bayesian Vector Autoregressive (BVAR) models tend to outperform DSGE models in forecasting key macroeconomic variables. Nonetheless, Steinbach *et al.* (2009); Gupta and Kabundi (2010); Alpanda *et al.* (2011); and Gupta and Steinbach (2013) show that DSGE models can compete with BVAR models when the DSGE framework allows for open economy features and various kinds of nominal and real rigidities. Also, Balcilar *et al.* (2013) apply a nonlinear DSGE model to South African macroeconomic data, to show that the nonlinear DSGE model outperforms the linear model as well as a selection of VAR models, thus highlighting the importance of incorporating nonlinearities in a DSGE framework to account for regular structural changes in an emerging economy like South Africa.

The remainder of the paper is structured as follows: Section 2 presents the theoretical framework of the DSGE model used in the empirical analysis. In Sections 3 and 4, we discuss the econometric methodology and empirical results, respectively. Section 5 concludes the paper.

2 Econometric Methodology

Consequently, the state space of the DSGE model[2] is given by equation (25) and equation (28).

We follow the same econometric procedure in three steps implemented by Schorfheide *et al.* (2010).

- We use Bayesian methods to estimate the linearized DSGE model on the seven core macroeconomic time series (output growth rate, consumption growth rate, investment growth rate, nominal wage growth rate, $100\times$ log hours, inflation, interest rates).

- We estimate so-called auxiliary regression equations that link the state variables associated with the DSGE model to various other macroeconomic variables which are of interest to the researcher but are not explicitly included in the structural DSGE model (non-core variables such as PCE inflation, core PCE inflation, unemployment rate, building plans passed).

- We use the estimated DSGE model to forecast its state variables, and then map these state forecasts into predictions for the core and non-core variables.

2.1 Bayesian Estimation of the DSGE model

Assuming that the innovations in equation (25), ϵ_t, are normally distributed, the likelihood function for the DSGE model ($p(Y^T|\theta)$, where Y^T is a sequence of observations) can be evaluated using the Kalman Filter.

The Kalman Filter also generates a sequence of estimates of the state vector ς_t:

$$\varsigma_{t|t}(\theta) = E\left[\varsigma_t|\theta, Y^t\right], \tag{1}$$

where $Y^t = [y_1, .., y_t]$. The Bayesian estimation of the DSGE model combines a prior $p(\theta)$ with the likelihood function $p(Y^T|\theta)$ to obtain a joint probability density function for data and parameters.

The posterior distribution is given by:

$$p(\theta|Y^T) = \frac{p(Y^T|\theta)p(\theta)}{p(Y^T)} \tag{2}$$

where

$$p(Y^T) = \int p(Y^T|\theta)p(\theta)d\theta \tag{3}$$

The posterior is solved using the Markov Chain Monte Carlo (MCMC) methods as employed in Schorfheide *et al.* (2010) and described in details in An and Schorfheide (2007). From the posterior distribution $p(\theta|Y^T)$, we generate draws using a random-walk Metropolis Hastings algorithm.

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[2]As suggested by the referee, we moved the section discussing the DSGE framework to the Appendix

Table 1: Non-modelled and related DSGE model variables

Non-core variable	DSGE model variable	Transformation
PCE Inflation	Final Good Inflation π_t	None
Core PCE Inflation	Final Good Inflation π_t	None
Unemployment Rate	Hours Worked L_t	$0.05L_t$
Building Plans Passed	Investment i_t	$-0.06i_t$

2.2 Non-core variables

The DSGE presented in Section 2 can predict for hours worked but does not include unemployment as variable in the model. We assume Z_t is a variable which is not included in the DSGE model (non-core variable) even if it is of interest in a forecasting exercise.

Considering equation (27), we can recover S_t from the larger vector ς_t using a selection matrix M with the property $S_t = M\varsigma_t$.

Using the Kalman Filter, we use $\widehat{\varsigma}_{t|t}$ to denote an estimate of the sequence $\varsigma_{t|t}(\theta)$, obtained by replacing θ with the posterior mean estimate $\widehat{\theta}_T$.

Hence, $\widehat{S}_{t|t} = M\widehat{\varsigma}_{t|t}$ and the auxiliary regression is as follows:

$$
\begin{aligned}
Z_t &= \alpha_0 + \widehat{S}'_{t|t}\alpha_1 + \xi_t \\
\xi_t &= \rho\xi_{t-1} + \eta_t \\
\eta_t &\sim N(0, \sigma_\eta^2)
\end{aligned}
\tag{4}
$$

where ξ_t is a variable-specific noise process. We estimate auxiliary regression using Bayesian methods.

This procedure follows the setup proposed by Schorfheide *et al.* (2010). The equation (4) can be re-written in a quasi-differenced form as:

$$
\begin{aligned}
Z_1 &= \alpha_0 + \widehat{S}'_{1|1}\alpha_1 + \xi_1 \\
Z_t &= \rho Z_{t-1} + \alpha_0(1-\rho) + [\widehat{S}'_{t|t} - \widehat{S}'_{t-1|t-1}\rho]\alpha_1 + \eta_t \\
t &= 2, ..., T
\end{aligned}
\tag{5}
$$

We use the DSGE model to derive a prior distribution for the α's for any Z_t variables which are linked to variables that are core in the DSGE. The prior means are based on the DSGE model's implied factor loadings for a model variable (the core variable) linked to the non-core variable. We link the two measures of PCE inflation to the final good inflation π_t, the unemployment rate to a scaled version of log hours worked L_t, and building plans passed to scaled percentage deviations i_t from its trend path.

Moreover, the priors take form as shown in Schorfheide *et al.* (2010):

$$\begin{aligned}
\alpha &\sim N(\mu_{\alpha,0}, V_{\alpha,0}) \\
\rho &\sim U(-1,1) \\
\sigma_\eta &\sim IG(\nu, \tau)
\end{aligned}$$

We construct $\mu_{\alpha,0}$ using a population regression of the form:

$$\mu_{\alpha,0} = \left(E_\theta^D \left[\widetilde{S}_t \widetilde{S}_t' \right] \right)^{-1} E_\theta^D \left[\widetilde{S}_t Z_t^{**} \right] \tag{6}$$

where $\widetilde{S}_t = [1, S_t']'$, θ is replaced by its posterior mean $\widehat{\theta}_T$. E_θ^D is the expectation taken under the probability distribution generated by the DSGE model, conditional on the parameter vector θ. Z_t^{**} is the variable from the DSGE chosen in the linkage between core and non-core variables.

The prior covariance matrix is diagonal with the following elements:

$$diag(V_{\alpha,0}) = \left[\lambda_0, \frac{\lambda_1}{\omega_1}, ..., \frac{\lambda_1}{\omega_j} \right]$$

where λ_0 and λ_1 are hyperparameters that determine the degree of shrinkage for the intercept α_0 and the loadings α_1 of the state variable. We scale the diagonal elements of $V_{\alpha,0}$ by ω_j^{-1}, $j = 1, ...J$, where ω_j denotes the DSGE model's implied variance of the j_{th} element of $\widehat{S}_{t|t}$, as shown in Schorfheide *et al.* (2010).

The procedure implemented can be interpreted as a factor model. The factors are given by the state variables of the DSGE model, while the measurement equation associated with the DSGE model describes how the core variables load on the factors. The random variable ξ_t in equation (4) is an idiosyncratic error term. This setup introduced by Schorfheide *et al.* (2010) is a simplified version of the approach introduced by Boivin and Giannoni (2006). The Boivin and Giannoni (2006) methodology implies that factors are estimated as endogenous in the DSGE model, with a computationally expensive procedure. Another way to introduce the idea of factors, combined with DSGE model, is the DSGE-FAVAR as shown in Consolo *et al.* (2009). The DSGE-FAVAR combines the hybrid DSGE-VAR à la Del Negro and Schorfheide (2004) with a Factor-Augmented VAR representation, considering factors as exogenous variable respect to the DSGE model. The auxiliary equations procedure of Schorfheide *et al.* (2010) is an alternative way to introduce the latent variables with the linkage between the core macroeconomic variables from the DSGE and the non-core variables.

2.2.1 Forecasting

Forecasts from the DSGE model are generated by sampling from the posterior predictive distribution of y_{T+h}. For each posterior draw $\theta^{(i)}$ we start from $\widehat{\varsigma}_{T|T}(\theta^{(i)})$ and draw a random sequence $\left\{ \epsilon_{T+1}^{(i)}, ..., \epsilon_{T+h}^{(i)} \right\}$. We then iterate the state transition equation forward to construct:

$$\begin{aligned}
S_{T+h|T}^{(i)} &= \Phi_1(\theta^{(i)}) S_{T+h-1|T}^{(i)} + \Phi_\epsilon(\theta^{(i)}) \epsilon_{T+h}^{(i)}, \\
h &= 1, ..., H \\
\varsigma_{T+h|T}^{(i)} &= [S_{T+h|T}^{(i)'}, S_{T+h-1|T}^{(i)'} M_s'(\theta^{(i)})]'
\end{aligned} \tag{7}$$

The measurement equation is employed to compute:

$$Y_{T+h|T}^{(i)} = A_0(\theta^{(i)}) + A_1(\theta^{(i)})\varsigma_{T+h|T}^{(i)}. \tag{8}$$

The posterior mean forecast $\widehat{Y}_{T+h|T}$ is given by averaging the $Y_{T+h|T}^{(i)}$s.

As regards the non-core variable Z_{T+h}, a draw from the posterior predictive distribution is obtained as follows. Using the sequence $S_{T+1|T}^{(i)}, ..., S_{T+H|T}^{(i)}$ constructed in equation (7), we iterate the quasi-differenced version, equation (5) of the auxiliary regression forward:

$$\begin{aligned} Z_{T+h|T}^{(i)} &= \rho^{(i)} Z_{T+h-1}^{(i)} + \alpha_0^{(i)}(1 - \rho^{(i)}) \\ &\quad + [S_{T+1|T}^{(i)'} - S_{T+h-1|T}^{(i)'}\rho^{(i)}]\alpha_1^{(i)} + \eta_{T+h}^{(i)} \end{aligned}$$

where the superscript i for the parameters of equation (4) refers to the i_{th} draw from the posterior distribution of ψ, and $\eta_{T+h}^{(i)}$ is a draw from a $N(0, \sigma_\eta^{2(i)})$. The point forecast $\widehat{Z}_{T+h|T}$ is given by averaging the $Z_{T+h|T}^{(i)}$s. We assume that the draws from the posterior distribution of θ and ψ are independent; instead, we assume correlation in the joint predictive distribution of Y_{T+h} and Z_{T+h}, because the i_{th} draw is computed from the same realization of the state vector $S_{T+h|T}^{(i)}$.

3   Empirical Results

3.1   Data Description

We include seven variables, measured at a quarterly frequency, in the vector of core variables that is used for the estimation of the DSGE model: the growth rate of output, household consumption, capital investment, nominal wages, hours worked, inflation and the nominal interest rate. Based on data availability, we use the period 1971Q2-2011Q4 for our analysis. The data for these variables was obtained from the South African Reserve Bank, Actuarial Society of South Africa and Statistics South Africa. Real output is computed by dividing current gross domestic product by the South African population of 16 years and older[3] as well as the GDP deflator. Household consumption is defined as nominal final consumption expenditure by households minus nominal final household consumption expenditure of durable goods. The resulting variable is then divided by the population of 16 years and older and deflated by the GDP deflator. Capital investment is defined as nominal gross fixed capital formation plus nominal final household consumption expenditure of durable goods. Investment is also deflated by using the population and the GDP deflator. Because there is no complete series of hours worked in South Africa, we follow a methodology used by Touna-Mama and Viegi (2012), where they use a production function method to extract hours worked from a series of capacity utilization in the manufacturing sector. The basic idea is linking capacity utilization to hours worked, considering that at business cycle frequency the only variable is labor. We then take the log of the series and multiplied by 100 so that all values can be interpreted as percentage deviations from the mean. Nominal wages are computed by dividing the total wage bill by the product of the population measure and the computed measure of average hours. The

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[3]The population data for South Africa is only available annually and as a result, the annual data was interpolated to obtain quarterly data.

inflation rate is derived as the log difference of the GDP deflator and converted into percentages. For the nominal interest rate, we use the annualized 91-day Treasury bill rate.

For non-core variables – which are also obtained from the South Africa Reserve Bank and Statistics South Africa, we consider household consumption expenditure inflation, core household expenditure inflation, the (first-differenced) unemployment rate, and the building plans passed, since unlike Shorfheide *et al.,* (2010), South Africa does not have data on housing starts. For the household consumption expenditure inflation, we use final consumption expenditure by households, while for core household consumption expenditure inflation; we use final consumption expenditure by households excluding food, non-alcoholic beverages and energy[4]. The unemployment rate measure is the official unemployment rate of which is the proportion of the labor force[5] that is unemployed. The building plans passed variable is defined as the quarter-on-quarter growth rate of the value of recorded building plans passed by large municipalities. Figures 1 and 2 provides the plots for the core and non-core variables.

We use the period 1971Q2-2011Q4 for our analysis, which corresponds to a total sample of 163 observations on each series. We use the first 115 observations (1971Q2-1999Q4) for in-sample estimation, while the remaining 48 observations (2000Q1-2011Q4) are used for out-of-sample forecasting, the starting point of which follows the literature on forecasting using DSGE-models, and corresponds to the start of the inflation targeting regime.

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[4]The data for disaggregated household consumption expenditure for South Africa is only available annually and as a result, the annual data for food, non-alcoholic beverages and energy were interpolated to obtain quarterly data.

[5]The labor force comprises all persons who are employed and unemployed aged 15 to 64 years.
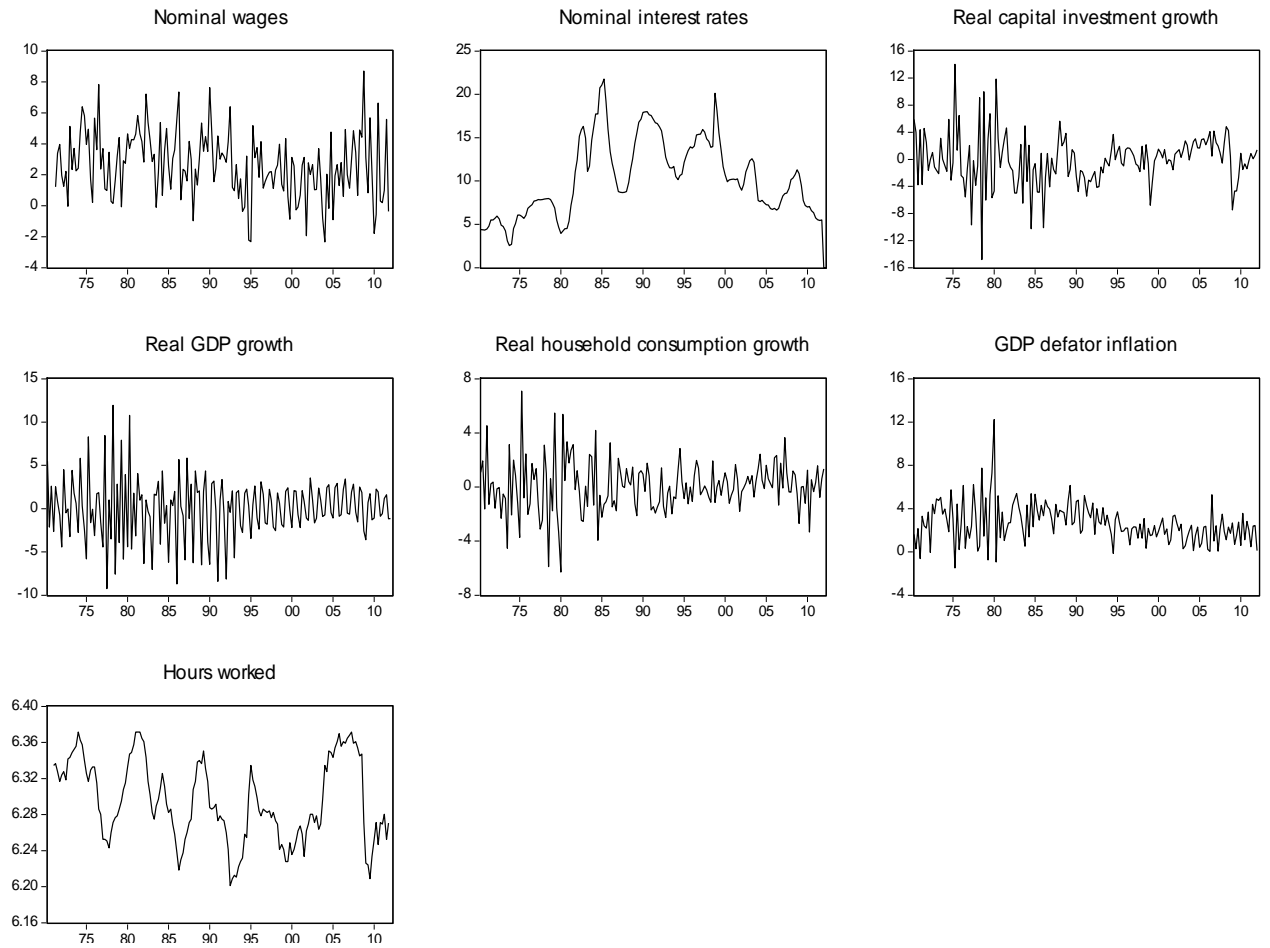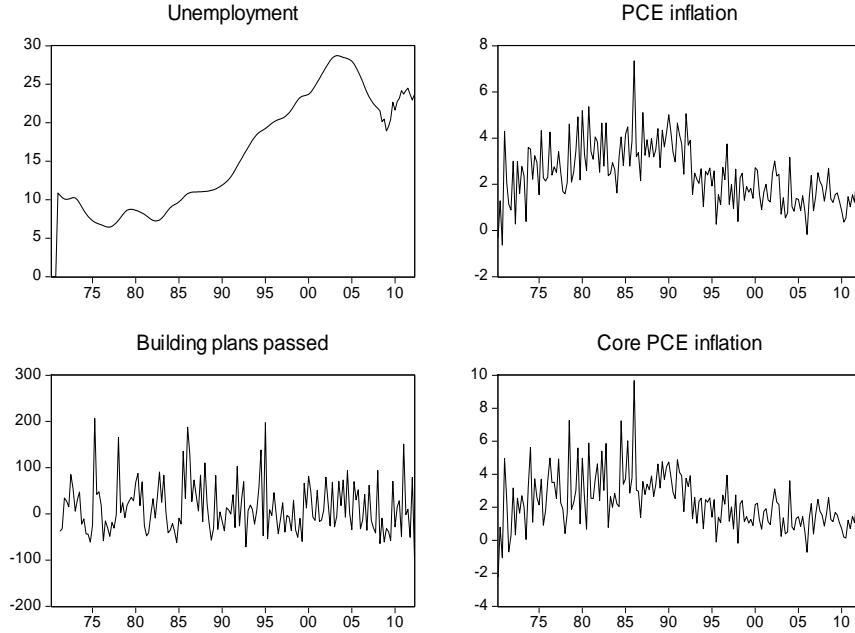
Figure 1: Core variables

Nominal wages

Nominal interest rates

Real capital investment growth

Real GDP growth

Real household consumption growth

GDP defator inflation

Hours worked

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Figure 2: Non-core variables

3.2 DSGE Priors

As far as the DSGE priors are concerned, we follow Schorfheide *et al.* (2010). We adjust priors of some parameters as in Alpanda *et al.* (2010) and Alpanda *et al.* (2011), considering the South African economy.

Para (1) and Para (2) in Tables 2 and 3 list the means and standard deviations for the Beta, Gamma, and Normal distributions: the upper and lower bound of the support for the Uniform distribution; and s and ν for the Inverse Gamma distribution where $p_{IG}\left(\sigma|\nu,s\right) \propto \sigma^{-(\nu+1)}e^{-\nu s^2/2\sigma^2}$. The joint prior distribution is obtained as a product of the marginal distributions tabulated in the table, with this product being truncated at the boundary of the determinacy region. Posterior summary statistics are computed based on the output of the posterior sampler. The following parameters are fixed: $\delta = 0.019$, $\lambda_w = 0.3$. Estimation sample: 1971Q2 to 2011Q4.

For the Bayesian Estimation of the equation (5), we set the τ hyperparameter (interpreted as the prior standard deviation of the idionsyncratic error ξ_1), to 0.12 (PCE inflation), 0.11 (core PCE inflation), 0.40 (unemployment rate), and 0.10 (housing starts). These values imply that the prior variance of ξ_1 is about 15% to 20% of the sample variance of Z_1. We set the degrees of freedom parameter ν of the inverted gamma prior for σ_η equal to 2, and we set for $\lambda_0 = \lambda_1 = \lambda$ three values: 1.00, 0.10, and 10^{-5} as implemented in Schorfheide *et al.* (2010). Note that the choice of $\lambda = 10^{-5}$ is to have a dogmatic prior under which the posterior estimate and prior mean coincide. Increasing λ, we allow the factor loading coefficients α to differ from the prior mean. In other words, higher is the value of λ, the more is the variation in the variable explained by the $s'_{t|t}\hat{\alpha}_1$, where $\hat{\alpha}_1$ is the posterior estimate of α_1. That is, higher is λ, the more information is contained in the recursively

estimated recovered states of the DSGE model based on the core variables. This is in fact what we ideally want when predicting the non-core variables, since otherwise a small value of λ would imply that the recovered states contain no information and the non-core variables can be predicted based on their AR(1) structure, as can be understood from equation 5.

Table 2: Prior and posterior of DSGE model parameters (Part 1)

Name	Prior			Posterior
	Density	Para (1)	Para (2)	Mean
Monetary Policy Parameters				
$400\pi_*$	Normal	3.00	1.50	4.71
ψ_1	Gamma	1.50	0.40	1.42
ψ_2	Gamma	0.20	0.10	0.003
ρ_R	Beta	0.20	2	0.40
Household				
h	Beta	0.70	0.05	0.72
a''	Gamma	0.20	0.10	0.16
ν_l	Gamma	2	0.75	0.42
ς_w	Beta	0.60	0.20	0.66
$400(1/\beta - 1)$	Gamma	2.00	1.00	1.11
Firms				
α	Beta	0.10	0.10	0.04
ς_p	Beta	0.60	0.20	0.65
S''	Gamma	10	1.50	12
λ_f	Gamma	0.08	0.15	0.10

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Table 3: Prior and posterior of DSGE model parameters (Part 2)

Name	Prior			Posterior
	Density	Para (1)	Para (2)	Mean
	Shocks			
400γ	Gamma	2	2	1.22
g_*	Gamma	0.30	0.10	0.27
ρ_a	Beta	0.2	0.1	0.02
ρ_μ	Beta	0.80	0.50	0.73
ρ_{λ_f}	Beta	0.60	0.20	0.99
ρ_g	Beta	0.80	0.05	0.86
ρ_b	Beta	0.60	0.20	0.98
ρ_ϕ	Beta	0.60	0.20	0.97
σ_a	Inverse Gamma	4	2	4.2
σ_μ	Inverse Gamma	0.75	2	0.92
σ_{λ_f}	Inverse Gamma	0.75	2	0.81
σ_g	Inverse Gamma	2	2	3.71
σ_b	Inverse Gamma	0.75	2	2.42
σ_ϕ	Inverse Gamma	4	2	4.80
σ_R	Inverse Gamma	0.20	2	0.40

3.3   Forecasting of Core Variables

Table 4 reports the out-of-sample root mean squared error (RMSE) statistics for the DSGE model's core variables, that is, growth rates of output, household consumption, capital investment as well as nominal wages, a measure of hours worked, the GDP deflator inflation and nominal interest rate for horizons $h = 1$, $h = 2$, $h = 4$, and $h = 12$.

We evaluate the DSGE model's forecasting performance by comparing the RMSEs associated with its forecasts to those associated with an AR(1) model recursively estimated by OLS.[6]

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[6]As in Schorfheide *et al.* (2010), the h-step forecast is generated by iterating one-step ahead predictions forward, ignoring parameter uncertainty: $\hat{y}_{i,T+h|T} = \hat{\beta}_{0,OLS} + \hat{\beta}_{1,OLS}\hat{y}_{i,T+h|T}$ where the OLS estimators are obtained from the regression $y_{i,t} = \beta_0 + \beta 1 y_{i,t-1} + u_{i,t}$.

Table 4: RMSEs comparison: DSGE model *vs.* AR(1)

Series	Model	h=1	h=2	h=4	h=12
Output growth (Q%)	DSGE	**1.78***	2.31	2.51***	2.35*
	AR(1)	2.12	2.13	1.78	1.86
Consumption growth (Q%)	DSGE	1.88***	2.00***	2.36***	1.88***
	AR(1)	1.32	1.31	1.33	1.41
Investment growth (Q%)	DSGE	**2.48**	3.34*	3.42	**2.35****
	AR(1)	2.72	2.60	2.67	2.93
Nominal wage growth (Q%)	DSGE	**0.02****	0.04**	0.07***	0.18***
	AR(1)	0.02	0.03	0.04	0.07
100xlog hours	DSGE	2.86**	2.74***	**2.24**	2.72
	AR(1)	2.43	2.43	2.45	2.53
Inflation (Q%)	DSGE	**1.50**	**1.46**	**1.39**	**1.28***
	AR(1)	1.53	1.53	1.58	1.67
Interest rates (A%)	DSGE	0.85*	1.47	2.61*	3.98*
	AR(1)	0.74	1.31	2.05	3.08

Note: We report RMSEs for the DSGE and AR(1) models. Numbers in boldface indicate a lower RMSE for the DSGE model. *, **, *** denote 10%, 5% or 1% significance of the two-sided modified Diebold-Mariano test of equal predictive accuracy under quadratic loss (DSGE vs AR(1)). The RMSEs are computed based on recursive estimates starting with the sample 1971Q2 to 1999Q4 and ending with the samples 1971Q2 to 2011Q3 ($h = 1$), 1971Q2 to 2011Q2 ($h = 2$), 1971Q2 to 2010Q4 ($h = 4$), and 1971Q2 to 2008Q4 (h=12), respectively. The h-step-ahead growth rate (inflation) forecasts refer to percentage changes between the periods $T+h$-1 and $T+h$.

Following Schorfheide *et al.* (2010), we use the Harvey, Leybourne, and Newbold (1998) variant of the Diebold-Mariano (1995) test for equal forecast accuracy of the DSGE and AR(1) models, using a quadratic loss function. On one hand, as shown in Table 4, the RMSE statistics for 1, 2, 4 and 12 quarters ahead forecasts of the GDP deflator, obtained from the estimated DSGE model are lower than the corresponding RMSEs of forecasts associated with the AR(1) model. Also, we note that, for the estimated DSGE models, the forecast accuracy improves at longer horizons. Therefore, compared to the AR(1) model, the DSGE model performs better in forecasting the GDP deflator inflation.

On the other hand, when comparing the DSGE model's forecasts of the other core variables and the corresponding forecasts associated with the AR(1), the outcome is mixed. For instance, the RMSE statistic for one quarter ahead forecast of the growth rate of output obtained from the estimated DSGE model is lower than the comparable RMSE statistic associated with the AR(1) model forecast. However, the AR(1) model performs better in forecasting the growth rate of output

at longer horizons, that is $h = 2$, $h = 4$, and $h = 12$. In the same vein, it appears that the DSGE model outperforms the AR(1) model in forecasting the growth rate of capital investment at $h = 1$ and $h = 12$. Considering $h = 1$, $h = 2$, $h = 4$, and $h = 12$, RMSE statistics for forecasts of household consumption growth obtained from the estimated DSGE model are higher than the comparable RMSEs associated with the AR(1) model's forecasts. Therefore, the DSGE models fails to outperform the AR(1) model in forecasting the growth in household consumption. Lastly the DSGE model's forecast of nominal wage growth and interest rate are better than the corresponding AR(1) model's forecast only for a one-quarter ahead horizon. Also, the DSGE model can forecast the hours worked better than the AR(1) model only for $h = 4$.[7]

3.4 Forecasting of Non-core Variables

Table 5 presents forecast error statistics for non-core variables - that is, personal consumption expenditure (PCE) inflation, core PCE inflation, the unemployment rate and building plans passed, obtained from estimated auxiliary regressions as in Schorfheide *et al.* (2010).

We compare the RMSE statistics of forecasts obtained from the auxiliary models to two alternative benchmark models: (1) an AR(1) model for Z_t estimated using OLS (we compute h-step forecasts by iterating one step ahead predictions forward) and (2) a multi-step least squares regression of the form: $z_t = \beta_0 + y'_{t-h}\beta_1 + z_{t-h}\beta_2 + u_t$ which we estimate for $h = 1$, $h = 2$, $h = 4$, and $h = 12$.

Generally, RMSE statistics associated with the auxiliary regression models forecasts of PCE inflation are consistently lower than the corresponding RMSEs associated with the AR(1) or multi-step regression model's forecast over longer horizons, that is $h = 4$, and $h = 12$. The preferred choice of λ varies depending on the horizon. To illustrate, $\lambda = 1$ is the preferred choice for a one-step ahead forecast. The larger the value of λ, the more of the variation in the variable is explained by the DSGE model's latent state variables. On the contrary, the lowest value of λ, that is 10^{-5} means that the idiosyncratic error term essentially captures the differences between the DSGE model variables and the related non-core variables. The other preferred choices and the corresponding forecast horizons are as follows: $\lambda = 0.1$ for $h = 2$ and $h = 4$, as well as $\lambda = 10^{-5}$ for $h = 12$. In this last case, the idiosyncratic error term essentially captures the discrepancies between the DSGE model variables and the related non-core variables for the 12 quarters-ahead forecast of PCE inflation.

The auxiliary regression models attain a lower RMSE than the AR(1)or multi-step regression benchmarks for core PCE inflation for $h = 1$, $h = 2$, $h = 4$, and $h = 12$. The preferred choice is $\lambda = 1$ for $h = 1$, $h = 2$ and $h = 12$. At these horizons, the variation in the variable is explained by the DSGE model's latent state variables. On the other hand $\lambda = 10^{-5}$ is the preferred choice for $h = 4$. Also, the auxiliary models (for $\lambda = 1$ and $\lambda = 0.1$) perform better than the AR(1)or multi-step regressions benchmark in forecasting building plans passed. Overall, the preferred choice is the specification with $\lambda = 1$. On the other hand, RMSE statistics associated with the auxiliary

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[7]In addition to comparing the DSGE model's performance in forecasting core variables to that of the benchmark autoregressive AR(1) model, we also considered the forecast performance of the Vector Autoregressive (VAR), Bayesian VAR (BVAR) models as well as forecast combination based on the simple mean of the forecasts from the AR, VAR and BVAR, all estimated with one lag. On the whole (for $h = 2$, $h = 4$ and $h = 12$), we find that DSGE-based forecasts for the inflation variable are still associated with smallest forecast errors compared to the other benchmarks. However, for the other variables the BVAR and the forecast combination method tends to outperform the other models. Details of these results are available upon request from the authors.

regression models' forecasts of the unemployment rate are lower than the RMSEs obtained for forecasts based on the AR(1)or multi-step regression only in the case involving a one quarter-ahead forecast with $\lambda = 1$.

Table 5: RMSEs for auxiliary regressions

Series	Model	λ	h=1	h=2	h=4	h=12
PCE inflation (Q%)	Aux	1.00	**0.97**	1.08	**0.91****	**0.79*****
	Aux	0.10	1.06	**0.93****	**0.86*****	**0.83*****
	Aux	10^{-5}	**0.98**	**0.98****	**0.98*****	**0.70*****
	Reg		1.03	1.06***	1.03***	1.48
	AR(1)		1.04	1.28	1.40	1.51
Core PCE inflation (Q%)	Aux	1.00	**1.09**	**1.06****	**1.17***	**0.97*****
	Aux	0.10	1.27	1.41	1.35	**1.28**
	Aux	10^{-5}	1.20	**1.30**	**1.15*****	**1.09****
	Reg		**1.20**	1.31*	1.22***	1.64
	AR(1)		1.29	1.46	1.52	1.63
Unemployment Rate (Q%)	Aux	1.00	**0.73**	0.63	0.74	0.89
	Aux	0.10	0.80	0.75*	0.81	0.95
	Aux	10^{-5}	0.95**	0.88**	0.92*	1.05
	Reg		0.66	0.58	0.59	0.76
	AR(1)		0.74	0.55***	0.64	0.90
Building Plans Passed (A%)	Aux	1.00	**0.41**	**0.40**	**0.41**	**0.38**
	Aux	0.10	**0.61**	**0.59**	**0.59**	**0.54**
	Aux	10^{-5}	1.95	1.94	1.87	1.94
	Reg		1.86	1.89	1.86	1.37
	AR(1)		1.51**	1.49**	1.49	1.52

We report RMSEs for the DSGE, AR(1) process, and an alternative model (a multi-step least squares regressions). Numbers in boldface indicate a lower RMSE of the DSGE model with respect AR(1) and/or the multi-step regression. *, **, *** denote 10%, 5% or 1% significance of the two-sided modified Diebold-Mariano test of equal predictive accuracy under quadratic loss (DSGE vs AR(1) and AR(1) vs multi-step). The RMSEs are computed based on recursive estimates starting with the sample 1971Q2 to 1999Q4 and ending with the samples 1971Q2 to 2011Q3 ($h = 1$), 1971Q2 to 2011Q2 ($h = 2$), 1971Q2 to 2010Q4 ($h = 4$), and 1971Q2 to 2008Q4 ($h = 12$), respectively. The h-step-ahead growth rate (inflation) forecasts refer to percentage changes between the periods $T+h\text{-}1$ and $T+h$.

3.5 *Ex-ante* forecasting of inflation variables

Given the fact that the hours worked series for South Africa is only available up to the fourth quarter of 2011, we estimate the models over the sample 1971Q2 to 2011Q4. Thereafter, we estimate the *ex-ante* forecasts[8] over the period 2012Q1 to 2013Q4. Note that here, we do not estimate the model recursively over 2012Q1-2013Q4, but produce forecasts based on a one-time estimation of the model till 2011Q4 beginning in 1971Q2. This is, in some sense, an acid test of the ability of the models in predicting possible turning points in the data.

Figure 3 plots the *ex-ante* forecasts for the inflation (GDP deflator), Core PCE as well as PCE inflations (obtained from the DSGE and other benchmark models) together with their actual (realised) values over the period 2011Q4 to 2013Q4. According to Figures 3(a) and 3(b), *ex-ante* forecasts obtained from the predictive regression model perform the best in closely tracking both actual PCE and actual Core PCE inflation series. On the other hand, Figure 3(c) shows that, in general, the DSGE model's *ex-ante* forecasts track actual GDP deflator inflation better than the benchmark AR(1) model.

Figure 3: *Ex-ante* inflation forecasts

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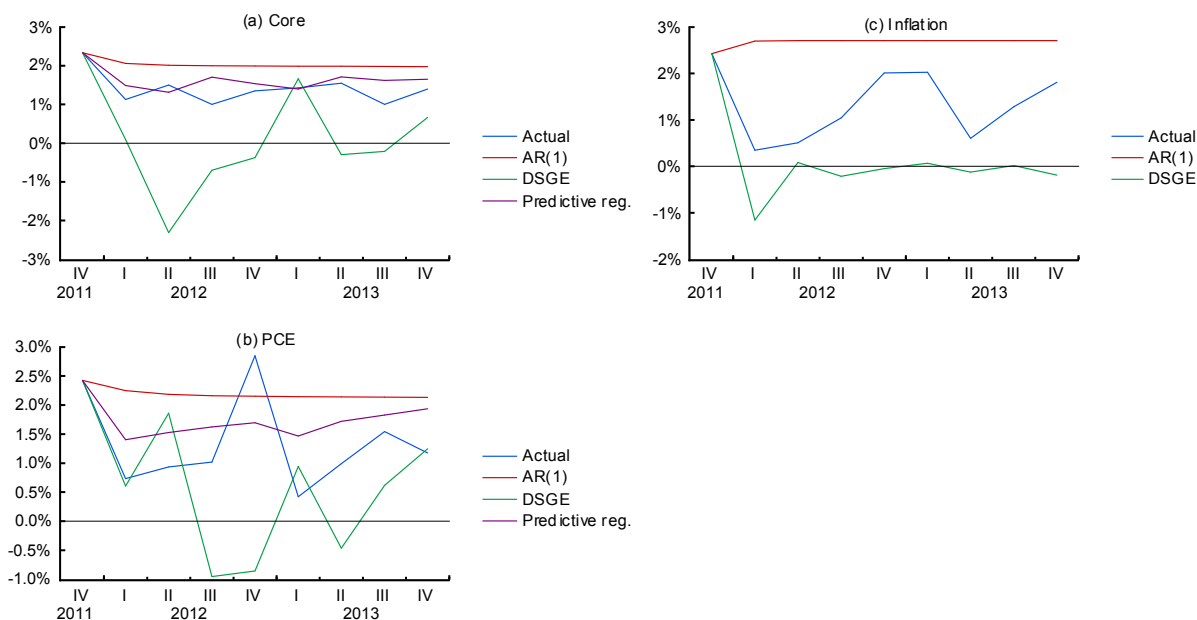
[8]We only report forecasts for inflation variables - our variables of concern in this paper. However, forecasts for all variables are available upon request from authors. However, it must be realized that we cannot compare the generated forecasts with actual data for hours worked over 2012Q1-2013Q4.

4   Conclusion

We follow Schorfheide *et al.* (2010) by applying a DSGE-based method for forecasting non-modeled variables on South African macroeconomic data. Our prime interest is forecasting inflation. The results show that forecasts of various measures of inflation (that is, GDP deflator inflation, PCE inflation and core PCE inflation) based on the DSGE-based procedure are superior to those obtained from statistical benchmark models. Essentially, the DSGE-based forecasts on inflation variables are associated with lower forecast errors compared to forecasts obtained from benchmark models.

Our findings are in line with Liu *et al.* (2009), Alpanda *et al.* (2011) and Gupta and Steinbach (2013). These studies develop more sophisticated frameworks that allow for various real and nominal rigidities in closed and small open economy NKDSGE models for South Africa and find that DSGE-based inflation forecasts tend to outperform those obtained from BVAR and VAR models. In addition, Balcilar *et al.* (2013) show that by including non-linearities in the DSGE framework to account for structural changes in South Africa's economy, Non-linear DSGE-based forecasts outperform those from the linear counterpart, as well as VAR models. In the same breadth, other studies (*e.g.* Kanyama and Thobejane, 2013; Gupta and Hartley, 2013; Aron and Muellbauer, 2012; Pretorius and van Rensburg, 1996) in the literature comparing the performance of various model in forecasting South African inflation conclude that model specifications beat benchmark time-series models (*e.g.* ARIMA and Vector Autoregressive (VAR)) models in forecasting South African inflation.

Given the inflation targeting regime in South Africa, the DSGE framework for forecasting non-modeled inflation variables proves to be a relevant tool in the conduct of monetary policy. In fact, a non-modeled variable such as the core PCE inflation contains important information about the underlying long-term inflation trend which is of interest to policy makers. On the other hand, the DSGE model's forecast of other macroeconomic variables are competitive when compared to the statistical benchmark models.

References

[1] Alpanda S, Kotzé K, Woglom G (2010) Should Central Banks of Small Open Economies Respond to Exchange Rate Fluctuations? The Case of South Africa, ERSA Working Paper, No. 174.

[2] Alpanda S, Kotzé K, Woglom G (2010) The Role of the Exchange Rate in a New Keynesian DSGE Model for the South African Economy, *South African Journal of Economics*, 78(2).

[3] Alpanda S, Kotzé K, Woglom G (2011) Forecasting Performance of an Estimated DSGE Model for the South African Economy, *South African Journal of Economics*, 79(1).

[4] Aron J and Muellbauer (2012) Improving Forecasting in an Emerging Economy, South Africa: Changing Trends, Long-run Restrictions and Disaggregation, *International Journal of Forecasting*, 28: 456-476.

[5] Amusa K, Gupta R, Karolia, S, Simo-Kengne, BD (2013) The Long-Run Impact of Inflation in South Africa, *Econometric Reviews*, 35(5), 798-812.

[6] An S, Schorfheide F (2007) Bayesian Analysis of DSGE Models, *Econometric Reviews*, 26(2-4), 113-172.

[7] Balcilar M, Gupta R and Kotzé K (2013) Forecasting South African Macroeconomic Data with a Nonlinear DSGE Model, Department of Economics, University of Pretoria, *Working Paper, No. 2013-13*.

[8] Aye GC, Balcilar M, Bosch A, Gupta R (2013) Housing and the Business Cycle in South Africa, Department of Economics, University of Pretoria, *Working Paper, No. 2013-23*.

[9] Calvo G (1983) Staggered Prices in a Utility-maximizing Framework, *Journal of Monetary Economics*, 12, 383-398.

[10] Christiano LJ, Eichenbaum M, Evans C (2005) Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy, *Journal of Political Economy*, 113, 1-45.

[11] Consolo A, Favero CA, Paccagnini A (2009) On the Statistical Identification of DSGE Models, *Journal of Econometrics*, 150, 99-115.

[12] Croce E, Khan MS (2000) Monetary Regimes and Inflation Targeting, *IMF Finance and Development*, 37(3).

[13] Del Negro M, Schorfheide F (2004) Priors from General Equilibrium Models for VARs, *International Economic Review*, 45, 643-673

[14] Del Negro M, Schorfheide F, Smets F, Wouters R (2007) On the Fit of New-Keynesisan Models, *Journal of Business, Economics and Statistics*, 25(2), 124-162

[15] Gupta R, Kabundi A (2010) Forecasting Macroeconomic Variables in a Small Open Economy: A Comparison between Small- and Large-Scale Models, *Journal of Forecasting*, 29(1-2), 168-185.

[16] Gupta R, Kabundi A (2011) A Large Factor Model for Forecasting Macroeconomic Variables in South Africa, *International Journal of Forecasting*, 27(4), 1076-1088.

[17] Gupta R and Hartley F (2013) The Role of Asset Prices in Forecasting Inflation and Output in South Africa, *Journal of Emerging Market Finance*, 12(3): 239-291.

[18] Gupta R, Steinbach R (2013) A DSGE-VAR Model for Forecasting Key South African Macroeconomic Variables, *Economic Modelling*, 33(1), 19-33.

[19] Liu G, Gupta R (2007) A Small-Scale DSGE Model for Forecasting the South African Economy, *South African Journal of Economics*, 75(2), 179-193.

[20] Liu G, Gupta R, Schaling E (2009) A New-Keynesian DSGE Model for Forecasting the South African Economy, *Journal of Forecasting*, 28, 387-404.

[21] Liu G, Gupta R, Schaling E (2010) Forecasting the South African Economy: A hybrid-DSGE Approach, *Journal of Economic Studies*, 37(2), 181-195.

[22] Kanyama I K and Thobejane B M (2013) Forecasting Macroeconomic Variables in South Africa: Parametric vs. Non-parametric Methods, Paper presented at the Economic Society of South Africa 2013 Conference.

[23] Liu D, Gupta R and Schaling E (2009) A New-Keynesian DSGE Model for Forecasting the South African Economy, *Journal of Forecasting*, 28(5): 387-404.

[24] Pretorius C J and Janse van Rensburg T N (1996) The Forecast Performance of Alternative Models of Inflation, South African Reserve Bank Occasional Paper No. 10.

[25] Schorfheide F, Sill K, Kryshko M (2010) DSGE model-based forecasting of non-modelled variables, *International Journal of Forecasting*, 26, 348-373.

[26] Sims, CA (2002) Solving Linear Rational Expectations Models, *Computational Economics*, 20 (1-2), 1-20

[27] Smets F, Wouters R (2003) An Estimated Dynamic Stochastic General Equilibrium Model of the Euro Area, *Journal of the European Economic Association, MIT Press*, 1(5), 1123-1175.

[28] Smets, Wouters R (2007) Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach, *American Economic Review*, 97(3), 586-606.

[29] Steinbach R, Mathuloe P, Smit BW (2009) An Open Economy New Keynesian DSGE Model of the South African Economy, *South African Journal of Economics*, 77(2), 207-227.

[30] Tovar CE (2009) DSGE Models and Central Banks, *Economics - The Open-Access, Open-Assessment E-Journal*, 3(16), 1-31.

[31] Touna-Mama A, Viegi, N (2012) Business Cycle Accounting for South Africa, *Mimeo*, Department of Economics, University of Pretoria.

Appendix: The DSGE Model

We consider a medium-scale New Keynesian model which features sticky nominal price and wage contracts, capital accumulation, investment adjustment costs, variable capital utilization, and habit formation. We follow Schorfheide *et al.* (2010) who use a medium-scale DSGE model based on the models proposed by Smets and Wouters (2003, 2007), Christiano *et al.* (2005), and Del Negro *et al.* (2007).

In the economy, there is a continuum of firms which combine capital and labor to produce differentiated intermediate goods. The production function is Cobb-Douglas in nature with capital elasticity α and total factor productivity (TFP) A_t. The TFP is assumed to be a non-stationary process and we take its growth rate, $a_t = \ln(A_t/A_{t-1})$, which is assumed to have a mean of γ. All variables of the model, (output, consumption, investment, capital, and real wage) are detrended by A_t. We define the log-deviation of each variables from the steady state of the model.

The intermediate goods producers hire labor and rent capital in competitive markets, and face identical real wages, w_t, and rental rates for capital, r_t^k. According to cost minimization, all firms produce with the same capital-labor ratio:

$$k_t - L_t = w_t - r_t^k \tag{9}$$

and the marginal costs are:

$$mc_t = (1 - \alpha)w_t + \alpha r_t^k \tag{10}$$

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The intermediate goods producers sell their output to perfectly competitive final good producers, which aggregate the inputs according to a CES function. The profit maximization of the final good producers implies that:

$$\widehat{y}_t(j) - \widehat{y} = -\left(1 + \frac{1}{\lambda_f e^{\widetilde{\lambda}_{f,t}}}\right)(p_t(j) - p_t) \tag{11}$$

where $\widehat{y}_t(j) - \widehat{y}$ and $p_t(j) - p_t$ are the quantity and price for the good j relative to the quantity and price of the final good. We consider the zero-profit condition for the final good producers to determine the price p_t of the final good. Since the price elasticity of the intermediate goods affects the mark-up that intermediate goods producers can charge over marginal costs, the mark-up shock, $\widetilde{\lambda}_{f,t}$, is assumed time-varying. As in Calvo (1983), we assume that a certain fraction of the intermediate goods producers ζ_p is unable to re-optimize their prices in each period. These firms adjust their prices mechanically according to steady state inflation π_*. Hence, there is no price dispersion in the steady-state. All other firms choose their price to maximize the expected discounted sum of future profits, which leads to the New Keynesian Phillips Curve:

$$\pi_t = \beta E_t[\pi_{t+1}] + \frac{(1 - \zeta_p\beta)(1 - \zeta_p)}{\zeta_p}mc_t + \frac{1}{\zeta_p}\lambda_{f,t} \tag{12}$$

where π_t is inflation and β is the discount rate.

The log-linearized aggregate production function is:

$$\widehat{y}_t = (1 - \alpha)L_t + \alpha k_t \tag{13}$$

Considering, equations (10), (9), and (13), the labor share lsh_t equals the marginal costs in terms of log-deviations, $lsh_t = mc_t$.

The economy is populated by a continuum of households with identical preferences, which are separable in consumption, leisure, and real money balances. The parameter h captures the degree of (internal) habit formation in consumption. The utility function at period t is a function of $\ln(C_t - hC_{t-1})$. Households supply monopolistically differentiated labor services which are aggregate according to a CES function that leads to a demand elasticity $1 + 1/\lambda_w$.

The composite labor services are then supplied to the intermediate goods producers at a real wage w_t.

We assume that in each period, a certain fraction ζ_w of households is unable to re-optimize their wages, in this way we introduce nominal wage rigidity. The households adjust their nominal wage by the steady state wage growth $e^{(\pi_* + \gamma)}$. All other households re-optimize their wages. The first-order conditions imply that:

$$
\begin{aligned}
\widetilde{w}_t = {} & \zeta_w\beta E_t[\widetilde{w}_{t+1} + \Delta w_{t+1} + \pi_{t+1} + a_{t+1}] \\
& + \frac{1 - \zeta_w\beta}{1 + \upsilon_l(1 + \lambda_w)/\lambda_w} \\
& \times \left(\upsilon_l L_t - w_t - \xi_t + \widetilde{b}_t + \frac{1}{1 - \zeta_w\beta}\phi_t\right),
\end{aligned} \tag{14}
$$

where $\widetilde{w}_t$ is the optimal real wage relative to the real wage for aggregate labor services, w_t, and υ_l is the inverse Frisch labor supply elasticity in a model without wage rigidity ($\zeta_w = 0$) and

differentiated labor. $\widetilde{b}_t$ is a shock to the household's discount factor; instead, ϕ_t is a preference shock that affects the household's intratemporal substitution between consumption and leisure.

The real wage paid by intermediate goods producers evolves according to:

$$w_t = w_{t-1} - \pi_t - a_t + \frac{1 - \zeta_w}{\zeta_w} \widetilde{w}_t. \tag{15}$$

Households share the same marginal utility of consumption ξ_t, which is given by the following expression:

$$
\begin{aligned}
(e^\gamma - h\beta)(e^\gamma - \beta)\xi_t &= -\left(e^{2\gamma} + \beta h^2\right)c_t \\
&\quad +\beta h e^\gamma E_t\left[c_{t+1} + a_{t+1}\right] \\
&\quad +h e^\gamma(c_{t-1} - a_t) + e^\gamma\left(e^\gamma - h\right)\widetilde{b}_t \\
&\quad -\beta h\left(e^\gamma - h\right)E_t\left[\widetilde{b}_{t+1}\right]
\end{aligned} \tag{16}
$$

where c_t is consumption. In addition to state-contingent claims, households accumulate three types of assets: one-period nominal bonds that yield the return R_t, capital $\overline{k}_t$, and real money balances. Since the preferences for real money balances are assumed to be additively separable and monetary policy is conducted through a nominal interest rate rule, money is block exogenous, hence in the empirical analysis there will not be the households' money demand equation as in Schorfheide *et al.* (2010).

The first order condition with respect to bond holdings delivers the standard Euler equation:

$$\xi_t = E_t\left[\xi_{t+1}\right] + R_t - E_t\left[\pi_{t+1}\right] - E_t\left[a_{t+1}\right]. \tag{17}$$

Capital accumulates according to the following law of motion:

$$
\begin{aligned}
\overline{k}_t &= (2 - e^\gamma - \delta)\left[\overline{k}_{t-1} - a_t\right] \\
&\quad +(e^\gamma + \delta - 1)\left[i_t + (1+\beta)S''e^{2\gamma}\mu_t\right],
\end{aligned} \tag{18}
$$

where i_t is investment (which is subject to adjustment costs), δ is the depreciation rate of capital, and μ_t can be interpreted as an investment-specific technology shock, and S'' denotes the second derivative of the investment adjustment cost function at the steady state.

The optimal investment satisfies the following first-order condition:

$$
\begin{aligned}
i_t &= \frac{1}{1+\beta}\left[i_{t-1} - a_t\right] + \frac{\beta}{1+\beta}E_t\left[i_{t+1} + a_{t+1}\right] \\
&\quad +\frac{1}{(1+\beta)S''e^{2\gamma}}\left(\xi_t^k - \xi_t\right) + \mu_t,
\end{aligned} \tag{19}
$$

where ξ_t^k is the value of the installed capital, which evolves according to:

$$
\begin{aligned}
\left(\xi_t^k - \xi_t\right) &= \beta e^{-\gamma}(1-\delta)E_t\left[\xi_{t+1}^k - \xi_{t+1}\right] \\
&\quad +E_t\left[\left(1 - (1-\delta)\beta e^{-\gamma}\right)r_{t+1}^k - (R_t - \pi_{t+1})\right].
\end{aligned} \tag{20}
$$

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The capital utilization u_t is variable, and r_t^k represents the rental rate of effective capital $k_t = u_t + \overline{k}_{t-1}$. The optimal degree of utilization is determined by:

$$u_t = \frac{r_*^k}{a''} r_t^k. \tag{21}$$

Here a'' is the derivative of the per-unit-of-capital cost function $a(u_t)$, evaluated at the steady state utilization rate. The central bank follows a standard feedback rule:

$$R_t = \rho_R R_{t-1} + (1 - \rho_R)(\psi_1 \pi_1 + \psi_2 \widehat{y}_t) + \sigma_R \epsilon_{R,t}, \tag{22}$$

where $\epsilon_{R,t}$ represents monetary policy shocks.

The aggregate resource constraint is given by:

$$\widehat{y}_t = (1 + g_*) \left[\frac{c_*}{y_*} c_t + \frac{i_*}{y_*} \left(i_t + \frac{r_*^k}{e^\gamma - 1 + \delta} u_t \right) \right] + g_t. \tag{23}$$

Here $\frac{c_*}{y_*}$ and $\frac{i_*}{y_*}$ are the steady state consumption-output and investment-output ratios, respectively, and $\frac{g_*}{(1+g_*)}$ corresponds to the government's share of the aggregate output. The process g_t can be interpreted as the exogenous government spending shock. We assume that fiscal policy is passive, i.e. the lump-sum taxes are used to satisfy its period budget constraint.

There are seven exogenous disturbances in the model, and six of them are assumed to follow AR(1) processes:

$$
\begin{aligned}
a_t &= \rho_a a_{t-1} + (1 - \rho_a)\gamma + \sigma_a \epsilon_{a,t} \\
\mu_t &= \rho_\mu \mu_{t-1} + \sigma_\mu \epsilon_{\mu,t} \\
\lambda_{f,t} &= \rho_{\lambda_f} \lambda_{f,t-1} + \sigma_{\lambda_f} \epsilon_{\lambda_f,t} \\
g_t &= \rho_g g_{t-1} + \sigma_g \epsilon_{g,t} \\
b_t &= \rho_b b_{t-1} + \sigma_b \epsilon_{b,t} \\
\phi_t &= \rho_\phi \phi_{t-1} + \sigma_\phi \epsilon_{\phi,t}.
\end{aligned}
\tag{24}
$$

We assume that the innovations of these exogenous processes, as well as the monetary policy shock $\epsilon_{R,t}$, are independent standard normal random variates, and collect them in the vector ϵ_t. We stack all of the DSGE model parameters in the vector θ.

All these equations are a linear rational expectations system solved numerically using the algorithm proposed by Sims (2002).

The solution is represented by the following transition equation:

$$S_t = \Phi_1(\theta) S_{t-1} + \Phi_\epsilon(\theta) \epsilon_t. \tag{25}$$

The coefficients of the matrices Φ_1 and Φ_ϵ are functions of the DSGE model parameters θ, and the vector S_t is given by:

$S_t = [c_t, i_t, \overline{k}_t, R_t, w_t, a_t, \phi_t, \mu_t, b_t, g_t, \lambda_{f,t}]'$.

The variables c_t, i_t, $\overline{k}_t$, R_t, and w_t are endogenous state variables, whereas the remaining elements of S_t are exogenous state variables.

The measurement equation which links the observables Y_t to the states S_t is composed of: quarter-to-quarter growth rates (measured in percentages) of real GDP, consumption, investment

and nominal wages, as well as a measure of the number of hours worked, GDP deflator inflation, and the federal funds rate. Since some of the observables include growth rates, the set of model states S_t is augmented by lagged values (S_{t-1}) of output, consumption, investment, and real wages.

According to the DSGE model solution, the lagged output, $\widehat{Y}_{t-1}$, can be expressed as a linear function of the elements of S_{t-1}. Hence, we can write:

$$\left[\widehat{Y}_{t-1},\ c_{t-1},\ i_{t-1},\ w_{t-1}\right]' = M_s\left(\theta\right)S_{t-1} \tag{26}$$

for a suitably chosen matrix $M_s\left(\theta\right)$, and we can define:

$$\varsigma_t = [S_t', S_{t-1}'M_s'\left(\theta\right)]'. \tag{27}$$

The measurement equations are:

$$Y_t = A_0\left(\theta\right) + A_1\left(\theta\right)\varsigma_t \tag{28}$$