Reflexivity of orthogonality in $\mathcal{A}$-modules

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Abstract

In this paper, as part of a project initiated by A. Mallios consisting of exploring new horizons for *Abstract Differential Geometry* (à la Mallios), [5, 6, 7, 8], such as those related to the *classical symplectic geometry*, we show that *essential* results pertaining to biorthogonality in pairings of vector spaces do hold for biorthogonality in pairings of $\mathcal{A}$-modules. We single out that *orthogonality is reflexive for orthogonally convenient pairings of free $\mathcal{A}$-modules of finite rank, governed by non-degenerate $\mathcal{A}$-morphisms, and where $\mathcal{A}$ is a PID* (Corollary 3.8). For the *rank formula* (Corollary 3.3), the algebra sheaf $\mathcal{A}$ is assumed to be a PID. The rank formula relates the rank of an $\mathcal{A}$-morphism and the rank of the kernel (sheaf) of the same $\mathcal{A}$-morphism with the rank of the source free $\mathcal{A}$-module of the $\mathcal{A}$-morphism concerned.

Key Words: convenient $\mathcal{A}$-modules, quotient $\mathcal{A}$-modules, free subpresheaf, orthogonally convenient $\mathcal{A}$-pairings.

1 Introduction

The present article could be inscribed within the framework of *Abstract Differential Geometry* (à la Mallios) for the simple reason that it is *sheaf-theoretic* by nature. To recall succinctly the main features of Abstract Differential Geometry (ADG as an acronym), one should retain the fact that

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ADG chiefly indicates the extent which one retrieves fundamental notions and results of the standard differential geometry of smooth manifolds with no use of any notion of classical differentiability. The aim of ADG is to bring out the sheaf-theoretic character of Calculus ([5]). Again, briefly speaking, ADG pursues an *axiomatic* approach to classical differential geometry; the basic tools are *sheaves of modules*, with respect to an appropriate *sheaf of* $\mathbb{C}$-*algebras* (alias, $\mathbb{C}$-*algebra sheaf*) over a given topological space X, which, in some circumstances, is assumed to be *paracompact*. It is worth noting that prominent features within the setting of ADG are *vector sheaves*, that is, locally free sheaves of $\mathcal{A}$-modules of finite rank over an arbitrary topolog-ical space X. Vector sheaves are deemed to be the counterparts/substitutes of *vector bundles* (of finite rank) in the classical theory. In [14], we proved that ADG has a possible extension in the field of symplectic geometry. For instance, the "*affine Darboux theorem*" is valid in the category of $\mathcal{A}$-modules, with $\mathcal{A}$ satisfying certain conditions. More precisely, *let $\mathcal{A}$ be an $\mathbb{R}$-algebra sheaf on a topological space X such that every positive section is invertible* (*this condition is called the* ***inverse-closed section condition***) *and every positive section has a square root, and let $(\mathcal{E}, \varphi)$ be a free $\mathcal{A}$-module of rank $2n$ on X, where $\varphi : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ is a skew-symmetric non-degenerate $\mathcal{A}$-bilinear form. Moreover, let I and J be two* (*possibly empty*) *subsets of $\{1, \dots, n\}$, $A = \{r_i \in \mathcal{E}(U) : i \in I\}$ and $B = \{s_j \in \mathcal{E}(U) : j \in J\}$ such that*

$$\varphi_U(r_i, r_j) = \varphi_U(s_i, s_j) = 0, \varphi_U(r_i, s_j) = \delta_{ij}, (i, j) \in I \times J.$$

Then, there exists a basis $\mathfrak{B}$ of $(\mathcal{E}(U), \varphi_U)$ that contains $A \cup B$. $\mathfrak{B}$ is called a symplectic basis of $(\mathcal{E}(U), \varphi_U)$. On using this result, we obtain a counterpart of the Darboux theorem (cf [13]). In fact, *let $\mathcal{A}$ be a $\mathbb{C}$-algebra sheaf on a given topological space X, satisfying the inverse-closed section condition, $(\mathcal{E}, \omega)$ a symplectic free $\mathcal{A}$-module of finite rank, say $n \in \mathbb{N}$. Then for any open set U in X,*

$$\omega_U = \sum_{k=1}^{n} t_U^{2k-1} \wedge t_{,U}^{2k}$$

where ${t^1}_U, \dots, {t^n}_U$ is a basis of $\mathcal{E}^*(U)$. On another side, *assume that $\mathcal{A}$ is a $\mathbb{C}$-algebra sheaf on X and has no zero-divisors* (we call it a "*strict integral domain*"), that is, for any open $U \subseteq X$, if r, $s \in \mathcal{A}(U)$ are nowhere-zero sections, then their product rs is nowhere zero. *If $\mathcal{E}$ is an $\mathcal{A}$-module and $\varphi : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ an orthosymmetric $\mathcal{A}$-bilinear form* (see, for instance, [4, p.97] for *orthosymmetric bilinear form*), that is, φ is such that $\mathcal{E}^{\perp} = \mathcal{E}^{\top}$; this is equivalent to saying that, for every open $U \subseteq X$ and sections $s \in \mathcal{E}(V)$, $t \in \mathcal{E}(U)$, where V is a subopen of U,

$$\varphi_V(s, t|_V) = 0 \text{ if and only if } \varphi_V(t|_V, s) = 0.$$

Then, componentwise, φ is either symmetric (that is, the ***geometry is orthogonal****) or skew-symmetric (the* ***geometry is symplectic****).*

This paper grew out of our earlier efforts to understand the conditions defining convenient $\mathcal{A}$-modules (cf [11], [15], and [16]), which, however, require notions of *free subpresheaf of modules* and of *PID-algebra sheaf.* The former, due to A. Mallios, is defined as follows. A subpresheaf F of a presheaf of modules (or more precisely, $A(U)$-modules) E on a topological space X is called a free subpresheaf if for every open U in X, $F(U)$ is a free sub-$A(U)$-module of $E(U)$. As to the notion of PID-algebra sheaf, we have the following definition. An algebra sheaf $\mathcal{A}$ is called a PID-algebra sheaf if for every open $U \subseteq X$, the algebra $\mathcal{A}(U)$ is a PID algebra. Consequently, all sub-$\mathcal{A}$-modules of a free $\mathcal{A}$-module are *section-wise free.* In the same vein, we recall that an $\mathcal{A}$-bilinear form $\phi : \mathcal{E} \oplus \mathcal{E} \longrightarrow \mathcal{A}$ on an $\mathcal{A}$-module $\mathcal{E}$ is called orthosymmetric if $\mathcal{E}^{\perp_\phi} = \mathcal{E}^{\top^\phi}$, cf [11]. Equivalently, for every open $U \subseteq X$ and (local) sections $s \in \mathcal{E}(V), t \in \mathcal{E}(U)$, where V is an open subset of U, we have $\phi_V(s, t|_V) = 0$ if, and only if, $\phi_V(t|_V, s) = 0$. (Given an $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{F}; \phi)$, $\mathcal{E}^{\perp_\phi}(U)$ consists of all sections $s \in \mathcal{F}(U)$ such that $\phi_V(\mathcal{E}(V), s|_V) = 0$ for every open $V \subseteq U$. Similarly, $\mathcal{F}^{\top^\phi}(U)$ consists of all $s \in \mathcal{E}(U)$ with $\phi_V(s|_V, \mathcal{F}(V)) = 0$ for any open $V \subseteq U$.)

Then, we have

Definition 1.1 *A* ***convenient $\mathcal{A}$-module*** *is a self $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{E}; \phi) \equiv (\mathcal{E}, \phi)$, where $\mathcal{E}$ is a free $\mathcal{A}$-module of finite rank and ϕ an orthosymmetric $\mathcal{A}$-bilinear form, such that the following conditions are satisfied:*

(1) *If $\mathcal{F}$ is a free subpresheaf of $\mathcal{A}(U)$-modules of $\mathcal{E}$, so is $\mathcal{F}^\perp \equiv \mathcal{F}^{\perp_\phi}$;*

(2) *Every free subpresheaf $\mathcal{F}$ of $\mathcal{A}(U)$-modules of $\mathcal{E}$ is orthogonally reflexive, i.e., $\mathcal{F}^{\perp\top} = \mathcal{F}^{\top\perp} = \mathcal{F}$;*

(3) *The intersection of any two free subpresheaves of $\mathcal{A}(U)$-modules of $\mathcal{E}$ is a free subpresheaf of $\mathcal{A}(U)$-modules.*

The notion of *convenient $\mathcal{A}$-module* was in fact an initial suggestion of A. Mallios that has been successfully applied in one of our joint articles with him (cf. [11]) and also since then. The motivation, of course, was to *iso-late characteristic properties* of an important notion that classically "works", and make it, eventually, independent of its classical "*environment*". Indeed, this is the general philosophy/moral of Mallios, and also the quintessence throughout the whole ADG. We do have applications of this notion in the "*geometry of gauge fields*", in general, where the *domain of coefficients* is no more the classical numerical fields, but suitable *function algebras* (e.g. wave functions), which are, in principle, *non-normed topological algebras.*

Now, concerning Definition 1.1, by supposing that the (*coefficient*) algebra sheaf $\mathcal{A}$ is a PID-*algebra sheaf*, we obtain that *every subpresheaf of $\mathcal{A}(U)$-modules of a free $\mathcal{A}$-module is free.* So in that context, conditions (1) and (3) of Definition 1.1 are satisfied. As for condition (2), the *orthogonality reflexivity* is a known situation in ordinary Functional Analysis: see, for instance, Hilbert spaces and structures having similar properties; we do have the so-called *complemented topological algebras*, *Hilbert algebras* and the like with the aforementioned property for *ideals*(:*modules*), and also analogous examples in *infinite-dimensional Hamiltonian systems.* The mentioned examples are particular cases of more general potential applications that can be "explored" in A. Mallios [9, p. 109ff, Chap. 3] within the same context of "geometry of gauge fields", this time however, in terms of ADG. (I am indebted to A. Mallios for this comment on convenient $\mathcal{A}$-modules.)

Let us also recall the notion of *orthogonally convenient $\mathcal{A}$-pairings,* which was introduced in our paper [15].

Definition 1.2 A free $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{F}; \phi)$ of (free) $\mathcal{A}$-modules $\mathcal{E}$ and $\mathcal{F}$ is called an **orthogonally convenient $\mathcal{A}$-pairing** if for all free sub-$\mathcal{A}$-modules $\mathcal{E}_0$ and $\mathcal{F}_0$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, their orthogonal $\mathcal{E}_0^{\perp_\phi}$ and $\mathcal{F}_0^{\top_\phi}$ are free sub-$\mathcal{A}$-modules of $\mathcal{F}$ and $\mathcal{E}$, respectively.

As an example of orthogonally convenient $\mathcal{A}$-pairings, one may consider *canonical free $\mathcal{A}$-pairings,* cf. [15]. An $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{E}^*; \nu)$ is called a canonical $\mathcal{A}$-pairing if the $\mathcal{A}$-bilinear form ν is defined such that, for every open $U \subseteq X$, $\psi \in \mathcal{E}^*(U)$ and $s \in \mathcal{E}(U)$, $\nu_U(s, \psi) := \psi_U(s)$. $(\mathcal{E}, \mathcal{E}^*; \nu)$ is called a *canonical free $\mathcal{A}$-pairing* if $\mathcal{E}$ is a free $\mathcal{A}$-module. As a convention, we will denote by ν the $\mathcal{A}$-bilinear form in the canonical $\mathcal{A}$-pairing determined by an $\mathcal{A}$-module $\mathcal{E}$.

In this paper, we show that given an orthogonally convenient $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{F}; \phi)$, where *$\mathcal{E}$ and $\mathcal{F}$ have finite rank* and *$\mathcal{A}$ is a PID-algebra sheaf, if ϕ is non-degenerate, orthogonality is reflexive on free sub-$\mathcal{A}$-modules.* More exactly, for any free sub-$\mathcal{A}$-modules $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, $(\mathcal{G}^{\perp_\phi})^{\top^\phi}$ is $\mathcal{A}$-isomorphic to $\mathcal{G}$, and $(\mathcal{H}^{\top^\phi})^{\perp_\phi}$ is $\mathcal{A}$-isomorphic to $\mathcal{H}$. It is worth noting at this place, that we do actually realize here the crucial "*property* (2)" of Definition 1.1. The proof of this result is based on the *rank formula,* which is the analogous of the *dimension formula of vector spaces.* See [3, p. 54, Corollaire 2].

As a general remark, all sheaves and presheaves in the paper are defined on a fixed topological space X. Also, if $\mathcal{E}$ is an $\mathcal{A}$-module, $\mathcal{E}(U)$ will denote the $\mathcal{A}(U)$-module of sections of $\mathcal{E}$ over an open subset U of X.

2 Universal property of quotient $\mathcal{A}$-modules

This section contains proofs of the basic results on biorthogonality in canonical pairings of $\mathcal{A}$-modules, namely Proposition 2.6 and Theorems 2.9 and 2.10.

Theorem 2.1 *Let $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{A}$-modules.*

1. *Let $\phi \in \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{F})$ be a surjective $\mathcal{A}$-morphism. Then, if $\psi \in \mathrm{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{G})$ such that $\ker\phi \subseteq \ker\psi$, there exists a unique $\theta \in \mathrm{Hom}_\mathcal{A}(\mathcal{F}, \mathcal{G})$ such that the diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\phi"] \arrow[dr, "\psi"'] & \mathcal{F} \arrow[d, dotted, "\theta"] \\ & \mathcal{G} \end{tikzcd}$$

commutes. In other words, the mapping $\theta \mapsto \theta \circ \phi$ *is an* $\mathcal{A}(X)$*-isomorphism from* $\mathrm{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ *onto the sub-*$\mathcal{A}(X)$*-module of* $\mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ *consisting of* $\mathcal{A}$*-morphisms whose kernel contains* $\ker \phi$*.*

2. *Let* $\phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ *be an injective* $\mathcal{A}$*-morphism. Then, if* $\psi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ *such that* $\mathrm{Im}\psi \subseteq \mathrm{Im}\phi$*, there exists a unique* $\theta \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ *making the diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[d, dotted, "\theta"'] \arrow[dr, "\psi"] & \\ \mathcal{F} \arrow[r, "\phi"'] & \mathcal{G} \end{tikzcd}$$

commute. More precisely, the mapping $\theta \mapsto \phi \circ \theta$ *is an* $\mathcal{A}(X)$*-isomorphism from* $\mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ *onto the sub-*$\mathcal{A}(X)$*-module of* $\mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{G})$ *consisting of* $\mathcal{A}$*-morphism whose image is contained in* $\mathrm{Im}\phi$*.*

Proof. Assertion 1. *Uniqueness.* Let θ_1, $\theta_2 \in \mathrm{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$ be such that $\psi = \theta_1 \circ \phi$ and $\psi = \theta_2 \circ \phi$. Fix an open subset U in X; since ϕ_U is surjective, the equation $\theta_{1,U} \circ \phi_U = \theta_{2,U} \circ \phi_U$ implies that $\theta_{1,U} = \theta_{2,U}$. Thus, $\theta_1 = \theta_2$.

Existence. Fix an open subset U in X and consider an element (section) $t \in \mathcal{F}(U)$. Since ϕ_U is surjective, there exists an element $s \in \mathcal{E}(U)$ such that $t = \phi_U(s)$. Now, suppose there exists an $r \in \mathcal{F}(U)$ with $u \in \ker \psi_U$ and $v \notin \ker \psi_U$ as its pre-images by ϕ_U, i.e.

$$\phi_U(v) = r = \phi_U(u)$$

with $u \in \ker \psi_U$ and $v \notin \ker \psi_U$. Since ϕ_U is linear, $\phi_U(v - u) = 0$; so $v - u \in \ker \phi_U \subseteq \ker \psi_U$. But $u \in \ker \psi_U$, so $v \in \ker \psi_U$, which yields a *contradiction*. We conclude that such a situation cannot occur. Furthermore, the element $\psi_U(s)$ does only depend on t. Let θ_U be the $\mathcal{A}(U)$-morphism sending $\mathcal{F}(U)$ into $\mathcal{G}(U)$ and such that

$$\theta_U(t) = \psi_U(s);$$

that

$$\psi_U = \theta_U \circ \phi_U$$

is clear.

Next, let us consider the *complete presheaves of sections* of $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, respectively, viz.

$$\Gamma(\mathcal{E}) \equiv (\Gamma(U,\mathcal{E}), \alpha^U_V), \quad \Gamma(\mathcal{F}) \equiv (\Gamma(U,\mathcal{F}), \beta^U_V), \quad \Gamma(\mathcal{G}) \equiv (\Gamma(U,\mathcal{G}), \delta^U_V).$$

Given open subsets U and V of X such that $V \subseteq U$, since $\psi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{G})$, one has

$$\psi_V \circ \alpha^U_V = \delta^U_V \circ \psi_U. \tag{1}$$

But $\psi_U = \theta_U \circ \phi_U$ and $\psi_V = \theta_V \circ \phi_V$, therefore, (1) becomes

$$\theta_V \circ \phi_V \circ \alpha^U_V = \delta^U_V \circ \theta_U \circ \phi_U$$

or

$$\theta_V \circ \beta^U_V \circ \phi_U = \delta^U_V \circ \theta_U \circ \phi_U. \tag{2}$$

Since ϕ_U is surjective, it follows from (2) that

$$\theta_V \circ \beta^U_V = \delta^U_V \circ \theta_U,$$

which means that $\theta \equiv (\theta_U)_{X \supseteq U,\ open}$ is an $\mathcal{A}$-morphism of $\mathcal{F}$ into $\mathcal{G}$ such that

$$\psi = \theta \circ \phi,$$

as required.

Finally, for **Assertion** 2 one applies dualization as it is the dual of **Assertion** 1. ■

The *universal property of quotient $\mathcal{A}$-modules* is then obtained as a corollary of Theorem 2.1. More precisely, one has

Corollary 2.2 (***Universal property of quotient $\mathcal{A}$-modules***) *Let $\mathcal{E}$ be an $\mathcal{A}$-module, $\mathcal{E}'$ a sub-$\mathcal{A}$-module of $\mathcal{E}$, and ϕ the canonical $\mathcal{A}$-morphism of $\mathcal{E}$ onto $\mathcal{E}/\mathcal{E}'$. The pair $(\mathcal{E}/\mathcal{E}', \phi)$ satisfies the following universal property:*

Given any pair $(\mathcal{F}, \psi)$ consisting of an $\mathcal{A}$-module $\mathcal{F}$ and an $\mathcal{A}$-morphism $\psi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{F})$ such that $\mathcal{E}' \subseteq \ker\psi$, there exists a unique $\mathcal{A}$-morphism $\widetilde{\psi} \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}/\mathcal{E}',\mathcal{F})$ such that the diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\phi"] \arrow[dr, "\psi"'] & \mathcal{E}/\mathcal{E}' \arrow[d, dotted, "\widetilde{\psi}"] \\ & \mathcal{F} \end{tikzcd}$$

commutes, i.e.

$$\psi = \widetilde{\psi} \circ \phi.$$

The kernel of $\widetilde{\psi}$ *equals the image by* ϕ *of the kernel of* ψ*, and the image of* $\widetilde{\psi}$ *equals the image of* ψ*.*

The mapping $\theta \mapsto \theta \circ \phi$ *is an* $\mathcal{A}(X)$*-isomorphism of the* $\mathcal{A}(X)$*-module* $\mathcal{H}om_{\mathcal{A}}(\mathcal{E}/\mathcal{E}', \mathcal{F})$ *onto the sub-*$\mathcal{A}(X)$*-module of* $\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ *consisting of* $\mathcal{A}$*-morphisms of* $\mathcal{E}$ *into* $\mathcal{F}$ *whose kernel contains* $\mathcal{E}'$*.*

Proof. Apply assertion 1 of Theorem 2.1. ■

Similarly to the classical case (cf. [3, p. 15, Corollary 1]), we also have the following corollary, the proof of which is an easy exercise and is, for that reason, omitted.

Corollary 2.3 *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be* $\mathcal{A}$*-modules and* $\phi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$*. Then,*

(1) $\mathcal{E}/\ker\phi = \mathrm{Im}\phi$ *within* $\mathcal{A}$*-isomorphism.*

(2) *Given a sub-*$\mathcal{A}$*-module* $\mathcal{F}'$ *of* $\mathcal{F}$*,* $\mathcal{E}' \equiv \phi^{-1}(\mathcal{F}')$ *is a sub-*$\mathcal{A}$*-module of* $\mathcal{E}$ *containing* $\ker\phi$*; moreover,* $\mathcal{F}' = \phi(\mathcal{E}')$ *if* ϕ *is surjective.*

(3) *Conversely, if* $\mathcal{E}'$ *is a sub-*$\mathcal{A}$*-module of* $\mathcal{E}$ *containing* $\ker\phi$*, then* $\mathcal{F}' \equiv \mathrm{Im}\ \mathcal{E}'$ *is a sub-*$\mathcal{A}$*-module of* $\mathcal{F}$ *such that* $\mathcal{E}' = \phi^{-1}(\mathcal{F}')$*.*

As a further application of the universal property of quotient $\mathcal{A}$-modules, we have

Corollary 2.4 *Let* $\mathcal{E}$ *be a free* $\mathcal{A}$*-module, and* $\mathcal{E}_1$ *a free sub-*$\mathcal{A}$*-module of* $\mathcal{E}$*. Then, the* $\mathcal{A}$*-morphism* $\phi \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}^*, \mathcal{E}_1^*)$ *such that every* ϕ_U *maps an element* $(\psi_V)_{U\supseteq V,\ open}$ *of* $\mathcal{E}^*(U)$ *onto its restriction* $(\psi_V|_{\mathcal{E}_1(V)})_{U\supseteq V,\ open} \in \mathcal{E}_1^*(U)$ *is surjective, and has* $\mathcal{E}_1^{\perp} \subseteq \mathcal{E}^*$ *as its kernel, where* $\mathcal{E}_1^{\perp}$ *is the orthogonal of* $\mathcal{E}_1$ *in the canonical* $\mathcal{A}$*-pairing* $(\mathcal{E}, \mathcal{E}^*; \nu)$*. Moreover,*

$$\mathcal{E}^*/\mathcal{E}_1^{\perp} = \mathcal{E}_1^*$$

within $\mathcal{A}$-isomorphism.

Proof. That $\ker\phi = \mathcal{E}_1^{\perp}$ is clear. Now, let $\mathcal{E}_2$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$ complementing $\mathcal{E}_1$. It follows (cf. [6, p. 137, relation (6.21)] that

$$\mathcal{E}^* = \mathcal{E}_1^* \oplus \mathcal{E}_2^*,$$

so that if U is open in X and

$$\psi \equiv (\psi_V)_{U\supseteq V,\ open} \in \mathcal{E}_1^*(U) \quad \text{and} \quad \theta \equiv (\theta_V)_{U\supseteq V,\ open} \in \mathcal{E}_2^*(U),$$

then

$$\Omega \equiv \psi + \theta \in \mathcal{E}^*(U).$$

Consequently,

$$\phi_U(\Omega) = (\Omega_V|_{\mathcal{E}_1(V)})_{U\supseteq V,\ open} = \psi;$$

thus ϕ_U is surjective. Hence, applying Corollary 2.3 (1), we obtain an $\mathcal{A}$-isomorphism

$$\mathcal{E}^*/\mathcal{E}_1^{\perp} \simeq \mathcal{E}_1^*.$$

■

Now, let us introduce the notion of $\mathcal{A}$-*projection.*

Definition 2.5 Let $\mathcal{E}$ be an $\mathcal{A}$-module, $\mathcal{F}$ and $\mathcal{G}$ two *supplementary sub-$\mathcal{A}$-modules* of $\mathcal{E}$. An $\mathcal{A}$-*endomorphism* $\pi^{\mathcal{F}} \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E}) := \mathcal{E}nd_{\mathcal{A}}(\mathcal{E})$ such that

$$\pi^{\mathcal{F}}(\mathcal{E}) = \pi^{\mathcal{F}}(\mathcal{F}\oplus\mathcal{G}) = \mathcal{F}$$

is called an **$\mathcal{A}$-projection onto $\mathcal{F}$** (**parallel to** $\mathcal{G}$). In a similar way, one defines **$\mathcal{A}$-projections onto $\mathcal{G}$** (**parallel to** $\mathcal{F}$). An $\mathcal{A}$-projection $\pi^{\mathcal{F}}$ is called **orthogonal** if, for any section $s \in \mathcal{E}(U)$,

$$\pi_U^{\mathcal{F}}(s) \equiv \pi_U^{\mathcal{F}}(r+t) = r,$$

where $s = r + t$ with $r \in \mathcal{F}(U)$ and $t \in \mathcal{G}(U)$. Likewise, one defines the orthogonal $\mathcal{A}$-projection onto $\mathcal{G}$.

Proposition 2.6 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module, $\mathcal{E}_1$ and $\mathcal{E}_2$ two supplementary free sub-$\mathcal{A}$-modules of $\mathcal{E}$, $\pi_1 \equiv \pi^{\mathcal{E}_1}$, $\pi_2 \equiv \pi^{\mathcal{E}_2}$ the corresponding orthogonal $\mathcal{A}$-projections. Then,*

$$\mathcal{E}^* = \mathcal{E}_1^\perp \oplus \mathcal{E}_2^\perp,$$

and the orthogonal $\mathcal{A}$-projections $\pi_1' \equiv \pi^{\mathcal{E}_1^\perp}$, $\pi_2' \equiv \pi^{\mathcal{E}_2^\perp}$ associated with this direct decomposition are given by setting

$$\pi'_{1,U}(\alpha) := (\alpha_V \circ \pi_{2,V})_{U \supseteq V,\ open} \quad \text{and} \quad \pi'_{2,U}(\alpha) := (\alpha_V \circ \pi_{1,V})_{U \supseteq V,\ open}$$

for any $\alpha \equiv (\alpha_V)_{U \supseteq V,\ open} \in \mathcal{E}^*(U)$.

The proof of Proposition 2.6 requires some part of [10, p. 404, Theorem 2.2], which we restate here for easy referencing.

Theorem 2.7 *Let $(\mathcal{E}, \mathcal{E}^*; \mathcal{A})$ be the canonical free $\mathcal{A}$-pairing determined by $\mathcal{E}$. Then, for any open subset $U \subseteq X$,*

$$\text{rank } \mathcal{E}^*(U) = \text{rank } \mathcal{E}(U).$$

If $\phi \in \mathcal{E}^*(U)$ and $\phi_U(s) = 0$ for all $s \in \mathcal{E}(U)$, then $\phi = 0$; on the other hand, if $\phi(s) = 0$ for all $\phi \in \mathcal{E}^*(U)$, then $s = 0$.

Now, let us get to the proof of Proposition 2.6.

Proof. (**Proposition 2.6**) Fix an open set U in X. That $(\alpha_V \circ \pi_{2,V})_{U \supseteq V,\ open}$ and $(\alpha_V \circ \pi_{1,V})_{U \supseteq V,\ open}$ belong to $\mathcal{E}_1^\perp(U)$ and $\mathcal{E}_2^\perp(U)$, respectively, is obvious. For any open $V \subseteq U$, the relation

$$\alpha_V = \alpha_V \circ \pi_{1,V} + \alpha_V \circ \pi_{2,V}$$

shows that

$$\mathcal{E}^*(U) = \mathcal{E}_1^\perp(U) + \mathcal{E}_2^\perp(U).$$

Finally, suppose that there exists $\beta \equiv (\beta_V)_{U \supseteq V,\ open}$ in $\mathcal{E}_1^\perp(U) \cap \mathcal{E}_2^\perp(U)$; since $\beta_V(s) = 0$ for any open $V \subseteq U$ and any $s \in \mathcal{E}(V) = \mathcal{E}_1(V) \oplus \mathcal{E}_2(V)$, it follows that $\beta = 0$ (cf. Theorem 2.7). Thus,

$$\mathcal{E}^*(U) = \mathcal{E}_1^\perp(U) \oplus \mathcal{E}_2^\perp(U)$$

and hence

$$\mathcal{E}^* = \mathcal{E}_1^\perp \oplus \mathcal{E}_2^\perp$$

as claimed. ■

From [10], we also recall the following result a particular case of which will be needed below.

Theorem 2.8 *Let $(\mathcal{E}, \mathcal{F}; \mathcal{A})$ be an $\mathcal{A}$-pairing such that the right $\mathcal{A}$-kernel, i.e. $\mathcal{E}^\perp$, is identically 0. Moreover, let $\mathcal{E}_0$ and $\mathcal{F}_0$ be sub-$\mathcal{A}$-modules of $\mathcal{E}$ and $\mathcal{F}$, respectively. Then, there exist natural $\mathcal{A}$-isomorphisms **into**:*

$$\mathcal{E}/\mathcal{F}_0^\perp \longrightarrow \mathcal{F}_0^* \quad \textit{and} \quad \mathcal{E}_0^\perp \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*.$$

An interesting result may be derived from Theorem 2.8, viz.:

Theorem 2.9 *Let $\mathcal{E}$ be a free $\mathcal{A}$-module, $\mathcal{E}_1$ a free sub-$\mathcal{A}$-module of $\mathcal{E}$, and ϕ the canonical $\mathcal{A}$-morphism of $\mathcal{E}$ onto (the free sub-$\mathcal{A}$-module) $\mathcal{E}/\mathcal{E}_1$. The $\mathcal{A}$-morphism*

$$\Lambda : (\mathcal{E}/\mathcal{E}_1)^* \longrightarrow \mathcal{E}^*$$

such that, given any open subset $U \subseteq X$ and a section $\psi \in (\mathcal{E}/\mathcal{E}_1)^*(U) := \mathrm{H}om_{\mathcal{A}|_U}((\mathcal{E}/\mathcal{E}_1)|_U, \mathcal{A}|_U)$,

$$\Lambda_U(\psi) := (\psi_V \circ \phi_V)_{U \supseteq V,\ open}$$

is an $\mathcal{A}$-isomorphism of $(\mathcal{E}/\mathcal{E}_1)^*$ onto $\mathcal{E}_1^\perp$, where $\mathcal{E}_1^\perp$ is the $\mathcal{A}$-orthogonal of $\mathcal{E}_1$ in the canonical $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{E}^*; \mathcal{A})$.

Proof. It is clear that Λ is indeed an $\mathcal{A}$-morphism. Now, let us fix an open set U in X and let us consider a section $\psi \equiv (\psi_V)_{U \supseteq V,\ open} \in (\mathcal{E}/\mathcal{E}_1)^*(U)$. Then, $\Lambda_U(\psi) = 0$ if for any open V in U and $s \in \mathcal{E}(V)$,

$$\Lambda_U(\psi)(s) = 0.$$

But

$$\Lambda_U(\psi)(s) = (\psi_V \circ \phi_V)(s) = \psi_V(\phi_V(s)) = 0,$$

therefore, by Theorem 2.7,

$$\psi_V = 0.$$

It follows that

$$\ker \Lambda_U = 0,$$

and consequently

$$\ker \Lambda = 0;$$

in other words, Λ is injective.

Next, for every $\psi \equiv (\psi_V)_{U\supseteq V,\ open} \in (\mathcal{E}/\mathcal{E}_1)^*(U)$, where the open set U is fixed in X,

$$\Lambda_U(\psi)(s) = (\psi_V \circ \phi_V)(s) = 0,$$

where s is any element in $\mathcal{E}_1(V)$; that is

$$\Lambda_U(\psi) \in \mathcal{E}_1^{\perp}(U),$$

from which we deduce that

$$\mathrm{Im}\ \Lambda \subseteq \mathcal{E}_1^{\perp}.$$

Finally, still under the assumption that U is an open set fixed in X, let us consider, for every open $V \subseteq U$, the following commutative diagram

$$\begin{tikzcd} \mathcal{E}(V) \arrow[r, "\phi_V"] \arrow[dr, "\psi_V\circ\phi_V"'] & (\mathcal{E}/\mathcal{E}_1)(V) \arrow[d, dotted, "\psi_V"] \\ & \mathcal{A}(V) \end{tikzcd} .$$

The *universal property of quotient $\mathcal{A}$-modules* (cf. Corollary 2.2) shows that, given an element $\sigma_V \in \mathrm{H}om_{\mathcal{A}(V)}(\mathcal{E}(V), \mathcal{A}(V))$ such that $\ker \phi_V \subseteq \ker \sigma_V$, i.e., $\sigma_V(\mathcal{E}_1(V)) = 0$, there is a unique $\psi_V \in \mathrm{H}om_{\mathcal{A}(V)}((\mathcal{E}/\mathcal{E}_1)(V), \mathcal{A}(V))$ such that

$$\sigma_V = \psi_V \circ \phi_V.$$

It is clear that the family $\sigma \equiv (\sigma_V)_{U\supseteq V,\ open}$ is an $\mathcal{A}$-morphism $\mathcal{E}|_U \longrightarrow \mathcal{A}|_U$ satisfying the property that:

$$\sigma = \psi \circ \phi.$$

Thus, Λ is surjective and the proof is finished. ■

As a result, based essentially on everything above, we have

Theorem 2.10 *Let $(\mathcal{E}, \mathcal{E}^*; \mathcal{A})$ be the canonical free $\mathcal{A}$-pairing and $\mathcal{E}_1$ a free sub-$\mathcal{A}$-module of $\mathcal{E}$. Then,*

(1) $(\mathcal{E}_1^\perp)^\top = \mathcal{E}_1$ *within $\mathcal{A}$-isomorphism.*

(2) $\mathcal{E}_1$ *has finite rank if and only if $\mathcal{E}_1^\perp$ has finite corank in $\mathcal{E}^*$, and then one has*

$$\text{rank } \mathcal{E}_1 = \text{corank}_{\mathcal{E}^*} \mathcal{E}_1^\perp.$$

(3) $\mathcal{E}_1$ *has finite corank in $\mathcal{E}$ if and only if $\mathcal{E}_1^\perp$ has finite rank, and*

$$\text{corank}_{\mathcal{E}} \mathcal{E}_1 = \text{rank } \mathcal{E}_1^\perp.$$

Proof. *Assertion* (1). Let $\mathcal{E}_2$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$, complementing $\mathcal{E}_1$. By Proposition 2.6,

$$\mathcal{E}^* = \mathcal{E}_1^\perp \oplus \mathcal{E}_2^\perp.$$

Yet, we know that $\mathcal{E}_1 \subseteq (\mathcal{E}_1^\perp)^\top$. Now, consider a section $s \in (\mathcal{E}_1^\perp)^\top(U)$; there exist $r \in \mathcal{E}_1(U)$ and $t \in \mathcal{E}_2(U)$ such that $s = r + t$. The section t is orthogonal to $\mathcal{E}_2^\perp(U)$, and since r and s are orthogonal to $\mathcal{E}_1^\perp(U)$, we then have that t is orthogonal to $\mathcal{E}_1^\perp(U) \oplus \mathcal{E}_2^\perp(U) = \mathcal{E}^*(U)$. It follows from Theorem 2.7 that $t = 0$; thus $(\mathcal{E}_1^\perp)^\top(U) \subseteq \mathcal{E}_1(U)$, and hence $(\mathcal{E}_1^\perp)^\top \subseteq \mathcal{E}_1$.

Assertion (2). Since $\mathcal{E}_1$ is free, it follows that $\mathcal{E}_1^* \simeq \mathcal{E}_1$ (cf. [6, p. 298, (5.2)]). Thus, $\mathcal{E}_1$ has finite rank if and only if $\mathcal{E}_1^*$ has finite rank, and

$$\text{rank } \mathcal{E}_1^* = \text{rank } \mathcal{E}_1.$$

But, by Corollary 2.4, $\mathcal{E}^*/\mathcal{E}_1^\perp$ is $\mathcal{A}$-isomorphic to $\mathcal{E}_1^*$, therefore

$$\text{rank } \mathcal{E}_1 = \text{corank}_{\mathcal{E}^*} \mathcal{E}_1^\perp.$$

Assertion (3). Let $\mathcal{E}_2$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$ complementing $\mathcal{E}_1$, that is $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. But $\mathcal{E}/\mathcal{E}_1$ is $\mathcal{A}$-isomorphic to $\mathcal{E}_2$ (cf. [12]), therefore $\mathcal{E}/\mathcal{E}_1$ is free; consequently $\mathcal{E}/\mathcal{E}_1$ has finite rank if and only if $(\mathcal{E}/\mathcal{E}_1)^*$ has finite rank, and one has

$$(\mathcal{E}/\mathcal{E}_1)^* \simeq \mathcal{E}/\mathcal{E}_1$$

so that

$$\text{corank}_{\mathcal{E}}\mathcal{E}_1 = \text{rank } \mathcal{E}/\mathcal{E}_1 = \text{rank } (\mathcal{E}/\mathcal{E}_1)^*.$$

But, by Theorem 2.9, $(\mathcal{E}/\mathcal{E}_1)^* \simeq \mathcal{E}_1^\perp$ within $\mathcal{A}$-isomorphism, so the assertion is corroborated. ■

3 Biorthogonality in $\mathcal{A}$-modules

In the theory of vector spaces (cf. [3, p.67, Théorème 7.5]), *if E is K-vector space, E^* its dual, and F a subspace of E^*, then F is finite dimensional if and only if F' (F' is the subspace of E consisting of vectors that are orthogonal to F) is finite codimensional in E. Moreover, one has*

$$\dim\ F = codim_E F', \quad (F')^\perp = F.$$

In this section, we investigate this result (Theorem 3.4) and those of the previous sections in a more general setting, that is, $\mathcal{A}$-pairings defined by arbitrary $\mathcal{A}$-bilinear morphisms.

For the purpose of the main results of this section, we recall that given an $\mathcal{A}$-pairing $(\mathcal{E}, \mathcal{F}; \phi)$, the $\mathcal{A}$-bilinear morphism ϕ is said to be *non-degenerate* if $\mathcal{E}^{\perp_\phi} \equiv \mathcal{E}^\perp = \mathcal{F}^{\top\phi} \equiv \mathcal{F}^\top = 0$, and *degenerate* otherwise. The $\mathcal{A}$-morphism

$$\phi^R \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{E}^*) \equiv \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}))$$

such that, for any open subset $U \subseteq X$ and sections $t \in \mathcal{F}(U)$ and $s \in \mathcal{E}(V)$, where $V \subseteq U$ is open,

$$\phi^R_U(t)(s) \equiv (\phi^R)_U(t)(s) := \phi_V(s, t|_V)$$

is called the *right insertion $\mathcal{A}$-morphism* associated with ϕ. Similarly, for every open subset $U \subseteq X$ and sections $s \in \mathcal{E}(U)$ and $t \in \mathcal{F}(V)$, where V is open in U,

$$\phi^L_U(s)(t) \equiv (\phi^L)_U(s)(t) := \phi_V(s|_V, t)$$

defines an $\mathcal{A}$-morphism, denoted ϕ^L, of $\mathcal{E}$ into $\mathcal{F}^*$, i.e.,

$$\phi^L \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F}^*) \equiv \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{H}om_{\mathcal{A}}(\mathcal{F}, \mathcal{A})).$$

The $\mathcal{A}$-morphism ϕ^L is called the *left insertion $\mathcal{A}$-morphism* associated with ϕ.

Definition 3.1 Let $\mathcal{E}$ and $\mathcal{F}$ be free $\mathcal{A}$-modules. An $\mathcal{A}$-morphism $\phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ is called **free** if Im ϕ is a free sub-$\mathcal{A}$-module of $\mathcal{F}$. The rank of Im ϕ is called the **rank** of ϕ, and is denoted **rank** ϕ.

We may now state the counterpart of the *fundamental theorem* "*of the whole* [standard] *theory*"; see, e.g., [3, p. 54, Théorème 6.4].

Theorem 3.2 *Let $\phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ be a free $\mathcal{A}$-morphism mapping a free $\mathcal{A}$-module $\mathcal{E}$ into a free $\mathcal{A}$-module $\mathcal{F}$. Then, the rank of ϕ is finite if and only if the kernel of ϕ has finite corank in $\mathcal{E}$. Moreover, one has*

$$\mathrm{rank}\phi := \mathrm{rank}\ \mathrm{Im}\phi = \mathrm{corank}_{\mathcal{E}} \ker\phi.$$

Proof. Corollary 2.3(1) shows that the quotient free $\mathcal{A}$-module $\mathcal{E}/\ker\phi$ is $\mathcal{A}$-isomorphic to Im ϕ. ■

Corollary 3.3 *Let $\mathcal{A}$ be a PID algebra sheaf and $\mathcal{E}$, $\mathcal{F}$ free $\mathcal{A}$-modules. Then, if rank $\mathcal{E}$ is finite, every free $\mathcal{A}$-morphism $\phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$ has finite rank, and*

$$\mathrm{rank}(\phi) + \mathrm{rank}\ \ \ker(\phi) = \mathrm{rank}\ \mathcal{E}. \tag{3}$$

*The formula above is called the **rank formula.***

Proof. Indeed, given that every $\mathcal{A}(U)$, where U is open in X, is a PID algebra, it follows that $\ker(\phi_U)$ is a free sub-$\mathcal{A}(U)$- module of the free $\mathcal{A}(U)$-module $\mathcal{E}(U)$. By elementary module theory (see, for instance, [1, p. 173, Proposition 8.8] or [2, p. 105, Corollary 2]), we have

$$\mathrm{rank}\ \ \ker(\phi_U) + \mathrm{rank}\ \mathrm{Im}(\phi_U) = \mathrm{rank}\ \mathcal{E}(U).$$

Since for any subsets U and V of X, rank $\ker(\phi_U) =$ rank $\ker(\phi_V)$, it follows that $\ker(\phi)$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}$, and therefore

$$\text{rank } \ker(\phi) + \text{rank Im}(\phi) = \text{rank } \mathcal{E},$$

or

$$\text{rank } \ker(\phi) + \text{rank}(\phi) = \text{rank } \mathcal{E}.$$

■

The *rank formula* being a spin-off of the so-called *fundamental theorem* of the whole classical theory (: Linear Algebra) contributes of course to the study of the *geometry* of a vector space through the *projective geometry* of its subspaces; hence, it significance for the analogous situation in our generalized case (: $\mathcal{A}$-modules).

Theorem 3.4 *Let $(\mathcal{E}, \mathcal{E}^*; \mathcal{A})$ be the canonical free $\mathcal{A}$-pairing, and $\mathcal{F}$ a free sub-$\mathcal{A}$-module of $\mathcal{E}^*$. $\mathcal{F}$ has finite rank if and only if $\mathcal{F}^\top$ has finite corank in $\mathcal{E}$; moreover, one has*

$$\text{rank } \mathcal{F} = \text{corank}_{\mathcal{E}} \mathcal{F}^\top; \quad (\mathcal{F}^\top)^\perp = \mathcal{F}.$$

Proof. The case $\mathcal{F} = 0$ is trivial.

Suppose that $\mathcal{F}$ has finite rank; let U be an open subset of X, $(e_1^{U*}, \ldots, e_n^{U*})$ a canonical (local) gauge of $\mathcal{F}$ (cf. [6, p. 291, (3.11) along with p. 301, (5.17) and (5.18)]), and $\phi \in \mathrm{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}^n)$ be such that if $s \in \mathcal{E}(U)$,

$$\phi_U(s) := (e_1^{U*}(s), \ldots, e_n^{U*}(s)).$$

It is clear that ϕ is indeed an $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{A}^n$ whose kernel is $\mathcal{F}^\top$, which is a free sub-$\mathcal{A}$-module of $\mathcal{E}$ for the simple reason that *canonical free $\mathcal{A}$-pairings are orthogonally convenient,* see [15]. It is also clear that Im ϕ is $\mathcal{A}$-isomorphic to the free $\mathcal{A}$-module $\mathcal{A}^n$; thus, by Theorem 3.2, one has

$$\text{rank}(\phi) := \text{corank}_{\mathcal{E}} \mathcal{F}^\top = \text{rank } \mathcal{F}. \tag{4}$$

According to Theorem 2.10(3), $(\mathcal{F}^\top)^\perp$ has finite rank, and

$$\text{rank } (\mathcal{F}^\top)^\perp = \text{corank}_{\mathcal{E}} \mathcal{F}^\top. \tag{5}$$

Since $\mathcal{F}$ is contained in $(\mathcal{F}^\top)^\perp$, we deduce from (4) and (5) that

$$\mathcal{F} = (\mathcal{F}^\top)^\perp.$$

Conversely, suppose that $\mathcal{F}^\top$ has finite corank in $\mathcal{E}$; then $(\mathcal{F}^\top)^\perp$ has finite rank, and thus $\mathcal{F}$ as well, as $\mathcal{F}$ is contained in $(\mathcal{F}^\top)^\perp$. ■

It is clear that if the $\mathcal{A}$-bilinear morphism $\phi : \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{A}$ is non-degenerate, then both insertion $\mathcal{A}$-morphisms ϕ^R and ϕ^L are injective. Moreover, if $\mathcal{E}$ and $\mathcal{F}$ are free $\mathcal{A}$-modules of finite rank, then

$$\mathcal{E} = \mathcal{F}$$

within $\mathcal{A}$-isomorphism.

While the notion of orthogonality with respect to arbitrary $\mathcal{A}$-bilinear forms generalizes orthogonality in canonical $\mathcal{A}$-pairings, the former may relate with the latter through the following lemma.

Lemma 3.5 *Let $(\mathcal{E}, \mathcal{F}; \phi)$ be a free $\mathcal{A}$-pairing, $\mathcal{G}$ and $\mathcal{H}$ free sub-$\mathcal{A}$-modules of $\mathcal{E}$ and $\mathcal{F}$, respectively. Then,*

$$\mathcal{G}^{\perp_\phi} \simeq (\phi^L(\mathcal{G}))^\top, \tag{6}$$

and

$$\mathcal{H}^{\top^\phi} \simeq (\phi^R(\mathcal{H}))^\top. \tag{7}$$

Proof. Let U be an open subset of X. Since $\mathcal{G}$ is free, it is clear that for a section $t \in \mathcal{F}(U)$ to be in $\mathcal{G}^{\perp_\phi}$ it is necessary and sufficient that

$$\phi_U(\mathcal{G}(U), t) = 0.$$

But

$$(\phi^L_U(\mathcal{G}(U)))^\top = \{t \in \mathcal{F}(U) : \ \phi^L_U(\mathcal{G}(U))(t) := \phi_U(\mathcal{G}(U), t) = 0\},$$

therefore (6) holds as required.

In a similar way, one shows (7). ■

The case where $(\mathcal{E}, \mathcal{F}; \phi)$ is an *orthogonally convenient pairing* and ϕ is *non-degenerate* is interesting, yielding e.g. the following result.

Theorem 3.6 *Let $(\mathcal{E}, \mathcal{F}; \phi)$ be an orthogonally convenient pairing , where $\mathcal{E}$ and $\mathcal{F}$ are free $\mathcal{A}$-modules of finite rank. Then, the free quotient $\mathcal{A}$-modules $\mathcal{E}/\mathcal{F}^{\top_\phi}$ and $\mathcal{F}/\mathcal{E}^{\perp_\phi}$ are isomorphic; i.e. one has*

$$\mathcal{E}/\mathcal{F}^{\top_\phi} = \mathcal{F}/\mathcal{E}^{\perp_\phi}$$

within $\mathcal{A}$-isomorphism. Hence, they also have the same rank.

Proof. Since $(\mathcal{E}, \mathcal{F}; \phi)$ is *orthogonally convenient,* kernels $\mathcal{E}^{\perp_\phi}$ and $\mathcal{F}^{\top_\phi}$ are free sub-$\mathcal{A}$-modules of $\mathcal{F}$ and $\mathcal{E}$, respectively. By [12], it follows that the quotient $\mathcal{A}$-modules $\mathcal{E}/\mathcal{F}^{\top_\phi}$ and $\mathcal{F}/\mathcal{E}^{\perp_\phi}$ are free, and for any open subset U of X,

$$(\mathcal{E}/\mathcal{F}^{\top_\phi})(U) = \mathcal{E}(U)/\mathcal{F}^{\top_\phi}(U) = \mathcal{E}(U)/\mathcal{F}(U)^{\top_\phi}$$

and

$$(\mathcal{F}/\mathcal{E}^{\perp_\phi})(U) = \mathcal{F}(U)/\mathcal{E}^{\perp_\phi}(U) = \mathcal{F}(U)/\mathcal{E}(U)^{\perp_\phi}$$

within $\mathcal{A}(U)$-isomorphism. Clearly, for a fixed open $U \subseteq X$, if $s \in \mathcal{E}(U)$ and $t, t_1 \in \mathcal{F}(U)$ such that $t - t_1 \in \mathcal{E}^{\perp_\phi}(U)$, then

$$\phi_U(s, t) = \phi_U(s, t_1).$$

In the same vein, if $s = s_1 \mod \mathcal{F}^{\top_\phi}(U)$ and $t = t_1 \mod \mathcal{E}^{\perp_\phi}(U)$, then

$$\phi_U(s, t) = \phi_U(s_1, t_1).$$

Now, let us consider the $\mathcal{A}$-bilinear morphism

$$\overline{\phi} \equiv (\overline{\phi}_U)_{X\supseteq U,\ open} \equiv ((\overline{\phi})_U)_{X\supseteq U,\ open} : \mathcal{E}/\mathcal{F}^{\top_\phi} \oplus \mathcal{F}/\mathcal{E}^{\perp_\phi} \longrightarrow \mathcal{A},$$

induced by the $\mathcal{A}$-bilinear morphism ϕ, which is such that, for any open $U \subseteq X$ and sections $\overline{s} := \mathrm{cl}(s) \mod \mathcal{F}^{\top_\phi}(U)$, $\overline{t} := \mathrm{cl}(t) \mod \mathcal{E}^{\perp_\phi}(U)$ ($\mathrm{cl}(s)$ stand for the *equivalence class containing* s), one has

$$\overline{\phi}_U(\overline{s}, \overline{t}) := \phi_U(s, t).$$

It is clear that $\overline{\phi}_U(\overline{s},\overline{t}) = 0$ for any $\overline{s} \in (\mathcal{E}/\mathcal{F}^{\top\phi})(U) = \mathcal{E}(U)/\mathcal{F}^{\top\phi}(U)$ is equivalent to $\phi_U(s,t) = 0$ for any $s \in \mathcal{E}(U)$; therefore $t \in \mathcal{E}^{\perp\phi}(U) = 0$ and hence $\overline{t} = 0$. This implies that $(\mathcal{E}/\mathcal{F}^{\top\phi})^{\perp\phi} = 0$. Similarly, that $\overline{\phi}_U(\overline{s},\overline{t}) = 0$ for any $\overline{t} \in (\mathcal{F}/\mathcal{E}^{\perp\phi})(U) = \mathcal{F}(U)/\mathcal{E}^{\perp\phi}(U)$ is equivalent to $\overline{s} = 0$, from which we deduce that $(\mathcal{F}/\mathcal{E}^{\perp\phi})^{\top\phi} = 0$. Hence, $\overline{\phi}$ is non-degenerate; so

$$\mathcal{E}/\mathcal{F}^{\top\phi} = \mathcal{F}/\mathcal{E}^{\perp\phi}$$

within $\mathcal{A}$-isomorphism. ■

We also have

Theorem 3.7 *Let $\mathcal{A}$ be a PID algebra sheaf, $(\mathcal{E},\mathcal{F};\phi)$ an orthogonally convenient $\mathcal{A}$-pairing with $\mathcal{E}$, $\mathcal{F}$ of finite rank. Then,*

(1) *For every free sub-$\mathcal{A}$-modules $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, one has*

1.1) $\phi^L(\mathcal{G}) \simeq (\mathcal{G}^{\perp\phi})^{\perp}$ *and* $\phi^R(\mathcal{H}) \simeq (\mathcal{H}^{\top\phi})^{\perp}$.

1.2) $\operatorname{rank} \phi^L(\mathcal{G}) = \operatorname{corank}_{\mathcal{F}}\mathcal{G}^{\perp\phi}$ *and* $\operatorname{rank} \phi^R(\mathcal{H}) = \operatorname{corank}_{\mathcal{E}}\mathcal{H}^{\top\phi}$.

(2) *$\mathcal{A}$-morphisms ϕ^L and ϕ^R have the same rank:*

$$\operatorname{rank}(\phi^L) = \operatorname{rank}(\phi^R), \tag{8}$$

which is (cf. Definition 3.1) *the rank of ϕ.*

Proof. *Assertion* (1). Since $(\mathcal{E},\mathcal{F};\phi)$ is orthogonally convenient, the sub-$\mathcal{A}$-module $\mathcal{G}^{\perp\phi}$ is free, and thus

$$\mathcal{G}^{\perp\phi}(U) \simeq \mathcal{G}(U)^{\perp\phi}$$

for every open $U \subseteq X$. By Lemma 3.5,

$$\mathcal{G}^{\perp\phi} = (\phi^L(\mathcal{G}))^{\top}$$

within $\mathcal{A}$-isomorphism. Applying Theorem 3.4, and since rank $\mathcal{F}$ is finite, we have

$$(\mathcal{G}^{\perp\phi})^{\perp} = \phi^L\mathcal{G}$$

within $\mathcal{A}$-isomorphism. By the same theorem along with Theorem 2.10, it follows that

$$\text{rank } \mathcal{G}^{\perp_\phi} + \text{rank } \phi^L\mathcal{G} = \text{rank } \mathcal{F},$$

from which we deduce that

$$\text{rank } \phi^L\mathcal{G} = \text{corank}_{\mathcal{F}}\mathcal{G}^{\perp_\phi}.$$

In particular,

$$\text{rank}(\phi^L) = \text{corank}_{\mathcal{F}}\mathcal{E}^{\perp_\phi}. \tag{9}$$

In a similar way, one shows the claims related to the induced $\mathcal{A}$-morphism ϕ^R by using the fact that rank $\mathcal{E}$ is finite. The analog of (9) is

$$\text{rank}(\phi^L) = \text{corank}_{\mathcal{E}}\mathcal{F}^{\top^\phi}. \tag{10}$$

Assertion (2). That

$$\ker(\phi^L) \simeq \mathcal{E}^{\perp_\phi} \quad \text{and} \quad \ker(\phi^R) \simeq \mathcal{F}^{\top^\phi}$$

is immediate. Applying the *rank formula* (Corollary 3.3), we obtain

$$\text{rank}(\phi^R) := \text{rank } \phi^R(\mathcal{F}) = \text{rank } \mathcal{F} - \text{rank } \mathcal{E}^{\perp_\phi} = \text{corank}_{\mathcal{F}}\mathcal{E}^{\perp_\phi}, \tag{11}$$

and

$$\text{rank}(\phi^L) := \text{rank } \phi^L(\mathcal{E}) = \text{rank } \mathcal{E} - \text{rank } \mathcal{F}^{\top^\phi} = \text{corank}_{\mathcal{E}}\mathcal{F}^{\top^\phi}. \tag{12}$$

From (9), (10), (11) and (12), one gets (8). ■

Corollary 3.8 *Let $\mathcal{A}$ be a PID algebra sheaf and $(\mathcal{E}, \mathcal{F}; \phi)$ an orthogonally convenient $\mathcal{A}$-pairing with $\mathcal{E}$, $\mathcal{F}$ free $\mathcal{A}$-modules of finite rank. Then,*

(1) *For every free sub-$\mathcal{A}$-modules $\mathcal{G}$ and $\mathcal{H}$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, one has*

1.1) rank $\mathcal{G}^{\perp_\phi} \geq$ rank $\mathcal{F}$ − rank $\mathcal{G}$ *and* rank $\mathcal{H}^{\top^\phi} \geq$ rank $\mathcal{E}$ − rank $\mathcal{H}$

1.2) $(\mathcal{G}^{\perp_\phi})^{\top^\phi} \supseteq \mathcal{G}$ *and* $(\mathcal{H}^{\top^\phi})^{\perp_\phi} \supseteq \mathcal{H}$.

(2) *If* ϕ *is nondegenerate, then*

2.1) $\operatorname{rank} \mathcal{G}^{\perp_\phi} + \operatorname{rank} \mathcal{G} = \operatorname{rank} \mathcal{F} = \operatorname{rank} \mathcal{E} = \operatorname{rank} \mathcal{H}^{\top^\phi} + \operatorname{rank} \mathcal{H}$

2.2) $(\mathcal{G}^{\perp_\phi})^{\top^\phi} \simeq \mathcal{G}$ *and* $(\mathcal{H}^{\top^\phi})^{\perp_\phi} \simeq \mathcal{H}$.

Proof. *Assertion* (1). Theorem 3.7 shows that

$$\operatorname{rank} \phi^L(\mathcal{G}) = \operatorname{corank}_{\mathcal{F}} \mathcal{G}^{\perp_\phi} = \operatorname{rank} \mathcal{F} - \operatorname{rank} \mathcal{G}^{\perp_\phi}.$$

On the other hand, by virtue of Corollary 3.3, one has

$$\operatorname{rank} \phi^L(\mathcal{G}) = \operatorname{rank} \mathcal{G} - \operatorname{rank} (\ker \phi^L \cap \mathcal{G}).$$

It follows, in particular, that

$$\operatorname{rank} \mathcal{G} \geq \operatorname{rank} \phi^L(\mathcal{G}),$$

from which we have

$$\operatorname{rank} \mathcal{G}^{\perp_\phi} \geq \operatorname{rank} \mathcal{F} - \operatorname{rank} \mathcal{G}.$$

Likewise, one shows the second inequality of 1.1).

Assertion (2). If ϕ is nondegenerate, $\operatorname{rank} \mathcal{E} = \operatorname{rank} \mathcal{F}$; therefore ϕ^L is an $\mathcal{A}$-isomorphism of $\mathcal{E}$ onto $\mathcal{F}^*$. Thus, $\operatorname{rank} \phi^L(\mathcal{G}) = \operatorname{rank} \mathcal{G}$, and

$$\operatorname{rank} \mathcal{G}^{\perp_\phi} = \operatorname{rank} \mathcal{F} - \operatorname{rank} \mathcal{G}.$$

Likewise, one has

$$\operatorname{rank} \mathcal{H}^{\top^\phi} = \operatorname{rank} \mathcal{E} - \operatorname{rank} \mathcal{H}.$$

Applying relation 2.1) to the free sub-$\mathcal{A}$-modules $\mathcal{G}$ and $\mathcal{G}^{\perp_\phi}$ of $\mathcal{E}$ and $\mathcal{F}$, respectively, we see that

$$\operatorname{rank} (\mathcal{G}^{\perp_\phi})^{\top^\phi} = \operatorname{rank} \mathcal{G}.$$

Since $\mathcal{G}$ is contained in $(\mathcal{G}^{\perp_\phi})^{\top^\phi}$, it follows that

$$(\mathcal{G}^{\perp_\phi})^{\top^\phi} = \mathcal{G}$$

within $\mathcal{A}$-isomorphism. In a similar way, we show that $(\mathcal{H}^{\top^\phi})^{\perp_\phi} = \mathcal{H}$ within $\mathcal{A}$-isomorphism. ■

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